lsernau/grh-lean · edition grh-compact · commit 72fead1

The condensed Lean proof of RH and GRH: a reading

First handout, working draft of November 26, 2026

What this is, and what it is not. The Lean repository `lsernau/grh-lean` states the Riemann hypothesis and the generalized Riemann hypothesis for Dirichlet L-functions in Mathlib's terms and proves them. Its authors report that Lean's kernel accepts the proofs with the three standard axioms only. That is a *claim* about a formal text: no mathematician has read the argument through, and no paper accompanies it. This handout is a reading of the compact edition at the commit above, from the final theorems down, in the order of the argument: each unit of the formal text is rewritten as mathematics from structured proofs that a program compared with the Lean source line by line. Classical results that the formal text proves in its background library are quoted from the literature instead of read. Nothing here says that RH is proved.

State of the reading. The argument is divided into 24 units in eight chapters (2 621 declarations). Units read (note built, structured proofs checked): 24. Structured proofs checked by program, note in preparation: 0. Dossier written, not yet passing its check: 0. Being read: 0. Entries of the catalogue of quoted results so far: 2 944.

Readers: Claude agents (dossiers and structured proofs, checked by `rb.py check`). Writer: Claude (notes). Second reader: an independent AI reader. Method: Readback. Every count in this handout is computed from the workspace at build time.

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Overview of the argument

1. What is claimed

The formal text ends with five theorems. Let ζ be the Riemann zeta function, $M \geq 1$ an integer, χ a Dirichlet character modulo M, primitive or not, and $L(s, \chi)$ its L-function. The five theorems say:

(1) every $s \in \mathbb{C}$ with $\zeta(s) = 0$, $s \notin \{-2, -4, -6, \dots\}$ and $s \neq 1$ has $\operatorname{Re} s = \frac{1}{2}$; this is the proposition that Mathlib defines as the Riemann hypothesis;

(2) the same statement, written out in the statement file of the repository; its formal proof is theorem (1) itself, so the two propositions agree by definition;

(3) $L(s, \chi) \neq 0$ whenever $\operatorname{Re} s > \frac{1}{2}$, unless χ is principal and $s = 1$;

(4) every zero s of $L(\,\cdot\,, \chi)$ with $0 < \operatorname{Re} s < 1$ has $\operatorname{Re} s = \frac{1}{2}$;

(5) statement (4) for primitive χ. It is a special case of (4): its proof does not use primitivity.

In Mathlib ζ and $L(\,\cdot\,, \chi)$ are functions on all of $\mathbb{C}$: the meromorphic continuations, with a conventional finite value at the pole $s = 1$ of ζ and of $L(\,\cdot\,, \chi)$ for principal χ. This is why $s = 1$ is excluded by hand. A Dirichlet character modulo M is a multiplicative map $\mathbb{Z}/M \to \mathbb{C}$ that vanishes off the units, and the principal character is 1 on the units. An imprimitive character lacks finitely many Euler factors of the primitive one, which adds zeros on the line $\operatorname{Re} s = 0$ only; hence the open strip in (4), and (3)–(5) say nothing about $\operatorname{Re} s \leq 0$.

What one must trust. Three things, and no more: the statement file, which imports only Mathlib and defines the propositions (2)–(5); Mathlib's definitions of ζ, of Dirichlet characters and of $L(s, \chi)$, at the pinned version; and Lean's kernel with its three standard axioms (propositional extensionality, choice, soundness of quotients). Every other definition of the repository, and so every object of this overview, occurs only inside proofs.

Status. The authors report that all files compile with no proof omitted and that the kernel accepts the five theorems from these axioms; the record of checks at the end of this handout says who checked what, including how far our own replay has got. That is a fact about a formal text. No mathematician has read the argument through, and no paper accompanies it. The result is therefore a *claim*: this handout describes what the formal proof asserts and how it argues, and it does not say that the Riemann hypothesis is proved. The repository attributes the proof to Levent Alpöge, with Claude, and says that its Lean text was written by Claude; the compact edition read here rewrites an earlier, longer formal proof and follows its method.

Three grades of statement. *Kernel-checked, as reported*: every statement of the formal text, so every theorem in a box below. *Read*: rewritten by us as mathematics from structured proofs that a program compared with the source; the chapter list after this overview shows the state of each unit. *Quoted*: results that the formal text proves in its background library and that we state without reading their proofs: classical results, cited from the literature, and one result that the literature covers only in part (Section 6). In this overview every theorem in a box was compared, hypothesis by hypothesis, with its Lean statement, and the closing table gives the places; the objects of Setting 2 are described after their Lean definitions and the authors' comments, and the chapters define them. The deductions carried out in Sections 2 to 4 follow short Lean proofs that were read in the source. The paragraphs headed *Idea* are not readings of proofs: they report

the authors' comments and the statements of intermediate lemmas, and the chapters will replace them. The argument runs as follows; A, B, C are the three main theorems:

$$\text{A, B, C} \overset{\S4}{\Longrightarrow} \text{main inequality} \overset{\S3}{\Longrightarrow} \text{one-sided bound for primes} \overset{\S2}{\Longrightarrow} (3) \overset{\S2}{\Longrightarrow} (1)\text{–}(5).$$

2. From primes to zeros

The weighted prime sum. Let d be a prime and $M \geq 1$ an integer. Put $K = \mathbb{Q}(\sqrt{-d})$, and let χ_K be the quadratic character of the field $\mathbb{Z}/d$, a Dirichlet character modulo d (the Legendre symbol $(\,\cdot\,/d)$ when d is odd); for $d \equiv 3 \bmod 4$ it is the quadratic character of K. With Λ the von Mangoldt function, define for integers $\nu \geq 1$

$$\lambda_{d,M}(\nu) = \Lambda(\nu)\big(1 - \chi_K(\nu)\big) \ \text{ if } \nu \equiv 1 \bmod M, \qquad \lambda_{d,M}(\nu) = 0 \ \text{ otherwise.}$$

So $\lambda_{d,M} \geq 0$, and, when $d \equiv 3 \bmod 4$ (as for every d used below, all $\equiv 7 \bmod 8$), $\lambda_{d,M}$ vanishes at the powers of the primes that split in K. For real x the *remainder* is

$$R_{d,M}(x) = \sum_{1 \leq \nu \leq x} \lambda_{d,M}(\nu)\Big(\frac{1}{\nu} - \frac{1}{x}\Big) - \frac{\log x}{\varphi(M)} - \kappa_{d,M},$$
$$\kappa_{d,M} = \frac{1}{\varphi(M)} \operatorname{Re}\Big(\sum_{\xi}\Big(\frac{\tilde{L}'}{\tilde{L}}(1, \xi\chi_K) - \frac{\tilde{L}'}{\tilde{L}}(1,\xi)\Big) - 1\Big).$$

Here φ is Euler's function, ξ runs over the Dirichlet characters modulo M, $\xi\chi_K$ is the character modulo Md with the values $\xi(\nu)\chi_K(\nu)$, and $\tilde{L}(s,\xi)$ is the entire function $L(s,\xi)$, or $(s-1)L(s,\xi)$ if ξ is principal. The term $\log x/\varphi(M) + \kappa_{d,M}$ is the expected value of the sum. (A remark of ours: by the explicit formula, the generalized Riemann hypothesis gives $R_{d,M}(x) = O(x^{-1/2})$.)

Zeros as poles. Let the prime d be $\equiv 7 \bmod 8$ and prime to M. The Laplace transform $\Phi_{d,M}(s) = \int_0^\infty R_{d,M}(e^v)\, e^{(1-s)v}\, dv$, defined for $\operatorname{Re} s > 2$, continues to a meromorphic function on $\mathbb{C}$ that is analytic at every point of the closed half-plane $\operatorname{Re} s \geq 1$, the point $s = 1$ included: the constant $\kappa_{d,M}$ is the one that cancels the pole at $s = 1$. In $\operatorname{Re} s > 0$ its poles are at most simple, and for $\operatorname{Re}\rho > 0$, $\rho \neq 1$, the product $(s-\rho)\Phi_{d,M}(s)$ tends, as $s \to \rho$, to

$$\operatorname{res}_{d,M}(\rho) = \frac{1}{\varphi(M)\,\rho(1-\rho)} \sum_{\xi}\Big(\operatorname{ord}_\rho \tilde{L}(\,\cdot\,,\xi) - \operatorname{ord}_\rho \tilde{L}(\,\cdot\,,\xi\chi_K)\Big),$$

where $\operatorname{ord}_\rho f$ is the order of vanishing at ρ of an entire function $f \neq 0$; one puts $\operatorname{res}_{d,M}(1) = 0$, the limit at $\rho = 1$. A zero of some $L(\,\cdot\,,\xi)$, ξ modulo M, counts positively, and a zero of a twist $L(\,\cdot\,,\xi\chi_K)$ negatively.

One-sided criterion. Let $M \geq 1$ be an integer, c a unit of $\mathbb{Z}/M$ and $\beta > \frac{1}{2}$ a real number. Suppose that every prime $d \equiv c \bmod M$ satisfies $d \equiv 7 \bmod 8$, is prime to M, and has

$$R_{d,M}(x) \leq x^{\beta-1} \quad \text{for all sufficiently large real } x.$$

Then $\tilde{L}(\rho,\chi) \neq 0$ for every Dirichlet character χ modulo M and every $\rho \in \mathbb{C}$ with $\operatorname{Re}\rho > \beta$.

The proof has two inputs from the background library; one may assume $\operatorname{Re}\rho < 1$, the rest being classical. Given ρ, the first input provides one prime $d \equiv c \bmod M$ for which no $\tilde{L}(s,\xi\chi_K)$, ξ modulo M, vanishes in the box $\operatorname{Re} s \geq \beta$, $|\operatorname{Im} s| \leq |\operatorname{Im}\rho|$; it rests on estimates

of zero-density type for quadratic twists, and Section 6 says what of it the literature covers. This is why the bound is asked of every prime of the class: the prime depends on the height of ρ. For this d the numbers $\operatorname{res}_{d,M}(s)$ at real points $s > \beta$ are non-negative, and a constant multiple of $e^{(\beta-1)v}$ exceeds $R_{d,M}(e^v)$ for all $v \geq 0$. Now Landau's theorem applies: the Laplace transform of a non-negative function has a singularity at the real point of its abscissa of convergence. This is the integral form of his theorem on Dirichlet series with non-negative coefficients, used in a version that tolerates real poles of the favourable sign. So $\Phi_{d,M}$ has no pole in $\operatorname{Re} s > \beta$, and $\operatorname{res}_{d,M}(\rho) = 0$. At ρ the twisted terms vanish by the choice of d; the remaining terms are non-negative with sum zero, so each is zero.

One-sided bound. Let N_0 be a positive multiple of 18480, let d be a prime with $d \equiv N_0/2 - 1 \bmod N_0$, and let ε be real with $0 < \varepsilon < \frac{1}{100}$. Then

$$R_{d,N_0}(x) \leq x^{8\varepsilon - 1/2} \quad \text{for all sufficiently large real } x.$$

How the five theorems follow. Let χ be a character modulo M and suppose $L(\rho, \chi) = 0$ with $\operatorname{Re}\rho > \frac{1}{2}$, the pole excepted. Put $N_0 = 18480\,M$ and $c = N_0/2 - 1$. Then $c^2 \equiv 1 \bmod N_0$, so c is a unit, and every prime $d \equiv c \bmod N_0$ is prime to N_0 and is $\equiv 7 \bmod 8$, because 16 divides $18480 = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. The character modulo N_0 induced by χ still has $\tilde{L}(\rho, \cdot) = 0$, the two L-functions differing by Euler factors without zeros in $\operatorname{Re} s > 0$. Choose $\varepsilon = (\min(\operatorname{Re}\rho, 1) - \frac{1}{2})/51$, so that $\varepsilon < \frac{1}{100}$ and $\beta = 8\varepsilon + \frac{1}{2} < \operatorname{Re}\rho$. The one-sided bound, for every such d, is the hypothesis of the criterion with modulus N_0, and the criterion contradicts the zero. This is theorem (3). The functional equation, taken from Mathlib for primitive characters and extended to all characters through their missing Euler factors, pairs a zero s in the strip with the zero $1 - s$ of $L(\cdot, \bar{\chi})$, which gives (4) and (5); for $M = 1$, with the trivial zeros and the pole treated by the functional equation of ζ, it gives (1) and (2). Everything else in the formal text serves the one-sided bound.

3. The main inequality

Setting 1 (parameters and data). *Parameters* are a pair $P = (N_0, d)$: a positive multiple N_0 of 18480 and a prime $d \equiv N_0/2 - 1 \bmod N_0$. Then $d \equiv 7 \bmod 8$, and every prime dividing N_0 splits in $K = \mathbb{Q}(\sqrt{-d})$. *Admissible data* for P are a record D with these components: real scales x, x_1, b (stored by their logarithms); natural numbers μ, m, r: m a power of 2 with $\sqrt{x} \leq m \leq 2\sqrt{x}$, the exponent μ with $2^\mu = m^{22}$ and $\mu \leq 32 \log x$, and r a power of 3 with $x^5 \leq r \leq 3x^5$; a finite set $\operatorname{thick}(D)$ of infinite places of the field $E(D)$ defined below; two distinct primes g_0, g_1 and an integer π_g of K with $\pi_g \bar{\pi}_g = g_0 g_1$. The scales satisfy

$$(\log b)^{100} \leq x, \qquad x^{10^6} \leq x_1, \qquad x_1^{10^6} \leq b, \qquad (\log x_1)^6 \leq \log b,$$

and there are further conditions, stated in Chapter III: a lower bound for $\log\log b$ by a constant of P, congruences and sizes for g_0, g_1 (both between b^{10} and b^{11}), conditions on $\operatorname{thick}(D)$, and the existence of primes in certain short windows.

So $\log b \ll x \ll x_1 \ll b$, with large gaps, and $m \approx \sqrt{x}$ is the number of theta lifts: the saving $x^{-1/2}$ of the one-sided bound will be $1/m$. The following objects are functions of D.

The field and its places. $N_1(D)$ is $N_0\,2^\mu = N_0\,m^{22}$ times a power of 3 fixed by μ, and $E(D)$ is K with the roots of unity of order $N_1(D)$ adjoined: a CM field, abelian over $\mathbb{Q}$. Its infinite places form a principal homogeneous space under $\operatorname{Gal}(E(D)/K)$. Restriction to the roots of unity identifies this group with $(\mathbb{Z}/N_1(D))^\times$: it is onto because K is not contained in the field that these roots of unity generate over $\mathbb{Q}$, the prime d being ramified in K and prime to $N_1(D)$. A base place identifies the places with this group. So a place τ has a class

modulo $N_1(D)$, and $\nu \cdot \tau$ is defined for integers ν prime to $N_1(D)$. The set $\mathrm{tops}(D)$ consists of the places of class $\equiv 1 \bmod N_0$; it has $N_1(D)/N_0$ elements, and $f(D) = \varphi(N_0) \cdot \#\mathrm{tops}(D)$ is the number of all infinite places. The places of $\mathrm{thick}(D)$ have class $\equiv -1 \bmod N_0$.

Sizes. $k(D) = r + 1 - m$ and $t(D) = 5r$; $a(D)$ is the least multiple of $N_1(D)$ that is at least $x\,b$; and $\mathrm{budget}(D) = f(D)\,a(D)\,(\log b)^3$.

Prime sums. The *nominal weight* of an infinite place τ is $w_D(\tau) = a(D)/x$ if $\tau \in \mathrm{thick}(D)$ and $w_D(\tau) = a(D)/x_1$ otherwise. For real z and an infinite place τ, with $(\,\cdot\,)_+$ the positive part, put

$$\mathrm{Ask}_D(z,\tau) = \sum_{\nu=1}^{a(D)} \lambda_{d,1}(\nu)\Big(\frac{z}{\nu} - w_D((-\nu)\cdot\tau)\Big)_+,$$

$$\mathrm{rem}(D) = \sum_{\tau \in \mathrm{tops}(D)} \Big(\mathrm{Ask}_D(a(D),\tau) - a(D)\big(\log a(D) + \kappa_{d,1} - \bar{w}_D\big)\Big).$$

Here $(-\nu)\cdot\tau$ is the image of τ under the class of $-\nu$ modulo $N_1(D)$ when ν is prime to $N_1(D)$, and τ itself otherwise; the terms with ν not prime to $N_1(D)$ vanish in any case, since every prime dividing $N_1(D)$ splits in K (Setting 1), so that $\Lambda(\nu) = 0$ or $1 - \chi_K(\nu) = 0$. Further, $\bar{w}_D$ is the mean of $\log w_D(\tau)$ over all infinite places τ of $E(D)$. (The formal text writes $\kappa_{d,1}$ as -1 minus Euler's constant plus $\mathrm{Re}(L'/L)(1,\chi_K)$; a lemma of Chapter II identifies the two.) In Section 4 the number $\mathrm{Ask}_D(z,\tau)$ will be the divisibility that the primes not split in K give to a derivative of order z at τ, and $\mathrm{rem}(D)$ is what the tops receive beyond the expected value. Finally the *main amount* is $h_{\mathrm{main}}(D) = m\,k(D)\,\mathsf{U}(D)$, where $\mathsf{U}(D)$, the *cost of one direction*, is a sum over all infinite places τ: the size charged to a derivative of order about $w_D(\tau)$, a factorial against a power of a period, less the divisibility $\mathrm{Ask}_D(w_D(\tau),\tau)$ that such a derivative is sure to have. Chapter III has the formula. Its value plays no role here: it cancels.

The remainder is a combination of the remainders of Section 2. As τ runs over $\mathrm{tops}(D)$, the place $(-\nu)\cdot\tau$ meets every thick place exactly once if $\nu \equiv 1 \bmod N_0$, and none otherwise. This is where the progression $1 \bmod N_0$ enters, and it gives the identity

$$\mathrm{rem}(D) = a(D)\Big(\#\mathrm{thick}(D)\,\big(R_{d,N_0}(x) - R_{d,N_0}(x_1)\big) + \#\mathrm{tops}(D)\,R_{d,1}(x_1)\Big). \tag{1}$$

Main inequality. In Setting 1, for all parameters P and all admissible data D for P, with number of lifts m and third scale b,

$$m \cdot \mathrm{rem}(D) \;\le\; (\log b)^2 \cdot \mathrm{budget}(D) \;=\; f(D)\,a(D)\,(\log b)^5.$$

From the main inequality to the one-sided bound. Fix ε and let x be large. Data exist with first scale x, with $\log b = x^{\varepsilon}$, $\log x_1 = x^{\varepsilon/6}$, and with $\#\mathrm{thick}(D) \ge \#\mathrm{tops}(D)/\log b$ (Chapter III; the thick set is a construction of Behrend type). Solving (1) for $R_{d,N_0}(x)$ leaves the combination $R_{d,N_0}(x_1) - X\,R_{d,1}(x_1)$ with $X = \#\mathrm{tops}(D)/\#\mathrm{thick}(D) \le \log b$. No pointwise prime number theorem in progressions is used to bound it. Instead, since both remainders have transforms without poles on $\mathrm{Re}\,s \ge 1$, every long window contains a good point: for $v_0 = 10^6 \log x$ large and every $X \in [0, e^{v_0}]$ there is $v_1 \in [v_0, v_0^2]$ with $R_{d,N_0}(e^{v_1}) - X\,R_{d,1}(e^{v_1}) \le e^{-v_0}$. Replacing x_1 by e^{v_1} keeps the data admissible. For the new data D' the main inequality and (1) give

$$R_{d,N_0}(x) \;\le\; \frac{f(D')\,(\log b)^5}{m \cdot \#\mathrm{thick}(D')} + x^{-10^6} \;\le\; \frac{\varphi(N_0)\,(\log b)^6}{\sqrt{x}} + x^{-10^6} \;\le\; x^{8\varepsilon - 1/2}.$$

4. Theorems A, B and C

One covolume is bounded from both sides. Theorems A and B produce many automorphic forms whose theta lifts are short and almost integral; Theorem C says that integral forms cannot be that short. The two bounds differ by $m \cdot \mathrm{rem}(D)$.

Setting 2 (forms and lifts). Let P and D be as in Setting 1. Chapters III, V and VII attach to D the following objects.

Levels. The number $g(D) = g_0 g_1$; a prime $N(D)$ that splits completely in $E(D)$, with $m \log b \le \log N(D) \le 30\, m \log b$; an integer $\pi_N(D)$ of $E(D)$ whose product with its complex conjugate is $N(D)$. The *chosen primes* are the primes of $E(D)$ that divide $\bar{\pi}_g$, and $\mathsf{G}(D)$ is the product of the groups GL_r of their residue fields, a finite group.

Columns. $\mathrm{Col}(D)$ is a finite-dimensional complex space of algebraic modular forms on the unitary group of a definite hermitian space of rank $k(D)$ over $E(D)$, of a fixed weight, with level at the chosen primes and at the primes dividing $\pi_N(D)$. Its subspace $\mathrm{Col}^{\mathrm{ram}}(D)$ of *ramified* columns is the image of the product, over the primes $\mathfrak{p}$ dividing $\pi_N(D)$, of the operators $1-$(translation by a central element at $\mathfrak{p}$). For a prime ℓ and a ring homomorphism $j\colon \mathbb{C} \to \mathbb{C}_\ell$, let O_j be the ring of the complex numbers whose image under j has absolute value at most 1. The values of these forms are vectors of a weight representation, and integrality is measured in a lattice that moves with the point: a form is *j-integral* if at every finite adelic point its value lies in the weight lattice at that point, that is, after j is applied to its coordinates and then the local weight action of the inverse of the point, all coordinates have absolute value at most 1. A column is *integral* if it is j-integral for every prime ℓ and every j.

Rows. $\mathrm{Row}(D)$ is a finite-dimensional complex space of cusp forms on the unitary group of an indefinite hermitian space of rank r over $E(D)$: of a fixed discrete series type at infinity, of principal level at the chosen primes and with no level at $N(D)$, cut out by a fixed idempotent of the group algebra of $\mathsf{G}(D)$.

Poincaré series. For $\gamma \in \mathsf{G}(D)$, the linear map $\mathcal{P}_D(\gamma)\colon \mathrm{Col}^{\mathrm{ram}}(D) \to \mathrm{Row}(D)$ multiplies a ramified column by a fixed non-zero constant, forms its Poincaré series on the indefinite group, translates it by γ and applies $\#\mathsf{G}(D)$ times the idempotent.

Twists and lifts. An *m-fold twist* of D is a continuous character ϑ of the norm-one adeles of $E(D)$, trivial on the norm-one elements of $E(D)$ and at the infinite places, at most tamely ramified, such that the set $Q(\vartheta)$ of primes where it ramifies contains the complex conjugate of none of its members, consists of primes of norm prime to $g(D)N(D)$, and is also the set of ramified primes of each of $\vartheta^2, \dots, \vartheta^m$. The trivial character 1 is one. For an m-fold twist ϑ, the map $\Theta_{D,\vartheta}\colon \mathrm{Row}(D) \to \mathcal{A}_{D,\vartheta}$ is the composite of m theta lifts, the i-th formed with ϑ^i; it is injective, and $\mathcal{A}_{D,\vartheta}$ is a space of algebraic modular forms on a definite unitary group of rank $r + m\, t(D)$. For $F, F' \in \mathcal{A}_{D,\vartheta}$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$, $\langle F, F' \rangle_\sigma$ is the Petersson product, a finite sum over classes, of the forms whose values are the σ-conjugates of those of F and F'. Put $\|F\|_\sigma^2 = \langle F, F \rangle_\sigma$ and $\mathrm{covol}_\sigma^2(F_1, \dots, F_n) = |\det(\langle F_i, F_{i'} \rangle_\sigma)_{i,i'}|$. For forms in $\mathcal{A}_{D,\vartheta}$, j-integrality is defined as for columns, in the weight lattice at each point.

Index. Rows $y_1, \dots, y_n$ have *D-index at most δ* if there are real numbers η_ℓ, $\ell \in \mathbb{N}$, with $\sum_{\ell \in J} \eta_\ell \le \delta$ for every finite $J \subset \mathbb{N}$, such that for every prime ℓ, every $j\colon \mathbb{C} \to \mathbb{C}_\ell$ and every m-fold twist ϑ of D with ℓ prime to the norms of the primes in $Q(\vartheta)$ there is an $n \times n$ matrix u over O_j with $|j(\det u)|_\ell \ge e^{-\eta_\ell}$ for which $\Theta_{D,\vartheta}(\sum_{i'} u_{ii'}\, y_{i'})$ is j-integral for $i = 1, \dots, n$.

Theorem A (short forms). In Settings 1 and 2 there is a rational number $q > 0$ with the following two properties.

(a) Size. There is a set S of integral ramified columns that spans $\mathrm{Col}^{\mathrm{ram}}(D)$ over $\mathbb{C}$ such that, for all $\psi \in S$, all $\gamma \in \mathsf{G}(D)$ and all $\sigma \in \mathrm{Aut}(\mathbb{C})$,

$$\big\|\Theta_{D,1}(q\,\mathcal{P}_D(\gamma)\psi)\big\|_\sigma^2 \;\le\; \exp\big(h_{\mathrm{main}}(D) - m\cdot\mathrm{rem}(D) + \log b\cdot\mathrm{budget}(D)\big).$$

(b) Index. For every finite set $\mathcal{T} \subseteq \mathsf{G}(D)$, every $n \ge 0$ and all rows $y_1, \dots, y_n$, each of the form $q\,\mathcal{P}_D(\gamma)\psi$ with $\gamma \in \mathcal{T}$ and ψ an integral ramified column, the family $y_1, \dots, y_n$ has D-index at most $\#\mathcal{T}\cdot\dim\mathrm{Col}(D)\cdot\mathrm{budget}(D)$.

Idea (the authors' comments; Chapter V). A seesaw identity moves the problem to definite groups: for every twist, the chain of lifts of the Poincaré series of a ramified column is a chain of m theta lifts of the column itself between definite spaces, restricted at the end. There a lift is a finite sum of Taylor coefficients at the origin of theta functions with complex multiplication by K, of order $a(D)$ at the tops. Their size at every conjugate comes from Stirling's formula and a period. Their divisibility at a prime p not split in K comes from the formal group of the CM curve, of height two at p: a derivative of order z against a partner place of weight w gains about $(z/p - w)_+$ powers of p. Summed over p with weights $\log p$ this is Ask_D, which is why $\lambda_{d,M}$ appears. The rational q divides the gain out, a power of the prime $N(D)$, chosen in a short window, pays the excess of size over expected divisibility, and $h_{\mathrm{main}}(D) - m\cdot\mathrm{rem}(D)$ is left. What some primes lose is regained on the image of scaled Hecke operators, at the price of the D-index in (b).

Theorem B (enough forms). In Settings 1 and 2, $\dim\mathrm{Col}(D) > 0$ and $\dim\mathrm{Row}(D) > 0$, and there is a finite set $\mathcal{T} \subseteq \mathsf{G}(D)$ with

$$\#\mathcal{T}\cdot\dim\mathrm{Col}(D) \;\le\; 5\,\dim\mathrm{Row}(D)\cdot\log(1+b^2) \qquad \text{and}$$
$$\dim\sum_{\gamma\in\mathcal{T}}\mathcal{P}_D(\gamma)(\mathrm{Col}^{\mathrm{ram}}(D)) \;\ge\; \Big(1-\frac{4}{b^2}\Big)\dim\mathrm{Row}(D).$$

Idea (Chapter VI). The translates are spread: for every subspace W of $\mathrm{Row}(D)$ some γ pulls W back to at most the share $\frac{1}{2}(1 + \dim W/\dim\mathrm{Row}(D)) + b^{-2}$ of the columns. This follows from a second moment of the Poincaré series over $\mathsf{G}(D)$, of Petersson type: its main term is dominated, with constant one, by a positive kernel of Bessel integrals at the real places (the place $A = \mathbb{C}$ of Chapter VI, which the notes of that chapter call the complex place) and of local Bessel integrals at a large finite set of finite places, followed by a sieve; the chosen primes enter through the lines that the level group must fix. A variance inequality turns the moment into the bound. A greedy choice of translates then gains at each step in proportion to the dimension still missing, whence the logarithm.

Theorem C (integral forms are long). In Settings 1 and 2, let $n \ge 0$ be an integer, let $y_1, \dots, y_n$ be rows in $\mathrm{Row}(D)$ that are linearly independent over $\mathbb{C}$, and let δ and V be real numbers. Suppose that $n \ge (1 - 4/b^2)\dim\mathrm{Row}(D)$, that $y_1, \dots, y_n$ have D-index at most δ, and that $\mathrm{covol}_\sigma^2(\Theta_{D,1}y_1, \dots, \Theta_{D,1}y_n) \le e^V$ for all σ in $\mathrm{Aut}(\mathbb{C})$. Then

$$\dim\mathrm{Row}(D)\cdot\big(h_{\mathrm{main}}(D) - 3\log b\cdot\mathrm{budget}(D)\big) \;\le\; V + 2\delta.$$

Idea (Chapters IV, VII and VIII). The tool is a height for families $Z = (Z_{\ell,\iota})$ of complex numbers indexed by the pairs of a prime ℓ and a field isomorphism $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$. Such a family

has *height at most* h if there are $H \in \mathbb{C}^\times$, $\alpha \in \mathbb{R}$ and real numbers η_ℓ with $|\sigma(H)| \le e^\alpha$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, $|\iota(H\, Z_{\ell,\iota})|_\ell \le e^{\eta_\ell}$ for all (ℓ, ι), and such that α plus the sum of the η_ℓ over the primes ℓ below any sufficiently large bound is at most h. Heights add in products, and a height bound for the constant family 1 is non-negative: this is the product formula, applied to the norm of H. Let $\mathfrak{z}_D(\vartheta) = \det(\Theta^*_{D,\vartheta}\Theta_{D,\vartheta})$. For a fit family of twists θ (Section 5), two bounds hold. First, $\mathfrak{z}_D(\theta_{\ell,\iota})/\mathfrak{z}_D(1)$ has height at most $V + 2\delta + b \dim \mathrm{Row}(D)$: the family is completed to a basis by crude rows, V bounds the Gram determinant at the trivial twist at every conjugate, and the D-index bounds the denominators at each ℓ. Second, $\mathfrak{z}_D(1)^2/\mathfrak{z}_D(\theta_{\ell,\iota})^2$ has height at most $2 \dim \mathrm{Row}(D)\,(2 \log b \cdot \mathrm{budget}(D) - h_{\mathrm{main}}(D))$: the volume of the chain is a constant times the volumes of the m single lifts, a single lift is governed by Hecke L-values, and each ratio of an untwisted to a twisted value repays one direction, $\mathsf{U}(D)$. The square of the first family times the second is the constant family 1, so twice the first bound plus the second is non-negative. This is Theorem C, since $b \le \log b \cdot \mathrm{budget}(D)$.

A, B and C give the main inequality. Take q and S from A and $\mathcal{T}$ from B. The rows $q\,\mathcal{P}_D(\gamma)\psi$, $\gamma \in \mathcal{T}$, $\psi \in S$, span $\sum_{\gamma\in\mathcal{T}} \mathcal{P}_D(\gamma)(\mathrm{Col}^{\mathrm{ram}}(D))$; choose a basis $y_1, \dots, y_n$ of this space among them, so that $n \ge (1-4/b^2) \dim \mathrm{Row}(D)$. By Hadamard's inequality and A(a), $\mathrm{covol}^2_\sigma(\Theta_{D,1}y_1, \dots, \Theta_{D,1}y_n) \le e^V$ with $V = n\,(h_{\mathrm{main}}(D) - m\,\mathrm{rem}(D) + \log b \cdot \mathrm{budget}(D))$. By A(b) and B their D-index is at most $\delta = 5 \dim \mathrm{Row}(D) \log(1+b^2)\,\mathrm{budget}(D)$. Theorem C compares the two. A crude bound, $|h_{\mathrm{main}}(D)| + m\,|\mathrm{rem}(D)| \le b^2 x^{99}$, shows that replacing n by $\dim \mathrm{Row}(D)$ costs at most $4x^{99}$ for each row. Dividing by $\dim \mathrm{Row}(D)$, the main amount cancels:

$$m \cdot \mathrm{rem}(D) \;\le\; (4 \log b + 10 \log(1+b^2))\,\mathrm{budget}(D) + 4x^{99} \;\le\; (\log b)^2\,\mathrm{budget}(D).$$

5. The twists

Theorem C compares the lifts at the trivial twist with the lifts at twists where they have few congruences with other forms. The formal proof does not find one such twist for all primes: it finds one for each embedding, which is what the height of Section 4 allows.

Twists. In Settings 1 and 2, for all parameters P and all admissible data D for P there is a family $\theta = (\theta_{\ell,\iota})$ of m-fold twists of D, one for each prime ℓ and each field isomorphism $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$, that is *fit for* D.

Fitness is defined in Chapter VIII; it has three clauses. The prime ℓ is prime to the norms of the primes in $Q(\theta_{\ell,\iota})$. For each i with $1 \le i \le m$ (Chapter VIII, like the formal text, calls this index j), the determinants that measure the ℓ-adic denominators of the orthogonal projection of integral forms to the space Z_i (the image, in the forms of the auxiliary pair, of the adjoint of the i-th lift at $\theta^i_{\ell,\iota}$ restricted to good forms), and to certain lines of lifts from U(1), are at most $e^{r'\eta_\ell}$ for each dimension, where r' is the rank of the auxiliary space ($r(D)$ for the good lifts, 1 for the lines), with real numbers η_ℓ whose sum over any finite set of primes is at most $\mathrm{budget}(D)/(m\,k(D))$. And for finitely many Hecke characters of U(1), the inverse ratios of twisted to untwisted L-values have height at most x^{999}.

Idea (the authors' comments; Chapter VIII). A denominator of the projection to theta lifts is a congruence between a lift and a form that is not a lift. By Ribet's method, in a form with congruent blocks and in any rank, such a congruence produces classes in Selmer groups of the Galois representations attached to the forms (attached in the weak sense of Chapter VII); these representations are an irreducible block plus a string of powers of the cyclotomic character. The even classes of these groups, where in the authors' words

Vandiver's conjecture lives, are never bounded. Twisting by an anticyclotomic character of ℓ-power order, ramified at new primes where a tie between inertia and Frobenius is imposed (an intermediate device, which the final twist releases), makes all the relevant Selmer groups short for the first m powers at once; Taylor–Wiles primes in a triangular arrangement remove the even classes, the odd parts of Tate–Shafarevich groups of abelian CM fields are bounded by one explicit integer (for the untwisted character the module is the unramified part, with no oddness condition), and the conditions at ℓ cost a loss that is controlled by lifting crystalline extension classes in large weight.

6. What is quoted

The repository separates the argument from a background library, which proves from Mathlib the known results that the argument needs, often in a weaker form that suffices. This write-up reads the argument and quotes the background. That is a choice of exposition, not an extra hypothesis: the formal proof contains proofs of all of it. The quoted results come from *analytic number theory* (Landau's theorem, the zero-free box for quadratic twists that is discussed below, a weak form of the Siegel–Walfisz theorem; from Mathlib, the functional equation and the non-vanishing on $\operatorname{Re} s = 1$); from *class field theory and Galois cohomology* (Artin reciprocity, Chebotarev's theorem, local Tate duality, the Greenberg–Wiles formula); from *complex multiplication* (theta functions of CM curves and their formal groups, Kronecker's limit formula, the Chowla–Selberg formula); from *automorphic forms* (reduction theory, the Weil representation and theta lifts, the Siegel–Weil and Rallis inner product formulas, discrete series); and from *Galois representations* (those of eigenforms on unitary groups, through Shimura varieties; p-adic Hodge theory with the Bloch–Kato conditions and Kisin modules; bounds of Kolyvagin–Rubin type). The catalogue at the end of this handout lists each result with a precise statement, its reference where the place is confirmed, and the relation of the formal statement to the published one: the theorem itself, a special case, or a weaker form.

Some quoted results stand outside these three relations: no published source in the form needed has been identified, so they are quoted from the formal library, with their statements in the chapters, and not from the literature. The first is the zero-free box of Section 2, one prime d in a class whose twists $\tilde{L}(s, \xi\chi_K)$ have no zero in a box for all characters ξ modulo M at once. The formal library proves it in two steps. The first is an estimate of zero-density type for each fixed character ξ, real or not: the number of the primes $d \le x$ with $d \equiv 7 \bmod 8$, prime to M, for which $\tilde{L}(\,\cdot\,, \xi\chi_K)$, $K = \mathbb{Q}(\sqrt{-d})$, has a zero in a given box to the right of the critical line is at most a constant times a power of x with exponent less than 1. The second adds these bounds over the characters ξ and sets them against the divergence of the sum of $1/p$ over the primes p of a residue class. The source that the formal text names, a zero-density theorem of Jutila, concerns real characters. It does not give the first step for a non-real ξ, whose twists are not real, and no published place for that step has been confirmed; its statement is in Chapter II. The others are in Chapter VIII: a bound for the odd parts of the Tate–Shafarevich groups of an abelian CM field by one explicit integer, uniform in the prime, the twist and the precision (Greither's theorem is an identity of characteristic ideals, and how it would give such a bound is not established); a finite set of Galois elements that detects the relevant cocycles; and the relations at split primes, with a lower bound for the number of relation families, together with the count of trivialised cocycles that supplies its hypothesis.

7. A map

ch.	what it proves	units (folders)
I	The five theorems from (3); the main inequality from A, B, C; the one-sided bound from the main inequality	statement file, main file, head file of the argument
II	The remainder and its residues; the one-sided criterion; a good point in every window; identity (1); the last steps of A, B and C	Endgame
III	Parameters and data; the field $E(D)$ and its places; comparison of sizes; existence of data at every large x	Data
IV	Heights and the product formula; few translates nearly span; the height of one Gram determinant	Heights
V	Theorem A: the seesaw identity, divisibility of derivatives of theta functions, sizes, the level	Seesaw, Curve, Columns, TwistedHecke, Core
VI	Theorem B: the second moment of Poincaré series, Bessel functions at real places	Filling, RealPlace, DiscreteSeries
VII	Theorem C: volume of the chain as a product of lift volumes, the exchange identity, heights of ratios of L-values	Rows, Exchange, Trace, Forms, SiegelWeil, Weil, GaloisRep; DiscreteSeries (Theorems 2, 3, 6)
VIII	The twists: Ribet's method, Selmer groups under twisting, Taylor–Wiles primes, crystalline lifts	Selmer, Patching, Lift

Where each statement lives. Paths are relative to the source folder of the compact edition at commit `72fead1`; the numbers are the lines of the declaration. The hypotheses of Theorem C stand in the `variable` line 13–14 of its file.

statement	Lean name	file, lines

Theorems (1), (2), (3), (4), (5)	`riemannHypothesis`
	`riemannZeta_eq_zero_re`
	`LFunction_ne_zero_of_half_lt_re`
	`LFunction_eq_zero_re`
	`LFunction_eq_zero_re_of_isPrimitive`
		`Main.lean`, 4, 6, 8, 10, 12–13

The propositions (2) to (5)	the definitions of the statement file	`Statements.lean`, 3–16

(3) inside the argument; from (3) to the line, for characters and for ζ	`New.grh`
	`New.zeros_on_line`
	`New.zeta_zeros_on_line`
		`New/Main.lean`, 174–190, 194–196, 200–204

The weight $\lambda_{d,M}$; the remainder $R_{d,M}$; zeros as poles	`lambdaClass`
	`primeError`
	`isPolar_primeError`
		`New/Endgame/PrimeError.lean`, 25, 38–39, 69–111

One-sided criterion; a good point in every window	`entireL_ne_zero_of_primeError_le` `IsPolar.exists_window_sub_mul_le`	`New/Endgame/OneSidedCriterion.lean`, 28–36, 73–99
Landau's theorem (quoted)	`IsPolar.landau`	`Lib/Analytic/Landau/PolarLandau.lean`, 88–136
Zero-free twists for one prime (quoted)	`exists_entireL_quadTwist_ne_zero`	`Lib/Analytic/QuadraticTwist/ZeroFreeTwist.lean`, 193–206
One-sided bound; main inequality	`eventually_primeError_le_rpow` `star_inequality`	`New/Main.lean`, 149–167, 121–145

Setting 1: parameters; data; their existence	`Parameters`
	`Data`
	`exists_data`
		`New/Data/Different.lean`, 23–28
		`New/Data/AdmissibleData.lean`, 22–45
		`New/Data/DataExist.lean`, 49–108

The main amount; the remainder; the crude bound; identity (1)	`Data.mainHeight`
	`Data.remainder`
	`Data.abs_mainHeight_add_remainder_le`
	`Data.remainder_eq`
		`New/Endgame/ShortAndEnough.lean`, 32, 42, 74–82, 86–112

statement	Lean name	file, lines

Setting 2: rows; columns; the chain of lifts	`Data.rows`
	`Data.Column`
	`Data.0`
		`New/Trace/GoodIdempotent.lean`, 118
		`New/Columns/ColumnDefs.lean`, 71
		`New/Rows/RowVolume.lean`, 27

Poincaré maps; integral ramified columns; index	`Data.projectedPoincare`
	`Data.integralRamified`
	`Data.IndexLe`
		`New/Endgame/ShortAndEnough.lean`, 120–121, 125–126, 130–132

Theorems A and B	`Data.short_forms` `Data.enough_forms`	`New/Endgame/ShortAndEnough.lean`, 138–168, 173–178
Theorem C; the family given by A and B	`Data.integral_forms_long` `Data.exists_family_covolumeSq_le`	`New/Endgame/LongForms.lean`, 63–71, 78–92
Height; the product formula	`HeightLe` `HeightLe.nonneg`	`New/Heights/Heights.lean`, 21–23, 65–77
Fit families of twists; their existence	`Data.IsFitFamily` `exists_isFitFamily`	`New/Selmer/FitTwists.lean`, 256–263 `New/Selmer/FitTwistsExist.lean`, 24–65

Chapter I

The theorems, and what one must trust

- **The five final theorems: the Riemann hypothesis and the generalised Riemann hypothesis for Dirichlet L-functions** (unit 155)
  read: note built, structured proofs checked. 5 declarations.

- **The statements: the Riemann hypothesis and the generalised Riemann hypothesis for Dirichlet L-functions, in three forms** (unit 153)
  read: note built, structured proofs checked. 4 declarations.

- **The inequality ($\star$), the one-sided bound for the remainder of the weighted prime sum, the generalised Riemann hypothesis, and the passage to the critical line** (unit 154)
  read: note built, structured proofs checked. 5 declarations, 62 proof lines.

READBACK NOTE 3

The five final theorems, and what one must trust

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The root of the artifact is a file of five theorems, each proved in one line: the Riemann hypothesis, in the formulation of the library and in the artifact's own; the non-vanishing of every Dirichlet L-function on the half-plane $\operatorname{Re} s > \frac{1}{2}$; and the statement that the zeros of these functions in the critical strip lie on the line $\operatorname{Re} s = \frac{1}{2}$, for all characters and for primitive ones. All five follow from one theorem, the zero-free half-plane, by the functional equations. This note states the five theorems, shows how each is obtained, and then says what one has to trust in order to believe them, and what that trust does not give.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, whose final theorems are stated below. That headline is a claim. The authors report that every file compiles with Lean v4.35.0 against a pinned version of the library Mathlib and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note treats one unit of the artifact, its root: all 5 of its 5 declarations. Their proofs are given in full; they are deductions of one line from three results of another unit, which are stated as Inputs 1 to 3 and not proved here. Every statement was compared with its Lean text. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 12 declarations.

1 The five theorems

Throughout, ζ is the Riemann zeta function and $L(s, \chi)$ the L-function of a Dirichlet character χ, as the mathematical library of Lean defines them (§2).

Theorem 1 (the Riemann hypothesis, in the library's formulation). *For every complex number s with $\zeta(s) = 0$, such that there is no natural number n with $s = -2(n+1)$, and with $s \neq 1$:* $\operatorname{Re} s = \frac{1}{2}$.

Theorem 2 (the Riemann hypothesis, in the artifact's formulation). *For every complex number s: if $\zeta(s) = 0$, if there is no natural number n with $s = -2(n+1)$, and if $s \neq 1$, then* $\operatorname{Re} s = \frac{1}{2}$.

LEAN `riemannHypothesis` · `riemannZeta_eq_zero_re`

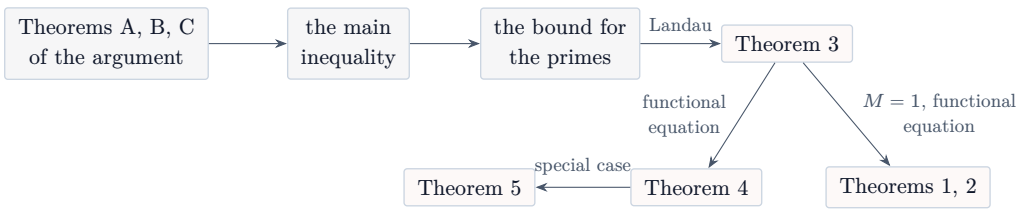
Theorem 3 (the zero-free half-plane). *For every integer $M \geq 1$, every Dirichlet character χ modulo M and every complex number s with $\operatorname{Re} s > \frac{1}{2}$: if χ is not the principal character modulo M or $s \neq 1$, then $L(s, \chi) \neq 0$.*

Theorem 4 (zeros in the critical strip). *For every integer $M \geq 1$, every Dirichlet character χ modulo M and every complex number s: if $0 < \operatorname{Re} s$, $\operatorname{Re} s < 1$ and $L(s, \chi) = 0$, then $\operatorname{Re} s = \frac{1}{2}$.*

Theorem 5 (zeros in the critical strip, primitive characters). *For every integer $M \geq 1$ and every Dirichlet character χ modulo M that is primitive, and for every complex number s: if $0 < \operatorname{Re} s$, $\operatorname{Re} s < 1$ and $L(s, \chi) = 0$, then $\operatorname{Re} s = \frac{1}{2}$.*

LEAN `LFunction_ne_zero_of_half_lt_re` · `LFunction_eq_zero_re` · `LFunction_eq_zero_re_of_isPrimitive`

Theorems 1 and 2 say the same thing: the first is stated with the proposition that the library itself calls the Riemann hypothesis, the second with the artifact's copy of it. Theorem 4 is the generalised Riemann hypothesis for Dirichlet L-functions in its usual form. Theorem 3 is the form in which the artifact proves it; it carries the whole argument, and the other four are deduced from it in §3.



The five theorems and what lies behind them. The grey boxes and the two arrows marked with functional equations are the subject of the note *The main inequality, the bound for the primes, and the zero-free half-plane*; Theorems A, B and C are the three main theorems of the argument (short forms, enough forms, integral forms are long) and are read in later chapters.

2 The setting

Conventions. The natural numbers are $0, 1, 2, \ldots$, and $\operatorname{Re} s$ is the real part of a complex number s. The declarations of this unit have no implicit arguments, so there is no ambient object to fix. The functions ζ and $L(s, \chi)$, Dirichlet characters, principal and primitive characters are those of Mathlib, the mathematical library of Lean, at the commit that the artifact pins. The note *The four statements: the Riemann hypothesis and its generalisation to Dirichlet L-functions* says what these objects are and what determines them. In short: ζ and $L(\cdot, \chi)$ are the analytic continuations of the series $\sum_{n \geq 1} n^{-s}$ and $\sum_{n \geq 1} \chi(n) n^{-s}$, with a conventional value at the pole $s = 1$ of ζ and of the L-function of a principal character; a Dirichlet character modulo M is a multiplicative map from $\mathbb{Z}/M$ to $\mathbb{C}$ that vanishes outside the units, the principal character is the one with value 1 at every unit, and a character is primitive when no character of a smaller modulus induces it.

Definition 2.1 (the four statements)**.** The artifact defines four propositions.

(1) *The Riemann hypothesis for ζ*: for every complex number s, if $\zeta(s) = 0$, if there is no natural number n with $s = -2(n + 1)$, and if $s \neq 1$, then $\operatorname{Re} s = \frac{1}{2}$.

(2) *The half-plane statement*: for every integer $M \geq 1$, every Dirichlet character χ modulo M and every complex number s with $\operatorname{Re} s > \frac{1}{2}$, if χ is not the principal character modulo M or $s \neq 1$, then $L(s, \chi) \neq 0$.

(3) *The strip statement*: for every integer $M \geq 1$, every Dirichlet character χ modulo M and every complex number s, if $0 < \operatorname{Re} s$, $\operatorname{Re} s < 1$ and $L(s, \chi) = 0$, then $\operatorname{Re} s = \frac{1}{2}$.

(4) *The strip statement for primitive characters*: for every integer $M \geq 1$ and every primitive Dirichlet character χ modulo M, and for every complex number s, if $0 < \operatorname{Re} s$, $\operatorname{Re} s < 1$ and $L(s, \chi) = 0$, then $\operatorname{Re} s = \frac{1}{2}$.

LEAN `RiemannZetaZerosOnLine` · `DirichletLNonvanishingRightOfHalf` · `DirichletLZerosOnLine` · `PrimitiveDirichletLZerosOnLine` · these four definitions form another unit, the subject of the note *The four statements*

Theorems 2, 3, 4 and 5 assert propositions (1), (2), (3) and (4), in this order. Theorem 1 asserts a proposition of the library, whose text agrees with (1) word for word, up to the way in which the number $n + 1$ is read as a complex number.

Inputs from another unit. The three results below are taken for granted in this note. They belong to the unit that ends the argument; they are proved in the note *The main inequality, the bound for the primes, and the zero-free half-plane*, the first from the main theorems of the argument, the other two completely.

Input 1 (the zero-free half-plane). *The half-plane statement holds.*

Input 2 (from the half-plane to the strip). *If the half-plane statement holds, then the strip statement holds.*

Input 3 (from the half-plane to ζ). *If the half-plane statement holds, then the Riemann hypothesis for ζ holds.*

LEAN `New.grh` · `New.zeros_on_line` · `New.zeta_zeros_on_line`

Input 1 is the main theorem of the artifact. Inputs 2 and 3 are classical: the functional equation of the completed L-function reflects a zero s with $0 < \operatorname{Re} s < \frac{1}{2}$ to the zero $1 - s$ of the L-function of the inverse character, and for $M = 1$ the L-function is ζ.

3 Proofs

Proof of Theorem 3. It is Input 1. □

Proof of Theorem 4. By Input 1 the half-plane statement holds, and by Input 2 the strip statement follows. □

Proof of Theorems 1 and 2. By Input 1 the half-plane statement holds, and by Input 3 the Riemann hypothesis for ζ follows. This is Theorem 2, and it is Theorem 1 because the proposition of the library and proposition (1) of Definition 2.1 are the same. □

Proof of Theorem 5. Let $M \geq 1$ be an integer and χ a primitive Dirichlet character modulo M. Theorem 4, applied to M and χ, gives the assertion for every complex number s. That χ is primitive is not used. □

LEAN each of the five formal proofs is one term · the terms use `New.grh`, `New.zeros_on_line` and `New.zeta_zeros_on_line` exactly as above

4 What one must trust

The five theorems are short to state, and their formal proof is very long. What does a reader have to accept in order to believe them, short of reading the proof? Four things, of different kinds.

1. That the statements mean what they say. The four propositions of Definition 2.1 are written in a file that uses the library and nothing else, and the root file states the five theorems with these propositions and with the library's own. So what is claimed can be read off two short files, and it rests on the library's definitions of ζ, of Dirichlet characters and of their L-functions. No definition made by the artifact enters a statement: all its other definitions occur only inside proofs. The library's functions are defined at every complex number, also at a pole, and the statements exclude the poles explicitly. The note *The four statements* goes through this.

2. The library. One trusts that the library's ζ and $L(s, \chi)$ are the classical functions. This is not a matter of faith in a construction: the library proves that they agree with the Dirichlet series for $\operatorname{Re} s > 1$ and are holomorphic away from the pole, and these two properties determine them. What one accepts is that these theorems of the library are themselves correctly checked, which is the next point.

3. The kernel and the axioms. A proof in Lean is a term that a small program, the kernel, checks against the statement. The authors report three things: every file of the artifact compiles with the pinned versions of Lean and of the library, without errors and without the placeholder that Lean accepts for a missing proof; the five theorems depend on no axiom other than the three standard ones of Lean with classical logic (propositional extensionality, the soundness of quotients, and the axiom of choice); and an independent tool, which compares a file holding only the five statements with the root file and checks the proofs again, accepted them. These are the authors' reports, read in the description that accompanies the artifact. On our side, a reading of the text of all files of the artifact, made when this workspace was set up, found no placeholder and no declaration of an axiom, and an independent build with the same versions is recorded in the handout's record of checks when it reports. Which axioms a proof rests on is for the kernel to say, not for a reading of the text. To accept the result on this ground is to trust the kernel of Lean, its implementation, and the three axioms.

4. Nothing about the argument. A kernel check establishes that the stated theorems follow from the axioms. It does not tell a mathematician why they hold, and it is not a reading: at the time of writing nobody has read the argument through. The purpose of this write-up is to make that reading possible. Each of its notes says what was compared with the Lean text by a program, what was read by a person, and what is taken for granted from other units; the map of the handout shows which units have been read.

5 Notes on the formal text

1. *The five proofs are terms.* Theorem 3 is proved by the name of Input 1 alone: it is the same theorem under a second name. Theorem 4 is Input 2 applied to Input 1. Theorem 1 is Input 3 applied to Input 1. Theorem 2 is proved by the name of Theorem 1. Theorem 5 takes the modulus, the assumption that it is not zero, the character and the assumption that it is primitive, and returns Theorem 4 for that modulus and character.
2. *Two formulations of the Riemann hypothesis.* Input 3 concludes proposition (1) of Definition 2.1, and Theorem 1 is stated with the proposition of the library; conversely Theorem 2 is stated with (1) and proved by Theorem 1. Lean accepts both steps only if the two propositions agree after unfolding their definitions. So the acceptance of the file is also a check that the artifact's copy is the library's formulation.
3. *The hypothesis of Theorem 5.* The assumption that the character is primitive is received and not used. Theorem 5 is stated because the generalised Riemann hypothesis is often formulated for primitive characters only; for the zeros with $0 < \operatorname{Re} s < 1$ the two formulations are equivalent, as the note *The four statements* explains.
4. *The modulus.* The formal text writes the modulus as a natural number N with the assumption $N \neq 0$, passed as an instance argument. Here it is an integer $M \geq 1$; the letter N is kept in this write-up for the level of the automorphic forms of the proof.
5. *Authorship.* The root file carries one comment: the proof is Levent Alpöge's, with Claude. The description of the artifact adds that the Lean text and the accompanying documents were written by Claude, and that this edition is a shorter rewriting of an earlier formal proof by Alpöge.

How this note was written. From the dossier of the unit. It has no structured proof, because each of the five formal proofs is a single term; the program check of the dossier covered 0 leaf steps of 0 proofs, found the five statements at their lines and compared the quotations of the imported declarations with the source. I read the whole root file and the whole file of the four statements, and the statements of the three inputs in their file. For §4 I read the description that accompanies the artifact, the file of five statements without proofs that it provides for the comparison tool, the configuration of that tool, and the files that pin the versions of Lean and of the library; I also read the library's formulation of the Riemann hypothesis in the library's public repository at the pinned commit. I did not run a build. The diagram and §4 are mine.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Theorem 1	`riemannHypothesis`	Main.lean	4–4
Theorem 2	`riemannZeta_eq_zero_re`	Main.lean	6–6
Theorem 3	`LFunction_ne_zero_of_half_lt_re`	Main.lean	8–8
Theorem 4	`LFunction_eq_zero_re`	Main.lean	10–10

In this note	Lean declaration	File	Lines
Theorem 5	`LFunction_eq_zero_re_of_isPrimitive`	Main.lean	12–13
Input 1	`New.grh`	Main.lean	174–190
Input 2	`New.zeros_on_line`	Main.lean	194–196
Input 3	`New.zeta_zeros_on_line`	Main.lean	200–204
Definition 2.1	`RiemannZetaZerosOnLine`	Statements.lean	3–4
Definition 2.1	`DirichletLNonvanishingRightOfHalf`	Statements.lean	6–8
Definition 2.1	`DirichletLZerosOnLine`	Statements.lean	10–12
Definition 2.1	`PrimitiveDirichletLZerosOnLine`	Statements.lean	14–16

The five theorems as they are written in the artifact.

```
theorem riemannHypothesis : RiemannHypothesis

theorem riemannZeta_eq_zero_re : RiemannZetaZerosOnLine

theorem LFunction_ne_zero_of_half_lt_re : DirichletLNonvanishingRightOfHalf

theorem LFunction_eq_zero_re : DirichletLZerosOnLine

theorem LFunction_eq_zero_re_of_isPrimitive : PrimitiveDirichletLZerosOnLine
```

The three inputs:

```
lemma New.grh : DirichletLNonvanishingRightOfHalf

lemma New.zeros_on_line (h : DirichletLNonvanishingRightOfHalf) : DirichletLZerosOnLine

lemma New.zeta_zeros_on_line (h : DirichletLNonvanishingRightOfHalf) :
    RiemannZetaZerosOnLine
```

In these texts `RiemannHypothesis` is the proposition of the library, in the namespace of the library; the four other propositions are those of Definition 2.1, and the table above gives their names in the order (1) to (4). The library's proposition is not quoted here, because the build of this note copies quotations from the artifact only; it is to be read in the library, in the file of the Riemann zeta function.

Notation

- s, $\operatorname{Re} s$ a complex number; its real part
- n a natural number, $n \geq 0$
- ζ the Riemann zeta function, as the library defines it
- M the modulus of a Dirichlet character, an integer ≥ 1
- χ a Dirichlet character modulo M
- $L(s, \chi)$ the L-function of the Dirichlet character χ, as the library defines it
- N the modulus in the formal text; in this note it is M

READBACK NOTE 2

The four statements: the Riemann hypothesis and its generalisation to Dirichlet L-functions

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The headline of the artifact is five theorems. Their statements are four propositions about the Riemann zeta function and the Dirichlet L-functions, written in one short file that uses nothing but the mathematical library of Lean: the Riemann hypothesis for ζ; the non-vanishing of every Dirichlet L-function on the half-plane $\operatorname{Re} s > \frac{1}{2}$; and the statement that the zeros of these functions in the critical strip lie on the line $\operatorname{Re} s = \frac{1}{2}$, for all characters and for primitive ones. This note states the four propositions in ordinary language, says which objects of the library they speak of and what determines those objects, and records how the four are related: the second implies the other three.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, whose final theorems are the Riemann hypothesis and its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles with Lean v4.35.0 against a pinned version of the library Mathlib and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note treats one unit of the artifact: all 4 of its 4 declarations, which are definitions of propositions. The unit proves nothing, so there is no proof to read and nothing is taken for granted from other units. Each of the four statements was compared with its Lean text, and the objects of the library that they use were compared with the library's source at the pinned commit (Appendix A says exactly what was read). The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 4 declarations.

1 The four statements

Throughout, ζ is the Riemann zeta function and $L(s, \chi)$ is the L-function of a Dirichlet character χ, as the library defines them; §2 says what these functions are. The first statement is about ζ, the other three are about the L-functions.

Statement 1 (the Riemann hypothesis for ζ). *For every complex number s: if $\zeta(s) = 0$, if there is no natural number n with $s = -2(n+1)$, and if $s \neq 1$, then* $\operatorname{Re} s = \frac{1}{2}$.

LEAN `RiemannZetaZerosOnLine`

Statement 2 (the half-plane statement). *For every integer $M \geq 1$, every Dirichlet character χ modulo M and every complex number s with $\operatorname{Re} s > \frac{1}{2}$: if χ is not the principal character modulo M or $s \neq 1$, then $L(s, \chi) \neq 0$.*

Statement 3 (the strip statement). *For every integer $M \geq 1$, every Dirichlet character χ modulo M and every complex number s: if $0 < \operatorname{Re} s$, $\operatorname{Re} s < 1$ and $L(s, \chi) = 0$, then $\operatorname{Re} s = \frac{1}{2}$.*

Statement 4 (the strip statement for primitive characters). *For every integer $M \geq 1$ and every Dirichlet character χ modulo M that is primitive, and for every complex number s: if $0 < \operatorname{Re} s$, $\operatorname{Re} s < 1$ and $L(s, \chi) = 0$, then $\operatorname{Re} s = \frac{1}{2}$.*

LEAN `DirichletLNonvanishingRightOfHalf` · `DirichletLZerosOnLine` · `PrimitiveDirichletLZerosOnLine`

Each of the four is a proposition, not a theorem: the unit defines them and asserts nothing. They are asserted by the five final theorems of the artifact, which are the subject of the note *The five final theorems, and what one must trust.* The argument is for Statement 2; its last steps, and the passage from Statement 2 to Statements 1 and 3, are the subject of the note *The main inequality, the bound for the primes, and the zero-free half-plane.*

In classical terms, Statement 1 is the Riemann hypothesis, and Statement 3 is the generalised Riemann hypothesis for Dirichlet L-functions in its usual form, for the zeros in the open critical strip. Statement 2 is the form in which the artifact proves it: a zero-free half-plane. It contains the classical non-vanishing on the closed half-plane $\operatorname{Re} s \geq 1$, and beyond that it says that there is no zero with $\frac{1}{2} < \operatorname{Re} s < 1$.

2 The objects of the statements

Conventions. The natural numbers are $0, 1, 2, \ldots$, and $\operatorname{Re} s$ is the real part of a complex number s. The declarations of this unit have no arguments, so there is no ambient object to fix. Everything that the four statements speak of is defined in Mathlib, the mathematical library of Lean, at the commit that the artifact pins; nothing in them is defined by the artifact. Appendix A gives the library's names for the objects and facts of this section.

The zeta function. The library's zeta function is a function $\zeta\colon \mathbb{C} \to \mathbb{C}$, defined at every complex number. It is constructed from Hurwitz zeta functions, and the library proves the following about it.

(a) For $\operatorname{Re} s > 1$, $\zeta(s) = \sum_{n \geq 1} n^{-s}$.
(b) ζ is complex differentiable at every $s \neq 1$, and $(s-1)\zeta(s) \to 1$ as $s \to 1$.
(c) $\zeta(-2(n+1)) = 0$ for every natural number n, and $\zeta(0) = -\frac{1}{2}$.

By (a), (b) and the identity theorem, the restriction of ζ to $\mathbb{C} \setminus \{1\}$ is *the* analytic continuation of the series: the two properties determine it. A function of Lean must have a value everywhere, so the library also assigns a value to ζ at the pole $s = 1$; that value is a convention, and no statement of this note depends on it, because Statement 1 excludes $s = 1$. The zeros in (c) are the trivial zeros $-2, -4, -6, \ldots$; they are what the second hypothesis of Statement 1 excludes.

Dirichlet characters. Let $M \geq 1$ be an integer. A *Dirichlet character modulo M* (with complex values) is a map $\chi\colon \mathbb{Z}/M \to \mathbb{C}$ with $\chi(1) = 1$ and $\chi(uv) = \chi(u)\chi(v)$ for all classes u and v, and with $\chi(u) = 0$ for every class u that is not a unit. It is evaluated at an integer through the class of the integer. The *principal character* modulo M takes the value 1 at every unit; in the library it is the unit element 1 of the group of characters. For $M = 1$ the ring $\mathbb{Z}/1$ has one element, which is a unit, and the principal character is the only character. If M_0 divides M, a character modulo M_0 *induces* a character modulo M, by composition with the reduction of units. The *conductor* of a character χ modulo M is the least divisor M_0 of M such that χ is induced by a character modulo M_0, and χ is *primitive* when its conductor is M. Every character modulo M is induced by exactly one primitive character, whose modulus is the conductor.

Dirichlet L-functions. For an integer $M \geq 1$ and a Dirichlet character χ modulo M, the library's L-function is a function $s \mapsto L(s, \chi)$ from $\mathbb{C}$ to $\mathbb{C}$, defined at every complex number. It is constructed as a linear combination of Hurwitz zeta functions, and the library proves the following.

(a) For $\operatorname{Re} s > 1$, $L(s, \chi) = \sum_{n \geq 1} \chi(n)\, n^{-s}$.

(b) $s \mapsto L(s, \chi)$ is complex differentiable at every s, except possibly at $s = 1$ when χ is principal.

(c) For $M = 1$, $L(s, \chi) = \zeta(s)$ for every complex number s.

(d) If χ_0 is a character modulo a divisor M_0 of M and χ is the character modulo M that it induces, then for every complex number s such that χ_0 is not principal or $s \neq 1$,

$$L(s, \chi) = L(s, \chi_0) \prod_{p \mid M} (1 - \chi_0(p)\, p^{-s}),$$

the product over the primes p that divide M.

So for a character that is not principal, $L(\,\cdot\,, \chi)$ is the entire function that continues the series. For a principal character it is the continuation to $\mathbb{C} \setminus \{1\}$, with a simple pole at 1, and its value at $s = 1$ is again a convention; this is why Statement 2 excepts the point $s = 1$ for a principal character, and only that point.

3 How the four statements are related

The unit contains no proof. What follows is said for orientation; the proofs are in the notes named.

Statement 2 implies the other three. Statement 2 implies Statement 3: a zero s with $0 < \operatorname{Re} s < \frac{1}{2}$ would be reflected by the functional equation to the zero $1 - s$ of the L-function of the inverse character, which lies in the half-plane of Statement 2. Statement 2 for $M = 1$ implies Statement 1, by property (c) of the L-functions and the functional equation of ζ. These two implications are Theorems 4 and 5 of the note *The main inequality, the bound for the primes, and the zero-free half-plane.* Statement 3 implies Statement 4, which is its special case for primitive characters. The artifact proves Statement 2 and deduces the other three in this way.

Remark 3.1 (what Statements 2 to 4 leave out)**.** Statements 2, 3 and 4 say nothing about zeros with $\operatorname{Re} s \leq 0$. There the L-function of a character has its trivial zeros, and the L-function of a character that is not primitive can have further zeros on the line $\operatorname{Re} s = 0$: those of the factors $1 - \chi_0(p)p^{-s}$ of property (d), with χ_0 the primitive character that induces it, for the primes p that divide the modulus but not the conductor. For the other primes p dividing the modulus, $\chi_0(p) = 0$ and the factor is 1; so a character whose modulus and conductor have the same prime divisors, for instance a character modulo 9 of conductor 3, has no further zero. Statement 1 does speak of all zeros of ζ: it says that a zero is a trivial zero or lies on the line $\operatorname{Re} s = \frac{1}{2}$.

Remark 3.2 (characters that are not primitive)**.** The statements are made for all characters, primitive or not. This costs nothing and gains nothing. Let χ be a character modulo M and χ_0 the primitive character that induces it. A factor $1 - \chi_0(p)p^{-s}$ of property (d) vanishes only if $|\chi_0(p)| = p^{\operatorname{Re} s}$; the left side is 0 or 1, so this forces $\operatorname{Re} s = 0$. Hence for $\operatorname{Re} s > 0$, and $s \neq 1$ when χ is principal, $L(s, \chi) = 0$ if and only if $L(s, \chi_0) = 0$. So Statement 3 is equivalent to Statement 4, and Statement 2 is equivalent to its restriction to primitive characters. For a principal character, χ_0 is the character modulo 1, whose L-function is ζ: Statements 2 and 3 contain the corresponding statements for ζ. This remark is ours and is not a declaration of the unit; the artifact proves only that Statement 3 implies Statement 4.

4 Notes on the formal text

1. *The file.* The four definitions are the whole content of one file, whose only import is the library. So the meaning of the four statements rests on the library alone, not on any definition of the artifact. This is by design: it is the file to read in order to know what is claimed.
2. *The modulus.* The formal text writes the modulus as a natural number N together with the assumption $N \neq 0$, which Lean passes as an instance argument; this is "every integer $M \geq 1$" above. The letter N is kept in this write-up for the level of the automorphic forms of the proof, and a modulus is written M throughout, as in the weighted prime sums of the argument.
3. *Statement 1 and the library's own formulation.* The library contains a formulation of the Riemann hypothesis as a proposition. Statement 1 is that proposition, written out again in the artifact: the two texts differ only in how the number $n + 1$ is read as a complex number. The first of the five final theorems has the library's proposition as its statement and the second has Statement 1; the artifact proves the second by citing the first, which Lean accepts only if the two propositions agree after unfolding.
4. *The trivial zeros.* They appear as "there is no natural number n with $s = -2(n + 1)$", that is, $s \notin \{-2, -4, -6, \dots\}$.
5. *The pole.* In Statement 1 the hypothesis $s \neq 1$ excludes the point where the value of ζ is a convention. In Statement 2 the hypothesis reads "χ is not the unit element of the group of characters, or $s \neq 1$"; it excludes exactly the pole of a principal character. Statements 3 and 4 need no such hypothesis, because $\operatorname{Re} s < 1$ there.

6. *The strip.* In Statements 3 and 4 the strip is open, and the two inequalities $0 < \operatorname{Re} s$ and $\operatorname{Re} s < 1$ are two separate hypotheses, in this order, before the hypothesis $L(s, \chi) = 0$.
7. *Order of the quantifiers.* Statement 4 takes the modulus, the character and the hypothesis that the character is primitive before the complex number s; Statements 2 and 3 take the modulus, the character and then s. Nothing depends on this order here.
8. *Real numbers.* The real part is a real number, and $\frac{1}{2}$ is the real number one half.

How this note was written. From the dossier of the unit, which has no structured proof because the unit has no proof; the program check of the dossier covered 0 leaf steps of 0 proofs and found the four statements at their lines. I read the whole file of the unit in the artifact and compared each of the four statements above with it, clause by clause. For the objects of the library I read, in the library's public repository at the commit that the artifact pins, the declarations listed in the second table of Appendix A: their statements and the comments above them, not their proofs, and not the construction of the Hurwitz zeta functions from which ζ and the L-functions are made. The dossier describes the same objects from memory and says so; its descriptions agree with what I read. The two remarks of §3 are mine.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Statement 1	`RiemannZetaZerosOnLine`	Statements.lean	3–4
Statement 2	`DirichletLNonvanishingRightOfHalf`	Statements.lean	6–8
Statement 3	`DirichletLZerosOnLine`	Statements.lean	10–12
Statement 4	`PrimitiveDirichletLZerosOnLine`	Statements.lean	14–16

The four statements as they are written in the artifact. Statement 1:

```
def RiemannZetaZerosOnLine : Prop :=
  ∀ s : ℂ, riemannZeta s = 0 → (¬∃ n : ℕ, s = -2 * ((n : ℂ) + 1)) → s ≠ 1 → s.re = 1 / 2
```

Statement 2:

```
def DirichletLNonvanishingRightOfHalf : Prop :=
  ∀ (N : ℕ) [NeZero N] (χ : DirichletCharacter ℂ N) (s : ℂ),
    1 / 2 < s.re → (χ ≠ 1 ∨ s ≠ 1) → DirichletCharacter.LFunction χ s ≠ 0
```

Statement 3:

```
def DirichletLZerosOnLine : Prop :=
  ∀ (N : ℕ) [NeZero N] (χ : DirichletCharacter ℂ N) (s : ℂ),
    0 < s.re → s.re < 1 → DirichletCharacter.LFunction χ s = 0 → s.re = 1 / 2
```

Statement 4:

```
def PrimitiveDirichletLZerosOnLine : Prop :=
  ∀ (N : ℕ) [NeZero N] (χ : DirichletCharacter ℂ N), χ.IsPrimitive →
    ∀ s : ℂ, 0 < s.re → s.re < 1 → DirichletCharacter.LFunction χ s = 0 → s.re = 1 / 2
```

In these texts the modulus is N, the letter `ℂ` stands for the complex numbers and `ℕ` for the natural numbers, `s.re` is $\operatorname{Re} s$, `χ ≠ 1` says that χ is not the principal character, and `[NeZero N]` is the assumption $N \neq 0$.

The objects of the library. The library is Mathlib at the commit that the artifact pins in its manifest. The declarations below were read there for this note; the left column says what each one is in §2.

In this note	Declaration of the library
the function ζ	`riemannZeta`
ζ, property (a)	`zeta_eq_tsum_one_div_nat_cpow`
ζ, property (b)	`differentiableAt_riemannZeta, riemannZeta_residue_one`
ζ, property (c)	`riemannZeta_neg_two_mul_nat_add_one, riemannZeta_zero`
the library's Riemann hypothesis	`RiemannHypothesis`
Dirichlet character modulo M	`DirichletCharacter, MulChar`
induced character	`DirichletCharacter.changeLevel`
conductor, primitive character	`DirichletCharacter.conductor, DirichletCharacter.IsPrimitive`
the function $L(s, \chi)$	`DirichletCharacter.LFunction`
L, property (a)	`DirichletCharacter.LFunction_eq_LSeries`
L, property (b)	`DirichletCharacter.differentiableAt_LFunction`
L, property (c)	`DirichletCharacter.LFunction_modOne_eq`
L, property (d)	`DirichletCharacter.LFunction_changeLevel`

Notation

s, $\operatorname{Re} s$ a complex number; its real part

n a natural number, $n \geq 0$

ζ the Riemann zeta function, as the library defines it

M, M_0 moduli of Dirichlet characters, integers ≥ 1; M_0 a divisor of M

χ, χ_0 Dirichlet characters modulo M and modulo M_0

u, v classes in $\mathbb{Z}/M$

$L(s, \chi)$ the L-function of the Dirichlet character χ, as the library defines it

p a prime number

N the modulus in the formal text; in this note it is M

READBACK NOTE 4

The main inequality, the bound for the primes, and the zero-free half-plane

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. This unit is the top of the argument. Its first result, the main inequality, bounds a remainder of weighted sums over primes: $m \cdot \mathrm{rem}(D) \le (\log b)^2 \cdot \mathrm{budget}(D)$ for all admissible data D of the proof. It comes from bounding one covolume from two sides: Theorems A and B of the argument give many independent automorphic forms with short theta lifts, Theorem C says that such lifts cannot be that short, and in the comparison the main amounts cancel. Letting the data grow, with the second scale at a good point of a long window, one gets the one-sided bound $R_{d,N_0}(x) \le x^{8\varepsilon-1/2}$ for the remainder of a smoothed sum over the prime powers $\nu \equiv 1$ modulo N_0, weighted by $\Lambda(\nu)(1-\chi_K(\nu))$ (Definition 2.5); the saving $x^{-1/2}$ is the reciprocal of the number m of lifts. A criterion of Landau turns this bound, for all primes d of one residue class, into the absence of zeros of Dirichlet L-functions in the half-plane $\mathrm{Re}\, s > \frac{1}{2}$, and the functional equations put the nontrivial zeros on the critical line.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, whose final theorems are the Riemann hypothesis and its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles with Lean v4.35.0 against a pinned version of the library Mathlib and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note treats one unit of the artifact: all 5 of its 5 declarations, Theorems 1 to 5 below. Their proofs are complete *given* eleven results of other units, which are stated as Inputs 1 to 11 with the unit that owns each and are not proved here; among them are the three main theorems of the argument (Inputs 1 and 2) and Landau's criterion (Input 10). Every statement of the unit and of the Inputs was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 48 declarations.

1 The statements

The objects in the five theorems are defined in §2: parameters and admissible data in Definitions 2.2 to 2.4, the remainders $R_{d,M}(x)$ and $\mathrm{rem}(D)$ in Definitions 2.5 and 2.6, the three statements in Definition 2.1.

Theorem 1 (the main inequality). *Let P be parameters and D admissible data for P, with number of lifts m and third scale b. Then*

$$m \cdot \mathrm{rem}(D) \;\le\; (\log b)^2 \cdot \mathrm{budget}(D).$$

Here $\mathrm{rem}(D)$ *is the remainder of the weighted prime sums of* D*, and* $\mathrm{budget}(D) = f(D) \cdot a(D) \cdot (\log b)^3$ *is its budget.*

LEAN `star_inequality`

Theorem 2 (the bound for the primes). *Let $P = (N_0, d)$ be parameters and ε a real number with $0 < \varepsilon/6$ and $\varepsilon < \frac{1}{100}$. Then for every sufficiently large real number x,*

$$R_{d,N_0}(x) \;\le\; x^{\,8\varepsilon+\frac{1}{2}-1}.$$

LEAN `eventually_primeError_le_rpow`

Theorem 3 (the zero-free half-plane). *The half-plane statement holds: for every integer $M \ge 1$, every Dirichlet character χ modulo M and every complex number s with $\mathrm{Re}\, s > \frac{1}{2}$, if χ is not the principal character modulo M or $s \ne 1$, then $L(s,\chi) \ne 0$.*

Theorem 4 (from the half-plane to the strip). *If the half-plane statement holds, then the strip statement holds: for every integer $M \ge 1$, every Dirichlet character χ modulo M and every complex number s with $0 < \mathrm{Re}\, s$, $\mathrm{Re}\, s < 1$ and $L(s,\chi) = 0$, one has $\mathrm{Re}\, s = \frac{1}{2}$.*

Theorem 5 (from the half-plane to ζ). *If the half-plane statement holds, then the Riemann hypothesis for ζ holds: every complex number s with $\zeta(s) = 0$, such that there is no natural number n with $s = -2(n+1)$, and with $s \ne 1$, has $\mathrm{Re}\, s = \frac{1}{2}$.*

LEAN `New.grh` · `New.zeros_on_line` · `New.zeta_zeros_on_line`

Theorem 3 is the main theorem of the artifact. Theorems 3, 4 and 5 together give the five final theorems, which are the subject of the note *The five final theorems, and what one must trust.*

The plan of the proof. The argument attaches to the data D a finite-dimensional space of automorphic forms, the *rows*, and a linear map on it, the composite of m theta lifts. Theorems A and B of the argument, taken together (Input 1), give linearly independent rows that almost span, have small index with respect to the integral structures, and have short lifts: the squared covolume of the lifts is at most $\exp(n\,(h_{\mathrm{main}}(D) - m \cdot \mathrm{rem}(D) + \mathrm{error}))$ at every conjugate, where n is the number of the rows and $h_{\mathrm{main}}(D)$ is the main amount. Theorem C (Input 2) is a bound in the other direction, a consequence of the product formula: integral forms are long. If the squared covolumes of the lifts of such rows are at most $\exp(V)$ at every conjugate, then V, plus twice a bound for the index, is at least $\dim \mathrm{Row}(D) \cdot (h_{\mathrm{main}}(D) - \mathrm{error})$. The two bounds can hold together only if $m \cdot \mathrm{rem}(D)$ is at most the sum of the

errors. This is Theorem 1, and it is all that the rest of the argument needs from the automorphic side.

The remainder $\mathrm{rem}(D)$ is a combination of remainders of weighted prime sums at the first two scales of the data (Input 6). For large x there are data with first scale x (Input 7), and the second scale can be moved to a point where the unwanted values are negligible (Input 9). Then Theorem 1, divided by m and by the other positive factors, is Theorem 2. Landau's criterion (Input 10) gives Theorem 3, and the functional equations give Theorems 4 and 5.

2 The setting

Conventions. The natural numbers are $0, 1, 2, \ldots$; $\mathrm{Re}\, s$ is the real part of a complex number s; log is the natural logarithm, and powers of positive real numbers with real exponents are defined through it; φ is Euler's function and Λ von Mangoldt's function. A statement holds *for every sufficiently large* real number if there is a real number such that it holds for every real number at least as large. The Riemann zeta function ζ, Dirichlet characters and their L-functions $L(s, \chi)$ are those of Mathlib, the mathematical library of Lean, at the commit that the artifact pins; the note *The four statements: the Riemann hypothesis and its generalisation to Dirichlet L-functions* says what determines them. A modulus is an integer $M \geq 1$. Parameters are called P and admissible data for P are called D; a statement about D is a statement for every P and all such D, and the components of P and of D are named where they are used.

The three statements.

Definition 2.1 (three propositions)**.** (1) The *Riemann hypothesis for* ζ: for every complex number s, if $\zeta(s) = 0$, if there is no natural number n with $s = -2(n+1)$, and if $s \neq 1$, then $\mathrm{Re}\, s = \frac{1}{2}$. (2) The *half-plane statement*: for every integer $M \geq 1$, every Dirichlet character χ modulo M and every complex number s with $\mathrm{Re}\, s > \frac{1}{2}$, if χ is not the principal character modulo M or $s \neq 1$, then $L(s, \chi) \neq 0$. (3) The *strip statement*: for every integer $M \geq 1$, every Dirichlet character χ modulo M and every complex number s, if $0 < \mathrm{Re}\, s$, $\mathrm{Re}\, s < 1$ and $L(s, \chi) = 0$, then $\mathrm{Re}\, s = \frac{1}{2}$.

LEAN `RiemannZetaZerosOnLine` · `DirichletLNonvanishingRightOfHalf` · `DirichletLZerosOnLine` · another unit; see the note *The four statements*

Parameters and data.

Definition 2.2 (parameters)**.** *Parameters* are a pair $P = (N_0, d)$ of natural numbers such that 18480 divides N_0, d is a prime number, N_0 divides $2(d+1)$, and N_0 does not divide $d+1$. For parameters $P = (N_0, d)$ we write $K = \mathbb{Q}(\sqrt{-d})$, a subfield of $\mathbb{C}$, and χ_K for the quadratic character of the field $\mathbb{Z}/d$, a Dirichlet character modulo d; it is the Legendre symbol $(\cdot/d)$ when d is odd, as it is for all parameters (Lemma 4.2).

LEAN `Parameters` · unit 152

The last two conditions say that $d \equiv N_0/2 - 1$ modulo N_0; they force $N_0 \geq 1$. Since $18480 = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11$, one has $d \equiv 7$ modulo 8 (Lemma 4.2); so K is an imaginary quadratic field in which 2 splits, and χ_K is its quadratic character.

Definition 2.3 (admissible data)**.** Let $P = (N_0, d)$ be parameters. The artifact attaches to P a real number $C(P)$, of which this note uses only Input 5. *Admissible data for P* are a record D with the following components.

(a) Three real numbers $\log x$, $\log x_1$ and $\log b$. Their exponentials x, x_1 and b are the first, second and third *scale* of D.

(b) Two natural numbers: m, the *number of lifts*, and r, a rank. By condition (ii) below, m is a power of 2; put

$$N_1(D) = N_0 \cdot m^{22} \cdot 3^{\lfloor \log_3(22 \log_2 m) \rfloor + 5},$$

and let $E(D)$ be the field generated over K by the roots of unity of order $N_1(D)$.

(c) A finite set $\mathrm{thick}(D)$ of infinite places of the field $E(D)$, the *thick* places.

(d) Two prime numbers and an element of the ring of integers of K.

They are subject to the following conditions.

(i) *The scales.*

$$10^{12}\, C(P) \le \log\log b, \qquad 100\, \log\log b \le \log x, \qquad 10^6\, \log x \le \log x_1,$$
$$10^6\, \log x_1 \le \log b, \qquad 6\, \log\log x_1 \le \log\log b.$$

(ii) *The number of lifts and the rank.* m is a power of 2 with exponent at least 1, $\sqrt{x} \le m \le 2\sqrt{x}$ and $22 \log_2 m \le 32 \log x$; r is a power of 3 with exponent at least 1, and $x^5 \le r \le 3\, x^5$.

(iii) *Further conditions,* on the classes of the thick places (a congruence, and the absence of certain relations between three of them), on the two primes and the integer of (d) (the primes are distinct, satisfy congruences and lie between b^{10} and b^{11}, and the norm of the integer is their product), and the existence of primes in certain short intervals.

The conditions (iii) are stated in full in the note on the data of the proof. In this note they matter only through the Inputs, and through one fact that was checked in the Lean text: $\log x_1$ occurs in the conditions (i) and nowhere else.

LEAN Data · Parameters.c · Data.N₁ · Data.E · unit 152

The conditions (i) put the scales far apart, $(\log b)^{100} \le x$, $x^{10^6} \le x_1$, $x_1^{10^6} \le b$, and make all of them very large (Lemma 3.1). The number $N_1(D)$ and the field $E(D)$ depend only on P and m. The formal record has one more component, a natural number called the exponent, which a condition makes equal to $22 \log_2 m$; this note does not give it a letter (§8, note 3). In the formal text the quotient $N_1(D)/N_0$ is the degree of a tower of fields, a function of the exponent.

Definition 2.4 (quantities derived from the data)**.** Let $P = (N_0, d)$ be parameters and D admissible data for P, with scales x, x_1, b.

(a) Every infinite place of $E(D)$ is the place of exactly one embedding $E(D) \to \mathbb{C}$ that is the identity on K. The Galois group of $E(D)$ over K permutes these embeddings simply transitively, and its action on the roots of unity identifies it with $(\mathbb{Z}/N_1(D))^\times$. (Restriction to the field generated over $\mathbb{Q}$ by these roots of unity is an isomorphism onto the Galois group of that field over $\mathbb{Q}$, because the quadratic field K is not contained in it: the prime d ramifies in K, and it

does not ramify in a field of roots of unity of order prime to d. And d is prime to $N_1(D)$: it is prime to N_0 by Lemma 4.2, and $N_1(D)$ has the same prime divisors as N_0, since m is a power of 2 and 6 divides N_0.) The artifact chooses one of the embeddings once. Each infinite place τ of $E(D)$ then has a *class* $\mathrm{cl}_D(\tau) \in (\mathbb{Z}/N_1(D))^\times$, the chosen embedding has class 1, and $\tau \mapsto \mathrm{cl}_D(\tau)$ is a bijection. For an integer ν prime to $N_1(D)$, $\nu \cdot \tau$ is the place whose class is ν times that of τ; for the other integers ν we put $\nu \cdot \tau = \tau$.

(b) $\mathrm{tops}(D)$, the set of *tops*, is the set of the infinite places τ of $E(D)$ whose class $\mathrm{cl}_D(\tau)$ is congruent to 1 modulo N_0.

(c) $f(D)$ is the number of infinite places of the field $E(D)$.

(d) $a(D)$ is the least multiple of $N_1(D)$ that is at least $x\,b$; it is a natural number.

(e) $\mathrm{budget}(D) = f(D) \cdot a(D) \cdot (\log b)^3$ is the *budget*.

LEAN Data.tops · Data.f · Data.a · Data.budget · unit 152

The weighted prime sums.

Definition 2.5 (the weight, and the remainder of the prime sum)**.** Let d be a prime number and $M \geq 1$ an integer; $K = \mathbb{Q}(\sqrt{-d})$, and χ_K is the quadratic character of the field $\mathbb{Z}/d$, a Dirichlet character modulo d (the Legendre symbol $(\,\cdot\,/d)$ when d is odd).

(a) For a natural number ν put $\lambda_{d,M}(\nu) = \Lambda(\nu)(1 - \chi_K(\nu))$ if $\nu \equiv 1$ modulo M, and $\lambda_{d,M}(\nu) = 0$ otherwise.

(b) For a Dirichlet character χ modulo an integer $M \geq 1$, let $s \mapsto \tilde{L}(s,\chi)$ be the following entire function: $\tilde{L}(s,\chi) = L(s,\chi)$ if χ is not principal; if χ is principal, $\tilde{L}(s,\chi) = (s-1)L(s,\chi)$ for $s \neq 1$, and $\tilde{L}(1,\chi) = \prod_{p \mid M}(1 - p^{-1})$, the product over the primes p dividing M.

(c) Put

$$\kappa_{d,M} = \frac{1}{\varphi(M)}\,\mathrm{Re}\Big(\sum_{\xi}\Big(\frac{\tilde{L}'}{\tilde{L}}(1,\xi\chi_K) - \frac{\tilde{L}'}{\tilde{L}}(1,\xi)\Big) - 1\Big),$$

where ξ runs over the Dirichlet characters modulo M, $\xi\chi_K$ is the character modulo Md with the values $\xi(\nu)\chi_K(\nu)$, and $\frac{\tilde{L}'}{\tilde{L}}(s,\chi)$ is the logarithmic derivative of $s \mapsto \tilde{L}(s,\chi)$.

(d) For a real number $x > 0$ put

$$R_{d,M}(x) = \sum_{1 \leq \nu \leq x} \lambda_{d,M}(\nu)\Big(\frac{1}{\nu} - \frac{1}{x}\Big) - \frac{\log x}{\varphi(M)} - \kappa_{d,M}.$$

LEAN lambdaClass · primeError · unit 151 · entireL · unit 106, background

The weight $\lambda_{d,M}(\nu)$ is never negative, and it vanishes unless ν is a power of a prime p with $\chi_K(p) \neq 1$; when $d \equiv 3$ modulo 4, as for every prime d to which the theorems below are applied, these are the primes that do not split in K. The sum in (d) is a smoothed count of such prime powers in the class of 1 modulo M; the term $\log x/\varphi(M) + \kappa_{d,M}$ is its expected value, and $R_{d,M}(x)$ is the *remainder*. That it tends to 0 is a form of the prime number theorem in progressions; under the generalised Riemann hypothesis it is bounded by a constant times $x^{-1/2}$, by the explicit formula, and Theorem 2 is an upper bound of nearly this strength.

Definition 2.6 (the remainder of the data)**.** Let $P = (N_0, d)$ be parameters and D admissible data for P, with scales x, x_1, b; in this definition τ is an infinite place of $E(D)$.

(a) The *nominal weight* of τ is $w_D(\tau) = a(D)/x$ if $\tau \in \mathrm{thick}(D)$, and $w_D(\tau) = a(D)/x_1$ otherwise.

(b) For a real number z,

$$\mathrm{Ask}_D(z, \tau) = \sum_{\nu=1}^{a(D)} \lambda_{d,1}(\nu) \left(\frac{z}{\nu} - w_D((-\nu) \cdot \tau) \right)_+ ,$$

where $(\,\cdot\,)_+$ is the positive part of a real number.

(c) For the prime number d, $\mathrm{cs}(d)$ is Euler's constant minus the real part of the logarithmic derivative of $s \mapsto \tilde{L}(s, \chi_K)$ at $s = 1$. Here χ_K is not principal, so this is the real part of $(L'/L)(1, \chi_K)$.

(d) $\mathrm{Ask}^{\mathrm{main}}(D) = a(D) \cdot (\log a(D) - 1 - \mathrm{cs}(d) - f(D)^{-1} \sum_{\tau'} \log w_D(\tau'))$, where τ' runs over all infinite places of $E(D)$; it is the *expected value* of the sums in (b) for $z = a(D)$.

(e) The *remainder* of D is

$$\mathrm{rem}(D) = \sum_{\tau \in \mathrm{tops}(D)} (\mathrm{Ask}_D(a(D), \tau) - \mathrm{Ask}^{\mathrm{main}}(D)).$$

LEAN `Data.nominalWeight` · `Data.primeSum` · unit 152 · `Data.mainAsk` · `Data.remainder` · unit 151 · `chowlaSelbergConst` · unit 106, background

So $\mathrm{rem}(D)$ compares weighted prime sums at the tops with their expected value. A lemma of unit 151 shows that $-1 - \mathrm{cs}(d) = \kappa_{d,1}$; the overview of this handout writes the expected value with $\kappa_{d,1}$. Input 6 expresses $\mathrm{rem}(D)$ through the remainders of Definition 2.5, and nothing else about Definition 2.6 is used in this note.

Notions of later chapters. Theorem 1 mentions none of the following notions, but its proof passes through them. They are named here with all their arguments; their constructions belong to later chapters and are not needed, because the proof only uses that the conclusion of Input 1 fits the hypothesis of Input 2.

Definition 2.7 (rows, lifts, covolume, index, main amount)**.** Let P be parameters and D admissible data for P, with number of lifts m and rank r.

(a) $\mathrm{Row}(D)$ is a complex vector space, the space of *rows*: certain cusp forms on a unitary group of rank r attached to the field $E(D)$. Its dimension over $\mathbb{C}$ is $\dim \mathrm{Row}(D)$; in the library the dimension of a space that is not finite-dimensional is 0.

(b) The data have *m-fold twists* ϑ: certain characters, defined in later chapters, each with a finite set of primes of $E(D)$ at which it ramifies. Among them is the trivial twist 1, which ramifies at no prime. For an m-fold twist ϑ, $\Theta_{D,\vartheta}$ is a $\mathbb{C}$-linear map from $\mathrm{Row}(D)$ to a space of algebraic modular forms: the composite of m theta lifts, formed with the first m powers of ϑ, restricted to the rows.

(c) A space of algebraic modular forms of the kind in (b) is given by a CM field, a rank, a set of level places, a level subgroup with a character of it, and weights at

the infinite places. For forms $F_1, \dots, F_n$ in one such space and an automorphism σ of the field $\mathbb{C}$, the *squared covolume* $\mathrm{covol}^2_\sigma(F_1, \dots, F_n)$ is the absolute value of the determinant of the matrix of the pairings of the σ-conjugates of $F_1, \dots, F_n$; it is a real number ≥ 0.

(d) For a natural number n, rows $y_1, \dots, y_n$ in $\mathrm{Row}(D)$ and a real number δ, the relation $y_1, \dots, y_n$ have D-index at most δ means: there are real numbers η_ℓ, one for each natural number ℓ, with $\sum_{\ell \in J} \eta_\ell \leq \delta$ for every finite set J of natural numbers, such that for every prime number ℓ, every ℓ-adic place of $\mathbb{C}$ and every m-fold twist ϑ to which ℓ is prime (that is, ℓ is prime to the norms of the primes at which ϑ ramifies), some $n \times n$ matrix over the valuation ring of that place, with determinant of absolute value at least $\exp(-\eta_\ell)$, transforms $y_1, \dots, y_n$ into rows all of whose images under $\Theta_{D,\vartheta}$ are integral at that place.

(e) $h_{\mathrm{main}}(D)$ is a real number, the *main amount*: the product of m, of the rank $r + 1 - m$, and of a real number called the cost of one direction, which is defined from the weights and the prime sums of D.

LEAN `Data.rows` · `Data.rowDim` · unit 140 · `Data.θ` · unit 143 · `LevelPlaces.LevelForm.covolumeSq` · unit 102, background · `Data.IndexLe` · `Data.mainHeight` · unit 151

3 The main inequality

The proof compares two bounds for one quantity. For rows $y_1, \dots, y_n$ as in Input 1 below, let V be the logarithm of a bound for the squared covolumes of their lifts at all automorphisms of $\mathbb{C}$, and δ a bound for their index. Input 1 bounds $V + 2\delta$ from above, Input 2 bounds it from below, and up to errors that are multiples of the budget,

$$h_{\mathrm{main}}(D) \;\leq\; \frac{V + 2\delta}{\dim \mathrm{Row}(D)} \;\leq\; h_{\mathrm{main}}(D) - m \cdot \mathrm{rem}(D).$$

The inequality between the two ends is Theorem 1. This display is a guide only; it stands in a comment at the head of the file of the unit, in which the authors also list the inputs and the sizes of the errors. The proof follows the Inputs.

Inputs from other units. The following results are taken for granted. They are stated as the dossier of this unit reads them, and each was compared with its Lean text.

Input 1 (Theorems A and B together; unit 151). *Let P be parameters and D admissible data for P, with number of lifts m and third scale b. Then there are a natural number n and rows $y_1, \dots, y_n$ in $\mathrm{Row}(D)$ such that: $y_1, \dots, y_n$ are linearly independent over $\mathbb{C}$; $(1 - 4/b^2) \cdot \dim \mathrm{Row}(D) \le n$; there is a real number δ with $\delta \le 5 \cdot \dim \mathrm{Row}(D) \cdot \log(1 + b^2) \cdot \mathrm{budget}(D)$ such that $y_1, \dots, y_n$ have D-index at most δ; and for every automorphism σ of $\mathbb{C}$,*

$$\mathrm{covol}_\sigma^2(\Theta_{D,1} y_1, \dots, \Theta_{D,1} y_n) \;\le\; \exp(n \cdot (h_{\mathrm{main}}(D) - m \cdot \mathrm{rem}(D) + \log b \cdot \mathrm{budget}(D))).$$

Input 2 (Theorem C: integral forms are long; unit 151). *Let P be parameters, D admissible data for P with third scale b, n a natural number, $y_1, \dots, y_n$ rows in $\mathrm{Row}(D)$, and δ and V real numbers. Assume that $y_1, \dots, y_n$ are linearly independent over $\mathbb{C}$, that $(1 - 4/b^2) \cdot \dim \mathrm{Row}(D) \le n$, that $y_1, \dots, y_n$ have D-index at most δ, and that $\mathrm{covol}_\sigma^2(\Theta_{D,1} y_1, \dots, \Theta_{D,1} y_n) \le \exp(V)$ for every automorphism σ of $\mathbb{C}$. Then*

$$\dim \mathrm{Row}(D) \cdot \left(h_{\mathrm{main}}(D) - 3 \cdot \log b \cdot \mathrm{budget}(D)\right) \;\le\; V + 2\delta.$$

LEAN `Data.exists_family_covolumeSq_le` · `Data.integral_forms_long`

In the words of the argument, Input 1 puts together Theorem A (short forms: there are integral forms whose lifts are short, with a bound for the index) and Theorem B (enough forms: few translates of them almost span the rows). Input 2 is Theorem C. Three smaller inputs are needed as well.

Input 3 (a crude bound; unit 151). *Let P be parameters and D admissible data for P, with number of lifts m and scales x and b. Then $|h_{\mathrm{main}}(D)| + m \cdot |\mathrm{rem}(D)| \le b^2 \cdot x^{99}$.*

Input 4 (there are rows; unit 148). *Let P be parameters and D admissible data for P. Then $\dim \mathrm{Row}(D) \neq 0$; in particular $\mathrm{Row}(D)$ has finite dimension.*

Input 5 (the constant and the places; unit 152). *Let $P = (N_0, d)$ be parameters. Then $64 \le C(P)$, $N_0 \le C(P)$, $d \le C(P)$ and $\varphi(N_0) \le C(P)$. Let moreover D be admissible data for P. Then N_0 divides $N_1(D)$,*

$$f(D) = \frac{N_1(D)}{N_0} \cdot \varphi(N_0) \qquad \text{and} \qquad \#\mathrm{tops}(D) = \frac{N_1(D)}{N_0}.$$

The formal text states the two equations for every exponent of the tower; they are used here for the exponent of D.

LEAN `Data.abs_mainHeight_add_remainder_le` · `Data.finrank_rows_ne_zero` · `Data.c_ge` · `Parameters.card_infinitePlace_E` · `Parameters.card_placeClass_eq`

Lemma 3.1 (sizes). *Let $P = (N_0, d)$ be parameters and D admissible data for P, with scales x, x_1, b. Then*

(a) $\log b \geq 10^6 \log x_1 \geq 10^{12} \log x$, $\log x \geq 100 \log\log b \geq 10^{14}\, C(P)$ *and* $C(P) \geq 64$*; in particular* $x \geq 2$, $\log b \geq 35$, $x^{99} \leq b$ *and* $\log b \leq x^{1/100}$*;*

(b) $a(D) \geq x\, b$, $f(D) \geq 1$, *and* $\mathrm{budget}(D) \geq b\,(\log b)^3 > 0$*;*

(c) $4 \cdot \log b \cdot \mathrm{budget}(D) + 4\, x^{99} + 10 \cdot \log(1 + b^2) \cdot \mathrm{budget}(D) \;\leq\; (\log b)^2 \cdot \mathrm{budget}(D)$.

Proof. (a) The four inequalities between the logarithms are conditions (i) of Definition 2.3, and $C(P) \geq 64$ is in Input 5. Hence $\log x \geq 64 \cdot 10^{14}$, which gives $x \geq 2$; then $\log b \geq 10^{12} \log x \geq 35$; $99 \log x \leq 10^{12} \log x \leq \log b$ gives $x^{99} \leq b$; and $100 \log\log b \leq \log x$ gives $\log b \leq x^{1/100}$.

(b) The first inequality is the definition of $a(D)$. A number field has an infinite place, so $f(D) \geq 1$. With $x \geq 1$ this gives $\mathrm{budget}(D) = f(D) \cdot a(D) \cdot (\log b)^3 \geq b\,(\log b)^3$, which is positive by (a).

(c) By (a) and (b), $4x^{99} \leq 4b \leq b\,(\log b)^3 \leq \mathrm{budget}(D)$. As $b \geq 1$ and $\log b \geq \log 2$, also $\log(1 + b^2) \leq \log(2b^2) \leq 3 \log b$. So the left side of (c) is at most

$$(4 \log b + 1 + 30 \log b) \cdot \mathrm{budget}(D) \;\leq\; 35\, \log b \cdot \mathrm{budget}(D),$$

and $35 \leq \log b$. □

LEAN in the formal text these comparisons are made by a calculus of sizes of unit 152, `Data.le_of_hasLogBound`; see §8, note 5

Proof of Theorem 1. Let n, $y_1, \dots, y_n$ and δ be as in Input 1, and put

$$h_{\mathrm{A}} = h_{\mathrm{main}}(D) - m \cdot \mathrm{rem}(D) + \log b \cdot \mathrm{budget}(D).$$

The last property in Input 1 says that the squared covolumes of $\Theta_{D,1} y_1, \dots, \Theta_{D,1} y_n$ are at most $\exp(n\, h_{\mathrm{A}})$. So Input 2 applies to n, $y_1, \dots, y_n$, δ and $V = n\, h_{\mathrm{A}}$, and gives

$$\begin{aligned} \dim \mathrm{Row}(D) \cdot \big(h_{\mathrm{main}}(D) - 3 \cdot \log b \cdot \mathrm{budget}(D)\big) \;&\leq\; n\, h_{\mathrm{A}} + 2\delta \\ &=\; \dim \mathrm{Row}(D) \cdot h_{\mathrm{A}} + \big(\dim \mathrm{Row}(D) - n\big) \cdot (-h_{\mathrm{A}}) + 2\delta. \end{aligned} \tag{1}$$

The two bounds are compared on families of different sizes, n and $\dim \mathrm{Row}(D)$, and the middle term on the right pays for the difference. It is small for the following reasons. By Input 4 the space of rows has finite dimension, so the linearly independent rows $y_1, \dots, y_n$ are at most $\dim \mathrm{Row}(D)$ in number, and with the second property in Input 1,

$$0 \;\leq\; \dim \mathrm{Row}(D) - n \;\leq\; \frac{4}{b^2} \cdot \dim \mathrm{Row}(D).$$

By Lemma 3.1(b), $\log b \cdot \mathrm{budget}(D) > 0$. With Input 3 and $m \geq 0$,

$$-h_{\mathrm{A}} = -h_{\mathrm{main}}(D) + m \cdot \mathrm{rem}(D) - \log b \cdot \mathrm{budget}(D) \;\leq\; |h_{\mathrm{main}}(D)| + m \cdot |\mathrm{rem}(D)| \;\leq\; b^2\, x^{99}.$$

Hence $(\dim \mathrm{Row}(D) - n) \cdot (-h_{\mathrm{A}}) \leq 4\, x^{99} \cdot \dim \mathrm{Row}(D)$. The bound for δ in Input 1 gives $2\delta \leq 10 \cdot \log(1 + b^2) \cdot \mathrm{budget}(D) \cdot \dim \mathrm{Row}(D)$. Putting these into (1),

$$h_{\mathrm{main}}(D) - 3 \cdot \log b \cdot \mathrm{budget}(D) \;\leq\; h_{\mathrm{A}} + 4\, x^{99} + 10 \cdot \log(1 + b^2) \cdot \mathrm{budget}(D),$$

after division by $\dim \mathrm{Row}(D)$, which is positive by Input 4. Inserting the definition of h_{A}, the main amount $h_{\mathrm{main}}(D)$ cancels, and

$$\begin{aligned} m \cdot \mathrm{rem}(D) \;&\leq\; 4 \cdot \log b \cdot \mathrm{budget}(D) + 4\, x^{99} + 10 \cdot \log(1 + b^2) \cdot \mathrm{budget}(D) \\ &\leq\; (\log b)^2 \cdot \mathrm{budget}(D), \end{aligned}$$

the last step by Lemma 3.1(c). □

LEAN star_inequality

The three terms of the error have separate sources: $4 \cdot \log b \cdot \mathrm{budget}(D)$ is the error of length in Input 1 and the error of covolume in Input 2, $4x^{99}$ pays for the dimensions that the rows do not span, and the last term is the index. All three are far below $(\log b)^2 \cdot \mathrm{budget}(D)$; the exponents 99 and 2 are slack.

4 The bound for the primes

Definition 4.1 (polar functions)**.** Let $\mathcal{E}\colon \mathbb{R} \to \mathbb{R}$ and $\omega\colon \mathbb{C} \to \mathbb{C}$ be functions. We say that $\mathcal{E}$ is polar with residue function ω, and write $\mathrm{Polar}(\mathcal{E}, \omega)$, if $\mathcal{E}$ is measurable and there is a function $\Phi\colon \mathbb{C} \to \mathbb{C}$, meromorphic on $\mathbb{C}$ and defined at every point (at a pole its value is arbitrary), such that: the function $v \mapsto |\mathcal{E}(v)| \exp(-v)$ is bounded on $v \geq 0$; $\Phi(s) = \int_0^\infty \mathcal{E}(v) \exp((1-s)v)\, dv$ for $\mathrm{Re}\, s > 2$; the function $s \mapsto (s-1)\Phi(s)$, whose value at $s = 1$ is 0 whatever $\Phi(1)$ is, is analytic at every point of the closed half-plane $\mathrm{Re}\, s \geq 1$; and for every complex number ρ with $\mathrm{Re}\, \rho > 0$, $(s-\rho)\Phi(s) \to \omega(\rho)$ as $s \to \rho$, $s \neq \rho$.

LEAN IsPolar · unit 107, background

So Φ continues the Laplace transform of $\mathcal{E}$. The third condition is to be read with its clause on the value at 1: it says that Φ is analytic at every point $s \neq 1$ of $\mathrm{Re}\, s \geq 1$, and that near $s = 1$ it is the quotient by $s - 1$ of an analytic function that vanishes at 1. So the singularity at 1 is removable, Φ has no pole on $\mathrm{Re}\, s \geq 1$, and $\omega(\rho) = 0$ for $\mathrm{Re}\, \rho \geq 1$. The clause cannot be dropped. Without it the constant function 1, whose transform is $1/(s-1)$, would be polar, and Input 9 below would be false for the constant functions 1 and 0: it would give $1 \leq \exp(-v_0)$ for large v_0. In the half-plane $\mathrm{Re}\, s > 0$ the poles of Φ are at most simple, with residue $\omega(\rho)$ at ρ. The definition, with a remark that derives these consequences, is also in Chapter II of this handout, the note *From a one-sided bound for primes to a zero-free half-plane*, in the same letters.

Inputs from other units.

Input 6 (the remainder of the data, through the remainders of the prime sums; unit 151). *Let $P = (N_0, d)$ be parameters and D admissible data for P, with first scale x and second scale x_1. Then*

$$\mathrm{rem}(D) = a(D) \cdot \Big(\#\mathrm{thick}(D) \cdot (R_{d,N_0}(x) - R_{d,N_0}(x_1)) + \#\mathrm{tops}(D) \cdot R_{d,1}(x_1) \Big).$$

Input 7 (data exist; unit 152). *Let $P = (N_0, d)$ be parameters and ε a real number with $0 < \varepsilon/6$ and $\varepsilon < \frac{1}{100}$. Then for every sufficiently large real number x there are admissible data D for P with first scale x, whose second scale x_1 and third scale b satisfy*

$$\log x_1 = x^{\varepsilon/6}, \qquad (10^6 \log x)^2 \leq \log x_1,$$
$$\log\log b = \varepsilon \log x, \qquad \frac{N_1(D)/N_0}{\log b} \leq \#\mathrm{thick}(D).$$

LEAN Data.remainder_eq · exists_data

Input 6 is where the prime sums of Definition 2.6 become the remainders of Definition 2.5: as τ runs over the tops, the places $(-\nu)\cdot\tau$ run over the classes $\equiv -\nu$ modulo N_0, which contain the thick places exactly when $\nu \equiv 1$ modulo N_0. So the thick places see the weight $\lambda_{d,N_0}(\nu)$ at the first scale, every place sees $\lambda_{d,1}(\nu)$ at the second scale, and the main terms are the expected values. Input 7 provides the data, and the next two inputs make it possible to move the second scale.

Input 8 (the remainders are polar; unit 151). *Let d be a prime number and $M \geq 1$ an integer such that $d \equiv 7$ modulo 8 and M is prime to d. For a complex number $\rho \neq 0, 1$ let*

$$\mathrm{res}_{d,M}(\rho) = \frac{1}{\varphi(M)\,\rho\,(1-\rho)} \sum_{\xi} (\mathrm{ord}_\rho\, \tilde{L}(\,\cdot\,,\xi) - \mathrm{ord}_\rho\, \tilde{L}(\,\cdot\,,\xi\chi_K)),$$

where ξ runs over the Dirichlet characters modulo M, $\xi\chi_K$ is as in Definition 2.5, and ord_ρ is the order of vanishing at ρ of an entire function that is not zero; and let $\mathrm{res}_{d,M}(0) = \mathrm{res}_{d,M}(1) = 0$. Then, as a function of the real variable v, $v \mapsto R_{d,M}(\exp(v))$ is polar with residue function $\rho \mapsto \mathrm{res}_{d,M}(\rho)$.

Input 9 (a good point in every long window; unit 151). *Let $\mathcal{E}_1$, $\mathcal{E}_2$ be functions from $\mathbb{R}$ to $\mathbb{R}$ and ω_1, ω_2 functions from $\mathbb{C}$ to $\mathbb{C}$ such that $\mathcal{E}_1$ is polar with residue function ω_1 and $\mathcal{E}_2$ is polar with residue function ω_2. Then for every sufficiently large real number v_0 and every real number X with $0 \leq X \leq \exp(v_0)$ there is a real number v_1 with $v_0 \leq v_1 \leq v_0^2$ and*

$$\mathcal{E}_1(v_1) - X \cdot \mathcal{E}_2(v_1) \;\leq\; \exp(-v_0).$$

LEAN `isPolar_primeError` · `IsPolar.exists_window_sub_mul_le`

In Input 8 the formal text gives the residue function by the one formula, for every complex ρ; a quotient by 0 is 0 in Lean, which is the value at $\rho = 0$ and $\rho = 1$ stated above. The values of the residue function play no role in this note, which uses only that the remainders are polar.

Lemma 4.2 (the parameters). *Let $P = (N_0, d)$ be parameters. Then $d \equiv 7$ modulo 8, and N_0 is prime to d.*

Proof. As 16 divides 18480, which divides N_0, and N_0 divides $2(d+1)$, the number 8 divides $d+1$. Let p be a prime that divides N_0. Then p divides $2(d+1)$. If $p = d$, then d divides 2, which is impossible since $d \equiv 7$ modulo 8. Hence no prime divides both N_0 and d. □

LEAN `Parameters.instHeegD` · `Parameters.coprime_N₁_d` · unit 152; the proof here is direct

Lemma 4.3 (moving the second scale). *Let P be parameters, D admissible data for P with scales x, x_1, b, and v_1 a real number with $10^6 \log x \leq v_1 \leq \log x_1$. Let D' be the record whose second scale is $\exp(v_1)$ and whose other components are those of D. Then D' are admissible data for P. Moreover $N_1(D') = N_1(D)$, $E(D') = E(D)$, $\mathrm{thick}(D') = \mathrm{thick}(D)$, $\mathrm{tops}(D') = \mathrm{tops}(D)$, $f(D') = f(D)$, $a(D') = a(D)$ and $\mathrm{budget}(D') = \mathrm{budget}(D)$.*

Proof. Only the conditions (i) of Definition 2.3 mention $\log x_1$, and of these the last three. For D' the first of the three is the lower bound for v_1. Since $0 < v_1 \le \log x_1$, the other two follow from those for D: $10^6 v_1 \le 10^6 \log x_1 \le \log b$ and $6 \log v_1 \le 6 \log\log x_1 \le \log\log b$. The quantities listed do not involve the second scale. □

Proof of Theorem 2. By Lemma 4.2, $d \equiv 7$ modulo 8 and N_0 is prime to d; also 1 is prime to d. So Input 8 applies with $M = N_0$ and with $M = 1$: the functions $\mathcal{E}_1(v) = R_{d,N_0}(\exp(v))$ and $\mathcal{E}_2(v) = R_{d,1}(\exp(v))$ are polar, and Input 9 applies to them. Let x be so large that Input 7 gives admissible data D for P with first scale x and the properties stated there, and that $v_0 = 10^6 \log x$ is large enough for the conclusion of Input 9. Let x_1 and b be the second and third scale of D and m its number of lifts. We show that $R_{d,N_0}(x) \le x^{8\varepsilon+\frac{1}{2}-1}$.

The window. Put $X = (N_1(D)/N_0)/\#\mathrm{thick}(D)$. The last property in Input 7, with $N_1(D)/N_0 \ge 1$ and $\log b > 0$, shows that $\#\mathrm{thick}(D) > 0$ and that $0 \le X \le \log b$. By Lemma 3.1(a), $\log b \le x^{1/100} \le \exp(v_0)$. So Input 9 gives a real number v_1 with

$$v_0 \le v_1 \le v_0^2 \qquad \text{and} \qquad R_{d,N_0}(\exp(v_1)) - X \cdot R_{d,1}(\exp(v_1)) \;\le\; \exp(-v_0) = x^{-10^6}. \tag{2}$$

The new data. By Input 7, $v_0^2 \le \log x_1$. So $10^6 \log x = v_0 \le v_1 \le \log x_1$, and Lemma 4.3 gives admissible data D' for P with second scale $\exp(v_1)$, which agree with D in the first and third scale, the number of lifts, the thick places, the tops, and in $f(D)$, $a(D)$ and the budget.

The identity. Input 6 for D', with $\#\mathrm{tops}(D) = N_1(D)/N_0$ from Input 5, reads

$$\mathrm{rem}(D') = a(D) \cdot \Big(\#\mathrm{thick}(D) \cdot (R_{d,N_0}(x) - R_{d,N_0}(\exp(v_1))) + \frac{N_1(D)}{N_0} \cdot R_{d,1}(\exp(v_1)) \Big).$$

Solving for $R_{d,N_0}(x)$, which is possible because $a(D)$ and $\#\mathrm{thick}(D)$ are positive,

$$R_{d,N_0}(x) = \frac{m \cdot \mathrm{rem}(D')}{m \cdot a(D) \cdot \#\mathrm{thick}(D)} + \Big(R_{d,N_0}(\exp(v_1)) - X \cdot R_{d,1}(\exp(v_1)) \Big). \tag{3}$$

The two bounds. The bracket in (3) is at most x^{-10^6} by (2). Theorem 1 for D' gives $m \cdot \mathrm{rem}(D') \le (\log b)^2 \cdot \mathrm{budget}(D) = f(D) \cdot a(D) \cdot (\log b)^5$. So the first term of (3) is at most

$$\frac{f(D) \cdot (\log b)^5}{m \cdot \#\mathrm{thick}(D)} \;\le\; \frac{\varphi(N_0)\,(\log b)^6}{\sqrt{x}} \;=\; \varphi(N_0)\, x^{6\varepsilon-\frac{1}{2}},$$

where the inequality uses the value $f(D) = (N_1(D)/N_0) \cdot \varphi(N_0)$ from Input 5, the bound $N_1(D)/N_0 \le \log b \cdot \#\mathrm{thick}(D)$ of Input 7, and $m \ge \sqrt{x}$; the equality is $\log b = x^{\varepsilon}$, again from Input 7.

The sizes. By Input 5 and the conditions (i), $\varphi(N_0) \le C(P) \le 10^{-12} \log\log b = 10^{-12}\, \varepsilon \log x$, and $\varepsilon \log x \le \frac{1}{2} \exp(2\varepsilon \log x) = \frac{1}{2} x^{2\varepsilon}$. So the first term is at most $\frac{1}{2} x^{8\varepsilon-\frac{1}{2}}$. As $x \ge 2$ by Lemma 3.1(a) and $\varepsilon > 0$, $x^{-10^6} \le \frac{1}{2} x^{8\varepsilon-\frac{1}{2}}$. Therefore $R_{d,N_0}(x) \le x^{8\varepsilon-\frac{1}{2}} = x^{8\varepsilon+\frac{1}{2}-1}$. □

LEAN `eventually_primeError_le_rpow`

The saving $x^{-1/2}$ comes from the division by m: Theorem 1 bounds $m \cdot \mathrm{rem}(D')$, not $\mathrm{rem}(D')$, by a quantity of the size of the trivial bound for the remainder, and m is about $\sqrt{x}$. The number of lifts is in this sense the source of the exponent $\frac{1}{2}$.

5 The zero-free half-plane

Input from another unit.

Input 10 (the one-sided criterion, after Landau; unit 151). *Let $M \geq 1$ be an integer, c an element of $\mathbb{Z}/M$ and β a real number, with $\frac{1}{2} < \beta$ and c a unit of the ring $\mathbb{Z}/M$. Assume that for every prime number d whose class modulo M is c: $d \equiv 7$ modulo 8, M is prime to d, and $R_{d,M}(x) \leq x^{\beta-1}$ for every sufficiently large real number x. Then for every complex number ρ with $\beta < \operatorname{Re}\rho$ and every Dirichlet character χ modulo M, $\tilde{L}(\rho, \chi) \neq 0$.*

LEAN `entireL_ne_zero_of_primeError_le`

The criterion is read in the note on the endgame. Its proof takes one suitable prime d of the class; by Input 8 the Laplace transform of $v \mapsto R_{d,M}(\exp(v))$ has a pole at every zero of an L-function modulo M that is not cancelled by a zero of a twisted L-function, and Landau's theorem on integrals of functions of one sign excludes poles to the right of β once the one-sided bound holds.

In the proof of Theorem 3 two facts of the library about characters are used. Let $M \geq 1$ be an integer, χ a Dirichlet character modulo M, and M' a multiple of M with $M' \geq 1$; let χ' be the character modulo M' induced by χ. First, χ' is principal if and only if χ is. Second, for every complex number s such that χ is not principal or $s \neq 1$,

$$L(s, \chi') = L(s, \chi) \cdot \prod_{p \mid M'} (1 - \chi(p)\, p^{-s}), \tag{4}$$

the product over the primes p dividing M'.

Proof of Theorem 3. Let $M \geq 1$ be an integer, χ a Dirichlet character modulo M and s a complex number with $\operatorname{Re} s > \frac{1}{2}$, such that χ is not principal or $s \neq 1$. Assume that $L(s, \chi) = 0$; we derive a contradiction. Put

$$\varepsilon = \frac{\min(\operatorname{Re} s, 1) - \frac{1}{2}}{51}, \qquad \beta = 8\varepsilon + \tfrac{1}{2}.$$

Then $0 < \varepsilon \leq \frac{1}{102} < \frac{1}{100}$ and $\frac{1}{2} < \beta$, and $\beta < \operatorname{Re} s$ because $8\varepsilon < \min(\operatorname{Re} s, 1) - \frac{1}{2}$.

The modulus. Let $N_0 = 18480\, M$, and let χ' be the character modulo N_0 induced by χ. Then $\tilde{L}(s, \chi') = 0$. Indeed, if χ is not principal, then χ' is not principal, and $\tilde{L}(s, \chi') = L(s, \chi') = 0$ by (4). If χ is principal, then χ' is principal and $s \neq 1$, so $\tilde{L}(s, \chi') = (s-1)L(s, \chi') = 0$, again by (4).

The class. Let c be the class of $9240\, M - 1$ modulo N_0, that is, of $N_0/2 - 1$. It is a unit, because $(9240\, M - 1)^2 = 18480\, M \cdot (4620\, M - 1) + 1$. Let d be a prime number in the class c. Then $P = (N_0, d)$ are parameters: 18480 divides N_0; and $d + 1 \equiv N_0/2$ modulo N_0, so N_0 divides $2(d+1)$ and does not divide $d + 1$. By Lemma 4.2, $d \equiv 7$ modulo 8 and N_0 is prime to d. By Theorem 2 for P and ε, $R_{d,N_0}(x) \leq x^{8\varepsilon + \frac{1}{2} - 1} = x^{\beta - 1}$ for every sufficiently large real number x.

The criterion. So the hypotheses of Input 10 hold for the modulus N_0, the class c and β. As $\beta < \operatorname{Re} s$, it gives $\tilde{L}(s, \chi') \neq 0$. This contradicts the first step. □

LEAN `New.grh`

Theorem 2 is used here for every prime d of one residue class modulo N_0, with one ε that depends on s; how large x has to be may depend on d. The modulus $18480\,M$ is chosen so that it can be the first component of parameters, and the passage from M to it costs nothing, because (4) is needed only in one direction: a zero of $L(\,\cdot\,,\chi)$ is a zero of $L(\,\cdot\,,\chi')$.

6 The critical line

Input from another unit.

Input 11 (symmetry of the zeros; unit 106, background). *Let $M \geq 1$ be an integer, χ a Dirichlet character modulo M, and s a complex number with $\operatorname{Re} s \neq 0$, $s \neq 1$ and $L(s,\chi) = 0$. Then $\Gamma_\chi(s) = 0$ or $L(1-s,\chi^{-1}) = 0$.*

LEAN `LFunction_one_sub_eq_zero`

Here Γ_χ is the gamma factor of χ in the library: $\Gamma_\chi(s) = \Gamma_{\mathbb{R}}(s)$ if $\chi(-1) = 1$ and $\Gamma_\chi(s) = \Gamma_{\mathbb{R}}(s+1)$ otherwise, where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\,\Gamma(s/2)$. The library's function Γ takes the value 0 at its poles $0, -1, -2, \dots$, so $\Gamma_\chi(s) = 0$ says that s is a pole of the gamma factor. Input 11 is the functional equation of the completed L-function, in the form that it takes for the zeros and for characters that need not be primitive. It is a classical result; the catalogue of quoted results of this write-up records it with its reference still to be confirmed. The library also provides the following facts, in which ζ^* is its completed zeta function.

(Z1) For the Dirichlet character χ modulo 1, $L(s,\chi) = \zeta(s)$ for every complex number s.

(Z2) $\zeta(0) = -\frac{1}{2}$.

(Z3) For $s \neq 0$, $\zeta(s) = \zeta^*(s)/\Gamma_{\mathbb{R}}(s)$, where a quotient by 0 is 0.

(Z4) $\zeta^*(1-s) = \zeta^*(s)$ for every complex number s.

(Z5) $\Gamma_{\mathbb{R}}(s) = 0$ if and only if $s = -2n$ for a natural number n; in particular $\Gamma_{\mathbb{R}}(s) \neq 0$ when $\operatorname{Re} s > 0$.

Proof of Theorem 4. Assume the half-plane statement. Let $M \geq 1$ be an integer, χ a Dirichlet character modulo M and s a complex number with $0 < \operatorname{Re} s < 1$ and $L(s,\chi) = 0$. If $\operatorname{Re} s > \frac{1}{2}$, the half-plane statement gives $L(s,\chi) \neq 0$, since $s \neq 1$; so $\operatorname{Re} s \leq \frac{1}{2}$. Suppose that $\operatorname{Re} s < \frac{1}{2}$. Input 11 applies, because $\operatorname{Re} s \neq 0$ and $s \neq 1$. The gamma factor does not vanish at s, by (Z5): $\operatorname{Re} s > 0$ and $\operatorname{Re}(s+1) > 0$. Hence $L(1-s,\chi^{-1}) = 0$. But $\operatorname{Re}(1-s) > \frac{1}{2}$ and $1-s \neq 1$, so the half-plane statement for the character χ^{-1} gives $L(1-s,\chi^{-1}) \neq 0$, a contradiction. Therefore $\operatorname{Re} s = \frac{1}{2}$. □

LEAN `New.zeros_on_line`

Proof of Theorem 5. Assume the half-plane statement, and let s be a complex number with $\zeta(s) = 0$, not of the form $-2(n+1)$ with n a natural number, and with $s \neq 1$. By (Z1), the half-plane statement for $M = 1$ says: $\zeta(\rho) \neq 0$ for every complex number ρ with $\operatorname{Re}\rho > \frac{1}{2}$ and $\rho \neq 1$. With $\rho = s$ this gives $\operatorname{Re} s \leq \frac{1}{2}$.

By (Z2), $s \neq 0$. So s is not of the form $-2n$ with n a natural number, and $\Gamma_{\mathbb{R}}(s) \neq 0$ by (Z5). By (Z3), $0 = \zeta(s) = \zeta^*(s)/\Gamma_{\mathbb{R}}(s)$, hence $\zeta^*(s) = 0$, and $\zeta^*(1-s) = 0$ by (Z4). As $1-s \neq 0$, (Z3) gives $\zeta(1-s) = \zeta^*(1-s)/\Gamma_{\mathbb{R}}(1-s) = 0$. Now $1-s \neq 1$;

so if $\operatorname{Re}(1-s) > \frac{1}{2}$, the half-plane statement with $\rho = 1-s$ would give $\zeta(1-s) \neq 0$. Hence $\operatorname{Re}(1-s) \leq \frac{1}{2}$, that is, $\operatorname{Re} s \geq \frac{1}{2}$, and so $\operatorname{Re} s = \frac{1}{2}$. □

LEAN `New.zeta_zeros_on_line`

7 Where the results are used

Theorem 1 is used only in the proof of Theorem 2, and Theorem 2 only in the proof of Theorem 3. Theorems 3, 4 and 5 are the three inputs of the unit that holds the five final theorems: the third final theorem is Theorem 3, the fourth is Theorem 4 applied to Theorem 3, and the first two are Theorem 5 applied to Theorem 3. Nothing else in the artifact uses this unit.

8 Notes on the formal text

1. *Names.* The authors call Theorem 1 the inequality ($\star$) and Theorem 2 the one-sided bound. In the formal text the parameters and the data of Theorem 1 are implicit arguments.

2. *Theorem 2.* The two inequalities for ε form one hypothesis, a conjunction, written $0 < \varepsilon/6$ and $\varepsilon < \frac{1}{100}$; the exponent is written $8\varepsilon + 2^{-1} - 1$. "For every sufficiently large real number" renders Lean's "eventually, as the number tends to infinity". In Definition 2.5(d) the formal definition is made for every real x; only large x matters. The constant $\kappa_{d,M}$ is the real part of the value at 1 of a function that the formal text defines for every complex argument.

3. *The record of the data.* The formal record stores the three logarithms, not the scales, and its conditions (i) are written with the library's logarithm, which is defined for all real numbers as the logarithm of the absolute value. The conditions force $\log x$, $\log x_1$ and $\log b$ to be positive (Lemma 3.1(a) is proved from them in the order in which (i) lists them), so the iterated logarithms are the usual ones. The record has one more component, a natural number that the formal text calls the exponent; a condition makes it equal to $22 \log_2 m$. This write-up keeps the formal text's letter for it free for the complex variable and does not name it: the degree of the tower, a function of the exponent in the formal text, is $N_1(D)/N_0$ here, and the conditions on the exponent are those on $22 \log_2 m$ in (ii).

4. *Input 7 is not used in full.* The proof of Theorem 2 does not use the equation $\log x_1 = x^{\varepsilon/6}$; it uses the inequality $(10^6 \log x)^2 \leq \log x_1$, which leaves room for the window.

5. *The comparisons of sizes.* The formal text does not carry out the estimates of Lemma 3.1(c) and of the last step of the proof of Theorem 2. It proves them by a calculus of unit 152: every quantity built from the data carries an upper or a lower bound that is the exponential of a real number, found by Lean's search for instances, and an inequality between two quantities is reduced to an inequality between two such exponents, which a decision procedure proves from the conditions (i). The estimates in this note are explicit, and they are mine; they use the same facts, which are the conditions (i) and (ii), the definition of $a(D)$ and Input 5. Of Input 5 the formal proofs of this unit cite by name only the number of the tops; the bounds for $C(P)$ and the number of the infinite places enter through the calculus. All three statements were read in the Lean text.

6. *The new data.* Lemma 4.3 is not a declaration. The formal text builds D' inside the proof of Theorem 2 by replacing one component, and proves the conditions (i) on the spot.
7. *Lemma 4.2.* The formal text cites two declarations of unit 152. The second says that the order of the roots of unity, for every exponent of the tower, is prime to d, and deduces this from the fact that χ_K takes the value 1 at every prime dividing that order; the formal proofs of Theorems 2 and 3 use it for the exponent 0 and keep the factor N_0. The proof given here is direct.
8. *The constant* 51. In the proof of Theorem 3 the formal text defines ε with the same divisor 51; any divisor larger than 50 would do.
9. *Theorems 4 and 5.* Each of the two formal proofs is one call of an automatic procedure that is given the facts used above: Input 11 and (Z5) for Theorem 4, (Z1) to (Z5) for Theorem 5. The case distinctions written out here are what that procedure has to find. They depend on the library's conventions, which are part of the statements (Z3) and (Z5): a quotient by 0 is 0, and Γ is 0 at its poles.
10. *What the unit does not use.* Every declaration of the unit is used by the five final theorems.

How this note was written. From the structured proofs of the unit: the program check of the dossier covered 27 leaf steps of 5 proofs and compared the quotations of the imported declarations with the source. In the Lean text I read the whole file of the unit, with the long comment at its head in which the authors list the inputs, the scales and the errors. I read the statements of all eleven Inputs, and the definitions behind every notion of §2: the parameters, the tower, the field, its places and their classes, every field of the record of admissible data, the derived quantities, the weight, the remainder of the prime sum with the functions it is built from, the prime sums and the remainder of the data, the polar functions, and the lines that define the rows, the lifts, the index, the squared covolume and the main amount. I read the proof of Input 11, which is short, and did not study the proofs of the other Inputs. In the library, at the pinned commit of its public repository, I read the statements listed in the second table of Appendix A. The letters and the forms of the notations are those of the overview of this handout. The proofs follow the structured proofs, with these differences, which are mine: Lemma 3.1 and the estimates at the end of the proof of Theorem 2 replace the calculus of sizes; Lemma 4.2 is proved directly; the proofs of Theorems 4 and 5 are written out; and the explanations between the proofs are mine.

Appendix A. Concordance with the Lean artifact

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Theorem 4	`New.zeros_on_line`	Main.lean	194–196
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Definition 2.1	`DirichletLNonvanishingRightOfHalf`	Statements.lean	6–8

In this note	Lean declaration	File	Lines
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Definition 2.3	`Data`	AdmissibleData.lean	22–45
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Input 5	`Parameters.card_infinitePlace_E`	Different.lean	168–170
Input 5	`Parameters.card_placeClass_eq`	Different.lean	174–176
Input 6	`Data.remainder_eq`	ShortAndEnough.lean	86–112
Input 7	`exists_data`	DataExist.lean	49–108
Definition 4.1	`IsPolar`	PolarLandau.lean	34–36
Input 8	`isPolar_primeError`	PrimeError.lean	69–111
Input 9	`IsPolar.exists_window_sub_mul_le`	OneSidedCriterion.lean	73–99
Input 10	`entireL_ne_zero_of_primeError_le`	OneSidedCriterion.lean	28–36
Input 11	`LFunction_one_sub_eq_zero`	ZeroSymmetry.lean	27–33
Lemma 4.2	`Parameters.instHeegD`	Different.lean	36–37
Lemma 4.2	`Parameters.coprime_N₁_d`	Different.lean	74–75
Lemma 3.1	`Data.le_of_hasLogBound`	Magnitudes.lean	121–121

The five statements of the unit as they are written in the artifact. Theorems 1 to 5:

```
lemma star_inequality {P} {D : Data P} : D.m * D.remainder ≤ D.logB ^ 2 * D.budget

lemma eventually_primeError_le_rpow (P : Parameters) {ε : ℝ} (h : 0 < ε / 6 ∧ ε < 1 / 100) :
    ∀ᶠ x in atTop, primeError P.d P.N₀ x ≤ x ^ (8 * ε + 2⁻¹ - 1)

lemma New.grh : DirichletLNonvanishingRightOfHalf

lemma New.zeros_on_line (h : DirichletLNonvanishingRightOfHalf) : DirichletLZerosOnLine

lemma New.zeta_zeros_on_line (h : DirichletLNonvanishingRightOfHalf) :
    RiemannZetaZerosOnLine
```

Inputs 1 and 2:

```
lemma Data.exists_family_covolumeSq_le (D : Data P) : ∃ n, ∃ y : Fin n → _, LinearIndependent ℂ y ∧
    (1 - 4 / D.b ^ 2) * D.rowDim ≤ n ∧
  (∃ δ ≤ 5 * D.rowDim * log (1 + D.b ^ 2) * D.budget, D.IndexLe y δ) ∧ ∀ σ,
  covolumeSq (D.θ 1 ∘ y) σ ≤ exp (n * (D.mainHeight - D.m * D.remainder + D.logB * D.budget))

lemma Data.integral_forms_long : D.rowDim * (D.mainHeight - 3 * (D.logB * D.budget)) ≤
  V + 2 * δ
```

Input 2 is stated in a part of its file where the rows, the two real numbers and its four hypotheses are declared beforehand; the hypotheses are the arguments `i`, `hn`, `I` and `hV`.

Legend. For data `D` of the formal text: `D.logX`, `D.logX₁`, `D.logB` are $\log x$, $\log x_1$, $\log b$, and `D.x`, `D.x₁`, `D.b` are the scales; `D.m` and `D.r` are m and r; `D.s` is the exponent $22 \log_2 m$, and `towerDeg D.s` is $N_1(D)/N_0$; `D.thick` is thick(D); `D.Φ` is the set of infinite places of $E(D)$; `P.N₀`, `P.d` and `P.c` are N_0, d and $C(P)$. The class of a place is given by `Parameters.placeMulEquiv` and, modulo a divisor, by `Parameters.placeClass`; the place $\nu \cdot \tau$ is the product of `D.placeOfInt v` and τ. The character $\xi\chi_K$ is `quadTwist ξ d`, and χ_K is `chiK d`; $\kappa_{d,M}$ is the real part of `logDerivMean d M 1`. In the formal proofs of this unit the modulus of Theorem 3 is `q` and the complex number is `ρ`; in the file of the statements they are `N` and `s`. The window of Input 9 is written there with `t`, `υ`, `E` and `E'` for v_0, v_1, $\mathcal{E}_1$ and $\mathcal{E}_2$, and the class of Input 10 with `a` for c. In the proof of Theorem 1 the formal text writes `d` for $\dim \mathrm{Row}(D)$, `L` for $\log b \cdot \mathrm{budget}(D)$ and `h` for h_{A}.

The facts of the library. The library is Mathlib at the commit that the artifact pins in its manifest.

In this note	Declaration of the library
induced character, principal or not	`DirichletCharacter.changeLevel, changeLevel_eq_one_iff`
the product formula (4)	`DirichletCharacter.LFunction_changeLevel`
$\tilde{L}(s,\chi)$ for principal χ	`DirichletCharacter.LFunctionTrivChar₁`
the gamma factors Γ_χ and $\Gamma_{\mathbb{R}}$	`DirichletCharacter.gammaFactor, Complex.Gammaℝ`
(Z1)	`DirichletCharacter.LFunction_modOne_eq`
(Z2)	`riemannZeta_zero`
(Z3)	`riemannZeta_def_of_ne_zero`
(Z4)	`completedRiemannZeta_one_sub`
(Z5)	`Complex.Gammaℝ_eq_zero_iff, Complex.Gammaℝ_ne_zero_of_re_pos`

Notation

$P = (N_0, d)$ parameters: 18480 divides N_0, d is a prime $\equiv N_0/2 - 1$ modulo N_0

K, χ_K the field $\mathbb{Q}(\sqrt{-d})$; the quadratic character of $\mathbb{Z}/d$, the Legendre symbol for odd d

D, D' admissible data for P

x, x_1, b the three scales of D; x also a real variable

m, r the number of lifts and the rank of D

$C(P)$ a constant attached to P

$N_1(D)$, $E(D)$ the order of the roots of unity; the field they generate over K

τ, τ', $\mathrm{cl}_D(\tau)$ infinite places of $E(D)$; the class in $(\mathbb{Z}/N_1(D))^\times$

thick(D), tops(D) the thick places; the tops

$f(D)$, $a(D)$ the number of infinite places of $E(D)$; the least multiple of $N_1(D)$ that is $\geq xb$

budget(D) the budget $f(D) \cdot a(D) \cdot (\log b)^3$

M, M' moduli, integers ≥ 1

ν, n, p an integer; a natural number; a prime number

Λ, φ von Mangoldt's function; Euler's function

$\lambda_{d,M}(\nu)$ the weight $\Lambda(\nu)(1-\chi_K(\nu))$ on $\nu \equiv 1$ modulo M

χ, χ', ξ Dirichlet characters; χ' induced by χ

s, ρ complex numbers

$L(s,\chi)$, $\tilde{L}(s,\chi)$ the L-function; the entire function equal to it, or to $(s-1)L(s,\chi)$ for principal χ

ζ, ζ^* the zeta function; the completed zeta function

Γ, $\Gamma_{\mathbb{R}}$, Γ_χ, π the gamma function; $\pi^{-s/2}\Gamma(s/2)$; the gamma factor of χ

$\kappa_{d,M}$, $R_{d,M}(x)$ a constant; the remainder of the weighted prime sum

$w_D(\tau)$, z the nominal weight; a real number

$\mathrm{Ask}_D(z,\tau)$, $\mathrm{Ask}^{\mathrm{main}}(D)$ a weighted prime sum; its expected value

$\mathrm{cs}(d)$ Euler's constant minus $\mathrm{Re}(L'/L)(1,\chi_K)$

$\mathrm{rem}(D)$ the remainder of the data D

$\mathrm{Row}(D)$, $\dim \mathrm{Row}(D)$ the space of rows; its dimension

ϑ, $\Theta_{D,\vartheta}$ an m-fold twist; the composite of the lifts

$F_1, \dots, F_n$, σ algebraic modular forms; an automorphism of $\mathbb{C}$

$\mathrm{covol}^2_\sigma(F_1, \dots, F_n)$ the squared covolume

$y_1, \dots, y_n$ rows

δ, V real numbers: a bound for the index, the logarithm of a bound for the covolume

ℓ, η_ℓ, J a natural number, mostly a prime; real numbers indexed by ℓ; a finite set of natural numbers (in the definition of the index)

$h_{\mathrm{main}}(D)$, h_{A} the main amount; the bound of Input 1 for one row, $h_{\mathrm{main}}(D) - m \cdot \mathrm{rem}(D) + \log b \cdot \mathrm{budget}(D)$

ε, β real numbers; $\beta = 8\varepsilon + \frac{1}{2}$ in §5

$\mathcal{E}$, $\mathcal{E}_1$, $\mathcal{E}_2$; ω, ω_1, ω_2; Φ functions $\mathbb{R} \to \mathbb{R}$; functions $\mathbb{C} \to \mathbb{C}$, residue functions; the transform, in the definition of polar functions

$\mathrm{res}_{d,M}(\rho)$, ord_ρ the residue of Input 8; the order of vanishing at ρ

v, v_0, v_1, X real numbers in the window of Input 9

c a class modulo M

Chapter II

From a one-sided bound for primes to a zero-free half-plane

- **The endgame: the weighted prime sum and its remainder, the one-sided criterion (Landau), the choice of the second scale, and the statements and assembly of Theorems A, B and C** (unit 151)

  read: note built, structured proofs checked. 29 declarations, 149 proof lines.

READBACK NOTE · CHAPTER II

From a one-sided bound for primes to a zero-free half-plane

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The unit read here closes the argument of the formal proof, and it has two parts. Part I is analytic number theory. For a prime $d \equiv 7 \pmod 8$ and a modulus M prime to d it studies the remainder $R_{d,M}(x)$ of a weighted sum over the prime powers $\equiv 1 \pmod M$ that sees only the primes not split in $\mathbb{Q}(\sqrt{-d})$. The Laplace transform of $v \mapsto R_{d,M}(\mathrm{e}^v)$ is meromorphic. In the half-plane $\operatorname{Re} s > 0$ its poles lie at zeros of the Dirichlet L-functions modulo M and of their twists by the Legendre symbol modulo d, and its residue at such a point ρ is $1/(\varphi(M)\rho(1-\rho))$ times the difference of two sums of orders of vanishing at ρ: that of the L-functions modulo M minus that of their twists; here φ is Euler's function (Theorem 1). By Landau's lemma, a one-sided bound $R_{d,M}(x) \le x^{\beta-1}$, valid for all primes d of one residue class, then forces $L(s,\chi) \neq 0$ for $\operatorname{Re} s > \beta$ and every character χ modulo M (Theorem 2); a companion finds, in every window $[v_0, v_0^2]$, a point where a combination of two such remainders is at most e^{-v_0} (Theorem 3). Part II belongs to the geometric side of the proof: it writes the remainder of the data as a combination of the $R_{d,M}(x)$ (Theorem 4), and it carries out the last steps of the three statements on theta lifts that the authors call Theorems A, B and C.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. Its authors report that it compiles and that the kernel of Lean accepts the final theorems from the three standard axioms; a replay by the owner of this write-up is recorded with the write-up, not here, and nothing was compiled for this note. No mathematician has read the argument through, so the headline is a *claim*, and this note does not say that the Riemann hypothesis is proved. It says what one unit of the formal text asserts and how it argues. The note treats 29 of the 29 declarations of the unit. *Proved here in full*: everything in Part I (Theorems 1 to 3 and the lemmas), from seven results of the background library that are quoted with their statements (Inputs 1 to 7: among them Landau's lemma, the non-vanishing of L-functions on $\operatorname{Re} s \ge 1$, and a zero-free box for quadratic twists); and Theorem 4, from the description of the data. *Deduced here from inputs that carry the substance*: Theorems A, B and C, whose proofs in this unit are short deductions from results of other units, stated as Inputs 8 to 25 with the unit that owns each. Every statement of the unit, and every quoted statement of the background library, was compared by the writer with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 44 declarations.

1 The statements of Part I

The objects first, in one paragraph; the definitions are in §2. Let d be a prime number and $M \geq 1$ a natural number. Write Λ for von Mangoldt's function, φ for Euler's function and χ_d for the quadratic character modulo d, the Legendre symbol when d is odd. The *weight* of a natural number ν is $\lambda_{d,M}(\nu) = \Lambda(\nu)(1 - \chi_d(\nu))$ if $\nu \equiv 1 \pmod M$, and 0 otherwise. For $d \equiv 3 \pmod 4$ it is carried by the powers of d and by the odd powers of the primes that are inert in $\mathbb{Q}(\sqrt{-d})$, in the residue class 1 modulo M. The *remainder* is, for real $x \geq 1$,

$$R_{d,M}(x) = \sum_{1 \leq \nu \leq x} \lambda_{d,M}(\nu)\Big(\frac{1}{\nu} - \frac{1}{x}\Big) - \frac{\log x}{\varphi(M)} - \kappa_{d,M},$$

where the constant $\kappa_{d,M}$ is formed from the logarithmic derivatives at $s = 1$ of the L-functions of the Dirichlet characters ξ modulo M and of their twists $\xi\chi_d$ (Definitions 2.6 and 2.7). For a Dirichlet character χ we write $\tilde{L}(s, \chi)$ for the entire function $L(s, \chi)$, or $(s-1)L(s, \chi)$ if χ is principal, and $\operatorname{ord}_s f$ for the order of vanishing at s of an entire function $f \neq 0$. A function $\mathcal{E}\colon \mathbb{R} \to \mathbb{R}$ is *polar with residue function* $\omega\colon \mathbb{C} \to \mathbb{C}$ (Definition 2.8) if $\mathcal{E}(v) = O(\mathrm{e}^v)$ and the transform $\int_0^\infty \mathcal{E}(v)\,\mathrm{e}^{(1-s)v}\,\mathrm{d}v$ continues to a meromorphic function Φ on $\mathbb{C}$ that has no pole in $\operatorname{Re} s \geq 1$ and satisfies $(s-\rho)\Phi(s) \to \omega(\rho)$ as $s \to \rho$, for every ρ with $\operatorname{Re}\rho > 0$.

Theorem 1 (zeros as poles)**.** Let d be a prime number with $d \equiv 7 \pmod 8$ and let $M \geq 1$ be a natural number with $\gcd(M, d) = 1$. For a complex number s put

$$\operatorname{res}_{d,M}(s) = \frac{1}{\varphi(M)\, s(1-s)} \sum_{\xi} \Big(\operatorname{ord}_s \tilde{L}(\,\cdot\,, \xi) - \operatorname{ord}_s \tilde{L}(\,\cdot\,, \xi\chi_d) \Big) \quad \text{for } s \neq 0, 1,$$

the sum over the Dirichlet characters ξ modulo M, and put $\operatorname{res}_{d,M}(0) = \operatorname{res}_{d,M}(1) = 0$. Then, as a function of the real variable v, $v \mapsto R_{d,M}(\mathrm{e}^v)$ is polar with residue function $s \mapsto \operatorname{res}_{d,M}(s)$.

Theorem 2 (the one-sided criterion)**.** Let $M \geq 1$ be a natural number, c an element of $\mathbb{Z}/M$ and β a real number. Assume that $\beta > \frac{1}{2}$, that c is a unit of $\mathbb{Z}/M$, and that every prime number d with $d \equiv c \pmod M$ has the following three properties: $d \equiv 7 \pmod 8$; $\gcd(M, d) = 1$; and

$$R_{d,M}(x) \leq x^{\beta - 1} \qquad \text{for all sufficiently large real } x.$$

Then for every complex number ρ with $\operatorname{Re}\rho > \beta$ and every Dirichlet character χ modulo M,

$$\tilde{L}(\rho, \chi) \neq 0.$$

Theorem 3 (a good point in every window)**.** Let $\mathcal{E}_1, \mathcal{E}_2 \colon \mathbb{R} \to \mathbb{R}$ and $\omega_1, \omega_2 \colon \mathbb{C} \to \mathbb{C}$ be functions such that $\mathcal{E}_1$ is polar with residue function ω_1 and $\mathcal{E}_2$ is polar with residue function ω_2. Then there is a real number v_* such that for every real $v_0 \geq v_*$ and every real X with $0 \leq X \leq \mathrm{e}^{v_0}$ there is a real number v_1 with $v_0 \leq v_1 \leq v_0^2$ and

$$\mathcal{E}_1(v_1) - X\,\mathcal{E}_2(v_1) \leq \mathrm{e}^{-v_0}.$$

In Theorem 2 the conclusion gives $L(\rho, \chi) \neq 0$, the point $\rho = 1$ excepted when χ is principal. The three theorems are used as follows in the last unit of the proof (Chapter I of the write-up). The geometric part of the proof bounds $R_{d,N_0}(x)$ from above by $x^{8\varepsilon - 1/2}$ for every $\varepsilon \in (0, \frac{1}{100})$, for a multiple N_0 of 18480 and every prime $d \equiv N_0/2 - 1 \pmod{N_0}$; Theorems 1 and 3 enter that deduction at one point, the choice of the second scale. Theorem 2 with $M = N_0$ and $c = N_0/2 - 1$ then turns the bound into $\tilde{L}(\rho, \chi) \neq 0$ for $\operatorname{Re}\rho > \frac{1}{2}$. The bound is one-sided, and it is needed for every prime of a class: §6 explains both points. Part II (§§8 to 11) needs the vocabulary of the data and of the forms, and its statements (Theorems 4, A, B and C) stand there.

2 The setting of Part I

Conventions. $\mathbb{N} = \{0, 1, 2, \dots\}$. For a natural number $M \geq 1$, a *Dirichlet character modulo M* is a multiplicative map $\mathbb{Z}/M \to \mathbb{C}$ that vanishes off the units; it is read as a function on $\mathbb{Z}$, and it is *principal* if it is 1 on the units. Sums over ξ run over all Dirichlet characters modulo M, and $\varphi(M)$ is their number. $\mathbf{1}$ is the Dirichlet character modulo 1. Λ is von Mangoldt's function, with $\Lambda(0) = 0$; log is the natural logarithm, e its base, and $(z)_+ = \max(z, 0)$ for real z. For an entire function $f \neq 0$ and $s \in \mathbb{C}$, $\operatorname{ord}_s f \in \mathbb{N}$ is the order of vanishing of f at s. A function is *analytic at a point* if it is holomorphic in a neighbourhood of it. *For all sufficiently large x* means: for all $x \geq x_*$, for some real x_*. Integrals are Lebesgue integrals; a family of complex numbers is *summable* if it is absolutely summable.

Definition 2.1 (the quadratic character and the twists)**.** Let d be a prime number. The Dirichlet character χ_d modulo d is the quadratic character of the field $\mathbb{Z}/d$, with values in $\{-1, 0, 1\}$: $\chi_d(n) = 0$ if $d \mid n$, $\chi_d(n) = 1$ if n is a non-zero square modulo d, and $\chi_d(n) = -1$ otherwise. For odd d it is the Legendre symbol, and for $d \equiv 3 \pmod 4$ it is the quadratic character χ_K of the field $K = \mathbb{Q}(\sqrt{-d})$. Let moreover $M \geq 1$ be a natural number and ξ a Dirichlet character modulo M. The *twist* $\xi\chi_d$ is the Dirichlet character modulo Md with $(\xi\chi_d)(n) = \xi(n)\,\chi_d(n)$ for every integer n.

LEAN `chiK` · `quadTwist` · `quadTwist_apply`

Definition 2.2 (the entire L-function)**.** Let $M \geq 1$ be a natural number and χ a Dirichlet character modulo M. The function $\tilde{L}(\,\cdot\,, \chi) \colon \mathbb{C} \to \mathbb{C}$, with values written $\tilde{L}(s, \chi)$, is the Dirichlet L-function $s \mapsto L(s, \chi)$ if χ is not principal, and the entire function that equals $(s-1)L(s, \chi)$ for $s \neq 1$ if χ is principal (its value at $s = 1$ is then $\prod_{p \mid M}(1 - 1/p)$, the residue of $L(s, \chi)$). In both cases it is entire and not identically zero. Its derivative is $\tilde{L}'$, and $\frac{\tilde{L}'}{\tilde{L}}(s, \chi)$ is the quotient of the two values at s.

LEAN entireL

Definition 2.3 (the Chowla–Selberg constant)**.** Let d be a prime number. Then

$$\mathrm{cs}(d) = \mathrm{C}_{\mathrm{Eu}} - \operatorname{Re} \frac{\tilde{L}'}{\tilde{L}}(1, \mathbf{1}\chi_d),$$

where C_{Eu} is Euler's constant and $\mathbf{1}\chi_d$ is the twist of the character modulo 1, that is, the character $n \mapsto \chi_d(n)$ modulo d.

LEAN chowlaSelbergConst

Definition 2.4 (the weight)**.** Let d be a prime number and let M and ν be natural numbers. Then

$$\lambda_{d,M}(\nu) = \Lambda(\nu)(1 - \chi_d(\nu)) \quad \text{if } \nu \equiv 1 \pmod{M}, \qquad \lambda_{d,M}(\nu) = 0 \quad \text{otherwise.}$$

LEAN lambdaClass

So $\lambda_{d,M}(\nu) \neq 0$ only if $\nu = p^e$ is a power of a prime, $\nu \equiv 1 \pmod{M}$, and either $p = d$ (the value is $\log d$) or $\chi_d(p) = -1$ and e is odd (the value is $2 \log p$).

Definition 2.5 (the weighted sums)**.** Let d be a prime number, let M and i be natural numbers and x a real number. Then

$$A_{d,M}(i, x) = x^{-i} \sum_{1 \le \nu \le x} \lambda_{d,M}(\nu)\, \nu^{\,i-1},$$

the sum over the natural numbers ν with $1 \le \nu \le x$; it is empty, and the value is 0, for $x < 1$.

LEAN lambdaSum

Definition 2.6 (the mean of the logarithmic derivatives)**.** Let d be a prime number, $M \ge 1$ a natural number and s a complex number. Then

$$G_{d,M}(s) = \frac{1}{\varphi(M)} \Big(\sum_{\xi} \Big(\frac{\tilde{L}'}{\tilde{L}}(s, \xi\chi_d) - \frac{\tilde{L}'}{\tilde{L}}(s, \xi) \Big) - 1 \Big), \qquad \kappa_{d,M} = \operatorname{Re} G_{d,M}(1),$$

the sum over the Dirichlet characters ξ modulo M. As a function of s it is meromorphic on $\mathbb{C}$, and it is analytic at every point where none of the functions $\tilde{L}(\,\cdot\,, \xi)$ and $\tilde{L}(\,\cdot\,, \xi\chi_d)$ vanishes.

LEAN logDerivMean

Definition 2.7 (the remainder)**.** Let d be a prime number, $M \ge 1$ a natural number and x a real number. Then

$$R_{d,M}(x) = A_{d,M}(0, x) - A_{d,M}(1, x) - \frac{\log x}{\varphi(M)} - \kappa_{d,M} = \sum_{1 \le \nu \le x} \lambda_{d,M}(\nu) \Big(\frac{1}{\nu} - \frac{1}{x} \Big) - \frac{\log x}{\varphi(M)} - \kappa_{d,M},$$

with $\kappa_{d,M} = \operatorname{Re} G_{d,M}(1)$ as in Definition 2.6.

LEAN primeError

Definition 2.8 (polar functions)**.** Let $\mathcal{E}\colon \mathbb{R} \to \mathbb{R}$ and $\omega\colon \mathbb{C} \to \mathbb{C}$ be functions. We say that $\mathcal{E}$ is polar with residue function ω, and write $\mathrm{Polar}(\mathcal{E}, \omega)$, if $\mathcal{E}$ is Borel measurable and there are a real number C and a function $\Phi\colon \mathbb{C} \to \mathbb{C}$, meromorphic on $\mathbb{C}$ and defined at every point (at a pole its value is arbitrary), with the following four properties.

(P1) $|\mathcal{E}(v)| \le C\,\mathrm{e}^{v}$ for every real $v \ge 0$.

(P2) $\Phi(s) = \int_0^\infty \mathcal{E}(v)\,\mathrm{e}^{(1-s)v}\,\mathrm{d}v$ for every $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 2$.

(P3) The function $s \mapsto (s-1)\Phi(s)$, whose value at $s = 1$ is 0 whatever $\Phi(1)$ is, is analytic at every point of the closed half-plane $\mathrm{Re}\, s \ge 1$.

(P4) For every $\rho \in \mathbb{C}$ with $\mathrm{Re}\, \rho > 0$: $(s-\rho)\Phi(s) \to \omega(\rho)$ as $s \to \rho$, $s \ne \rho$.

LEAN `IsPolar`

Remark 2.9 (how to read the definition)**.** By (P1) and the measurability, the integral in (P2) converges absolutely for $\mathrm{Re}\, s > 2$, so Φ is the meromorphic continuation of the Laplace transform of $v \mapsto \mathcal{E}(v)\mathrm{e}^{v}$, and it is unique. (P4) says that the poles of Φ in the half-plane $\mathrm{Re}\, s > 0$ are at most simple, and that $\omega(\rho)$ is the residue at ρ; so $\omega(\rho) = 0$ exactly when Φ is analytic at ρ (after correcting its value there). The clause of (P3) on the value at $s = 1$ is essential: (P3) says that Φ is analytic at every point $s \ne 1$ of $\mathrm{Re}\, s \ge 1$, and that near $s = 1$ it is the quotient by $s - 1$ of an analytic function that vanishes at 1: the singularity at $s = 1$ is removable. Hence a polar function has $\omega(\rho) = 0$ for $\mathrm{Re}\, \rho \ge 1$. Read without the clause, for a meromorphic function with its removable singularities filled in, (P3) would allow a simple pole of Φ at $s = 1$; the constant function 1, with $\Phi(s) = 1/(s-1)$, would then be polar, and Theorem 3 would be false for the constant functions 1 and 0. The values of ω on $\mathrm{Re}\, \rho \le 0$ play no role.

Remark 2.10 (what the remainder measures; not part of the formal text)**.** By the orthogonality of characters, for $\mathrm{Re}\, s > 1$ and $\gcd(M, d) = 1$, $d \ne 2$,

$$\sum_{\nu \ge 1} \lambda_{d,M}(\nu)\,\nu^{-s} = G_{d,M}(s) + \frac{s}{\varphi(M)(s-1)}$$

(this is step 3 of the proof of Theorem 1). The only singularity on $\mathrm{Re}\, s \ge 1$ is the simple pole at $s = 1$ with residue $1/\varphi(M)$, which comes from the principal character. Accordingly one expects $\sum_{\nu \le x} \lambda_{d,M}(\nu)/\nu = \log x/\varphi(M) + \text{constant} + o(1)$ and $x^{-1}\sum_{\nu \le x} \lambda_{d,M}(\nu) = 1/\varphi(M) + o(1)$, and the constant $\kappa_{d,M}$ is the one for which $R_{d,M}(x)$ has no constant term: in the language of Definition 2.8, the one for which the transform has no pole at $s = 1$. A zero ρ of some $\tilde{L}(\,\cdot\,, \xi)$ with $0 < \mathrm{Re}\, \rho < 1$ gives the transform a pole with residue $\mathrm{ord}_\rho\, \tilde{L}(\,\cdot\,, \xi)/(\varphi(M)\rho(1-\rho))$, hence, formally, a term of size $x^{\mathrm{Re}\,\rho - 1}$ in $R_{d,M}(x)$; a zero of a twist $\tilde{L}(\,\cdot\,, \xi\chi_d)$ does the same with the opposite sign. So the bound $R_{d,M}(x) = O(x^{\beta-1})$ is what one expects if there is no zero with $\mathrm{Re}\, \rho > \beta$, and the generalised Riemann hypothesis corresponds to $\beta = \frac{1}{2}$. Theorem 2 goes in the other direction, and it asks for an upper bound only.

3 Quoted results

Part I rests on seven results that the artifact proves in its background library, from Mathlib, and we quote all seven. Six of them are classical or elementary. The seventh, Input 5, is not: the theorem that its source names treats real characters, and the formal statement also covers the quadratic twists of non-real characters, by an estimate for each fixed character that the library proves itself; its box says exactly what is proved and what the cited theorem gives. Each box gives the statement as the formal text has it, the unit that owns it, and what is known about its source: the comparison with the literature that the second reader of this write-up made for three of them, and otherwise the entry of the catalogue of quoted results. *Reference to be confirmed* means that there is no reference with a confirmed place.

Input 1 (non-vanishing on $\operatorname{Re} s \geq 1$; unit 107)**.** Let $M \geq 1$ be a natural number, χ a Dirichlet character modulo M and s a complex number with $\operatorname{Re} s \geq 1$. Then $\tilde{L}(s, \chi) \neq 0$.

Classical: Hadamard and de la Vallée Poussin on the line $\operatorname{Re} s = 1$, Dirichlet at $s = 1$. In the formal text it is a direct consequence of two theorems of Mathlib, at the version that the artifact pins, after the principal case is split off: for non-principal χ, $L(s, \chi) \neq 0$ on $\operatorname{Re} s \geq 1$; for principal χ and $s \neq 1$ both factors of $(s-1)L(s, \chi)$ are non-zero; and at $s = 1$ the entire function has the value $\prod_{p \mid M}(1 - 1/p) \neq 0$. Relation: the theorem itself. The comparison with Mathlib at the pin was made by the second reader of this write-up.

LEAN `entireL_ne_zero`

Input 2 (the logarithmic derivative; unit 107)**.** Let $M \geq 1$ be a natural number, χ a Dirichlet character modulo M and s a complex number.
(a) If $\operatorname{Re} s > 1$, the family $\left(\chi(\nu)\Lambda(\nu)\nu^{-s}\right)_{\nu \geq 1}$ is summable, with sum

$$\sum_{\nu \geq 1} \chi(\nu)\Lambda(\nu)\, \nu^{-s} = \frac{[\chi \text{ principal}]}{s-1} - \frac{\tilde{L}'}{\tilde{L}}(s, \chi),$$

where the bracket $[\chi \text{ principal}]$ is the number 1 if χ is principal and 0 otherwise.
(b) $\operatorname{ord}_s \tilde{L}(\,\cdot\,, \chi) = 0$ if and only if $\tilde{L}(s, \chi) \neq 0$; and $(s'-s)\,\frac{\tilde{L}'}{\tilde{L}}(s', \chi) \to \operatorname{ord}_s \tilde{L}(\,\cdot\,, \chi)$ as $s' \to s$, $s' \neq s$.

Elementary: (a) is the logarithmic derivative of the Euler product, (b) the residue of the logarithmic derivative of an analytic function. The catalogue names Davenport, *Multiplicative Number Theory*, for (a), without a place: reference to be confirmed.

LEAN `LSeriesHasSum_mul_vonMangoldt` · `tendsto_sub_mul_logDeriv_entireL`

Input 3 (twists are not principal; unit 107)**.** Let M be a natural number, ξ a Dirichlet character modulo M and d a prime number with $d \neq 2$ and $\gcd(M, d) = 1$. Then the twist $\xi\chi_d$ is not the principal character modulo Md.

Elementary (the Chinese remainder theorem gives n with $n \equiv 1 \pmod{M}$ and $\chi_d(n) = -1$).

LEAN `quadTwist_ne_one`

Input 4 (Landau's lemma; unit 107). Let $\mathcal{E}\colon \mathbb{R} \to \mathbb{R}$ and $\omega\colon \mathbb{C} \to \mathbb{C}$ be functions such that $\mathcal{E}$ is polar with residue function ω, and let β be a real number with $\beta > 0$. Assume that $\operatorname{Re}\omega(s) \geq 0$ for every real number $s > \beta$, and that $\mathcal{E}(v) \leq \mathrm{e}^{(\beta-1)v}$ for all sufficiently large real v. Then $\omega(\rho) = 0$ for every complex number ρ with $\operatorname{Re}\rho > \beta$.

Relation to the literature, as compared by the second reader of this write-up from the pinned Lean text and the primary texts (the writer has not consulted them): a corollary of D. V. Widder, *The Laplace Transform*, Chapter II, §5, Theorem 5b, p. 58 [Wi], with a residue argument. Widder's theorem concerns the Laplace–Stieltjes transform of a monotone function with a finite abscissa of convergence, and gives a singularity at the real point of the abscissa; it applies to the primitive of the non-negative measurable function $v \mapsto C_1\mathrm{e}^{\beta v} - \mathcal{E}(v)\mathrm{e}^{v}$, for a suitable constant C_1. What the formal statement adds is the step at that real point, which compares the sign of the residue with positivity (Remark 3.1). The header of the formal file names H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I*, §15.1, Lemma 15.1, p. 463 [MV], as the source; that lemma is the same theorem for a density that is Riemann integrable on bounded intervals and of eventually constant sign, whereas $\mathcal{E}$ is only assumed measurable here, so it does not apply as it stands.

LEAN `IsPolar.landau`

Input 5 (a zero-free box for quadratic twists; unit 127). Let $M \geq 1$ be a natural number, β and z real numbers and c an element of $\mathbb{Z}/M$. Assume that $\beta > \frac{1}{2}$, that c is a unit of $\mathbb{Z}/M$, and that every prime number d with $d \equiv c \pmod{M}$ satisfies $d \equiv 7 \pmod{8}$ and $\gcd(M, d) = 1$. Then there is a prime number d with $d \equiv c \pmod{M}$ such that

$$\tilde{L}(s, \xi\chi_d) \neq 0$$

for every Dirichlet character ξ modulo M and every complex number s with $\operatorname{Re} s \geq \beta$ and $|\operatorname{Im} s| \leq z$.

The number z is any real number; for $z < 0$ the condition on s is empty, and the statement still gives a prime. *What the formal library proves*, in two steps that the writer read in the Lean text. First, an estimate for one character: for every Dirichlet character ξ modulo M, real or not, every real number $\Delta > 0$ and every real z, the number of the primes $d \leq x$ with $d \equiv 7 \pmod{8}$ and $\gcd(M, d) = 1$ for which $\tilde{L}(\,\cdot\,, \xi\chi_d)$ has a zero s with $\operatorname{Re} s \geq \frac{1}{2} + 9\Delta$ and $|\operatorname{Im} s| \leq z$ is $O(x^{1-\Delta/2})$ as $x \to \infty$. The proof passes to the primitive character that induces ξ, and uses a three-lines argument and a smoothed mean square of sums of Jacobi symbols. Second, the bounds for the $\varphi(M)$ characters ξ are added and set against the divergence of $\sum 1/p$ over the primes $p \equiv c \pmod{M}$: if every prime of the class failed for some ξ, that sum would converge. *What the literature gives*, as compared by the second reader of this write-up from the pinned Lean text and the primary text (the writer has not consulted it): the header of the formal file names M. Jutila, On mean values of Dirichlet polynomials with real characters, Acta Arith. 27 (1975), 191–198; its Theorem 2, equation (6), p. 193 [Ju], bounds the number of zeros in a box, summed over real characters. For a non-real character ξ the twists $\xi\chi_d$ are not real, so that theorem alone does not give the first step, and it does not contain the second. Input 5 is therefore not a classical theorem in the form used here: it is the combination of a variant of a classical zero-density estimate, for which no published place has been confirmed, with the divergence of a sum over primes. The catalogue of quoted results records it in this way.

LEAN `exists_entireL_quadTwist_ne_zero`

Input 6 (Laplace integrals; unit 107)**.** *(a)* Let s be a complex number with $\operatorname{Re} s > 0$ and i a natural number. Then $\int_0^\infty v^i\, \mathrm{e}^{-sv}\, \mathrm{d}v = i!/s^{i+1}$.
(b) Let $\mathcal{E}\colon \mathbb{R} \to \mathbb{R}$ be a Borel measurable function, C a real number with $|\mathcal{E}(v)| \le C\mathrm{e}^v$ for every real $v \ge 0$, and s a complex number with $\operatorname{Re} s > 2$. Then $v \mapsto \mathcal{E}(v)\, \mathrm{e}^{(1-s)v}$ is integrable on $(0, \infty)$.

Elementary: (a) is the integral of the Gamma function.

LEAN `laplace_pow` · `integrableOn_laplace_of_abs_le`

Input 7 (Taylor series across a removable singularity; unit 107)**.** Let $\Phi\colon \mathbb{C} \to \mathbb{C}$ be a function, $\mathfrak{r}_0$ a real number and s_0, s_1, s_2 complex numbers. Assume that the function $s \mapsto (s - s_1)\Phi(s)$ is analytic at every point of the open disc $|s - s_0| < \mathfrak{r}_0$, and that $s_0 \ne s_1$, $|s_1 - s_0| < \mathfrak{r}_0$ and $|s_2 - s_0| < \mathfrak{r}_0$. Then the family $\left((s_2 - s_0)^i\, \Phi^{(i)}(s_0)/i!\right)_{i \ge 0}$ is summable, where $\Phi^{(i)}$ is the i-th complex derivative.

Elementary. The function $s \mapsto (s - s_1)\Phi(s)$ vanishes at s_1, because it is a product with the factor $s - s_1$ whatever the value $\Phi(s_1)$ is; being analytic, it is divisible by $s - s_1$. So off s_1 the function Φ agrees with a function analytic in the disc, with the same derivatives at s_0, and the Taylor series of that function at s_0 converges absolutely in the disc. Analyticity of the product without the vanishing would allow a simple pole of Φ at s_1.

LEAN `summable_taylor_of_removable`

Remark 3.1 (from the classical lemma to Input 4; our derivation)**.** In the formal library Input 4 is proved directly, by the Taylor expansion at a real point that proves the classical lemma. Here is how it follows from the classical statement in this form (its relation to Widder's theorem is described in the box of Input 4): *if $\mathcal{E}_0 \ge 0$ on $(0, \infty)$ and the Laplace integral $\int_0^\infty \mathcal{E}_0(v)\mathrm{e}^{-sv}\mathrm{d}v$ converges for some real s but not for all, then the function that it defines to the right of its abscissa of convergence x_0 has no analytic continuation to a neighbourhood of the point x_0.* Let C and Φ be as in Definition 2.8, let $\mathcal{E}(v) \le \mathrm{e}^{(\beta-1)v}$ for $v \ge v_* \ge 0$, and put $C_1 = \max(1, |C|\mathrm{e}^{2v_*})$. Then $\mathcal{E}_0(v) = C_1\, \mathrm{e}^{\beta v} - \mathcal{E}(v)\mathrm{e}^v$ is ≥ 0 for all $v > 0$: for $v \ge v_*$ because $\mathcal{E}(v)\mathrm{e}^v \le \mathrm{e}^{\beta v}$, and for $v < v_*$ because $\mathcal{E}(v)\mathrm{e}^v \le |C|\mathrm{e}^{2v_*} \le C_1\mathrm{e}^{\beta v}$. Its Laplace integral converges for $\operatorname{Re} s > \max(2, \beta)$, where by (P2) it equals $C_1/(s-\beta) - \Phi(s)$, a function meromorphic on $\mathbb{C}$. Suppose that the abscissa of convergence x_0 is $> \beta$. To the right of x_0 the integral is analytic, so by the identity theorem it equals $C_1/(s-\beta) - \Phi(s)$ on all of $\operatorname{Re} s > x_0$; for real $s > x_0$ it is ≥ 0. Multiplying by $s - x_0 > 0$ and letting s decrease to x_0, (P4) gives $-\omega(x_0) \ge 0$. With the hypothesis $\operatorname{Re}\omega(x_0) \ge 0$ this forces $\omega(x_0) = 0$, so $C_1/(s-\beta) - \Phi(s)$ is analytic at x_0, against the classical lemma. Hence $x_0 \le \beta$: the integral is analytic in $\operatorname{Re} s > \beta$ and equals $C_1/(s-\beta) - \Phi(s)$ there, so Φ has no pole in $\operatorname{Re} s > \beta$, and $\omega(\rho) = 0$ for $\operatorname{Re}\rho > \beta$ by (P4). Note that the hypothesis does not ask that Φ be analytic at the real points to the right of β: poles there whose residue has non-negative real part are allowed, and the conclusion says that there are none. This is the form in which the lemma is used below.

4 Elementary properties of the weight and of the remainder

Lemma 4.1 (bounds for the weight)**.** *Let d be a prime number and let M and ν be natural numbers. Then $0 \le \lambda_{d,M}(\nu)$ and $|\lambda_{d,M}(\nu)| \le 2\nu$.*

Proof. $0 \le \Lambda(\nu) \le \log \nu \le \nu$ for $\nu \ge 1$, $\Lambda(0) = 0$, and $1 - \chi_d(\nu) \in \{0, 1, 2\}$. □

LEAN `lambdaClass_nonneg_and_abs_le`

Lemma 4.2 (bound for the weighted sums)**.** *Let d be a prime number, let M and i be natural numbers and v a real number. Then $|A_{d,M}(i, \mathrm{e}^v)| \le 2\,\mathrm{e}^v$.*

Proof. Put $x = \mathrm{e}^v$. By Lemma 4.1, $|\lambda_{d,M}(\nu)|\,\nu^{\,i-1} \le 2\nu^{\,i} \le 2x^i$ for $1 \le \nu \le x$, and the sum in Definition 2.5 has at most x terms. So $|A_{d,M}(i,x)| \le x^{-i} \cdot x \cdot 2x^i = 2x$. □

LEAN `abs_lambdaSum_exp_le`

Lemma 4.3 (the remainder as a sum of positive parts)**.** *Let d be a prime number, let $M \ge 1$ and a be natural numbers, and let x be a real number with $0 < x$ and $x \le a$. Then*

$$R_{d,M}(x) = \frac{1}{a}\sum_{\nu=1}^{a} \lambda_{d,M}(\nu)\Big(\frac{a}{\nu} - \frac{a}{x}\Big)_+ - \frac{\log x}{\varphi(M)} - \kappa_{d,M}.$$

Proof. $a \ge x > 0$, so $a \ge 1$. For $1 \le \nu \le a$ the number $\frac{1}{a}(\frac{a}{\nu} - \frac{a}{x})_+ = (\frac{1}{\nu} - \frac{1}{x})_+$ is $\frac{1}{\nu} - \frac{1}{x}$ if $\nu \le x$ and 0 if $\nu > x$. So the sum is $\sum_{1 \le \nu \le x} \lambda_{d,M}(\nu)(\frac{1}{\nu} - \frac{1}{x}) = A_{d,M}(0,x) - A_{d,M}(1,x)$, and the claim is Definition 2.7. □

LEAN `primeError_eq_sum`

This identity is the reason for the shape of the remainder. In Part II the prime sums of the data have the form $\sum_{\nu \le a} \lambda_{d,M}(\nu)(z/\nu - w)_+$ with a real *order* z and a real *weight* w; with $z = a$ and $w = a/x$ this is a times the sum of the lemma.

Lemma 4.4 (the constant for the modulus 1)**.** *Let d be a prime number. Then $\operatorname{Re} G_{d,1}(1) = -1 - \operatorname{cs}(d)$; in other words $\kappa_{d,1} = -1 - \operatorname{cs}(d)$.*

Proof. Modulo 1 there is one Dirichlet character, $\mathbf{1}$, and $\varphi(1) = 1$; so $G_{d,1}(s) = \frac{\tilde L'}{\tilde L}(s, \mathbf{1}\chi_d) - \frac{\tilde L'}{\tilde L}(s, \mathbf{1}) - 1$. The function $\tilde L(\,\cdot\,, \mathbf{1})$ is $s \mapsto (s-1)\zeta(s)$, where ζ is the Riemann zeta function; from the expansion $\zeta(s) = 1/(s-1) + \mathrm{C}_{\mathrm{Eu}} + O(s-1)$ at $s = 1$ its value at 1 is 1 and its derivative at 1 is C_{Eu}, so $\frac{\tilde L'}{\tilde L}(1, \mathbf{1}) = \mathrm{C}_{\mathrm{Eu}}$. Hence $\operatorname{Re} G_{d,1}(1) = \operatorname{Re} \frac{\tilde L'}{\tilde L}(1, \mathbf{1}\chi_d) - \mathrm{C}_{\mathrm{Eu}} - 1 = -\operatorname{cs}(d) - 1$ by Definition 2.3. □

LEAN `re_logDerivMean_one`

5 The transform of the remainder: proof of Theorem 1

Let d and M be as in Theorem 1: d is a prime with $d \equiv 7 \pmod 8$, $M \ge 1$ and $\gcd(M, d) = 1$. Put

$$\Phi_{d,M}(s) = \frac{1}{s-1}\Big(\frac{G_{d,M}(s)}{s} - \kappa_{d,M}\Big) \qquad (s \in \mathbb{C},\ s \ne 1), \tag{5.1}$$

and $\Phi_{d,M}(1) = 0$; the value at $s = 1$ plays no role. We verify Definition 2.8 for $\mathcal{E}(v) = R_{d,M}(\mathrm{e}^v)$ and $\omega(s) = \operatorname{res}_{d,M}(s)$, with the constant $C = |\kappa_{d,M}| + 5$ and the function $\Phi = \Phi_{d,M}$. The proof has seven steps.

Step 1: analyticity. By Input 1 none of the functions $\tilde{L}(\,\cdot\,,\xi)$ and $\tilde{L}(\,\cdot\,,\xi\chi_d)$, ξ modulo M, has a zero in $\operatorname{Re} s \geq 1$. So $s \mapsto G_{d,M}(s)$ is analytic at every point of $\operatorname{Re} s \geq 1$ (Definition 2.6), and so is the function $s \mapsto G_{d,M}(s)/s - \kappa_{d,M}$. As a quotient of meromorphic functions, $\Phi_{d,M}$ is meromorphic on $\mathbb{C}$.

Step 2: measurability and (P1). For $i = 0, 1$ the function $v \mapsto A_{d,M}(i, \mathrm{e}^v)$ is the product of e^{-iv} and a step function, and $v \mapsto v/\varphi(M) + \kappa_{d,M}$ is continuous; so $v \mapsto R_{d,M}(\mathrm{e}^v)$ is Borel measurable. For $v \geq 0$, Lemma 4.2 gives

$$|R_{d,M}(\mathrm{e}^v)| \leq 2\mathrm{e}^v + 2\mathrm{e}^v + \frac{v}{\varphi(M)} + |\kappa_{d,M}| \leq \bigl(|\kappa_{d,M}| + 5\bigr)\,\mathrm{e}^v,$$

because $v \leq \mathrm{e}^v$, $\varphi(M) \geq 1$ and $1 \leq \mathrm{e}^v$.

Step 3: the Dirichlet series of the weight. For $\operatorname{Re} s > 1$ the family $(\lambda_{d,M}(\nu)\nu^{-s})_{\nu \geq 1}$ is summable, and

$$\sum_{\nu \geq 1} \lambda_{d,M}(\nu)\,\nu^{-s} = G_{d,M}(s) + \frac{s}{\varphi(M)\,(s-1)}. \tag{5.2}$$

Indeed, by the orthogonality of characters the indicator of the class $\nu \equiv 1 \pmod M$ is $\varphi(M)^{-1}\sum_\xi \xi(\nu)$, so $\lambda_{d,M}(\nu) = \varphi(M)^{-1}\sum_\xi (\xi(\nu) - (\xi\chi_d)(\nu))\Lambda(\nu)$ by Definitions 2.4 and 2.1. Input 2(a), for the characters ξ modulo M and $\xi\chi_d$ modulo Md, gives

$$\sum_{\nu \geq 1} \lambda_{d,M}(\nu)\,\nu^{-s} = \frac{1}{\varphi(M)} \sum_{\xi} \Bigl(\frac{[\xi \text{ principal}] - [\xi\chi_d \text{ principal}]}{s-1} - \frac{\tilde{L}'}{\tilde{L}}(s,\xi) + \frac{\tilde{L}'}{\tilde{L}}(s,\xi\chi_d) \Bigr).$$

Here $[\xi\chi_d \text{ principal}] = 0$ for every ξ, by Input 3: $d \neq 2$ because $d \equiv 7 \pmod 8$, and $\gcd(M, d) = 1$. And $\sum_\xi [\xi \text{ principal}] = 1$. So the right side is $G_{d,M}(s) + \varphi(M)^{-1}(1 + \frac{1}{s-1})$, which is (5.2). This is the only place where the hypothesis $d \equiv 7 \pmod 8$ enters the proof.

Step 4: the transforms of the weighted sums. Let $\operatorname{Re} s > 2$ and $i \in \{0, 1\}$. By Lemma 4.2 and Input 6(b) the function $v \mapsto A_{d,M}(i, \mathrm{e}^v)\mathrm{e}^{(1-s)v}$ is integrable on $(0, \infty)$, and

$$(s + i - 1) \int_0^\infty A_{d,M}(i, \mathrm{e}^v)\,\mathrm{e}^{(1-s)v}\,\mathrm{d}v = \sum_{\nu \geq 1} \lambda_{d,M}(\nu)\,\nu^{-s}. \tag{5.3}$$

To see this, substitute $x = \mathrm{e}^v$: the integral is

$$\int_1^\infty A_{d,M}(i,x)\,x^{-s}\,\mathrm{d}x = \int_1^\infty \Bigl(\sum_{1 \leq \nu \leq x} \lambda_{d,M}(\nu)\,\nu^{\,i-1} \Bigr)\, x^{-(s+i-1)-1}\,\mathrm{d}x.$$

The coefficients $\lambda_{d,M}(\nu)\nu^{i-1}$ are ≥ 0, and their sum up to x is $x^i A_{d,M}(i,x) \leq 2x^{1+i}$ by Lemma 4.2. Partial summation, in the form that Mathlib provides, says that a Dirichlet series whose coefficients have absolute partial sums $O(x^{1+i})$ is, at every s' with $\operatorname{Re} s' > 1 + i$, equal to s' times the integral over $(1, \infty)$ of its partial sums against $x^{-s'-1}$. With $s' = s + i - 1$, whose real part exceeds $1 + i$ because $\operatorname{Re} s > 2$, this is (5.3).

Step 5: (P2). Let $\operatorname{Re} s > 2$. By Input 6(a) at the point $s-1$, $\int_0^\infty \mathrm{e}^{(1-s)v}\mathrm{d}v = \frac{1}{s-1}$ and $\int_0^\infty v\,\mathrm{e}^{(1-s)v}\mathrm{d}v = \frac{1}{(s-1)^2}$. By Definition 2.7, $R_{d,M}(\mathrm{e}^v) = A_{d,M}(0,\mathrm{e}^v) - A_{d,M}(1,\mathrm{e}^v) - v/\varphi(M) - \kappa_{d,M}$; each of the four terms, times $\mathrm{e}^{(1-s)v}$, is integrable on $(0,\infty)$. So by (5.3) for $i=0$ and $i=1$, and by (5.2),

$$\int_0^\infty R_{d,M}(\mathrm{e}^v)\,\mathrm{e}^{(1-s)v}\,\mathrm{d}v = \Big(\frac{1}{s-1} - \frac{1}{s}\Big)\Big(G_{d,M}(s) + \frac{s}{\varphi(M)(s-1)}\Big) - \frac{1}{\varphi(M)(s-1)^2} - \frac{\kappa_{d,M}}{s-1}$$
$$= \frac{G_{d,M}(s)}{s(s-1)} - \frac{\kappa_{d,M}}{s-1} = \Phi_{d,M}(s).$$

The term $\log x/\varphi(M)$ of the remainder has cancelled the double pole at $s=1$ that the pole of the Dirichlet series produces.

Step 6: the constant is real, and (P3). We claim that $G_{d,M}(1) = \kappa_{d,M}$, that is, that $G_{d,M}(1)$ is real. For real $s > 2$ the number $\Phi_{d,M}(s)$ is the integral of a real function, by Step 5; so $G_{d,M}(s)/s - \kappa_{d,M} = (s-1)\Phi_{d,M}(s)$ is real. By Step 1 the function $s \mapsto \operatorname{Im}(G_{d,M}(s)/s - \kappa_{d,M})$ of the real variable s is real analytic on a neighbourhood of the half-line $[1,\infty)$; it vanishes on $(2,\infty)$, hence on $[1,\infty)$, by the identity theorem. At $s=1$ this gives $\operatorname{Im} G_{d,M}(1) = 0$, and $\operatorname{Re} G_{d,M}(1) = \kappa_{d,M}$ by definition. Now (P3): for $s \neq 1$, $(s-1)\Phi_{d,M}(s) = G_{d,M}(s)/s - \kappa_{d,M}$, and at $s=1$ the left side is 0 and the right side is $G_{d,M}(1) - \kappa_{d,M} = 0$. So $s \mapsto (s-1)\Phi_{d,M}(s)$ is the function of Step 1, which is analytic at every point of $\operatorname{Re} s \geq 1$. This is the role of the constant $\kappa_{d,M}$ in the remainder: it removes the pole of the transform at $s=1$.

Step 7: (P4). Let $\operatorname{Re}\rho > 0$. If $\rho = 1$, then $(s-1)\Phi_{d,M}(s) \to 0 = \operatorname{res}_{d,M}(1)$ by Step 6. Let $\rho \neq 1$. By Input 2(b), applied to each ξ and to each $\xi\chi_d$,

$$(s-\rho)\,G_{d,M}(s) \;\longrightarrow\; \frac{1}{\varphi(M)}\sum_{\xi}\big(\operatorname{ord}_\rho \tilde{L}(\,\cdot\,,\xi\chi_d) - \operatorname{ord}_\rho \tilde{L}(\,\cdot\,,\xi)\big) \qquad (s\to\rho,\ s\neq\rho),$$

the constant -1 in $G_{d,M}(s)$ contributing nothing. Since $s \to \rho \neq 0$, $s-1 \to \rho-1 \neq 0$ and $(s-\rho)\kappa_{d,M} \to 0$, (5.1) gives

$$(s-\rho)\,\Phi_{d,M}(s) \;\longrightarrow\; \frac{1}{\varphi(M)\,\rho(\rho-1)}\sum_{\xi}\big(\operatorname{ord}_\rho \tilde{L}(\,\cdot\,,\xi\chi_d) - \operatorname{ord}_\rho \tilde{L}(\,\cdot\,,\xi)\big) = \operatorname{res}_{d,M}(\rho).$$

This proves Theorem 1. □

LEAN isPolar_primeError

6 The one-sided criterion: proof of Theorem 2

Let M, c, β be as in Theorem 2, let $\rho \in \mathbb{C}$ with $\operatorname{Re}\rho > \beta$, and let χ be a Dirichlet character modulo M. If $\operatorname{Re}\rho \geq 1$, then $\tilde{L}(\rho,\chi) \neq 0$ by Input 1. Assume $\operatorname{Re}\rho < 1$.

Step 1: the prime d. The hypotheses of Input 5 hold for M, β, c and $z = |\operatorname{Im}\rho|$: they are part of the hypotheses of Theorem 2. So there is a prime $d \equiv c \pmod M$ with

$$\tilde{L}(s,\xi\chi_d) \neq 0 \qquad \text{for all } \xi \text{ modulo } M \text{ and all } s \text{ with } \operatorname{Re} s \geq \beta,\ |\operatorname{Im} s| \leq |\operatorname{Im}\rho|. \tag{6.1}$$

By hypothesis this d satisfies $d \equiv 7 \pmod 8$, $\gcd(M,d) = 1$, and $R_{d,M}(x) \leq x^{\beta-1}$ for all sufficiently large x.

Step 2: the polar function. By Theorem 1, $v \mapsto R_{d,M}(\mathrm{e}^v)$ is polar with residue function $s \mapsto \mathrm{res}_{d,M}(s)$. We check the two hypotheses of Input 4 for it, with the exponent $\beta > \frac{1}{2} > 0$.

The one-sided bound. If v is large, so is e^v, and then $R_{d,M}(\mathrm{e}^v) \le (\mathrm{e}^v)^{\beta-1} = \mathrm{e}^{(\beta-1)v}$.

The sign of the residues at real points. Let s be a real number with $s > \beta$. If $s \ge 1$, no $\tilde{L}(\,\cdot\,,\xi)$ and no $\tilde{L}(\,\cdot\,,\xi\chi_d)$ vanishes at s (Input 1), all orders at s are 0 (Input 2(b)), and $\mathrm{res}_{d,M}(s) = 0$. If $\beta < s < 1$, the point s lies in the box of (6.1), so $\mathrm{ord}_s\, \tilde{L}(\,\cdot\,,\xi\chi_d) = 0$ for all ξ, and

$$\mathrm{res}_{d,M}(s) = \frac{1}{\varphi(M)\, s(1-s)} \sum_{\xi} \mathrm{ord}_s\, \tilde{L}(\,\cdot\,,\xi) \;\ge\; 0,$$

because $s(1-s) > 0$.

Step 3: conclusion. Input 4 gives $\mathrm{res}_{d,M}(\rho) = 0$. The point ρ lies in the box of (6.1), so again $\mathrm{ord}_\rho\, \tilde{L}(\,\cdot\,,\xi\chi_d) = 0$ for all ξ; and $\rho(1-\rho) \ne 0$ because $0 < \mathrm{Re}\,\rho < 1$. Hence $\sum_\xi \mathrm{ord}_\rho\, \tilde{L}(\,\cdot\,,\xi) = 0$. The summands are natural numbers, so each is 0; in particular $\mathrm{ord}_\rho\, \tilde{L}(\,\cdot\,,\chi) = 0$, that is, $\tilde{L}(\rho,\chi) \ne 0$ by Input 2(b). □

LEAN `entireL_ne_zero_of_primeError_le`

Remark 6.1 (the anatomy of the criterion)**.** *Where positivity comes from.* Landau's lemma needs a transform whose poles to the right of β, if real, have residues of one sign. In $\mathrm{res}_{d,M}(s)$ the zeros of the $\tilde{L}(\,\cdot\,,\xi)$ count positively and those of the twists $\tilde{L}(\,\cdot\,,\xi\chi_d)$ negatively. The twists are present because the weight $\lambda_{d,M}(\nu)$ carries the factor $1 - \chi_d(\nu)$; this factor is what the geometric part of the proof can bound, since it selects the primes that do not split in $K = \mathbb{Q}(\sqrt{-d})$. Input 5 removes the negative terms from a box of the height of ρ.

Why every prime of a class. The prime d is not chosen in advance: Input 5 produces it, and it depends on the height $|\,\mathrm{Im}\,\rho|$ of the point at which a zero is to be excluded. So the one-sided bound must be known for every prime of the class c.

Why an upper bound suffices. The lemma is applied to the non-negative function $C_1\mathrm{e}^{\beta v} - R_{d,M}(\mathrm{e}^v)\mathrm{e}^v$ (Remark 3.1); only the inequality $R_{d,M}(x) \le x^{\beta-1}$ is used, and no lower bound.

Real zeros. Zeros of the $\tilde{L}(\,\cdot\,,\xi)$ on the real segment $(\beta,1)$ are not excluded by hypothesis: they would give poles with positive residue, which Input 4 admits, and the conclusion excludes them. Real zeros of the twists in $(\beta,1)$ are excluded by (6.1). The header of the formal file notes that the classical forms of this argument assume that there are no real zeros, and names as related work M. Suzuki, *On variants of Chebyshev's conjecture* (2024), §§15.1–15.2 of [MV], and Landau's paper of 1905 on a theorem of Chebyshev; we have not compared the statements.

7 A good point in every window: proof of Theorem 3

The proof of Theorem 3 uses the analyticity of the transform on $\operatorname{Re} s \geq 1$ through the moments $\int_0^\infty v^i \mathrm{e}^{-3v} \mathcal{E}(v)\,\mathrm{d}v$, which are the derivatives of Φ at $s = 4$ up to sign.

Proposition 7.1 (moments of a polar function)**.** *Let $\mathcal{E}\colon \mathbb{R} \to \mathbb{R}$ and $\omega\colon \mathbb{C} \to \mathbb{C}$ be functions such that $\mathcal{E}$ is polar with residue function ω. Then there is a real number $\mathfrak{r}_0 > 3$ such that for every real number $\mathfrak{r}$ with $3 < \mathfrak{r} < \mathfrak{r}_0$ there is a real number v_* such that for every real $v_0 \geq v_*$:*

(i) $|\mathcal{E}(v)| \leq v_0\,\mathrm{e}^v$ *for every real $v \geq 0$;*

(ii) for every natural number i the function $v \mapsto v^i \mathrm{e}^{-3v}\mathcal{E}(v)$ is integrable on $(0,\infty)$, and

$$\Big|\int_0^\infty v^i\,\mathrm{e}^{-3v}\,\mathcal{E}(v)\,\mathrm{d}v\Big| \leq \frac{i!\,v_0}{\mathfrak{r}^i}.$$

Proof. Let C and Φ be as in Definition 2.8. Only the measurability of $\mathcal{E}$ and (P1), (P2), (P3) are used.

A disc of analyticity. The closed disc $|s-4| \leq 3$ lies in the half-plane $\operatorname{Re} s \geq 1$, and it contains the point $s = 1$. By (P3) the function $s \mapsto (s-1)\Phi(s)$, which vanishes at $s = 1$, is analytic at each of its points. The set of points at which a function is analytic is open and the disc is compact, so there is a real number $\mathfrak{r}_0 > 3$ such that $s \mapsto (s-1)\Phi(s)$ is analytic at every point of the open disc $|s-4| < \mathfrak{r}_0$. Let $3 < \mathfrak{r} < \mathfrak{r}_0$. Input 7, with $s_0 = 4$, $s_1 = 1$, $s_2 = 4 + \mathfrak{r}$ and the radius $\mathfrak{r}_0$, shows that the Taylor series of Φ at 4 converges absolutely at $4 + \mathfrak{r}$, beyond the point 1:

$$C_{\mathfrak{r}} = \sum_{i \geq 0} \frac{\mathfrak{r}^i\,|\Phi^{(i)}(4)|}{i!} < \infty.$$

Differentiation under the integral sign. We claim that for every natural number i and every $s \in \mathbb{C}$ with $\operatorname{Re} s > 2$ the function $v \mapsto (-v)^i \mathcal{E}(v) \mathrm{e}^{(1-s)v}$ is integrable on $(0,\infty)$, and

$$\Phi^{(i)}(s) = \int_0^\infty (-v)^i\,\mathcal{E}(v)\,\mathrm{e}^{(1-s)v}\,\mathrm{d}v. \tag{7.1}$$

For $i = 0$ this is Input 6(b) and (P2). Let the claim hold for i, let $\operatorname{Re} s > 2$, and choose $\Delta > 0$ with $2\Delta < \operatorname{Re} s - 2$. For $v > 0$ and $s' \in \mathbb{C}$ with $\operatorname{Re} s' > 2 + 2\Delta$,

$$\big|(-v)^{i+1}\mathcal{E}(v)\,\mathrm{e}^{(1-s')v}\big| \leq \frac{1}{\Delta}\,\big|(-v)^i\mathcal{E}(v)\,\mathrm{e}^{(1-(2+\Delta))v}\big|,$$

because $\Delta v \leq \mathrm{e}^{\Delta v}$ and $\Delta + 1 - \operatorname{Re} s' \leq -1 - \Delta$. The right side is integrable on $(0,\infty)$, by the claim for i at the point $2 + \Delta$, and it does not depend on s'. So the integral in (7.1) may be differentiated under the integral sign on the half-plane $\operatorname{Re} s' > 2 + 2\Delta$, which is a neighbourhood of s: its derivative at s is the integral for $i + 1$, which converges absolutely. Since the integral for i equals $\Phi^{(i)}$ on $\operatorname{Re} s' > 2$, this is the claim for $i + 1$.

Conclusion. At $s = 4$, (7.1) says that $v \mapsto v^i \mathrm{e}^{-3v}\mathcal{E}(v)$ is integrable on $(0,\infty)$ and that $\big|\int_0^\infty v^i \mathrm{e}^{-3v}\mathcal{E}(v)\,\mathrm{d}v\big| = |\Phi^{(i)}(4)| \leq i!\,C_{\mathfrak{r}}/\mathfrak{r}^i$. So (i) and (ii) hold for every $v_0 \geq v_* = \max(C, C_{\mathfrak{r}})$, (i) by (P1). □

LEAN `IsPolar.eventually_moment_le`

Proof of Theorem 3. Apply Proposition 7.1 to $(\mathcal{E}_1, \omega_1)$ and to $(\mathcal{E}_2, \omega_2)$, and choose a real number $\mathfrak{r}$ with $3 < \mathfrak{r} < 4$ that is below the bound $\mathfrak{r}_0$ of each of the two. Let v_* be so large that the conclusions (i) and (ii) hold for $\mathcal{E}_1$ and for $\mathcal{E}_2$ whenever $v_0 \geq v_*$, and that $v_* \geq 99/\log(\mathfrak{r}/3)$ and $v_* \geq 9$. Let $v_0 \geq v_*$ and $0 \leq X \leq \mathrm{e}^{v_0}$, and put $i = \lceil v_0^2/9 \rceil$. Since $v_0 + 1 \leq \mathrm{e}^{v_0}$ and $X \leq \mathrm{e}^{v_0}$,

$$v_0 + v_0 X + 1 \leq \mathrm{e}^{2v_0}. \tag{7.2}$$

Suppose, for a contradiction, that $\mathcal{E}_1(v) - X\mathcal{E}_2(v) > \mathrm{e}^{-v_0}$ for every v in the window $[v_0, v_0^2]$. Consider, for $v > 0$,

$$I(v) = \mathrm{e}^{7v_0 - \mathfrak{r}v} + \mathrm{e}^{2v_0 - v_0^2 - v} + \mathrm{e}^{-3v}\bigl(\mathcal{E}_1(v) - X\,\mathcal{E}_2(v) - \mathrm{e}^{-v_0}\bigr).$$

$I(v) \geq 0$ *for all* $v > 0$. In the window all three terms are positive. Outside it, by (i) for $\mathcal{E}_1$ and $\mathcal{E}_2$ and by (7.2), $|\mathcal{E}_1(v) - X\mathcal{E}_2(v) - \mathrm{e}^{-v_0}| \leq (v_0 + Xv_0 + 1)\mathrm{e}^{v} \leq \mathrm{e}^{2v_0 + v}$, so the third term is at least $-\mathrm{e}^{2v_0 - 2v}$. If $v < v_0$, then $7v_0 - \mathfrak{r}v \geq 2v_0 - 2v$, because $(\mathfrak{r} - 2)v < 2v_0$; so the first term compensates. If $v > v_0^2$, then $2v_0 - v_0^2 - v \geq 2v_0 - 2v$; so the second term compensates.

The contradiction. Multiply by v^i and integrate over $(0, \infty)$. By Input 6(a), and by (ii) for $\mathcal{E}_1$ and $\mathcal{E}_2$,

$$0 \leq \int_0^\infty v^i I(v)\,\mathrm{d}v \leq \frac{i!\,\mathrm{e}^{7v_0}}{\mathfrak{r}^{i+1}} + i!\,\mathrm{e}^{2v_0 - v_0^2} + \frac{i!\,(v_0 + Xv_0)}{\mathfrak{r}^i} - \frac{i!\,\mathrm{e}^{-v_0}}{3^{i+1}}.$$

With (7.2) and after division by $i!$,

$$\frac{\mathrm{e}^{-v_0}}{3^{i+1}} < \frac{\mathrm{e}^{7v_0}}{\mathfrak{r}^{i+1}} + \mathrm{e}^{2v_0 - v_0^2} + \frac{\mathrm{e}^{2v_0}}{\mathfrak{r}^i}. \tag{7.3}$$

But each term on the right is at most a third of the left side. Indeed $i \geq v_0^2/9$ and $v_0 \log(\mathfrak{r}/3) \geq 99$ give $(\mathfrak{r}/3)^i \geq \mathrm{e}^{11v_0}$, and $\mathrm{e}^{11v_0} \geq 3\mathrm{e}^{8v_0}$ and $\mathrm{e}^{11v_0} \geq 9\mathrm{e}^{3v_0}$: this settles the first and the third term. For the second term one needs $(i+2)\log 3 + 3v_0 \leq v_0^2$; it holds because $i + 2 \leq v_0^2/9 + 3$, $\log 3 < \frac{11}{10}$ and $v_0 \geq 9$. So the right side of (7.3) is at most the left side, a contradiction. Hence some $v_1 \in [v_0, v_0^2]$ has $\mathcal{E}_1(v_1) - X\mathcal{E}_2(v_1) \leq \mathrm{e}^{-v_0}$. □

LEAN `IsPolar.exists_window_sub_mul_le`

Remark 7.2 (what the window lemma replaces)**.** The weight $v^i\mathrm{e}^{-3v}$ has its mass near $v = i/3 \approx v_0^2/27$, inside the window, and the factor $\mathfrak{r}^{-i}$ with $\mathfrak{r} > 3$ in Proposition 7.1 says that the moments of a polar function are smaller, by a factor exponential in i, than those of a function of constant sign and size 1. So a polar function cannot stay above e^{-v_0} on a whole window. In the proof of the one-sided bound (Chapter I) the theorem is applied to $\mathcal{E}_1(v) = R_{d,N_0}(\mathrm{e}^v)$ and $\mathcal{E}_2(v) = R_{d,1}(\mathrm{e}^v)$, which are polar by Theorem 1, and e^{v_1} becomes the second scale of the data. A pointwise bound for these remainders, that is, a prime number theorem in progressions with a remainder term, is thus not needed; the header of the formal file names the work of Pintz and of Kaczorowski on sign changes of remainder terms as related, and we have not compared it.

8 Part II. The data, and what is taken from other units

From here on the note needs the vocabulary of the geometric part of the proof: the data, the rows and columns, the theta lifts. These objects belong to other units, and the chapters on those units define them. This section and the Settings of §§10 and 11 say what is used of them. The results of the present unit are then proved in full from the properties that are listed; the properties themselves are *taken for granted*, each with the unit that owns it, in the words of the reader's dossier, whose statements were compared with the Lean text.

Setting 8.1 (parameters and data; unit 152)**.** *Parameters* are a pair $P = (N_0, d)$: a positive multiple N_0 of 18480 and a prime number d with $d \equiv N_0/2 - 1 \pmod{N_0}$. *Admissible data* for P are a record D of unit 152. Among its components are three positive real numbers x, x_1, b, the *scales* (the record stores their logarithms), two natural numbers m and r, and a finite set $\mathrm{thick}(D)$ of infinite places of the field $E(D)$ defined below. To the parameters P and data D the following objects are attached.

The field and its places. $K = \mathbb{Q}(\sqrt{-d})$. The natural number $N_1(D)$ is a multiple of N_0 with the same prime divisors. The field $E(D)$ is K with the roots of unity of order $N_1(D)$ adjoined. Its infinite places form a principal homogeneous space under $\mathrm{Gal}(E(D)/K) \cong (\mathbb{Z}/N_1(D))^\times$; the isomorphism is restriction to the roots of unity, and it is onto because K is not contained in the field that they generate over $\mathbb{Q}$: the prime d ramifies in K and does not divide $N_1(D)$. A base place, fixed once by the artifact, identifies the places with this group, and so makes them a finite commutative group; $f(D)$ is their number. Reduction modulo N_0 gives each place τ a *class* in $(\mathbb{Z}/N_0)^\times$, multiplicatively in τ. For an integer ν and a place τ, the place $\nu \cdot \tau$ is the product of τ and the place identified with ν modulo $N_1(D)$ if ν is prime to $N_1(D)$, and $\nu \cdot \tau = \tau$ otherwise; if ν is prime to $N_1(D)$, the class of $\nu \cdot \tau$ is ν times the class of τ.

Tops and thick places. $\mathrm{tops}(D)$ is the set of the places of class 1 modulo N_0. Every place of $\mathrm{thick}(D)$ has class -1 modulo N_0. And $f(D) = \varphi(N_0) \cdot \#\mathrm{tops}(D)$.

Sizes. $k(D) = r + 1 - m$, $t(D) = 5r$, $a(D)$ is the least multiple of $N_1(D)$ that is at least $x\,b$, and $\mathrm{budget}(D) = f(D)\,a(D)\,(\log b)^3$.

Prime sums. The *nominal weight* of a place τ is $w_D(\tau) = a(D)/x$ if $\tau \in \mathrm{thick}(D)$ and $w_D(\tau) = a(D)/x_1$ otherwise; it is positive, and $\bar{w}_D$ is the mean of $\log w_D(\tau)$ over all $f(D)$ infinite places τ. For a real number z and a place τ,

$$\mathrm{Ask}_D(z, \tau) = \sum_{\nu=1}^{a(D)} \lambda_{d,1}(\nu) \left(\frac{z}{\nu} - w_D((-\nu) \cdot \tau) \right)_+ .$$

Periods, the cost of one direction, the level. $g(D)$ is the product of two primes of the data, and the *period term* is $\mathrm{per}_D = \log(g(D)\,\mathrm{rd}_{E(D)}/2\pi)$, with $\mathrm{rd}_{E(D)}$ the root discriminant of $E(D)$. For a natural number i the *jet size* is $\mathrm{jet}_D(i) = \log(i!) - i\,(\mathrm{cs}(d) - \mathrm{per}_D)$. The *cost of one direction* $\mathsf{U}(D)$ is a real number: a sum over all infinite places τ of a jet size minus the prime sum $\mathrm{Ask}_D(w_D(\tau), \tau)$; its formula is in Chapter III, and its value plays no role here. The *level* $N(D)$ is a prime number (unit 141). Finally (unit 145)

$$h_0(D) = m\Big(\sum_{\tau \in \mathrm{tops}(D)} (\mathrm{jet}_D(a(D)) - \mathrm{Ask}_D(a(D), \tau)) + k(D)\,\mathsf{U}(D) \Big) - \#\mathrm{tops}(D) \cdot a(D)\,\log N(D).$$

The formal text writes $\mathrm{cs}(d) - \mathrm{per}_D$ as $\log(\Omega/g(D))$ with the period $\Omega = \mathrm{e}^{\mathrm{cs}(d)} \cdot 2\pi/\mathrm{rd}_{E(D)}$. The next two boxes are what Part II takes from units 152 and 141 about sizes.

Input 8 (comparison of sizes; unit 152)**.** In Setting 8.1:

(S0) $a(D) \geq 1$, $x \leq a(D)$, $x_1 \leq a(D)$, $m\,k(D) \neq 0$, $b > 0$ and $\mathrm{budget}(D) > 0$;

(S1) $\#\mathrm{tops}(D) \cdot \big(m\,(1 + \frac{1}{2}\log a(D)) + a(D)\,(\mathrm{per}_D + \frac{15}{2}\log b)\big) \leq \mathrm{budget}(D)$;

(S2) if $|\mathrm{Ask}_D(z,\tau)| \leq 2\,a(D)\,|z|$ for all real z and all places τ, then $|h_{\mathrm{main}}(D)| + m\,|\mathrm{rem}(D)| \leq b^2 x^{99}$ (Definitions 9.1 and 9.3);

(S3) $x^{99} + 5\,\mathrm{budget}(D) \leq \log b \cdot \mathrm{budget}(D)$;

(S4) $x^{99} + x_1^2 b^2 + 4m^2\,t(D)\,x^{999} \leq \frac{1}{4}\,b^3$;

(S5) $m^2 \cdot 2\,t(D)\,x^{999} \leq 2\log b \cdot \mathrm{budget}(D)$;

(S6) $b \leq \log b \cdot \mathrm{budget}(D)$.

These are instances of one mechanism of unit 152. Every quantity attached to the data carries a bound of the form exp of a linear form in $\log x$, $\log x_1$, $\log b$ and $\log\log b$, found from the bounds of its factors, and two such bounds are compared through the conditions on the scales, among them $(\log b)^{100} \leq x$, $x^{10^6} \leq x_1$ and $x_1^{10^6} \leq b$. At (S2), (S3) and (S4) the source does not write out the inequality that it passes to this mechanism; the forms given here are those of the dossier, and the proofs below show that they are what is needed.

Input 9 (the level; unit 141)**.** In Setting 8.1: $N(D)$ is a prime number, and $\log N(D) \geq m\,(\bar{w}_D + \mathrm{per}_D) - \mathrm{per}_D - \frac{15}{2}\log b$.

The level is chosen, in a window of length $\sqrt{\log b}$, above this target.

9 The remainder of the data: Theorem 4

Definition 9.1 (the main amount)**.** Let P be parameters and D admissible data for P, with components x, x_1, b, m, r (Setting 8.1). Then $h_{\mathrm{main}}(D) = m\,k(D)\,\mathsf{U}(D)$.

LEAN `Data.mainHeight`

Definition 9.2 (the expected value of a prime sum)**.** Let P be parameters and D admissible data for P, with components x, x_1, b, m, r (Setting 8.1). Then

$$\mathrm{Ask}^{\mathrm{main}}(D) = a(D)\,\big(\log a(D) - 1 - \mathrm{cs}(d) - \bar{w}_D\big) = a(D)\,\big(\log a(D) + \kappa_{d,1} - \bar{w}_D\big),$$

the second form by Lemma 4.4.

LEAN `Data.mainAsk`

Definition 9.3 (the remainder of the data)**.** Let P be parameters and D admissible data for P, with components x, x_1, b, m, r (Setting 8.1). Then

$$\mathrm{rem}(D) = \sum_{\tau \in \mathrm{tops}(D)} \big(\mathrm{Ask}_D(a(D),\tau) - \mathrm{Ask}^{\mathrm{main}}(D)\big).$$

LEAN `Data.remainder`

Theorem 4 (the remainder of the data in terms of the remainders of Part I)**.** Let $P = (N_0, d)$ be parameters and D admissible data for P, with components x, x_1, b, m, r (Setting 8.1). Then

$$\operatorname{rem}(D) = a(D)\Big(\#\mathrm{thick}(D)\cdot\big(R_{d,N_0}(x) - R_{d,N_0}(x_1)\big) + \#\mathrm{tops}(D)\cdot R_{d,1}(x_1)\Big).$$

Proof. Write $a = a(D)$. By (S0), $a \geq 1$, $0 < x \leq a$ and $0 < x_1 \leq a$.

Step 1: counting. For every natural number ν,

$$\#\{\tau \in \mathrm{tops}(D) : (-\nu)\cdot\tau \in \mathrm{thick}(D)\} = \begin{cases} \#\mathrm{thick}(D) & \text{if } \nu \equiv 1 \pmod{N_0}, \\ 0 & \text{otherwise.} \end{cases} \tag{9.1}$$

Let $\nu \equiv 1 \pmod{N_0}$. Then ν is prime to N_0, hence to $N_1(D)$, which has the same prime divisors. So $\tau \mapsto (-\nu)\cdot\tau$ is a bijection of the set of places that multiplies the classes by $-\nu \equiv -1$: it maps $\mathrm{tops}(D)$ onto the set of the places of class -1 modulo N_0, which contains $\mathrm{thick}(D)$, and every thick place is met exactly once. Let $\nu \not\equiv 1 \pmod{N_0}$. If ν is prime to $N_1(D)$, the places $(-\nu)\cdot\tau$ with $\tau \in \mathrm{tops}(D)$ have class $-\nu \neq -1$, and none is thick. If ν is not prime to $N_1(D)$, then $(-\nu)\cdot\tau = \tau$ by the convention of Setting 8.1, and no place of $\mathrm{tops}(D)$ is thick, because the classes 1 and -1 modulo N_0 differ. This proves (9.1). By Definition 2.4, $\lambda_{d,N_0}(\nu)$ equals $\lambda_{d,1}(\nu)$ if $\nu \equiv 1 \pmod{N_0}$ and is 0 otherwise; so (9.1) gives, for every natural number ν,

$$\#\{\tau \in \mathrm{tops}(D) : (-\nu)\cdot\tau \in \mathrm{thick}(D)\}\cdot\lambda_{d,1}(\nu) = \#\mathrm{thick}(D)\cdot\lambda_{d,N_0}(\nu). \tag{9.2}$$

Step 2: the sum of the prime sums over the tops. For $\tau \in \mathrm{tops}(D)$ and $1 \leq \nu \leq a$ the weight $w_D((-\nu)\cdot\tau)$ is a/x if $(-\nu)\cdot\tau$ is thick and a/x_1 otherwise. Summing over τ and using (9.2),

$$\begin{aligned}\sum_{\tau\in\mathrm{tops}(D)} \mathrm{Ask}_D(a,\tau) = \#\mathrm{thick}(D)\sum_{\nu=1}^{a}\lambda_{d,N_0}(\nu)\Big(\big(\tfrac{a}{\nu}-\tfrac{a}{x}\big)_+ - \big(\tfrac{a}{\nu}-\tfrac{a}{x_1}\big)_+\Big) \\ + \#\mathrm{tops}(D)\sum_{\nu=1}^{a}\lambda_{d,1}(\nu)\big(\tfrac{a}{\nu}-\tfrac{a}{x_1}\big)_+.\end{aligned}$$

By Lemma 4.3, for $M \in \{N_0, 1\}$ and a real number z with $0 < z \leq a$, $\sum_{\nu=1}^{a}\lambda_{d,M}(\nu)(\frac{a}{\nu}-\frac{a}{z})_+ = a\big(R_{d,M}(z) + \log z/\varphi(M) + \kappa_{d,M}\big)$. We use it for $M = N_0$ at $z = x$ and $z = x_1$, where the constant κ_{d,N_0} drops out of the difference, and for $M = 1$ at $z = x_1$, where $\varphi(1) = 1$:

$$\begin{aligned}\sum_{\tau\in\mathrm{tops}(D)} \mathrm{Ask}_D(a,\tau) = a\Big(\#\mathrm{thick}(D)\big(R_{d,N_0}(x) - R_{d,N_0}(x_1)\big) + \#\mathrm{tops}(D)\,R_{d,1}(x_1)\Big) \\ + a\Big(\#\mathrm{thick}(D)\,\frac{\log x - \log x_1}{\varphi(N_0)} + \#\mathrm{tops}(D)\,(\log x_1 + \kappa_{d,1})\Big).\end{aligned}$$

Step 3: the expected value. Since $\#\mathrm{thick}(D)$ places have weight a/x and the other $f(D) - \#\mathrm{thick}(D)$ have weight a/x_1, $\log a - \bar{w}_D = \log x_1 + \frac{\#\mathrm{thick}(D)}{f(D)}(\log x - \log x_1)$. With $f(D) = \varphi(N_0)\cdot\#\mathrm{tops}(D)$ and Definition 9.2,

$$\#\mathrm{tops}(D)\cdot\mathrm{Ask}^{\mathrm{main}}(D) = a\Big(\#\mathrm{thick}(D)\,\frac{\log x - \log x_1}{\varphi(N_0)} + \#\mathrm{tops}(D)\,(\log x_1 + \kappa_{d,1})\Big).$$

Step 4. By Definition 9.3, $\mathrm{rem}(D) = \sum_{\tau \in \mathrm{tops}(D)} \mathrm{Ask}_D(a,\tau) - \#\mathrm{tops}(D) \cdot \mathrm{Ask}^{\mathrm{main}}(D)$, and Steps 2 and 3 give the claim: the terms $\log x$, $\log x_1$ and $\kappa_{d,1}$ cancel. □

LEAN `Data.remainder_eq`

Step 1 is the place where the progression 1 modulo N_0 enters the proof: the thick places sit in one class, and the tops are the stabiliser of that class. The convention for the integers ν that are not prime to $N_1(D)$ does not affect the prime sums: for such ν the weight $\lambda_{d,1}(\nu)$ is 0, because every prime that divides $N_1(D)$ splits in K (unit 152).

Proposition 9.4 (the level pays the excess)**.** *Let P be parameters and D admissible data for P, with components x, x_1, b, m, r (Setting 8.1). Then*

$$h_0(D) \le h_{\mathrm{main}}(D) - m \cdot \mathrm{rem}(D) + \mathrm{budget}(D).$$

Proof. Write $a = a(D)$. By the definitions of $h_0(D)$, $h_{\mathrm{main}}(D)$ and $\mathrm{rem}(D)$,

$$h_0(D) = h_{\mathrm{main}}(D) - m \cdot \mathrm{rem}(D) + \#\mathrm{tops}(D) \cdot \Big(m\,(\mathrm{jet}_D(a) - \mathrm{Ask}^{\mathrm{main}}(D)) - a\,\log N(D) \Big). \tag{9.3}$$

The excess of one row. By the definitions of the jet size and of $\mathrm{Ask}^{\mathrm{main}}(D)$, in which the constant $\mathrm{cs}(d)$ cancels, $\mathrm{jet}_D(a) - \mathrm{Ask}^{\mathrm{main}}(D) = \log(a!) - a\log a + a + a\,(\bar{w}_D + \mathrm{per}_D)$. Stirling's bound $\log(a!) \le a\log a - a + 1 + \frac{1}{2}\log a$, valid for $a \ge 1$, gives

$$\mathrm{jet}_D(a) - \mathrm{Ask}^{\mathrm{main}}(D) \le a\,(\bar{w}_D + \mathrm{per}_D) + 1 + \tfrac{1}{2}\log a.$$

The level. By Input 9, $a \log N(D) \ge a\,m\,(\bar{w}_D + \mathrm{per}_D) - a\,(\mathrm{per}_D + \frac{15}{2}\log b)$. So the term $m\,a\,(\bar{w}_D + \mathrm{per}_D)$ cancels in the bracket of (9.3), which is at most $m\,(1 + \frac{1}{2}\log a) + a\,(\mathrm{per}_D + \frac{15}{2}\log b)$. The claim follows from (S1). □

LEAN `Data.heightPerRow_le`

The number $h_0(D)$ will bound the logarithm of the size of a chain of m theta lifts of a short column (Input 12). The proposition says that the level $N(D)$, whose logarithm is subtracted in $h_0(D)$, has been chosen so that it cancels the excess of size over expected divisibility of the m lifts; what remains is the main amount, less m times the remainder of the data.

Lemma 9.5 (a crude bound)**.** *Let P be parameters and D admissible data for P, with components x, x_1, b, m, r (Setting 8.1). Then*

$$|h_{\mathrm{main}}(D)| + m \cdot |\mathrm{rem}(D)| \le b^2\, x^{99}.$$

Proof. For every real number z and every place τ, $|\mathrm{Ask}_D(z,\tau)| \le 2\,a(D)\,|z|$: the sum has $a(D)$ terms, and $0 \le \lambda_{d,1}(\nu)\,(z/\nu - w)_+ \le 2\nu \cdot |z|/\nu$ for every weight $w \ge 0$, by Lemma 4.1. The claim is then (S2). In the formal text the comparison of sizes also uses that $|\,\mathrm{cs}(d)|$ is at most a constant of the parameters, which the conditions on the scales make negligible. □

LEAN `Data.abs_mainHeight_add_remainder_le`

The bound is far from sharp, and it is only used to pay for a small number of missing dimensions in the proof of the main inequality. To see that there is room (this is not the formal argument): unit 152 has $m \le 2\sqrt{x}$, $r \le 3x^5$ and $f(D) \le x^{11}(\log b)^{1/5}$ up to a constant of the parameters, and $a(D)$ is about xb; so the $f(D)$ prime sums in $\mathsf{U}(D)$, each at most $2a(D)^2/x$, give $|h_{\mathrm{main}}(D)|$ the order $x^{18}b^2$ at most, and $m\,|\mathrm{rem}(D)|$ has the order $x^{14}b^2$ at most.

10 Theorems A and B

Setting 10.1 (forms and lifts; units 102, 140, 141, 143, 145, 147, 148)**.** Let P be parameters and D admissible data for P, with components x, x_1, b, m, r, as in Setting 8.1. Other units attach to D the following objects.

Columns (unit 145). $\mathrm{Col}(D)$ is a finite-dimensional complex vector space of algebraic modular forms on a definite unitary group of rank $k(D)$ over $E(D)$; its complex dimension $\dim \mathrm{Col}(D)$ is a different number from the rank $k(D)$, and both occur below. The *ramifying operator* is a linear map $\mathrm{ram}_D \colon \mathrm{Col}(D) \to \mathrm{Col}(D)$, and its image $\mathrm{Col}^{\mathrm{ram}}(D)$ is the space of *ramified columns*.

Rows (units 140, 141). $\mathsf{G}(D)$ is a finite group. $\mathrm{PF}(D)$ is a complex vector space of cusp forms on an indefinite unitary group of rank r over $E(D)$, the *forms of principal level*, on which $\mathsf{G}(D)$ acts linearly. The space of *rows* $\mathrm{Row}(D) \subseteq \mathrm{PF}(D)$ is the image of a fixed idempotent of the group algebra of $\mathsf{G}(D)$; it is finite-dimensional. For $\gamma \in \mathsf{G}(D)$ the linear map $\varpi_D(\gamma) \colon \mathrm{PF}(D) \to \mathrm{Row}(D)$ translates a form by γ and applies $\#\mathsf{G}(D)$ times the idempotent.

Poincaré series (unit 148). A linear map $\mathrm{Poin}_D \colon \mathrm{Col}(D) \to \mathrm{PF}(D)$.

Twists and lifts (units 141, 143, 147). A *twist* of D is a character ϑ of a certain kind, with a finite set $Q(\vartheta)$ of primes of $E(D)$ where it ramifies; an *m-fold twist* is a twist whose powers $\vartheta^2, \dots, \vartheta^m$ are twists with the same set; the trivial character 1 is an m-fold twist with $Q(1) = \emptyset$. A prime number ℓ is *prime to* ϑ if it is prime to the norms of the primes in $Q(\vartheta)$. For a twist ϑ, the *chain of lifts* is a linear map $\Theta^{\flat}_{D,\vartheta} \colon \mathrm{PF}(D) \to \mathcal{A}_{D,\vartheta}$, the composite of m theta lifts, into a finite-dimensional complex vector space $\mathcal{A}_{D,\vartheta}$ of algebraic modular forms on a definite unitary group, which carries a positive definite hermitian product. For an m-fold twist ϑ, $\Theta_{D,\vartheta} \colon \mathrm{Row}(D) \to \mathcal{A}_{D,\vartheta}$ is the restriction of $\Theta^{\flat}_{D,\vartheta}$ to the rows, and $\mathfrak{z}_D(\vartheta) = \det(\Theta^*_{D,\vartheta}\Theta_{D,\vartheta})$.

Conjugates and sizes (unit 102). The columns and the elements of the spaces $\mathcal{A}_{D,\vartheta}$ are functions with values in complex vector spaces that have a rational structure and a norm; so for $\sigma \in \mathrm{Aut}(\mathbb{C})$ a form F has a conjugate $\sigma \circ F$. We say that F is *pointwise at most* e^B *at* σ, for a real number B, if every value of $\sigma \circ F$ has squared norm at most e^B. For $F, F_1, \dots, F_n$ in $\mathcal{A}_{D,\vartheta}$, $\|F\|^2_\sigma$ is the Petersson norm of $\sigma \circ F$, a finite sum over classes of squared norms of values, and $\mathrm{covol}^2_\sigma(F_1, \dots, F_n)$ is the absolute value of the determinant of the matrix of Petersson products of the $\sigma \circ F_i$.

Integrality (unit 102). For a prime number ℓ and a ring homomorphism $j \colon \mathbb{C} \to \mathbb{C}_\ell$, O_j is the ring of the complex numbers H with $|j(H)|_\ell \leq 1$. A column, or an element of some $\mathcal{A}_{D,\vartheta}$, is *$j$-integral* if all its values are, for a lattice fixed in unit 102; the j-integral elements form an O_j-module, and if F is j-integral and $|j(H)|_\ell \leq 1$ for a complex number H, then HF is j-integral. A column is *integral* if it is j-integral for every prime ℓ and every $j \colon \mathbb{C} \to \mathbb{C}_\ell$.

Input 10 (the seesaw identity; unit 141)**.** In Settings 8.1 and 10.1 there is a non-zero complex number H such that for every twist ϑ of D there is a good datum Ψ for ϑ with

$$\Theta^{\flat}_{D,\vartheta}(\mathrm{Poin}_D(H\,\psi)) = \mathrm{Ch}_D(\vartheta,\Psi)(\psi) \qquad \text{for every } \psi \in \mathrm{Col}^{\mathrm{ram}}(D).$$

Here a *good datum* for ϑ is a notion of unit 145, and $\mathrm{Ch}_D(\vartheta,\Psi)\colon \mathrm{Col}(D) \to \mathcal{A}_{D,\vartheta}$ is the chain of m theta lifts between definite spaces that Ψ defines (unit 147). The formal text fixes one such number H, by choice; we write $\mathrm{cst}^{\mathrm{see}}_D$ for it.

The identity moves the lifts of Poincaré series to the definite side, where sizes and divisibility are computed; it is the subject of Chapter V.

Definition 10.2 (the scaled inclusion)**.** Let P be parameters and D admissible data for P, with components x, x_1, b, m, r (Settings 8.1 and 10.1). The linear map $\mathrm{inc}(D)\colon \mathrm{Col}^{\mathrm{ram}}(D) \to \mathrm{Col}(D)$ is the inclusion multiplied by the constant $\mathrm{cst}^{\mathrm{see}}_D$ of Input 10.

LEAN `Data.scaledInclusion`

Definition 10.3 (the Poincaré maps)**.** Let P be parameters and D admissible data for P, with components x, x_1, b, m, r (Settings 8.1 and 10.1), and let $\gamma \in \mathsf{G}(D)$. The linear map $\mathcal{P}_D(\gamma)\colon \mathrm{Col}^{\mathrm{ram}}(D) \to \mathrm{Row}(D)$ is the composite $\varpi_D(\gamma) \circ \mathrm{Poin}_D \circ \mathrm{inc}(D)$; its value at a ramified column ψ is written $\mathcal{P}_D(\gamma)\psi$.

LEAN `Data.projectedPoincare`

Definition 10.4 (integral ramified columns)**.** Let P be parameters and D admissible data for P, with components x, x_1, b, m, r (Settings 8.1 and 10.1). Then $\mathrm{Col}^{\mathrm{ram}}_{\mathrm{int}}(D)$ is the set of the ramified columns $\psi \in \mathrm{Col}^{\mathrm{ram}}(D)$ that are integral: j-integral for every prime number ℓ and every ring homomorphism $j\colon \mathbb{C} \to \mathbb{C}_\ell$.

LEAN `Data.integralRamified`

Definition 10.5 (the index of a family of rows)**.** Let P be parameters and D admissible data for P, with components x, x_1, b, m, r (Settings 8.1 and 10.1). Let n be a natural number, $y = (y_1, \dots, y_n)$ a family of rows in $\mathrm{Row}(D)$, and δ a real number. We say that y has D-index at most δ, and write $\mathrm{ind}_D(y) \le \delta$, if there are real numbers η_ℓ, $\ell \in \mathbb{N}$, with $\sum_{\ell \in J} \eta_\ell \le \delta$ for every finite set $J \subset \mathbb{N}$, such that for every prime number ℓ, every ring homomorphism $j\colon \mathbb{C} \to \mathbb{C}_\ell$ and every m-fold twist ϑ of D to which ℓ is prime, there is an $n \times n$ matrix $u = (u_{ii'})$ with entries in O_j such that $|j(\det u)|_\ell \ge \mathrm{e}^{-\eta_\ell}$ and such that $\Theta_{D,\vartheta}(\sum_{i'} u_{ii'}\, y_{i'})$ is j-integral for $i = 1, \dots, n$.

LEAN `Data.IndexLe`

The index bounds, prime by prime and for all twists at once, the denominators of the lifts of the family: after a change of the family by a matrix of small determinant, the lifts are integral.

Input 11 (short integral columns span; unit 145)**.** In Settings 8.1 and 10.1, $\mathrm{Col}^{\mathrm{ram}}(D)$ is spanned over $\mathbb{C}$ by the ramified columns that are integral and pointwise at most $\mathrm{e}^{\mathrm{budget}(D)}$ at every $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Input 12 (the size of the chain; units 145, 150)**.** In Settings 8.1 and 10.1, let $\mathrm{nonsplit}(D)$ be the set of the prime numbers $\ell < x_1$ with $\chi_d(\ell) \neq 1$, and let $\mathrm{gain}_D(\ell)$, for a natural number ℓ, be the real number that unit 150 calls the *gain* at ℓ: the divisibility by ℓ that is asked of a chain of lifts. Let Ψ be a good datum for the trivial twist, $\sigma \in \mathrm{Aut}(\mathbb{C})$, $\psi \in \mathrm{Col}^{\mathrm{ram}}(D)$ and B a real number. If ψ is pointwise at most e^B at σ, then $\mathrm{Ch}_D(1,\Psi)(\psi)$ is pointwise at most

$$\exp\Bigl(B + h_0(D) + 2\sum_{\ell \in \mathrm{nonsplit}(D)} \mathrm{gain}_D(\ell)\,\log\ell + \mathrm{budget}(D)\Bigr) \quad \text{at } \sigma.$$

Input 13 (passing to the rows; unit 143)**.** In Settings 8.1 and 10.1, let $F \in \mathrm{PF}(D)$ and $\gamma \in \mathsf{G}(D)$.
(a) Let $\sigma \in \mathrm{Aut}(\mathbb{C})$ and let B be a real number. If $\Theta^{\flat}_{D,1}(F)$ is pointwise at most e^B at σ, then $\|\Theta_{D,1}(\varpi_D(\gamma)F)\|^2_\sigma \leq \exp(x^{99} + B)$.
(b) Let ϑ be a twist of D, ℓ a prime number, $j\colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism and H a complex number. If $H \cdot \Theta^{\flat}_{D,\vartheta}(F)$ is j-integral, then so is $H \cdot \Theta^{\flat}_{D,\vartheta}(\varpi_D(\gamma)F)$.

Input 14 (the divisibility of the chain; units 145, 150)**.** In Settings 8.1 and 10.1, unit 145 defines for every natural number ℓ a real number $\mathrm{err}_D(\ell)$, the *allowed loss* at ℓ, with these properties.
(a) $\mathrm{err}_D(\ell) = 0$ unless $\ell \in \mathrm{nonsplit}(D)$ or $\ell = N(D)$; and $\sum_{\ell \in J} \mathrm{err}_D(\ell)\log\ell \leq \mathrm{budget}(D)$ for every finite set $J \subset \mathbb{N}$.
(b) Let ℓ be a prime number, $j\colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism, and e an integer with $e < \mathrm{gain}_D(\ell) - \mathrm{err}_D(\ell) + 1$ if $\ell \in \mathrm{nonsplit}(D)$, and $e < -\mathrm{err}_D(\ell) + 1$ otherwise. Then there are a natural number $i_0 \leq \mathrm{err}_D(\ell) \cdot \dim \mathrm{Col}(D)$ and an O_j-submodule $\mathcal{S}$ of $\mathrm{Col}(D)$ such that: every finite family $\psi_1, \dots, \psi_n$ of j-integral columns admits an $n \times n$ matrix u over O_j whose determinant divides ℓ^{i_0} in O_j and for which all $\sum_{i'} u_{ii'}\psi_{i'}$ lie in $\mathcal{S}$; and $\ell^{-e} \cdot \mathrm{Ch}_D(\vartheta,\Psi)(\psi)$ is j-integral for every twist ϑ of D to which ℓ is prime, every good datum Ψ for ϑ and every $\psi \in \mathcal{S} \cap \mathrm{Col}^{\mathrm{ram}}(D)$.

Input 15 (the index of a span; unit 149)**.** In Settings 8.1 and 10.1, let ℓ be a prime number, $j\colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism, i_0 a natural number and q a rational number. Let $\mathcal{M} \subseteq \mathrm{Col}^{\mathrm{ram}}(D)$ and $\mathcal{N} \subseteq \mathrm{Row}(D)$ be O_j-submodules. Assume that for every $\gamma \in \mathsf{G}(D)$ every finite family $\psi_1, \dots, \psi_n$ in $\mathcal{M}$ admits an $n \times n$ matrix u over O_j whose determinant divides ℓ^{i_0} and for which all $q\,\mathcal{P}_D(\gamma)(\sum_{i'} u_{ii'}\psi_{i'})$ lie in $\mathcal{N}$. Then for every finite set $\mathcal{T} \subseteq \mathsf{G}(D)$, every finite family $y_1, \dots, y_n$ in the O_j-span of $\mathcal{N}$ and of the rows $q\,\mathcal{P}_D(\gamma)\psi$ with $\gamma \in \mathcal{T}$, $\psi \in \mathcal{M}$, admits an $n \times n$ matrix u over O_j whose determinant divides $\ell^{i_0 \cdot \#\mathcal{T}}$ and for which all $\sum_{i'} u_{ii'}y_{i'}$ lie in $\mathcal{N}$.

A special case of a lemma on modules over a commutative ring: the index of a sum of images is at most the product of the indices.

Lemma 10.6 (a rational number with prescribed valuations)**.** *Let J be a finite set of natural numbers and $e\colon \mathbb{N} \to \mathbb{Z}$, $\ell \mapsto e_\ell$, a function. Assume that every element*

of J is a prime number and that $e_\ell = 0$ for every $\ell \notin J$. Then there is a rational number q with $q > 0$, $\log q = \sum_{\ell \in J} e_\ell \log \ell$, and $|q|_\ell = |\ell|_\ell^{e_\ell}$ for every prime number ℓ, where $|\cdot|_\ell$ is the absolute value of $\mathbb{C}_\ell$.

Proof. Take $q = \prod_{\ell \in J} \ell^{e_\ell}$. For a prime $\ell \in J$ the other factors are ℓ-adic units; for a prime $\ell \notin J$ all factors are, and $e_\ell = 0$. □

LEAN `exists_rat_log_eq_and_padic_norm_eq`

Theorem A (short forms)**.** Let P be parameters and D admissible data for P, with components x, x_1, b, m, r (Settings 8.1 and 10.1). There is a rational number $q > 0$ with the following two properties.
(a) Size. There is a subset S of $\mathrm{Col}^{\mathrm{ram}}_{\mathrm{int}}(D)$ that spans $\mathrm{Col}^{\mathrm{ram}}(D)$ over $\mathbb{C}$, such that for all $\psi \in S$, all $\gamma \in \mathsf{G}(D)$ and all $\sigma \in \mathrm{Aut}(\mathbb{C})$,

$$\|\Theta_{D,1}(q\,\mathcal{P}_D(\gamma)\psi)\|_\sigma^2 \le \exp(h_{\mathrm{main}}(D) - m \cdot \mathrm{rem}(D) + \log b \cdot \mathrm{budget}(D)).$$

(b) Index. For every finite set $\mathcal{T} \subseteq \mathsf{G}(D)$, every natural number n and every family $y = (y_1, \dots, y_n)$ of rows, each of the form $q\,\mathcal{P}_D(\gamma)\psi$ with $\gamma \in \mathcal{T}$ and $\psi \in \mathrm{Col}^{\mathrm{ram}}_{\mathrm{int}}(D)$: y has D-index at most $\#\mathcal{T} \cdot \dim \mathrm{Col}(D) \cdot \mathrm{budget}(D)$.

Proof. The number q. Let $J = \mathrm{nonsplit}(D) \cup \{N(D)\}$, a finite set of primes (Input 9), and for a natural number ℓ let

$$e_\ell = \lfloor \mathrm{err}_D(\ell) - \mathrm{gain}_D(\ell) \rfloor \ \text{ if } \ell \in \mathrm{nonsplit}(D), \qquad e_\ell = \lfloor \mathrm{err}_D(\ell) \rfloor \ \text{ otherwise.}$$

Then $e_\ell = 0$ for $\ell \notin J$ by Input 14(a). Let q be the rational number of Lemma 10.6: $q > 0$, $\log q = \sum_{\ell \in J} e_\ell \log \ell$, and $|q|_\ell = |\ell|_\ell^{e_\ell}$ for every prime ℓ. For every twist ϑ choose a good datum Ψ_ϑ as in Input 10. By Definitions 10.2 and 10.3, for $\gamma \in \mathsf{G}(D)$, $\psi \in \mathrm{Col}^{\mathrm{ram}}(D)$ and an m-fold twist ϑ,

$$\mathcal{P}_D(\gamma)\psi = \varpi_D(\gamma)(\mathrm{Poin}_D(\mathrm{cst}^{\mathrm{see}}_D\ \psi)), \qquad \Theta^{\flat}_{D,\vartheta}(\mathrm{Poin}_D(\mathrm{cst}^{\mathrm{see}}_D\ \psi)) = \mathrm{Ch}_D(\vartheta, \Psi_\vartheta)(\psi). \tag{10.1}$$

(a). Let S be the set of the ramified columns that are integral and pointwise at most $\mathrm{e}^{\mathrm{budget}(D)}$ at every $\sigma \in \mathrm{Aut}(\mathbb{C})$. Then $S \subseteq \mathrm{Col}^{\mathrm{ram}}_{\mathrm{int}}(D)$, and S spans $\mathrm{Col}^{\mathrm{ram}}(D)$ by Input 11. Let $\psi \in S$, $\gamma \in \mathsf{G}(D)$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$. Since $\sigma(q) = q$, the column $q\psi$ is pointwise at most $q^2\mathrm{e}^{\mathrm{budget}(D)} = \exp(2\log q + \mathrm{budget}(D))$ at σ. By (10.1) and Input 12, $\Theta^{\flat}_{D,1}(\mathrm{Poin}_D(\mathrm{cst}^{\mathrm{see}}_D\ q\psi)) = \mathrm{Ch}_D(1, \Psi_1)(q\psi)$ is pointwise at most $\exp(2\log q + \mathrm{budget}(D) + h_0(D) + 2\sum_{\ell \in \mathrm{nonsplit}(D)} \mathrm{gain}_D(\ell)\log\ell + \mathrm{budget}(D))$ at σ, and by Input 13(a) the number $\|\Theta_{D,1}(q\,\mathcal{P}_D(\gamma)\psi)\|_\sigma^2$ is at most exp of x^{99} plus this exponent. Now

$$\begin{aligned} 2\log q = 2\sum_{\ell \in J} e_\ell \log \ell &\le 2\sum_{\ell \in J} \mathrm{err}_D(\ell)\log\ell - 2\sum_{\ell \in \mathrm{nonsplit}(D)} \mathrm{gain}_D(\ell)\log\ell \\ &\le 2\,\mathrm{budget}(D) - 2\sum_{\ell \in \mathrm{nonsplit}(D)} \mathrm{gain}_D(\ell)\log\ell, \end{aligned}$$

by Input 14(a). The gains cancel: this is what q is for. With $h_0(D) \le h_{\mathrm{main}}(D) - m \cdot \mathrm{rem}(D) + \mathrm{budget}(D)$ (Proposition 9.4) the exponent is at most $h_{\mathrm{main}}(D) - m \cdot \mathrm{rem}(D) + x^{99} + 5\,\mathrm{budget}(D)$, and (S3) gives (a).

(b). Let $\mathcal{T}$, n and y be as in (b), and put $\eta_\ell = \#\mathcal{T} \cdot \dim \mathrm{Col}(D) \cdot \mathrm{err}_D(\ell)\log\ell$ for $\ell \in \mathbb{N}$. Then $\sum_{\ell \in J'} \eta_\ell \le \#\mathcal{T} \cdot \dim \mathrm{Col}(D) \cdot \mathrm{budget}(D)$ for every finite set $J' \subset \mathbb{N}$,

by Input 14(a). Let ℓ be a prime number, $j\colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism and ϑ an m-fold twist to which ℓ is prime. The integer $e = -e_\ell$ satisfies the hypothesis of Input 14(b), because $-\lfloor z \rfloor < -z + 1$ for real z. Let i_0 and $\mathcal{S}$ be as given there, and let $\mathcal{M}$ be the O_j-module of the j-integral ramified columns, $\mathcal{N}$ the O_j-module of the rows y' for which $\Theta_{D,\vartheta}(y')$ is j-integral. We check the hypothesis of Input 15. Let $\gamma \in \mathsf{G}(D)$ and let $\psi_1, \dots, \psi_n$ be in $\mathcal{M}$. Input 14(b) gives a matrix u over O_j whose determinant divides ℓ^{i_0} such that every $\psi_i' = \sum_{i'} u_{ii'}\psi_{i'}$ lies in $\mathcal{S} \cap \mathrm{Col}^{\mathrm{ram}}(D)$ and $\ell^{e_\ell}\,\mathrm{Ch}_D(\vartheta, \Psi_\vartheta)(\psi_i')$ is j-integral. Since $|q|_\ell = |\ell^{e_\ell}|_\ell$, also $q\,\mathrm{Ch}_D(\vartheta, \Psi_\vartheta)(\psi_i') = q\,\Theta^\flat_{D,\vartheta}(\mathrm{Poin}_D(\mathrm{cst}_D^{\mathrm{see}}\,\psi_i'))$ is j-integral, by (10.1); and then so is $q\,\Theta^\flat_{D,\vartheta}(\varpi_D(\gamma)\,\mathrm{Poin}_D(\mathrm{cst}_D^{\mathrm{see}}\,\psi_i')) = \Theta_{D,\vartheta}(q\,\mathcal{P}_D(\gamma)\psi_i')$, by Input 13(b). So $q\,\mathcal{P}_D(\gamma)\psi_i' \in \mathcal{N}$. Each y_i is of the form $q\,\mathcal{P}_D(\gamma)\psi$ with $\gamma \in \mathcal{T}$ and $\psi \in \mathrm{Col}^{\mathrm{ram}}_{\mathrm{int}}(D) \subseteq \mathcal{M}$. So Input 15 gives an $n \times n$ matrix u over O_j whose determinant divides $\ell^{i_0 \cdot \#\mathcal{T}}$ in O_j, with $\Theta_{D,\vartheta}(\sum_{i'} u_{ii'}y_{i'})$ j-integral for all i. Finally $|j(\det u)|_\ell \geq |\ell|_\ell^{i_0 \cdot \#\mathcal{T}} = \exp(-i_0 \cdot \#\mathcal{T} \cdot \log \ell) \geq \mathrm{e}^{-\eta_\ell}$, because $i_0 \leq \mathrm{err}_D(\ell) \cdot \dim \mathrm{Col}(D)$. This is Definition 10.5. □

LEAN `Data.short_forms`

Input 16 (columns and rows exist; units 141, 148)**.** In Settings 8.1 and 10.1: $\mathrm{Col}(D) \neq 0$ and $\mathrm{Row}(D) \neq 0$.

Input 17 (the filling theorem, and few translates; units 148, 135)**.** In Settings 8.1 and 10.1:
(a) For every non-zero complex number H the family of linear maps $\varpi_D(\gamma) \circ \mathrm{Poin}_D \circ (H \cdot \mathrm{ram}_D)\colon \mathrm{Col}(D) \to \mathrm{Row}(D)$, $\gamma \in \mathsf{G}(D)$, is *spread with constant* b^{-2}: for every subspace W of $\mathrm{Row}(D)$ there is $\gamma \in \mathsf{G}(D)$ such that the preimage of W under the map of index γ has dimension at most $(\frac{1}{2}(1 + \dim W / \dim \mathrm{Row}(D)) + b^{-2}) \cdot \dim \mathrm{Col}(D)$.
(b) If a family of linear maps from $\mathrm{Col}(D) \neq 0$ to $\mathrm{Row}(D)$, indexed by $\mathsf{G}(D)$, is spread with constant b^{-2}, then there is a finite set $\mathcal{T} \subseteq \mathsf{G}(D)$ with $\#\mathcal{T} \cdot \dim \mathrm{Col}(D) \leq 5 \dim \mathrm{Row}(D) \cdot \log(1 + b^2)$ such that the sum of the images of the maps of index $\gamma \in \mathcal{T}$ has dimension at least $(1 - 4b^{-2}) \dim \mathrm{Row}(D)$.

(a) is the filling theorem of unit 148, the subject of Chapter VI. (b) is a special case of a lemma of unit 135 on families of linear maps between finite-dimensional spaces, proved by a greedy choice of translates.

Theorem B (enough forms)**.** Let P be parameters and D admissible data for P, with components x, x_1, b, m, r (Settings 8.1 and 10.1). Then $\dim \mathrm{Col}(D) > 0$ and $\dim \mathrm{Row}(D) > 0$, and there is a finite set $\mathcal{T} \subseteq \mathsf{G}(D)$ with

$$\#\mathcal{T} \cdot \dim \mathrm{Col}(D) \leq 5 \dim \mathrm{Row}(D) \cdot \log(1 + b^2) \qquad \text{and}$$
$$\left(1 - \frac{4}{b^2}\right) \dim \mathrm{Row}(D) \leq \dim \sum_{\gamma \in \mathcal{T}} \mathcal{P}_D(\gamma)(\mathrm{Col}^{\mathrm{ram}}(D)).$$

Proof. The dimensions are positive by Input 16. Apply Input 17(a) with $H = \mathrm{cst}_D^{\mathrm{see}}$, and then (b), which is allowed because $b > 0$ by (S0): there is a finite set $\mathcal{T} \subseteq \mathsf{G}(D)$ with the first inequality, such that the images of

the maps $\varpi_D(\gamma) \circ \mathrm{Poin}_D \circ (\mathrm{cst}_D^{\mathrm{see}} \cdot \mathrm{ram}_D)$, $\gamma \in \mathcal{T}$, span dimension at least $(1 - 4/b^2) \dim \mathrm{Row}(D)$. For a column ψ_0 the element $\mathrm{ram}_D(\psi_0)$ is a ramified column, and $\varpi_D(\gamma)\,\mathrm{Poin}_D(\mathrm{cst}_D^{\mathrm{see}}\,\mathrm{ram}_D(\psi_0)) = \mathcal{P}_D(\gamma)\,\mathrm{ram}_D(\psi_0)$ by Definitions 10.2 and 10.3. So each of these images lies in $\mathcal{P}_D(\gamma)(\mathrm{Col}^{\mathrm{ram}}(D))$, and the second inequality follows. □

LEAN `Data.enough_forms`

11 Theorem C, and the family given by A and B

Setting 11.1 (heights and fit families; units 135, 138)**.** Let P and D be as in Settings 8.1 and 10.1. Families of complex numbers and of twists are indexed by the pairs (ℓ, ι) of a prime number ℓ and a field isomorphism $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$ onto an algebraic closure of the field of ℓ-adic numbers, with its absolute value $|\cdot|_\ell$. A family $Z = (Z_{\ell,\iota})$ of complex numbers has *height at most* h, for a real number h, if there are a non-zero complex number H, a real number α and real numbers η_ℓ, $\ell \in \mathbb{N}$, such that $|\sigma(H)| \le \mathrm{e}^{\alpha}$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, $|\iota(H\,Z_{\ell,\iota})|_\ell \le \mathrm{e}^{\eta_\ell}$ for all pairs (ℓ, ι), and $\alpha + \sum_{\ell<i,\ \ell \text{ prime}} \eta_\ell \le h$ for all sufficiently large natural numbers i (unit 135). A family $\theta = (\theta_{\ell,\iota})$ of m-fold twists of D may be *fit for* D; this property is defined in unit 138 and bounds, for each ℓ, the congruences of the lifts at the twists $\theta_{\ell,\iota}$. Of its definition we use only that ℓ is prime to $\theta_{\ell,\iota}$ for every pair (ℓ, ι); otherwise it enters through Inputs 20, 21 and 22.

Input 18 (heights; unit 135)**.** For families Z, Z' of complex numbers indexed by the pairs (ℓ, ι) and real numbers h, h':
(a) if Z has height at most h and Z' has height at most h', then the family of the products $Z_{\ell,\iota} Z'_{\ell,\iota}$ has height at most $h + h'$; the same holds for finite products, and the family of the n-th powers of Z has height at most $n\,h$;
(b) if Z has height at most h, $h \le h'$, and $|\iota(Z'_{\ell,\iota})|_\ell \le |\iota(Z_{\ell,\iota})|_\ell$ for all pairs, then Z' has height at most h';
(c) if the constant family 1 has height at most h, then $h \ge 0$.
(c) is the product formula, applied to the norm of the auxiliary number H. Chapter IV.

Input 19 (fit families exist; unit 138)**.** In Settings 8.1, 10.1 and 11.1 there is a family $\theta = (\theta_{\ell,\iota})$ of m-fold twists of D that is fit for D.
The subject of Chapter VIII.

Input 20 (the volume of the chain; units 143, 147)**.** In Settings 8.1, 10.1 and 11.1, say that a complex-valued function $\vartheta \mapsto \mathrm{cst}(\vartheta)$ on the m-fold twists of D *costs* n *squares*, for a natural number n, if for every family θ that is fit for D the family $(\ell, \iota) \mapsto |\mathrm{cst}(\theta_{\ell,\iota})/\mathrm{cst}(1)|^{-2}$ has height at most $n \cdot 2\,t(D)\,x^{999}$. There is a function $\mathrm{cst}_D^{\mathrm{vol}}$ that costs m^2 squares such that for every m-fold twist ϑ,

$$\mathfrak{z}_D(\vartheta) \neq 0 \qquad \text{and} \qquad \mathfrak{z}_D(\vartheta) = \left|\mathrm{cst}_D^{\mathrm{vol}}(\vartheta)\right|^{\dim \mathrm{Row}(D)} \cdot \prod_{i=1}^{m} \mathrm{vol}_D(\vartheta, i),$$

where $\mathrm{vol}_D(\vartheta, i)$ is the volume of the i-th lift of the chain at the twist ϑ^i (unit 143).

Input 21 (the height of one lift; unit 143)**.** In Settings 8.1, 10.1 and 11.1, let θ be a family of m-fold twists that is fit for D, and let i be a natural number with $1 \le i \le m$. Then the family $(\ell, \iota) \mapsto \operatorname{vol}_D(1,i)/\operatorname{vol}_D(\theta_{\ell,\iota}, i)$ has height at most

$$\dim \operatorname{Row}(D) \cdot k(D) \cdot \Big(\log b \cdot \frac{\operatorname{budget}(D)}{m\, k(D)} - \mathsf{U}(D) \Big).$$

Each lift repays one direction: this is where $\mathsf{U}(D)$, and so the main amount, enters the lower bound. Chapter VII.

Input 22 (crude rows; unit 147)**.** In Settings 8.1, 10.1 and 11.1 there is a function $\operatorname{cst}^{\mathrm{cr}}_D$ on the m-fold twists of D, with non-zero complex values, that costs $2m^2$ squares, such that $\mathrm{PF}(D)$ is spanned over $\mathbb{C}$ by the forms F with these two properties: for every m-fold twist ϑ, every prime number ℓ that is prime to ϑ and every ring homomorphism $j\colon \mathbb{C} \to \mathbb{C}_\ell$, the form $(\operatorname{cst}^{\mathrm{cr}}_D(\vartheta)/\operatorname{cst}^{\mathrm{cr}}_D(1)) \cdot \Theta^{\flat}_{D,\vartheta}(F)$ is j-integral; and $\Theta^{\flat}_{D,1}(F)$ is pointwise at most $\exp(x_1^2 b^2)$ at every $\sigma \in \operatorname{Aut}(\mathbb{C})$. We call such forms *crude*.

Input 23 (one determinant; unit 135)**.** In Settings 8.1, 10.1 and 11.1, let $\theta = (\theta_{\ell,\iota})$ be a family of m-fold twists of D and let $y_1, \dots, y_{n'}$ be a basis of $\operatorname{Row}(D)$. Let B, δ, h be real numbers, n'' a natural number, $(\eta_\ell)_{\ell \in \mathbb{N}}$ real numbers and $(H_{\ell,\iota})$ a family of non-zero complex numbers. Assume:

(1) $\Theta_{D,1}$ is injective, and $|\sigma(\det(\langle \Theta_{D,1} y_i, \Theta_{D,1} y_{i'} \rangle)_{i,i'})| \le \mathrm{e}^{B}$ for every $\sigma \in \operatorname{Aut}(\mathbb{C})$;

(2) the family $(\ell, \iota) \mapsto |H_{\ell,\iota}|^{-2}$ has height at most h;

(3) $\sum_{\ell \in J} \eta_\ell \le \delta$ for every finite set $J \subset \mathbb{N}$;

(4) for every pair (ℓ, ι) and every ring homomorphism $j\colon \mathbb{C} \to \mathbb{C}_\ell$: if $F, F' \in \mathcal{A}_{D,\theta_{\ell,\iota}}$, F' is j-integral and F is integral for the composite of j with complex conjugation, then $|j(\langle F, F' \rangle)|_\ell \le 1$;

(5) for every pair (ℓ, ι) and every ring homomorphism $j\colon \mathbb{C} \to \mathbb{C}_\ell$ there is a complex $n' \times n'$ matrix u with $\mathrm{e}^{-\eta_\ell} |j(H_{\ell,\iota})|_\ell^{n''} \le |j(\det u)|_\ell$ such that all $\Theta_{D,\theta_{\ell,\iota}}(\sum_{i'} u_{ii'} y_{i'})$ are j-integral.

Then the family $(\ell, \iota) \mapsto \mathfrak{z}_D(\theta_{\ell,\iota})/\mathfrak{z}_D(1)$ has height at most $B + 2\delta + n'' h$.

A special case: the lemma of unit 135 is stated for an arbitrary family of linear maps from one space into inner product spaces. Chapter IV.

Input 24 (integral forms pair integrally; units 102, 152, 147)**.** In Settings 8.1 and 10.1, hypothesis (4) of Input 23 holds for every twist: for a twist ϑ, a prime number ℓ, a ring homomorphism $j\colon \mathbb{C} \to \mathbb{C}_\ell$ and $F, F' \in \mathcal{A}_{D,\vartheta}$ with F' j-integral and F integral for the composite of j with complex conjugation, $|j(\langle F, F' \rangle)|_\ell \le 1$.

Unit 102 proves this with a factor: a common multiple of the orders of the stabilisers that occur in the Petersson product. Units 152 and 147 show that at the level of $\mathcal{A}_{D,\vartheta}$ all these stabilisers are trivial.

Input 25 (Hadamard and Fischer; unit 102)**.** In Settings 8.1 and 10.1, let ϑ be a twist, $\sigma \in \mathrm{Aut}(\mathbb{C})$, B and V real numbers, and $F_1, \dots, F_n, F_1', \dots, F_{n''}' \in \mathcal{A}_{D,\vartheta}$.
(a) If $\|F_i'\|_\sigma^2 \le \mathrm{e}^B$ for all i, then $\mathrm{covol}_\sigma^2(F_1', \dots, F_{n''}') \le \mathrm{e}^{n''B}$.
(b) If moreover $\mathrm{covol}_\sigma^2(F_1, \dots, F_n) \le \mathrm{e}^V$, then $\mathrm{covol}_\sigma^2(F_1, \dots, F_n, F_1', \dots, F_{n''}') \le \mathrm{e}^{V+n''B}$.
(c) If $\mathrm{covol}_{\sigma'}^2(F_1, \dots, F_n) \le \mathrm{e}^V$ for every $\sigma' \in \mathrm{Aut}(\mathbb{C})$, then $|\sigma(\det(\langle F_i, F_{i'}\rangle)_{i,i'})| \le \mathrm{e}^V$.

Lemma 11.2 (the height of the inverse ratio of volumes)**.** *Let P be parameters and D admissible data for P, with components x, x_1, b, m, r (Settings 8.1, 10.1 and 11.1), and let $\theta = (\theta_{\ell,\iota})$ be a family of m-fold twists of D that is fit for D. Then the family $(\ell, \iota) \mapsto \mathfrak{z}_D(1)^2/\mathfrak{z}_D(\theta_{\ell,\iota})^2$ has height at most*

$$2 \dim \mathrm{Row}(D) \cdot (2 \log b \cdot \mathrm{budget}(D) - h_{\mathrm{main}}(D)).$$

Proof. By Input 20, for every pair (ℓ, ι),

$$\frac{\mathfrak{z}_D(1)^2}{\mathfrak{z}_D(\theta_{\ell,\iota})^2} = \Big(\prod_{i=1}^{m} \frac{\mathrm{vol}_D(1,i)}{\mathrm{vol}_D(\theta_{\ell,\iota}, i)}\Big)^2 \cdot \Big(\Big|\frac{\mathrm{cst}_D^{\mathrm{vol}}(\theta_{\ell,\iota})}{\mathrm{cst}_D^{\mathrm{vol}}(1)}\Big|^{-2}\Big)^{\dim \mathrm{Row}(D)}.$$

By Inputs 21 and 18(a) the product over i has height at most m times the bound of Input 21, which is $\dim \mathrm{Row}(D) \cdot (\log b \cdot \mathrm{budget}(D) - h_{\mathrm{main}}(D))$ by Definition 9.1; here $m\,k(D) \neq 0$ by (S0). The function $\mathrm{cst}_D^{\mathrm{vol}}$ costs m^2 squares, so the last family in brackets has height at most $m^2 \cdot 2\,t(D)\,x^{999} \le 2 \log b \cdot \mathrm{budget}(D)$, by (S5) and Input 18(b). By Input 18(a) the height of the product is at most $2 \dim \mathrm{Row}(D)\,(\log b \cdot \mathrm{budget}(D) - h_{\mathrm{main}}(D)) + \dim \mathrm{Row}(D) \cdot 2 \log b \cdot \mathrm{budget}(D)$. □

LEAN `Data.heightLe_volume_ratio_sq`

Setting 11.3 (an almost spanning family of small index and small covolume)**.** Let P be parameters and D admissible data for P, with components x, x_1, b, m, r (Settings 8.1 and 10.1). Let n be a natural number, $y = (y_1, \dots, y_n)$ a family of rows in $\mathrm{Row}(D)$, and δ and V real numbers, such that:

(i) the family y is linearly independent over $\mathbb{C}$;
(ii) $(1 - 4/b^2) \cdot \dim \mathrm{Row}(D) \le n$;
(iii) y has D-index at most δ;
(iv) $\mathrm{covol}_\sigma^2(\Theta_{D,1} y_1, \dots, \Theta_{D,1} y_n) \le \mathrm{e}^V$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Lemma 11.4 (the height of the ratio of volumes)**.** *In Setting 11.3, let $\theta = (\theta_{\ell,\iota})$ be a family of m-fold twists of D that is fit for D (Setting 11.1). Then the family $(\ell, \iota) \mapsto \mathfrak{z}_D(\theta_{\ell,\iota})/\mathfrak{z}_D(1)$ has height at most $V + 2\delta + \dim \mathrm{Row}(D) \cdot b$.*

Proof. Completion to a basis. Let $\mathrm{cst}_D^{\mathrm{cr}}$ be as in Input 22. The map $\varpi_D(1)$, for the unit element of $\mathsf{G}(D)$, maps $\mathrm{PF}(D)$ onto $\mathrm{Row}(D)$, so the rows $\varpi_D(1)F$ with F crude span $\mathrm{Row}(D)$. Since y is linearly independent (i), there are $n'' = \dim \mathrm{Row}(D) - n$ crude forms $F_1, \dots, F_{n''}$ such that $y_1, \dots, y_n, \varpi_D(1)F_1, \dots, \varpi_D(1)F_{n''}$ is a basis of $\mathrm{Row}(D)$. We apply Input 23 to this basis, to the family θ, and to $H_{\ell,\iota} = \mathrm{cst}_D^{\mathrm{cr}}(\theta_{\ell,\iota})/\mathrm{cst}_D^{\mathrm{cr}}(1)$.

Hypothesis (1). $\Theta_{D,1}$ is injective because $\mathfrak{z}_D(1) \neq 0$ (Input 20). Each F_i is crude, so $\|\Theta_{D,1}(\varpi_D(1)F_i)\|_\sigma^2 \le \exp(x^{99} + x_1^2 b^2)$ for every σ, by Input 13(a). With (iv) and

Input 25(b), (c), the conjugates of the Gram determinant of the images of the basis under $\Theta_{D,1}$ have absolute value at most e^{B} with $B = V + n''\,(x^{99} + x_1^2 b^2)$.

Hypotheses (2), (3), (4). The function $\mathrm{cst}_D^{\mathrm{cr}}$ costs $2m^2$ squares and θ is fit, so (2) holds with $h = 2m^2 \cdot 2\,t(D)\,x^{999}$. Let (η_ℓ) be as in Definition 10.5 for the family y, by (iii); this is (3). And (4) is Input 24.

Hypothesis (5). Let (ℓ, ι) be a pair and $j\colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism. The prime ℓ is prime to $\theta_{\ell,\iota}$, because θ is fit. By (iii) there is an $n \times n$ matrix u_0 over O_j with $|j(\det u_0)|_\ell \geq \mathrm{e}^{-\eta_\ell}$ such that all $\Theta_{D,\theta_{\ell,\iota}}(\sum_{i'}(u_0)_{ii'}y_{i'})$ are j-integral. Let u be the block matrix with the blocks u_0 and $H_{\ell,\iota}$ times the identity matrix of size n''. Then $|j(\det u)|_\ell \geq \mathrm{e}^{-\eta_\ell}|j(H_{\ell,\iota})|_\ell^{n''}$. The new rows of the second block are the $H_{\ell,\iota}\,\varpi_D(1)F_i$, and $\Theta_{D,\theta_{\ell,\iota}}(H_{\ell,\iota}\,\varpi_D(1)F_i) = H_{\ell,\iota}\Theta^{\flat}_{D,\theta_{\ell,\iota}}(\varpi_D(1)F_i)$ is j-integral, by the first property of crude forms and Input 13(b).

Conclusion. Input 23 gives the height bound $V + 2\delta + n''\,(x^{99} + x_1^2 b^2 + 4m^2 t(D) x^{999})$. By (ii), $n'' \leq 4b^{-2}\dim \mathrm{Row}(D)$, and by (S4) the bracket is at most $\frac{1}{4}b^3$. So the bound is at most $V + 2\delta + \dim \mathrm{Row}(D) \cdot b$, and Input 18(b) concludes. □

LEAN `Data.heightLe_volume_rows`

Theorem C (integral forms are long)**.** Let P be parameters and D admissible data for P, with components x, x_1, b, m, r (Settings 8.1 and 10.1). Let n be a natural number, $y = (y_1, \dots, y_n)$ a family of rows in $\mathrm{Row}(D)$, and δ and V real numbers. Assume that the family y is linearly independent over $\mathbb{C}$, that $(1 - 4/b^2) \cdot \dim \mathrm{Row}(D) \leq n$, that y has D-index at most δ, and that $\mathrm{covol}_\sigma^2(\Theta_{D,1}y_1, \dots, \Theta_{D,1}y_n) \leq \mathrm{e}^V$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$. Then

$$\dim \mathrm{Row}(D) \cdot (h_{\mathrm{main}}(D) - 3\log b \cdot \mathrm{budget}(D)) \leq V + 2\delta.$$

Proof. The hypotheses are those of Setting 11.3. Let θ be a family of m-fold twists that is fit for D (Input 19). For every pair (ℓ, ι) the numbers $\mathfrak{z}_D(1)$ and $\mathfrak{z}_D(\theta_{\ell,\iota})$ are not zero (Input 20), so

$$\Big(\frac{\mathfrak{z}_D(\theta_{\ell,\iota})}{\mathfrak{z}_D(1)}\Big)^2 \cdot \frac{\mathfrak{z}_D(1)^2}{\mathfrak{z}_D(\theta_{\ell,\iota})^2} = 1.$$

By Lemma 11.4 and Input 18(a) the first factor has height at most $2\,(V + 2\delta + \dim \mathrm{Row}(D) \cdot b)$; by Lemma 11.2 the second has height at most $2\dim \mathrm{Row}(D)\,(2\log b \cdot \mathrm{budget}(D) - h_{\mathrm{main}}(D))$. So the constant family 1 has height at most the sum, and by the product formula, Input 18(c),

$$0 \leq 2\,(V + 2\delta + \dim \mathrm{Row}(D) \cdot b) + 2\dim \mathrm{Row}(D) \cdot (2\log b \cdot \mathrm{budget}(D) - h_{\mathrm{main}}(D)).$$

With $b \leq \log b \cdot \mathrm{budget}(D)$, which is (S6), this is the claim. □

LEAN `Data.integral_forms_long`

Corollary 11.5 (Theorems A and B together)**.** *Let P be parameters and D admissible data for P, with components x, x_1, b, m, r (Settings 8.1 and 10.1). There are a natural number n and a family $y = (y_1, \dots, y_n)$ of rows in $\mathrm{Row}(D)$ such that: the family y is linearly independent over $\mathbb{C}$; $(1 - 4/b^2) \cdot \dim \mathrm{Row}(D) \leq n$; there is a*

real number $\delta \le 5 \dim \mathrm{Row}(D) \cdot \log(1+b^2) \cdot \mathrm{budget}(D)$ *such that* y *has* D*-index at most* δ*; and for every* $\sigma \in \mathrm{Aut}(\mathbb{C})$*,*

$$\mathrm{covol}_\sigma^2(\Theta_{D,1} y_1, \dots, \Theta_{D,1} y_n) \le \exp\Big(n \cdot \big(h_{\mathrm{main}}(D) - m \cdot \mathrm{rem}(D) + \log b \cdot \mathrm{budget}(D)\big)\Big).$$

Proof. Let q and S be as in Theorem A, and $\mathcal{T}$ as in Theorem B. Since S spans $\mathrm{Col}^{\mathrm{ram}}(D)$ and $q \neq 0$, the rows $q\mathcal{P}_D(\gamma)\psi$ with $\gamma \in \mathcal{T}$, $\psi \in S$ span the space $\sum_{\gamma \in \mathcal{T}} \mathcal{P}_D(\gamma)(\mathrm{Col}^{\mathrm{ram}}(D))$. Choose a basis $y_1, \dots, y_n$ of this space among them. Then y is linearly independent and $n \ge (1 - 4/b^2) \dim \mathrm{Row}(D)$ by Theorem B. By Theorem A(b), with $S \subseteq \mathrm{Col}^{\mathrm{ram}}_{\mathrm{int}}(D)$, the family y has D-index at most $\delta = \#\mathcal{T} \cdot \dim \mathrm{Col}(D) \cdot \mathrm{budget}(D)$, and $\delta \le 5 \dim \mathrm{Row}(D) \cdot \log(1+b^2) \cdot \mathrm{budget}(D)$ by Theorem B and $\mathrm{budget}(D) > 0$. Finally each $\|\Theta_{D,1} y_i\|_\sigma^2$ is at most $\exp(h_{\mathrm{main}}(D) - m \cdot \mathrm{rem}(D) + \log b \cdot \mathrm{budget}(D))$ by Theorem A(a), and Hadamard's inequality, Input 25(a), bounds the covolume. □

LEAN `Data.exists_family_covolumeSq_le`

12 Where the results are used

All consumers are in the last unit of the proof (unit 154, Chapter I), which has three steps.

The main inequality $m \cdot \mathrm{rem}(D) \le (\log b)^2 \cdot \mathrm{budget}(D)$. Take the family of Corollary 11.5 and apply Theorem C to it, with $V = n\,(h_{\mathrm{main}}(D) - m \cdot \mathrm{rem}(D) + \log b \cdot \mathrm{budget}(D))$. The main amount occurs on both sides, once with the factor $\dim \mathrm{Row}(D)$ and once with the factor n; Lemma 9.5 pays for the difference, since $\dim \mathrm{Row}(D) - n \le 4b^{-2} \dim \mathrm{Row}(D)$. After division by $\dim \mathrm{Row}(D)$ the main amount has cancelled, and what remains is the inequality.

The one-sided bound $R_{d,N_0}(x) \le x^{8\varepsilon - 1/2}$ for large x. Theorem 4 solves for $R_{d,N_0}(x)$, and leaves the combination $R_{d,N_0}(x_1) - X\,R_{d,1}(x_1)$ with $X = \#\mathrm{tops}(D)/\#\mathrm{thick}(D)$ to be bounded from above. Theorem 1, for the moduli N_0 and 1, and Theorem 3 provide a point v_1 in the window $[v_0, v_0^2]$, $v_0 = 10^6 \log x$, at which this combination is at most e^{-v_0}, and the second scale is moved to $x_1 = \mathrm{e}^{v_1}$; the main inequality bounds the rest.

The zero-free half-plane. Theorem 2 with $M = N_0$ and $c = N_0/2 - 1$.

Inside the unit, Lemmas 4.1 to 4.4 serve Theorems 1 and 4 and Lemma 9.5; Proposition 7.1 serves Theorem 3; Proposition 9.4 and Lemma 10.6 serve Theorem A; Lemmas 11.2 and 11.4 serve Theorem C.

13 Notes on the formal text

Conventions of Lean that the statements hide. Division by zero gives zero. So the residue function of Theorem 1 is defined at $s = 0$ and $s = 1$, with the value 0, as stated; at $s = 1$ the numerator vanishes anyway, by Input 1. The logarithmic derivative of a function is 0 at its zeros, so $G_{d,M}(s)$ is defined for every s; the proofs use it only as a meromorphic function and at points where no $\tilde{L}(\,\cdot\,, \xi)$, $\tilde{L}(\,\cdot\,, \xi\chi_d)$ vanishes. The remainder $R_{d,M}(x)$ is defined for every real x, with Lean's logarithm of a number ≤ 0; only $x > 0$ is used. The integral of a function that is not integrable

is 0, which is why Proposition 7.1 and Input 6 assert integrability separately. The weight $\lambda_{d,M}(\nu)$ and the sums $A_{d,M}(i,x)$ are defined for every natural number M, also $M=0$, where $\mathbb{Z}/0=\mathbb{Z}$ and the weight vanishes identically.

Definition 2.8. The formal definition asks that Φ be meromorphic at every point of $\mathbb{C}$ as a function on $\mathbb{C}$; its values at its poles are arbitrary. Condition (P3) is stated for the product $(s-1)\Phi(s)$ as a function, so its value at $s=1$ is 0; the clause on that value in (P3) and Remark 2.9 are our reading of it, and the comment above the formal definition says the same: no pole at 1 is allowed.

Theorem 1. The hypothesis $d\equiv 7 \pmod 8$ is used only through $d\neq 2$ (Step 3). The formal proof orders the steps differently: it proves the Laplace identity (our Steps 3 to 5) after it has used it, as a pending goal, to show that the constant is real (Step 6); the order of the goals was inferred by the reader of the dossier from the structure of the term, and the writer agrees with that reading. The formal statement speaks of the function $v\mapsto R_{d,M}(\mathrm{e}^{v})$ and of a residue function given by a formula; the names $\Phi_{d,M}$ and $\mathrm{res}_{d,M}$ are ours, after the overview of the write-up, and the transform (5.1) and the constant $|\kappa_{d,M}|+5$ are the witnesses of the formal proof.

Theorem 2. The conclusion is about $\tilde{L}(\rho,\chi)$; for principal χ this is $(\rho-1)L(\rho,\chi)$. The hypothesis on the primes $d\equiv c$ is one conjunction in the formal text, and it is used three times: for Input 5, for Theorem 1, and for the one-sided bound. In the formal statement d runs over the type of prime numbers and $d\equiv c \pmod M$ is written as an equality in $\mathbb{Z}/M$.

Proposition 7.1: the boundary of the statement. The formal statement contains a local definition of the function $v\mapsto v^{i}\mathrm{e}^{-3v}\mathcal{E}(v)$, written with a sign of definition. The census of the workspace, which reads the text line by line, takes that sign for the end of the statement; the dossier corrects this, and the writer read the complete statement in the source: it ends after the bound $i!\,v_0/\mathfrak{r}^{i}$. The quantifiers of the formal statement are filters, *for all* $\mathfrak{r}>3$ *close to* 3 and *for all large* v_0; the proposition writes them out, in this order. Its proof uses neither the meromorphy of Φ nor (P4); the residues of a polar function enter the proof only through Landau's lemma.

Arguments that the census lists and Lean does not take. The census reads the arguments of a declaration from its text and counts every variable of the surrounding section whose name occurs in the statement. In three places a bound variable of the statement has the name of such a section variable, and Lean, which the writer follows, does not take it as an argument. *(1)* Corollary 11.5: its statement binds its own n, y and δ under existential quantifiers, and its only arguments are the parameters and the data. The reader of the dossier could not confirm this with Lean; the writer confirms it from the use of the corollary in the proof of the main inequality, where it is applied to the data alone and nothing could determine further arguments. *(2)* Input 5: the character ξ is bound by a universal quantifier inside the statement and is not an argument. *(3)* Definition 10.3: the formal declaration takes the data and γ, and its value is a linear map; the census reads the arrow in the type of that map as

one more argument. The notation of this note has a third place for that reason, and it is left empty when the map itself is meant.

Theorem A. The formal statement names the family of maps $\gamma \mapsto q\,\mathcal{P}_D(\gamma)$: it says that there are q and a family equal to this one, with the two properties. We have substituted the family. In (b) the finite sets J of Definition 10.5 are arbitrary finite sets of natural numbers, not only sets of primes, as in the formal text. The formal proof of (b) obtains the bound for the determinant from divisibility in the valuation ring O_j, as we do.

Theorem B. Its first two assertions, that there are columns and rows, are part of the formal statement. The filling theorem is stated for columns and the operator ram_D, and Theorem B for ramified columns; the proof passes from one to the other.

Theorem C. In the formal text the four hypotheses of Setting 11.3 are variables of the section, shared by Lemma 11.4 and Theorem C, in the order (i), (ii), (iii), (iv); Lemma 11.2 takes none of them. The formal proof of Lemma 11.4 gets the injectivity of $\Theta_{D,1}$ from a lemma that is applied automatically and that we have not located; we derive it from $\mathfrak{z}_D(1) \neq 0$, which is the same fact.

Two names for the chain. The formal text has two maps for the chain of lifts on $\mathrm{PF}(D)$: one with values in forms that are functions with vector values, for which pointwise sizes and integrality of values are defined, and one with values in the space that carries the Petersson product. They have the same values, and the proofs of Theorems A and C pass from one to the other without comment; we write $\Theta^{\flat}_{D,\vartheta}$ for both.

What the reader of the dossier was unsure of. Besides the three points above (the boundary of the statement of Proposition 7.1, the arguments of Corollary 11.5 and of Definition 10.3): the facts of Mathlib that Lemma 4.4 needs, namely that $(s-1)\zeta(s)$ has the value 1 and the derivative C_{Eu} at $s = 1$, were inferred by the first reader from the proof going through, because the pinned version of Mathlib was not at hand. The second reader of this write-up has since found both at the pin, and also the value $\prod_{p\mid M}(1-1/p)$ of Definition 2.2 and the two statements of Mathlib that Step 3 and Step 4 of the proof of Theorem 1 quote (orthogonality of characters against von Mangoldt's function, and partial summation). For this note the writer did not open Mathlib: every fact of Mathlib that the note uses rests on the second reader's check at the pin. This is said of this note only; the three notes of Chapter I list in their appendices the declarations of Mathlib that their writer read at the pinned commit. The readings of the Inputs 8 to 25 come from their formal statements and comments, not from their proofs; the chapters on the units that own them are the reference.

The dossier, and its second reading. The second reader of this write-up compared the dossier with the source and asked for six corrections; this note agrees with all of them. Two concern words of the dossier that were wrong as mathematics while its quotations were exact. In the structured proof of Theorem 4, one step says that $(-\nu)\cdot\tau$ runs over the places of class $-\nu$ as τ runs over the tops; this holds for ν

prime to $N_1(D)$ only, and for the other ν the formal text makes $(-\nu)\cdot\tau=\tau$ (Step 1 above treats both cases). In the structured proof of Lemma 9.5, an intermediate bound had absolute values around $z/\nu-w$ where the positive part stands; with the positive part the bound $|z|/\nu$ is correct. The other four are points of exposition: the quadratic character is that of K only for $d\equiv 3 \pmod 4$ (Definition 2.1); the value of the entire L-function of a principal character at $s=1$ (Definition 2.2); the vanishing at $s=1$ that makes the singularity removable (Remark 2.9, Input 7); and the difference between the rank $k(D)$ and the dimension of $\mathrm{Col}(D)$ (Setting 10.1).

What of the unit the final theorems use. The last unit of the proof uses Theorems 1, 2, 3, 4 and C, Corollary 11.5 and Lemma 9.5 directly; every other declaration of the unit is used by one of these.

How this note was written. From the dossier of unit 151: its statements, its 18 structured proofs with 104 leaf steps, which a program compared with the source, and its readings of the declarations of other units. The writer read in the Lean text itself: the four files of the unit completely, statements and proofs; the file of the background library with the definition of polar functions, Landau's lemma and the elementary inputs, completely; the definitions of the characters, of the entire L-function and of the Chowla–Selberg constant; the statement and the last step of the proof of Input 5; the record of the data and the definitions attached to it that Setting 8.1 uses; the statements, as quoted in the dossier, of every Input of Part II, and in the source the definitions of the height, of fit families, of twists, of the index relation of unit 149 and of the mechanism that compares sizes; and the last unit of the proof, to see how the results are used. Every statement of the note was compared with its formal statement, argument by argument and in the order of the quantifiers. The relation of Inputs 1, 4 and 5 to the literature is the second reader's comparison, as their boxes say. Not opened for this note: the proofs of the Inputs of Part II, and Mathlib, whose facts were checked at the pin by the second reader, as said above.

The arguments are those of the formal proofs, with these differences. In Theorem 3 the final numerical comparison is done by hand here and by an arithmetic tactic there. In Theorem 4 the counting of Step 1 is written out here; the formal proof reindexes the sum by a translation of the group of places and uses a lemma of unit 152, which says that the place named by an integer ν has class -1 modulo N_0 exactly when $\nu\equiv -1$, also for the integers that are not prime to $N_1(D)$. Remarks 2.10, 3.1, 6.1 and 7.2, and the estimate of orders of magnitude after Lemma 9.5, are ours and have no formal counterpart.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Definition 2.1	`chiK`	QuadField.lean	20–20
Definition 2.1	`quadTwist`	QuadField.lean	26–27
Definition 2.1	`quadTwist_apply`	PolarLandau.lean	20–22
Definition 2.2	`entireL`	QuadField.lean	31–31
Definition 2.3	`chowlaSelbergConst`	QuadField.lean	35–35
Definition 2.4	`lambdaClass`	PrimeError.lean	25–25
Definition 2.5	`lambdaSum`	PrimeError.lean	29–29

In this note	Lean declaration	File	Lines
Definition 2.6	`logDerivMean`	PrimeError.lean	33–34
Definition 2.7	`primeError`	PrimeError.lean	38–39
Definition 2.8	`IsPolar`	PolarLandau.lean	34–36
Input 1	`entireL_ne_zero`	PolarLandau.lean	55–56
Input 2	`LSeriesHasSum_mul_vonMangoldt`	PolarLandau.lean	58–62
Input 2	`tendsto_sub_mul_logDeriv_entireL`	PolarLandau.lean	64–73
Input 3	`quadTwist_ne_one`	PolarLandau.lean	26–29
Input 4	`IsPolar.landau`	PolarLandau.lean	88–136
Input 5	`exists_entireL_quadTwist_ne_zero`	ZeroFreeTwist.lean	193–206
Input 6	`laplace_pow`	PolarLandau.lean	47–50
Input 6	`integrableOn_laplace_of_abs_le`	PolarLandau.lean	40–43
Input 7	`summable_taylor_of_removable`	PolarLandau.lean	78–82
Lemma 4.1	`lambdaClass_nonneg_and_abs_le`	PrimeError.lean	41–43
Lemma 4.2	`abs_lambdaSum_exp_le`	PrimeError.lean	45–48
Lemma 4.3	`primeError_eq_sum`	PrimeError.lean	60–64
Lemma 4.4	`re_logDerivMean_one`	PrimeError.lean	52–54
Theorem 1	`isPolar_primeError`	PrimeError.lean	69–111
Theorem 2	`entireL_ne_zero_of_primeError_le`	OneSidedCriterion.lean	28–36
Proposition 7.1	`IsPolar.eventually_moment_le`	OneSidedCriterion.lean	41–67
Theorem 3	`IsPolar.exists_window_sub_mul_le`	OneSidedCriterion.lean	73–99
Definition 9.1	`Data.mainHeight`	ShortAndEnough.lean	32–32
Definition 9.2	`Data.mainAsk`	ShortAndEnough.lean	36–37
Definition 9.3	`Data.remainder`	ShortAndEnough.lean	42–42
Theorem 4	`Data.remainder_eq`	ShortAndEnough.lean	86–112
Proposition 9.4	`Data.heightPerRow_le`	ShortAndEnough.lean	48–70
Lemma 9.5	`Data.abs_mainHeight_add_remainder_le`	ShortAndEnough.lean	74–82
Definition 10.2	`Data.scaledInclusion`	ShortAndEnough.lean	116–116
Definition 10.3	`Data.projectedPoincare`	ShortAndEnough.lean	120–121
Definition 10.4	`Data.integralRamified`	ShortAndEnough.lean	125–126
Definition 10.5	`Data.IndexLe`	ShortAndEnough.lean	130–132
Lemma 10.6	`exists_rat_log_eq_and_padic_norm_eq`	ShortAndEnough.lean	21–26
Theorem A	`Data.short_forms`	ShortAndEnough.lean	138–168
Theorem B	`Data.enough_forms`	ShortAndEnough.lean	173–178
Lemma 11.2	`Data.heightLe_volume_ratio_sq`	LongForms.lean	19–32
Lemma 11.4	`Data.heightLe_volume_rows`	LongForms.lean	39–58
Theorem C	`Data.integral_forms_long`	LongForms.lean	63–71
Corollary 11.5	`Data.exists_family_covolumeSq_le`	LongForms.lean	78–92

Legend. The note changes some letters of the formal text, never the number of arguments. The modulus is M here, and M or N there. The unit class c of Theorem 2 and Input 5 is `a`; the height z of Input 5 is `y`. The exponent i of the weighted sums is `r`, and their real argument x is `y`; the natural number a and the point x of Lemma 4.3 are `A` and `z`. A polar function $\mathcal{E}$ with residue function ω is `E` with `r`; in Theorem 3, $\mathcal{E}_1, \omega_1, \mathcal{E}_2, \omega_2$ are `E`, `r`, `E'`, `r'`, the parameter v_0 is `t`, and the point v_1 is `u`. In Proposition 7.1 the radius $\mathfrak{r}$ is `R`, the number v_0 is `t` and the order i is `k`. In Input 7, s_0, s_1, s_2 and Φ are `z`, `a`, `σ` and `Ψ`, and the radius $\mathfrak{r}_0$ is `ε`. In Lemma 10.6 the set J is `T`, the exponents e_ℓ are `g l`, and primes are `l` throughout. In Part II the scales x, x_1, b are the exponentials of the components `logX`, $\texttt{logX}_1$, `logB` of the data; a place τ is `σ` or `τ`; an element γ of $\mathsf{G}(D)$ is `s`; a finite subset $\mathcal{T}$ is `t`; a ramified column ψ is `φ`; a ring homomorphism j is `ι`, of the class `PadicPlace`; the numbers η_ℓ are `e l`, and a finite set J of natural numbers is `U`. The objects of other units have these formal names: $\mathrm{tops}(D)$, $\mathrm{thick}(D)$, $\mathrm{budget}(D)$, $\mathsf{U}(D)$, $h_0(D)$ are `Data.tops`,

`Data.thick`, `Data.budget`, `Data.dirCost`, `Data.heightPerRow`; $w_D(\tau)$, $\mathrm{Ask}_D(z,\tau)$ and $\mathrm{jet}_D(i)$ are `Data.nominalWeight`, `Data.primeSum` and `Data.jetSize`; $\mathrm{Col}(D)$, $\mathrm{Col}^{\mathrm{ram}}(D)$, $\mathrm{Row}(D)$, $\mathsf{G}(D)$ are `Data.Column`, `Data.ramifiedColumns`, `Data.rows`, `Data.G`; Poin_D and $\varpi_D(\gamma)$ are `Data.poincare` and `Data.goodProjTranslate`; $\Theta_{D,\vartheta}$ and $\mathfrak{z}_D(\vartheta)$ are `Data.θ` and `Data.𝔷`; the height is `HeightLe`, and fitness is `Data.IsFitFamily`.

Six statements, as the artifact has them. The remainder; Theorem 1; Theorem 2; polar functions; Landau's lemma; Theorem 4.

```
def primeError [NeZero M]
    x := lambdaSum d M 0 x - lambdaSum d M 1 x - x.log / φ M - (logDerivMean d M 1).re

lemma isPolar_primeError (hd : d % 8 = 7) (h : M.Coprime d) : IsPolar (primeError d M ·.exp) fun s ↦
    (∑ ξ, (analyticOrderNatAt (@entireL M _ ξ) s - analyticOrderNatAt (entireL (quadTwist ξ d))
  s : ℂ)) / φ M / (s * (1 - s))

lemma entireL_ne_zero_of_primeError_le {M} [NeZero M] {a : ZMod M} {β} (hβ : 1 / 2 < β)
  (ha : IsUnit a) (h : ∀ d : Primes, d = a → (d.1 % 8 = 7 ∧ M.Coprime d) ∧ ∀ᶠ x in atTop, primeError
  d M x ≤ x ^ (β - 1)) ρ (hρ : β < ρ.re) (χ : DirichletCharacter ℂ M) : entireL χ ρ ≠ 0

def IsPolar (E : ℝ → ℝ) (r : ℂ → ℂ) := Measurable E ∧ ∃ C Φ, Meromorphic Φ ∧ (∀ v ≥ 0, |E v| ≤ C *
  v.exp) ∧ (∀ s, 2 < re s → Φ s = ∫ v in Ioi 0, E v * ((1 - s) * v).exp) ∧ AnalyticOnNhd ℂ (fun s ↦
  (s - 1) * Φ s) {s | 1 ≤ s.re} ∧ ∀ ρ, 0 < ρ.re → (𝓝[≠] ρ).Tendsto (fun z ↦ (z - ρ) * Φ z) (𝓝 (r ρ))

lemma IsPolar.landau {r} (h : IsPolar E r) {β : ℝ} (hβ : 0 < β) (hr : ∀ x, β < x → 0 ≤ (r x).re)
    (hw : ∀ᶠ v in atTop, E v ≤ ((β - 1) * v).exp) {ρ} (hρ : β < ρ.re) : r ρ = 0

lemma Data.remainder_eq : D.remainder = D.a * (#D.thick *
  (primeError P.d P.N₀ D.x - primeError P.d P.N₀ D.x₁) + #D.tops * primeError P.d 1 D.x₁)
```

Notation

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ natural numbers with 0, integers, rational, real, complex numbers

$\mathbb{C}_\ell, \overline{\mathbb{Q}}_\ell$ the completion of an algebraic closure of the ℓ-adic numbers; an algebraic closure; $|\cdot|_\ell$ their absolute value

Re, Im real and imaginary part

e, $\log$, $(z)_+$ base of the natural logarithm; natural logarithm; $\max(z, 0)$

$O(\cdot)$, $o(1)$ Landau's symbols

d a prime number; in Part II the second parameter

M a natural number, the modulus of the characters

ν a natural number, the variable of the prime sums

p, e a prime; an integer exponent; e_ℓ the exponents of Lemma 10.6

Λ, φ von Mangoldt's function; Euler's function

χ_d, χ_K the quadratic character modulo d (Definition 2.1); it is the character of $K = \mathbb{Q}(\sqrt{-d})$ if $d \equiv 3 \pmod 4$

ξ, χ, $\mathbf{1}$ Dirichlet characters modulo M; the character modulo 1

$\xi\chi_d$ the twist of ξ, a character modulo Md (Definition 2.1)

$[\chi \text{ principal}]$ the number 1 if χ is principal, 0 otherwise

$L(s,\chi)$, $\tilde{L}(s,\chi)$, $\tilde{L}(\cdot,\chi)$ the Dirichlet L-function; the entire L-function, its values and the function (Definition 2.2); $\tilde{L}'$ its derivative

ζ the Riemann zeta function

$\mathrm{ord}_s f$ order of vanishing at s of an entire function $f \neq 0$

C_{Eu}, $\mathrm{cs}(d)$ Euler's constant; the Chowla–Selberg constant (Definition 2.3)

$\lambda_{d,M}(\nu)$ the weight (Definition 2.4)

$A_{d,M}(i,x)$ the weighted sums (Definition 2.5)

$G_{d,M}(s)$, $\kappa_{d,M}$ the mean of the logarithmic derivatives and its real part at $s = 1$ (Definition 2.6)

$R_{d,M}(x)$ the remainder (Definition 2.7)

$\Phi_{d,M}(s)$, $\mathrm{res}_{d,M}(s)$ the transform of the remainder, (5.1); its residue function (Theorem 1)

s, s', s_0, s_1, s_2 complex numbers, the variable of the transforms

ρ a complex number with $\mathrm{Re}\,\rho > 0$; in Theorem 2 the point where a zero is excluded

β a real number, the exponent of the one-sided bound

c a unit of $\mathbb{Z}/M$, the class of the primes d

x, z, v real numbers: x the argument of the remainder and of the weighted sums, v its

logarithm; x_0 (an abscissa of convergence), x_*, v_* particular values

i a natural number: an exponent, the order of a moment or of a derivative; in Part II also the index of a row or of a lift, with i', i_0

a a natural number (Lemma 4.3); in the proofs of §9 it abbreviates $a(D)$

$\mathcal{E}, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2$; $\omega, \omega_1, \omega_2$ functions $\mathbb{R} \to \mathbb{R}$; functions $\mathbb{C} \to \mathbb{C}$, residue functions

$\mathrm{Polar}(\mathcal{E}, \omega)$ $\mathcal{E}$ is polar with residue function ω (Definition 2.8)

C, C_1, $C_\mathfrak{r}$; Φ real constants; a meromorphic function on $\mathbb{C}$, the transform; $\Phi^{(i)}$ its i-th derivative

v_0, v_1, X the parameter of a window, a point of it, a real factor (Theorem 3)

$\mathfrak{r}$, $\mathfrak{r}_0$, Δ radii of discs: $\mathfrak{r}_0$ the radius in Input 7 and the bound of Proposition 7.1, $3 < \mathfrak{r} < \mathfrak{r}_0$; a positive real number, in the box of Input 5 and in the proof of that proposition

ε a real number with $0 < \varepsilon < \frac{1}{100}$, in the exponent of the one-sided bound of Chapter I

$I(v)$ the auxiliary function in the proof of Theorem 3

$P = (N_0, d)$, D parameters and admissible data (Setting 8.1)

x, x_1, b; m, r the scales of D; its two natural numbers

K, $E(D)$, $N_1(D)$, $f(D)$ $\mathbb{Q}(\sqrt{-d})$; the field of the data, the order of its roots of unity, the number of its infinite places

τ, $\nu \cdot \tau$ an infinite place of $E(D)$; its translate by the integer ν

$\mathrm{tops}(D)$, $\mathrm{thick}(D)$ the places of class 1 modulo N_0; the thick places

$k(D)$, $t(D)$, $a(D)$ $r + 1 - m$; $5r$; the least multiple of $N_1(D)$ that is $\geq xb$

$\mathrm{budget}(D)$ $f(D)\, a(D)\, (\log b)^3$

$w_D(\tau)$, $\bar{w}_D$, w the nominal weight, the mean of its logarithm; a real weight

$\mathrm{Ask}_D(z, \tau)$, $\mathrm{Ask}^{\mathrm{main}}(D)$ the prime sum of order z at τ; its expected value (Definition 9.2)

$g(D)$, $\mathrm{rd}_{E(D)}$, per_D, Ω, π product of two primes of the data; root discriminant; the period term $\log(g(D)\, \mathrm{rd}_{E(D)}/2\pi)$; the period; the number $3.14\ldots$

$\mathrm{jet}_D(i)$, $\mathsf{U}(D)$, $N(D)$, $h_0(D)$ jet size, cost of one direction, level, height per row (Setting 8.1)

$h_{\mathrm{main}}(D)$, $\mathrm{rem}(D)$ the main amount and the remainder of the data (Definitions 9.1, 9.3)

$\mathrm{Col}(D)$, $\mathrm{Col}^{\mathrm{ram}}(D)$, $\mathrm{Col}^{\mathrm{ram}}_{\mathrm{int}}(D)$ columns, ramified columns, integral ramified columns

ψ, ψ_i, ψ'_i, ψ_0 columns

$\mathsf{G}(D)$, γ, $\mathcal{T}$ the finite group of the data, an element, a finite subset

$\mathrm{PF}(D)$, $\mathrm{Row}(D)$, $y = (y_i)$, n forms of principal level, rows, a family of n rows; n', n'' further numbers of rows

$\varpi_D(\gamma)$, Poin_D, ram_D translation followed by the projection to rows; Poincaré series; the ramifying operator

$\mathrm{inc}(D)$, $\mathcal{P}_D(\gamma)\psi$ the scaled inclusion and the Poincaré maps (Definitions 10.2, 10.3)

ϑ, $Q(\vartheta)$, $\theta = (\theta_{\ell,\iota})$ a twist, its set of ramified primes; a family of m-fold twists

$\Theta^\flat_{D,\vartheta}$, $\Theta_{D,\vartheta}$, $\mathcal{A}_{D,\vartheta}$ the chain of lifts on $\mathrm{PF}(D)$ and on the rows; its target

$\mathfrak{z}_D(\vartheta)$, $\mathrm{vol}_D(\vartheta, i)$ the volume of the chain; of its i-th lift

F, F', F_i forms, in $\mathcal{A}_{D,\vartheta}$ or in $\mathrm{PF}(D)$

σ, $\mathrm{Aut}(\mathbb{C})$ an automorphism of the field $\mathbb{C}$; their group

$\|F\|^2_\sigma$, $\mathrm{covol}^2_\sigma(F_1, \ldots, F_n)$ Petersson norm and squared covolume of σ-conjugates

ℓ, j, O_j a prime number (in Lemma 10.6 and in sums, a natural number); a ring homomorphism $\mathbb{C} \to \mathbb{C}_\ell$; its valuation ring in $\mathbb{C}$

ι a field isomorphism $\mathbb{C} \to \overline{\mathbb{Q}}_\ell$

η_ℓ, J real numbers indexed by $\ell \in \mathbb{N}$; a finite set of natural numbers

u, u_0 square matrices

$\mathrm{ind}_D(y) \leq \delta$, δ y has D-index at most δ (Definition 10.5); a real number

V, B, h, h' real numbers: bounds for logarithms of covolumes, of sizes, of heights

q, S the rational number and the spanning set of Theorem A

Ψ, $\mathrm{Ch}_D(\vartheta, \Psi)$ a good datum; the chain of lifts between definite spaces

$\mathrm{gain}_D(\ell)$, $\mathrm{err}_D(\ell)$, $\mathrm{nonsplit}(D)$ gain and allowed loss at ℓ; the primes $\ell < x_1$ with $\chi_d(\ell) \neq 1$

$\mathcal{S}$, $\mathcal{M}$, $\mathcal{N}$ O_j-modules of columns and of rows

W a subspace of $\mathrm{Row}(D)$

H, $H_{\ell,\iota}$ complex numbers used as factors

Z, Z', α families of complex numbers indexed by the pairs (ℓ, ι); a real number in the definition of the height

$\mathrm{cst}^{\mathrm{see}}_D$, $\mathrm{cst}^{\mathrm{vol}}_D$, $\mathrm{cst}^{\mathrm{cr}}_D$ the constants of Inputs 10, 20, 22

References

The places are those found by the second reader of this write-up (Inputs 4 and 5); the writer has not consulted the texts.

[Ju] M. Jutila, On mean values of Dirichlet polynomials with real characters, Acta Arithmetica 27 (1975), 191–198. Theorem 2, equation (6), p. 193.

[MV] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I. Classical Theory*, Cambridge, 2007. §15.1, Lemma 15.1, p. 463.

[Wi] D. V. Widder, *The Laplace Transform*, 1941; the second printing, of 1946, was the one compared. Chapter II, §5, Theorem 5b, p. 58.

Chapter III

The data: parameters, fields and scales

- **The data of the proof: the parameters N_0 and d, the CM field $E = K(\zeta)$ and its places, the scales x, x_1, b, the ranks m, k, r, t, the order a, the comparison of sizes on the logarithmic scale, and the existence of admissible data at every large x** (unit 152)

  read: note built, structured proofs checked. 239 declarations, 213 proof lines.

The parameters and the CM field: places, Frobenius elements, the different

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The argument works over one family of number fields. The parameters $P = (N_0, d)$ are a multiple N_0 of 18480 and a prime $d \equiv N_0/2 - 1 \bmod N_0$; they give the imaginary quadratic field $K(P) = \mathbb{Q}(\sqrt{-d})$ and, for every exponent μ, the CM field $E(P, \mu)$ obtained by adjoining a root of unity of order $N_1(P, \mu) = N_0 \cdot 3^{\lfloor \log_3 \mu \rfloor + 5} \cdot 2^{\mu}$. This note sets up the arithmetic of these fields: the infinite places of $E(P, \mu)$ form a group, identified with its Galois group over $K(P)$ and with the units modulo $N_1(P, \mu)$ (Theorem A); the decomposition of a prime is read from an explicit Frobenius element (Theorem B); the ring of integers is generated by the root of unity and $(1 + \sqrt{-d})/2$, and the different is principal, generated by a purely imaginary integer the geometric mean of whose absolute values at the infinite places is the root discriminant (Theorem C). From these the note builds the data that the Weil representation over $E(P, \mu)$ needs.

What this note is. The artifact grh-compact is a body of Lean code whose final theorems state the Riemann hypothesis and its analogue for Dirichlet L-functions. That is a claim: its authors report that Lean's kernel accepts the code; a replay of that check by the owner of this write-up was started, and the handout's record of checks gives its state; nothing was compiled for this note; no mathematician has read the proof through. This note treats one part of one unit (the folder of the data of the proof): 96 of its 239 declarations, those of the two files on the parameters, the CM field and its Weil data; the note *Admissible data* treats the rest. Every statement below was compared, argument by argument, with the Lean text, and every proof is given in full, except that results of the background library are quoted as *Inputs* with their place in the artifact. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 96 declarations.

1 The statements

The notation is that of Section 2: $P = (N_0, d)$ are parameters, μ is a natural number, $N_1(P, \mu) = N_0 \operatorname{tw}(\mu)$ is the order of the root of unity $\varsigma(P, \mu)$ of the field $E(P, \mu)$ over $K(P) = \mathbb{Q}(\sqrt{-d})$, and $\mathrm{Pl}(P, \mu)$ is the set of infinite places of $E(P, \mu)$, a group by Definition 4.4.

Theorem A (places, automorphisms, classes). Let $P = (N_0, d)$ be parameters and μ a natural number.

(a) For every infinite place $v \in \mathrm{Pl}(P,\mu)$, the automorphism $\mathrm{gal}_{P,\mu}(v)$ of $E(P,\mu)$ over $K(P)$ sends $\varsigma(P,\mu)$ to $\varsigma(P,\mu)^n$, where $n \in \{0, \dots, N_1(P,\mu)-1\}$ represents the class $\mathrm{cyc}_{P,\mu}(v) \in (\mathbb{Z}/N_1(P,\mu))^\times$.

(b) For every integer ν prime to N_0, the automorphism $\mathrm{gal}_{P,\mu}([\nu]_{P,\mu})$ sends $\varsigma(P,\mu)$ to $\varsigma(P,\mu)^\nu$.

(c) $E(P,\mu)$ has exactly $\mathrm{tw}(\mu)\,\varphi(N_0)$ infinite places, φ being Euler's function.

(d) Exactly $\mathrm{tw}(\mu)$ places $v \in \mathrm{Pl}(P,\mu)$ have class 1 modulo N_0, that is $\mathrm{cl}_P^{N_0 | N_1(P,\mu)}(v) = 1$.

Theorem B (decomposition of primes). Let $P = (N_0, d)$ be parameters, μ a natural number, p a prime number with $p \nmid N_1(P,\mu)\,d$, and w a finite place of $E(P,\mu)$ with $p \in w$. Let $v \in \mathrm{Pl}(P,\mu)$, and let $(\mathfrak{c}, e)$ be one of the two pairs $(\mathrm{id}, 1)$ and $(\text{complex conjugation}, -1)$, where $\mathfrak{c}$ is an automorphism of the ring of integers of $E(P,\mu)$ and e an integer.

(a) $v \cdot w = \mathfrak{c}^{-1}(w)$ if and only if there is a natural number n with $(v, e) = \mathrm{Fr}_\mu(p)^n$ in $\mathrm{Pl}(P,\mu) \times \mathbb{Z}$. Here $\mathfrak{c}^{-1}(w)$ is w for the first pair and the conjugate place $\overline{w}$ for the second.

(b) The norm of w is p^o, where o is the order of the Frobenius element $\mathrm{Fr}_\mu(p)$ in $\mathrm{Pl}(P,\mu) \times \mathbb{Z}$.

Theorem C (integers and the different). Let $P = (N_0, d)$ be parameters and μ a natural number; write $\mathcal{O}$ for the ring of integers of $E(P,\mu)$ and $\mathcal{O}_K$ for that of $K(P)$.

(a) Let $z \in \mathcal{O}$ be a primitive root of unity of order $N_1(P,\mu)$ and $y = \Psi'_{N_1(P,\mu)}(z)$ the value at z of the derivative of the cyclotomic polynomial of that order. Then y divides $N_1(P,\mu)$ in $\mathcal{O}$, the ideal (y) is the different of $\mathcal{O}$ over $\mathcal{O}_K$, and $\mathcal{O} = \mathcal{O}_K[z]$.

(b) The different of $\mathcal{O}$ over $\mathbb{Z}$ is the principal ideal generated by $\partial(P,\mu) \cdot \sqrt{-d}_{P,\mu}$, and also by the purely imaginary integer $\varpi(P,\mu)$ of Definition 5.5.

(c) $\prod_{v \in \mathrm{Pl}(P,\mu)} |\iota_v(\varpi(P,\mu))| = \mathrm{rd}_{E(P,\mu)}^{\#\mathrm{Pl}(P,\mu)}$, where $\mathrm{rd}_{E(P,\mu)}$ is the root discriminant of $E(P,\mu)$.

Why these parameters. The explanations of this paragraph are motivation, read from the comments of the formal text and from the places where later units use the results; they are not statements of this unit. The congruence on d does two things. First, $d \equiv -1$ modulo $N_0/2$ and $8 \mid N_0/2$ make every prime p that divides $N_1(P,\mu)$ a divisor of $(d+1)/4$, the norm of $(1+\sqrt{-d})/2$. So all these primes split in $K(P)$, into the prime above p that contains $(1+\sqrt{-d})/2$ and its conjugate, and the different of $E(P,\mu)$ over $K(P)$, which lives above these primes, can be divided between an idele and its conjugate (Lemma 7.3); this is the self-duality that the Weil representation asks for. Second, $d \not\equiv -1$ modulo N_0 itself makes the powers of d modulo N_0 avoid

-1 (Lemma 3.2). For a natural number n and a place v, the place $[-d^n]_{P,\mu} \cdot v$ then never has the class of v modulo N_0: the ramified prime d never relates two places of the same class, and later units use this. (The prime d does occur in the prime sums of the main inequality: $d^2 \equiv 1 \bmod N_0$ by the proof of Lemma 3.2.) The divisibility by $18480 = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ makes every prime below 13 split in $K(P)$ (Lemma 3.1(c)); with the factor 3^5 of $\mathrm{tw}(\mu)$ it puts the roots of unity of order $5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$ into $E(P,\mu)$. The order of the root of unity grows with μ only by powers of 2 and 3, so that the count of places is exact (Theorem A(c), (d)) and the group of places has a large cyclic 2-part, where the note *Admissible data* builds the thick places.

2 The setting

Conventions. $\mathbb{N} = \{0, 1, 2, \dots\}$. For a number field F, a *finite place* is a nonzero prime ideal of its ring of integers, and an *infinite place* is an archimedean absolute value; the place of an embedding $\psi\colon F \to \mathbb{C}$ is $x \mapsto |\psi(x)|$. A CM field carries its complex conjugation $x \mapsto \overline{x}$, which every embedding into $\mathbb{C}$ turns into complex conjugation; it acts on the finite places, $\overline{w} = \{\overline{x} : x \in w\}$. Ψ_M is the cyclotomic polynomial of order M, the monic integral polynomial whose complex roots are the primitive roots of unity of order M. Three letters are local to this note: ψ, ψ' are embeddings of fields and homomorphisms of rings, and Ψ occurs only in Ψ_M; the columns ψ of the overview of the handout and the additive character Ψ of Chapter IV do not occur here. In a statement about a commutative ring R of characteristic p, or an integral domain R, these properties of R are part of the statement. Divisibility, floors and subtraction of natural numbers are those of $\mathbb{N}$, as in the formal text.

Definition 2.1 (Parameters)**.** *Parameters* are a pair $P = (N_0, d)$ of natural numbers such that 18480 divides N_0, d is a prime number, N_0 divides $2(d+1)$, and N_0 does not divide $d+1$. Since N_0 is even, the last two conditions say together that $d \equiv N_0/2 - 1 \bmod N_0$.

LEAN `Parameters`

Definition 2.2 (The order of the root of unity)**.** For a natural number μ put $\mathrm{tw}(\mu) = 3^{\lfloor \log_3 \mu \rfloor + 5} \cdot 2^\mu$, where $\lfloor \log_3 0 \rfloor = 0$. For parameters $P = (N_0, d)$ and a natural number μ put

$$N_1(P,\mu) = N_0 \cdot \mathrm{tw}(\mu), \qquad M_\omega(P,\mu) = (N_0\, d)^{\mu+6},$$

and let $K(P) = \mathbb{Q}(\mathrm{i}\sqrt{d}) \subset \mathbb{C}$ be the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, as a subfield of the complex numbers. Its quadratic character $\chi_{K(P)}$ is, on the integers, the Legendre symbol modulo d (so $\chi_{K(P)}(d) = 0$); in the formal text it is a function of d.

LEAN `towerDeg` · `Parameters.N₁` · `Parameters.K` · `Parameters.Mω`

The power of 3 in $\mathrm{tw}(\mu)$ lies between 81μ and 243μ for $\mu \geq 1$. The number $M_\omega(P,\mu)$ will be the order of certain characters of residue fields in later units; here only Lemma 3.1(d) concerns it.

Definition 2.3 (The CM field, its places, its root of unity)**.** Let $P = (N_0, d)$ be parameters and μ a natural number. $E(P,\mu)$ is the splitting field over $K(P)$ of the cyclotomic polynomial $\Psi_{N_1(P,\mu)}$ (in it $X^{N_1(P,\mu)} - 1$ splits as well), and $\varsigma(P,\mu) \in E(P,\mu)$

is a chosen primitive root of unity of order $N_1(P,\mu)$; so $E(P,\mu) = K(P)(\varsigma(P,\mu))$. $\mathrm{Pl}(P,\mu)$ is the set of infinite places of $E(P,\mu)$. Finally

$$\Omega(P,\mu) = \exp(\mathrm{cs}(d)) \cdot \frac{2\pi}{\mathrm{rd}_{E(P,\mu)}},$$

where $\mathrm{rd}_{E(P,\mu)}$ is the root discriminant of $E(P,\mu)$ (the absolute value of its discriminant, raised to the power one over its degree) and $\mathrm{cs}(d)$ is Euler's constant minus the real part of the logarithmic derivative at 1 of the entire Dirichlet series of $\chi_{K(P)}$ (its L-function made entire) (a definition of the library, unit 106).

LEAN Parameters.E · Parameters.Φ · Parameters.ζ · Parameters.isPrimitiveRoot_zeta · Parameters.bigOmega

The number $\Omega(P,\mu)$ is a period: later units measure theta functions against it. This unit only defines it.

3 The parameters

Lemma 3.1 (Arithmetic of the parameters)**.** *Let $P = (N_0, d)$ be parameters and μ a natural number.*

(a) $d \equiv 7 \bmod 8$, and $64 < d$.

(b) If p is a prime number dividing $N_1(P,\mu)$, then p divides $(d+1)/4$, and $\chi_{K(P)}(p) = 1$.

(c) If p is a prime number with $\chi_{K(P)}(p) \neq 1$, then $13 \leq p$.

(d) $N_1(P,\mu)$ and d are coprime; $2^{\mu+1}$ divides $N_1(P,\mu)$; and $N_1(P,\mu)\,d$ divides $M_\omega(P,\mu)$.

Proof. (a) 16 divides N_0 and N_0 divides $2(d+1)$, so $8 \mid d+1$. Also $N_0 \neq 0$, because 0 divides only 0, so $18480 \leq N_0 \leq 2(d+1)$.

(b) The primes 2 and 3 divide N_0, so p divides N_0, hence p divides $2(d+1) = 8 \cdot \frac{d+1}{4}$ (the quotient is exact by (a)). Thus p divides 8 or $(d+1)/4$; in the first case $p = 2$, and 2 divides $(d+1)/4$ by (a). So $4p$ divides $d+1$. Now $\chi_{K(P)}(p)$ is the Jacobi symbol of p over the odd number d, and a Jacobi symbol with upper entry p depends only on the lower entry modulo $4p$. As $d \equiv 4p-1 \bmod 4p$, it equals the symbol of p over $4p-1$, which is the symbol of $4p$ over $4p-1$ because 4 is a square, that is the symbol of 1 over $4p-1$, which is 1.

(c) The primes below 13 divide 18480, hence $N_1(P,0)$, and (b) applies with the exponent 0.

(d) If the prime d divided $N_1(P,\mu)$, then $\chi_{K(P)}(d) = 1$ by (b), but $\chi_{K(P)}(d) = 0$. Next $2 \mid N_0$ and $2^\mu \mid \mathrm{tw}(\mu)$. Finally $\lfloor \log_3 \mu \rfloor \leq \mu$, so $\mathrm{tw}(\mu)$ divides $6^{\mu+5}$, which divides $(N_0 d)^{\mu+5}$ because $6 \mid N_0$; multiplying by $N_0\,d$ gives the claim. □

LEAN Parameters.instHeegD · Parameters.lt_d · Parameters.dvd_norm_c₀ · Parameters.chiK_eq_one_of_dvd · Parameters.thirteen_le_of_chiK_ne_one · Parameters.coprime_N₁_d · Parameters.two_pow_dvd_N₁ · Parameters.N₁_mul_d_dvd_M_omega

Lemma 3.2 (Powers of d modulo N_0)**.** *Let $P = (N_0, d)$ be parameters and n a natural number. Then $d^n \not\equiv -1 \bmod N_0$.*

Proof. In $\mathbb{Z}/N_0$ one has $2(d+1)=0$ and $d+1\neq 0$. As d is odd, $d-1=2i$ with an integer i, and $d^2-1=i\cdot 2(d+1)=0$. So every power of d is 1 or d. Here $d\neq -1$ by assumption, and $1\neq -1$, since $2=0$ would give $d+1=2(i+1)=0$. □

LEAN `Parameters.pow_d_ne_neg_one`

4 The field and its places

Lemma 4.1 (The cyclotomic polynomial stays irreducible)**.** *Let $P=(N_0,d)$ be parameters and μ a natural number. Then $\Psi_{N_1(P,\mu)}$ is irreducible over $K(P)$.*

Proof. Let F be the cyclotomic field of order d. As d is a prime with $d\equiv 3 \bmod 4$ (Lemma 3.1(a)), $K(P)$ embeds into F by Input (I0) of Section 5. It suffices that the polynomial is irreducible over F. Let x be a root of it in an extension of F. The minimal polynomial of x over F divides $\Psi_{N_1(P,\mu)}$, and its degree is $[F(x):F]$. The field $F(x)$ contains roots of unity of the coprime orders $N_1(P,\mu)$ and d (Lemma 3.1(d)), hence of order $N_1(P,\mu)d$, so its degree over $\mathbb{Q}$ is at least $\varphi(d)\,\varphi(N_1(P,\mu))$ by the irreducibility of cyclotomic polynomials over $\mathbb{Q}$. As $[F:\mathbb{Q}]=\varphi(d)$, we get $[F(x):F]\geq\varphi(N_1(P,\mu))$, the degree of $\Psi_{N_1(P,\mu)}$. So $\Psi_{N_1(P,\mu)}$ is the minimal polynomial of x. □

LEAN `Parameters.irreducible_cyclotomic`

Lemma 4.2 (The CM field lies in a cyclotomic field)**.** *Let $P=(N_0,d)$ be parameters and μ a natural number. There is an embedding of fields of $E(P,\mu)$ into the cyclotomic field of order $N_1(P,\mu)\,d$. Consequently $E(P,\mu)$ is abelian over $\mathbb{Q}$ and over $K(P)$, it is totally complex, and it is a CM field.*

Proof. The cyclotomic field of order $N_1(P,\mu)d$ contains a primitive root of unity of order d, hence a copy of $K(P)$ by (I0), and over it $\Psi_{N_1(P,\mu)}$ splits; a splitting field embeds into every extension in which the polynomial splits. A subfield of an abelian extension of $\mathbb{Q}$ is abelian over $\mathbb{Q}$, and $E(P,\mu)$ is totally complex because it contains the imaginary quadratic field $K(P)$. A totally complex abelian number field is a CM field: because the Galois group is abelian, complex conjugation is the same automorphism for every embedding into $\mathbb{C}$, and its fixed field is totally real. □

LEAN `Parameters.toCycloField`; the consequences are instances without names, stated after it in the formal text.

Lemma 4.3 (An embedding over the quadratic field is determined by its place)**.** *Let $P=(N_0,d)$ be parameters, μ a natural number, and ψ, ψ' two embeddings of $E(P,\mu)$ into $\mathbb{C}$ that are the identity on $K(P)$. If ψ and ψ' have the same infinite place, then $\psi=\psi'$.*

Proof. Two embeddings with the same place are equal or complex conjugate. If ψ' were the conjugate of ψ, then every $x\in K(P)$ would be equal to its complex conjugate, which fails for $\mathrm{i}\sqrt{d}$. □

LEAN `Parameters.algHom_eq_of_mk_eq`

Definition 4.4 (The group of places and its three descriptions)**.** Let $P=(N_0,d)$ be parameters and μ a natural number. Fix an embedding $\iota_1 : E(P,\mu)\to\mathbb{C}$ that is the

identity on $K(P)$ (the formal text takes the one that the algebraic closedness of $\mathbb{C}$ provides). The map

$$\mathrm{Gal}(E(P,\mu)/K(P)) \longrightarrow \mathrm{Pl}(P,\mu), \qquad \gamma \longmapsto \text{the place of } \iota_1 \circ \gamma,$$

is a bijection: it is injective by Lemma 4.3, and surjective because the Galois group permutes transitively the places of $E(P,\mu)$ above the one infinite place of $K(P)$. The set $\mathrm{Pl}(P,\mu)$ is given the structure of a commutative group that makes this bijection an isomorphism; its unit element is the place of ι_1.

(i) $\mathrm{gal}_{P,\mu}\colon \mathrm{Pl}(P,\mu) \to \mathrm{Gal}(E(P,\mu)/K(P))$ is the inverse isomorphism.

(ii) $\mathrm{cyc}_{P,\mu}\colon \mathrm{Pl}(P,\mu) \to (\mathbb{Z}/N_1(P,\mu))^\times$ is $\mathrm{gal}_{P,\mu}$ followed by the cyclotomic character, which is an isomorphism by Lemma 4.1.

(iii) For an integer ν, the place $[\nu]_{P,\mu} \in \mathrm{Pl}(P,\mu)$ is the one with $\mathrm{cyc}_{P,\mu}([\nu]_{P,\mu}) = \nu \bmod N_1(P,\mu)$ if ν is prime to $N_1(P,\mu)$, and the unit element otherwise.

(iv) For a natural number m' with $m' \mid N_1(P,\mu)$, the homomorphism

$$\mathrm{cl}_P^{m'|N_1(P,\mu)}\colon \mathrm{Pl}(P,\mu) \longrightarrow (\mathbb{Z}/m')^\times$$

is $\mathrm{cyc}_{P,\mu}$ followed by reduction modulo m': the *class modulo* m' of a place. (In the formal text its explicit arguments are P and the proof of the divisibility; the divisibility is written in the place of that proof.)

(v) For $v \in \mathrm{Pl}(P,\mu)$ the *chosen embedding at* v is $\iota_v = \iota_1 \circ \mathrm{gal}_{P,\mu}(v)$; its place is v. This is the CM type of $E(P,\mu)$ that all later units use.

LEAN `Parameters.galOfPlace` · `Parameters.placeMulEquiv` · `Parameters.placeOfInt` · `Parameters.placeClass`; the group structure and the CM type are instances without names.

Lemma 4.5 (The dictionary)**.** *Let $P = (N_0, d)$ be parameters and μ a natural number; let $v, v' \in \mathrm{Pl}(P,\mu)$, let ν be an integer and i a natural number.*

(a) $\iota_v(x) = x$ *for every* $x \in K(P)$*, and* $\iota_{v\,v'}(x) = \iota_v(\mathrm{gal}_{P,\mu}(v')(x))$ *for every* $x \in E(P,\mu)$*.*

(b) $\overline{\varsigma(P,\mu)} = \varsigma(P,\mu)^{-1}$*.*

(c) $[\nu]_{P,\mu}^{i} = [\nu^i]_{P,\mu}$*.*

(d) If ν is prime to N_0, then for every natural number m' with $m' \mid N_1(P,\mu)$ the class $\mathrm{cl}_P^{m'|N_1(P,\mu)}([\nu]_{P,\mu})$ is the class of ν modulo m'.

(e) $\mathrm{cl}_P^{N_0|N_1(P,\mu)}([\nu]_{P,\mu}) = -1$ *if and only if* $\nu \equiv -1 \bmod N_0$*.*

Proof. (a) holds by the definition of ι_v, since $\mathrm{gal}_{P,\mu}$ is a homomorphism into a group of automorphisms over $K(P)$. (b) The injective map ι_1 commutes with conjugation and sends $\varsigma(P,\mu)$ to a complex number of absolute value 1, whose conjugate is its inverse. (c) If ν is prime to $N_1(P,\mu)$, both sides have the class ν^i. If not, then ν^i is not prime to $N_1(P,\mu)$ for $i \geq 1$ and both sides are the unit element, and for $i = 0$ both sides are the unit element as well. (d) The numbers N_0 and $N_1(P,\mu)$ have the same prime divisors, so ν is prime to $N_1(P,\mu)$ and the class of $[\nu]_{P,\mu}$ is ν. (e) If the class is -1, then ν is prime to $N_1(P,\mu)$, because the unit element has class $1 \neq -1$ (Lemma 3.2 with $n = 0$); then (d) applies. Conversely $\nu \equiv -1 \bmod N_0$ is prime to N_0, and (d) applies. □

LEAN Parameters.phi_algebraMap · Parameters.phi_mul · Parameters.star_ζ · Parameters.placeOfInt_pow · Parameters.placeClass_placeOfInt · Parameters.placeClass_placeOfInt_eq_neg_one

Proof of Theorem A. (a) is the definition of the cyclotomic character, and (b) follows from (a) and Lemma 4.5(d): the exponent n represents ν modulo $N_1(P,\mu)$, and $\varsigma(P,\mu)$ has that order. (c) By Definition 4.4 the number of places is the order of $(\mathbb{Z}/N_1(P,\mu))^\times$, which is the degree of $\Psi_{N_1(P,\mu)}$. For a prime p dividing M one has $\Psi_M(X^{p^k}) = \Psi_{Mp^k}(X)$ (induction on k); as 2 and 3 divide N_0, this gives $\Psi_{N_1(P,\mu)}(X) = \Psi_{N_0}(X^{\mathrm{tw}(\mu)})$, of degree $\mathrm{tw}(\mu)\,\varphi(N_0)$. (d) The places of class 1 modulo N_0 form the kernel of the reduction from $(\mathbb{Z}/N_1(P,\mu))^\times$ onto $(\mathbb{Z}/N_0)^\times$, whose order is the quotient of the two orders. □

LEAN Parameters.galOfPlace_zeta · Parameters.galOfPlace_placeOfInt_zeta · Parameters.card_infinitePlace_E · Parameters.card_placeClass_eq · Parameters.expand_pow_cyclotomic · Parameters.expand_towerDeg_cyclotomic

Definition 4.6 (The permutation of the places by an automorphism of $\mathbb{C}$)**.** Let $P = (N_0, d)$ be parameters, μ a natural number and σ a field automorphism of $\mathbb{C}$ that is the identity on $K(P)$. Then $\mathrm{perm}_\mu(\sigma)$ is the permutation of $\mathrm{Pl}(P,\mu)$ that multiplies every place, on the left, by the place of the embedding $\sigma \circ \iota_1$. (Its explicit arguments in the formal text are μ and σ; the parameters are read from σ.)

LEAN Parameters.placePerm

Lemma 4.7. *Let $P = (N_0, d)$ be parameters, μ a natural number, σ a field automorphism of $\mathbb{C}$ that is the identity on $K(P)$, $v \in \mathrm{Pl}(P,\mu)$ and $x \in E(P,\mu)$. Then $\sigma(\iota_v(x)) = \iota_{v^*}(x)$ with $v^* = \mathrm{perm}_\mu(\sigma)(v)$.*

Proof. Let u be the place of $\sigma \circ \iota_1$. The embeddings $\sigma \circ \iota_1$ and ι_u are the identity on $K(P)$ and have the same place, so they are equal (Lemma 4.3). Hence $\sigma(\iota_v(x)) = \iota_u(\mathrm{gal}_{P,\mu}(v)(x)) = \iota_{u\,v}(x)$ by Lemma 4.5(a). □

LEAN Parameters.placePerm_spec

Lemma 4.8 (Roots of unity, and the norm to the quadratic field)**.** *Let $P = (N_0, d)$ be parameters, μ a natural number and $y \in E(P,\mu)$.*

(a) If y is a root of unity, then there is a natural number $n < N_1(P,\mu)$ with $\varsigma(P,\mu)^n = y$.

(b) $\prod_{v \in \mathrm{Pl}(P,\mu)} \iota_v(y)$ is the norm of y from $E(P,\mu)$ to $K(P) \subset \mathbb{C}$.

Proof. (a) It suffices that $y^{N_1(P,\mu)} = 1$. By Lemma 4.2 the roots of unity of $E(P,\mu)$ embed into those of the cyclotomic field of the even order $N_1(P,\mu)d$, a group of that order; so $z = y^{N_1(P,\mu)}$ satisfies $z^d = 1$. If $z \neq 1$, then z has order d, and as in the proof of Lemma 4.1 the degree of $E(P,\mu)$ over $\mathbb{Q}$ is at least $\varphi(d)\,\varphi(N_1(P,\mu))$. That degree is twice the number of infinite places, $2\,\varphi(N_1(P,\mu))$, so $d - 1 \leq 2$, against $d > 64$. (b) The norm is the product of the conjugates $\gamma(y)$ over the Galois group; apply ι_1, which is the identity on $K(P)$. □

LEAN Parameters.exists_zeta_pow_eq · Parameters.prod_phi_eq_norm

5 The different

Definition 5.1 (Four integers)**.** Let $P = (N_0, d)$ be parameters and μ a natural number, and let $\mathcal{O}$ be the ring of integers of $E(P,\mu)$. The root of unity $\varsigma(P,\mu)$ lies in $\mathcal{O}$ (the formal text has a second name for it as an element of $\mathcal{O}$, written here in the same way). $c_0(P,\mu) \in \mathcal{O}$ is the integer $(1+\sqrt{-d})/2$ of $K(P)$, and $\sqrt{-d}_{P,\mu} = 2\,c_0(P,\mu) - 1$ is the square root of $-d$ that lies in $K(P) \subset \mathbb{C}$ on the positive imaginary axis. Finally $\partial(P,\mu) = \Psi'_{N_1(P,\mu)}(\varsigma(P,\mu)) \in \mathcal{O}$.

LEAN `Parameters.zetaInt` · `Parameters.c₀InE` · `Parameters.sqrtNegDInE` · `Parameters.relDifferentGen` · `Parameters.coe_zetaInt` · `Parameters.isPrimitiveRoot_zetaInt` · `Parameters.coe_sqrtNegDInE`

Lemma 5.2. *Let $P = (N_0, d)$ be parameters and μ, n natural numbers. Then, in the ring of integers of $E(P,\mu)$: $\overline{\sqrt{-d}_{P,\mu}} = -\sqrt{-d}_{P,\mu}$, and $\sqrt{-d}_{P,\mu}$ divides d; $c_0(P,\mu)\,(1 - c_0(P,\mu)) = (d+1)/4$ and $\overline{c_0(P,\mu)} = 1 - c_0(P,\mu)$; and $\overline{\varsigma(P,\mu)}^{\,n}\,\varsigma(P,\mu)^n = 1$.*

Proof. The square of $\sqrt{-d}_{P,\mu}$ is $-d$, its conjugate is the other root $-\sqrt{-d}_{P,\mu}$, and $c_0(P,\mu) = (1+\sqrt{-d}_{P,\mu})/2$. The last identity is Lemma 4.5(b). □

LEAN `Parameters.star_sqrtNegDInE` · `Parameters.sqrtNegDInE_dvd` · `Parameters.c₀InE_mul_one_sub` · `Parameters.star_c₀InE` · `Parameters.star_zetaInt_pow_mul`

Inputs. In (I0) to (I1′), d is any prime number with the congruence stated. (I0), unit 106: for a prime $d \equiv 3 \bmod 4$, the field $\mathbb{Q}(\sqrt{-d})$ embeds into every field of characteristic 0 that contains a primitive root of unity of order d (a quadratic Gauss sum). (I1), unit 106: for a prime $d \equiv 7 \bmod 8$ the different of $\mathbb{Q}(\sqrt{-d})$ over $\mathbb{Q}$ is generated by $\sqrt{-d}$, and the ring of integers is $\mathbb{Z}[(1+\sqrt{-d})/2]$. (I1′), unit 106: let $d \equiv 7 \bmod 8$ be prime, p an odd prime number, ψ' a homomorphism from the ring of integers of $\mathbb{Q}(\sqrt{-d})$ to a commutative ring R of characteristic p, n a natural number and $x \in R$. Then $x = \psi'((1+\sqrt{-d})/2)^{p^n}$ if and only if $2x - 1 = \chi(p)^n\,\psi'(\sqrt{-d})$, where $\chi(p)$ is the Legendre symbol of p modulo d (Euler's criterion with the law of quadratic reciprocity). (I2), unit 10: let $F \subset F'$ be number fields with rings of integers $\mathcal{O}_F \subset \mathcal{O}_{F'}$, let $z \in \mathcal{O}_{F'}$ generate F' over F, and let a polynomial over $\mathcal{O}_F$ vanish at z and have derivative y at z; if the ideal (y) divides the different of $\mathcal{O}_{F'}$ over $\mathcal{O}_F$, then that different is (y) and $\mathcal{O}_{F'} = \mathcal{O}_F[z]$. (I3), unit 10: a prime ideal whose square contains a prime number divides the different over $\mathbb{Z}$.

References, as the catalogue of the write-up and the dossier of the unit record them; none was opened for this note. For (I3): K. Conrad [1], Theorem 4.13 and its proof, of which (I3) is a consequence (the lower bound for the exponent of the different at a ramified prime; the reading agent's record), and Neukirch [2], Ch. III, §2, place not confirmed. For (I1): Neukirch [2], Ch. I, place not confirmed. For (I2): Serre [3], Ch. III, place not confirmed. For (I0) and (I1′) the catalogue names no work. From Mathlib we use that the ring of integers of $\mathbb{Q}(z)$, for a root of unity z, is $\mathbb{Z}[z]$, and that the different is multiplicative in a tower of number fields.

Proof of Theorem C(a). Write $X^{N_1(P,\mu)} - 1 = \Psi_{N_1(P,\mu)} \cdot Q$ with an integral polynomial Q. Differentiating and evaluating at z gives $y\,Q(z) = N_1(P,\mu)\,z^{N_1(P,\mu)-1}$, so

$y\,Q(z)\,z = N_1(P,\mu)$: thus y divides $N_1(P,\mu)$. The subfield $\mathbb{Q}(z)$ is the cyclotomic field of order $N_1(P,\mu)$; its ring of integers is $\mathbb{Z}[z]$, so its different over $\mathbb{Q}$ is generated by y. By (I1) the different of $K(P)$ over $\mathbb{Q}$ is generated by $\sqrt{-d}_{P,\mu}$, which divides d (Lemma 5.2); as $N_1(P,\mu)$ and d are coprime (Lemma 3.1(d)), the ideals (y) and $(\sqrt{-d}_{P,\mu})$ of $\mathcal{O}$ are coprime. The different of $\mathcal{O}$ over $\mathbb{Z}$ is a multiple of (y), by transitivity through $\mathbb{Q}(z)$, and it is the product of the different of $\mathcal{O}$ over $\mathcal{O}_K$ and $(\sqrt{-d}_{P,\mu})$, by transitivity through $K(P)$. Hence (y) divides the different of $\mathcal{O}$ over $\mathcal{O}_K$, and (I2), for the polynomial $\Psi_{N_1(P,\mu)}$, gives both remaining assertions. (I2) applies because z generates $E(P,\mu)$ over $K(P)$: the root of unity $\varsigma(P,\mu)$ is a power of z, both being primitive of order $N_1(P,\mu)$. □

LEAN `Parameters.relDifferent_spec`

Lemma 5.3. *Let $P = (N_0, d)$ be parameters and μ a natural number. Then $\partial(P,\mu)$ divides $N_1(P,\mu)$ in the ring of integers of $E(P,\mu)$, so $\partial(P,\mu) \neq 0$; and $N_1(P,\mu)\,d$ lies in the different of that ring over $\mathbb{Z}$.*

Proof. The first assertion is Theorem C(a) for $z = \varsigma(P,\mu)$. By Theorem C(a) and transitivity the different over $\mathbb{Z}$ is generated by $\partial(P,\mu)\cdot\sqrt{-d}_{P,\mu}$, which is the first half of Theorem C(b); this product divides $N_1(P,\mu)\,d$ by Lemma 5.2. □

LEAN `Parameters.relDifferentGen_dvd_N₁` · `Parameters.relDifferentGen_ne_zero` · `Parameters.different_eq_span_relDifferentGen_mul` · `Parameters.cycloConductor_mem_different`

Lemma 5.4 (A unit that makes the generator real)**.** *Let $P = (N_0, d)$ be parameters and μ a natural number. There is an integer u_0 of $E(P,\mu)$ with $\overline{u_0}\,u_0 = 1$ and $\overline{u_0\,\partial(P,\mu)} = u_0\,\partial(P,\mu)$.*

Proof. The derivative of a product $\prod(X-x)$ of distinct linear factors at one of its roots is the product of the differences of that root and the others. So $\partial(P,\mu)$ is the product of the differences $\varsigma(P,\mu) - x$ over the primitive roots of unity $x \neq \varsigma(P,\mu)$ of order $N_1(P,\mu)$. Each of them is $x = \varsigma(P,\mu)^{\,l}$ with l prime to $N_1(P,\mu)$, hence odd, $l = 2i+1$, and

$$\varsigma(P,\mu) - \varsigma(P,\mu)^{\,2i+1} = \varsigma(P,\mu)^{\,i+1}\bigl(\varsigma(P,\mu)^{-i} - \varsigma(P,\mu)^{\,i}\bigr).$$

The second factor changes sign under conjugation (Lemma 4.5(b)). As 4 divides $N_1(P,\mu)$, the ring of integers contains a square root i_* of -1, and $\overline{\mathrm{i}_*} = -\mathrm{i}_*$. Put $u_x = \mathrm{i}_*\,\overline{\varsigma(P,\mu)}^{\,i+1}$; then $\overline{u_x}\,u_x = 1$ by Lemma 5.2, and $u_x\,(\varsigma(P,\mu) - x) = \mathrm{i}_*\bigl(\varsigma(P,\mu)^{-i} - \varsigma(P,\mu)^{\,i}\bigr)$ is fixed by conjugation. The product u_0 of the u_x has the two properties. (The formal proof is this induction over the factors.) □

LEAN `Parameters.exists_unit_mul_relDifferentGen_real`

Definition 5.5 (The purely imaginary generator of the different)**.** Let $P = (N_0, d)$ be parameters and μ a natural number. With an integer u_0 chosen as in Lemma 5.4, $\varpi(P,\mu) = u_0\cdot\partial(P,\mu)\cdot\sqrt{-d}_{P,\mu}$.

LEAN `Parameters.ϖ`

Lemma 5.6. *Let $P = (N_0, d)$ be parameters, μ a natural number and $v \in \mathrm{Pl}(P,\mu)$. Then $\overline{\varpi(P,\mu)} = -\varpi(P,\mu)$, and $|\iota_v(\varpi(P,\mu))| = |\iota_v(\partial(P,\mu))|\cdot\sqrt{d}$.*

Proof. $u_0\,\partial(P,\mu)$ is fixed by conjugation and $\sqrt{-d}_{P,\mu}$ changes sign. Further $|\iota_v(u_0)|^2 = \iota_v(\overline{u_0}\,u_0) = 1$ and $|\iota_v(\sqrt{-d}_{P,\mu})| = \sqrt{d}$. □

LEAN Parameters.star_ϖ · Parameters.infinitePlace_apply_varpi

Proof of Theorem C(b), (c). The first half of (b) was shown with Lemma 5.3; the second follows because u_0 is a unit. For (c), all infinite places of $E(P,\mu)$ are complex, so the square of the product in question is the absolute value of the norm of $\varpi(P,\mu)$ to $\mathbb{Q}$, that is the absolute norm of the different, that is the absolute value of the discriminant, which is $\mathrm{rd}_{E(P,\mu)}$ to the power twice the number of infinite places. Take square roots. □

LEAN Parameters.different_eq_span_varpi · Parameters.prod_varpi_eq_rootDiscr_pow

6 Frobenius elements

Definition 6.1 (The action on finite places; the Frobenius element)**.** Let $P=(N_0,d)$ be parameters and μ a natural number, and let $\mathcal{O}$ be the ring of integers of $E(P,\mu)$. For $v\in\mathrm{Pl}(P,\mu)$ let $\mathrm{gal}_v\colon\mathcal{O}\to\mathcal{O}$ be the restriction of $\mathrm{gal}_{P,\mu}(v)$ (its one explicit argument in the formal text is v). The group $\mathrm{Pl}(P,\mu)$ acts on the finite places of $E(P,\mu)$ by $v\cdot w=\{x\in\mathcal{O}:\mathrm{gal}_v(x)\in w\}$. For a natural number p,

$$\mathrm{Fr}_\mu(p)=\left([\chi_{K(P)}(p)\,p]_{P,\mu},\ \chi_{K(P)}(p)\right)\in\mathrm{Pl}(P,\mu)\times\mathbb{Z},$$

and powers of such pairs are taken componentwise. (The parameters P are an implicit argument of the formal definition; the notation shows the two explicit ones, the exponent μ and p.)

LEAN Parameters.galOfPlaceInt · Parameters.frobElem · Parameters.coe_galOfPlaceInt · Parameters.mem_smul; the action is an instance without a name.

Setting 6.2. $P=(N_0,d)$ are parameters, μ is a natural number, $\mathcal{O}$ is the ring of integers of $E(P,\mu)$, p is a prime number, $v\in\mathrm{Pl}(P,\mu)$, and $(\mathfrak{c},e)$ is one of the two pairs $(\mathrm{id},1)$ and $(\text{complex conjugation},-1)$, where $\mathfrak{c}$ is an automorphism of the ring $\mathcal{O}$ and e an integer. Further R is a commutative ring of characteristic p, $\psi\colon\mathcal{O}\to R$ is a homomorphism of rings, and n is a natural number.

Lemma 6.3 (A Frobenius congruence is tested on two generators)**.** *In Setting 6.2 assume $p\nmid N_1(P,\mu)$. Then $\psi(\mathrm{gal}_v(x))=\psi(\mathfrak{c}(x))^{p^n}$ holds for all $x\in\mathcal{O}$ if and only if both*

$$\psi(\mathrm{gal}_v(\varsigma(P,\mu)))=\psi(\mathrm{gal}_u(\varsigma(P,\mu)))\quad\text{with } u=[e\,p^n]_{P,\mu}$$

and

$$\chi_{K(P)}(p)^n\,e\,\psi(\sqrt{-d}_{P,\mu})=\psi(\sqrt{-d}_{P,\mu}).$$

Proof. Both $x\mapsto\psi(\mathrm{gal}_v(x))$ and $x\mapsto\psi(\mathfrak{c}(x))^{p^n}$ are homomorphisms of rings, the second because R has characteristic p. By (I1) and Theorem C(a) the ring $\mathcal{O}$ is generated by $\varsigma(P,\mu)$ and $c_0(P,\mu)$, so the two maps agree if and only if they agree at these two elements. At the root of unity: $\mathfrak{c}$ raises it to the power e (Lemma 4.5(b)), and $e\,p^n$ is prime to N_0 because $p\nmid N_1(P,\mu)$, so $\mathfrak{c}(\varsigma(P,\mu))^{p^n}=\mathrm{gal}_u(\varsigma(P,\mu))$ by Theorem A(b). At $c_0(P,\mu)$, which gal_v fixes: here $d\equiv7\bmod8$ and p is odd (Lemma 3.1(a) and the hypothesis $p\nmid N_1(P,\mu)$), so (I1′) applies, with $\chi(p)=\chi_{K(P)}(p)$, to the restriction ψ' of $\psi\circ\mathfrak{c}$ to the integers of $K(P)$, where $\psi'(\sqrt{-d})=e\,\psi(\sqrt{-d}_{P,\mu})$, and to $x=\psi(c_0(P,\mu))$, where $2x-1=\psi(\sqrt{-d}_{P,\mu})$. This turns the condition at $c_0(P,\mu)$ into the second equation. □

LEAN Parameters.frob_ext · Parameters.isCoprime_N₀_of_not_dvd

Lemma 6.4 (Frobenius congruences and powers of the Frobenius element)**.** *In Setting 6.2 assume that R is an integral domain and that $p \nmid N_1(P,\mu)\,d$. Then $\psi(\mathrm{gal}_v(x)) = \psi(\mathfrak{c}(x))^{p^n}$ holds for all $x \in \mathcal{O}$ if and only if $(v,e) = \mathrm{Fr}_\mu(p)^n$ in $\mathrm{Pl}(P,\mu) \times \mathbb{Z}$.*

Proof. Use Lemma 6.3. The image of the root of unity in the domain R is a root of the cyclotomic polynomial of an order that is not zero in R, so its powers with exponents below that order are distinct; by Theorem A(a) the first condition says that v and u have the same class, that is $v = [e\,p^n]_{P,\mu}$. In the second condition $\psi(\sqrt{-d}_{P,\mu})$ can be cancelled, because its square $-d$ is not zero in R; it says $\chi_{K(P)}(p)^n\,e = 1$ in R, hence $e = \chi_{K(P)}(p)^n$, as both are ± 1 and $p \neq 2$. Together with Lemma 4.5(c) the two conditions say $v = [\chi_{K(P)}(p)\,p]^n_{P,\mu}$ and $e = \chi_{K(P)}(p)^n$. □

LEAN Parameters.frob_iff_eq_frobElem_pow

Proof of Theorem B. The residue ring $\mathcal{O}/w$ is a finite field of characteristic p; let ψ be the reduction. (a) If $v \cdot w = \mathfrak{c}^{-1}(w)$, then $\mathrm{gal}_v \circ \mathfrak{c}^{-1}$ maps w into itself and induces an endomorphism of the finite field, which is a power of the Frobenius map: for some n, $\psi(\mathrm{gal}_v(x)) = \psi(\mathfrak{c}(x))^{p^n}$ for all x, and Lemma 6.4 gives $(v,e) = \mathrm{Fr}_\mu(p)^n$. Conversely, such an n gives that congruence by Lemma 6.4, and then $x \in v \cdot w$ if and only if $\mathfrak{c}(x)^{p^n} \in w$, if and only if $\mathfrak{c}(x) \in w$. (b) The norm of w is p^i for the residue degree i. By Lemma 6.4, for the unit place and the first pair, $\mathrm{Fr}_\mu(p)^n = 1$ if and only if $x^{p^n} = x$ on the residue field, if and only if i divides n. So i is the order of $\mathrm{Fr}_\mu(p)$. □

LEAN Parameters.smul_eq_iff_exists_frobElem_pow · Parameters.q_eq_pow

Lemma 6.5 (No ramification)**.** *Let $P = (N_0, d)$ be parameters, μ a natural number, p a prime number with $p \nmid N_1(P,\mu)\,d$ and w a finite place of $E(P,\mu)$ with $p \in w$. Then $p \notin w^2$, and p has valuation exactly 1 at w.*

Proof. If $p \in w^2$, then w divides the different by (I3), which contains $N_1(P,\mu)\,d$ (Lemma 5.3); a natural number in w is divisible by p. □

LEAN Parameters.natCast_notMem_sq · Parameters.intValuation_p

Lemma 6.6 (Transitivity up to conjugation)**.** *Let $P = (N_0, d)$ be parameters, μ a natural number, p a prime number, and w, w' finite places of $E(P,\mu)$ with $p \in w$ and $p \in w'$. Then there is $v \in \mathrm{Pl}(P,\mu)$ with $v \cdot w = w'$ or $\overline{v \cdot w} = w'$.*

Proof. The Galois group of $E(P,\mu)$ over $\mathbb{Q}$ permutes the primes above p transitively: there is an automorphism γ with $\gamma(w') = w$, that is $w' = \{x : \gamma(x) \in w\}$. Let v be the place of the embedding $\iota_1 \circ \gamma$. The embedding $\iota_v = \iota_1 \circ \mathrm{gal}_{P,\mu}(v)$ has the same place, so γ is $\mathrm{gal}_{P,\mu}(v)$, or $\mathrm{gal}_{P,\mu}(v)$ followed by complex conjugation, which commutes with $\mathrm{gal}_{P,\mu}(v)$ because $E(P,\mu)$ is abelian over $\mathbb{Q}$ (Lemma 4.2). In the first case $w' = v \cdot w$ by the definition of the action, which takes preimages; in the second $w' = \{x : \overline{\mathrm{gal}_v(x)} \in w\} = \{x : \mathrm{gal}_v(\overline{x}) \in w\} = \overline{v \cdot w}$. □

LEAN Parameters.exists_smul_eq_or_star

Lemma 6.7 (Primes inert in the quadratic field)**.** *Let $P = (N_0, d)$ be parameters, μ a natural number and p a prime number with $\chi_{K(P)}(p) = -1$. Then $p \nmid N_1(P,\mu)\,d$. Let moreover w be a finite place of $E(P,\mu)$ with $p \in w$, and $v \in \mathrm{Pl}(P,\mu)$. Then*

$v \cdot w = w$ if and only if $v = [-p]^n_{P,\mu}$ for an even natural number n; $v \cdot w = \overline{w}$ if and only if $v = [-p]^n_{P,\mu}$ for an odd n; and the norm of w is p to the least common multiple of 2 and the order of $[-p]_{P,\mu}$ in $\mathrm{Pl}(P,\mu)$.

Proof. The primes dividing $N_1(P,\mu)\,d$ have $\chi_{K(P)} = 1$ or 0 (Lemma 3.1(b)). Here $\mathrm{Fr}_\mu(p) = ([-p]_{P,\mu}, -1)$, and Theorem B applies with either pair. □

LEAN `Parameters.not_dvd_N₁_mul_d_of_inert` · `Parameters.smul_eq_iff_inert` · `Parameters.smul_eq_star_iff_inert` · `Parameters.q_eq_inert`

Lemma 6.8 (Numbers congruent to 1 modulo the conductor)**.** *Let $P = (N_0, d)$ be parameters, μ a natural number and p a natural number with $p \equiv 1 \bmod N_1(P,\mu)\,d$. Then $\chi_{K(P)}(p) = 1$ and $\mathrm{Fr}_\mu(p) = 1$. If moreover p is a prime number, then $p \nmid N_1(P,\mu)\,d$; and for every finite place w of $E(P,\mu)$ with $p \in w$ and every $v \in \mathrm{Pl}(P,\mu)$: $v \cdot w \neq \overline{w}$; $v \cdot w = w$ if and only if $v = 1$; and the norm of w is p.*

Proof. $p \equiv 1$ modulo d and modulo $N_1(P,\mu)$, so the symbol is 1 and $[p]_{P,\mu}$ is the unit element. A prime dividing the modulus would divide 1. The rest is Theorem B with $\mathrm{Fr}_\mu(p) = 1$. □

LEAN `Parameters.chiK_eq_one_of_modEq_one` · `Parameters.frobElem_eq_one` · `Parameters.not_dvd_N₁_mul_d_of_modEq_one` · `Parameters.smul_ne_star` · `Parameters.smul_eq_self_iff_eq_one` · `Parameters.q_eq_of_modEq_one`

Lemma 6.9 (An integer of norm p times a number prime to p)**.** *Let $P = (N_0, d)$ be parameters, μ a natural number, p a prime number with $p \equiv 1 \bmod N_1(P,\mu)\,d$, w a finite place of $E(P,\mu)$ with $p \in w$, and a an integer of $E(P,\mu)$ such that $a\,\overline{a} = p\,k$ for a natural number k prime to p. Then $\overline{a} \in w$ if and only if $a \notin w$. If $a \in w$, then a has valuation exactly 1 at w.*

Proof. $a\,\overline{a} \in w$, so one of the two lies in w. If both did, then $p\,k \in w^2$ with $k \notin w$, so $p \in w^2$, against Lemma 6.5. The valuations of a and $\overline{a}$ at w add up to that of $p\,k$, which is 1. □

LEAN `Parameters.star_mem_iff` · `Parameters.intValuation_eq_of_mul_star`

Proposition 6.10 (As many places above p and a as infinite places)**.** *Let $P = (N_0, d)$ be parameters, μ a natural number, p a prime number with $p \equiv 1 \bmod N_1(P,\mu)\,d$, and a an integer of $E(P,\mu)$ such that $a\,\overline{a} = p\,k$ for a natural number k prime to p. Then there is a bijection from $\mathrm{Pl}(P,\mu)$ onto the set of finite places of $E(P,\mu)$ that contain both a and p.*

Proof. Fix a finite place w containing p. Send v to $v \cdot w$ if $a \in v \cdot w$ and to $\overline{v \cdot w}$ otherwise. Both places contain p, and exactly one of them contains a (Lemma 6.9), so the map is well defined. If u and v have the same image, then $u \cdot w$ and $v \cdot w = (u^{-1}v) \cdot (u \cdot w)$ are equal or conjugate; Lemma 6.8 excludes the second and turns the first into $u = v$. A place containing a and p is $v \cdot w$ or $\overline{v \cdot w}$ for some v (Lemma 6.6), and then it is the image of v. □

LEAN `Parameters.nonempty_infinitePlace_equiv_finPlace`

7 Weil data

Lemma 7.1. *Let $P = (N_0, d)$ be parameters, μ a natural number, f an injective endomorphism of the ring of integers of $E(P,\mu)$ and w a finite place of $E(P,\mu)$. Then $f(\partial(P,\mu))$ and $\partial(P,\mu)$ have the same valuation at w.*

Proof. $f(\partial(P,\mu))$ is the derivative of the cyclotomic polynomial at the primitive root $f(\varsigma(P,\mu))$, and by Theorem C(a) it generates the same ideal as $\partial(P,\mu)$. □

LEAN `Parameters.valuation_map_relDifferentGen`

Definition 7.2 (The idele of the different)**.** Let $P = (N_0, d)$ be parameters and μ a natural number. $\alpha(P,\mu)$ is the finite idele of $E(P,\mu)$ whose component is $\partial(P,\mu)^{-1}$ at the finite places that contain $c_0(P,\mu)$, and 1 at all others.

LEAN `Parameters.α`

Lemma 7.3. *Let $P = (N_0, d)$ be parameters, μ a natural number and w a finite place of $E(P,\mu)$. Then the inverse of $\alpha(P,\mu)$ and the product $N_1(P,\mu)\cdot\alpha(P,\mu)$ are integral at every finite place, and the component at w of $\alpha(P,\mu)\cdot\overline{\alpha(P,\mu)}\cdot\partial(P,\mu)$ is a unit.*

Proof. The first two assertions hold because $\partial(P,\mu)$ is an integer that divides $N_1(P,\mu)$. For the third, the component of the conjugate idele at w is $\overline{\partial(P,\mu)}^{-1}$ if $\overline{c_0(P,\mu)} = 1 - c_0(P,\mu)$ lies in w, and 1 otherwise. Not both $c_0(P,\mu)$ and $1-c_0(P,\mu)$ lie in w. If exactly one does, the component in question is 1 or $\partial(P,\mu)/\overline{\partial(P,\mu)}$, a unit by Lemma 7.1 for complex conjugation. If neither does, the component is $\partial(P,\mu)$, and $\partial(P,\mu) \notin w$: otherwise $N_1(P,\mu) \in w$, and as $N_1(P,\mu)$ divides a power of $(d+1)/4 = c_0(P,\mu)(1-c_0(P,\mu))$ by Lemma 3.1(b), one of the two factors would lie in w. □

LEAN `Parameters.inv_α_mem` · `Parameters.v_alpha_mul_relDifferentGen` · `Parameters.N₁_mul_α_mem`

Inputs. In (I4) to (I6), F is a CM field with a chosen embedding ι_v at every infinite place v. (I4), unit 106: *Weil data* for F consist of a primitive root of unity of order 5040 in F, a *splitting character* (a continuous unitary character v of the ideles of F, trivial on $F^\times$, with $v(\overline{x}) = v(x)^{-1}$ for every idele x, and with $v(x) = \prod_v x_v/|x_v|$ for every idele x whose finite components are all local units), an element $\jmath \in F$, and a finite idele, such that: $\overline{\jmath} = -\jmath$; the imaginary part of $\iota_v(\jmath)$ is positive for every infinite place v; the product of $\jmath$, the idele and its conjugate generates the inverse of the different over $\mathbb{Z}$ at every finite place; $(1-\jmath^{-1})/2$ is an algebraic integer; and v takes the value -1 at a uniformiser of every finite place w with $\overline{w} = w$. (I5), unit 104: let F contain a primitive root of unity of order 5040 (the formal text calls such a CM field *stably unramified*: then F contains a square root of -1 and $(1+\sqrt{-7})/2$, so it is unramified over its maximal totally real subfield at every finite place). If every root of unity y of F is $y = u/\overline{u}$ for a unit u of the ring of integers with $\prod_v \iota_v(u) = 1$, then F has a splitting character. (I6), unit 97: every splitting character takes the value -1 at a uniformiser of every finite place w with $\overline{w} = w$.

Proposition 7.4 (Weil data for the CM field)**.** *Let $P = (N_0, d)$ be parameters and μ a natural number. The field $E(P,\mu)$, with the embeddings of Definition 4.4, has Weil data in the sense of (I4), with $\jmath = -1/\sqrt{-d}_{P,\mu}$ and the idele $\alpha(P,\mu)$.*

Proof. As $5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$ divides $N_1(P,\mu)$ ($2^4 \cdot 5 \cdot 7$ divides 18480, hence N_0, and 3^2 divides 3^5, hence $\mathrm{tw}(\mu)$), a power of $\varsigma(P,\mu)$ has order 5040: this is the first datum of (I4) and the first hypothesis of (I5). For the second hypothesis of (I5), let y be a root of unity; $y = \varsigma(P,\mu)^n$ by Lemma 4.8(a). Put $i = N_1(P,\mu)/4$, a natural number, and

$$u = \varsigma(P,\mu)^i\,(\varsigma(P,\mu) - 1).$$

Its norm to $K(P)$ is 1: the norm of $\varsigma(P,\mu)$ is 1, and the norm of $\varsigma(P,\mu) - 1$ is the value $\Psi_{N_1(P,\mu)}(1)$, by Lemma 4.1 and because the degree is even; that value is 1 since $N_1(P,\mu)$ is not a power of a prime. So u is a unit, and $\prod_v \iota_v(u) = 1$ by Lemma 4.8(b). Further $\overline{u} = \varsigma(P,\mu)^{-i}(\varsigma(P,\mu)^{-1} - 1) = -\varsigma(P,\mu)^{-i-1}(\varsigma(P,\mu) - 1)$, and $\varsigma(P,\mu)^{2i} = -1$, so $\varsigma(P,\mu)\,\overline{u} = u$; hence $y\,\overline{u^n} = u^n$, and (I5) gives a splitting character. Now $\jmath$: its conjugate is $-\jmath$ by Lemma 5.2; $\iota_v(\jmath) = \mathrm{i}/\sqrt{d}$ for every v, because ι_v is the identity on $K(P)$; and $(1 - \jmath^{-1})/2 = c_0(P,\mu)$. By Lemma 7.3 and Theorem C(b), $\jmath\,\alpha(P,\mu)\,\overline{\alpha(P,\mu)}$ has at every finite place the valuation of the inverse of $\partial(P,\mu) \cdot \sqrt{-d}_{P,\mu}$, a generator of the different. The last condition is (I6). □

LEAN `Parameters.instWeilDataE`

8 Notes on the formal text

Names and arguments. The formal text writes N_0 and d as the two components of P, and every other object of this note as a function of P and the exponent; the note keeps all explicit arguments. Three notations show less than the full list of arguments, and say so where they are defined: the Frobenius element and the permutation of places have P as an implicit argument, and the restriction gal_v has both P and μ implicit. The class map takes P and a proof of a divisibility; the note writes the divisibility. The formal text calls the exponent `s` and the root of unity `ζ`; this write-up keeps those letters for the complex variable and for Riemann's function.

Signatures that the reader of the unit derived. For eleven declarations of this note the list of arguments in the dossier is derived from Lean's rule for section variables and not printed by Lean: Theorem B(a), the statements of Lemmas 6.6, 6.9 and Proposition 6.10, the three statements on places in Lemma 6.8, the two equivalences of Lemma 6.7, and Theorem C(a). I read each of them in the Lean text and agree: where the statement binds a variable itself (the place v in Lemma 6.6, the exponent in Theorem B(a) and Lemma 6.7, the element y of Theorem C(a), the cofactor k of Lemma 6.9), the section variable of the same name is not an argument; and the hypothesis $p \in w$ is an argument of the three statements of Lemma 6.8 that mention w, because the text includes it again after having omitted it. This follows the documented rule; only Lean's own output can confirm it, and nothing was compiled here.

The embedding into the cyclotomic field. The census of the workspace records one explicit argument for the embedding of Lemma 4.2, an element of $E(P,\mu)$. In the Lean text the declaration is a homomorphism of $\mathbb{Q}$-algebras with the two implicit arguments P and μ and no explicit one; the arrow of the homomorphism was read as an argument. The note accounts for the entry by naming the field.

Where the formal statements differ from the natural ones. Theorem A(a) is stated in Lean with the class map for the divisor $N_1(P,\mu)$ of itself, which is $\mathrm{cyc}_{P,\mu}$. Lemma 3.1(b) is stated in Lean with the quotient $(d+1)/4$ of natural numbers; it is exact. In Theorem B and Lemmas 6.3, 6.4 the pair $(\mathfrak{c}, e)$ is a ring automorphism and an integer with the hypothesis that they form one of the two pairs; the group $\mathrm{Pl}(P,\mu)\times\mathbb{Z}$ has the integers as a multiplicative monoid, so the order in Theorem B(b) is the order in a monoid. Valuations are multiplicative in Lean: "valuation exactly 1" is there the value, in the value group of the discrete valuation, that Lean writes `.exp (-1)`, the value of a uniformiser (not the real number e^{-1}). The census of the workspace cuts the statement of Theorem C(a) at the word `let`; the statement runs two lines further, as the reader's corrected list of statement spans records and as the Lean file shows, and for this reason it is not among the statements quoted in Appendix A. In Lemma 6.8 the first two assertions do not assume that p is prime; the formal text removes that hypothesis explicitly. Lemma 4.5(c) and (e) hold for all integers ν, by the convention that an integer not prime to the order names the unit place. Proposition 7.4 is an instance: it provides the data rather than asserting their existence, and it depends on the choices of Lemma 5.4 and of (I5). The formal definition behind (I4) lists the root of unity of order 5040 as the property that (I5) assumes, and it writes $\jmath$ as `j`; in the overview of the handout the letter j with its dot is a homomorphism of rings from $\mathbb{C}$ to a non-archimedean field. The first hypothesis of (I5) cannot be dropped. The field $\mathbb{Q}(\sqrt{3},\sqrt{-2})$ has the roots of unity ±1, and $u=(\sqrt{-2}+\sqrt{-6})/2$ is a unit in it with $u/\overline{u}=-1$ and, for the two embeddings that send $\sqrt{-2}$ to the upper half-plane, with $\prod_v \iota_v(u)=1$. But a splitting character of a CM field restricts to the ideles of the maximal totally real subfield as the quadratic character of the extension, and its values on the ideles with unit finite components show that this character is unramified at every finite place; the field $\mathbb{Q}(\sqrt{3},\sqrt{-2})$ is ramified over $\mathbb{Q}(\sqrt{3})$ above 2. (This example is due to the referee of the note and was checked by hand; it is not in the formal text.)

What is used where. Inside the unit, the note *Admissible data* uses Theorem A, Lemmas 3.1 and 4.5, Lemmas 6.5 and 6.9, and Proposition 6.10. The remaining results serve later units: Lemma 4.7 the conjugates of algebraic parts, Lemmas 6.7 and 3.2 the local analysis at the primes that do not split in $K(P)$, Theorem C and Proposition 7.4 the Weil representation. This was seen from the places of use in the source, not from a kernel export.

How this note was written. From the dossier of the unit: the texts of its members and the structured proofs of the declarations of the two files treated here. I read both Lean files completely and compared every statement of this note with the Lean statement, argument by argument, with the lists of arguments of the census and of the dossier. The proofs follow the structured proofs; those of Lemma 3.1(d), Lemma 4.5(c) and Theorem A(c), (d), which the formal text leaves to automation,

are mine. The Inputs were taken from the dossier's reading of the library and from the library's own comments; I opened the definition (I4) and the statements of (I0), (I1′) and (I5), and none of the proofs behind (I0) to (I6). In the revision of 26 November 2026, after a referee's report, the hypothesis of (I5) on the root of unity of order 5040 was restored from the formal statement, and (I0) and (I1′) were moved from the proofs into the box of Inputs. A second reading of the unit by the reading agent arrived while this note was written; its three corrections that touch this note (the polynomial that defines the field, the direction of the automorphism in Lemma 6.6, the value of a uniformiser) were checked against the Lean text and adopted. The program check of the unit's dossier covered 145 leaf steps of 39 proofs.

Appendix A. Concordance with the Lean artifact

The formal names of the objects: `P.N₀`, `P.d` are N_0, d; `towerDeg` applied to μ is $\mathrm{tw}(\mu)$; the exponent `s` is μ; `P.ζ s` and `P.zetaInt s` are the root of unity; `P.Φ s` is the group of places and `Φ v` the embedding ι_v; `star` is complex conjugation. Three statements as the artifact has them:

```
@[local grind] structure Parameters where
  (N₀ d : ℕ)
  dvd_N₀ : 18480 ∣ N₀
  d_prime : d.Prime
  /-- `d ≡ N₀/2 - 1 mod N₀`. -/
  d_congr : N₀ ∣ 2 * (d + 1) ∧ ¬N₀ ∣ d + 1

lemma prod_varpi_eq_rootDiscr_pow : ∏ v : P.Φ s, v (P.ϖ s).val =
  rootDiscr (P.E s) ^ numInfPlaces (P.E s)

lemma smul_eq_iff_exists_frobElem_pow : v • w = comap c c.surjective w ↔ ∃ n, (v, e) =
  P.frobElem s p ^ n
```

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Notation

$P = (N_0, d)$ parameters (Definition 2.1)

μ the exponent, a natural number

$\mathrm{tw}(\mu)$, $N_1(P,\mu)$ the tower degree; the order of the root of unity

$M_\omega(P,\mu)$ the number $(N_0 d)^{\mu+6}$

$K(P)$, $\chi_{K(P)}$ $\mathbb{Q}(\sqrt{-d})$ and its quadratic character

$E(P,\mu)$, $\varsigma(P,\mu)$ the CM field and its chosen root of unity

$\mathrm{Pl}(P,\mu)$ the group of infinite places of $E(P,\mu)$

$\Omega(P,\mu)$, $\mathrm{rd}_{E(P,\mu)}$ the period of Definition 2.3; the root discriminant; $\mathrm{cs}(d)$ as there

$\mathrm{gal}_{P,\mu}$, $\mathrm{cyc}_{P,\mu}$ from places to automorphisms and to classes

$[\nu]_{P,\mu}$ the place named by the integer ν

$\mathrm{cl}_P^{m' \mid N_1(P,\mu)}$ the class of a place modulo a divisor m' of $N_1(P,\mu)$

ι_v, ι_1 the chosen embedding at the place v, at the unit place

$\mathrm{perm}_\mu(\sigma)$ the permutation of places by $\sigma \in \mathrm{Aut}(\mathbb{C})$

Ψ_M the cyclotomic polynomial of order M (the letter Ψ is local to this note); X its variable; Q a cofactor

$\mathcal{O}$, $\mathcal{O}_K$, $\mathcal{O}_F$ rings of integers

$c_0(P,\mu)$, $\sqrt{-d}_{P,\mu}$ $(1+\sqrt{-d})/2$ and $\sqrt{-d}$ in $\mathcal{O}$

$\partial(P,\mu)$, $\varpi(P,\mu)$ generators of the different over $K(P)$ and over $\mathbb{Q}$

gal_v the restriction of $\mathrm{gal}_{P,\mu}(v)$ to $\mathcal{O}$

$v \cdot w$, $\overline{w}$ the action of a place v on a finite place w; the conjugate place

$\mathrm{Fr}_\mu(p)$ the Frobenius element at p

$\alpha(P,\mu)$ the idele of Definition 7.2

p a prime number, or a natural number where said

v, v', u, v^* infinite places

w, w' finite places

$\mathfrak{c}, e$ an automorphism of $\mathcal{O}$ and a sign (Setting 6.2)

R, ψ, ψ' a ring; homomorphisms and embeddings (local to this note)

$\chi(p)$ in (I1′): the Legendre symbol of p modulo d

φ, π, L Euler's function; the circle constant; in "L-function"

F, F' auxiliary number fields

x, y, z, a elements of fields and rings

n, i, l, k, o natural numbers and integers: exponents, orders, degrees, a cofactor

ν an integer naming a place

σ, γ an automorphism of $\mathbb{C}$; an element of a Galois group

$\jmath$ the purely imaginary element of the Weil data

υ a splitting character

f an endomorphism of $\mathcal{O}$

u_0, u_x the unit of Lemma 5.4 and its factors

u also a unit, in (I5) and in the proof of Proposition 7.4

References

References

[1] K. Conrad, *The different ideal*, expository notes on the author's web page. Theorem 4.13 and its proof, as recorded by the reading agent for (I3); not opened for this note.

[2] J. Neukirch, *Algebraic Number Theory*, 1999. As recorded in the catalogue of the write-up, places not confirmed.

[3] J.-P. Serre, *Local Fields*. As recorded in the catalogue of the write-up, place not confirmed.

Admissible data: scales, ranks and sizes, and their existence

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The whole argument is run on one record of numbers and places, the *admissible data* D: three scales $x(D)$, $x_1(D)$, $b(D)$ with $\log b(D) \ll x(D) \ll x_1(D) \ll b(D)$, the number of lifts $m(D) \approx \sqrt{x(D)}$ and the rank $r(D) \approx x(D)^5$, a set of *thick* infinite places of the CM field $E(D)$ that satisfies no multiplicative relation with small exponents, and two auxiliary primes. This note states the record, names what the later chapters compute from it (ranks, the order $a(D)$, weights, the prime sum, the budget), and describes the device by which the formal text compares sizes: every quantity carries an upper and a lower exponent, and two quantities are compared by their exponents (Theorem B). It then proves that admissible data exist at every large scale (Theorem A), by a weak form of the Siegel–Walfisz theorem, a pigeonhole in the class group and a construction of Behrend type.

What this note is. The artifact grh-compact is a body of Lean code whose final theorems state the Riemann hypothesis and its analogue for Dirichlet L-functions. That is a claim: its authors report that Lean's kernel accepts the code; a replay of that check by the owner of this write-up was started, and the handout's record of checks gives its state; nothing was compiled for this note; no mathematician has read the proof through. This note treats the second part of one unit (the folder of the data of the proof): 98 of its 239 declarations, among them nine definitions recalled from the first part, the note *The parameters and the CM field*. Every statement was compared, argument by argument, with the Lean text. Proofs are complete, with two exceptions that are said where they occur: results of other units and of the background library are quoted as *Inputs*, and the comparisons of exponents that the formal text leaves to a decision procedure are carried out here by hand only in the cases listed in Section 7. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 98 declarations.

1 The statements

Theorem A (admissible data exist). Let $P = (N_0, d)$ be parameters and ε a real number with $0 < \varepsilon/6$ and $\varepsilon < 1/100$. Then for every sufficiently large real number y there are admissible data D for P (Definition 2.3) such that

$$x(D) = y, \qquad \log x_1(D) = y^{\varepsilon/6}, \qquad (10^6 \log y)^2 \le \log x_1(D),$$

$$\log\log b(D) = \varepsilon \log y, \qquad \mathrm{tw}(\mu(D))/\log b(D) \le \#\,\mathrm{thick}(D).$$

By Theorem A(d) of the first part, $\mathrm{tw}(\mu(D))$ is the number of the places of class 1 modulo N_0, the set $\mathrm{tops}(D)$. So the last inequality says that the thick places are at least as many as the tops divided by $\log b(D)$.

Theorem B (comparison by exponents). Let D be admissible data for parameters $P = (N_0, d)$, and let l, l' be real numbers with $l < l'$.

(a) If F, G are real numbers with $\mathrm{Bd}_D^{\top}(F; l)$ and $\mathrm{Bd}_D^{\perp}(G; l')$, that is $|F| \le e^{l}$ and $e^{l'} \le G$, then $F < G$, and so $F \le G$.

(b) If A, B are natural numbers with $\mathrm{Bd}_D^{\top}(A; l)$ and $\mathrm{Bd}_D^{\perp}(B; l')$, then $A < B$, and so $A \le B$.

The statement is immediate. What it stands for is a method. In the formal text the two bounds are found mechanically from the shape of the expressions F and G (Proposition 3.4 and Table 1), and the inequality $l < l'$ between two linear expressions in $\log x(D)$, $\log x_1(D)$, $\log b(D)$, $\log\log b(D)$ is found from the conditions on the scales. All comparisons of sizes of the later chapters are made in this way.

2 The setting

Conventions. log is the natural logarithm and e its base; for natural numbers, subtraction stops at 0 and $\lfloor\ \rfloor$ is the floor, as in the formal text. A *finite place* of a number field is a nonzero prime ideal of its ring of integers, written like the ideal, with norm $\mathrm{N}(w)$; an *infinite place* is an archimedean absolute value. The bar is complex conjugation on a CM field and on its finite places. An index set I is a finite set.

Definition 2.1 (Recalled from the first part)**.** *Parameters* are a pair $P = (N_0, d)$ of natural numbers with $18480 \mid N_0$, d prime, $N_0 \mid 2(d+1)$ and $N_0 \nmid d+1$. For a natural number μ: $\mathrm{tw}(\mu) = 3^{\lfloor \log_3 \mu \rfloor + 5} 2^{\mu}$, $N_1(P,\mu) = N_0\ \mathrm{tw}(\mu)$ and $M_\omega(P,\mu) = (N_0 d)^{\mu+6}$; $K(P) = \mathbb{Q}(\sqrt{-d}) \subset \mathbb{C}$, with quadratic character $\chi_{K(P)}$; $E(P,\mu)$ is $K(P)$ with the roots of unity of order $N_1(P,\mu)$ adjoined, a CM field; $\mathrm{Pl}(P,\mu)$ is the set of its infinite places, a commutative group isomorphic to $(\mathbb{Z}/N_1(P,\mu))^{\times}$; for an integer ν, $[\nu]_{P,\mu} \in \mathrm{Pl}(P,\mu)$ is the place whose class is ν if ν is prime to $N_1(P,\mu)$, and the unit element otherwise; for a natural number m' with $m' \mid N_1(P,\mu)$, $\mathrm{cl}_P^{m' \mid N_1(P,\mu)} \colon \mathrm{Pl}(P,\mu) \to (\mathbb{Z}/m')^{\times}$ is the class modulo m'; and $\Omega(P,\mu) = \exp(\mathrm{cs}(d)) \cdot 2\pi/\mathrm{rd}$, with rd the root discriminant of $E(P,\mu)$ and $\mathrm{cs}(d)$ Euler's constant minus the real part of the logarithmic derivative at 1 of the L-function of $\chi_{K(P)}$.

LEAN towerDeg · Parameters.N₁ · Parameters.Mω · Parameters.K · Parameters.E · Parameters.Φ · Parameters.placeOfInt · Parameters.placeClass · Parameters.bigOmega

Definition 2.2 (The constant of the parameters)**.** For parameters $P = (N_0, d)$,

$$C(P) = C_{\mathrm{jet}} + (2d+6) + \Big(N_0 + |\mathrm{cs}(d)| + \sum_{x} \big|\log|\Psi'_{N_0}(x)|\big|\Big),$$

where x runs over the complex primitive roots of unity of order N_0, Ψ_{N_0} is the cyclotomic polynomial of that order, and C_{jet} is a natural number chosen from an existence statement of unit 150 (there it bounds the size of the conjugates of certain

algebraic numbers of the Weil representation; this unit uses nothing about it). What this unit uses of $C(P)$ is Lemma 3.5.

LEAN `Parameters.c`

Definition 2.3 (Admissible data)**.** Let $P = (N_0, d)$ be parameters. *Admissible data for P* are a record D with the following components and properties. Components: three real numbers, the logarithms $\log x(D)$, $\log x_1(D)$, $\log b(D)$ of the scales of Definition 2.4; natural numbers $\mu(D)$, $m(D)$, $r(D)$; a finite set $\mathrm{thick}(D)$ of infinite places of $E(P, \mu(D))$, the *thick places*; natural numbers $g_0(D)$, $g_1(D)$; and an integer $\pi_g(D)$ of $K(P)$. Properties:

(D1) $10^{12}\, C(P) \le \log\log b(D)$, $100 \log\log b(D) \le \log x(D)$, $10^6 \log x(D) \le \log x_1(D)$, $10^6 \log x_1(D) \le \log b(D)$ and $6 \log\log x_1(D) \le \log\log b(D)$.

(D2) $\mu(D) \le 32 \log x(D)$.

(D3) $\sqrt{x(D)} \le m(D) \le 2\sqrt{x(D)}$, and there is a natural number n with $m(D) = 2^{n+1}$ and $\mu(D) = 22(n+1)$.

(D4) $x(D)^5 \le r(D) \le 3\, x(D)^5$, and there is a natural number n with $r(D) = 3^{n+1}$.

(D5) For every natural number m' with $m' \mid N_1(P, \mu(D))$ that divides $N_0 \cdot 3^{\lfloor \log_3 \mu(D) \rfloor + 5}$, every thick place has class -1 modulo m': $\mathrm{cl}_P^{m' \mid N_1(P,\mu(D))}(\tau) = -1$ for all $\tau \in \mathrm{thick}(D)$.

(D6) Let $\psi = \mathrm{cl}_P^{2^{\mu(D)+1} \mid N_1(P,\mu(D))}$ be the class modulo $2^{\mu(D)+1}$. For all natural numbers i, j with $0 < i < j \le 324\, \mu(D)$ and all $\tau, \tau', \tau'' \in \mathrm{thick}(D)$: if $\psi(\tau'/\tau)^j = \psi(\tau''/\tau)^i$, then $\psi(\tau'/\tau) = 1$.

(D7) $g_0(D) \ne g_1(D)$; and for $i \in \{0, 1\}$: $g_i(D)$ is a prime number, $g_i(D) \equiv 1 \bmod N_1(P, \mu(D))\, d$, the square of $M_\omega(P, \mu(D))$ divides $g_i(D) - 1$, and $b(D)^{10} \le g_i(D) \le b(D)^{11}$.

(D8) $\pi_g(D) \cdot \overline{\pi_g(D)} = g_0(D)\, g_1(D)$.

(D9) For every real number y with $100 \log b(D) \le y \le 2^{\mu(D)}$ there is a prime number p with $p \equiv 1$ modulo the square of $M_\omega(P, \mu(D))$ and $y \le \log p \le y + \sqrt{\log b(D)}$.

LEAN `Data`

Definition 2.4 (The scales)**.** For admissible data D for parameters $P = (N_0, d)$, the three *scales* $x(D)$, $x_1(D)$, $b(D)$ are the exponentials of the three real numbers that D stores. In terms of them (D1) reads

$$\exp(10^{12} C(P)) \le \log b(D), \qquad (\log b(D))^{100} \le x(D), \qquad x(D)^{10^6} \le x_1(D),$$
$$x_1(D)^{10^6} \le b(D), \qquad (\log x_1(D))^6 \le \log b(D).$$

LEAN `Data.x` · `Data.x₁` · `Data.b`

Definition 2.5 (The field and its places)**.** Let D be admissible data for parameters $P = (N_0, d)$. Put $N_1(D) = N_1(P, \mu(D))$, $E(D) = E(P, \mu(D))$ and $\mathrm{Pl}(D) = \mathrm{Pl}(P, \mu(D))$; let $f(D)$ be the number of elements of $\mathrm{Pl}(D)$ and $\mathrm{rd}(D)$ the root discriminant of $E(D)$. The formal text also names the product $N_1(D) \cdot d$, the order of a cyclotomic field that contains $E(D)$. For an integer j, $[j]_D = [j]_{P,\mu(D)}$. The *tops* are the places of class 1 modulo N_0,

$$\mathrm{tops}(D) = \{\tau \in \mathrm{Pl}(D) : \mathrm{cl}_P^{N_0 \mid N_1(P,\mu(D))}(\tau) = 1\},$$

and $g(D) = g_0(D)\, g_1(D)$.

LEAN Data.N₁ · Data.E · Data.Φ · Data.f · Data.rootDisc · Data.cycloConductor · Data.placeOfInt · Data.tops · Data.gProd

Definition 2.6 (Ranks and gaps)**.** Let D be admissible data for parameters $P = (N_0, d)$, and j a natural number. Put

$$k(D) = (r(D) + 1) - m(D), \qquad t(D) = 5\, r(D),$$

$$n_j(D) = k(D) + j\,(t(D) + 1), \qquad \mathrm{pos}_j(D) = k(D) + j - 1,$$

$$h_2(D) = r(D)^8, \qquad h(D) = 2\, h_2(D), \qquad h_2'(D) = h_2(D) - \lfloor (t(D) + 1)/2 \rfloor.$$

LEAN Data.k · Data.t · Data.n · Data.posRank · Data.h₂ · Data.h · Data.reducedHalfGap

Definition 2.7 (The order and the weights)**.** Let D be admissible data for parameters $P = (N_0, d)$, y a real number, j a natural number and $\tau \in \mathrm{Pl}(D)$. $\lceil y \rceil_D$ is the least natural multiple of $N_1(D)$ that is at least y (it is 0 for $y \le 0$). Put

$$a(D) = \lceil x(D)\, b(D) \rceil_D, \qquad b_1(D) = \lceil a(D)/x(D) \rceil_D, \qquad b_2(D) = \lceil a(D)/x_1(D) \rceil_D.$$

The *nominal weight* $w_D(\tau)$ is $a(D)/x(D)$ if $\tau \in \mathrm{thick}(D)$ and $a(D)/x_1(D)$ otherwise. The *weight* $w_{D,j}(\tau)$ is $b_1(D) + j\, h(D) + h_2(D)$ if $\tau \in \mathrm{thick}(D)$ and $b_2(D) + j\, h(D) + h_2(D)$ otherwise.

LEAN Data.roundUp · Data.a · Data.b₁ · Data.b₂ · Data.nominalWeight · Data.w

Definition 2.8 (The prime sum, the budget, the cost of one direction)**.** Let D be admissible data for parameters $P = (N_0, d)$, y a real number, j a natural number and $\tau \in \mathrm{Pl}(D)$. With $\lambda_{d,1}(\nu) = \Lambda(\nu)\,(1 - \chi_{K(P)}(\nu))$, where Λ is von Mangoldt's function (a definition of unit 151), and $(\ )_+$ the positive part,

$$\mathrm{Ask}_D(y, \tau) = \sum_{\nu=1}^{a(D)} \lambda_{d,1}(\nu) \Big(\frac{y}{\nu} - w_D([-\nu]_D \cdot \tau) \Big)_+,$$

$$\mathrm{budget}(D) = f(D)\, a(D)\, (\log b(D))^3, \qquad \mathrm{jet}_D(j) = \log(j!) - j \log \frac{\Omega(P, \mu(D))}{g(D)},$$

$$\mathsf{U}(D) = \sum_{\tau \in \mathrm{Pl}(D)} \Big(\mathrm{jet}_D(w_{D,0}(\tau)) - \mathrm{Ask}_D(w_D(\tau), \tau) \Big).$$

LEAN Data.primeSum · Data.budget · Data.jetSize · Data.dirCost

Why these choices. This paragraph is motivation, taken from the comments of the formal text and from the places where later units use the data; nothing in it is a statement of this unit. The one-sided bound for primes is proved at the scale $x(D)$, and the saving is $1/m(D)$, about $x(D)^{-1/2}$: $m(D)$ is the number of theta lifts. At the much larger scale $x_1(D)$ a second remainder enters, which Chapter II makes harmless by moving $x_1(D)$ inside a window; the conditions leave room for that. The scale $b(D)$ is the size of the weights, which are orders of vanishing: $a(D) \approx x(D)b(D)$ at the tops and about $b(D)$, or $b(D)x(D)/x_1(D)$, at the other places, according to whether the place is thick. All losses of the argument are powers of $\log b(D)$, the unit of the budget, and $(\log b(D))^{100} \le x(D)$ makes them small against the saving. The ranks grow in $2m(D)$ steps of $t(D) + 1$ from $k(D)$; $m(D)$ is a power of 2 and

$r(D)$ a power of 3 so that the parities of Lemma 3.7 hold and $2^{\mu(D)} = m(D)^{22}$ is about $x(D)^{11}$. The thick places all have class -1 modulo N_0, so for a top τ the place $[-\nu]_D \cdot \tau$ can be thick only for $\nu \equiv 1 \bmod N_0$: this is how the progression 1 modulo N_0 enters the prime sum, with the number of thick places as its coefficient, and why Theorem A must provide many of them. Condition (D6) says that the thick places, seen in the 2-part of the group of places, satisfy no relation $\psi(\tau'/\tau)^j = \psi(\tau''/\tau)^i$ with small exponents; a later unit applies it to $\tau' = [-p]_D^i\tau$ and $\tau'' = [-p]_D^j\tau$ for a prime p, so that the powers of one Frobenius class carry a thick place to at most one other thick place. The primes $g_0(D)$, $g_1(D)$ split completely in $E(D)$; they carry an auxiliary level (Section 5), and (D9) is used in unit 141 to choose a prime level $N(D)$ of prescribed size.

3 The calculus of sizes

Definition 3.1 (Bounds on the logarithmic scale)**.** Let D be admissible data for parameters $P = (N_0, d)$, let F and l be real numbers and $*$ one of the two signs $\top$, $\bot$ (a truth value in the formal text). The statement $\mathrm{Bd}^*_D(F;\, l)$ means $|F| \le e^l$ if $* = \top$, and $e^l \le F$ if $* = \bot$. The two cases are also written $\mathrm{Bd}^\top_D(F;\, l)$ for $\mathrm{Bd}^\top_D(F;\, l)$ and $\mathrm{Bd}^\bot_D(F;\, l)$ for $\mathrm{Bd}^\bot_D(F;\, l)$; in the formal text $\mathrm{Bd}^\top_D$ and $\mathrm{Bd}^\bot_D$ are the relation with its sign fixed. The data D do not occur in the meaning: they are an index, which lets the rules below name the quantities of D. We call l an *upper exponent*, respectively a *lower exponent*, of F.

LEAN `Data.HasLogBound` · `Data.HasLogUpper` · `Data.HasLogLower`

Lemma 3.2 (First rules)**.** *Let D be admissible data for parameters $P = (N_0, d)$, let F, G, l, l' be real numbers and A, B natural numbers.*

(a) If $\mathrm{Bd}^\bot_D(F;\, l)$, then $0 < F$.

(b) If $\mathrm{Bd}^\top_D(F;\, l)$ and $|G| \le F$, then $\mathrm{Bd}^\top_D(G;\, l)$. If $\mathrm{Bd}^\top_D(F;\, l)$, $G \le F$ and $0 \le G$, then $\mathrm{Bd}^\top_D(G;\, l)$.

(c) If $\mathrm{Bd}^\top_D(A;\, l)$ and $B \le A$, then $\mathrm{Bd}^\top_D(B;\, l)$.

(d) If $\mathrm{Bd}^\bot_D(G;\, l')$ and $G \le F$, then $\mathrm{Bd}^\bot_D(F;\, l')$.

Proof. Each is a chain of two inequalities, with $e^l > 0$ in (a). □

LEAN `Data.HasLogBound.pos` · `Data.hasLogUpper_of_abs_le` · `Data.hasLogUpper_of_le_of_nonneg` · `Data.hasLogUpper_of_le` · `Data.hasLogLower_of_le`

Lemma 3.3 (The interval rule)**.** *Let D be admissible data for parameters $P = (N_0, d)$, let F, G, H, l, l', l'' be real numbers with $\mathrm{Bd}^\bot_D(G;\, l)$, $\mathrm{Bd}^\top_D(H;\, l')$ and $G \le F \le H$, and let $*$ be one of the two signs. Assume $l' \le l''$ or $l'' \le l$, according to whether $*$ is $\top$ or $\bot$. Then $\mathrm{Bd}^*_D(F;\, l'')$.*

Proof. $0 < e^l \le G \le F \le H \le e^{l'}$, so $|F| \le e^{l'} \le e^{l''}$ in the first case and $e^{l''} \le e^l \le F$ in the second. □

LEAN `Data.HasLogBound.of_mem_Icc`. In the formal text G and H are F by default, and the two last hypotheses are found by automation.

Proof of Theorem B. $F \le |F| \le e^l < e^{l'} \le G$; in (b) the natural numbers are compared as real numbers. □

LEAN Data.lt_of_hasLogBound · Data.le_of_hasLogBound · Data.nat_lt_of_hasLogBound · Data.nat_le_of_hasLogBound

Proposition 3.4 (The rules, and the exponents of the quantities of the data)**.** *Let D be admissible data for parameters $P = (N_0, d)$. Let F, G be real numbers with upper exponents l, l', or with lower exponents l, l'; in the second case they are positive. Then, in the same sense:*

(a) FG has the exponent $l + l'$, and F^A the exponent $A\,l$ for every natural number A;

(b) $F + G$ has the upper exponent $\max(l, l') + 1$, respectively the lower exponent $\max(l, l')$;

(c) $\sqrt{F}$ has the exponent $l/2$; and if F has the lower exponent l and the upper exponent l', then $1/F$ has the upper exponent $-l$ and the lower exponent $-l'$;

(d) a natural number $A \neq 0$ has the upper exponent A and the lower exponent 0, and e^l has both exponents l.

Moreover every line of Table 1 holds: the quantity in the first column has the upper exponent in the second column and, where one is given, the lower exponent in the third.

quantity	upper exponent	lower exponent
$\log b(D)$	$\log\log b(D)$	$\log\log b(D)$
$\log x_1(D)$	$\log\log x_1(D)$	$\log\log x_1(D)$
$\mu(D)$, $C(P)$	$\log\log x_1(D)$, $\log\log b(D)$	
$m(D)$	$\frac{1}{2}\log x(D) + 2$	$\frac{1}{2}\log x(D)$
$r(D)$; $t(D)$	$5\log x(D) + 3$; $5\log x(D) + 8$	$5\log x(D)$
$k(D)$	$5\log x(D) + 3$	$5\log x(D) - 2$
$n_{m(D)}(D)$	$5.5\,\log x(D) + 12$	$5.5\,\log x(D)$
$h_2(D)$; $h(D)$	$40\log x(D) + 25$; $40\log x(D) + 27$	$40\log x(D)$
$\mathrm{tw}(\mu(D))$	$11\log x(D) + \frac{1}{5}\log\log b(D)$	$11\log x(D)$
$f(D)$	$11\log x(D) + \frac{1}{5}\log\log b(D) + C(P)$	$11\log x(D)$
$g_i(D)$; $g(D)$	$11\log b(D)$; $22\log b(D)$	$10\log b(D)$; $20\log b(D)$
$a(D)$	$\log b(D) + \log x(D) + 1$	$\log b(D) + \log x(D)$
$b_1(D)$	$\log b(D) + 2$	$\log b(D) - 2$
$b_2(D)$	$\log b(D) - \log x_1(D) + \log x(D) + 2$	$\log b(D) - \log x_1(D) + \log x(D)$
$w_D(\tau)$	$\log b(D) + 1$	$\log b(D) - \log x_1(D)$
$w_{D,0}(\tau)$	$\log b(D) + 2$	that of $w_D(\tau)$, for every $w_{D,j}(\tau)$
$M_\omega(P, \mu(D))$	$\frac{1}{9}\log b(D)$	
$\log \mathrm{ord}_{\mathrm{ch}}(D)$	$32\log x(D)$	

Table 1. Exponents recorded in the formal text; $\tau \in \mathrm{Pl}(D)$, $i \in \{0, 1\}$, j a natural number; $\mathrm{ord}_{\mathrm{ch}}(D)$ is defined in Definition 5.3.

Proof. (a) to (d) are the rules $e^l e^{l'} = e^{l+l'}$, $2 < e$, and $A \leq e^A$. A line of the table follows from the interval rule, Lemma 3.3, applied to the bounds that the definitions give, and from (a) to (d). For example (D3) gives $e^{\log x(D)/2} \leq m(D) \leq 2\,e^{\log x(D)/2}$, and $2 \leq e^2$. For $\mathrm{tw}(\mu(D))$: the factor $2^{\mu(D)} = m(D)^{22}$ has the exponents $11\log x(D) + 44$ and $11\log x(D)$; the factor $3^{\lfloor \log_3 \mu(D) \rfloor}$ is at most $\mu(D) + 1$ and 3^5 has the upper exponent 15, so an upper exponent is $11\log x(D) + \log\log x_1(D) + 60$, which is below the one in the table because $\log\log x_1(D) \leq \frac{1}{6}\log\log b(D)$ and

$\log\log b(D) \geq 10^{12}\, C(P)$ with $C(P) \geq 64$ (Lemma 3.5). The line for $f(D)$ follows from this one by $f(D) = \mathrm{tw}(\mu(D))\,\varphi(N_0)$ (Theorem A(c) of the first part) and $\varphi(N_0) \leq C(P)$. For $a(D)$ use Lemma 3.6: $x(D)b(D) \leq a(D) \leq x(D)b(D) + N_1(D)$. The remaining lines are verified in the same way; we did not redo all of them by hand (Section 7). □

LEAN `Data.hasLogBound_mul` · `Data.hasLogBound_add` · `Data.hasLogBound_natCast` · `Data.hasLogBound_f`; the other rules and lines are instances without names.

The formal text has a few more rules of the same kind, used rarely: for the natural ceiling, the absolute value, a difference, a finite sum, and for upper bounds of logarithms ($\log F$, $\log(FG)$, $\log F^A$, $\log A!$).

Lemma 3.5. *Let $P = (N_0, d)$ be parameters. Then $64 \leq C(P)$, $N_0 \leq C(P)$, $d \leq C(P)$ and $\varphi(N_0) \leq C(P)$, with φ Euler's function.*

Proof. All summands in Definition 2.2 are nonnegative, $64 < d$, and $\varphi(N_0) \leq N_0$. □

LEAN `Data.c_ge`

Lemma 3.6 (Rounding up)**.** *Let D be admissible data for parameters $P = (N_0, d)$, and let F, l be real numbers with $\mathrm{Bd}^{\perp}_D(F; l)$. Then $F \leq \lceil F \rceil_D \leq F + N_1(D)$.*

Proof. $F > 0$, and the least integer $n \geq F/N_1(D)$ satisfies $n < F/N_1(D) + 1$; multiply by $N_1(D)$. □

LEAN `Data.roundUp_mem_Icc`

Lemma 3.7 (Ranks)**.** *Let D be admissible data for parameters $P = (N_0, d)$. Then:*

(a) $r(D)$ is odd, $m(D)$ is even, $k(D)$ is even and $t(D)$ is odd;

(b) $10000 \leq m(D)$; $r(D) + 1 = k(D) + m(D)$; $k(D) \leq r(D)$; $100\, m(D) \leq k(D)$; $0 < k(D)$;

(c) $r(D) + 6 \leq t(D)$; $k(D) + t(D) + 1 \leq h_2(D)$; and $j \mapsto n_j(D)$ is monotone.

Proof. By (D3) and (D4), $m(D)$ is a power of 2 and $r(D)$ a power of 3, both with positive exponent. From Table 1 and Theorem B: $10000 < m(D)$, because $10000 < \frac{1}{2}\log x(D)$ by (D1) and Lemma 3.5; and $2\, m(D) \leq r(D)$, because $\frac{1}{2}\log x(D) + 4 < 5\log x(D)$. So the subtraction in $k(D)$ is exact, $r(D)/2 \leq k(D) \leq r(D)$, and $100\, m(D) \leq k(D)$ because $\frac{1}{2}\log x(D) + 102 < 5\log x(D) - 2$. The parities follow from $k(D) = r(D) + 1 - m(D)$ and $t(D) = 5\, r(D)$. Finally $r(D) \geq 3$ gives $r(D) + 6 \leq 5\, r(D)$ and $k(D) + t(D) + 1 \leq 6\, r(D) + 1 \leq r(D)^8$. □

LEAN `Data.r_odd` · `Data.m_even` · `Data.k_even` · `Data.t_odd` · `Data.le_m` · `Data.r_add` · `Data.k_le_r` · `Data.hundred_mul_m_le_k` · `Data.k_pos` · `Data.r_add_six_le_t` · `Data.k_add_t_add_one_le_h₂` · `Data.n_mono`

Lemma 3.8. *Let D be admissible data for parameters $P = (N_0, d)$, and $\tau \in \mathrm{Pl}(D)$ with $\tau \in \mathrm{thick}(D)$. Then $\mathrm{cl}_P^{N_0 | N_1(P, \mu(D))}(\tau) = -1$.*

Proof. This is (D5) for the divisor N_0. □

LEAN `Data.placeClass_of_thick`

4 Weights

Lemma 4.1 (Nominal weights and weights)**.** *Let D be admissible data for parameters $P = (N_0, d)$, $\tau \in \mathrm{Pl}(D)$ and j a natural number. Then*

(a) $a(D)/x_1(D) \le w_D(\tau) \le a(D)/x(D)$*; in particular* $0 \le w_D(\tau)$*;*

(b) $w_D(\tau) + x(D)^{42} \le 2\,b(D)$*;*

(c) $w_D(\tau) \le w_{D,j}(\tau) - h_2(D)$*;*

(d) if $j \le 2\,m(D)$*, then* $w_{D,j}(\tau) \le w_D(\tau) + x(D)^{42}$*, hence* $w_{D,j}(\tau) \le 2\,b(D)$*, and* $\log w_{D,j}(\tau) \le \log w_D(\tau) + 1/x(D)$*.*

Proof. (a) $x(D) \le x_1(D)$. (b) By (a) and Lemma 3.6, $w_D(\tau) \le (x(D)b(D) + N_1(D))/x(D) = b(D) + N_1(D)/x(D)$, and $N_1(D)/x(D) + x(D)^{42} \le b(D)$ by Theorem B: an upper exponent of the left side is $42\log x(D) + 1$. (c) $b_1(D) \ge a(D)/x(D)$ and $b_2(D) \ge a(D)/x_1(D)$ by Lemma 3.6. (d) By the same lemma the difference $w_{D,j}(\tau) - w_D(\tau)$ is at most $N_1(D) + 2\,m(D)\,h(D) + h_2(D)$, with upper exponent $40.5\log x(D) + 33$, below $42\log x(D)$. With (b) this gives the second inequality. For the third, $w_D(\tau)\,e^{1/x(D)} \ge w_D(\tau) + w_D(\tau)/x(D)$, and $x(D)^{42} \le w_D(\tau)/x(D)$ by Theorem B, since $42\log x(D) < \log b(D) - \log x_1(D) - \log x(D)$. □

LEAN `Data.nominalWeight_mem_Icc` · `Data.nominalWeight_nonneg` · `Data.nominalWeight_add_le` · `Data.nominalWeight_le_w_sub` · `Data.w_le_nominalWeight_add` · `Data.w_le_two_mul_b` · `Data.log_w_le`

5 Chosen places and neat levels

Inputs. Let D be admissible data for parameters $P = (N_0, d)$, and I an index set. (J1), unit 141: a finite place w of $E(D)$ *is chosen* if it contains $\overline{\pi_g(D)}$; for $i \in \{0, 1\}$, $\mathrm{Ch}_i(D)$ is the set of chosen places that contain $g_i(D)$, and every chosen place lies in one of the two. A place containing $g_i(D)$ has norm $g_i(D)$, and $\pi_g(D)\,\overline{\pi_g(D)}$ is $g_i(D)$ times a number prime to $g_i(D)$. (J2), unit 141: the *level* $N(D)$ is a prime number with $N(D) \equiv 1 \bmod N_1(D)\,d$, and $\pi_N(D)$ is an integer of $E(D)$ with $\pi_N(D)\,\overline{\pi_N(D)} = N(D)$. (J3), unit 141: ω_D attaches to every finite place w of $E(D)$ a character $\omega_D(w)$ of its residue field, of order exactly $M_\omega(P, \mu(D))$ if w is chosen, and trivial otherwise. (J4), unit 122: $G_{\mathrm{f}}(D, I)$ is the unitary group of the standard hermitian form with index set I over the finite adeles of $E(D)$, $K_0(D, I)$ its subgroup of matrices that are integral with their inverses, $\Gamma(D, I)$ the image of the unitary group over $E(D)$; for an integer a of $E(D)$ the principal level of a consists of the elements of $K_0(D, I)$ congruent to 1 modulo a. The general linear group with index set I over a finite ring with n elements has at most $n^{(\#I)^2}$ elements. (J5), unit 50, a lemma of Minkowski and Serre: if finitely many distinct finite places of a CM field with more than one infinite place have norms whose product exceeds $2\,\#I$ raised to the degree of the field over $\mathbb{Q}$, then for every subgroup $\mathcal{K}$ of the group of elements of the integral subgroup that reduce to 1 at these places, and every element x of the adelic group, the intersection of the rational group with $x\mathcal{K}x^{-1}$ is trivial. The catalogue of the write-up refers for (J5) to Serre [3] and to Minkowski, as a special case, with the place not confirmed.

Lemma 5.1 (The chosen places)**.** *Let D be admissible data for parameters $P = (N_0, d)$, $i \in \{0,1\}$, and w a finite place of $E(D)$.*

(a) There is a bijection from $\mathrm{Pl}(D)$ onto $\mathrm{Ch}_i(D)$ (the formal text fixes one); so $\mathrm{Ch}_i(D)$ has $f(D)$ elements, and the product of the norms of all chosen places is $g(D)^{f(D)}$.

(b) If $g_i(D) \in w$, then $\overline{w}$ is chosen if and only if w is not. If w is chosen, then $\overline{w}$ is not chosen.

(c) The family of all chosen places, indexed by the pairs of an index i and an element of $\mathrm{Ch}_i(D)$, has as its range the set of chosen places; it is injective, and no member is the conjugate of a member.

Proof. (a) Proposition 6.10 of the first part, for the prime $g_i(D)$, which is $\equiv 1$ mod $N_1(D)\,d$ by (D7), and the integer $\overline{\pi_g(D)}$, with (J1); the places of $\mathrm{Ch}_i(D)$ have norm $g_i(D)$. (b) Lemma 6.9 of the first part, for the same prime and integer; a chosen place contains $g_0(D)$ or $g_1(D)$. (c) Places above $g_0(D) \neq g_1(D)$ have different norms, and (b). □

LEAN `Data.chosenEquiv` · `Data.card_chosenPlace` · `Data.prod_q_chosenPlace` · `Data.isChosen_star_iff` · `Data.IsChosen.not_star` · `Data.range_chosen` · `Data.isConjApart_chosen`

Lemma 5.2 (The level is unramified)**.** *Let D be admissible data for parameters $P = (N_0, d)$ and w a finite place of $E(D)$. If $N(D) \in w$, then $N(D)$ has valuation exactly 1 at w. If $\pi_N(D) \in w$, then $\pi_N(D)$ has valuation exactly 1 at w.*

Proof. Lemmas 6.5 and 6.9 of the first part, for the prime $N(D)$ of (J2), which does not divide $N_1(D)\,d$ because it is $\equiv 1$ mod $N_1(D)\,d$, and for the integer $\pi_N(D)$, whose norm is $N(D)$ times 1. □

LEAN `Data.v_N_eq_exp_neg_one` · `Data.intValuation_pi`

Definition 5.3 (The order and the level of the chosen places)**.** Let D be admissible data for parameters $P = (N_0, d)$. $\mathrm{ord}_{\mathrm{ch}}(D)$ is the order of the product, over all chosen places, of the general linear groups of rank $r(D)\,t(D)$ over the residue fields. For an index set I, which is finite, $K_{\mathrm{ch}}(D, I)$ is the subgroup of the elements of $K_0(D, I)$ that reduce to 1 at every chosen place.

LEAN `Data.chosenGroupOrder` · `Data.chosenLevel`

Lemma 5.4. *Let D be admissible data for parameters $P = (N_0, d)$, and fix a finite index set I. If I has at most $r(D)\,t(D)$ elements, then the order of the product, over all chosen places, of the general linear groups with index set I over the residue fields divides $\mathrm{ord}_{\mathrm{ch}}(D)$. If a is an integer of $E(D)$ divisible by $\overline{\pi_g(D)}$, then the principal level of a is contained in $K_{\mathrm{ch}}(D, I)$.*

Proof. Embed I into a set with $r(D)\,t(D)$ elements and extend matrices by the identity: this is an injective homomorphism of groups, and the order of a subgroup divides the order of the group. Every chosen place contains $\overline{\pi_g(D)}$, hence a. □

LEAN `Data.card_GL_dvd_chosenGroupOrder` · `Data.principalLevel_le_chosenLevel`

Proposition 5.5 (Neat levels, and Minkowski's bound)**.** *Let D be admissible data for parameters $P = (N_0, d)$; fix a finite index set I with at most $2\,n_{m(D)}(D)$ elements and a subgroup $\mathcal{K}$ of $G_{\mathrm{f}}(D, I)$.*

(a) *If $\mathcal{K}$ is contained in $K_{\mathrm{ch}}(D,I)$, then $\Gamma(D,I)\cap x\mathcal{K}x^{-1}$ is trivial for every element x of $G_{\mathrm{f}}(D,I)$.*

(b) *If $\mathcal{K}$ is contained in $K_0(D,I)$, then the order of $\Gamma(D,I)\cap x\mathcal{K}x^{-1}$ divides $\mathrm{ord}_{\mathrm{ch}}(D)$ for every element x of $G_{\mathrm{f}}(D,I)$.*

Proof. (a) is (J5) for the family of chosen places, which is injective (Lemma 5.1). Its hypotheses: $E(D)$ is a CM field with $f(D)=\mathrm{tw}(\mu(D))\,\varphi(N_0)>1$ infinite places (Theorem A(c) of the first part), its degree is $2f(D)$, and the product of the norms is $g(D)^{f(D)}$, so it suffices that $(4\,n_{m(D)}(D))^2<g(D)$, which holds by Theorem B and Table 1: $11\log x(D)+32<20\log b(D)$. (b) Conjugation by x^{-1} followed by reduction at the chosen places is a homomorphism from $\Gamma(D,I)\cap x\mathcal{K}x^{-1}$ to the product of the general linear groups over the residue fields. Its kernel lies in $\Gamma(D,I)\cap x\,K_{\mathrm{ch}}(D,I)\,x^{-1}$, which is trivial by (a). So the order divides that of the product, which divides $\mathrm{ord}_{\mathrm{ch}}(D)$ by Lemma 5.4, since $2\,n_{m(D)}(D)\le r(D)\,t(D)$ by Theorem B: the upper exponent $5.5\,\log x(D)+14$ of $2\,n_{m(D)}(D)$ is below the lower exponent $10\log x(D)$ of $r(D)\,t(D)$. □

LEAN `Data.classAut_eq_bot_of_le_chosenLevel` · `Data.card_classAut_dvd_chosenGroupOrder`

Lemma 5.6 (Powers of the residue characters)**.** *Let D be admissible data for parameters $P=(N_0,d)$ and w a finite place of $E(D)$, and fix an integer ν with $0<|\nu|\le 8\,m(D)$. Then $\omega_D(w)^\nu\neq 1$ if and only if w is chosen.*

Proof. At a place that is not chosen the character is trivial. At a chosen place its order $M_\omega(P,\mu(D))$ is a multiple of $N_1(D)\,d$ (Lemma 3.1(d) of the first part), and $8\,m(D)<N_1(D)\,d$ by Theorem B: $8\,m(D)$ has the upper exponent $\frac12\log x(D)+10$, the number $N_1(D)\,d\ge\mathrm{tw}(\mu(D))$ has the lower exponent $11\log x(D)$, and $\frac12\log x(D)+10<11\log x(D)$. So the order does not divide ν. □

LEAN `Data.omega_zpow_ne_one`. Two instances without names follow in the formal text: the last two lines of Table 1.

For the line of $\log\mathrm{ord}_{\mathrm{ch}}(D)$ in Table 1: by (J4) and Lemma 5.1(a),

$$\mathrm{ord}_{\mathrm{ch}}(D)\le g(D)^{\,f(D)\,(r(D)\,t(D))^2},$$

and the exponent times $\log g(D)$ has an upper exponent below $32\log x(D)$.

6 Existence of admissible data

Proposition 6.1 (A construction of Behrend type)**.** *Let Γ_1, Γ_2 be commutative groups, $\psi\colon\Gamma_1\to\Gamma_2$ a homomorphism, $\tau_0,\gamma\in\Gamma_1$, and n, M, L natural numbers with $L>0$ and $(LM)^n\le o$, where o is the order of $\psi(\gamma)$ (it is 0 in the formal text if that order is infinite). Then there is a finite subset $T\subseteq\Gamma_1$ such that*

(i) *every element of T is $\tau_0\gamma^{\,i}$ for a natural number i;*

(ii) *$M^n/(n\,M^2)\le\#T$;*

(iii) *for all natural numbers i, j with $0<i<j\le L$ and all $\tau,\tau',\tau''\in T$: if $\psi(\tau'/\tau)^j=\psi(\tau''/\tau)^i$, then $\psi(\tau'/\tau)=1$.*

Proof. Among the M^n vectors $u=(u_1,\dots,u_n)$ with integer coordinates $0\le u_k<M$, the sum of squares takes at most $n(M-1)^2+1\le nM^2$ values (for $n,M\ge1$; for $n=0$ or $M=0$ the bound (ii) is 0, Section 7); so for some value q the sphere S of the

vectors with $\sum_k u_k^2 = q$ has at least $M^n/(nM^2)$ elements (this count is in Mathlib). For a vector u with natural coordinates put $[u] = \sum_k u_k (LM)^{k-1}$; the map $u \mapsto [u]$ is additive, and on the vectors with all coordinates below LM it is injective with values below $(LM)^n \le o$. Hence $u \mapsto \psi(\gamma)^{[u]}$ is injective on these vectors. Let T be the set of the $\tau_0 \gamma^{[u]}$, $u \in S$. Then (i) holds, and (ii) because $L \ge 1$ makes the coordinates of S smaller than LM. For (iii) let τ, τ', τ'' come from $u, v, z \in S$ and write $j = i + i'$ with $i' > 0$. The hypothesis says $\psi(\gamma)^{(i+i')[v]} = \psi(\gamma)^{i[z]+i'[u]}$, and both exponents are values of the bracket at vectors with coordinates at most $(i + i')(M - 1) < LM$. So $(i + i')\, v = i\, z + i'\, u$: the point v lies between z and u on a segment, and all three lie on one sphere. Expanding $i^2 z_k^2 = ((i + i')v_k - i' u_k)^2$ and summing,

$$\begin{aligned} i\,(i\,q + i'q) &= \sum_k i\,(i\,z_k^2 + i' u_k^2) = \sum_k (i + i')(i'(u_k - v_k)^2 + i\,v_k^2) \\ &= (i + i')\, i' \sum_k (u_k - v_k)^2 + (i + i')\, i\, q, \end{aligned}$$

so $u = v$, hence $\tau' = \tau$. □

LEAN `exists_three_term_free`. The header of the Lean file cites Shapira [4], Theorem 1, after Behrend [1]; the artifact records these citations as found, with no place compared.

Lemma 6.2 (Pigeonhole in the class group)**.** *Fix a number field F with class number n, and let $\mathfrak{p}_i$, $\mathfrak{q}_i$, for $i \in \{0, \dots, n\}$, be finite places of F such that $\mathfrak{p}_i\mathfrak{q}_i$ is principal for every i. Then there are indices $i \ne j$ such that $\mathfrak{p}_i\mathfrak{q}_j$ is principal.*

Proof. Two of the $n + 1$ classes of the $\mathfrak{p}_i$ are equal, say those of $\mathfrak{p}_i$ and $\mathfrak{p}_j$ with $i \ne j$; then the class of $\mathfrak{p}_i\mathfrak{q}_j$ is that of $\mathfrak{p}_j\mathfrak{q}_j$, the trivial one. □

LEAN `exists_mul_principal`

Inputs. (J6), unit 126, a weak form of the Siegel–Walfisz theorem: let $M > 0$ be a multiple of 4 and A a natural number. For all sufficiently large real W, every natural number $k > 1$ with $M^k \le W^A$ and every real y with $W \le y \le W^A$, there is a prime number $p \equiv 1 \bmod M^k$ with $y \le \log p \le y + W$. This is a weaker consequence of the Siegel–Walfisz theorem, in the form of Montgomery and Vaughan [2], §11.3, Corollary 11.19 (p. 381, for the sum of von Mangoldt's function over a residue class) and Corollary 11.21 (p. 382, for the primes of a residue class): for a modulus that is at most a fixed power of the logarithm of the bound, the primes of a residue class prime to the modulus are counted with an error that saves more than every power of that logarithm. Applied to the primes between e^y and $2e^y$ and to the modulus $M^k \le W^A \le y^A$, this gives, once W is large in terms of M and A, a prime $p \equiv 1 \bmod M^k$ with $y < \log p \le y + \log 2 \le y + W$; for $A = 0$ no y is in the range once $W > 1$. The two places were located and the deduction was made by the reading agent; the book was not opened for this note. The formal proof goes another way: it starts from a small disc around 1 without zeros of the L-functions of the characters modulo M^k. (J7), unit 106: for a prime $d \equiv 7 \bmod 8$ and a prime number p whose Legendre symbol modulo d is 1, the ideal (p) of $\mathbb{Q}(\sqrt{-d})$ is a product of two prime ideals of norm p; and the absolute norm of a principal ideal (y) is $y\,\overline{y}$. (J8), unit 106: if q is an integer with $|q| > 1$ and $q^o \equiv 1 \bmod 2^{\mu+1}$ for a natural number $o > 0$, then $2^\mu \le (|q| + 1)\, o$.

Proof of Theorem A. Write $Y = \log y$ and $W = y^{\varepsilon/6} = e^{\varepsilon Y/6}$, and let $\#\,\mathrm{Cl}(K(P))$ be the class number of $K(P)$. Fix a natural number n with $n > 6$ and $n\,\varepsilon/6 > 99\,N_0 d$. Since W exceeds every fixed multiple of every fixed power of Y when y is large, we may assume all of the following: $Y \geq 2$;

$$W \geq 10^{12}\,Y^2, \qquad W \geq (8019N_0 + 2)\,10368^{\,n}\,Y^{n+1}, \qquad W \geq 8019 \cdot 2^n\,n\,Y,$$

$$W \geq 2\,\#\,\mathrm{Cl}(K(P)) + 10^6, \qquad W \geq \exp(10^{12}\,C(P));$$

and the conclusion of (J6) for the modulus $N_0 d$ and $A = n$ at this W.

Ranks and scales. Choose natural numbers i_2, i_3 with $2^{i_2} \leq e^{Y/2} < 2^{i_2+1}$ and $3^{i_3} \leq e^{5Y} < 3^{i_3+1}$, and put

$$m = 2^{i_2+1}, \qquad r = 3^{i_3+1}, \qquad \mu = 22\,(i_2 + 1),$$

and take Y, W, W^6 for the logarithms of the three scales. Then (D3) and (D4) hold, and $2^\mu = m^{22}$. As $i_2 \log 2 \leq Y/2$ and $Y \geq 2$, we get $\mu \leq 16\,Y + 22 \leq 32\,Y$, which is (D2). For (D1): log log of the third scale is $\log W^6 = \varepsilon Y$, which is at least $10^{12}\,C(P)$ by the last assumption and at most $Y/100$; $10^6 Y \leq W$ and $10^6 W \leq W^6$ hold as $W \geq 10^{12} Y^2$ and $W \geq 10^6$; and $6 \log W = \varepsilon Y$. The same computations give the first four assertions of the theorem.

Primes. First $(N_0 d)^{2(\mu+6)} \leq W^n$: the logarithm of the left side is at most $2(32Y + 6)\,N_0 d \leq 99\,N_0 d\,Y < n \log W$. Hence also $2^\mu \leq W^n$. By (J6) with $k = 2(\mu + 6)$: for every real y' with $W \leq y' \leq W^n$ there is a prime $p \equiv 1$ modulo the square of $M_\omega(P, \mu)$ with $y' \leq \log p \leq y' + W$. This gives (D9), because $W \leq 100\,W^6$, $2^\mu \leq W^n$ and $W \leq \sqrt{W^6}$. For $i \in \{0, \dots, \#\,\mathrm{Cl}(K(P))\}$ apply it to $y' = 10W^6 + 2iW$, which is at most $W^7 \leq W^n$: there are primes $p_i \equiv 1$ modulo the square of $M_\omega(P, \mu)$ with $10W^6 + 2iW \leq \log p_i \leq 10W^6 + (2i + 1)W$. They increase strictly with i, and $10W^6 \leq \log p_i \leq 11W^6$ because $(2\,\#\,\mathrm{Cl}(K(P)) + 1)W \leq W^2$. As $N_1(P, \mu)\,d$ divides $M_\omega(P, \mu)$, each p_i is $\equiv 1 \bmod N_1(P, \mu)\,d$; so its Legendre symbol modulo d is 1 and $(p_i) = \mathfrak{p}_i \mathfrak{q}_i$ in $K(P)$ with prime ideals of norm p_i by (J7). By Lemma 6.2 there are $i \neq j$ and an integer π of $K(P)$ with $\mathfrak{p}_i \mathfrak{q}_j = (\pi)$, so that $\pi\overline{\pi} = p_i p_j$. With the two primes p_i, p_j and the integer π, (D7) and (D8) hold.

Thick places. Put $M_3 = N_0 \cdot 3^{\lfloor \log_3 \mu \rfloor + 5}$, so that $N_1(P, \mu) = M_3\,2^\mu$, and let ψ be the class modulo $2^{\mu+1}$ on $\mathrm{Pl}(P, \mu)$. The integers -1 and $M_3 + 1$ are prime to N_0; put $\tau_0 = [-1]_{P,\mu}$ and $\gamma = [M_3 + 1]_{P,\mu}$. By Lemma 4.5(d) of the first part, $\psi(\gamma)$ is the class of $M_3 + 1$, so its order o satisfies $2^\mu \leq (M_3 + 2)\,o$ by (J8). Choose a natural number i_0 with $2^{n\,i_0} \leq 2^\mu / W < 2^{n(i_0+1)}$; this is possible because $W \leq e^{Y/2} < m \leq 2^\mu$. Now $M_3 \leq 243\,N_0(\mu + 1) \leq 8019\,N_0 Y$ and $324\mu \leq 10368\,Y$, so

$$(M_3 + 2)\,(324\mu \cdot 2^{i_0})^n \leq (8019N_0 + 2)\,10368^{\,n}\,Y^{n+1} \cdot \frac{2^\mu}{W} \leq 2^\mu \leq (M_3 + 2)\,o.$$

Thus Proposition 6.1 applies with $L = 324\mu$ and the side 2^{i_0} in the role of its M. It gives a set T of places with (D6), with at least $2^{n\,i_0}/(n\,2^{2i_0})$ elements, all of the form $[-1]_{P,\mu} \cdot [M_3 + 1]^i_{P,\mu}$. For a divisor m' of M_3, such a place has class $-(M_3 + 1)^i = -1$ modulo m', again by Lemma 4.5(d) of the first part: this is (D5). So the record with thick set T is admissible.

Many thick places. It remains that $\mathrm{tw}(\mu)/W^6 \leq 2^{n\,i_0}/(n\,2^{2i_0})$. Here $\mathrm{tw}(\mu) \leq 243(\mu + 1)2^\mu \leq 8019\,Y \cdot 2^{n(i_0+1)}\,W$, and $2^{i_0} < W$ because $2^{n\,i_0} \leq 2^\mu / W < W^n$. So

$$\frac{\mathrm{tw}(\mu)}{W^6} \cdot \frac{n\,2^{2i_0}}{2^{n\,i_0}} \leq \frac{8019\,Y 2^n\,n\,W^3}{W^6} \leq \frac{W^4}{W^6} \leq 1. \qquad \square$$

7 Notes on the formal text

Names and arguments. In the formal text the scales, the ranks $k(D)$, $t(D)$, the field and every other object of Definitions 2.4 to 2.8 are functions of the data, and the note writes them so. The components of the record are written in the same way, $m(D)$, $r(D)$, $\mu(D)$, $\mathrm{thick}(D)$, $g_i(D)$, $\pi_g(D)$; the overview of the write-up writes some of them without the argument. The record stores the logarithms of the scales, and the scales are their exponentials: the note writes $\log x(D)$ for the stored number. The component that the formal text calls `s` is $\mu(D)$ here. In Lean the hypotheses $\mathrm{Bd}_D^{\top}(F;\, l)$ and $\mathrm{Bd}_D^{\perp}(G;\, l')$ of Theorem B and of Lemmas 3.2 to 3.6 are instance arguments: Lean finds them by search, from the rules of Proposition 3.4 and the lines of Table 1, each of which is an instance; and the hypothesis $l < l'$ of Theorem B has a default proof by a decision procedure.

What was not redone by hand. The formal text proves each line of Table 1, and each use of Theorem B, by one call of a decision procedure on the conditions (D1). In this note the inequalities between exponents are written out for the lines of $m(D)$, $\mathrm{tw}(\mu(D))$, $f(D)$ and for the uses in Lemmas 3.7, 4.1, 5.6 and Proposition 5.5; the other lines of the table are stated as the formal text has them. The exponents given in the proofs are mine and may differ from those that Lean finds. The proof of Theorem A is complete here; the formal proof closes several of its numerical steps by automation, and which goal each tactic settles there was inferred by the reader of the unit from the order of the conditions, not from Lean's output.

Signatures that could not be settled without Lean. (1) For five declarations treated here the list of arguments is derived by the reader of the unit and not printed by Lean: Lemma 3.5, the range and the conjugation statement of Lemma 5.1(c), and the lines of $f(D)$ and of $C(P)$ in Table 1. I read them in the Lean text and agree: the letter that looks like a section variable is in each case another object (the constant of the parameters, written with a dot after P; a variable bound inside the statement). (2) The two lines of Table 1 for $w_{D,0}(\tau)$ and $w_{D,j}(\tau)$ are instances that stand, in the Lean file, after a command that includes the hypothesis $j \le 2\,m(D)$ in what follows. Lean's documentation says that this command concerns theorems and no other declarations; and the proof of Lemma 4.1(d) needs the second instance to be found by search, which could not supply such a hypothesis. So I read both instances without the hypothesis, and the first one without j, against the census of the workspace; only Lean can settle it. (3) The rule for quotients is an instance that takes bounds for F and for $1/G$. The census lists a further bound for G as a possible argument; by Lean's rule for definitions, which keeps a section variable only if the term uses it, it is not one. This too is a reading of the rule, not Lean's answer.

Where the formal statements differ from the natural ones. (D5) quantifies over a number and a proof that it divides the order of the root of unity; the formal text calls this number `m`, a name that hides the component $m(D)$ there, and the note writes m'. The subtraction in $k(D)$, $\mathrm{pos}_j(D)$ and $h_2'(D)$ is that of natural numbers;

Lemma 3.7 shows that it is exact for $k(D)$. In (D7) the formal text says that the pair of primes is injective and that the square of $M_\omega(P, \mu(D))$ divides $g_i(D) - 1$ in natural numbers. In Theorem A the hypothesis is $0 < \varepsilon/6$, and "for every sufficiently large y" is a statement along the filter at infinity. In Proposition 6.1 the bound (ii) is a real quotient, which Lean reads as 0 when $n = 0$ or $M = 0$, and the order of an element of infinite order is 0; then the hypothesis forces $M = 0$ and the statement is empty. Lemma 3.2(a) is all that the formal text uses of positivity. The definition of $h_2'(D)$ and of $\mathrm{pos}_j(D)$ are not used in this unit.

How this note was written. From the dossier of the unit: the texts of its members and the structured proofs of the five files treated here, of which the longest is the proof of Theorem A. I read the five Lean files completely and compared every statement of this note with the Lean statement, argument by argument. The proof of Theorem A follows the formal one step by step, with its constants; the proofs of Lemmas 3.7 and 4.1 and of Proposition 3.4 replace calls of automation by explicit exponents. Of the Inputs I opened only the comments of the formal text and the reading of the dossier, not the proofs. The references are those of the catalogue of the write-up and of the artifact's own list; I did not open the cited works. For (J6) the places in the book and the deduction are those of the reading agent's second reading of the catalogue. In the revision of 26 November 2026, after a referee's report, the infinite places received the letter τ of the overview of the handout, the hypotheses of (J5) and the exponents in Proposition 5.5 and Lemma 5.6 were written out, and (J6) received its deduction. A second reading of the unit by the reading agent arrived while this note was written; I compared its remarks on this part with the Lean text, and they agree with what is stated here. The program check of the unit's dossier covered 145 leaf steps of 39 proofs.

Appendix A. Concordance with the Lean artifact

The formal names of the objects: `D.logX`, `D.logX₁`, `D.logB` are $\log x(D)$, $\log x_1(D)$, $\log b(D)$; `D.s`, `D.m`, `D.r` are $\mu(D)$, $m(D)$, $r(D)$; `D.g i` is $g_i(D)$ and `D.piG` is $\pi_g(D)$; `D.Φ` is $\mathrm{Pl}(D)$; `P.c` is $C(P)$; `primeSum` is the prime sum, `dirCost` the cost of one direction. Three statements as the artifact has them:

```
structure Data (P : Parameters) where
  /-- `logX = log x`, `logB = log b`, `logX₁ = log x₁`: the sizes are chosen on the logarithmic
  scale. -/
  (logX logB logX₁ : ℝ) (s m r : ℕ) s_le : s ≤ 32 * logX
  thick : Finset _
  g : Fin 2 → ℕ
  piG : 𝒪 P.K
  scale : 1e12 * P.c ≤ log logB ∧ 100 * log logB ≤ logX ∧ 1e6 * logX ≤ logX₁ ∧ 1e6 * logX₁ ≤ logB ∧
    6 * log logX₁ ≤ log logB
  m_mem : ↑m ∈ Icc (exp (logX / 2)) (2 * exp (logX / 2))
  r_mem : ↑r ∈ Icc (exp (5 * logX)) (3 * exp (5 * logX))
  m_pow : ∃ e, m = 2 ^ (e + 1) ∧ s = (e + 1) * 22
  r_pow : ∃ e, r = 3 ^ (e + 1)
  thick_eq_neg_one_mod {m} (d : m ∣ P.N₁ s) : m ∣ P.N₀ * 3 ^ (.log 3 s + 5) → ∀ σ ∈ thick, P.placeClass d σ = -1
  three_term : let φ := P.placeClass P.two_pow_dvd_N₁; ∀ i j, 0 < i → i < j → j ≤ 324 * s →
        ∀ σ ∈ thick, ∀ τ ∈ thick, ∀ ρ ∈ thick, φ (τ / σ) ^ j = φ (ρ / σ) ^ i → φ (τ / σ) = 1
  g_prime i : (g i).Prime
  g_modEq_one i : g i ≡ 1 [MOD P.N₁ s * P.d]
  dvd_g_sub_one i : P.Mω s ^ 2 ∣ g i - 1
  g_injective : g.Injective
```

```
  g_mem i : ↑(g i) ∈ Icc (exp logB ^ 10) (exp logB ^ 11)
  norm_piG : piG * star piG = g 0 * g 1
  exists_prime_window : ∀ U ∈ Icc (100 * logB) (2 ^ s), ∃ p, p.Prime ∧ p ≡ 1 [MOD P.Mω s ^ 2] ∧
    log p ∈ Icc U (U + √logB)

lemma exists_data P {ε : ℝ} (e0 : 0 < ε / 6) (e1 : ε < 1 / 100) : ∀ᶠ x in atTop, ∃ D : Data P,
    D.x = x ∧ log D.x₁ = x ^ (ε / 6) ∧ (1e6 * log x) ^ 2 ≤ log D.x₁ ∧
    log (log D.b) = ε * log x ∧ towerDeg D.s / log D.b ≤ #D.thick

protected lemma le_of_hasLogBound : F ≤ G
```

In this note	Lean declaration	File	Lines
Definition 2.1	`towerDeg`	Different.lean	41–41
Definition 2.1	`Parameters.N₁`	Different.lean	44–44
Definition 2.1	`Parameters.K`	Different.lean	64–64
Definition 2.1	`Parameters.Mω`	Different.lean	68–68
Definition 2.1	`Parameters.E`	Different.lean	79–79
Definition 2.1	`Parameters.Φ`	Different.lean	83–83
Definition 2.1	`Parameters.placeOfInt`	Different.lean	135–135
Definition 2.1	`Parameters.placeClass`	Different.lean	139–139
Definition 2.1	`Parameters.bigOmega`	Different.lean	87–87
Definition 2.2	`Parameters.c`	AdmissibleData.lean	15–17
Definition 2.3	`Data`	AdmissibleData.lean	22–45
Definition 2.4	`Data.x`	AdmissibleData.lean	52–52
Definition 2.4	`Data.x₁`	AdmissibleData.lean	55–55
Definition 2.4	`Data.b`	AdmissibleData.lean	58–58
Definition 2.5	`Data.E`	AdmissibleData.lean	62–62
Definition 2.5	`Data.Φ`	AdmissibleData.lean	66–66
Definition 2.5	`Data.N₁`	AdmissibleData.lean	69–69
Definition 2.5	`Data.f`	AdmissibleData.lean	72–72
Definition 2.5	`Data.cycloConductor`	AdmissibleData.lean	75–75
Definition 2.5	`Data.rootDisc`	AdmissibleData.lean	78–78
Definition 2.5	`Data.placeOfInt`	AdmissibleData.lean	113–113
Definition 2.5	`Data.tops`	AdmissibleData.lean	127–127
Definition 2.5	`Data.gProd`	AdmissibleData.lean	124–124
Definition 2.6	`Data.k`	AdmissibleData.lean	82–82
Definition 2.6	`Data.t`	AdmissibleData.lean	85–85
Definition 2.6	`Data.n`	AdmissibleData.lean	116–116
Definition 2.6	`Data.posRank`	AdmissibleData.lean	119–119
Definition 2.6	`Data.h₂`	AdmissibleData.lean	92–92
Definition 2.6	`Data.h`	AdmissibleData.lean	95–95
Definition 2.6	`Data.reducedHalfGap`	AdmissibleData.lean	98–98
Definition 2.7	`Data.roundUp`	AdmissibleData.lean	88–88
Definition 2.7	`Data.a`	AdmissibleData.lean	101–101
Definition 2.7	`Data.b₁`	AdmissibleData.lean	105–105
Definition 2.7	`Data.b₂`	AdmissibleData.lean	109–109
Definition 2.7	`Data.w`	AdmissibleData.lean	131–131
Definition 2.7	`Data.nominalWeight`	AdmissibleData.lean	134–134
Definition 2.8	`Data.primeSum`	AdmissibleData.lean	138–139
Definition 2.8	`Data.budget`	AdmissibleData.lean	143–143
Definition 2.8	`Data.jetSize`	AdmissibleData.lean	147–147
Definition 2.8	`Data.dirCost`	AdmissibleData.lean	151–151
Definition 3.1	`Data.HasLogBound`	Magnitudes.lean	18–19
Definition 3.1	`Data.HasLogUpper`	Magnitudes.lean	38–38
Definition 3.1	`Data.HasLogLower`	Magnitudes.lean	41–41
Lemma 3.2	`Data.HasLogBound.pos`	Magnitudes.lean	43–43
Lemma 3.2	`Data.hasLogUpper_of_abs_le`	Magnitudes.lean	94–94

In this note	Lean declaration	File	Lines
Lemma 3.2	`Data.hasLogUpper_of_le_of_nonneg`	Magnitudes.lean	96–97
Lemma 3.2	`Data.hasLogUpper_of_le`	Magnitudes.lean	111–111
Lemma 3.2	`Data.hasLogLower_of_le`	Magnitudes.lean	113–113
Lemma 3.3	`Data.HasLogBound.of_mem_Icc`	Magnitudes.lean	49–51
Theorem B	`Data.lt_of_hasLogBound`	Magnitudes.lean	117–117
Theorem B	`Data.le_of_hasLogBound`	Magnitudes.lean	121–121
Theorem B	`Data.nat_lt_of_hasLogBound`	Magnitudes.lean	123–123
Theorem B	`Data.nat_le_of_hasLogBound`	Magnitudes.lean	125–125
Proposition 3.4	`Data.hasLogBound_natCast`	Magnitudes.lean	53–54
Proposition 3.4	`Data.hasLogBound_mul`	Magnitudes.lean	63–63
Proposition 3.4	`Data.hasLogBound_add`	Magnitudes.lean	65–66
Proposition 3.4	`Data.hasLogBound_f`	Magnitudes.lean	155–156
Lemma 3.5	`Data.c_ge`	Magnitudes.lean	21–22
Lemma 3.6	`Data.roundUp_mem_Icc`	Magnitudes.lean	169–171
Lemma 3.7	`Data.r_odd`	Magnitudes.lean	27–27
Lemma 3.7	`Data.m_even`	Magnitudes.lean	30–30
Lemma 3.7	`Data.le_m`	Magnitudes.lean	137–137
Lemma 3.7	`Data.r_add`	Magnitudes.lean	139–139
Lemma 3.7	`Data.k_le_r`	Magnitudes.lean	141–141
Lemma 3.7	`Data.hundred_mul_m_le_k`	Magnitudes.lean	195–195
Lemma 3.7	`Data.k_pos`	Magnitudes.lean	197–197
Lemma 3.7	`Data.n_mono`	Magnitudes.lean	199–199
Lemma 3.7	`Data.k_even`	Magnitudes.lean	201–201
Lemma 3.7	`Data.t_odd`	Magnitudes.lean	205–205
Lemma 3.7	`Data.r_add_six_le_t`	Magnitudes.lean	207–207
Lemma 3.7	`Data.k_add_t_add_one_le_h₂`	Magnitudes.lean	210–210
Lemma 3.8	`Data.placeClass_of_thick`	Magnitudes.lean	34–35
Lemma 4.1	`Data.nominalWeight_mem_Icc`	Weights.lean	14–16
Lemma 4.1	`Data.nominalWeight_nonneg`	Weights.lean	21–21
Lemma 4.1	`Data.nominalWeight_add_le`	Weights.lean	25–27
Lemma 4.1	`Data.nominalWeight_le_w_sub`	Weights.lean	29–30
Lemma 4.1	`Data.w_le_nominalWeight_add`	Weights.lean	36–37
Lemma 4.1	`Data.w_le_two_mul_b`	Weights.lean	40–41
Lemma 4.1	`Data.log_w_le`	Weights.lean	51–54
Lemma 5.1	`Data.chosenEquiv`	ChosenPlaceLevels.lean	18–19
Lemma 5.1	`Data.card_chosenPlace`	ChosenPlaceLevels.lean	23–24
Lemma 5.1	`Data.prod_q_chosenPlace`	ChosenPlaceLevels.lean	38–39
Lemma 5.1	`Data.isChosen_star_iff`	ChosenPlaceLevels.lean	46–47
Lemma 5.1	`Data.IsChosen.not_star`	ChosenPlaceLevels.lean	81–82
Lemma 5.1	`Data.range_chosen`	ChosenPlaceLevels.lean	59–60
Lemma 5.1	`Data.isConjApart_chosen`	ChosenPlaceLevels.lean	56–57
Lemma 5.2	`Data.v_N_eq_exp_neg_one`	ChosenPlaceLevels.lean	26–28
Lemma 5.2	`Data.intValuation_pi`	ChosenPlaceLevels.lean	30–32
Definition 5.3	`Data.chosenGroupOrder`	ChosenPlaceLevels.lean	36–36
Definition 5.3	`Data.chosenLevel`	ChosenPlaceLevels.lean	44–44
Lemma 5.4	`Data.card_GL_dvd_chosenGroupOrder`	ChosenPlaceLevels.lean	51–54
Lemma 5.4	`Data.principalLevel_le_chosenLevel`	ChosenPlaceLevels.lean	62–63
Proposition 5.5	`Data.classAut_eq_bot_of_le_chosenLevel`	ChosenPlaceLevels.lean	67–70
Proposition 5.5	`Data.card_classAut_dvd_chosenGroupOrder`	ChosenPlaceLevels.lean	75–79
Lemma 5.6	`Data.omega_zpow_ne_one`	ChosenPlaceLevels.lean	86–90
Proposition 6.1	`exists_three_term_free`	DataExist.lean	20–36
Lemma 6.2	`exists_mul_principal`	DataExist.lean	41–45
Theorem A	`exists_data`	DataExist.lean	49–108

Notation

$P = (N_0, d)$ parameters
D admissible data for P (Definition 2.3)
$x(D)$, $x_1(D)$, $b(D)$ the three scales
$\mu(D)$, $m(D)$, $r(D)$ the exponent, the number of lifts, the rank of the rows
$\mathrm{thick}(D)$ the set of thick places
$g_i(D)$, $g(D)$, $\pi_g(D)$ the two auxiliary primes, their product, an integer of that norm
$C(P)$, C_{jet} the constant of the parameters and one of its summands
$\mathrm{tw}(\mu)$, $N_1(P,\mu)$, $M_\omega(P,\mu)$ as in the first part
$K(P)$, $\chi_{K(P)}$, $E(P,\mu)$, $\mathrm{Pl}(P,\mu)$ as in the first part
$[\nu]_{P,\mu}$, $\mathrm{cl}_P^{m'|N_1(P,\mu)}$ the place named by ν; the class modulo a divisor m'
$\Omega(P,\mu)$ the period
$N_1(D)$, $E(D)$, $\mathrm{Pl}(D)$ order, field and places for the exponent of D
$f(D)$, $\mathrm{rd}(D)$ the number of infinite places; the root discriminant
$[j]_D$, $\mathrm{tops}(D)$ the place named by j; the tops
$k(D)$, $t(D)$, $n_j(D)$, $\mathrm{pos}_j(D)$ ranks
$h_2(D)$, $h(D)$, $h_2'(D)$ the half-gap, the gap, the reduced half-gap
$\lceil y \rceil_D$ y rounded up to a multiple of $N_1(D)$
$a(D)$, $b_1(D)$, $b_2(D)$ the order and the two rounded quotients
$w_D(\tau)$, $w_{D,j}(\tau)$ nominal weight; weight
$\mathrm{Ask}_D(y,\tau)$, $\lambda_{d,1}$, Λ the prime sum; its coefficients; von Mangoldt's function
$\mathrm{budget}(D)$, $\mathrm{jet}_D(j)$, $\mathsf{U}(D)$ budget; jet size; cost of one direction
$\mathrm{Bd}^*_D(F;l)$, $\mathrm{Bd}^\top_D$, $\mathrm{Bd}^\bot_D$ bounds on the logarithmic scale; $* \in \{\top, \bot\}$
$N(D)$, $\pi_N(D)$ the level and an integer of that norm
$\mathrm{Ch}_i(D)$, ω_D chosen places above $g_i(D)$; residue characters
$G_{\mathrm{f}}(D,I)$, $K_0(D,I)$, $\Gamma(D,I)$, $K_{\mathrm{ch}}(D,I)$ adelic unitary group, integral subgroup, rational points, level of the chosen places
$\mathrm{ord}_{\mathrm{ch}}(D)$ the order of Definition 5.3
$\#\,\mathrm{Cl}(K(P))$ the class number of $K(P)$
F, G, H real numbers; F also a number field in Lemma 6.2
l, l', l'' exponents
A, B natural numbers; A also the exponent in (J6)
y, y' real numbers; y the scale in Theorem A
Y, W $\log y$ and $y^{\varepsilon/6}$ in the proof of Theorem A
ε the exponent of Theorem A
τ, τ', τ'' infinite places, or elements of the set T
ν an integer
i, j, k, n, o, q natural numbers and integers: indices, exponents, orders
i', i_0, i_2, i_3 exponents in two proofs
m, r, μ in the proof of Theorem A, the components under construction
m' a divisor of the order of the root of unity ((D5), Definition 2.1, proof of Theorem A)
p, p_i prime numbers
M, L a side and a bound (Proposition 6.1); M also the modulus of (J6)
M_3 in the proof of Theorem A: N_0 times the power of 3 in the order of the root of unity
$\Gamma_1, \Gamma_2, \psi, \tau_0, \gamma$ groups, a homomorphism, two elements; ψ also a class map; τ_0 also the place named by -1
T, S a finite set of group elements; a sphere of integer vectors
u, v, z, $[u]$ integer vectors and the bracket of the proof of Proposition 6.1
$\mathfrak{p}_i, \mathfrak{q}_i$, π prime ideals; a generator
w, $\overline{w}$, $\mathrm{N}(w)$ a finite place, its conjugate, its norm
I, $\mathcal{K}$, x, a an index set, a subgroup, a group element (also a root of unity in Definition 2.2), an integer
φ, π, e, L Euler's function; the circle constant in 2π; the base of exp; in "L-function"
$\mathrm{cs}(d)$, rd, Ψ_{N_0} as in the first part

References

References

[1] F. A. Behrend, Proc. Nat. Acad. Sci. 32 (1946). As cited by the artifact; not opened.

[2] H. L. Montgomery, R. C. Vaughan, *Multiplicative Number Theory I. Classical Theory*, Cambridge University Press, 2007. §11.3, Corollary 11.19 (p. 381) and Corollary 11.21 (p. 382), located by the reading agent; not opened for this note.

[3] J.-P. Serre, Rigidité du foncteur de Jacobi d'échelon $n \geq 3$, Séminaire Henri Cartan 13 (1960/61), appendix. As recorded in the catalogue of the write-up, place not confirmed.

[4] A. Shapira, Behrend-type constructions for sets of linear equations, Acta Arith. (2006), Theorem 1. As cited by the artifact; not opened.

Chapter IV

Heights and twisted lattices

- **Heights of families indexed by ℓ-adic embeddings, the product formula, and the height of one Gram determinant** (unit 135)
  read: note built, structured proofs checked. 18 declarations, 71 proof lines.

Heights and the product formula; few translates nearly span; the height of one Gram determinant

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. A *height bound* is defined for families $Z = (Z_{\ell,\iota})$ of complex numbers indexed by the pairs of a prime ℓ and a field isomorphism $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$: one auxiliary number $H \neq 0$ may be multiplied in, and the bound adds the logarithmic size of all conjugates of H to the ℓ-adic sizes of the numbers $H\,Z_{\ell,\iota}$, summed over ℓ. Such bounds add in products, and the product formula, applied to the norm of H, shows that a bound for the constant family 1 is non-negative (Theorem 1). This lets the argument compare ℓ-adic estimates that are obtained with a different auxiliary object at every embedding. Theorem 2 bounds the height of a family of ratios of Gram determinants, a twisted one at each embedding over one untwisted one, by the size of the untwisted determinant, plus twice an index, plus ν times a height bound for a family of scalars. The unit also holds linear algebra for another part of the argument: if the translates of a linear map are spread, few of them nearly span the target (Theorem 3); and two inequalities, one of Fourier type over a finite field, one of Hilbert–Schmidt type.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles with Lean v4.35.0 against a pinned version of the library Mathlib and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note treats one unit of the artifact, the folder on heights: all 18 of its 18 declarations. Their proofs are given in full. Taken for granted are three results of the background library of the artifact, stated as Inputs 1 to 3 with the units that own them, and standard facts of Mathlib, which are named where they are used. Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 27 declarations.

1 Heights, and the three main statements

Throughout, ℓ is a prime number, $\mathbb{Q}_\ell$ is the field of ℓ-adic numbers, $\bar{\mathbb{Q}}_\ell$ is an algebraic closure of $\mathbb{Q}_\ell$, and $\mathbb{C}_\ell$ is the completion of $\bar{\mathbb{Q}}_\ell$. On these three fields $|\cdot|_\ell$ is the absolute value with $|\ell|_\ell = 1/\ell$, and $|\cdot|$ is the usual absolute value of $\mathbb{C}$. The group $\mathrm{Aut}(\mathbb{C})$ consists of all field automorphisms of $\mathbb{C}$, continuous or not.

Definition 1.1 (the index set)**.** $\mathcal{I}$ is the set of all pairs (ℓ, ι) of a prime number ℓ and an isomorphism of rings $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$. In this note a *family indexed by* $\mathcal{I}$ is a

family $Z = (Z_{\ell,\iota})$ of complex numbers, one for each pair (ℓ, ι) in $\mathcal{I}$. Such families are multiplied entry by entry, and 1 also denotes the family all of whose entries are 1.

LEAN `PrimePadicIso`

Isomorphisms ι exist for every ℓ, because $\mathbb{C}$ and $\bar{\mathbb{Q}}_\ell$ are algebraically closed fields of characteristic 0 and of the same cardinality (Inputs 2 and 3 in §2). None of them is continuous and none is distinguished. The index set is a way to speak of all embeddings into $\bar{\mathbb{Q}}_\ell$ of all number fields at once: the restrictions of the isomorphisms ι to a number field contained in $\mathbb{C}$ are exactly its field embeddings into $\bar{\mathbb{Q}}_\ell$, again by Input 2. So for a complex number that is algebraic over $\mathbb{Q}$, its images under all ι are its conjugates in $\bar{\mathbb{Q}}_\ell$, just as its images under $\mathrm{Aut}(\mathbb{C})$ are its conjugates in $\mathbb{C}$. The entries of a family may depend on (ℓ, ι), and in the argument they do.

Definition 1.2 (height at most h)**.** Let h be a real number and $Z = (Z_{\ell,\iota})$ a family indexed by $\mathcal{I}$. We write $\mathrm{ht}(Z) \le h$, and say that Z *has height at most* h, if there are a complex number $H \neq 0$, a real number α and a sequence $\eta = (\eta_0, \eta_1, \eta_2, \dots)$ of real numbers such that

(i) $|\sigma(H)| \le \exp(\alpha)$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$;
(ii) $|\iota(H\, Z_{\ell,\iota})|_\ell \le \exp(\eta_\ell)$ for every $(\ell, \iota) \in \mathcal{I}$;
(iii) there is a natural number n_0 such that $\alpha + \sum_{\ell<n} \eta_\ell \le h$ for every natural number $n \ge n_0$, the sum being over the primes ℓ below n.

LEAN `HeightLe`

The sign $\mathrm{ht}(Z) \le h$ is the name of a relation between h and Z; no number called the height of Z is defined or needed. With $H = 1$ and $\alpha = 0$ the definition asks that the ℓ-adic sizes of the entries, in logarithms and summed over ℓ, be at most h (Lemma 3.2). The number H allows more: it is one number for all (ℓ, ι), a common factor whose size at the archimedean places is paid for in (i). The sequence η is indexed by all natural numbers, as in the formal text; only its terms at the primes matter. Two examples: for an integer $n \ge 1$, the constant family $1/n$ has height at most $\log n$ (take $H = n$, $\alpha = \log n$, $\eta = 0$), and the constant family n has height at most $-\log n$ (take $H = 1/n$, $\alpha = -\log n$, $\eta = 0$). So a height bound says how small a family is ℓ-adically, all primes together, and it may be negative. It is not the height of Diophantine geometry: no positive parts are taken, the archimedean places enter only through the auxiliary number H, and the entries of a family need not be the conjugates of one number. How negative a bound can be is limited by the first main result.

Theorem 1 (the product formula). *Let h be a real number. If the constant family 1 has height at most h, that is, if $\mathrm{ht}(1) \le h$, then $0 \le h$.*

LEAN `HeightLe.nonneg`

This is the only place of the unit where arithmetic enters. The proof (§3) shows that the number H of the definition is algebraic and applies the product formula of $\mathbb{Q}$ to its norm q, a non-zero rational number: the logarithms of the absolute values of q at all places of $\mathbb{Q}$ add up to zero, condition (i) bounds the logarithm at the archimedean place by α times the degree of H, and condition (ii) bounds the logarithm at ℓ by

η_ℓ times the degree. So $\alpha + \sum_{\ell<n} \eta_\ell$ cannot be negative once n exceeds the primes that divide the numerator or the denominator of q. The argument uses Theorem 1 together with the additivity of height bounds in products: if two families Z and Z' have entrywise product 1, then a bound for Z and a bound for Z' add up to a non-negative number (Corollary 3.5). An upper bound for the height of Z' thus becomes a lower bound for every height bound of Z, and this is how a covolume is bounded from below in the proof of Theorem C of the argument (§7).

Gram determinants. Inner products are conjugate-linear in the first variable and linear in the second, as in the library. For a finite family $v = (v_i)_{i\in I}$ of vectors of a complex inner product space, $\mathrm{Gram}(v) = (\langle v_i, v_{i'}\rangle)_{i,i'\in I}$ is its Gram matrix, and for a linear map T defined on these vectors Tv is the family $(Tv_i)_{i\in I}$. For a $\mathbb{C}$-linear map $T\colon W \to \mathcal{H}$ from a finite-dimensional complex inner product space to a complex inner product space, T^*T is the endomorphism of W whose matrix in an orthonormal basis v of W is $\mathrm{Gram}(Tv)$; it does not depend on v. Its determinant $\det(T^*T)$ is a non-negative real number, the square of the factor by which T multiplies volumes, and it is non-zero exactly when T is injective (Lemma 4.1).

Setting 1.3 (a family of linear maps, a basis, and integral structures)**.** Let W be a finite-dimensional complex inner product space, Ξ a set, and $\theta = (\theta_{\ell,\iota})$ a family of elements of Ξ, one for each pair (ℓ, ι) in $\mathcal{I}$. For every $\vartheta \in \Xi$ let $\mathcal{H}_\vartheta$ be a complex inner product space and $T_\vartheta\colon W \to \mathcal{H}_\vartheta$ a $\mathbb{C}$-linear map, and let ϑ_0 be an element of Ξ. Let I be a finite set and $w = (w_i)_{i\in I}$ a family of vectors of W. Let α, δ and h be real numbers, $c = (c_{\ell,\iota})$ a family of complex numbers indexed by $\mathcal{I}$, $\eta = (\eta_0, \eta_1, \eta_2, \dots)$ a sequence of real numbers, and ν a natural number. For every $(\ell, \iota) \in \mathcal{I}$ and every homomorphism of rings $j\colon \mathbb{C} \to \mathbb{C}_\ell$ let $\mathcal{L}_{\ell,\iota}(j)$ be a subset of $\mathcal{H}_{\theta_{\ell,\iota}}$. For such j, the homomorphism $\bar{j}\colon \mathbb{C} \to \mathbb{C}_\ell$ is j preceded by complex conjugation.

Theorem 2 (the height of one Gram determinant). *In Setting 1.3, suppose that*

(a) *the map T_{ϑ_0} is injective;*

(b) *the family w is linearly independent and $\dim W = \#I$, so that w is a basis of W;*

(c) $|\sigma(\det \mathrm{Gram}(T_{\vartheta_0} w))| \leq \exp(\alpha)$ *for every* $\sigma \in \mathrm{Aut}(\mathbb{C})$*;*

(d) $c_{\ell,\iota} \neq 0$ *for every* $(\ell,\iota) \in \mathcal{I}$*, and* $\mathrm{ht}(|c|^{-2}) \leq h$ *for the family* $|c|^{-2} = (|c_{\ell,\iota}|^{-2})$*;*

(e) $\sum_{n \in J} \eta_n \leq \delta$ *for every finite set J of natural numbers;*

(f) *for every $(\ell,\iota) \in \mathcal{I}$, every homomorphism of rings $j\colon \mathbb{C} \to \mathbb{C}_\ell$, every $F \in \mathcal{L}_{\ell,\iota}(\bar{j})$ and every $F' \in \mathcal{L}_{\ell,\iota}(j)$, one has $|j(\langle F, F' \rangle)|_\ell \leq 1$;*

(g) *for every $(\ell,\iota) \in \mathcal{I}$ and every homomorphism of rings $j\colon \mathbb{C} \to \mathbb{C}_\ell$ there is a matrix $u = (u_{ii'})_{i,i' \in I}$ of complex numbers such that*

$$\exp(-\eta_\ell) \cdot |j(c_{\ell,\iota})|_\ell^\nu \;\leq\; |j(\det u)|_\ell$$

and

$$\sum_{i' \in I} u_{ii'}\, T_{\theta_{\ell,\iota}}(w_{i'}) \in \mathcal{L}_{\ell,\iota}(j) \quad \text{for all } i \in I.$$

Then the family $Z = (Z_{\ell,\iota})$ of the ratios

$$Z_{\ell,\iota} \;=\; \frac{\det(T^*_{\theta_{\ell,\iota}} T_{\theta_{\ell,\iota}})}{\det(T^*_{\vartheta_0} T_{\vartheta_0})}$$

satisfies $\mathrm{ht}(Z) \leq \alpha + 2\delta + \nu h$.

LEAN `one_determinant`

In words: when T_ϑ is injective, the vectors $T_\vartheta w_i$ form a basis of $T_\vartheta(W)$, and $\det \mathrm{Gram}(T_\vartheta w)$ is the square of the volume of the parallelotope that they span (the squared covolume of the overview of the handout, at the identity of $\mathrm{Aut}(\mathbb{C})$). Hypothesis (c) bounds this number at the base index ϑ_0, at all its conjugates. The sets $\mathcal{L}_{\ell,\iota}(j)$ play the part of the j-integral vectors: by (f) the pairing of a $\bar{j}$-integral vector with a j-integral one is j-integral. Hypothesis (g) says that, at the index $\theta_{\ell,\iota}$ chosen for the pair (ℓ,ι), suitable combinations of the vectors $T_{\theta_{\ell,\iota}} w_i$ are integral, with a matrix of combination whose determinant is not too small: η_ℓ bounds the logarithm of an index, and $c_{\ell,\iota}$ is a scalar that ν of the vectors are allowed to carry. The conclusion compares the covolume at $\theta_{\ell,\iota}$ with the covolume at ϑ_0: the ratio of their squares is ℓ-adically small, all primes together, up to the archimedean size α, twice the index δ and the cost h of each of the ν scalars. The index counts twice because the proof needs one integral combination for the embedding j_ι of Definition 2.1 and one for its conjugate. The square of an index appears as it does for lattices over $\mathbb{Z}$, where a lattice with integral Gram matrix that has index n in another one forces n^2 times the Gram determinant of the latter to be an integer.

Spread families. The third statement is linear algebra over $\mathbb{C}$.

Definition 1.4 (spread families of linear maps)**.** Let U and W be complex vector spaces, Ω a set, $B = (B_\gamma)_{\gamma \in \Omega}$ a family of $\mathbb{C}$-linear maps $B_\gamma\colon U \to W$, and ε a real

number. The family B is *ε-spread*, in symbols $\mathrm{Spread}_\varepsilon(B)$, if for every linear subspace $W' \subseteq W$ there is a $\gamma \in \Omega$ with

$$\dim B_\gamma^{-1}(W') \;\le\; \Big(\frac{1}{2}\Big(1+\frac{\dim W'}{\dim W}\Big)+\varepsilon\Big)\cdot \dim U.$$

LEAN `IsSpread`

If B_γ is injective and its image is in general position with respect to W', the preimage of W' has about the share $\dim W'/\dim W$ of the dimension of U. The definition asks less: for every W' some member of the family must pull W' back to at most the mean of this share and 1, up to ε.

Theorem 3 (few translates nearly span). *Let U be a finite-dimensional complex vector space with $U \neq 0$, let W be a finite-dimensional complex inner product space, Ω a set, $B = (B_\gamma)_{\gamma\in\Omega}$ a family of $\mathbb{C}$-linear maps $B_\gamma\colon U \to W$, and $\varepsilon > 0$ a real number. If $\mathrm{Spread}_\varepsilon(B)$, then there is a finite subset $\mathcal{T} \subseteq \Omega$ with*

$$\#\mathcal{T}\cdot\dim U \;\le\; 5\,\dim W\cdot\log(1+\varepsilon^{-1})$$

and

$$(1-4\varepsilon)\,\dim W \;\le\; \dim\sum_{\gamma\in\mathcal{T}} B_\gamma(U).$$

LEAN `exists_few_translates`

Each image $B_\gamma(U)$ has dimension at most $\dim U$, so at least $(1-4\varepsilon)\dim W/\dim U$ members are needed to span a subspace of dimension $(1-4\varepsilon)\dim W$. The theorem says that this least number, times $5\log(1+\varepsilon^{-1})/(1-4\varepsilon)$, suffices. In the argument the maps B_γ are the translates of one map by the elements γ of a finite group, whence the name.

2 The setting

Conventions. The natural numbers are $0, 1, 2, \dots$, and $\#I$ is the number of elements of a finite set I. The letter ℓ always denotes a prime number; a sum $\sum_{\ell<n}$ runs over the primes below n. Vector spaces are over $\mathbb{C}$ unless a finite field is named. A *complex inner product space* is a complex vector space with a positive definite hermitian form $\langle\cdot,\cdot\rangle$, conjugate-linear in the first variable, and $\|F\|^2 = \langle F, F\rangle$; completeness is not assumed. For a finite set I, the space $\mathbb{C}^I$ has the inner product for which the standard basis $\mathbf{1} = (\mathbf{1}_i)_{i\in I}$ is orthonormal. The tensor product of two complex inner product spaces carries the inner product with $\langle F_1\otimes F_2, F_1'\otimes F_2'\rangle = \langle F_1, F_1'\rangle\langle F_2, F_2'\rangle$. For a finite-dimensional subspace $\mathcal{H}'$ of an inner product space $\mathcal{H}$, $p_{\mathcal{H}'}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}'$. The functions of the library are defined everywhere: the logarithm of a real number q is $\log|q|$ for $q \neq 0$ and 0 for $q = 0$; a quotient with denominator 0 is 0; and the dimension of a vector space that is not finite-dimensional is 0. These conventions enter two statements only, Lemma 3.4 for $q = 0$ and Definition 1.4, which the formal text states without assuming finite dimension. The declarations of this unit take no ambient object that is fixed once for the whole note: each statement below introduces the spaces, sets and maps it is about.

Letters local to this note. Sections 1 to 6 do not depend on the data of the proof, and ten letters have there a meaning of their own, recorded in the notation list: ξ and ρ (vectors with coordinates in a finite set), τ and τ_0 (homomorphisms of rings), c (a family of scalars), q (a rational number), Ψ (an additive character), ω (a function on a group), Γ_ϑ (a Gram determinant) and Ω (an index set). In the overview of the handout and in Chapter III these letters denote Dirichlet characters, zeros, infinite places, a residue class, the rational number of Theorem A, cyclotomic polynomials, residue characters, groups of rational points and a period. None of those objects occurs in these sections, and of the local letters only c returns in Section 7, in its role of Theorem 2.

Definition 2.1 (the embedding attached to ι)**.** For a prime number ℓ and an isomorphism of rings $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$, the homomorphism of rings $j_\iota\colon \mathbb{C} \to \mathbb{C}_\ell$ is ι followed by the inclusion of $\bar{\mathbb{Q}}_\ell$ in $\mathbb{C}_\ell$. So $|j_\iota(H)|_\ell = |\iota(H)|_\ell$ for every complex number H.

LEAN `complexToCp` · a definition of the background library, in the unit *Lib.ClassField.Places*

Inputs from the background library. The three results below are taken for granted in this note. They are theorems of the background library of the artifact, which proves them from Mathlib; the units that own them are named, and none of these units has been read yet.

Input 1 (a number bounded at all 2-adic identifications is algebraic; unit *Lib.CM.Genus*). *Let H be a complex number and Y a real number. If $|j_\iota(H)|_2 \le Y$ for every isomorphism of rings $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_2$, then H is algebraic over $\mathbb{Q}$.*

Input 2 (embeddings extend to isomorphisms; unit *Lib.General.LinearAlgebra*). *Let L be an algebraically closed field of the same cardinality as $\mathbb{C}$, and let τ_0 and τ be injective homomorphisms of rings from one countable commutative ring to $\mathbb{C}$ and to L, respectively. Then there is an isomorphism of rings $\iota\colon \mathbb{C} \to L$ with $\iota \circ \tau_0 = \tau$.*

Input 3 (cardinality; unit *Lib.PadicHodge.PeriodRings*). *For every prime number ℓ, the field $\bar{\mathbb{Q}}_\ell$ has the same cardinality as $\mathbb{C}$.*

LEAN `isAlgebraic_of_padic` · `exists_ringEquiv_extend` · `mk_complex_eq_mk_padicAlgCl`

Input 2 is a form of Steinitz's theorem that an algebraically closed field is determined by its characteristic and its transcendence degree: the catalogue of this handout records it under Steinitz [St] as a special case, that of two algebraically closed fields of characteristic 0 and of the cardinality of the continuum, with an isomorphism that extends a given one between countable subrings; the place in [St] is not confirmed. For Inputs 1 and 3 the catalogue has no reference: reference to be confirmed. Both are elementary, the first once Input 2 is granted. Input 3 holds because $\mathbb{Q}_\ell$ has the cardinality of the continuum and an algebraic extension of an infinite field has the cardinality of that field. Input 1 has a short formal proof, which was read: if H were transcendental, one would choose an element of $\bar{\mathbb{Q}}_2$ that is transcendental over $\mathbb{Q}$ and of absolute value greater than Y (the algebraic elements are countable, so the others are dense), and Input 2, applied to the two embeddings of the polynomial ring over $\mathbb{Q}$ in one variable that send the variable to H and to this element, would give an ι violating the bound.

Facts of Mathlib. The proofs below also use standard facts that the formal proofs take from Mathlib, the mathematical library of Lean: unique factorisation of integers; that the image of the norm of a finite separable field extension in an algebraically closed field is the product over the embeddings; that $|\cdot|_\ell$ on $\bar{\mathbb{Q}}_\ell$ restricts to the ℓ-adic absolute value on $\mathbb{Q}$; the properties of Gram matrices collected in Lemma 4.1, where a short proof is given; Bessel's inequality, Parseval's identity and Pythagoras' theorem in inner product spaces; the trace of an endomorphism as the sum of its diagonal coefficients in an orthonormal basis; the existence of a non-trivial additive character of a finite field; and elementary linear algebra. Their formal statements were not read: Mathlib is not part of the artifact, and its pinned version was not at hand when this note was written. The dossier of the unit records these facts from their names and from the way they are used.

3 Height bounds and the product formula

In this section Z and Z' are families indexed by $\mathcal{I}$ and h, h' are real numbers; a triple (H, α, η) as in Definition 1.2 is called a *witness* of $\mathrm{ht}(Z) \le h$.

Lemma 3.1 (monotonicity)**.** *Let h and h' be real numbers and Z, Z' families indexed by $\mathcal{I}$. If $\mathrm{ht}(Z) \le h$, if $h \le h'$, and if $|\iota(Z'_{\ell,\iota})|_\ell \le |\iota(Z_{\ell,\iota})|_\ell$ for every $(\ell,\iota) \in \mathcal{I}$, then $\mathrm{ht}(Z') \le h'$.*

Proof. A witness (H, α, η) of $\mathrm{ht}(Z) \le h$ is a witness of $\mathrm{ht}(Z') \le h'$: conditions (i) and (iii) are unchanged or weaker, and

$$|\iota(HZ'_{\ell,\iota})|_\ell = |\iota(H)|_\ell |\iota(Z'_{\ell,\iota})|_\ell \le |\iota(HZ_{\ell,\iota})|_\ell \le \exp(\eta_\ell),$$

because ι is a homomorphism of rings and $|\cdot|_\ell$ is multiplicative. □

LEAN `HeightLe.mono`

Lemma 3.2 (families of bounded entries)**.** *Let h be a real number, Z a family indexed by $\mathcal{I}$, and $\eta = (\eta_0, \eta_1, \eta_2, \dots)$ a sequence of real numbers. If $\sum_{\ell \in J} \eta_\ell \le h$ for every finite set J of prime numbers, and if $|\iota(Z_{\ell,\iota})|_\ell \le \exp(\eta_\ell)$ for every $(\ell,\iota) \in \mathcal{I}$, then $\mathrm{ht}(Z) \le h$.*

Proof. The triple $(1, 0, \eta)$ is a witness: $\sigma(1) = 1$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, condition (ii) is the second hypothesis, and (iii) holds for every n by the first hypothesis, applied to the set J of the primes below n. □

LEAN `HeightLe.of_le`

Proposition 3.3 (height bounds add in products)**.** *Height bounds have the following three properties.*

(a) *Let h and h' be real numbers and Z, Z' families indexed by $\mathcal{I}$. If $\mathrm{ht}(Z) \le h$ and $\mathrm{ht}(Z') \le h'$, then $\mathrm{ht}(ZZ') \le h + h'$.*

(b) *Let Ω be a set and $I \subseteq \Omega$ a finite subset, and for every $i \in \Omega$ let h_i be a real number and Z_i a family indexed by $\mathcal{I}$. If $\mathrm{ht}(Z_i) \le h_i$ for every $i \in I$, then $\mathrm{ht}(\prod_{i \in I} Z_i) \le \sum_{i \in I} h_i$.*

(c) *Let h be a real number, Z a family indexed by $\mathcal{I}$ with $\mathrm{ht}(Z) \le h$, and n a natural number. Then $\mathrm{ht}(Z^n) \le n\,h$.*

Proof. (a) Let (H, α, η) and (H', α', η') be witnesses, with thresholds n_0 and n_0' in (iii). Then $HH' \neq 0$, $|\sigma(HH')| \leq \exp(\alpha + \alpha')$ for every σ, and

$$|\iota(HH' Z_{\ell,\iota} Z'_{\ell,\iota})|_\ell = |\iota(H Z_{\ell,\iota})|_\ell \cdot |\iota(H' Z'_{\ell,\iota})|_\ell \leq \exp(\eta_\ell + \eta'_\ell) \qquad \text{for every } (\ell, \iota) \in \mathcal{I}.$$

For $n \geq n_0 + n_0'$ both inequalities (iii) hold, and their sum is the inequality (iii) for the triple $(HH', \alpha + \alpha', \eta + \eta')$ and the bound $h + h'$.

(b) By induction on the number of elements of I. If I is empty, the product is the constant family 1 and the sum is 0, and $\mathrm{ht}(1) \leq 0$ by Lemma 3.2 with $\eta = 0$. The step from I to I with one more element is (a).

(c) This is (b) for a set I of n elements, with all h_i equal to h and all Z_i equal to Z. □

LEAN `HeightLe.mul` · `HeightLe.prod` · `HeightLe.pow`

Lemma 3.4 (the product formula for $\mathbb{Q}$)**.** *Let n be a natural number and q a rational number, with numerator q_{num} and denominator $q_{\mathrm{den}} \geq 1$ in lowest terms (for $q = 0$ these are 0 and 1). If $\max(|q_{\mathrm{num}}|, q_{\mathrm{den}}) < n$, then*

$$\log|q| + \sum_{\ell < n} \log|q|_\ell = 0,$$

where $|q|_\ell$ is the ℓ-adic absolute value of q, the sum is over the primes ℓ below n, and for $q = 0$ all the logarithms are read as 0.

Proof. For $q = 0$ there is nothing to prove. Let $q \neq 0$. A prime that divides q_{num} or q_{den} is at most $\max(|q_{\mathrm{num}}|, q_{\mathrm{den}})$, hence below n. Write ord_ℓ for the exponent of ℓ in an integer. By unique factorisation,

$$\log|q_{\mathrm{num}}| = \sum_{\ell < n} \mathrm{ord}_\ell(q_{\mathrm{num}}) \log \ell \qquad \text{and} \qquad \log q_{\mathrm{den}} = \sum_{\ell < n} \mathrm{ord}_\ell(q_{\mathrm{den}}) \log \ell.$$

Since $\log|q|_\ell = (\mathrm{ord}_\ell(q_{\mathrm{den}}) - \mathrm{ord}_\ell(q_{\mathrm{num}})) \log \ell$, the sum of the numbers $\log|q|_\ell$ over the primes $\ell < n$ is $\log q_{\mathrm{den}} - \log|q_{\mathrm{num}}| = -\log|q|$. □

LEAN `log_product_formula`

Proof of Theorem 1. Let (H, α, η) be a witness of $\mathrm{ht}(1) \leq h$, with threshold n_0 in (iii). Here (ii) reads $|\iota(H)|_\ell \leq \exp(\eta_\ell)$ for all $(\ell, \iota) \in \mathcal{I}$.

The number H is algebraic. By (ii) for $\ell = 2$ and Definition 2.1, $|j_\iota(H)|_2 \leq \exp(\eta_2)$ for every isomorphism $\iota \colon \mathbb{C} \to \bar{\mathbb{Q}}_2$, and Input 1 applies. So $\mathbb{Q}(H)$ is a number field. Write $\deg H$ for its degree over $\mathbb{Q}$, and let $q = \mathrm{Nm}_{\mathbb{Q}(H)/\mathbb{Q}}(H)$ be the norm of H, a rational number, non-zero because $H \neq 0$.

Local bounds. We claim that

$$\log|q| \leq \deg H \cdot \alpha \qquad \text{and} \qquad \log|q|_\ell \leq \deg H \cdot \eta_\ell \quad \text{for every prime } \ell. \tag{1}$$

Fix ℓ. The image of q in $\bar{\mathbb{Q}}_\ell$ is the product of the numbers $\tau(H)$ over the $\deg H$ field embeddings $\tau \colon \mathbb{Q}(H) \to \bar{\mathbb{Q}}_\ell$. Each τ is the restriction of an isomorphism $\iota \colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$: this is Input 2 for the field $L = \bar{\mathbb{Q}}_\ell$, which has the cardinality of $\mathbb{C}$ by Input 3, for the countable ring $\mathbb{Q}(H)$, for its inclusion τ_0 in $\mathbb{C}$, and for τ. Hence $|\tau(H)|_\ell = |\iota(H)|_\ell \leq \exp(\eta_\ell)$ by (ii), and $|q|_\ell \leq \exp(\deg H \cdot \eta_\ell)$, because $|\cdot|_\ell$ on $\bar{\mathbb{Q}}_\ell$ restricts to the ℓ-adic absolute value on $\mathbb{Q}$. The first inequality of (1) is proved in the same way with $\mathbb{C}$ in the

place of $\bar{\mathbb{Q}}_\ell$: the embeddings $\mathbb{Q}(H) \to \mathbb{C}$ are the restrictions of the automorphisms $\sigma \in \mathrm{Aut}(\mathbb{C})$ by Input 2 for $L = \mathbb{C}$, and (i) bounds each $|\sigma(H)|$ by $\exp(\alpha)$.

The product formula. Put $n = n_0 + \max(|q_{\mathrm{num}}|, q_{\mathrm{den}}) + 1$. By (iii), by (1) and by Lemma 3.4,

$$\deg H \cdot h \;\geq\; \deg H \cdot \Big(\alpha + \sum_{\ell<n} \eta_\ell\Big) \;\geq\; \log|q| + \sum_{\ell<n} \log|q|_\ell \;=\; 0,$$

and $\deg H \geq 1$. □

LEAN `HeightLe.nonneg` · uses `isAlgebraic_of_padic`, `exists_ringEquiv_extend`, `mk_complex_eq_mk_padicAlgCl` and `log_product_formula`

Corollary 3.5 (the form in which Theorem 1 is used)**.** *Let h and h' be real numbers and Z, Z' families indexed by $\mathcal{I}$ with $Z_{\ell,\iota} Z'_{\ell,\iota} = 1$ for every $(\ell, \iota) \in \mathcal{I}$. If $\mathrm{ht}(Z) \leq h$ and $\mathrm{ht}(Z') \leq h'$, then $h + h' \geq 0$.*

Proof. By Proposition 3.3(a), $\mathrm{ht}(1) \leq h + h'$; now apply Theorem 1. □

The corollary is not a declaration of the formal text; it is the last step of the formal proof of Theorem C of the argument, where it is applied to the square of one family and to a second family (§7).

Remark 3.6 (constant families)**.** The proof of Theorem 1 gives more, by the same steps. Let Z be the constant family with value a non-zero algebraic number, and let $\bar{h}$ be the mean of the logarithms of the absolute values of its conjugates in $\mathbb{C}$. If $\mathrm{ht}(Z) \leq h$, then $h \geq -\bar{h}$. Indeed, for a witness (H, α, η) the number H is again algebraic, and one applies the product formula to the norm of H times the value of Z, taken from the number field that the two generate. So a height bound for a constant family cannot fall below what the product formula allows, and the two examples of §1 are best possible. This remark is ours; the formal text has only the case of the constant family 1.

4 The height of one Gram determinant

Lemma 4.1 (Gram matrices)**.** *Let I be a finite set and $F = (F_i)_{i \in I}$ a family of vectors of a complex inner product space.*

(1) $\det \mathrm{Gram}(F) \neq 0$ *if and only if the family F is linearly independent.*

(2) *For matrices $u = (u_{ii'})$ and $u' = (u'_{ii'})$ of complex numbers indexed by $I \times I$, put $F'_i = \sum_{i_1 \in I} u'_{ii_1} F_{i_1}$ and $F''_i = \sum_{i_1 \in I} u_{ii_1} F_{i_1}$ for $i \in I$. Then*

$$\big(\langle F'_i, F''_{i'} \rangle\big)_{i,i' \in I} \;=\; \overline{u'} \cdot \mathrm{Gram}(F) \cdot u^{\mathrm{t}},$$

where $\overline{u'}$ is the matrix of the complex conjugates of the entries of u' and u^{t} is the transpose of u.

(3) *Let W be a finite-dimensional complex inner product space with $\dim W = \#I$, let $w = (w_i)_{i \in I}$ be a family of vectors of W, and let T be a $\mathbb{C}$-linear map from W to a complex inner product space. Then $\det \mathrm{Gram}(Tw) = \det(T^*T) \cdot \det \mathrm{Gram}(w)$. In particular $\det(T^*T) \neq 0$ if and only if T is injective.*

Proof. Write $[F]$ for the linear map from $\mathbb{C}^I$ that sends the standard basis vector $\mathbf{1}_i$ to F_i. Then $\mathrm{Gram}(F)$ is the matrix of the hermitian form $(\xi,\xi') \mapsto \langle [F]\xi, [F]\xi'\rangle$ on $\mathbb{C}^I$. This form is positive semi-definite, and it is definite exactly when $[F]$ is injective, which is (1). Statement (2) is the expansion of $\langle F_i', F_{i'}''\rangle$ by sesquilinearity. For (3), choose an orthonormal basis v of W indexed by I, and let u be the matrix with $w_{i'} = \sum_{i\in I} u_{i'i} v_i$. By (2), applied to the family Tv and then to the family v, $\mathrm{Gram}(Tw) = \overline{u}\cdot \mathrm{Gram}(Tv)\cdot u^{\mathrm{t}}$ and $\mathrm{Gram}(w) = \overline{u}\cdot u^{\mathrm{t}}$. As $\mathrm{Gram}(Tv)$ is the matrix of T^*T in the basis v, taking determinants gives the formula. The last assertion follows from (1) for the family Tv. □

LEAN facts of Mathlib; the formal text writes $\det(T^*T)$ as the square of a function of Mathlib that is the volume factor of a linear map

Proof of Theorem 2. For every ϑ in Ξ put $\Gamma_\vartheta = \det \mathrm{Gram}(T_\vartheta w)$. By (b) and Lemma 4.1(1) the number $\det\mathrm{Gram}(w)$ is not zero, and by Lemma 4.1(3)

$$\Gamma_\vartheta \;=\; \det(T_\vartheta^* T_\vartheta)\cdot \det \mathrm{Gram}(w) \qquad \text{for every } \vartheta\in\Xi.$$

By (a), $\Gamma_{\vartheta_0}\neq 0$. So the family of the theorem is $Z_{\ell,\iota} = \Gamma_{\theta_{\ell,\iota}}/\Gamma_{\vartheta_0}$.

Reduction. Let Z' be the family with entries $Z'_{\ell,\iota} = Z_{\ell,\iota}\,|c_{\ell,\iota}|^{2\nu}$. Then Z is the product of Z' and of the ν-th power of the family $|c|^{-2}$, and this power has height at most νh by (d) and Proposition 3.3(c). By Proposition 3.3(a) it suffices to show that $\mathrm{ht}(Z') \le \alpha + 2\delta$. We claim that the triple $(\Gamma_{\vartheta_0}, \alpha, 2\eta)$ is a witness. Condition (i) is (c). Condition (iii) holds for every n, by (e) for the set J of the primes below n. Condition (ii) is the inequality

$$\big|\iota(\Gamma_{\theta_{\ell,\iota}}\cdot |c_{\ell,\iota}|^{2\nu})\big|_\ell \;\le\; \exp(2\eta_\ell) \qquad \text{for every } (\ell,\iota)\in\mathcal{I}, \tag{2}$$

which remains to be proved.

The ℓ-adic bound. Fix $(\ell,\iota)\in\mathcal{I}$ and put $j = j_\iota$, so that $|\iota(\,\cdot\,)|_\ell = |j(\,\cdot\,)|_\ell$. Let u be a matrix as in (g) for (ℓ,ι) and j, and u' a matrix as in (g) for (ℓ,ι) and $\bar{j}$. Since $\bar{j}$ is j preceded by complex conjugation, the two inequalities of (g) read

$$\exp(-\eta_\ell)\,|j(c_{\ell,\iota})|_\ell^\nu \le |j(\det u)|_\ell \qquad\text{and}\qquad \exp(-\eta_\ell)\,|j(\overline{c_{\ell,\iota}})|_\ell^\nu \le |j(\det \overline{u'})|_\ell. \tag{3}$$

Put $F_i' = \sum_{i_1\in I} u'_{ii_1} T_{\theta_{\ell,\iota}}(w_{i_1})$ and $F_i'' = \sum_{i_1\in I} u_{ii_1} T_{\theta_{\ell,\iota}}(w_{i_1})$ for $i\in I$. By (g) the vectors F_i' lie in $\mathcal{L}_{\ell,\iota}(\bar{j})$ and the vectors F_i'' lie in $\mathcal{L}_{\ell,\iota}(j)$, so by (f) every entry of the matrix $(\langle F_i', F_{i'}''\rangle)_{i,i'\in I}$ lies in the set O_j of the complex numbers whose image under j has absolute value at most 1. This set is a subring of $\mathbb{C}$, because $|\,\cdot\,|_\ell$ is non-archimedean; hence the determinant of the matrix lies in O_j as well. By Lemma 4.1(2) the matrix is $\overline{u'}\cdot \mathrm{Gram}(T_{\theta_{\ell,\iota}} w)\cdot u^{\mathrm{t}}$, with determinant $\det\overline{u'}\cdot\Gamma_{\theta_{\ell,\iota}}\cdot\det u$. Therefore

$$|j(\det\overline{u'})|_\ell \cdot |j(\Gamma_{\theta_{\ell,\iota}})|_\ell \cdot |j(\det u)|_\ell \;\le\; 1. \tag{4}$$

Now $|c_{\ell,\iota}|^2 = \overline{c_{\ell,\iota}}\cdot c_{\ell,\iota}$, so by (3) and then (4)

$$\big|j(\Gamma_{\theta_{\ell,\iota}}|c_{\ell,\iota}|^{2\nu})\big|_\ell \;\le\; |j(\Gamma_{\theta_{\ell,\iota}})|_\ell\cdot\exp(2\eta_\ell)\cdot|j(\det\overline{u'})|_\ell\cdot|j(\det u)|_\ell \;\le\; \exp(2\eta_\ell).$$

This is (2). □

LEAN `one_determinant` · uses `HeightLe.mul`, `HeightLe.pow`, `complexToCp`, and from the background library `IsIntMatrix` and `IsIntMatrix.det_mem`: a determinant of a matrix with entries in a subring lies in the subring

The proof uses hypotheses (f) and (g) only for the two homomorphisms j_ι and its conjugate, and it uses nothing about the sets $\mathcal{L}_{\ell,\iota}(j)$ beyond (f) and (g). The inner product of W does not matter either: by Lemma 4.1(3) the ratio $Z_{\ell,\iota}$ is a ratio of Gram determinants of images of the basis w.

Completing a family to a basis. Theorem 2 needs a basis of W. In the argument a linearly independent family is given, and it is completed by vectors of a second kind, for which hypothesis (g) holds with the scalars $c_{\ell,\iota}$. The following lemma provides the completion.

Lemma 4.2 (extension to a basis from a spanning set)**.** *Let W be a finite-dimensional complex vector space, n a natural number, S a subset of W that spans W, and $y_1, \dots, y_n$ linearly independent vectors of W. Then there are a natural number n' and vectors $z_1, \dots, z_{n'}$ in S such that $y_1, \dots, y_n, z_1, \dots, z_{n'}$ are linearly independent and $\dim W = n + n'$.*

Proof. Let W_y be the span of $y_1, \dots, y_n$; it has dimension n. The images of the elements of S span the quotient W/W_y, so there are $z_1, \dots, z_{n'}$ in S whose images form a basis of W/W_y, and $n' = \dim W - n$. A linear relation among $y_1, \dots, y_n, z_1, \dots, z_{n'}$ becomes, modulo W_y, a relation among the images of the z_i, so its coefficients at the z_i vanish; what is left is a relation among the y_i, whose coefficients vanish too. □

LEAN `exists_extend_basis_of_span_eq_top`

5 Few translates nearly span

Proof of Theorem 3. If $\dim W = 0$, the empty set has the two properties. Let $\dim W \geq 1$. For a finite subset $\mathcal{T}' \subseteq \Omega$ put

$$W_{\mathcal{T}'} = \sum_{\gamma \in \mathcal{T}'} B_\gamma(U), \qquad \operatorname{codim} W_{\mathcal{T}'} = \dim W - \dim W_{\mathcal{T}'},$$

$$\Pi(\mathcal{T}') \;=\; \#\mathcal{T}' \cdot \dim U \;+\; 5\,\dim W \cdot \log\Bigl(1 + \frac{\operatorname{codim} W_{\mathcal{T}'}}{\varepsilon\,\dim W}\Bigr).$$

Call $\mathcal{T}'$ *admissible* if $\Pi(\mathcal{T}') \leq 5\,\dim W \cdot \log(1+\varepsilon^{-1})$. The empty set is admissible, with equality. Among the admissible sets choose one, $\mathcal{T}$, for which $\dim W_{\mathcal{T}}$ is largest. The logarithm in $\Pi(\mathcal{T})$ is non-negative, so

$$\#\mathcal{T} \cdot \dim U \;\leq\; \Pi(\mathcal{T}) \;\leq\; 5\,\dim W \cdot \log(1+\varepsilon^{-1})\,:$$

this is the first inequality.

For the second, suppose that $\dim W_{\mathcal{T}} < (1-4\varepsilon)\dim W$, that is,

$$\operatorname{codim} W_{\mathcal{T}} \;>\; 4\,\varepsilon\,\dim W. \tag{5}$$

Since B is ε-spread, there is a $\gamma \in \Omega$ with

$$\dim B_\gamma^{-1}(W_{\mathcal{T}}) \;\leq\; \Bigl(\frac{1}{2}\Bigl(1 + \frac{\dim W_{\mathcal{T}}}{\dim W}\Bigr) + \varepsilon\Bigr)\dim U \;=\; \Bigl(1 - \frac{\operatorname{codim} W_{\mathcal{T}}}{2\dim W} + \varepsilon\Bigr)\dim U.$$

Let rk be the rank of the map $U \to W/W_{\mathcal{T}}$ induced by B_γ. Its kernel is $B_\gamma^{-1}(W_{\mathcal{T}})$, so

$$\mathrm{rk} \;=\; \dim U - \dim B_\gamma^{-1}(W_{\mathcal{T}}) \;\geq\; \Bigl(\frac{\operatorname{codim} W_{\mathcal{T}}}{2\dim W} - \varepsilon\Bigr)\dim U \;>\; \frac{\operatorname{codim} W_{\mathcal{T}}}{4\dim W}\cdot\dim U \;>\; 0, \tag{6}$$

the last two inequalities by (5) and $\dim U \geq 1$. Put $\mathcal{T}'' = \mathcal{T} \cup \{\gamma\}$. Then $W_{\mathcal{T}''} = W_{\mathcal{T}} + B_\gamma(U)$ and $\operatorname{codim} W_{\mathcal{T}''} = \operatorname{codim} W_{\mathcal{T}} - \mathrm{rk}$, so that the quotient of the arguments of the two logarithms is $1 - \mathrm{rk}/(\varepsilon \dim W + \operatorname{codim} W_{\mathcal{T}})$. This number lies in the interval from 0 to 1, and the logarithm of a positive number is at most that number minus 1. As $\mathcal{T}''$ has at most one element more than $\mathcal{T}$,

$$\begin{aligned} \Pi(\mathcal{T}'') - \Pi(\mathcal{T}) &\leq \dim U + 5 \dim W \cdot \log\Bigl(1 - \frac{\mathrm{rk}}{\varepsilon \dim W + \operatorname{codim} W_{\mathcal{T}}}\Bigr) \\ &\leq \dim U - \frac{5 \dim W \cdot \mathrm{rk}}{\varepsilon \dim W + \operatorname{codim} W_{\mathcal{T}}}. \end{aligned}$$

By the first inequality of (6) and by (5),

$$5 \dim W \cdot \mathrm{rk} \;\geq\; \Bigl(\frac{5}{2} \operatorname{codim} W_{\mathcal{T}} - 5\varepsilon \dim W\Bigr) \dim U \;\geq\; (\operatorname{codim} W_{\mathcal{T}} + \varepsilon \dim W) \dim U,$$

the second step because $\frac{3}{2} \operatorname{codim} W_{\mathcal{T}} \geq 6\varepsilon \dim W$. Hence $\Pi(\mathcal{T}'') \leq \Pi(\mathcal{T})$, and $\mathcal{T}''$ is admissible. By the choice of $\mathcal{T}$, $\dim W_{\mathcal{T}''} \leq \dim W_{\mathcal{T}}$, so $B_\gamma(U) \subseteq W_{\mathcal{T}}$ and $\mathrm{rk} = 0$, against (6). □

LEAN `exists_few_translates` · uses `IsSpread`

The potential Π explains the two constants. Adding a member costs $\dim U$ in the first term. The spread property guarantees that a suitable member lowers the codimension by a fixed share of what is left, as long as (5) holds, and the logarithm turns this into a drop of the second term by at least $\dim U$. The factor 5 is what makes the last display true under (5), and 4ε is where the process stops.

6 Two inequalities behind the spread property

The hypothesis of Theorem 3 is proved in another chapter, by a second moment over a finite group. The unit on heights holds the two general inequalities of that proof. Both concern a finite group G and a map from G to an inner product space whose inner products depend only on the quotient of the two arguments.

Lemma 6.1 (inner products of sums over a group)**.** *Let G be a finite group, $\mathcal{H}$ a complex inner product space, $v\colon G \to \mathcal{H}$ and $w\colon G \to \mathcal{H}$ two maps, and $\omega\colon G \to \mathbb{C}$ a function such that $\langle v(\gamma), w(\gamma')\rangle = \omega(\gamma^{-1}\gamma')$ for all $\gamma, \gamma' \in G$. Then*

$$\Bigl\langle \sum_{\gamma \in G} v(\gamma), \sum_{\gamma \in G} w(\gamma) \Bigr\rangle \;=\; \#G \cdot \sum_{\gamma \in G} \omega(\gamma).$$

Proof. The left side is $\sum_\gamma \sum_{\gamma'} \omega(\gamma^{-1}\gamma')$, and for each γ the map $\gamma' \mapsto \gamma^{-1}\gamma'$ is a bijection of G. □

LEAN `inner_sum_sum_eq_card_mul`

In the next statement $\mathbb{F}$ is a finite field, $\mathbb{F}^I$ is the space of vectors indexed by a finite set I, and $\mathrm{GL}_I(\mathbb{F})$ is the group of invertible matrices indexed by $I \times I$, which acts on $\mathbb{F}^I$. A function ω as in (b) below is a function of positive type on G.

Proposition 6.2 (singular minors are rare)**.** *Let G be a finite group, $\mathcal{H}$ a complex inner product space and $v\colon G \to \mathcal{H}$ a map. Let $\mathbb{F}$ be a finite field, I and I' finite sets, $\pi\colon G \to \mathrm{GL}_I(\mathbb{F})$ a homomorphism of groups, and $A = (A_1, A_0)$ a pair of matrices over $\mathbb{F}$ with rows indexed by I and columns indexed by I'. Let $\omega\colon G \to \mathbb{R}$ be a function and Y a real number. Suppose that*

(a) $\omega(\gamma) \geq 0$ *for every* $\gamma \in G$*;*

(b) $\langle v(\gamma), v(\gamma')\rangle = \omega(\gamma^{-1}\gamma')$ *for all* $\gamma, \gamma' \in G$*;*

(c) $Y \geq 0$*;*

(d) *for every non-zero vector* $\xi \in \mathbb{F}^{I'}$*, the sum of* $\omega(\gamma)$ *over all* $\gamma \in G$ *such that* $(\pi(\gamma)^{\mathrm{t}})^{-1} A_1 \xi = A_1 \xi$ *is at most* Y*, and so is the sum of* $\omega(\gamma)$ *over all* $\gamma \in G$ *such that* $\pi(\gamma) A_0 \xi = A_0 \xi$*.*

Then

$$\#\mathbb{F} \cdot \sum_{\substack{\gamma \in G \\ \det(A_1^{\mathrm{t}} \pi(\gamma) A_0) = 0}} \omega(\gamma) \;\leq\; 2 \sum_{\gamma \in G} \omega(\gamma) \;+\; (\#\mathbb{F})^{\#I'} \sqrt{(\#\mathbb{F})^{\#I}} \cdot Y.$$

The matrix $A_1^{\mathrm{t}} \pi(\gamma) A_0$ is square, indexed by $I' \times I'$. If the two stabilisers in (d) carry little of the mass of ω, the proposition says that the set where this matrix is singular carries at most about the share $2/\#\mathbb{F}$ of the mass.

Proof. Fix a non-trivial character Ψ of the additive group of $\mathbb{F}$; its values have absolute value 1. For a finite set I and ρ, ρ' in $\mathbb{F}^I$ write $\rho \cdot \rho' = \sum_{i \in I} \rho_i \rho'_i$. The sum of a non-trivial character over a finite group is zero, and multiplication by a non-zero element of $\mathbb{F}$ is a bijection of $\mathbb{F}$. Hence, coordinate by coordinate,

$$\sum_{\rho' \in \mathbb{F}^I} \Psi(\rho \cdot \rho') \;=\; (\#\mathbb{F})^{\#I} \;\text{ if } \rho = 0, \qquad \text{and} \qquad = 0 \;\text{ otherwise.} \tag{7}$$

For $\xi_1, \xi_0 \in \mathbb{F}^{I'}$ and $\gamma \in G$ put $\omega_{\xi_1, \xi_0}(\gamma) = \Psi\big(\xi_1 \cdot A_1^{\mathrm{t}} \pi(\gamma) A_0\, \xi_0\big)\, \omega(\gamma)$.

Step 1. Fix γ and ξ_1. By (7) for the set I', the mean over ξ_0 of the numbers $\Psi(\xi_1 \cdot A_1^{\mathrm{t}} \pi(\gamma) A_0\, \xi_0)$ is 1 if $\xi_1^{\mathrm{t}} A_1^{\mathrm{t}} \pi(\gamma) A_0 = 0$ and 0 otherwise. The ξ_1 with this property form a subspace of $\mathbb{F}^{I'}$; it has at least one element, and at least $\#\mathbb{F}$ elements when the matrix is singular. So the sum over ξ_1 of the mean over ξ_0 of $\sum_\gamma \omega_{\xi_1, \xi_0}(\gamma)$ is the sum over γ of $\omega(\gamma)$ times the number of elements of this subspace. It is a real number, and as $\omega \geq 0$ by (a),

$$\#\mathbb{F} \cdot \sum_{\substack{\gamma \in G \\ \det(A_1^{\mathrm{t}} \pi(\gamma) A_0) = 0}} \omega(\gamma) \;\leq\; \sum_{\xi_1 \in \mathbb{F}^{I'}} (\#\mathbb{F})^{-\#I'} \sum_{\xi_0 \in \mathbb{F}^{I'}} \operatorname{Re} \sum_{\gamma \in G} \omega_{\xi_1, \xi_0}(\gamma).$$

Step 2. If ξ_1 or ξ_0 is zero, then $\omega_{\xi_1, \xi_0} = \omega$. We claim that

$$\operatorname{Re} \sum_{\gamma \in G} \omega_{\xi_1, \xi_0}(\gamma) \;\leq\; \sqrt{(\#\mathbb{F})^{\#I}} \cdot Y \qquad \text{for all non-zero } \xi_1, \xi_0 \in \mathbb{F}^{I'}. \tag{8}$$

In the inner product space $\mathbb{C}^{\mathbb{F}^I}$, with standard basis $(\mathbf{1}_\rho)_{\rho \in \mathbb{F}^I}$, put

$$\Psi_\rho \;=\; (\#\mathbb{F})^{-\#I/2} \sum_{\rho' \in \mathbb{F}^I} \Psi(-\rho \cdot \rho')\, \mathbf{1}_{\rho'} \qquad \text{for } \rho \in \mathbb{F}^I.$$

By (7) these vectors form a second orthonormal family, and

$$\langle \Psi_\rho, \mathbf{1}_{\rho'} \rangle = (\#\mathbb{F})^{-\#I/2}\, \Psi(\rho \cdot \rho') \qquad \text{for all } \rho, \rho' \in \mathbb{F}^I.$$

Consider in $\mathbb{C}^{\mathbb{F}^I} \otimes \mathcal{H}$ the two vectors

$$F = \sum_{\gamma \in G} \Psi_{(\pi(\gamma)^{\mathrm{t}})^{-1} A_1 \xi_1} \otimes v(\gamma), \qquad F' = \sum_{\gamma \in G} \mathbf{1}_{\pi(\gamma) A_0 \xi_0} \otimes v(\gamma).$$

The dot product of $(\pi(\gamma)^{\mathrm{t}})^{-1}A_1\xi_1$ and $\pi(\gamma')A_0\xi_0$ is $\xi_1 \cdot A_1^{\mathrm{t}}\pi(\gamma^{-1}\gamma')A_0\,\xi_0$. So by (b) the inner product of the term of F at γ with the term of F' at γ' is $(\#\mathbb{F})^{-\#I/2}\,\omega_{\xi_1,\xi_0}(\gamma^{-1}\gamma')$, and Lemma 6.1 gives

$$\langle F, F'\rangle \;=\; \#G \cdot (\#\mathbb{F})^{-\#I/2} \sum_{\gamma\in G} \omega_{\xi_1,\xi_0}(\gamma).$$

In the same way, the inner product of the terms of F at γ and γ' is $\omega(\gamma^{-1}\gamma')$ if $(\pi(\gamma^{-1}\gamma')^{\mathrm{t}})^{-1}A_1\xi_1 = A_1\xi_1$ and 0 otherwise, because the family (Ψ_ρ) is orthonormal and $\gamma \mapsto (\pi(\gamma)^{\mathrm{t}})^{-1}$ is a homomorphism. By Lemma 6.1 and (d), $\|F\|^2 \le \#G \cdot Y$; likewise $\|F'\|^2 \le \#G \cdot Y$. Hence $\mathrm{Re}\langle F, F'\rangle \le \|F\|\,\|F'\| \le \frac{1}{2}(\|F\|^2 + \|F'\|^2) \le \#G \cdot Y$, which is (8).

Step 3. In the right side of Step 1, the terms with $\xi_1 = 0$ contribute $\sum_\gamma \omega(\gamma)$. The terms with $\xi_1 \neq 0$ and $\xi_0 = 0$ contribute at most the same, because there are fewer than $(\#\mathbb{F})^{\#I'}$ of them and $\sum_\gamma \omega(\gamma) \ge 0$. The remaining terms contribute at most $(\#\mathbb{F})^{\#I'}\sqrt{(\#\mathbb{F})^{\#I}} \cdot Y$, by (8) and (c). This is the assertion. □

LEAN `card_mul_sum_det_le` · uses `inner_sum_sum_eq_card_mul`, and from the background library `sum_addChar_dotProduct` (the relation (7)), `invTransposeGL` and `transposeGL` (the map $\gamma \mapsto (\pi(\gamma)^{\mathrm{t}})^{-1}$)

Definition 6.3 (the dyad sum of a family)**.** Let $\mathcal{H}$ be a complex inner product space with an orthonormal basis $e = (e_{i'})_{i'\in I'}$ indexed by a finite set I', and let $v = (v_i)_{i\in I}$ be a family of vectors of $\mathcal{H}$ indexed by a finite set I. For $i \in I$ let $\bar{v}_i = (\langle v_i, e_{i'}\rangle)_{i'\in I'}$ in $\mathbb{C}^{I'}$ be the vector of the complex conjugates of the coordinates of v_i in the basis e. The *dyad sum* of v is

$$\Delta_e(v) \;=\; \sum_{i\in I} \bar{v}_i \otimes v_i \quad \text{in } \mathbb{C}^{I'} \otimes \mathcal{H}.$$

LEAN `dyadSum`

Lemma 6.4 (inner products of dyad sums)**.** *Let $\mathcal{H}$ be a complex inner product space with an orthonormal basis $e = (e_{i'})_{i'\in I'}$ indexed by a finite set I', and let $v = (v_i)_{i\in I}$ and $w = (w_{i''})_{i''\in I''}$ be families of vectors of $\mathcal{H}$ indexed by finite sets I and I''. Then*

$$\langle \Delta_e(v), \Delta_e(w)\rangle \;=\; \sum_{i\in I} \sum_{i''\in I''} |\langle w_{i''}, v_i\rangle|^2.$$

Proof. The inner product of $\bar{v}_i \otimes v_i$ and $\bar{w}_{i''} \otimes w_{i''}$ is $\langle \bar{v}_i, \bar{w}_{i''}\rangle \cdot \langle v_i, w_{i''}\rangle$, and by Parseval's identity

$$\langle \bar{v}_i, \bar{w}_{i''}\rangle \;=\; \sum_{i'\in I'} \langle w_{i''}, e_{i'}\rangle\,\langle e_{i'}, v_i\rangle \;=\; \langle w_{i''}, v_i\rangle.$$

The product of $\langle w_{i''}, v_i\rangle$ and its complex conjugate $\langle v_i, w_{i''}\rangle$ is $|\langle w_{i''}, v_i\rangle|^2$. □

LEAN `inner_dyadSum`

Under the identification of $\mathbb{C}^{I'} \otimes \mathcal{H}$ with the endomorphisms of $\mathcal{H}$ that the basis e provides, the dyad sum of v is the positive operator $F \mapsto \sum_{i\in I}\langle v_i, F\rangle\, v_i$, and the inner product of the lemma is the trace of the product of two such operators. This remark is ours and is not used.

Proposition 6.5 (a variance inequality)**.** *Let $\mathcal{H}$ be a finite-dimensional complex inner product space, $e = (e_{i'})_{i'\in I'}$ an orthonormal basis of $\mathcal{H}$ indexed by a finite set I',*

and I a finite set. Let $T\colon \mathbb{C}^I \to \mathcal{H}$ and $C\colon \mathcal{H} \to \mathcal{H}$ be $\mathbb{C}$-linear maps, $\mathcal{U} \subseteq \mathbb{C}^I$ and $\mathcal{H}' \subseteq \mathcal{H}$ linear subspaces, and $\mathcal{E}$ a real number. Suppose that $\sum_{i\in I}\|(T^*T-1)\mathbf{1}_i\| \le \mathcal{E}$, where T^* is the adjoint of T, and that $(T-CT)(\mathcal{U}) \subseteq \mathcal{H}'$. Then

$$\dim \mathcal{U} \;-\; \mathcal{E} \;-\; \sum_{i\in I}\|CT\mathbf{1}_i\|^2 \;\le\; \mathrm{Re}\langle \Delta_e(T\mathbf{1}), \Delta_e(e')\rangle,$$

where $T\mathbf{1}$ is the family $(T\mathbf{1}_i)_{i\in I}$ and e' is the orthonormal basis of $\mathcal{H}'$, indexed by $\{1,\dots,\dim \mathcal{H}'\}$, that the library provides.

By Lemma 6.4 and Parseval's identity in $\mathcal{H}'$ the right side is the real number $\sum_{i\in I}\|p_{\mathcal{H}'}T\mathbf{1}_i\|^2$, the square of the Hilbert–Schmidt norm of $p_{\mathcal{H}'}T$; it does not depend on the choice of e'. So the proposition says: if T is nearly an isometry, and if on a subspace $\mathcal{U}$ it lands in $\mathcal{H}'$ up to the small map CT, then the part of T in $\mathcal{H}'$ has Hilbert–Schmidt norm squared at least about $\dim \mathcal{U}$.

Proof. Let $v = (v_{i''})_{i''\in I''}$ be an orthonormal basis of $\mathcal{U}$, so that $\#I'' = \dim \mathcal{U}$.

Step 1. The endomorphism $p_{\mathcal{U}}(1-T^*T)$ of $\mathbb{C}^I$ maps into $\mathcal{U}$, so its trace can be computed in the basis v of $\mathcal{U}$ and also in the standard basis:

$$\dim \mathcal{U} - \sum_{i''\in I''}\|Tv_{i''}\|^2 \;=\; \sum_{i''\in I''}\langle v_{i''}, (1-T^*T)v_{i''}\rangle \;=\; \sum_{i\in I}\langle p_{\mathcal{U}}\mathbf{1}_i, (1-T^*T)\mathbf{1}_i\rangle.$$

The left side is real, and the real part of the right side is at most

$$\sum_{i\in I}\|p_{\mathcal{U}}\mathbf{1}_i\|\,\|(1-T^*T)\mathbf{1}_i\| \;\le\; \mathcal{E},$$

because $\|p_{\mathcal{U}}\mathbf{1}_i\| \le 1$. Hence $\dim \mathcal{U} - \mathcal{E} \le \sum_{i''\in I''}\|Tv_{i''}\|^2$.

Step 2. For each i'' the vector $Tv_{i''} - CTv_{i''}$ lies in $\mathcal{H}'$, so $Tv_{i''}$ and $CTv_{i''}$ have the same component orthogonal to $\mathcal{H}'$. By Pythagoras' theorem,

$$\|Tv_{i''}\|^2 \;\le\; \|p_{\mathcal{H}'}Tv_{i''}\|^2 + \|CTv_{i''}\|^2.$$

Step 3. For every $\mathbb{C}$-linear map $C'\colon \mathcal{H} \to \mathcal{H}$,

$$\sum_{i''\in I''}\|C'Tv_{i''}\|^2 = \sum_{i'\in I'}\sum_{i''\in I''}|\langle (C'T)^*e_{i'}, v_{i''}\rangle|^2 \;\le\; \sum_{i'\in I'}\|(C'T)^*e_{i'}\|^2 = \sum_{i\in I}\|C'T\mathbf{1}_i\|^2,$$

by Parseval's identity in $\mathcal{H}$, Bessel's inequality for the orthonormal family v, and Parseval's identity in $\mathbb{C}^I$.

Steps 1 and 2, and Step 3 for $C' = p_{\mathcal{H}'}$ and for $C' = C$, give

$$\dim \mathcal{U} - \mathcal{E} \;\le\; \sum_{i\in I}\|p_{\mathcal{H}'}T\mathbf{1}_i\|^2 + \sum_{i\in I}\|CT\mathbf{1}_i\|^2,$$

and the first sum is the right side of the assertion, as was said after the statement. □

LEAN `finrank_sub_le_re_inner_dyadSum` · uses `inner_dyadSum`

7 Where the results are used

The heights serve Theorem C of the argument, and the linear algebra of §§5 and 6 serves Theorem B. The notation of this section is that of the overview of the handout: D is admissible data, with third scale $b(D)$ and number of lifts $m(D)$; $\mathrm{Row}(D)$ is the space of rows; for an $m(D)$-fold twist ϑ of D, $\Theta_{D,\vartheta}\colon \mathrm{Row}(D) \to \mathcal{A}_{D,\vartheta}$ is the chain of lifts and $\mathfrak{z}_D(\vartheta) = \det(\Theta^*_{D,\vartheta}\Theta_{D,\vartheta})$. What follows was read in the statements and in the last steps of the formal proofs; the chapters named give the proofs.

Theorem C: a covolume is bounded from below. Let $y_1, \dots, y_n$ be linearly independent rows with $n \geq (1 - 4/b(D)^2)\dim \mathrm{Row}(D)$, of D-index at most δ, whose lifts at the trivial twist satisfy $\mathrm{covol}^2_\sigma(\Theta_{D,1}y_1, \dots, \Theta_{D,1}y_n) \leq \exp(V)$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$. Chapter VIII provides a family $\theta = (\theta_{\ell,\iota})$ of $m(D)$-fold twists, one for each pair in $\mathcal{J}$, that is fit for D. Two height bounds are then proved.

First bound. The family $\mathfrak{z}_D(\theta_{\ell,\iota})/\mathfrak{z}_D(1)$ has height at most

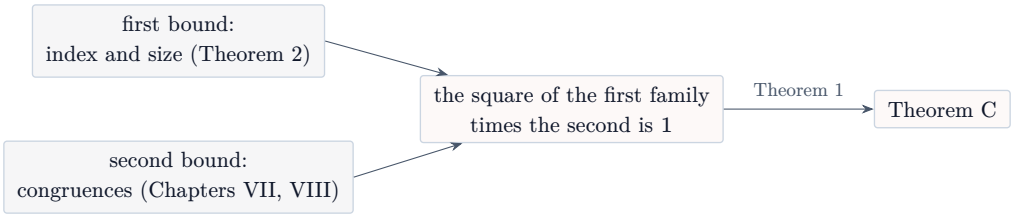
$$V + 2\delta + b(D) \cdot \dim \mathrm{Row}(D).$$

This is Theorem 2 for $W = \mathrm{Row}(D)$, for the set Ξ of the $m(D)$-fold twists with $\vartheta_0 = 1$, for $T_\vartheta = \Theta_{D,\vartheta}$, and for the sets $\mathcal{L}_{\ell,\iota}(j)$ of the j-integral forms, for which (f) is the integrality of the pairing. The basis w consists of $y_1, \dots, y_n$ and of n' further rows $z_1, \dots, z_{n'}$ given by Lemma 4.2, taken from a spanning set of crude rows that Chapter VII constructs. These come with scalars $c_{\ell,\iota}$, ratios of a twisted to an untwisted constant, for which the family $|c|^{-2}$ has a height bound. The matrix of (g) is made of two blocks: the matrix that the D-index provides for the y_i, with $|j(\det u)|_\ell \geq \exp(-\eta_\ell)$, and $c_{\ell,\iota}$ times the unit matrix for the z_i; so $\nu = n'$, and hypothesis (e) holds with the δ of the D-index. The number α is V plus a term for the z_i. There are at most $(4/b(D)^2)\dim \mathrm{Row}(D)$ vectors z_i, and the last lines of the formal proof bound what each of them adds to α and to νh by $b(D)^3/4$: this is the term $b(D) \cdot \dim \mathrm{Row}(D)$.

Second bound. The family $\mathfrak{z}_D(1)^2/\mathfrak{z}_D(\theta_{\ell,\iota})^2$ has height at most

$$2\,\dim \mathrm{Row}(D) \cdot (2 \log b(D) \cdot \mathrm{budget}(D) - h_{\mathrm{main}}(D)).$$

Here $\mathfrak{z}_D(\vartheta)$ is a constant times the product of the volumes of the $m(D)$ single lifts, each ratio of an untwisted to a twisted volume has a height bound (Chapter VII), and Proposition 3.3 and Lemma 3.1 combine them.



The square of the first family times the second is the constant family 1. By Proposition 3.3 and Theorem 1, twice the first bound plus the second is non-negative:

$$\dim \mathrm{Row}(D) \cdot (h_{\mathrm{main}}(D) - 2 \log b(D) \cdot \mathrm{budget}(D) - b(D)) \;\leq\; V + 2\delta,$$

which gives Theorem C because $b(D) \le \log b(D) \cdot \mathrm{budget}(D)$. The twist $\theta_{\ell,\iota}$ depends on the pair (ℓ, ι), so the numbers $\mathfrak{z}_D(\theta_{\ell,\iota})$ are not the conjugates of one algebraic number, and no product formula applies to them. The height carries the ℓ-adic bounds of the two families through the multiplication, and the product formula is applied at the end to the auxiliary numbers H alone. The complete deduction belongs to the note on the unit *New.Endgame*, which ends the argument.

Other height bounds. Height bounds for families of ratios of a twisted to an untwisted quantity (values of Hecke L-functions, volumes of single lifts, the constants above) are proved in the units *New.Rows*, *New.Exchange* and *New.Selmer*, and combined by Lemma 3.1 and Proposition 3.3.

Theorem B: enough forms. Theorem 3 is applied with $W = \mathrm{Row}(D)$, with U the space of columns $\mathrm{Col}(D)$, with the finite group $\mathsf{G}(D)$ as index set and with $\varepsilon = b(D)^{-2}$. The map B_γ is a fixed operator from the columns onto the ramified columns followed by the Poincaré map $\mathcal{P}_D(\gamma)$, in the normalisation of Chapter VI. That this family is $b(D)^{-2}$-spread is the filling theorem of that chapter. Its proof is a second moment over the group: Proposition 6.5 is applied to each translate and summed, with Lemmas 6.1 and 6.4, and Proposition 6.2 is applied inside an induction, to a function of positive type and to the set where a minor is singular. Theorem 3 then gives the finite set $\mathcal{T}$ of Theorem B, with $\#\mathcal{T} \cdot \dim \mathrm{Col}(D) \le 5 \dim \mathrm{Row}(D) \cdot \log(1 + b(D)^2)$ and a span of dimension at least $(1 - 4/b(D)^2) \dim \mathrm{Row}(D)$.

8 Notes on the formal text

1. *Two ways to add up the local exponents.* Definition 1.2 asks for the bound on the sums of η over the primes below n, for all large n. Lemma 3.2 assumes the bound for all finite sets of primes, and hypothesis (e) of Theorem 2 for all finite sets of natural numbers. The formal text has these three forms, and they are kept here. In the definition the formal text says that the bound holds eventually; the threshold n_0 is this statement written out.
2. *Functions defined everywhere.* The formal Lemma 3.4 includes $q = 0$, by the convention $\log 0 = 0$. Definition 1.4 is stated for arbitrary complex vector spaces, with the library's dimension; Theorem 3 uses it for finite-dimensional spaces only.
3. *Theorem 2.* The formal statement asks (f) and (g) for every homomorphism $j \colon \mathbb{C} \to \mathbb{C}_\ell$, and its proof uses two of them for each (ℓ, ι). The sets $\mathcal{L}_{\ell,\iota}(j)$ are arbitrary subsets. The formal conclusion is written with the volume factor of the library, squared, and the quotient is formed in $\mathbb{C}$. The cost 2δ comes from the two matrices of the proof, one for j_ι and one for its conjugate, and νh from the family $|c|^{-2}$, which is bounded independently. The base map T_{ϑ_0} must be injective, so that the witness Γ_{ϑ_0} is not zero.
4. *Theorem 3.* The formal statement takes W to be an inner product space; the proof does not use the inner product. It chooses an admissible set whose span is maximal for inclusion; the proof above takes the largest dimension, which amounts to the same.
5. *Proposition 6.2.* The formal statement gives the pair A as a function on the two truth values; A_1 is its value at "true" and A_0 its value at "false". The condition

in (d) is written with one formula for both, through an automorphism of $\mathrm{GL}_I(\mathbb{F})$ that is the identity for A_0 and the passage to the inverse of the transpose for A_1, a definition of the background library. The formal proof takes a character from Mathlib and the relation (7) from the background library, where it is stated for finite commutative rings; here it is proved for a field in one line.

6. *Proposition 6.5.* The basis e enters the statement only through the dyad sums, and the basis e' is the one that the library attaches to a finite-dimensional inner product space.
7. *Arguments read from the text.* The list of arguments of each declaration was made by a program from the text of the files. For some declarations of the second file it contains variables of the surrounding section whose names are bound again inside the statement (a basis, a second map on G, a field, an index set). Lean does not take them as arguments, as the dossier notes, and where such a variable is explicit the uses of the lemma in the artifact confirm it. In the accounts of this note they are matched with the bound variables of the same name.
8. *What the final theorems use.* All declarations of the unit are stated here. Whether each of them is reached from the final theorems is for the kernel-based ledger of the handout to say.
9. *Literature named by the formal text.* The file on heights calls the product formula through a norm and adelic norms changed place by place standard, and names as related work Gaudron (2006), Bombieri and Gubler (2006, ch. 1) and notes of K. Conrad. The file on determinants names Wolsey (1982) for greedy covering with a logarithmic loss, and Bost (2001) and Laurent (1994) for size against divisibility of a determinant. The references are listed below as the artifact gives them, with the state that its own list of references records; we have not compared them.

How this note was written. From the dossier of the unit: its statements in words, its structured proofs, and its imports. The program check of the dossier covered 47 leaf steps of 7 proofs, found the 18 statements at their lines and compared the quotations of the imports with the source; a second reading of the dossier by the reading agent asked for no correction. I read the files of the unit completely, statements and proofs, and compared every statement of this note with its Lean text argument by argument, in Lean's order of quantifiers. I also read: the statements of the imported declarations in the background library and the proof of Input 1; the formal proofs of Theorem C and of its two height bounds, the statement and the last lines of the proof of Theorem B, the statement of the filling theorem, and the places where the two inequalities of §6 are applied; the authors' list of references. I did not read Mathlib, and I did not run a build. Shorter than, or different from, the formal proofs: Lemma 3.4 (the formal proof is one call of automation on the prime factorisation); the bound for $\|F\|^2$ in Proposition 6.2, which the formal proof obtains from one general identity for an orthonormal family and an automorphism of the linear group; Step 1 of Proposition 6.5, which follows the same two traces; the arithmetic at the end of Theorem 3, which the formal proof leaves to automation and which is written out here. Lemma 4.1, Corollary 3.5, Remark 3.6, the examples and the diagram are mine.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Definition 1.1	`PrimePadicIso`	Heights.lean	17–17
Definition 1.2	`HeightLe`	Heights.lean	21–23
Lemma 3.4	`log_product_formula`	Heights.lean	28–30
Lemma 3.1	`HeightLe.mono`	Heights.lean	36–37
Lemma 3.2	`HeightLe.of_le`	Heights.lean	42–43
Proposition 3.3	`HeightLe.mul`	Heights.lean	47–50
Proposition 3.3	`HeightLe.prod`	Heights.lean	54–55
Proposition 3.3	`HeightLe.pow`	Heights.lean	59–60
Theorem 1	`HeightLe.nonneg`	Heights.lean	65–77
Lemma 6.1	`inner_sum_sum_eq_card_mul`	OneDeterminant.lean	22–23
Proposition 6.2	`card_mul_sum_det_le`	OneDeterminant.lean	28–54
Definition 6.3	`dyadSum`	OneDeterminant.lean	58–58
Lemma 6.4	`inner_dyadSum`	OneDeterminant.lean	60–62
Proposition 6.5	`finrank_sub_le_re_inner_dyadSum`	OneDeterminant.lean	67–82
Definition 1.4	`IsSpread`	OneDeterminant.lean	90–92
Theorem 3	`exists_few_translates`	OneDeterminant.lean	97–111
Lemma 4.2	`exists_extend_basis_of_span_eq_top`	OneDeterminant.lean	115–122
Theorem 2	`one_determinant`	OneDeterminant.lean	129–149
Input 1	`isAlgebraic_of_padic`	QuadraticZeta.lean	17–24
Input 2	`exists_ringEquiv_extend`	CommonEigenvectors.lean	38–47
Input 3	`mk_complex_eq_mk_padicAlgCl`	Ainf.lean	16–18
Definition 2.1	`complexToCp`	UnramifiedCharacters.lean	44–44
Theorem 2	`IsIntMatrix`	AdelicRingDefs.lean	53–53
Theorem 2	`IsIntMatrix.det_mem`	GeneralisedIndex.lean	10–11
Proposition 6.2	`sum_addChar_dotProduct`	LocalHomUnit.lean	42–46
Proposition 6.2	`invTransposeGL`	ColumnLattices.lean	109–110
Proposition 6.2	`transposeGL`	ValuationPair.lean	136–136

Letters of the formal text and of this note. The letters of this note follow the overview of the handout where it has fixed them, and differ from those of the formal text. Heights: the formal text writes `c`, `y` for our h, Z; `e` for η; `M` for the bound n of the primes. Section 6: it writes `H` for $\mathcal{H}$; `k` for $\mathbb{F}$; `n`, `m`, `ι` for the index sets I, I', I''; `u`, `v` for the maps v, w on G; `a`, `b`, `s` for the elements of G; `f` and `θ` for ω; `b` for the basis e; `P`, `R`, `U`, `A`, `ε` for T, C, $\mathcal{U}$, $\mathcal{H}'$, $\mathcal{E}$. Theorem 3: `V`, `W` for U, W; `θ` or `S` for Ω; `n` for ε; `F` for W'; `t` for $\mathcal{T}$. Lemma 4.2: `V` for W and `k` for n'. Theorem 2: `θ` for Ξ; `H θ` for $\mathcal{H}_\vartheta$; `u` for the maps T; `o` for ϑ_0; `S` and `x` for I and w; `g` for h; `y` for c; `e` for η; `U` for J; `Λ` for $\mathcal{L}$; `k` for ν; `m` for the matrix u.

Statements as they are written in the artifact. The definition of a height bound and Theorem 1:

```
def HeightLe c (y : _ → ℂ) := ∃ᵉ (H) (α) (e : ℕ → ℝ), H ≠ 0 ∧ (∀ σ : ℂ ≃+* ℂ, ‖σ H‖ ≤ α.exp) ∧
  (∀ ι : PrimePadicIso, ‖ι.2 (H * y ι)‖ ≤ (e ι.1).exp) ∧ ∀ᶠ M in atTop,
  α + ∑ l ∈ range M with l.Prime, e l ≤ c

lemma HeightLe.nonneg (h : HeightLe c 1) : 0 ≤ c
```

Theorem 2:

```
lemma one_determinant {θ : _ → 0} {H : θ → Type} [∀ θ, NormedAddCommGroup (H θ)] [∀ θ,
  InnerProductSpace ℂ (H θ)] (u : ∀ θ, W →ₗ[ℂ] H θ) {o} (hu : (⇑(u o)).Injective) [Fintype S] {x :
  S → W} (i : LinearIndependent ℂ x) (hk : finrank ℂ W = card S) {α δ g} (hα : ∀ σ : ℂ ≃+* ℂ, ‖σ
  (gram ℂ (u o ∘ x)).det‖ ≤ Real.exp α) {γ : _ → ℂ} (γ0 : ∀ ι, γ ι ≠ 0) (hγ : HeightLe g fun ι ↦ (‖γ
```

```
  ι‖ ^ 2 :)⁻¹) {e : ℕ → ℝ} (he : ∀ U, ∑ l ∈ U, e l ≤ δ) (Λ : ∀ ι : PrimePadicIso, (ℂ →+* ℂ_[ι.1]) →
  Set (H (θ ι))) (k : ℕ) (w : ∀ ι j, ∀ x ∈ Λ ι (j.comp (starRingEnd ℂ)), ∀ y ∈ Λ ι j, ‖j ⟪x, y⟫_ℂ‖ ≤
  1) (l : ∀ ι j, ∃ m, Real.exp (-e ι.1) * ‖j (y ι)‖ ^ k ≤ ‖j (det m)‖ ∧ ∀ i, ∑ l, m i l • u _ (x
  l) ∈ Λ ι j) : HeightLe (α + 2 * δ + k * g) fun ι ↦ (u (θ ι)).normDet ^ 2 / (u o).normDet ^ 2
```

The spread property and Theorem 3:

```
def IsSpread {W} [AddCommMonoid W] [Module ℂ W] (B : S → V →ₗ[ℂ] W) n :=
  ∀ F : Submodule .., let φ : ℝ := finrank ℂ F / finrank ℂ W
  ∃ s, finrank ℂ (comap (B s) F) ≤ ((1 + φ) / 2 + n) * finrank ℂ V

lemma exists_few_translates {B : θ → V →ₗ[ℂ] W} {n} (hn : 0 < n) (h : IsSpread B n) [Nontrivial V] :
    ∃ t, #t * finrank ℂ V ≤ 5 * finrank ℂ W * Real.log (1 + n⁻¹) ∧
    (1 - 4 * n) * finrank ℂ W ≤ finrank ℂ ↥(⨆ s ∈ t, (B s).range)
```

Propositions 6.2 and 6.5:

```
lemma card_mul_sum_det_le [Fintype k] (π : G →* GL n k) (A : _ → Matrix n m k) {θ} (hθ : ∀ s, 0 ≤ θ
  s) (hu : ∀ a b, ⟪u a, u b⟫_ℂ = θ (a⁻¹ * b)) Y (hY : 0 ≤ Y) (h : ∀ b, ∀ v ≠ 0, ∑ s with A b *ᵥ v =
  cond b invTransposeGL 1 (π s) • A b *ᵥ v, θ s ≤ Y) : card k * ∑ s with ((A true)ᵀ *
  (π s).1 * A false).det = 0, θ s ≤ 2 * univ.sum θ + card k ^ card m * √(card k ^ card n) * Y

lemma finrank_sub_le_re_inner_dyadSum [FiniteDimensional ℂ H] (P : EuclideanSpace ℂ n →ₗ[ℂ] H) (R :
  End ℂ H) (U A : Submodule ℂ _) {ε} (k : ∑ x, ‖(P.adjoint ∘ₗ P - 1) (.single x 1)‖ ≤ ε)
  (h : ∀ v ∈ U, P v - R (P v) ∈ A) : finrank ℂ U - ε - ∑ x, ‖R (P (.single x 1))‖ ^ 2 ≤
  re ⟪dyadSum b (P <| .single · 1), dyadSum b (stdOrthonormalBasis ℂ A ·)⟫_ℂ
```

Notation

ℓ a prime number

$\mathbb{Q}_\ell$, $\bar{\mathbb{Q}}_\ell$, $\mathbb{C}_\ell$ the ℓ-adic numbers, an algebraic closure, its completion

$|\cdot|_\ell$, $|\cdot|$ the absolute value with $|\ell|_\ell = 1/\ell$; the absolute value of $\mathbb{C}$

ι an isomorphism of rings from $\mathbb{C}$ to $\bar{\mathbb{Q}}_\ell$, or to a field L

$\mathcal{I}$ the set of the pairs (ℓ, ι)

σ, $\mathrm{Aut}(\mathbb{C})$ a field automorphism of $\mathbb{C}$; the group of all of them

Z, Z', Z_i families of complex numbers indexed by $\mathcal{I}$, with entries $Z_{\ell,\iota}$

h, h', h_i real numbers, bounds for heights

$\mathrm{ht}(Z) \le h$ Z has height at most h

H, α, η a witness: a non-zero complex number, a real number, a sequence $(\eta_0, \eta_1, \dots)$ of real numbers

n, n_0, n' natural numbers; a threshold; the number of added vectors

q, q_{num}, q_{den} a rational number, its numerator and denominator

ord_ℓ, Nm, $\deg H$ exponent of ℓ; norm of a field extension; degree of $\mathbb{Q}(H)$ over $\mathbb{Q}$

$\bar{h}$ in Remark 3.6: a mean of logarithms

j, $\bar{j}$, j_ι a homomorphism of rings $\mathbb{C} \to \mathbb{C}_\ell$; j after complex conjugation; ι followed by the inclusion

O_j the complex numbers with $|j(\cdot)|_\ell \le 1$

L; τ, τ_0 an algebraically closed field; injective homomorphisms of rings

Y a real number, a bound

I, I', I''; i, i', i'', i_1 finite sets; their elements (in Theorem 2 and Lemma 4.1, i and i' both run over I)

$\#I$ the number of elements

$\mathcal{H}$, $\mathcal{H}'$, $\mathcal{H}_\vartheta$ complex inner product spaces; $\mathcal{H}'$ a subspace of $\mathcal{H}$

$\langle\cdot,\cdot\rangle$, $\|\cdot\|$ inner product, conjugate-linear in the first variable; its norm

F, F', F'', F_i vectors of an inner product space

v, w families of vectors, or maps from G to $\mathcal{H}$; Tv the family (Tv_i)

$\mathrm{Gram}(v)$ the Gram matrix $(\langle v_i, v_{i'}\rangle)$

T, T_ϑ, T^* $\mathbb{C}$-linear maps into inner product spaces; the adjoint; T^*T as in §1

W, W', W_y, $W_{\mathcal{T}}$ a finite-dimensional complex vector space; subspaces

Ξ, ϑ, ϑ_0, θ a set; its elements; a family $(\theta_{\ell,\iota})$ of elements

c a family of non-zero complex numbers $c_{\ell,\iota}$

δ, ν a real number, a bound for an index; a natural number

J a finite set of natural numbers

$\mathcal{L}_{\ell,\iota}(j)$ a subset of $\mathcal{H}_{\theta_{\ell,\iota}}$

u, u', $\overline{u'}$, u^{t} matrices of complex numbers; conjugate entries; transpose

Γ_ϑ $\det \mathrm{Gram}(T_\vartheta w)$

U, Ω, B, B_γ a complex vector space; a set; a family of linear maps $U \to W$

ε, $\mathrm{Spread}_\varepsilon(B)$ a real number; B is ε-spread

$\mathcal{T}$, $\mathcal{T}'$, $\mathcal{T}''$ finite subsets of Ω

Π, rk, codim the potential, a rank, a codimension, in the proof of Theorem 3

S; y_i, z_i a spanning set; vectors

G, γ, γ' a finite group; elements of G, or of Ω

ω, ω_{ξ_1,ξ_0} a function on G; its product with character values

$\mathbb{F}$, $\mathrm{GL}_I(\mathbb{F})$, π a finite field; invertible matrices; a homomorphism $G \to \mathrm{GL}_I(\mathbb{F})$

$A = (A_1, A_0)$ a pair of matrices over $\mathbb{F}$

ξ, ξ', ξ_1, ξ_0, ρ, ρ' vectors with coordinates indexed by a finite set

Ψ, Ψ_ρ a non-trivial additive character of $\mathbb{F}$; the normalised characters of $\mathbb{F}^I$

$\mathbf{1}$, $\mathbf{1}_i$ the standard basis of $\mathbb{C}^I$

e, e' orthonormal bases of $\mathcal{H}$ and of $\mathcal{H}'$

$\Delta_e(v)$, $\bar{v}_i$ the dyad sum; conjugate coordinates

C, C', $\mathcal{U}$, $\mathcal{E}$ linear maps $\mathcal{H} \to \mathcal{H}$; a subspace of $\mathbb{C}^I$; a real number, a bound

$p_{\mathcal{U}}$, $p_{\mathcal{H}'}$ orthogonal projections

D, $b(D)$, $m(D)$, V in §7: admissible data, its third scale, its number of lifts; a real number

$\mathrm{Row}(D)$, $\mathrm{Col}(D)$, $\mathsf{G}(D)$ rows, columns, the finite group of the data

$\Theta_{D,\vartheta}$, $\mathcal{A}_{D,\vartheta}$, $\mathfrak{z}_D(\vartheta)$ the chain of lifts, its target, $\det(\Theta^*_{D,\vartheta}\Theta_{D,\vartheta})$

$\mathcal{P}_D(\gamma)$ the Poincaré map translated by γ

covol_σ in §7: the covolume at σ, as in the overview of the handout

$\mathrm{budget}(D)$, $h_{\mathrm{main}}(D)$ the budget and the main amount of the data

References

[St] E. Steinitz, Algebraische Theorie der Körper, J. reine angew. Math. 137 (1910) 167–309. Taken from the catalogue of quoted results of this handout; place not confirmed.

Named by the formal text as related work, copied from its list of references with the state recorded there; not compared by us. For the file on heights: Gaudron, Pentes des fibrés vectoriels adéliques sur un corps global, 2006 (work found, no place given); Bombieri–Gubler, Heights in Diophantine Geometry, 2006 (place confirmed at second hand); K. Conrad, Zorn's lemma and some applications II, notes (not compared, no place given). For the file on determinants: Wolsey, Greedy algorithm for the submodular set covering problem, 1982; Bost, Algebraic leaves of algebraic foliations over number fields, 2001; Laurent, Linear forms in two logarithms and interpolation determinants, 1994 (for these three: work found, no place given).

Chapter V

Theorem A: short forms

- **The seesaw identity for the chain of theta lifts, and the index types of the data** (unit 141)
  read: note built, structured proofs checked. 312 declarations, 117 proof lines.

- **Theta functions of the CM curve: periods and scales, tight frames, the integral vacuum, Galois conjugates of jets, and divisibility of jets at non-split primes** (unit 150)
  read: note built, structured proofs checked. 89 declarations, 719 proof lines.

- **Columns and the chain of theta lifts: levels and branching on the spaces X_j, short ramified columns, divisibility of the chain at each prime (marked primes and scaled Hecke operators), and size bounds** (unit 145)
  read: note built, structured proofs checked. 105 declarations, 141 proof lines.

- **Elementary divisors of a Frobenius-twisted Hecke operator: twisting data, Frobenius inequalities for least terms, interlacing-cone volumes, flag counts and an approximate projector** (unit 144)
  read: note built, structured proofs checked. 78 declarations.

- **A crude bound for the class number of the standard definite hermitian genus, and divisibility of indices of submodules** (unit 149)
  read: note built, structured proofs checked. 11 declarations, 26 proof lines.

READBACK NOTE 21

The seesaw identity for the chain of theta lifts

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The rows of Theorem A of the write-up are Poincaré series on an indefinite unitary group of rank r, and the forms whose size and divisibility must be controlled are their images under a chain of m theta lifts through hermitian spaces of growing rank, the last of them definite. This note proves the identity that makes that control possible: up to one non-zero constant, the chain of lifts of the Poincaré series of a ramified column is a finite combination of chains of m theta lifts of the column itself between *definite* spaces, restricted at the end, and the coefficients of the combination have denominators only at the conductor of the twist. The proof unfolds the Poincaré series and then one lift at a time against Bessel periods; each unfolded period is a finite sum over cosets whose weights are reciprocals of numbers dividing a power of the norm of the conductor.

What this note is. The artifact grh-compact is a body of Lean code whose final theorems state the Riemann hypothesis and its analogue for Dirichlet L-functions. Its authors report that it compiles and that Lean's kernel accepts these theorems from the three standard axioms; the record of checks of the write-up says who replayed what. That is a fact about a formal text: no mathematician has read the argument through, and the result is a *claim*. This note is one of four notes of the unit *Seesaw* (Chapter V, Theorem A: short forms) and treats 80 of its 312 declarations: the indefinite chain of lifts, the stage functions of its unfolding, the lattice estimates and the identity itself. The proofs of this note's own declarations are written out; the statements taken from other units, from the other three notes of the unit and from the background library are marked as *Inputs*, with their owners. Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 80 declarations.

1 The statement

Setting 1.1 (the data)**.** Let $P = (N_0, d)$ be parameters and D admissible data for P (Chapter III of the write-up). The components of the record D are written with bare letters: the scales x, x_1, b, the exponent μ, the number of lifts m, the rank r, the thick set, the primes $g_0 \neq g_1$ and the integer π_g of $K = \mathbb{Q}(\sqrt{-d})$ with $\pi_g\overline{\pi_g} = g_0 g_1$. Everything else attached to D is written as a function of D:

(a) from Chapter III: the CM field $E(D)$ with its CM type, the set of its infinite places, and its Weil data (an element $\mathbf{j}$ with $\bar{\mathbf{j}} = -\mathbf{j}$, a finite idele α and the splitting character v); the number $f(D)$ of infinite places; the product $g(D) = g_0 g_1$; $t(D) = 5r$; the ranks $n_j(D)$, with $n_0(D) = r + 1 - m$;

(b) from the note *The characters, the level prime and the index sets of the data* of this unit: the number $\varrho(D)$, for which $m = 2\varrho(D) + 2$; the integer $\bar{\pi}_g(D)$

of $E(D)$, the conjugate of π_g, whose prime divisors are the *chosen places*; the level prime $N(D)$, which splits completely in $E(D)$ and is prime to $g(D)$, and an integer $\pi_N(D)$ of $E(D)$ with $\pi_N(D)\,\overline{\pi_N(D)} = N(D)$; *tame characters* ϑ of D, each with its finite set $Q(\vartheta)$ of places and its tame part ψ_ϑ, the number $\mathrm{n}(\vartheta) = \prod_{w\in Q(\vartheta)} \mathrm{N}\, w$, and the *twists* of D, the tame characters with $\psi_{\vartheta,w} \neq 1$ for all $w \in Q(\vartheta)$; the characters $\mu_{D,j}(\vartheta)$ of the norm-one adeles and their tame parts $\psi_{D,j}(\vartheta)$; the index sets $\underline{X}_j(D)$ and $\underline{Y}_j(D)$, where $\underline{Y}_j(D)$ is $\underline{Y}_0(D)$ followed by j blocks of $t(D)$ indices; the finite group $\mathsf{G}(D) = \prod \mathrm{GL}_{\underline{Y}_0(D)}(\mathbb{F}_w)$, the product over the chosen places w, and the element $e_0(D)$ of its group algebra;

(c) from the unit *Trace* (Chapter VII): for every j a hermitian space with frame $Y_j(D)$ on the index set $\underline{Y}_j(D)$ (Definition 2.1), of rank $r + j\,t(D)$, which for $j \geq m$ is the standard definite space; the space of forms of principal level on $Y_0(D)$ with its representation of $\mathsf{G}(D)$ (Definition 2.5);

(d) from the units *Columns* and *Exchange* (Chapters V and VII): the set $\Pi(D)$ of the places of $E(D)$ that divide $\pi_N(D)$; the standard definite spaces $X_j(D)$ on $\underline{X}_j(D)$, the columns $\mathrm{Col}(D)$ and the ramified columns $\mathrm{Col}^{\mathrm{ram}}(D)$, the definite chains and the maps $B^{\Psi}_{D,\vartheta}$ (Definitions 2.13 and 2.15);

(e) from the unit *Filling* (Chapter VI): the Poincaré series P_D (Definition 2.11).

The left side of the identity is *indefinite*: $\iota_{D,\vartheta}$ is the composite of m theta lifts $Y_0(D) \to Y_1(D) \to \ldots \to Y_m(D)$, the j-th formed with the character of the j-th power of ϑ, applied to forms of principal level on the indefinite space $Y_0(D)$ (Definition 2.8). The right side is *definite*: $B^{\Psi}_{D,\vartheta}$ is a finite combination, with coefficients Ψ, of composites of m theta lifts $X_0(D) \to \ldots \to X_m(D)$ of algebraic modular forms, followed by the restriction from $X_m(D)$ to $Y_m(D)$ (Definition 2.15).

Theorem A (the seesaw identity). In Setting 1.1 there is one non-zero complex number c_D, depending only on D, such that for every twist ϑ of D there is a finitely supported function $\Psi\colon \mathrm{Tr}(D) \to \mathbb{C}$ which is a good datum for ϑ (Definition 2.15) and satisfies

$$\iota_{D,\vartheta}(\mathrm{P}_D(c_D\,\varphi)) \;=\; B^{\Psi}_{D,\vartheta}(\varphi) \qquad \text{for every ramified column } \varphi \in \mathrm{Col}^{\mathrm{ram}}(D).$$

The constant c_D is chosen before ϑ, and Ψ before φ: one scalar for the datum D, one good datum for each twist, and the identity for all ramified columns at once; no condition is put on the column.

LEAN `Data.seesaw`

The proof has two engines. The first is an induction over the lifts. Its statement is abstract on purpose: the map S below will be specialised, in the proof of Theorem A, to a tensor of finite theta data.

Theorem B (unfolding). In Setting 1.1 let j be a natural number with $j \le 2\varrho(D)+2$, let $\mathcal{Y}$ be a grading of $Y_j(D)$ with norm $N(D)$, with a finite set X and a string of length l, where $l = 2\varrho(D)+1-j$ (and $l=0$ for $j = 2\varrho(D)+2$), and let a be a complex function of the tuples $\xi = (\xi_i)_{i\in\mathbb{N}}$ of real points $\xi_i \in G_\infty(Y_i(D))$. There is a complex number C such that for every tame character ϑ of D, every complex vector space M and every map S from the tuples $q = (q_i)_{i\in\mathbb{N}}$ of cosets $q_i \in G_f(Y_i(D))/\mathcal{J}_i(D,\vartheta)$ to M there is a vector $\mathsf{w} \in M$ such that:

(i) $\mathsf{w} \in \mathrm{Adm}(v_S, \mathrm{all}; \mathrm{n}(\vartheta), 1; Q(\vartheta)=\emptyset, \mathrm{cut}(D)^{\,j})$, where v_S sends a tuple $y=(y_i)$ of finite points $y_i \in K^{(\mathrm{n}(\vartheta))}(Y_i(D))$, integral at the places not dividing $\mathrm{n}(\vartheta)$, to $S\big((y_i\,\mathcal{J}_i(D,\vartheta))_i\big)$, and "all" is the set of all such tuples;

(ii) $\Omega(\mathsf{w}) = \mathsf{w}$ for every tuple $b=(b_i)$ with $b_i \in K_i^{\mathrm{col}}(D)$ and every linear map $\Omega\colon M\to M$ such that $\Omega(S(q)) = S(q')$ for all q, where $q_i' = i_{y_i(D),f}(b_i)\,q_i$ for $i<j$ and $q_i' = q_i$ for $i\ge j$;

(iii) for every form z on $Y_j(D)$ of the conjugate type $\overline{\mathcal{F}_j(D)}$, every ramified column $\varphi \in \mathrm{Col}^{\mathrm{ram}}(D)$ and every linear form $L\colon M\to\mathbb{C}$: *if* for all tuples ξ of real points and y of finite points with $y_{2\varrho(D)+2}=1$ and $y_0 \in K_D^{\mathrm{ch}}(Y_0(D))$ the function $F = \mathfrak{P}^{(j)}_{D,\vartheta,\varphi}[\xi,y;\mathcal{Y}]\cdot z$, in case it is integrable on $\Gamma_B(\mathcal{Y})\backslash G(Y_j(D))$, satisfies

$$\int_{\Gamma_B(\mathcal{Y})\backslash G(Y_j(D))} F \;=\; \mathsf{a}(\xi)\cdot L\big(S((y_i\,\mathcal{J}_i(D,\vartheta))_i)\big),$$

then

$$\int_{[G(Y_j(D))]} z\cdot\mathcal{L}^{(j)}(D,\vartheta)(\mathrm{P}_D\varphi) \;=\; C\cdot L(\mathsf{w}).$$

The notation $\mathrm{Adm}(\cdot,\cdot;\cdot,\cdot;\cdot,\cdot)$ for admissible combinations is recalled in Definition 2.12, the number $\mathrm{cut}(D)$ in Definition 2.17.

LEAN `Data.exists_isAdmissibleComb_unfolding`

The second engine cuts one period into a finite sum; the induction of Theorem B uses it once at every lift.

Theorem C (finite cuts). In Setting 1.1 let j be a natural number with $j \le 2\varrho(D)+1$, and let $\mathcal{Y} = \mathcal{Y}_j(D)$ be the grading of $Y_j(D)$ (Definition 2.6). There is a level A of $Y_j(D)$ with $i_{\mathcal{Y},f}(K_j^{\mathrm{col}}(D)) \subseteq A$ such that for every tame character ϑ of D there is a finite set T of cosets of $\mathcal{J}_j(D,\vartheta)$ in the finite adelic group of $Y_j(D)$ with these properties:

(i) for every $p \in B_f(\mathcal{Y}) \cap A$ and every coset t: $t \in T$ if and only if $p\,t \in T$;

(ii) if the coset of a finite point h lies in T, then h is integral at every place not dividing $\mathrm{n}(\vartheta)$, and for $j = 0$ it lies in the principal level $K_D^{\mathrm{ch}}(Y_0(D))$ at the chosen places;

(iii) let h be a finite adelic point whose coset lies in T, or for which there are a ramified column $\varphi \in \mathrm{Col}^{\mathrm{ram}}(D)$, tuples ξ, y of real and of finite points with $y_0 \in K_D^{\mathrm{ch}}(Y_0(D))$ and a real point g_∞ with $\mathfrak{P}^{(j)}_{D,\vartheta,\varphi}[\xi, y; \mathcal{Y}](g_\infty, h) \neq 0$; then the multiplicity $\mathrm{mult}_{B_f(\mathcal{Y}),A}(T; h\,\mathcal{J}_j(D,\vartheta))$ of Definition 2.20 divides a power of $\mathrm{n}(\vartheta)$.

Here $K_j^{\mathrm{col}}(D)$ is the level of stage j (Definition 2.13), $\mathcal{J}_j(D,\vartheta)$ the level of Definition 2.7, and $\mathfrak{P}^{(j)}_{D,\vartheta,\varphi}[\xi, y; \mathcal{Y}]$ the stage function of Definition 2.16.

LEAN `Data.exists_stage_cut`

2 The setting

Conventions. E is a CM field with a CM type, with complex conjugation $z \mapsto \overline{z}$; where theta functions occur, Weil data for E are fixed, and where integral Witt theory is used E is stably unramified. The letters I, I', X, X' and the letter H for an abstract group denote ambient objects: finite index sets and a group. A hermitian space with frame V over E has an index set and signs; a grading $\mathcal{Y}$ of V has a norm N, a finite set X and a string of length l; a second hermitian space with frame is V', with a grading $\mathcal{Y}'$ whose string has length l'. P denotes parameters and D admissible data for P; a tame character of D is ϑ. A commutative ring is R; index sets of blocks are R_0 and T_0. Measures are Haar measures or the measures they induce; a measurable group carries a left invariant measure. Matrices act on column vectors; g^* is the conjugate transpose of a matrix g. All statements about filling data concern filling data S of a shape, with a parameter, weights and a central element, as in Setting 2.10.

Hermitian spaces, their groups and levels. The following vocabulary belongs to the background library of the artifact; it is recalled here in the form in which this note uses it.

Definition 2.1 (spaces with frame; recalled from the background library)**.** Let $\mathbb{A}$ be the ring of adeles of E, $\mathbb{A}_f$ the ring of finite adeles and $\hat{\mathcal{O}}$ its subring of integral elements. A *hermitian space with frame* V over E consists of a finite index set I, signs $\mathrm{sgn}_V \colon I \to \{\pm 1\}$, an invertible hermitian matrix $J_V \in \mathrm{GL}_I(E)$, the Gram matrix of the basis of V, and a *frame* $x_V \in \mathrm{GL}_I(\mathbb{A})$ with $x_V^* J_V x_V$ equal to the diagonal matrix of the signs at every infinite place and to the identity at the finite places. Its finite part is $x_{V,f} \in \mathrm{GL}_I(\mathbb{A}_f)$, and $x_{V,f}\hat{\mathcal{O}}^I$ is the *lattice* of V; it is unimodular. The groups of V are written in the coordinates of the frame: $G_f(V) = \{g \in \mathrm{GL}_I(\mathbb{A}_f) : g^* g = 1\}$,

the product $G_\infty(V)$ over the infinite places of the unitary groups of the signs, and $G(V) = G_\infty(V) \times G_f(V)$; a finite point g is identified with $(1, g)$. In *rational coordinates* a point g is the matrix $x_V g x_V^{-1}$, which preserves J_V, and the *rational points* $\Gamma(V)$ are the g for which it has entries in E. The quotient $[G(V)] = \Gamma(V)\backslash G(V)$ carries the invariant measure fixed in the library, and for a subgroup Δ of $\Gamma(V)$ the quotient $\Delta\backslash G(V)$ carries the measure induced by the same Haar measure. V is *standard* if all signs are 1 and $x_V = 1$. The subgroup $K_0(V) = G_f(V) \cap \mathrm{GL}_I(\hat{\mathcal{O}})$ is the stabiliser of the lattice, and a *level* of V is an open subgroup of $G_f(V)$ contained in $K_0(V)$. For a finite place w, $K_w(V)$ is the subgroup of the elements of $K_0(V)$ that are 1 at all places other than w and $\overline{w}$. For a natural number n, $K^{(n)}(V)$ is the subgroup of the finite points whose component at w is integral with integral inverse for every finite place w not dividing n. For a finite idele β and a finite place w, $\mathrm{ord}_w\,\beta$ is the order of the component of β at w.

Definition 2.2 (gradings and Bessel groups; recalled from the background library)**.** Let V be a hermitian space with frame, with index set I, let $N \in E$ with $N \neq 0$ and $\overline{N} = N$, let X be a finite set and $l \geq 0$ an integer. A *grading* $\mathcal{Y}$ of V as X *plus a string of length* l *with norm* N consists of a bijection $e_{\mathcal{Y}} \colon X \sqcup \{0, \dots, l-1\} \to I$ and a constant $\mathfrak{c}(\mathcal{Y}) \in E$ with $\mathfrak{c}(\mathcal{Y}) \neq 0$ and $\overline{\mathfrak{c}(\mathcal{Y})} = \mathfrak{c}(\mathcal{Y})$, such that in the basis indexed through $e_{\mathcal{Y}}$ the Gram matrix J_V is the identity on X, pairs the string vectors z_a and $z_{a'}$ to 1 if $a + a' = l - 1$ and $a \neq a'$, to $-N$ if $a = a'$ and $2a = l - 1$, and to 0 otherwise, and pairs X with the string to 0. The *grade* of a basis vector is 0 on X and $\mathrm{gr}(z_a) = l - 1 - 2a$ on the string; z_{l-1-a} is the *reverse* of z_a. With matrices in rational coordinates:

- the *unipotent subgroup* $U(\mathcal{Y})$ of $G(V)$ consists of the points u for which $u - 1$ strictly raises grades;
- $i_{\mathcal{Y}} \colon G(X) \to G(V)$ is the inclusion of the group of the standard definite space on X, acting on the span of X and fixing the string vectors, and $i_{\mathcal{Y},f}$ is its finite part;
- the *Bessel group* $B(\mathcal{Y})$ consists of the points that fix every string vector of grade ≥ 0 up to vectors of strictly higher grade; it contains $U(\mathcal{Y})$ and the image of $i_{\mathcal{Y}}$. Its finite part is $B_f(\mathcal{Y}) = \{g \in G_f(V) : (1, g) \in B(\mathcal{Y})\}$, its rational points are $\Gamma_B(\mathcal{Y}) = \Gamma(V) \cap B(\mathcal{Y})$, and $[B(\mathcal{Y})]$ is the quotient of $B(\mathcal{Y})$ by $\Gamma_B(\mathcal{Y})$, with the finite measure induced by a Haar measure.

A function $F \colon G(V) \to \mathbb{C}$ is *generic for* $\mathcal{Y}$ if it is continuous, left invariant under $\Gamma_B(\mathcal{Y})$, and for every $u \in U(\mathcal{Y})$ such that $u - 1$ raises grades by at least 2 and every $g \in G(V)$ satisfies $F(ug) = \psi_E(\frac{\mathrm{j}}{2}\,\mathrm{tr}(\mathrm{sh}_{\mathcal{Y}}\,u))F(g)$; here ψ_E is the standard additive character of $\mathbb{A}/E$ and $\mathrm{sh}_{\mathcal{Y}}$ is the rational matrix that kills X and sends z_a to a multiple of z_{a+1} (and the last string vector to 0), the multiple being $\mathfrak{c}(\mathcal{Y})$ times $-N$ if z_a is the middle vector, times $-N\mathfrak{c}(\mathcal{Y})$ if l is even and z_a has grade 1, and times 1 otherwise.

Definition 2.3 (theta data, lifts and the unfolding step; recalled)**.** For hermitian spaces with frame V and V' the space $\mathrm{Hom}(V, V')$ is a hermitian space with frame. For a hermitian space with frame W the library provides the space $\mathcal{S}(W)$ of theta data with the Heisenberg operators $\varrho(y)$ of the adelic vectors y (and $\varrho_f(y)$ of the finite ones), the Weil representation ω of $G(W)$, the theta functional θ, the vacuum, the *finite part*

$\mathcal{W}_f(W)$ spanned by the translates $\varrho_f(y)$ of the vacuum, and the datum $v \otimes f$ made of a vector v of the Fock model at the infinite places and $f \in \mathcal{S}(W)$. For a character ν of the norm-one adeles the pair $G(V) \times G(V')$ acts on $\mathcal{S}(\mathrm{Hom}(V,V'))$ by the Weil representation twisted by ν and powers of v, the *twisted pair action* ω_ν^{pr} of the note *Two consecutive theta lifts*; its restriction to finite points acts on the finite part. The *theta lift* with datum Φ of a function F on $[G(V)]$ is $h \mapsto \int_{[G(V)]} \theta(\omega_\nu^{\mathrm{pr}}(g,h)\Phi)\, F(g)\, dg$. Let $\mathcal{Y}$, $\mathcal{Y}'$ be gradings of V, V' with the same norm and strings of lengths l and $l-1$. The *string map* $T_{\mathcal{Y},\mathcal{Y}'} \colon V \to V'$ is the rational matrix that kills X and sends the string vector z_a of $\mathcal{Y}$ to a multiple of the string vector of $\mathcal{Y}'$ with the same number a (and the last one to 0); the multiple is 1, $-N\mathfrak{c}(\mathcal{Y})$ or $\mathfrak{c}(\mathcal{Y})$ for z_a of positive, zero or negative grade if l is odd, and $\mathfrak{c}(\mathcal{Y})$ or 1 for positive or negative grade if l is even. A matrix $n \colon V \to V'$ is read as a vector of $\mathrm{Hom}(V,V')$ in the coordinates of the frames, and the *degree* of its position (i',i) is $\mathrm{gr}(i') - \mathrm{gr}(i)$. For $g \in G(V)$, $h \in G(V')$ the *constant term at the string map* $\mathrm{ct}_{\mathcal{Y},\mathcal{Y}'}(\nu,\Phi)(g,h)$ is the mean of

$$\theta\big(\varrho(n)\,\varrho(T_{\mathcal{Y},\mathcal{Y}'})\,\omega_\nu^{\mathrm{pr}}(g,h)\Phi\big)$$

over the adelic matrices $n \colon V \to V'$ with entries only in positions of positive degree, modulo the rational ones. For a function F on $G(V)$ and $x \in G(V)$ the *unfolded function* is

$$\big(\mathrm{unf}_{\mathcal{Y},\mathcal{Y}'}(\nu,\Phi;F,x)\big)(h) = \int_{[B(\mathcal{Y})]} F(\varsigma x)\ \mathrm{ct}_{\mathcal{Y},\mathcal{Y}'}(\nu,\Phi)(\varsigma x,h)\, d\varsigma.$$

Inputs 1 (the background library). In the notation of Definitions 2.2 and 2.3:

(a) *Genericity is inherited.* If F is generic for $\mathcal{Y}$ and $\mathfrak{c}(\mathcal{Y}') = \mathfrak{c}(\mathcal{Y})$, then $\mathrm{unf}_{\mathcal{Y},\mathcal{Y}'}(\nu,\Phi;F,x)$ is generic for $\mathcal{Y}'$.

(b) *One unfolding step.* If F is generic for $\mathcal{Y}$ and z is an automorphic, rapidly decreasing function on $G(V')$, then $\mathrm{unf}_{\mathcal{Y},\mathcal{Y}'}(\nu,\Phi;F,x)\cdot z$ is integrable on $\Gamma_B(\mathcal{Y}')\backslash G(V')$, and its integral there equals $\int_{[B(\mathcal{Y})]} F(\varsigma x) \int_{[G(V')]} \theta(\omega_\nu^{\mathrm{pr}}(\varsigma x,h)\Phi)\, z(h)\, dh\, d\varsigma$.

(c) *Lift and descent are transposes.* A datum Φ is an *arrow* from a type q on V to a type q' on V' if every form of type q is rapidly decreasing and its theta lift with datum Φ is a form of type q'. If Φ is an arrow from q to q' and, for the pair with the two groups exchanged, an arrow from $\overline{\mathsf{q}'}$ to $\overline{\mathsf{q}}$, then for a form F of type q and a form z of type $\overline{\mathsf{q}'}$ the integral over $[G(V')]$ of z times the lift of F equals the integral over $[G(V)]$ of F times the lift of z for the exchanged pair (the *descent* of z).

(d) *Adapted bases exist* (Hermite reduction over $\hat{\mathcal{O}}$), *and Witt's theorem holds over* $\hat{\mathcal{O}}$: if E is stably unramified and two matrices over $\hat{\mathcal{O}}$ with n rows and equally many columns have left inverses over $\hat{\mathcal{O}}$ and equal Gram matrices for the standard form, then a matrix in $\mathrm{GL}_n(\hat{\mathcal{O}})$ preserving the standard form carries one to the other.

Inputs 1, continued (the background library). In the same notation:

(e) *Discrete-series edges are arrows.* If the parameters, weights and signs of V and V' correspond in the sense of the unit *Exchange*, then the vector of the Fock model given by the product of the chosen joint harmonic polynomials is an *edge* between the two discrete-series types; and an edge, tensored with any element of the finite part, is an arrow in the sense of (c) from the first type to the second and, for the pair with the two groups exchanged, from the conjugate of the second to the conjugate of the first, for characters of the weights of the two spaces.

(f) *Data named modulo a modulus.* For a modulus $\mathfrak{m}$, integral finite points $k \in K_0(V)$, $k' \in K_0(V')$ and a residue matrix y modulo $\mathfrak{m}$, with $y' = \overline{k'}^{-1} y\, \overline{k}$ for the reductions $\overline{k}$, $\overline{k'}$ modulo $\mathfrak{m}$: the operator $\omega_\nu^{\mathrm{pr}}((1,k),(1,k'))$ carries the vector named on the vacuum by the indicator of y' to the vector named by the indicator of y, times the value of the twisting character of V at $\det k'$ and the inverse of the value of that of V' at $\det k$.

(g) *Constant terms are zero terms.* There is a splitting of $\mathrm{Hom}(V,V')$, in the sense of the note *Two consecutive theta lifts*, whose isotropic frame consists of the basis vectors at the positions of positive degree, such that for every datum Φ and every adelic matrix $y\colon V \to V'$ the mean of $\theta(\varrho(n)\varrho(y)\Phi)$ over the matrices n of positive degree, as in Definition 2.3, is the theta function of the zero term of $\varrho(y)\Phi$ along that splitting.

(h) *Kernels of definite pairs give theta lifts.* For standard definite spaces of ranks $0 < n < n'$, weights that fit, a level, a non-zero scale and characters of the two weights, the lifts of algebraic modular forms by the theta kernels of the pair form a family of theta lifts in the sense of Input 5 of the note *Two consecutive theta lifts*: the theta integral of a coefficient of a form against a datum with a polynomial vector at the infinite places is a coefficient of the lift of the form.

(i) *Cut-off functions and integration in stages.* A proper action of a group with a right invariant measure on a σ-compact Hausdorff space has a cut-off function, a non-negative function whose integral over every orbit is 1. If Δ is a subgroup of a subgroup H of a group G, and a map $(x,h) \mapsto h\,\mathsf{s}(x)$ carries the product of a measure on a space of parameters x and the measure of H to the measure of G, then for a measurable, left Δ-invariant function F that is integrable on $\Delta\backslash G$, the inner integrals $\int_{\Delta\backslash H} F(h\,\mathsf{s}(x))\,dh$ are integrable in x and their integral is $\int_{\Delta\backslash G} F$; the quotients carry the measures induced as in Definition 2.1.

Status of Inputs 1. For (a) and (b) the Lean file cites B. Wang Peng Jun, *Global theta lifting and automorphic periods associated to nilpotent orbits*, J. Number Theory 271 (2025), Prop. 3.2 and Thm. 4.5, and for the local case R. Gomez and C.-B. Zhu, GAFA 24 (2014); the authors mark the first as compared, with the place confirmed. Our catalogue has not confirmed these places: *reference to be confirmed.* For (d) the catalogue lists Witt's theorem over $\hat{\mathcal{O}}$ in the special case of orthonormal columns, which the library deduces from the statement with equal Gram matrices used here,

and gives for it U. A. First, *Witt's extension theorem for quadratic spaces over semiperfect rings*, Remark 2.1 and Corollary 4.9, applied on each orbit of conjugation on the finite places; for Hermite reduction it records "Hermite normal form over a valuation ring" without a place. For the first statement of (e) the catalogue cites Ichino, *Theta lifting for discrete series representations of real unitary groups* (arXiv:2002.09148), Theorem 4.1 and Lemma 7.8, pp. 35–36, for the compact types, and records the verdict of its second reading: the formal matching condition selects witnesses for a two-sided analytic statement beyond the published joint-type criterion. So the formal statement is not a special case or a weaker form of the cited ones: it goes beyond them, and its relation to the literature is unresolved; it is used here as the library proves it. Statement (c), the second statement of (e) and (f) are classed elementary. For (h) the library's heading is "Theta lift for a definite unitary pair as a class sum on algebraic modular forms"; for the class-sum formula behind it the catalogue names B. Gross, *Algebraic modular forms*, Israel J. Math. 113 (1999), Kashiwara–Vergne (1978) and R. Howe, *Transcending classical invariant theory*, J. Amer. Math. Soc. 2 (1989), place not confirmed. For (g) and (i) the catalogue has no entry: *reference to be confirmed.*

The data.

Definition 2.4 (index sets and blocks)**.** In Setting 1.1, for a natural number j the map $\mathrm{in}_{D,j}\colon \underline{Y}_j(D) \to \underline{Y}_{j+1}(D)$ is the inclusion of the index set of $Y_j(D)$ as the first summand of $\underline{Y}_{j+1}(D) = \underline{Y}_j(D) \sqcup \{0, \dots, t(D)-1\}$. Every $\underline{Y}_j(D)$ is non-empty. For sets R_0 and T_0 and a natural number j, let R_0 followed by j copies of T_0 be the disjoint union built by adding one copy of T_0 at a time; for an index i of it the *block number* $\mathrm{bl}_j(i) \in \{0, \dots, j\}$ is 0 if i lies in R_0 and ν if i lies in the ν-th copy of T_0. A square matrix indexed by such a set is *block upper triangular* if its entry at (i, i') vanishes whenever $\mathrm{bl}_j(i) > \mathrm{bl}_j(i')$, and for $\nu \in \{0, \dots, j\}$ its ν-th *diagonal block* is the square submatrix on the indices of block number ν.

LEAN `Data.inclY` · `Data.instNonemptyY` · `blockNumber`

Definition 2.5 (forms of principal level; recalled from the unit *Trace*)**.** In Setting 1.1, for a hermitian space with frame V over $E(D)$, $K_D^{\mathrm{ch}}(V)$ is the level of the elements of $K_0(V)$ that are congruent to 1 modulo every chosen place, the *principal level at the chosen places*. The forms on $Y_0(D)$ of the type of the rows that are fixed by $K_D^{\mathrm{ch}}(Y_0(D))$ are the *forms of principal level*; the group $\mathsf{G}(D)$ acts on them through the reductions of $K_0(Y_0(D))$ at the chosen places, a form F being sent by $\gamma \in \mathsf{G}(D)$ to γF.

Definition 2.6 (the gradings of the spaces $Y_j(D)$)**.** In Setting 1.1:

(a) for a natural number j with $j \le 2\varrho(D)+1$, $\mathcal{Y}_j(D)$ is the grading of $Y_j(D)$ as $\underline{X}_j(D)$ plus a string of length $2\varrho(D)+1-j = m-1-j$ with norm $N(D)$, with the bijection onto $\underline{Y}_j(D)$ fixed in the unit *Trace* and with the constant $\mathfrak{c}(\mathcal{Y}_j(D)) = -\left(\mathbf{j}\, g(D)\, \varpi(P,\mu)\right)^{-1}$, where $\varpi(P,\mu)$ is the generator of the different of $E(D)$ fixed in Chapter III;

(b) $\mathcal{Y}_{\mathrm{last}}(D)$ is the grading of the standard space $Y_m(D)$ as a set of $n_m(D)-1$ elements, the set $\underline{X}_m(D)$ without its last element, plus a string of length 0, with

norm $N(D)$ and constant 1. In the formal text it is given for every number M together with a proof that $m = M$.

The last element of $\underline{X}_j(D)$ is called the *point* of $\underline{X}_j(D)$, and $\underline{X}_j^\circ(D)$ is the set $\underline{X}_j(D)$ without its point.

LEAN `Data.gradingY` · `Data.lastGrading`

For $j \le m-1$ the space $Y_j(D)$ is thus the orthogonal sum of the definite space $X_j(D)$ and a string of length $m-1-j$, one vector shorter after each lift. A non-empty string makes the space indefinite (by its hyperbolic planes and, for odd length, its middle vector of norm $-N(D)$). For $j = m-1$ the string is empty and $Y_{m-1}(D)$ is definite; $Y_m(D)$ is the standard definite space of rank $n_m(D) - 1$, whose index set the grading $\mathcal{Y}_{\mathrm{last}}(D)$ identifies with $\underline{X}_m^\circ(D)$.

Definition 2.7 (one lift of the indefinite chain)**.** In Setting 1.1 let ϑ be a tame character of D and j a natural number.

(a) $\mathcal{J}_j(D,\vartheta)$ is the level of $Y_j(D)$ of the elements of $K_0(Y_j(D))$ that are congruent to 1 modulo every chosen place and modulo every place of $Q(\vartheta)$.

(b) $\psi_j^{\uparrow}(D,\vartheta) = \psi_{D,j+1}(\vartheta^{j+1})$ is the tame part of the character $\mu_{D,j+1}(\vartheta^{j+1})$ of the lift number $j+1$: a family of characters of the residue fields, equal to $\omega_{D,w}^{j+1}\psi_{\vartheta,w}^{j+1}$ at w.

(c) $\mathcal{K}_j(D,\vartheta)$ is the theta kernel of the pair $(Y_j(D), Y_{j+1}(D))$: the space of theta data of the space of homomorphisms from $Y_j(D)$ to $Y_{j+1}(D)$, with the twisted pair action ω_ν^{pr} of Definition 2.3 for $\nu = \mu_{D,j+1}(\vartheta^{j+1})$ and with the theta functional.

(d) $\Upsilon_j(D,\vartheta)$ is the element of the finite part of this space obtained from the vacuum by the operator named by the indicator function of the matrices that are integral and congruent to the block inclusion matrix of $\mathrm{in}_{D,j}$ modulo the product of the chosen places and of the places of $Q(\vartheta)$.

(e) $\mathcal{F}_j(D)$ is the space of the forms on $Y_j(D)$ of the type given by the parameter of $Y_j(D)$ fixed in the unit *Exchange*, with the Petersson product; $\overline{\mathcal{F}_j(D)}$ is the space of the forms of the conjugate type.

(f) $\mathfrak{d}_j(D,\vartheta)$ is the datum $v \otimes \Upsilon_j(D,\vartheta)$, where v is the vector of the Fock model given by the joint harmonic polynomial of the pair fixed in the unit *Exchange*.

(g) For $F \in \mathcal{F}_j(D)$, $\mathcal{L}_j(D,\vartheta)(F) \in \mathcal{F}_{j+1}(D)$ is $[K_0(Y_j(D)) : \mathcal{J}_j(D,\vartheta)]$ times the theta lift of F with the kernel $\mathcal{K}_j(D,\vartheta)$ and the datum $\mathfrak{d}_j(D,\vartheta)$ (the lift as a map between spaces of forms, as chosen in the library).

LEAN `Data.levelY` · `Data.liftTame` · `Data.thetaKernelY` · `Data.upsilonY` · `Data.FormY` · `Data.datumY` · `Data.liftY`

Definition 2.8 (the chain of lifts)**.** In Setting 1.1 let ϑ be a tame character of D. For a natural number j and a form F of principal level on $Y_0(D)$,

$$\mathcal{L}^{(j)}(D,\vartheta)(F) \;=\; \mathcal{L}_{j-1}(D,\vartheta)(\cdots \mathcal{L}_0(D,\vartheta)(F)\cdots) \in \mathcal{F}_j(D)$$

is the composite of the first j lifts. The space $Y_m(D)$ is definite and standard, and its forms of the type of $\mathcal{F}_m(D)$ are identified with algebraic modular forms on $Y_m(D)$; $\iota_{D,\vartheta}$ is the linear map that sends a form F of principal level on $Y_0(D)$ to the algebraic modular form that corresponds to $\mathcal{L}^{(m)}(D,\vartheta)(F)$.

LEAN Data.liftChainY · Data.ι

Definition 2.9 (the level of the image)**.** In Setting 1.1 let ϑ be a tame character of D.

(a) $\mathsf{P}_{\mathrm{last}}(D,\vartheta)$ is the group of the families $\kappa=(\kappa_w)$, indexed by the chosen places and the places of $Q(\vartheta)$, of matrices $\kappa_w \in \mathrm{GL}_{\underline{Y}_m(D)}(\mathbb{F}_w)$ that are block upper triangular for the blocks of $\underline{Y}_m(D)$ and equal to 1 at the chosen places.

(b) For $\kappa \in \mathsf{P}_{\mathrm{last}}(D,\vartheta)$, $\mathrm{lch}_{D,\vartheta}(\kappa)$ is the product, over these places w and over $\nu \in \{1,\dots,m\}$, of the values of $\psi^{\uparrow}_{\nu-1}(D,\vartheta)$ at w on the determinant of the ν-th diagonal block of κ_w.

(c) For a form F of principal level on $Y_0(D)$, $\iota^{\mathrm{lev}}_{D,\vartheta}(F)$ is $\iota_{D,\vartheta}(F)$, regarded as an element of the space of the algebraic modular forms on $Y_m(D)$ on which every element of $K_0(Y_m(D))$ with reductions $\kappa \in \mathsf{P}_{\mathrm{last}}(D,\vartheta)$ acts by $\mathrm{lch}_{D,\vartheta}(\kappa)$ (Proposition 3.4 shows that it lies there).

(d) For $\gamma \in \mathsf{G}(D)$, $\mathrm{lift}_{D,\vartheta}(\gamma)$ is a chosen element of $K_0(Y_m(D))$ whose reduction is the block diagonal matrix with blocks γ_w and 1 at every chosen place w, and 1 at the places of $Q(\vartheta)$.

LEAN Data.lastResLevel · Data.lastLevelChar · Data.iotaAtLevel · Data.groupLift

Setting 2.10 (filling data that fit a space; recalled from the unit *Filling*)**.** Let S be filling data over E of a shape with a number of hyperbolic planes (Chapter VI): among its components are a prime N, two further primes with product g_{S}, their chosen places, a generator ϖ_{S} of the different, a level K_{S} on the definite part X_0, and Bessel systems at the infinite places with their vectors $W(g_\infty)$ of Bessel functions. Let V be a hermitian space with frame, $\mathcal{Y}$ a grading of V as the set of the shape plus a string, and $\eta \in G_f(X_0)$; assume that S *fits* V with a parameter, weights, the grading $\mathcal{Y}$ and η: the Bessel systems have the types of the parameter and the weights, the bijection of $\mathcal{Y}$ is the standard one of the shape, the type of the parameter is an idempotent kernel, η is central, and $x_{V,f} = u\cdot\eta\cdot\Lambda\cdot k$ for a finite idele u, the lattice basis Λ of the filling data, and an integral matrix k with integral inverse that is congruent to 1 at the chosen places. For a vector $\phi=(\phi_a)$ of coefficients, indexed by the pairs a of a class of X_0 for K_{S} and a Bessel coordinate, and a subgroup Δ of $\Gamma(V)$, the *Poincaré series* is

$$\mathrm{P}_{\mathsf{S}}(\phi,\Delta)(g) = \sum_a \phi_a \sum_{\gamma\in\Delta} \mathrm{cb}_a(\gamma g),$$

where the cut-off Bessel function cb_a is a cut-off times a Bessel function of the real component if a translate of the finite component, determined by the class of a and by η, lies in the principal level $K^{\mathrm{pr}}_{\mathsf{S}}$ at the chosen places, and 0 otherwise. The *column coordinates* cc_{S} send an algebraic modular form on X_0 of level K_{S} to the vector of the Bessel coordinates of its values at representatives of the classes, the Weyl coordinates being changed to Bessel coordinates by a fixed linear isomorphism A.

Definition 2.11 (columns, the Poincaré series and the projection to rows)**.** In Setting 1.1, $\mathrm{Col}(D)$ is the space of the algebraic modular forms on $X_0(D)$ of the weight fixed in the unit *Columns* that are fixed by the level $K^{\mathrm{col}}_0(D)$, and $\mathrm{Col}^{\mathrm{ram}}(D)$ is its subspace of ramified columns (Definition 2.13). The filling data of D fit $Y_0(D)$ with the grading $\mathcal{Y}_0(D)$ (Proposition 4.1).

(a) cc_D is the linear isomorphism from $\mathrm{Col}(D)$ to the space of coefficient vectors that first translates a column by the central element $u_0(D)$ and then takes the column coordinates of Setting 2.10.

(b) P_D is the linear map from $\mathrm{Col}(D)$ to the forms of principal level on $Y_0(D)$ that sends φ to the Poincaré series of $\mathrm{cc}_D(\varphi)$ over all rational points (recalled from the unit *Filling*).

(c) For $\gamma \in \mathsf{G}(D)$, $\mathrm{pr}_D(\gamma)$ is the linear map from the forms of principal level on $Y_0(D)$ to the image of $e_0(D)$ that sends F to $|\mathsf{G}(D)| \cdot e_0(D)(\gamma F)$.

LEAN `Data.columnCoords` · `Data.goodProjTranslate`

The definite side.

Definition 2.12 (admissible combinations; recalled from the note *Two consecutive theta lifts* of this unit)**.** For a natural number n, $O^{(n)}$ is the ring of the complex numbers z with $|\jmath(z)|_\ell \le 1$ for every prime ℓ not dividing n and every ring homomorphism $\jmath\colon \mathbb{C} \to \mathbb{C}_\ell$. Let v be a family of vectors of a complex vector space, $\mathcal{I}$ a set of its indices, n and d natural numbers, p a proposition and B a real number. Then $\mathrm{Adm}(v, \mathcal{I}; n, d; p, B)$ is the set of the vectors $\sum_x r(x)v(x)$, where r is a finitely supported complex function with support in $\mathcal{I}$, with $d \cdot r(x) \in O^{(n)}$ for all x and, if p holds, with $\sum_x |\sigma(r(x))| \le B$ for every field automorphism σ of $\mathbb{C}$. Such a vector is an *admissible combination* of the $v(x)$, $x \in \mathcal{I}$, away from n, with denominator d and, under p, size B.

Definition 2.13 (columns, stages and definite chains; recalled from the units *Columns* and *Exchange* and from the note *Chains of definite theta lifts and the Gauss-sum operator* of this unit)**.** In Setting 1.1, for a natural number j:

(a) $K_j^{\mathrm{col}}(D)$ is the level of $X_j(D)$ of the elements that are congruent to 1 modulo the chosen places and whose last row, the row of the point, is congruent to $(0, \dots, 0, 1)$ modulo $\pi_N(D)$; $\mathrm{St}_j(D)$ is the space of the smooth algebraic modular forms on $X_j(D)$ of the weight of stage j; the ramified columns are the image on $\mathrm{Col}(D)$ of $\prod_{w \in \Pi(D)} (1 - z_w(D))$, where $z_w(D)$ is the translation by the scalar matrix of an integral adele of norm one, fixed in the unit *Columns*, that generates the units modulo w and is 1 at the places other than w and $\overline{w}$;

(b) for a tame character ϑ of D, $k_{D,\vartheta,j}$ is the theta kernel of the definite pair $(X_j(D), X_{j+1}(D))$ with the character $\mu_{D,j+1}(\vartheta^{j+1})$; its data are the elements of the finite part of $\mathcal{S}(\mathrm{Hom}(X_j(D), X_{j+1}(D)))$, and the lift with a datum Φ is a linear map $\mathrm{St}_j(D) \to \mathrm{St}_{j+1}(D)$;

(c) $\mathrm{Tr}(D)$ is the set of the tuples $t = (t_j)_{j \in \mathbb{N}}$ of finite adelic matrices t_j with rows $\underline{X}_{j+1}(D)$ and columns $\underline{X}_j(D)$; for a natural number q, $\mathrm{Tr}_q(D)$ is the set of the t for which $\pi_g \cdot t_j$ is integral at every place not dividing q, for all j (the weight is π_g, the conjugate of $\bar{\pi}_g(D)$); $\mathrm{v}_D(t)_j$ is the normalised vacuum of $\mathrm{Hom}(X_j(D), X_{j+1}(D))$ translated by $\varrho_f(t_j)$;

(d) for data $\Phi = (\Phi_j)$, $\mathrm{C}^{(j)}_{D,\vartheta}(\Phi)\colon \mathrm{St}_0(D) \to \mathrm{St}_j(D)$ is the composite of the first j lifts with the data $\Phi_0, \dots, \Phi_{j-1}$; for paired data $\Phi = (\Phi_i)$ with Φ_i in the tensor product of the data spaces of the lifts $2i+1$ and $2i+2$, $\mathrm{C}^{[p]}_{D,\vartheta}(\Phi)$ is the composite of the first p pairs of lifts, where a pair with datum $s \otimes s'$ is the lift with s'

after the lift with s, extended linearly; for a finitely supported $\Psi\colon \mathrm{Tr}(D)\to\mathbb{C}$, $\mathrm{C}^{(j)}_{D,\vartheta}[\Psi]=\sum_t \Psi(t)\,\mathrm{C}^{(j)}_{D,\vartheta}(\mathrm{v}_D(t).)$;

(e) $\mathrm{res}_D\colon \mathrm{St}_m(D)\to\{\text{algebraic modular forms on } Y_m(D)\}$ is the restriction of the unit *Columns*: restriction to the block diagonal elements $\mathrm{diag}(h,1)$, followed by a branching matrix and multiplication by a power of the conjugates of $\pi_N(D)$.

Definition 2.14 (translate sums, pair translates and reduced vacua)**.** In Setting 1.1 let i be a natural number. The *translate sum* of the note on the Gauss-sum operator, for the index sets $\underline{X}^{\circ}_{2i}(D)$, $\underline{X}_{2i+1}(D)$, $\underline{X}^{\circ}_{2i+2}(D)$, the set of places $\Pi(D)$ and the prime $N(D)$, is the endomorphism

$$\mathsf{T}_{N(D),\Pi(D)}(\underline{X}^{\circ}_{2i}(D),\underline{X}_{2i+1}(D),\underline{X}^{\circ}_{2i+2}(D)) \;=\; \sum_{y}\varrho_f(\mathrm{c}(y))\otimes\varrho_f(\mathrm{r}(y))$$

of $\mathcal{W}_f(\mathrm{Hom}(X_{2i}(D),X_{2i+1}(D)))\otimes\mathcal{W}_f(\mathrm{Hom}(X_{2i+1}(D),X_{2i+2}(D)))$: the sum over the vectors y with entries in $\prod_{w\in\Pi(D)}\mathbb{Z}/N(D)$, indexed by $\underline{X}_{2i+1}(D)$, where $\mathrm{c}(y)$ is the matrix whose last column, that of the point of $\underline{X}_{2i}(D)$, is $-y/N(D)$ at the places of $\Pi(D)$ and which vanishes elsewhere, and $\mathrm{r}(y)$ the matrix whose last row, that of the point of $\underline{X}_{2i+2}(D)$, is the conjugate of that column. For a family $v=(v_j)_{j\in\mathbb{N}}$ of finite data $v_j\in\mathcal{W}_f(\mathrm{Hom}(X_j(D),X_{j+1}(D)))$ the *pair translates* are

$$\tau_D(v)_i \;=\; \mathsf{T}_{N(D),\Pi(D)}(\underline{X}^{\circ}_{2i}(D),\underline{X}_{2i+1}(D),\underline{X}^{\circ}_{2i+2}(D))(v_{2i}\otimes v_{2i+1}),$$

and for $t\in\mathrm{Tr}(D)$ (Definition 2.13(c)) the *reduced vacuum* of t is the pure tensor

$$\mathrm{rv}_D(t)=\tau_D(\mathrm{v}_D(t).)_{\varrho(D)}\otimes\ldots\otimes\tau_D(\mathrm{v}_D(t).)_0,$$

formed with the family $\mathrm{v}_D(t).=(\mathrm{v}_D(t)_j)_j$ of the translated vacua of t.

LEAN `Data.reducedVacuum`

Definition 2.15 (good data and the map $B^{\Psi}_{D,\vartheta}$; recalled from the units *Columns* and *Exchange*)**.** In Setting 1.1 let ϑ be a tame character of D and $\Psi\colon\mathrm{Tr}(D)\to\mathbb{C}$ finitely supported. The map $B^{\Psi}_{D,\vartheta}$ from $\mathrm{Col}(D)$ to the algebraic modular forms on $Y_m(D)$ is $\varphi\mapsto\mathrm{res}_D(\mathrm{C}^{(m)}_{D,\vartheta}[\Psi](\varphi))$. For a twist ϑ of D the function Ψ is a *good datum for* ϑ if

(i) its support lies in $\mathrm{Tr}_{\mathrm{n}(\vartheta)N(D)}(D)$, its values lie in $O^{(\mathrm{n}(\vartheta))}$, and if $Q(\vartheta)=\emptyset$, then $\sum_t|\sigma(\Psi(t))|\le\exp(x^{60})$ for every field automorphism σ of $\mathbb{C}$;

(ii) it is *jointly invariant*: for every tuple (a_j) with $a_j\in K^{\mathrm{col}}_j(D)$ and $a_j=1$ for $j\ge m$, and every linear map β on the tensor product of the data spaces of the first m lifts, $\sum_t\Psi(t)\,\beta(\bigotimes_{j<m}\omega(a_j,a_{j+1})\mathrm{v}_D(t)_j)=\sum_t\Psi(t)\,\beta(\bigotimes_{j<m}\mathrm{v}_D(t)_j)$;

(iii) for every $\varphi\in\mathrm{St}_0(D)$:

$$\mathrm{C}^{(m)}_{D,\vartheta}[\Psi](\varphi)=\sum_t\Psi(t)\,\mathrm{C}^{[\varrho(D)+1]}_{D,\vartheta}(\tau_D(\mathsf{P}_{\Pi}\mathrm{v}_D(t).).)(\varphi),$$

where P_{Π} is the projection of the finite part to the vectors that are vacua at the places of $\Pi(D)$ and their conjugates, applied to every member of the family $\mathrm{v}_D(t).$, and the paired data are the pair translates of Definition 2.14.

Stage functions.

Definition 2.16 (stage functions)**.** In Setting 1.1 let ϑ be a tame character of D, φ a column in $\mathrm{Col}(D)$, $\xi = (\xi_i)_{i\in\mathbb{N}}$ a tuple of real points $\xi_i \in G_\infty(Y_i(D))$ and $y = (y_i)_{i\in\mathbb{N}}$ a tuple of finite points $y_i \in G_f(Y_i(D))$. For a natural number j with $j \le 2\varrho(D)+2$, a grading $\mathcal{Y}$ of $Y_j(D)$ with norm $N(D)$ and a point g of $G(Y_j(D))$, the complex number $\mathfrak{P}^{(j)}_{D,\vartheta,\varphi}[\xi, y; \mathcal{Y}](g)$ is defined by recursion on j:

- $\mathfrak{P}^{(0)}_{D,\vartheta,\varphi}[\xi, y; \mathcal{Y}]$ is the Poincaré series $\mathrm{P}_{\mathsf{S}}(\mathrm{cc}_D(\varphi), \Gamma_B(\mathcal{Y}))$ of Setting 2.10 over the rational points of the Bessel group of $\mathcal{Y}$, for the filling data of D;
- $\mathfrak{P}^{(j+1)}_{D,\vartheta,\varphi}[\xi, y; \mathcal{Y}]$ is the unfolded function $\mathrm{unf}_{y_{j(D)},\mathcal{Y}}(\nu, \mathfrak{d}_j(D,\vartheta); F, (\xi_j, y_j))$ of Definition 2.3, with $\nu = \mu_{D,j+1}(\vartheta^{j+1})$ and $F = \mathfrak{P}^{(j)}_{D,\vartheta,\varphi}[\xi, y; \mathcal{Y}_j(D)]$.

LEAN `Data.unfoldStage`

Definition 2.17 (evaluated data)**.** In Setting 1.1:

(a) for a natural number i, $\mathrm{ev}_{D,i}$ is the *finite evaluation* of the pair of lifts $Y_{2i}(D) \to Y_{2i+1}(D) \to Y_{2i+2}(D)$, a linear map

$$\mathcal{S}(\mathrm{Hom}(Y_{2i+1}(D), Y_{2i}(D)^- \oplus Y_{2i+2}(D))) \longrightarrow \mathcal{W}_f(\mathrm{Hom}(X_{2i}(D), X_{2i+1}(D))) \otimes \mathcal{W}_f(\mathrm{Hom}(X_{2i+1}(D), X_{2i+2}(D))),$$

where the exponent $^-$ denotes the space with the opposite form: it is the finite evaluation, in the sense of the note *Two consecutive theta lifts*, of the middle arrow pair of the gradings $\mathcal{Y}_{2i}(D), \mathcal{Y}_{2i+1}(D), \mathcal{Y}_{2i+2}(D)$ if $i < \varrho(D)$, of the last arrow pair of $\mathcal{Y}_{2\varrho(D)}(D), \mathcal{Y}_{2\varrho(D)+1}(D), \mathcal{Y}_{\mathrm{last}}(D)$ formed with $\pi_N(D)$ if $i = \varrho(D)$, and 0 if $i > \varrho(D)$;

(b) $\mathrm{cut}(D) = g(D)^{\,f(D)\,n_m(D)^2}$;

(c) for a tame character ϑ, a tuple $y = (y_j)$ of finite points $y_j \in G_f(Y_j(D))$ and a natural number j, $\Upsilon_j(D, \vartheta; y) = \omega_\nu^{\mathrm{pr}}(y_j, y_{j+1})\, \Upsilon_j(D, \vartheta)$ with $\nu = \mu_{D,j+1}(\vartheta^{j+1})$;

(d) for a natural number i, $\mathrm{ed}_i(D, \vartheta; y) = \mathrm{ev}_{D,i}(\Upsilon_{2i}(D, \vartheta; y) \boxtimes \Upsilon_{2i+1}(D, \vartheta; y))$, where $\boxtimes$ is the arrow product of two data, a datum on $\mathrm{Hom}(Y_{2i+1}(D), Y_{2i}(D)^- \oplus Y_{2i+2}(D))$.

LEAN `Data.pairEvalY` · `Data.cutBound` · `Data.translatedUpsilon` · `Data.evaluatedDatum`

Lattices along a string.

Definition 2.18 (adapted bases)**.** Let $e \colon X \sqcup \{0, \dots, l-1\} \to I$ be a bijection. The *flag order* of an index $i \in I$ is $\mathrm{fl}_e(i) = 2l$ if i comes from X and $\mathrm{fl}_e(i) = 4a+1$ if i is the a-th string index; a square matrix indexed by I is *triangular for the flag* if its entry at (i, i') vanishes whenever $\mathrm{fl}_e(i) > \mathrm{fl}_e(i')$. Let $\mathcal{Y}$ be a grading of a hermitian space with frame V, let $h, b \in \mathrm{GL}_I(\mathbb{A}_f)$ and let $\beta = (\beta_a)_{a<l}$ be a family of finite ideles. Then (b, β) is adapted to h for $\mathcal{Y}$ means: b is triangular for the flag of e_y; $b^{-1}h \in \mathrm{GL}_I(\hat{\mathcal{O}})$, so that the columns of b and of h span the same lattice; and the diagonal entry of b at the a-th string index is β_a, for every a. For a finite point $y \in G_f(V)$, a pair (b, β) is adapted at y for $\mathcal{Y}$ if (b, β) is adapted to $x_{V,f}\, y$ for $\mathcal{Y}$; then b is an *adapted basis* of the lattice of y and β its *string diagonal*.

LEAN `flagOrder` · `Grading.IsAdaptedBasis` · `Grading.IsAdaptedAt`

Definition 2.19 (near lines)**.** Let $\mathcal{Y}$ be a grading of V with a string of length l, $\mathfrak{a}$ a finite idele, Q a finite set of finite places, n and k natural numbers and $h \in G_f(V)$. Then h has near lines for $(\mathcal{Y}; \mathfrak{a}, Q; n, k)$ means that there is a pair (b, β) that is adapted at h for $\mathcal{Y}$ such that for every string index a and every finite place w

$$\left|\operatorname{ord}_w \beta_a - \operatorname{ord}_w\left((\mathfrak{a}/\overline{\mathfrak{a}})^n\, \epsilon_a\right)\right| \;\le\; \begin{cases} k & \text{if } w \in Q \text{ or } \overline{w} \in Q, \\ 0 & \text{otherwise,} \end{cases}$$

where $\epsilon_a = 1$ if l is odd and, if l is even, $\epsilon_a = \mathfrak{a}^{-1}$ for $2a < l$ and $\epsilon_a = \overline{\mathfrak{a}}$ otherwise. In words: the lattice of h has an adapted basis whose string diagonal has the orders of the reference ideles exactly outside Q and its conjugate, and within k on them.

LEAN `Grading.HasNearLines`

Definition 2.20 (multiplicity of a cut)**.** Let H be a group with subgroups S, A and J, let T be a finite set of left cosets of J in H and t a left coset of J. The *multiplicity* $\operatorname{mult}_{S,A}(T; t)$ is the number of the left cosets of $S \cap A$ in S that contain an element ς with $\varsigma^{-1} t \in T$; it is 0 if there are infinitely many. When T is stable under left multiplication by $S \cap A$, as in all uses, every element of such a coset has this property.

LEAN `coverMultiplicity`

Inputs 2 (other units and the other notes of this unit). In Setting 1.1:

(a) (*Trace*, unit 140) for $j < m$ the frame of $Y_j(D)$ has finite part $\Lambda_j\, k_j$ with Λ_j the graded lattice basis of $\mathcal{Y}_j(D)$ for the ideles $\mathfrak{a}(D) = \alpha/\overline{\hat{\pi}_g(D)}$, the principal idele $\hat{\pi}_N(D)$ of $\pi_N(D)$ and the scale $u_j(D) = (\mathfrak{a}(D)/\overline{\mathfrak{a}(D)})^{\lceil (m-j)/2 \rceil}$, and k_j integral with integral inverse and congruent to 1 at the chosen places; here $\hat{\pi}_g(D)$ is a finite idele generating the ideal of $\bar{\pi}_g(D)$, equal to 1 at the places not dividing it, and $\hat{\pi}_N(D)$ is the finite idele with the components of $\pi_N(D)$. If w or $\overline{w}$ lies in $Q(\vartheta)$ for a tame character ϑ, then w divides $\mathrm{n}(\vartheta)$ and divides neither $\bar{\pi}_g(D)$ nor $N(D)$. Moreover $m = 2\varrho(D) + 2$.

(b) (*Columns*, unit 145) $\Pi(D)$ is a split level for $N(D)$: it is disjoint from its conjugate, the places above $N(D)$ are those of $\Pi(D)$ and their conjugates, and $N(D)$ is a uniformiser at each of them. For every $w \in \Pi(D)$, every ramified column has the form $\psi - z_w(D)\psi$ with $\psi \in \mathrm{Col}(D)$. If $t \in \mathrm{Tr}_q(D)$ and S is a set of places prime to q whose conjugates are not chosen, then $\mathrm{v}_D(t)_j$ is fixed by the projection to the vectors that are vacua at the places of S and their conjugates.

(c) (*Exchange*, unit 147) $|\underline{Y}_j(D)| \le n_m(D)$ for $j \le m$. For $j < m$ the parameters, weights and signs of $Y_j(D)$ and $Y_{j+1}(D)$ correspond, in the sense required by Inputs 1(e).

Inputs 2, continued: the unit *Filling*. In Setting 1.1:

(d) (*Filling*, unit 148) $K_0^{\mathrm{col}}(D)$ is the preimage of the principal level at the chosen places under the inclusion of the finite points of $X_0(D)$ into those of $Y_0(D)$, and the principal level $K_{\mathsf{S}}^{\mathrm{pr}}$ of the filling data of D is $K_D^{\mathrm{ch}}(Y_0(D))$ (in the formal text, by definition). *Terms of the series*, for admissible filling data as in Setting 2.10: (1) if a term $\mathrm{cb}_a(\gamma' g)$, with γ' a rational point of the Bessel group, does not vanish at a point g whose finite component, translated by the representative of a class and by η, lies in the principal level, then a belongs to that class and γ' to the group of the rational unipotent elements that the class conjugates into the principal level; (2) over that group the cut-offs sum to 1, so that the terms $\mathrm{cb}_a(\gamma' g)$ sum to the Bessel function of the real component; (3) *strong approximation for the unipotent group*: for a finite point u of the unipotent group of the shape and any finite point h there is a rational unipotent ς with $h^{-1}u^{-1}\varsigma h$ in the principal level. *Unfolding of Poincaré series*: for admissible filling data as in Setting 2.10, a form z on V and a subgroup Δ of $\Gamma(V)$, the product of z and $\mathrm{P}_{\mathsf{S}}(\phi,\Delta)$ is integrable on $\Delta\backslash G(V)$, with integral $\sum_a \phi_a \int_{G(V)} z\cdot \mathrm{cb}_a$. *The last stage*: there is a linear map R from $\mathrm{St}_m(D)$ to $\mathcal{F}_m(D)$ such that (1) for all tame ϑ, finitely supported Ψ, forms F of principal level and columns φ, if $\mathcal{L}^{(m)}(D,\vartheta)(F) = R(\mathrm{C}^{(m)}_{D,\vartheta}[\Psi](\varphi))$ then $\iota_{D,\vartheta}(F) = B^{\Psi}_{D,\vartheta}(\varphi)$; (2) for every linear functional $\mathtt{t}$ on the values of forms of stage m there is a complex number $C_{\mathtt{t}}$ such that for every $\Xi\in\mathcal{F}_m(D)$, every $\psi\in\mathrm{St}_m(D)$ and every function F on $G(Y_m(D))$ with $F(i_{y_{\mathrm{last}}(D)}(h)) = \langle \mathtt{t},\psi\rangle(\mathrm{diag}(h,1))$ for all h: $\int_{[G(Y_m(D))]} F\overline{\Xi} = C_{\mathtt{t}}\cdot\langle \Xi, R\psi\rangle$. Here $\langle \mathtt{t},\psi\rangle(g) = \mathtt{t}(g_\infty^{-1}\psi(g_f))$ is the *coefficient* of ψ at $\mathtt{t}$, and $\langle\,,\rangle$ the Petersson product.

Inputs 2, continued: the unit *Curve* and the other notes of this unit. In Setting 1.1:

(e) (*Curve*, unit 150) *named data are admissible combinations*: let W be a hermitian space with frame, with index set I, let $\mathfrak{m}$ be a modulus, g' an integer of E and q a natural number with $g'q \in \mathfrak{m}$, let $\mathfrak{e}$ be an integral finite adele with $\mathfrak{e}q \equiv 0$ modulo $\mathfrak{m}$ (when none is named, $\mathfrak{e} = g'$), let Ψ_0 be a function of residue vectors modulo $\mathfrak{m}$ indexed by I (residue matrices, for a space of homomorphisms) with values in $O^{(q)}$, p a proposition and B a real number such that, if p holds, $\sum_x |\sigma(\Psi_0(x))| \leq B$ for every field automorphism σ of $\mathbb{C}$, and let u be a vector of $\mathcal{S}(W)$ and u_0 its projection to the vectors that are vacua at the places of $\mathfrak{m}$ and their conjugates. Then the vector named at the places of $\mathfrak{m}$ by Ψ_0 on u is an admissible combination of the translates $\varrho_f(t)u_0$, over the finite vectors t with $\bar{\mathfrak{e}}\,t$ integral at the places not dividing q, away from q, with denominator the absolute norm of the ideal of g' to the power $|I|$ (which is 0 for $g' = 0$) and, under p, size B. The weight $\bar{\mathfrak{e}}$ of the translates is thus set by $\mathfrak{e}$ and not by g'.

(f) (the note *Two consecutive theta lifts* of this unit) the calculus of admissible combinations (substitution multiplies denominators and sizes; linear and bilinear images; pure tensors; multiplication by a root of unity; clearing a constant); the Weil translate of a translated vacuum by a finite point g is an admissible combination of translated vacua, with coefficient governed by the finite multiplier of g; the finite multiplier and the values of the characters $\mu_{D,j}(\vartheta)$ on determinants have ℓ-adic absolute value 1 at points integral above ℓ; *support of the zero term*, *the double unfolding*, *the double lift*, *equivariance of the finite evaluation* and *admissible evaluation*, each quoted where it is used (§5, §9).

(g) (the note *Chains of definite theta lifts and the Gauss-sum operator* of this unit) for finite points a_j of the spaces $X_j(D)$ and paired data Φ_i, $\mathrm{C}^{[p]}_{D,\vartheta}((\omega(a_{2i}, a_{2i+1}) \otimes \omega(a_{2i+1}, a_{2i+2}))\Phi_i)(a_0\varphi) = a_{2p}\,\mathrm{C}^{[p]}_{D,\vartheta}(\Phi)(\varphi)$, the points acting on forms by right translation; the characters $\mu_{D,j}(\vartheta^j)$ are unramified at the places above $N(D)$, and a scalar pair acts on the finite part through its quotient.

3 The chain of lifts and its level

Lemma 3.1. *In Setting 1.1 let ϑ be a tame character of D and j a natural number.*

(i) *For every place $w \in Q(\vartheta)$, the component of $\psi_{D,j}(\vartheta)$ at w is $\psi_{\vartheta,w}$.*

(ii) *$\mathcal{J}_j(D,\vartheta) \subseteq K^{\mathrm{ch}}_D(Y_j(D))$.*

(iii) *If $j < m$, then the datum $\mathfrak{d}_j(D,\vartheta)$ is an arrow, for the kernel $\mathcal{K}_j(D,\vartheta)$, from the type of $\mathcal{F}_j(D)$ to the type of $\mathcal{F}_{j+1}(D)$, and for the pair with the two groups exchanged it is an arrow from the type of $\overline{\mathcal{F}_{j+1}(D)}$ to the type of $\overline{\mathcal{F}_j(D)}$ (Inputs 1(c)).*

Proof. (i) $\psi_{D,j}(\vartheta) = \omega_D^j \psi_\vartheta$, and $\omega_{D,w}$ is trivial at the places that are not chosen; a place of $Q(\vartheta)$ does not divide $\bar{\pi}_g(D)$ (Inputs 2(a)). (ii) The condition that defines

$\mathcal{J}_j(D,\vartheta)$ contains the condition at the chosen places. (iii) The parameters, weights and signs of $Y_j(D)$ and $Y_{j+1}(D)$ correspond (Inputs 2(c)). By the first statement of Inputs 1(e) the vector v of the Fock model given by the joint harmonic is an edge between the two types, and by the second statement the datum $\mathfrak{d}_j(D,\vartheta) = v \otimes \Upsilon_j(D,\vartheta)$, an edge tensored with an element of the finite part, is an arrow in both directions; the character of the kernel has the weights of the two spaces (the note on the characters of the data). The status of this input, on which every lift of the chain rests, is given after Inputs 1. □

LEAN `Data.liftTamePart_eq_of_mem_Q` · `Data.levelY_le_chosenLevel` · `Data.isArrow_datumY`

Lemma 3.2 (the finite datum is an eigenvector of the level)**.** *In Setting 1.1 let ϑ be a tame character of D and j a natural number. Let $k \in K_0(Y_j(D))$ and $k' \in K_0(Y_{j+1}(D))$, and let $g = (g_w)$ and $g' = (g'_w)$ be the families of their reductions at the chosen places and the places of $Q(\vartheta)$. Assume that for every such place w the first columns of g'_w, those indexed by $\underline{Y}_j(D)$ through $\mathrm{in}_{D,j}$, form the matrix $\binom{g_w}{0}$. Then, for the Weil representation ω of the kernel $\mathcal{K}_j(D,\vartheta)$,*

$$\omega\big((1,k),(1,k')\big)\,\Upsilon_j(D,\vartheta) \;=\; \Big(\prod_w \frac{\psi_j^{\uparrow}(D,\vartheta)_w(\det g'_w)}{\psi_j^{\uparrow}(D,\vartheta)_w(\det g_w)}\Big)\cdot \Upsilon_j(D,\vartheta).$$

Proof. $\Upsilon_j(D,\vartheta)$ is named on the vacuum by the indicator of one residue matrix, the block inclusion $\binom{1}{0}$ modulo the product $\mathfrak{m}$ of the places in question. The Weil operator of an integral pair (k,k') carries the vector named by the indicator of a residue matrix y to the vector named by the indicator of $\overline{k'}\,y\,\overline{k}^{-1}$, the bars denoting reduction modulo $\mathfrak{m}$, times the values of the two twisting characters at $\det k$ and $\det k'$ (Inputs 1(f)). The hypothesis says exactly that $\overline{k'}\binom{1}{0}\overline{k}^{-1} = \binom{1}{0}$ at every place. The twisting character is $\mu_{D,j+1}(\vartheta^{j+1})$ times a power of υ; it has tame part $\psi_j^{\uparrow}(D,\vartheta)$, so its value at the determinant of an element of $K_0(\,\cdot\,)$ is the product of the values of the tame part at the reductions of the determinant, and υ is unramified. This gives the quotient. □

LEAN `Data.thetaKernelY_omega_upsilonY`

Lemma 3.3 (blocks and counts)**.** *(i) For sets R_0, T_0, a natural number j and an index i of R_0 followed by j copies of T_0: $\mathrm{bl}_j(i) = 0$ if and only if i lies in the first block R_0.*

(ii) For finite sets R_0 and T_0 and a natural number j there is a natural number ν such that for every commutative ring R the group of the invertible matrices over R, indexed by R_0 followed by j copies of T_0, that are block upper triangular has $|\,\mathrm{GL}_{R_0}(R)| \cdot |\,\mathrm{GL}_{T_0}(R)|^j \cdot |R|^{\nu}$ elements (the cardinality of an infinite set being counted as 0).

(iii) Let $\mathfrak{X}$ be a system of level places of E: two finite families of finite places, the chosen places and the twist places, no two of them equal or conjugate. For a finite set R_0, the kernel of the reduction of $K_0(\,\cdot\,)$ on R_0 modulo the product $\mathfrak{m}_1$ of all these places equals the kernel of the family of the reductions at these places.

(iv) For $\mathfrak{X}$ as in (iii) and a natural number e, the residue ring of E modulo the e-th power of the product of the chosen places times the product of the twist places has $(\prod_w \mathrm{N}\,w)^e \cdot \prod_v \mathrm{N}\,v$ elements, w running over the chosen and v over the twist places.

(v) For a finite set R_0 and $\mathfrak{X}$ as in (iii), the order of GL_{R_0} of the residue ring modulo $\mathfrak{m}_1$ equals the index in $K_0(\,\cdot\,)$ of the level of the elements congruent to 1 at all places of $\mathfrak{X}$.

Proof. (i) By induction on j: a new copy of T_0 gets the new number $j+1 \neq 0$, and old indices keep their number. (ii) By induction on j, with $\nu = 0$ for $j = 0$. A block upper triangular matrix for $j+1$ has the block form $\left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$ with A block upper triangular for j, and it is invertible exactly when A and C are; this is a bijection onto triples (A, B, C) with B arbitrary, so ν grows by the number of entries of B. (iii) Two integral matrices are congruent modulo $\mathfrak{m}_1$ if and only if they are congruent modulo each of its prime factors, which are distinct. (iv) The norm is multiplicative. (v) By (iii) the level is the kernel of the reduction modulo $\mathfrak{m}_1$, which is surjective onto GL_{R_0} of the residue ring. □

LEAN `blockNumber_eq_zero` · `card_parabolic_blockNumber` · `LevelPlaces.ker_reductionK₀_eq` · `LevelPlaces.card_levelModulus_resRing` · `LevelPlaces.card_GL_levelModulus`

Proposition 3.4 (the chain is equivariant)**.** *In Setting 1.1 let ϑ be a tame character of D, $\gamma \in \mathsf{G}(D)$, F a form of principal level on $Y_0(D)$ and $k \in K_0(Y_m(D))$, and let $g = (g_w)$ be the family of the reductions of k at the chosen places and the places of $Q(\vartheta)$. Assume that every g_w is block upper triangular for the blocks of $\underline{Y}_m(D)$, and that at every chosen place w the diagonal block of number 0 of g_w is γ_w. Then*

$$k \cdot \iota_{D,\vartheta}(F) \;=\; \Big(\prod_{w}\prod_{\nu=1}^{m} \psi^{\uparrow}_{\nu-1}(D,\vartheta)_w\big(\det g_w^{(\nu)}\big)\Big) \cdot \iota_{D,\vartheta}(\gamma F),$$

where $k\cdot$ is right translation of algebraic modular forms and $g_w^{(\nu)}$ the ν-th diagonal block of g_w.

Proof. The identification of forms on the definite space with algebraic modular forms commutes with right translation, so the claim is an identity for $\mathcal{L}^{(m)}(D,\vartheta)(F)$. We prove it for $\mathcal{L}^{(j)}(D,\vartheta)(F)$ and every j for which the data of the first j lifts are arrows, by induction on j; for $j = m$ this is Lemma 3.1(iii). For $j = 0$ the chain is the identity, the product is empty, and k acts on a form of principal level through its reductions at the chosen places, which are γ. For the step let g'_w be the upper left block of g_w, on the indices of $\underline{Y}_j(D)$. It is block upper triangular, and $\det g_w = \det g'_w \cdot \det g_w^{(j+1)}$, so g'_w is invertible; choose $k' \in K_0(Y_j(D))$ with the reductions g'_w. For an arrow, right translation by k of the lift of a form equals the lift, with the datum moved by $\omega((1,k'),(1,k))$, of the form translated by k'. The Weil operator acts on $\mathfrak{d}_j(D,\vartheta) = v \otimes \Upsilon_j(D,\vartheta)$ through the finite datum, and by Lemma 3.2 (its hypothesis is the block triangularity) it multiplies $\Upsilon_j(D,\vartheta)$ by $\prod_w \psi^{\uparrow}_j(D,\vartheta)_w(\det g_w)/\psi^{\uparrow}_j(D,\vartheta)_w(\det g'_w) = \prod_w \psi^{\uparrow}_j(D,\vartheta)_w(\det g_w^{(j+1)})$. The induction hypothesis for k' supplies the factors of the blocks $1,\dots,j$, the diagonal blocks of g'_w being those of g_w. □

LEAN `Data.algFormRep_iota`

Taking $\gamma = 1$ and k with reductions in $\mathsf{P}_{\mathrm{last}}(D,\vartheta)$ shows that $\iota_{D,\vartheta}(F)$ lies in the space of Definition 2.9(c), so that $\iota^{\mathrm{lev}}_{D,\vartheta}(F)$ is defined.

Corollary 3.5 (the chain commutes with the finite group)**.** *In Setting 1.1, for a tame character ϑ of D, $\gamma \in \mathsf{G}(D)$ and a form F of principal level on $Y_0(D)$: $\iota_{D,\vartheta}(\gamma F) = \mathrm{lift}_{D,\vartheta}(\gamma) \cdot \iota_{D,\vartheta}(F)$.*

Proof. Apply Proposition 3.4 to $k = \mathrm{lift}_{D,\vartheta}(\gamma)$. Its reductions are block diagonal with blocks γ_w and 1; by Lemma 3.3(i) the entries outside the block of number 0 are those of the identity, so every diagonal block of number $\nu \geq 1$ is the identity and the product of character values is 1. □

LEAN `Data.iota_principalRep`

4 The Poincaré series at Bessel points

Proposition 4.1. *In Setting 1.1 the filling data of D fit the space $Y_0(D)$, with the parameter of the rows, the weights of the columns, the grading $\mathcal{Y}_0(D)$ and the central element $\eta = u_0(D)^{-1}$, in the sense of Setting 2.10.*

Proof. Four of the five conditions are properties of the chosen objects: the Bessel systems were chosen of these types, the bijection of $\mathcal{Y}_0(D)$ is by definition the standard one of the shape, the type of the parameter of the rows is an idempotent kernel (unit *Trace*), and a scalar is central. For the frame, Inputs 2(a) gives $\Lambda_0 k_0$ with Λ_0 the graded lattice basis for the scale $u_0(D)$; an entrywise comparison, using that $u_0(D)$ has norm one, identifies Λ_0 with $u_0(D) \cdot \eta \cdot \Lambda$. □

LEAN `Data.filling_fits`

Lemma 4.2. *In Setting 2.10 let ϕ be a vector of coefficients.*

(i) For a subgroup Δ of $\Gamma(V)$, $\gamma \in \Delta$ and $g \in G(V)$: $\mathrm{P}_{\mathsf{S}}(\phi, \Delta)(\gamma g) = \mathrm{P}_{\mathsf{S}}(\phi, \Delta)(g)$.

(ii) If $g \in G(V)$ and $\mathrm{P}_{\mathsf{S}}(\phi, \Gamma_B(\mathcal{Y}))(g) \neq 0$, then there is $\varsigma \in B_f(\mathcal{Y})$ such that ςg_f lies in the principal level $K_{\mathsf{S}}^{\mathrm{pr}}$, g_f being the finite component of g.

Proof. (i) Reindex the sum over Δ by $\gamma' \mapsto \gamma'\gamma$. (ii) A term $\mathrm{cb}_a(\gamma g)$ is non-zero only if a translate of the finite component of γg by an element of the image of $i_{y,f}$ lies in $K_{\mathsf{S}}^{\mathrm{pr}}$; this translate is ςg_f with ς the finite part of an element of the Bessel group, because γ and the image of i_y lie in it. If no such ς existed, all terms would vanish. □

LEAN `Filling.Fits.poincare_rat_mul` · `Filling.Fits.exists_besselFin_mul_mem_of_ne_zero`

Proposition 4.3 (normal form)**.** *In Setting 2.10, with Weil data for E and S admissible in the sense of the unit* Filling*, let ψ be an algebraic modular form on X_0 of level K_{S}, let $u \in U(\mathcal{Y})$, $a \in G(X_0)$ and $h \in G(V)$ with finite component in $K_{\mathsf{S}}^{\mathrm{pr}}$. Then*

$$\mathrm{P}_{\mathsf{S}}(\mathrm{cc}_{\mathsf{S}}(\psi), \Gamma_B(\mathcal{Y}))(u\, i_y(a)\, h) = \psi_E\Big(-\frac{\sigma(u)}{g_{\mathsf{S}}\varpi_{\mathsf{S}}}\Big) \cdot \Big\langle \mathsf{A}(a_\infty^{-1}\psi(a_f\eta)),\, W(h_\infty)\Big\rangle,$$

where $\sigma(u)$ is the sum of the entries of u just above the diagonal along the frame of the shape, in the coordinates of the filling data, a_∞^{-1} acts on the Weyl coordinates of the weight, and $\langle\,,\rangle$ is the bilinear dot product of coordinate vectors.

Proof. Write $a_f\eta = \gamma\, \mathsf{g}\, k$ with γ a rational point of X_0, g the chosen representative of the class of $a_f\eta$ and $k \in K_{\mathsf{S}}$. *First*, if a point g has finite component, translated by g and η, in the principal level, then $\mathrm{P}_{\mathsf{S}}(\mathrm{cc}_{\mathsf{S}}(\psi), \Gamma_B(\mathcal{Y}))(g) = \langle \mathsf{A}(\psi(\mathsf{g})), W(g_\infty)\rangle$: a term $\mathrm{cb}_{a'}(\gamma' g)$ with $\gamma' \in \Gamma_B(\mathcal{Y})$ can be non-zero only if a' belongs to the class of g and γ' comes from the group of the rational unipotent elements that this class conjugates into the principal level; so the sum over the classes reduces to this class, and over

that group the cut-offs sum to 1 (Inputs 2(d), facts (1) and (2)). *Then*, by strong approximation for the unipotent group (Inputs 2(d), fact (3)), there is a rational unipotent ς such that the finite part of ςu, conjugated by $a_f\eta$, lies in the principal level. By Lemma 4.2(i) the series is unchanged if its argument is multiplied on the left by ς and by $i_y(\gamma^{-1})$, and the first step applies to the new point. The Bessel vector transforms under the real points of X_0 by the Weyl representation, which A intertwines, and under the real unipotent part by the character of σ; the level of ψ absorbs k; and for a unipotent element whose finite part lies in the principal level the value of the character at the real part equals $\psi_E(-\sigma/(g_{\mathsf{S}}\varpi_{\mathsf{S}}))$ of the whole element. Since σ is additive on products of unipotent elements and vanishes on rational points under ψ_E, the factor is the one stated. The remaining level condition holds because K_{S} is the preimage of the principal level and h_f lies in it. □

LEAN `Filling.Fits.poincare_bessel_normal_form`

Proposition 4.4 (the Poincaré series is generic)**.** *In Setting 2.10, with Weil data for E and S admissible, let ψ be an algebraic modular form on X_0 of level K_{S}, and assume $\mathbf{j}\cdot\mathfrak{c}(y) = -(g_{\mathsf{S}}\varpi_{\mathsf{S}})^{-1}$. Then $\mathrm{P}_{\mathsf{S}}(\mathrm{cc}_{\mathsf{S}}(\psi), \Gamma_B(y))$ is generic for y.*

Proof. Left invariance under $\Gamma_B(y)$ is Lemma 4.2(i). For every point g there is a continuous function F_g with $\mathrm{P}(u\,h) = \psi_E(-\sigma(u)/(g_{\mathsf{S}}\varpi_{\mathsf{S}}))\,F_g(h)$ for all $u \in U(y)$ and all h whose finite part differs from that of g by an element of the principal level: if the series vanishes at all such points, take $F_g = 0$; otherwise Lemma 4.2(ii) moves one of these points into the principal level by an element of $B_f(y)$, which is a product of a unipotent element and an element of the image of i_y, and Proposition 4.3 gives F_g explicitly, a continuous function of the real component. The sets of such h are open and cover, and $F_g = \mathrm{P}$ there (take $u = 1$); so the series is continuous. For u as in the definition of genericity, $\mathrm{P}(ug) = \psi_E(-\sigma(u)/(g_{\mathsf{S}}\varpi_{\mathsf{S}}))\mathrm{P}(g)$. Finally $\mathrm{sh}_y = \mathfrak{c}(y)\cdot\mathsf{C}$ for the matrix C of the shape with $\mathrm{tr}(\mathsf{C}u) = \sigma(u) + \overline{\sigma(u^{-1})}$ (an entrywise computation, by cases on the shape). Moreover $\sigma(u^{-1}) = -\sigma(u)$, because σ is additive on products of unipotent elements; so $\mathrm{tr}(\mathrm{sh}_y u) = \mathfrak{c}(y)\,(\sigma(u) - \overline{\sigma(u)})$. Put $\varkappa = -(g_{\mathsf{S}}\varpi_{\mathsf{S}})^{-1}$; by the hypothesis $\varkappa = \mathbf{j}\,\mathfrak{c}(y)$, and $\overline{\varkappa} = -\varkappa$ because $\bar{\mathbf{j}} = -\mathbf{j}$ and $\overline{\mathfrak{c}(y)} = \mathfrak{c}(y)$. Hence

$$\tfrac{\mathbf{j}}{2}\,\mathrm{tr}(\mathrm{sh}_y u) = \tfrac{1}{2}(\varkappa\,\sigma(u) - \varkappa\,\overline{\sigma(u)}) = \tfrac{1}{2}(\varkappa\,\sigma(u) + \overline{\varkappa\,\sigma(u)}),$$

and $\psi_E(\frac{1}{2}(x+\overline{x})) = \psi_E(x)$ for every adele x, the standard character being invariant under conjugation. So the character of the normal form, $\psi_E(\varkappa\,\sigma(u))$, is $\psi_E(\frac{\mathbf{j}}{2}\,\mathrm{tr}(\mathrm{sh}_y u))$. □

LEAN `Filling.Fits.isGeneric_poincare`

5 Stage functions

Lemma 5.1. *In Setting 1.1 let ϑ be a tame character of D, $\varphi \in \mathrm{Col}(D)$, and ξ, y tuples of real and of finite points as in Definition 2.16.*

(i) Let i and j be natural numbers with $j \le i$ and $j \le 2\varrho(D) + 2$, and let ξ', y' be obtained from ξ, y by replacing the entries of index i by a real point ξ'_i and a finite point y'_i. Then $\mathfrak{P}^{(j)}_{D,\vartheta,\varphi}[\xi', y'; \mathcal{Y}] = \mathfrak{P}^{(j)}_{D,\vartheta,\varphi}[\xi, y; \mathcal{Y}]$ for every grading $\mathcal{Y}$ of $Y_j(D)$.

(ii) For every natural number j with $j \le 2\varrho(D) + 1$, the function $\mathfrak{P}^{(j)}_{D,\vartheta,\varphi}[\xi, y; \mathcal{Y}_j(D)]$ is generic for $\mathcal{Y}_j(D)$.

(iii) If y° is the tuple of the chosen representatives of the cosets $y_j\,\mathcal{J}_j(D,\vartheta)$, then $\mathrm{ed}_i(D,\vartheta;y^\circ)=\mathrm{ed}_i(D,\vartheta;y)$ for every natural number i.

Proof. (i) By the recursion, the function of stage j uses only the points of index $<j$. (ii) By the recursion: for $j=0$ it is Proposition 4.4 for the filling data of D (Proposition 4.1), whose hypothesis is the definition of the constant of $\mathcal{Y}_0(D)$; the step is Inputs 1(a), all gradings $\mathcal{Y}_j(D)$ having the same constant. (iii) The representative is $y_j\,v_j$ with $v_j\in\mathcal{J}_j(D,\vartheta)$, so $\Upsilon_j(D,\vartheta;y^\circ)=\omega_\nu^{\mathrm{pr}}(y_j,y_{j+1})\,\omega_\nu^{\mathrm{pr}}(v_j,v_{j+1})\Upsilon_j(D,\vartheta)$, and $\omega_\nu^{\mathrm{pr}}(v_j,v_{j+1})\Upsilon_j(D,\vartheta)=\Upsilon_j(D,\vartheta)$ by Lemma 3.2: the reductions of v_j and v_{j+1} are 1, so its hypothesis holds and the character factor is 1. □

LEAN Data.unfoldStage_update · Data.isGeneric_unfoldStage · Data.evaluatedDatum_out

The next statement is the bridge to the definite side. It uses two results of the note *Two consecutive theta lifts*, stated for an *arrow pair* $\mathcal{R}$ of three gradings $\mathcal{Y}\to\mathcal{Y}'\to\mathcal{Y}''$ with sets X, X', X_0'': such a pair has a finite set X'', a homomorphism $\mathrm{emb}_{\mathcal{R}}\colon G(X_0'')\to G(X'')$ and a finite evaluation. *The double unfolding*: for a function F generic for $\mathcal{Y}$, the iterated integral over $[B(\mathcal{Y}')]$ and $[B(\mathcal{Y})]$ of F times the two constant terms at the string maps is a constant times the integral over $[G(X)]\times[G(X')]$ of F on the image of i_y times the theta function of the zero term of the arrow product of the two data. *The double lift*: such an integral of a coefficient of an algebraic modular form φ against the theta kernel of an arrow tensor is a coefficient of the composite of the two definite lifts of φ.

Proposition 5.2 (even stages are coefficients of definite chains)**.** *In Setting 1.1 let $\xi=(\xi_j)$ be a tuple of real points as in Definition 2.16. Let i be a natural number with $2i<2\varrho(D)+1$, let $\mathcal{Y}$ be a grading of $Y_{2i+2}(D)$ with norm $N(D)$, a finite set X and a string of length l, let $\mathcal{R}$ be an arrow pair, with a finite set Q_0 of hyperbolic planes, of the gradings $\mathcal{Y}_{2i}(D)\to\mathcal{Y}_{2i+1}(D)\to\mathcal{Y}$ whose finite evaluation is $\mathrm{ev}_{D,i}$, and let $\xi^\star\in G_\infty(Y_{2i+2}(D))$. Then there is a linear functional t such that for every tame character ϑ of D, every column $\varphi\in\mathrm{Col}(D)$, every tuple y of finite points with $y_0\in K_D^{\mathrm{ch}}(Y_0(D))$ and every $g\in G(X)$:*

$$\mathfrak{P}^{(2i+2)}_{D,\vartheta,\varphi}[\xi,y;\mathcal{Y}](i_y(g)\cdot(\xi^\star,y_{2i+2}))=\langle\mathsf{t},\mathrm{C}^{[i+1]}_{D,\vartheta}(\mathrm{ed}_\cdot(D,\vartheta;y))(\varphi)\rangle(\mathrm{emb}_{\mathcal{R}}(g)).$$

Proof. By strong induction on i, for all $\mathcal{Y}$, $\mathcal{R}$, $\xi^\star$. *Claim*: there is a functional t_0 such that the function of stage $2i$, on the image of $i_{y_{2i}(D)}$ translated by (ξ_{2i},y_{2i}), is the coefficient at t_0 of the chain of i pairs. For $i=0$ the stage function is the Poincaré series, and Proposition 4.3 with $u=1$ (so that the character is 1) and the hypothesis on y_0 presents it as $\langle\mathsf{A}(a_\infty^{-1}\varphi(a_f)),W(\xi_0)\rangle$, a linear functional of the value of φ; the central translation in cc_D cancels η. For $i>0$ the claim is the induction hypothesis for $i-1$, with the grading $\mathcal{Y}_{2i}(D)$ and the middle arrow pair. Now unfold twice: by Definition 2.16 the function of stage $2i+2$ at the given point is an iterated integral over $[B(\mathcal{Y}_{2i+1}(D))]$ and $[B(\mathcal{Y}_{2i}(D))]$ of the function of stage $2i$, generic by Lemma 5.1(ii), times two constant terms; the real parts of the Weil operators are evaluated at the fixed real points and absorbed in the Fock vectors. By the double unfolding this is a constant times an integral over $[G(X_{2i}(D))]\times[G(X_{2i+1}(D))]$ of the stage function, replaced by its coefficient from the claim, times the theta function of the zero term of the arrow product $\Upsilon_{2i}(D,\vartheta;y)\boxtimes\Upsilon_{2i+1}(D,\vartheta;y)$ translated by the stacked string map. The zero term of an integrated datum is an integrated datum whose finite part is the finite zero term, and by the defining property of the finite

evaluation and the hypothesis on $\mathcal{R}$ this finite part is the arrow tensor of $\mathrm{ed}_i(D,\vartheta;y)$. The double lift, applied to the lifts $2i+1$ and $2i+2$ (theta lifts in the sense required, by Inputs 1(h)), turns the integral into a coefficient of the composite of the two lifts with datum $\mathrm{ed}_i(D,\vartheta;y)$ applied to the chain of i pairs. The functional t so obtained depends on ξ, $\xi^\star$, $\mathcal{R}$ only. $\square$

LEAN `Data.unfoldStage_even`

6 Lattices along a string

Throughout this section, ord_w of an idele is as in Definition 2.1; the formal text works with $-\mathrm{ord}_w$, the logarithm of the valuation.

Lemma 6.1 (adapted bases)**.** *Let $\mathcal{Y}$ be a grading of a hermitian space with frame V, with index set I, a finite set X and a string of length l.*

(i) Let $e\colon X \sqcup \{0,\dots,l-1\} \to I$ be a bijection and A_1, A_2 square matrices over a commutative ring R, indexed by I and triangular for the flag of e. Then for every string index a, with $i = e(a)$: $(A_1A_2)_{ii} = (A_1)_{ii}(A_2)_{ii}$.

(ii) For every $y \in G_f(V)$ there is a pair (b,β) that is adapted at y for $\mathcal{Y}$.

(iii) Let $y \in G_f(V)$ and let (b,β) be adapted at y for $\mathcal{Y}$. Put $\mathsf{k} = (b^{-1}\, x_{V,f}\, y)^{-1}$. For string indices a, a' with $a' \le l-1-a$, the entry of $\mathsf{k}^*\mathsf{k}$ at the string indices (a,a') is $J_V(z_a, z_{a'}) \cdot \overline{\beta_a}\beta_{a'}$.

(iv) If (b,β) is adapted to h for $\mathcal{Y}$ for some matrix $h \in \mathrm{GL}_I(\mathbb{A}_f)$, then the inverses satisfy the same relation with themselves: (b^{-1},β^{-1}) is adapted to b^{-1} for $\mathcal{Y}$.

(v) Let $h, h' \in \mathrm{GL}_I(\mathbb{A}_f)$ with (b,β) is adapted to h for $\mathcal{Y}$ and (b',β') is adapted to h' for $\mathcal{Y}$, and let w be a finite place at which the matrix $h^{-1}h'$ is integral. Then $\mathrm{ord}_w \beta'_a \ge \mathrm{ord}_w \beta_a$ for every string index a.

(vi) In the situation of (v), if instead h and h' have the same component at w, then $\mathrm{ord}_w \beta'_a = \mathrm{ord}_w \beta_a$ for every string index a.

Proof. (i) A string index is alone in its class of the flag order, since the numbers $4a+1$ are distinct and different from $2l$; in the sum $\sum_{i'} (A_1)_{ii'}(A_2)_{i'i}$ triangularity leaves only $i' = i$. (ii) Hermite reduction over $\hat{\mathcal{O}}$ (Inputs 1(d)), applied to the inverse of $x_{V,f}\, y$, gives an invertible b, triangular for the flag, such that $(x_{V,f}\, y)^{-1} b$ is integral with integral inverse. The inverse of a matrix that is triangular for the flag is triangular for the flag, so by (i), applied to $b\, b^{-1} = 1$, the diagonal entries of b at the string indices are units of $\mathbb{A}_f$; let β be this diagonal. (iii) Since $x_{V,f}^* J_V x_{V,f} = 1$ and y is unitary, $\mathsf{k}^*\mathsf{k} = b^* J_V b$. In the flag order the string vectors of grade ≥ 0 come first, by decreasing grade, then X, then the other string vectors. By triangularity the column of b at a string index a' is $\beta_{a'} z_{a'}$ plus a combination of basis vectors that precede $z_{a'}$ in the flag order, and J_V pairs a string vector $z_{a''}$ only with its reverse. For $a' \le l-1-a$ a comparison of flag orders shows that the only term of $b^* J_V b$ at (a,a') is the product of the two diagonal entries with $J_V(z_a, z_{a'})$. (iv) The inverse of a triangular matrix is triangular; $(b^{-1})^{-1} b^{-1} = 1$; and by (i) the string diagonal of b^{-1} is β^{-1}. (v) $b^{-1}b' = (b^{-1}h)(h^{-1}h')(h'^{-1}b')$ is integral at w and triangular, and by (i) and (iv) its entry at the a-th string index is $\beta_a^{-1}\beta'_a$. (vi) Apply (v) twice. $\square$

LEAN mul_apply_string_of_blockTriangular · Grading.exists_isAdaptedAt · Grading.IsAdaptedAt.gram_string_entry · Grading.IsAdaptedBasis.inv · Grading.IsAdaptedBasis.ideleLogVal_le · Grading.IsAdaptedBasis.ideleLogVal_eq

Proposition 6.2 (lattices with the same lines lie in one Bessel orbit)**.** *Let E be stably unramified, $\mathcal{Y}$ a grading of a hermitian space with frame V with a finite set X and a string of length l, and let $h, h' \in G_f(V)$. Let (b, β) be adapted at h for $\mathcal{Y}$ and (b', β') be adapted at h' for $\mathcal{Y}$, and assume $\mathrm{ord}_w \beta'_a = \mathrm{ord}_w \beta_a$ for every finite place w and every string index a with $a \le l - 1 - a$. Then there is $k \in K_0(V)$ with $h'\, k\, h^{-1} \in B_f(\mathcal{Y})$.*

Proof. Multiplying b' on the right by the diagonal matrix that is 1 on X and $\beta'^{-1}_a \gamma_a$ on the string, with $\gamma_a = \beta_a$ for $a \le l - 1 - a$ and $\gamma_a = \beta'_a$ otherwise, gives an adapted basis at h' with string diagonal γ: the factor is integral with integral inverse by the hypothesis. So we may assume $\beta'_a = \beta_a$ on the first half $a \le l - 1 - a$. Let $\mathsf{k} = (b^{-1} x_{V,f} h)^{-1}$ and k' likewise; both lie in $\mathrm{GL}_I(\hat{\mathcal{O}})$. The columns of k at the string indices of the first half form a matrix with an integral left inverse, and so do those of k'. By Lemma 6.1(iii) the two matrices have the same Gram matrix. Witt's theorem over $\hat{\mathcal{O}}$ (Inputs 1(d)) gives $k \in K_0(V)$ carrying the first to the second. In rational coordinates this says that $h'kh^{-1}$, applied to the column of b at a first-half string index, gives the column of b' there. By the description of the Bessel group it remains to see that $h'kh^{-1} - 1$, in rational coordinates, has zero entry (i, i') whenever i' is a string index of grade ≥ 0 and $\mathrm{gr}(i) \le \mathrm{gr}(i')$; the string indices of grade ≥ 0 are those of the first half. Writing the matrix as its product with $b\, b^{-1}$, the entry is a sum over the columns of b at first-half string indices of smaller or equal number (by triangularity of b^{-1}), on which the matrix acts as $b \mapsto b'$; the columns of b and b' at such an index a'' are $\beta_{a''} z_{a''}$ plus string vectors of higher grade, with the same diagonal. A check of the cases gives zero. □

LEAN Grading.IsAdaptedAt.exists_mem_besselFin

For the next three statements recall from the note *Two consecutive theta lifts*: for finite ideles $\mathfrak{a}$, π and a finite adele u of norm one, a grading $\mathcal{Y}$ of V as $X \sqcup \{\mathrm{pt}\}$ plus a string of length l *spans the lattice of* $(\mathfrak{a}, \pi, u)$ if $x_{V,f} = \Lambda\, k$ with $k \in \mathrm{GL}_I(\hat{\mathcal{O}})$ and Λ the *graded lattice basis*: for even l the diagonal matrix with 1 on $X \sqcup \{\mathrm{pt}\}$, $u\,\mathfrak{a}^{-1}$ at the string indices a with $2a < l$ and $u\,\overline{\mathfrak{a}}$ at the others; for odd l the matrix with 1 on X, u on the string, and the column of the point equal to $\overline{\pi}^{-1}$ at the point plus $u(\pi\overline{\pi})^{-1}$ at the middle string vector. And N is *split and squarefree along* a finite set Σ of finite places with the idele π if $N = \pi\overline{\pi}$, π is a uniformiser at the places of Σ and a unit elsewhere, and Σ is disjoint from its conjugate.

Lemma 6.3 (the standard lattice has near lines)**.** *Let $\mathcal{Y}$ be a grading of V as $X \sqcup \{\mathrm{pt}\}$ plus a string of length l, let $\mathfrak{a}$, π be finite ideles and n a natural number such that $\mathcal{Y}$ spans the lattice of $(\mathfrak{a}, \pi, (\mathfrak{a}/\overline{\mathfrak{a}})^n)$, let K be a subgroup of $K_0(V)$, Q a finite set of finite places and $h \in G_f(V)$ such that $\varsigma h \in K$ for some $\varsigma \in B_f(\mathcal{Y})$. Then h has near lines for $(\mathcal{Y}; \mathfrak{a}, Q; n, 0)$.*

Proof. Let $x_{V,f} = \Lambda k$. In rational coordinates ς^{-1} is triangular for the flag with string diagonal 1, by the description of the Bessel group, and so is Λ, with string diagonal $(\mathfrak{a}/\overline{\mathfrak{a}})^n \epsilon_a$. Their product b satisfies $b^{-1} x_{V,f} h = k \cdot (\varsigma h) \in \mathrm{GL}_I(\hat{\mathcal{O}})$ in the coordinates of the frame, and by Lemma 6.1(i) its string diagonal is that of Λ. So all the differences of orders are 0. □

LEAN `Grading.hasNearLines_zero`

Lemma 6.4. *Let $\mathcal{Y}$ be a grading of V as $X \sqcup \{\mathrm{pt}\}$ plus a string of length l with norm N that spans the lattice of $(\mathfrak{a}, \pi, u)$, let N be split and squarefree along Σ with the idele π, and let $h \in G_f(X \sqcup \{\mathrm{pt}\})$ be integral with integral inverse and with the row of the point congruent to $(0, \dots, 0, 1)$ modulo π. Then $i_{\mathcal{Y},f}(h) \in K_0(V)$.*

Proof. In the coordinates of the frame $x_{V,f} = \Lambda k$, $i_{\mathcal{Y},f}(h) - 1 = k^{-1}\Lambda^{-1}(\tilde{h} - 1)\Lambda k$ with $\tilde{h}$ the block matrix of h and 1. For even l the rows and columns $X \sqcup \{\mathrm{pt}\}$ of Λ are those of the identity, and $h - 1$ is integral. For odd l, $\Lambda^{-1}(\tilde{h} - 1)\Lambda$ has the entries of $h - 1$, except that the column of the point is divided by $\overline{\pi}$ and the row of the point is multiplied by $\overline{\pi}$ (so that the entry at the point itself is unchanged); moreover the row of the point of $h - 1$, divided by $-\pi$, stands in the row of the middle string vector, whose entry in the column of the point is the entry of $h - 1$ at the point divided by $-N$. So three congruences are needed. The row of the point of $h - 1$ is $\equiv 0$ modulo π by hypothesis. The integral matrices with integral inverse and with this property form a group, and h is unitary; so $h^{-1} = h^*$ has the property too, which says that the column of the point of $h - 1$ is $\equiv 0$ modulo $\overline{\pi}$. In particular the entry of $h - 1$ at the point is $\equiv 0$ modulo π and modulo $\overline{\pi}$, hence modulo $N = \pi\overline{\pi}$, as π is a uniformiser on Σ, $\overline{\pi}$ on $\overline{\Sigma}$, both are units elsewhere and Σ is disjoint from $\overline{\Sigma}$. So the matrix is integral, and conjugating by $k \in \mathrm{GL}_I(\hat{\mathcal{O}})$ keeps integrality. □

LEAN `Grading.inclXFin_mem_Kf`

Proposition 6.5 (the two halves of a string diagonal)**.** *Let E be stably unramified, $\mathcal{Y}$ a grading of V with a finite set X, a string of length l and norm N, let $h \in G_f(V)$ and let (b, β) be adapted at h for $\mathcal{Y}$. Let N be split and squarefree along Σ with the idele π. Assume: if l is odd, then for no $\varsigma \in B_f(\mathcal{Y})$ and no $w \in \Sigma$ does $(\varsigma h)^{-1}\, i_{\mathcal{Y},f}(K_w(X))\,(\varsigma h) \subseteq K_w(V)$ hold. Then for every string index a and every finite place w: $\mathrm{ord}_w \beta_a + \mathrm{ord}_{\overline{w}} \beta_{l-1-a} \geq 0$.*

Proof. For any adapted pair, Lemma 6.1(iii) with $a' = l - 1 - a$ shows that the number $\overline{\beta_a}\beta_{l-1-a} J_V(z_a, z_{l-1-a})$ is an entry of the Gram matrix of an element of $\mathrm{GL}_I(\hat{\mathcal{O}})$, hence integral; and $J_V(z_a, z_{l-1-a})$ is 1, or $-N$ at the middle index. This is the claim, except when l is odd, a is the middle index and w or $\overline{w}$ lies in Σ, where it gives only $\mathrm{ord}_w \beta_a + \mathrm{ord}_{\overline{w}} \beta_a \geq -1$. For the middle index the claim is symmetric in w and $\overline{w}$, so let $w \in \Sigma$; then $\overline{w} \neq w$. Suppose the claim fails, so that the sum is -1. Let γ be the family with $\gamma_{a'} = \beta_{a'}$ up to the middle index and $\gamma_{a'} = \overline{\beta_{l-1-a'}}^{-1}$ beyond, and d the diagonal matrix with 1 on X and γ on the string; it is an adapted basis of itself. Since $\overline{w} \neq w$, there is a finite point y, equal to 1 outside w and $\overline{w}$, such that $x_{V,f}\, y\, h$ and d have the same component at w. Choose a pair (b', β') adapted at yh (Lemma 6.1(ii)). By Lemma 6.1(vi) its diagonal has the orders of γ at w and those of β away from w and $\overline{w}$. At $\overline{w}$ its orders are at most those of γ, by Lemma 6.1(v) for the matrices $x_{V,f}\, yh$ and d. Indeed, the matrix that compares the two lattices is $(x_{V,f}\, yh)^{-1}\mathsf{d} = (x_{V,f}\, yh)^* J_V \mathsf{d}$, and since $x_{V,f}\, yh$ agrees with d at w, its component at $\overline{w}$ is that of $\mathsf{d}^* J_V \mathsf{d}$. This matrix is the identity on X, and its only non-zero entries on the string are $\overline{\gamma_{a'}}\gamma_{l-1-a'} = 1$ off the middle index and $-N\overline{\gamma_a}\gamma_a$ at the middle index a, whose order at $\overline{w}$ is $1 + \mathrm{ord}_w \beta_a + \mathrm{ord}_{\overline{w}} \beta_a = 0$ by the assumed failure. So the comparison matrix is integral at $\overline{w}$. And the orders at $\overline{w}$ are at least those of β on the first half, by the Gram condition of the first step for (b', β') and, at the middle index, by the failure of the inequality. As $\gamma = \beta$ on the first half, the two

diagonals have the same orders there at every place. So Proposition 6.2 applies to h and yh: there is $k \in K_0(V)$ with $\varsigma = yhkh^{-1} \in B_f(\mathcal{Y})$. For this ς and w the forbidden inclusion holds: for $a_0 \in K_w(X)$, the conjugate $(\varsigma h)^{-1} i_{y,f}(a_0)(\varsigma h)$ is 1 outside w, $\overline{w}$, and at w it is k^{-1} times the conjugate of the block matrix of a_0 and 1 by $\mathfrak{d}^{-1} x_{V,f}$, times k; the lattice of $\mathfrak{d}$ is stable under such block matrices, so this is integral. This contradicts the assumption. □

LEAN `Grading.IsAdaptedAt.ideleLogVal_add_rev_nonpos`

Proposition 6.6 (one lift moves the lines by one step)**.** *Let E have Weil data, with idele α. Let $\mathcal{Y}$ be a grading of V with a string of length l and $\mathcal{Y}'$ a grading of V' with a string of length l', with the same norm N and $l' + 1 = l$. Let $\mathfrak{a}$ be a finite idele, Q a finite set of finite places, n, n', k natural numbers with $n = n'$ if l is odd and $n = n' + 1$ if l is even, let $h \in G(V)$, $h' \in G(V')$, ν a character, v a vector of the Fock model and $\Phi \in \mathcal{S}(\mathrm{Hom}(V, V'))$, and $\mathfrak{b}$ a finite idele. Assume:*

(a) h_f has near lines for $(\mathcal{Y}; \mathfrak{a}, Q; n, k)$;

(b) $\mathfrak{c}(\mathcal{Y})$ and $\mathfrak{a}\overline{\mathfrak{a}}$ have the same order at every finite place;

(c) $\mathrm{ct}_{y,y'}(\nu, v \otimes \Phi)(h, h') \neq 0$;

(d) $\mathrm{ord}_w \mathfrak{b} \leq \mathrm{ord}_w \overline{(\alpha/\mathfrak{a})} + [w \in Q]$ for every finite place w;

(e) $\varrho_f(\mathfrak{b}\, y)\Phi = \Phi$ for every integral finite vector y;

(f) every pair (b', β') that is adapted at h'_f for $\mathcal{Y}'$ has $\mathrm{ord}_w \beta'_a + \mathrm{ord}_{\overline{w}} \beta'_{l'-1-a} \geq 0$ for all string indices a and all finite places w.

Then h'_f has near lines for $(\mathcal{Y}'; \mathfrak{a}, Q; n', k+1)$.

Proof. Let (b, β) be the adapted pair of (a) and (b', β') any adapted pair at h'_f. *Claim*: for every string index a of $\mathcal{Y}'$,

$$\mathfrak{c}(\mathcal{Y})^{\delta_a} \cdot \frac{\overline{\mathfrak{b}}}{\overline{\beta_{l'-a}}\ \alpha\ \beta'_a} \in \hat{\mathcal{O}}, \qquad \delta_a = 1 \text{ if } (l' \text{ odd}) \Leftrightarrow (a \leq l'-1-a), \quad \delta_a = 0 \text{ otherwise.}$$

Granting the claim, take orders at every place: with (b) the claim bounds $\mathrm{ord}_w \beta'_a$ from above by $-\mathrm{ord}_{\overline{w}} \beta_{l'-a}$ plus the order of $\overline{\mathfrak{b}}/\alpha$ and of the power of $\mathfrak{a}\overline{\mathfrak{a}}$; (a) controls $\beta_{l'-a}$ within k on $Q \cup \overline{Q}$ and exactly elsewhere; (d) costs one more unit on Q; and (f) turns the upper bound for one half of β' into the lower bound for the other half. Linear arithmetic gives the near-lines bound for β' with n' and $k+1$.

Proof of the claim. By Inputs 1(g) there is a splitting of $\mathrm{Hom}(V, V')$ along the maps of positive degree for which the constant term at the string map $T_{y,y'}$ is the theta function of the zero term of the Heisenberg translate by this map. By (c) and the support of the zero term (note *Two consecutive theta lifts*: if a datum is fixed by the translations $\varrho_f(\mathsf{M}y)$, y integral, and the finite zero term of its translate by a vector t does not vanish, then t lies in $\mathsf{M}'\hat{\mathcal{O}} + k^{\perp}$, where $\mathsf{M}\mathsf{M}'^{*} = 1$ and k is the isotropic frame of the splitting), applied to the datum moved by the finite parts of (h, h'), which by (e) is fixed by the translations of $\mathfrak{b}$ times the lattice moved by (h, h'): there is an integral vector λ_0 such that $T_{y,y'}$ and $\overline{\mathfrak{b}}^{-1}(h, h')\lambda_0$ have the same pairing with the isotropic frame. Reading this relation in the frames of V and V', for the entries (i', i) with $\mathrm{gr}(i') < \mathrm{gr}(i)$, expresses the entries of $\overline{\mathfrak{b}}\, T_{y,y'}$ there through the matrix $\mathsf{W} = x_{V',f} h'_f \cdot \lambda_0 \cdot (x_{V,f} h_f)^{-1}$. Now $b'^{-1}\, \mathsf{W}\, J_V^{-1} (b^{-1})^{*}$ is integral, because b' and b are adapted and λ_0 is integral. Its entry at the a-th string index of $\mathcal{Y}'$ and the string

index $l'-a$ of $\mathcal{Y}$ is a double sum in which, by the triangularity of b'^{-1} and b^{-1}, only one term survives: the product of the diagonal entries $\beta_a'^{-1}$ and $\overline{\beta_{l'-a}}^{-1}$, of $\overline{\mathfrak{b}}/\alpha$, of the multiple of the string map at z_a (which is 1, $\mathfrak{c}(\mathcal{Y})$ or $-N\mathfrak{c}(\mathcal{Y})$, by Definition 2.3) and of the entry of J_V^{-1} at $(z_a, z_{l'-a})$, which is 1, or $-N^{-1}$ when z_a is the middle vector of $\mathcal{Y}$. The multiple $-N\mathfrak{c}(\mathcal{Y})$ occurs exactly at the middle vector, where the factor $-N^{-1}$ cancels $-N$; so the product of the last two factors is $\mathfrak{c}(\mathcal{Y})$ or 1, and comparing the cases of Definition 2.3 with the definition of δ_a shows that it is $\mathfrak{c}(\mathcal{Y})^{\delta_a}$. This is the claim. □

LEAN `Grading.HasNearLines.step`

Lemma 6.7 (valuations of the data)**.** *In Setting 1.1:*

(i) the ideal of $E(D)$ generated by $\bar{\pi}_g(D)$ has absolute norm $g(D)^{f(D)}$;

(ii) for every finite place w, the idele $\hat{\pi}_g(D)$ of Inputs 2(a) and $\bar{\pi}_g(D)$ have the same order at w;

(iii) for every natural number j with $j \le 2\varrho(D)+1$ and every finite place w, the constant $\mathfrak{c}(\mathcal{Y}_j(D))$ and the idele $\mathfrak{a}(D)\overline{\mathfrak{a}(D)}$ have the same order at w;

(iv) for every natural number j with $j \le 2\varrho(D)+1$, the grading $\mathcal{Y}_j(D)$ spans the lattice of $(\mathfrak{a}(D), \hat{\pi}_N(D), u_j(D))$, $\hat{\pi}_N(D)$ the principal idele of $\pi_N(D)$;

(v) for a tame character ϑ of D and a hermitian space with frame V over $E(D)$, with a finite index set: every element of $K_D^{\mathrm{ch}}(V)$ that is 1 at the places of $Q(\vartheta)$ is congruent to 1 modulo every chosen place and every place of $Q(\vartheta)$.

Proof. (i) The ideal is the product of the chosen places, and those above g_i have norm g_i and are $f(D)$ in number. (ii) Both orders are the multiplicity of w in the ideal of $\bar{\pi}_g(D)$. (iii) By the self-duality of the Weil data, $\mathbf{j}\alpha\overline{\alpha}$ has at w the order of the inverse different, which $\varpi(P,\mu)^{-1}$ generates; so $(\mathbf{j}\, g(D)\, \varpi(P,\mu))^{-1}$ has the order of $\alpha\overline{\alpha}/g(D)$, and $\mathfrak{a}(D)\overline{\mathfrak{a}(D)} = \alpha\overline{\alpha}/(\hat{\pi}_g(D)\overline{\hat{\pi}_g(D)})$, where $\hat{\pi}_g(D)\overline{\hat{\pi}_g(D)}$ has the order of $\bar{\pi}_g(D)\overline{\bar{\pi}_g(D)} = g(D)$ by (ii). (iv) This is Inputs 2(a), by the definition of the spaces and of the gradings. (v) At a chosen place this is the definition of $K_D^{\mathrm{ch}}(V)$; at a place of $Q(\vartheta)$ an element with component 1 reduces to 1. □

LEAN `Data.absNorm_piGBar` · `Data.v_chosenIdele` · `Data.valuation_shift_const_eq_ideleA_norm` · `Data.spansLattice_gradingY` · `Data.chosenLevel_inf_trivialOn_le`

Proposition 6.8 (where a stage function lives)**.** *In Setting 1.1 let ϑ be a tame character of D, ξ and y tuples of real and of finite points as in Definition 2.16 with $y_0 \in K_D^{\mathrm{ch}}(Y_0(D))$, let $\varphi \in \mathrm{Col}^{\mathrm{ram}}(D)$ be a ramified column, j a natural number with $j \le 2\varrho(D)+1$, and $g \in G(Y_j(D))$ a point with $\mathfrak{P}^{(j)}_{D,\vartheta,\varphi}[\xi, y; \mathcal{Y}_j(D)](g) \neq 0$. Then*

$$g_f \text{ has near lines for } (\mathcal{Y}_j(D); \mathfrak{a}(D), Q(\vartheta); \lfloor (m-j+1)/2 \rfloor, j).$$

Proof. By induction on j. For $j=0$, Lemma 4.2(ii) moves g_f into the principal level by an element of the finite Bessel group, and Lemma 6.3 applies, by Lemma 6.7(iv) with $u_0(D) = (\mathfrak{a}(D)/\overline{\mathfrak{a}(D)})^{\lfloor (m+1)/2 \rfloor}$. For the step from j to $j+1$: the value at g is an integral over $\varsigma \in [B(\mathcal{Y}_j(D))]$ of the stage function at $\varsigma \cdot (\xi_j, y_j)$ times a constant term, so both factors are non-zero for some ς. The induction hypothesis at that point is hypothesis (a) of Proposition 6.6, which we apply with $\mathfrak{a} = \mathfrak{a}(D)$, $Q = Q(\vartheta)$, the datum $\mathfrak{d}_j(D,\vartheta) = v \otimes \Upsilon_j(D,\vartheta)$ and $\mathfrak{b}$ a generator of the product $\mathfrak{m}$ of the chosen places and the places of $Q(\vartheta)$; the parities of the string lengths match the exponents.

(b) is Lemma 6.7(iii). (d): the order of $\mathfrak{b}$ at w is the multiplicity of w in the ideal of $\bar{\pi}_g(D)$ plus $[w \in Q(\vartheta)]$, and $\alpha/\mathfrak{a}(D) = \overline{\hat{\pi}_g(D)}$. (e): $\Upsilon_j(D,\vartheta)$ is named on the vacuum by an indicator modulo $\mathfrak{m}$; by Inputs 2(e) with $g' = 0$, $q = 1$ and $\mathfrak{e} = \mathfrak{b}$ (so that $\mathfrak{e}\,q \equiv 0$ modulo $\mathfrak{m}$) it is a combination of translates $\varrho_f(t)$ of the vacuum with $\overline{\mathfrak{b}}t$ integral at every place. Only this support is used: the denominator that the Input gives for $g' = 0$ is 0 and says nothing about the coefficients. Now $\varrho_f(\mathfrak{b}y)$ fixes the vacuum for integral y and commutes with $\varrho_f(t)$ up to the value of the additive character at the pairing of $\mathfrak{b}y$ and t, which is 1 because $\overline{\mathfrak{b}}t$ is integral (Input 1(a) and (e) of the note *Two consecutive theta lifts*: the standard lattice is self-dual). (f) follows from Proposition 6.5 for $\mathcal{Y}_{j+1}(D)$, with $\Sigma = \Pi(D)$ and the principal idele $\hat{\pi}_N(D)$ of $\pi_N(D)$ (Inputs 2(b)), once its hypothesis is verified.

So let the string of $\mathcal{Y}_{j+1}(D)$ be odd, that is $j+1 = 2i+2$ for a natural number i, and suppose that for some $\varsigma \in B_f(\mathcal{Y}_{j+1}(D))$ and $w \in \Pi(D)$ the inclusion of Proposition 6.5 holds at the point g_f. Write $\varsigma^{-1} = u\, i_{\mathcal{Y}_{j+1}(D)}(a_0)$ with u unipotent, and $\varphi = \psi - z_w(D)\psi$ with $\psi \in \mathrm{Col}(D)$ (Inputs 2(b)). By Proposition 5.2 for the pair index i, with the real point of g and after replacing the entry of ξ and y of index $2i+2$ (Lemma 5.1(i),(iii)), the function of stage $2i+2$ on the middle grading is a coefficient of the chain of $i+1$ pairs. The central element of $z_w(D)$ is a scalar of norm one supported at w and $\overline{w}$; by Inputs 2(g) the chain of pairs of $z_w(D)\psi$ is the right translate by this scalar of the chain of pairs of ψ, the character factors being trivial because the characters are unramified above $N(D)$. Moving u out by genericity (a string of odd length makes every unipotent element raise grades by 2), the stage function at g becomes the difference of two values of the same coefficient, at a point and at its translate by the scalar; by the assumed inclusion this translate differs from the point by an element of $\mathcal{J}_{j+1}(D,\vartheta)$ — a place of $\Pi(D)$ is neither chosen nor in $Q(\vartheta)$ — and the evaluated data depend only on cosets modulo these levels (Lemma 5.1(iii)). The two values agree and the stage function vanishes at g, against the assumption. □

LEAN `Data.hasNearLines_of_unfoldStage_ne_zero`

7 Periods as finite sums

Lemma 7.1. *Let H be a group with subgroups S, A, J.*

(i) For a finite set T of left cosets of J, an element $p \in S$ and a left coset t of J: $\mathrm{mult}_{S,A}(T; p\,t) = \mathrm{mult}_{S,A}(T; t)$.

(ii) Let H be a measurable group with a left invariant measure, written vol*, let J be a measurable subgroup and T_0 a non-empty finite set of left cosets of J. Then the union of the cosets in T_0 has measure $|T_0| \cdot \mathrm{vol}(J)$.*

Proof. (i) Left multiplication by p^{-1} is a bijection of the cosets of $S \cap A$ in S that carries the set counted for $p\,t$ to the set counted for t. (ii) The union is disjoint and every coset is a left translate of J. □

LEAN `coverMultiplicity_smul` · `measure_union_cosets`

Proposition 7.2 (a cut with multiplicities of prime-power type)**.** *Let V be a hermitian space with frame, let S, A, J, H_0 be subgroups of $G_f(V)$ with A and J levels, $J \subseteq A \subseteq H_0$, let K be a compact subset of $G_f(V)$, q a natural number and W a set*

of finite points. Assume: every $h \in \mathsf{K}$ lies in H_0 and has a natural number n with $(huh^{-1})^{q^n} \in A$ for all $u \in J$; and for every $h \in \mathsf{W}$ there is $\varsigma \in S$ with $\varsigma h \in \mathsf{K}$. Then there is a finite set T of left cosets of J such that

(i) for every $p \in S \cap A$ and every coset t: $t \in T$ if and only if $p\,t \in T$;

(ii) every h with $hJ \in T$ lies in H_0;

(iii) for every h with $hJ \in T$ or $h \in \mathsf{W}$, the multiplicity $\mathrm{mult}_{S,A}(T; hJ)$ divides a power of q.

Proof. Let C be a set of cosets hJ, $h \in \mathsf{K}$, that meets every S-orbit of such cosets exactly once, and let T be the set of the cosets $p\,t$ with $p \in S \cap A$, $t \in \mathsf{C}$. It is contained in the image of the compact set $A\mathsf{K}$ in the discrete space of cosets of the open subgroup J, hence finite. (i) holds by construction. (ii): if $hJ = p\,h_0 J$ with $h_0 \in \mathsf{K}$, then $h \in A\,H_0\,J \subseteq H_0$. (iii): a coset hJ in T, and by the second hypothesis also the coset of a point of W, lies in the S-orbit of some $h_0 J \in \mathsf{C}$, $h_0 \in \mathsf{K}$; by Lemma 7.1(i) it has the multiplicity of $h_0 J$. Let S_0 be the stabiliser of $h_0 J$ in S. Since C meets the orbit once, the elements $\varsigma \in S$ with $\varsigma^{-1} h_0 J \in T$ are those of $S_0\,(S \cap A)$, so the multiplicity is the index of $S_0 \cap A$ in S_0, which is finite because $h_0 A h_0^{-1}$ is commensurable with A. This index divides the order of the quotient of S_0 by the normal core of $S_0 \cap A$. Every element of S_0 lies in $h_0 J h_0^{-1}$, so by the first hypothesis a fixed q^n-th power of it lies in A, and the same holds for its conjugates in S_0: every element of the finite quotient has order dividing q^n. A finite group all of whose elements have order a power of q has order dividing a power of q. □

LEAN `exists_cut_coverMultiplicity_dvd`

Proposition 7.3 (a period over the Bessel quotient as a finite sum)**.** *Let E have Weil data, $\mathcal{Y}$ a grading of a hermitian space with frame V, with a finite set X and a string, A a level of V and a a complex function on $G_\infty(V)$. There is a complex number C such that the following holds for every function F_1 on $G(V)$ that is generic for $\mathcal{Y}$, every automorphic function F_2 on $G(V)$, every level J of V, every finite set T of cosets of J in $G_f(V)$ and every complex function b on the cosets of J; write $F = F_1 F_2$. If*

(a) F is integrable on $\Gamma_B(\mathcal{Y})\backslash G(V)$;

(b) for every $p \in B_f(\mathcal{Y}) \cap A$ and every coset t: $t \in T$ if and only if $p\,t \in T$;

(c) $\mathrm{mult}_{B_f(\mathcal{Y}),A}(T; g_f J) \neq 0$ for every $g \in G(V)$ with $F(g) \neq 0$;

(d) $\int_{[B(\mathcal{Y})]} F(\varsigma g)\,d\varsigma = [K_0(V) : J] \cdot \mathsf{a}(g_\infty)\,\mathsf{b}(g_f J)$ for every $g \in G(V)$ with $g_f J \in T$,

then

$$\int_{\Gamma_B(\mathcal{Y})\backslash G(V)} F \;=\; C \cdot \sum_{t \in T} \frac{\mathsf{b}(t)}{\mathrm{mult}_{B_f(\mathcal{Y}),A}(T; t)}.$$

Proof. Normalise the Haar measure of $B_f(\mathcal{Y})$ so that the compact open subgroup $B_f(\mathcal{Y}) \cap A$ has measure 1. The Bessel group acts properly on $G(V)$, so there is a cut-off function for this action: a function β_0 on $G(V)$ whose integral over each orbit of $B(\mathcal{Y})$ is 1. Let $\alpha_0(g_\infty)$ be the integral of β_0 over the finite part of the orbit, and let C be the integral of a against α_0 times the measure of $G_\infty(V)$; it depends on $\mathcal{Y}$, A, a only. Let ψ_T be the function on the cosets of J equal to the inverse of the multiplicity on T and to 0 elsewhere. By (c) the integral of F may be restricted to the set where the multiplicity does not vanish. Integration in stages over $\Gamma_B(\mathcal{Y})\backslash G(V)$ (a result

of the background library on quotient measures, of the type of Weil's integration formula) writes it as the integral, against the density $\alpha_0(g_\infty)\psi_T(g_f J)$, of the inner integrals $\int_{[B(y)]} F(\varsigma g)\, d\varsigma$; this uses the identity

$$\int_{B_f(y)} \psi_T(\varsigma^{-1} t)\, d\varsigma = 1 \qquad \text{for every coset } t \text{ of non-zero multiplicity,}$$

whether t lies in T or not. It holds because the multiplicity is invariant under all of $B_f(y)$ (Lemma 7.1(i)), while the indicator of T is invariant only under $B_f(y) \cap A$, by (b): the integrand is the inverse of the multiplicity of t on the set of the ς with $\varsigma^{-1} t \in T$, which is a union of exactly that many cosets of $B_f(y) \cap A$ (Lemma 7.1(ii)), and it vanishes elsewhere. For a coset of multiplicity zero the integrand vanishes identically; by (c) such cosets carry none of the integral of F. By (d) the inner integrals are $[K_0(V) : J]\, \mathsf{a}(g_\infty)\mathsf{b}(g_f J)$. The density is a product, the real part gives C, and the finite part is a sum over the finite set T of $\psi_T(t)\mathsf{b}(t)$ times the measure of a coset of J, which is $[K_0(V) : J]^{-1}$ (Lemma 7.1(ii)); the two indices cancel. □

LEAN `Grading.exists_integral_eq_sum_over_cut`

Proof of Theorem C. The case $j = 0$. Take $A = K_D^{\mathrm{ch}}(Y_0(D))$; by Inputs 2(d) it contains $i_{y_0(D),f}(K_0^{\mathrm{col}}(D))$. For a tame character ϑ apply Proposition 7.2 with $S = B_f(y_0(D))$, $J = \mathcal{J}_0(D, \vartheta) \subseteq A$ (Lemma 3.1(ii)), $\mathsf{K} = A$, $H_0 = K^{(\mathrm{n}(\vartheta))}(Y_0(D)) \cap A$, $q = \mathrm{n}(\vartheta)$ and W the set of the points where some function of stage 0 does not vanish. For $h \in A$ and $u \in J \subseteq A$ the conjugate huh^{-1} lies in the group A, so the exponent $n = 0$ works; and Lemma 4.2(ii) moves a point of W by an element of S into the principal level of the filling data of D, which is A (Inputs 2(d)).

The case $j \neq 0$. Take $A = K_0(Y_j(D))$; it contains $i_{y_j(D),f}(K_j^{\mathrm{col}}(D))$ by Lemma 6.4, whose hypotheses are Lemma 6.7(iv), Inputs 2(b) and the congruence condition in the definition of $K_j^{\mathrm{col}}(D)$. For every finite point h choose an adapted pair $(b(h), \beta(h))$ (Lemma 6.1(ii)). Fix ϑ, put $Q = Q(\vartheta)$ and call the *profile* of h the family of the integers $\mathrm{ord}_w \beta(h)_a - \mathrm{ord}_w \beta(1)_a$, for the string indices a and the places w in $Q \cup \overline{Q}$. For every profile with values between $-j$ and j choose a point, with this profile if there is one, and keep its components at the places of $Q \cup \overline{Q}$ only, setting the others to 1; let K be the union of the finitely many cosets $h_0\, K_0(Y_j(D))$ of these points h_0. It is compact. Apply Proposition 7.2 with $S = B_f(y_j(D))$, $J = \mathcal{J}_j(D, \vartheta)$, $H_0 = K^{(\mathrm{n}(\vartheta))}(Y_j(D))$, $q = \mathrm{n}(\vartheta)$ and W the set of the finite parts of the points where a function of stage j of a ramified column, with $y_0 \in K_D^{\mathrm{ch}}(Y_0(D))$, does not vanish. The hypotheses: a point of K is integral at the places not dividing $\mathrm{n}(\vartheta)$, because the places of $Q \cup \overline{Q}$ divide it (Inputs 2(a)); for $u \in J$ the conjugate huh^{-1} agrees with an element of $K_0(\,\cdot\,)$ outside $Q \cup \overline{Q}$, and at a place w of Q the component u_w of u is congruent to 1 modulo w, that is, $u_w \in 1 + \mathfrak{p}_w \mathrm{M}_I(\mathcal{O}_w)$, where $\mathfrak{p}_w$ is the maximal ideal of $\mathcal{O}_w$ and $\mathrm{M}_I(\mathcal{O}_w)$ the ring of the matrices over $\mathcal{O}_w$. Let ℓ be the residue characteristic of w. By the binomial expansion, the ℓ-th power maps $1 + \mathfrak{p}_w^\nu \mathrm{M}_I(\mathcal{O}_w)$ into $1 + \mathfrak{p}_w^{\nu+1} \mathrm{M}_I(\mathcal{O}_w)$ for every $\nu \geq 1$; so $u_w^{\ell^\nu} \in 1 + \mathfrak{p}_w^{\nu+1} \mathrm{M}_I(\mathcal{O}_w)$ for all $u \in J$ and all ν. These subgroups are stable under powers, and ℓ divides $\mathrm{n}(\vartheta)$. As they form a basis of neighbourhoods of 1, one exponent n, chosen for the finitely many places of Q, puts $u_w^{\mathrm{n}(\vartheta)^n}$ into the open subgroup $h_w^{-1}\, \mathrm{GL}_I(\mathcal{O}_w) h_w$ for every $u \in J$ at once; at the conjugate place the component of $(huh^{-1})^{\mathrm{n}(\vartheta)^n}$ is then integral with integral inverse as well, the element being unitary. Finally let $h \in \mathsf{W}$. By Proposition 6.8 the lattice of h has near lines; by Lemma 6.3 for $h = 1$ and $Q = \emptyset$ the string diagonal at 1 has exactly

the reference orders; so the profile of h has values in $[-j, j]$ and the diagonal of h has the orders of that of 1 outside $Q \cup \overline{Q}$. Let h_0 be the chosen point with this profile. By Lemma 6.1(vi), adapted bases at h_0 and at h have string diagonals of the same orders at every place, and Proposition 6.2 gives $k \in K_0(\,\cdot\,)$ with $\varsigma = h_0\, k\, h^{-1} \in B_f(\mathcal{Y}_j(D))$; then $\varsigma h = h_0 k \in \mathsf{K}$. The three conclusions of Proposition 7.2 are (i), (ii) and (iii) of the theorem; for $j = 0$, (ii) includes $h \in A$. □

8 The unfolding

Proof of Theorem B. By induction on j, for all $\mathcal{Y}$ and all functions a.

$j = 0$. Take $C = \mathsf{a}(1)$, the value at the tuple of trivial points, and $\mathsf{w} = S(1)$, the value at the tuple of trivial cosets: a combination with one coefficient 1, which is (i). (ii) holds because no index is moved: $q' = q$ for every q, so that $\Omega(S(1)) = S(1)$. For (iii), unfold the Poincaré series twice (Inputs 2(d)): the integral over $[G(Y_0(D))]$ of z times $\mathrm{P}_D\varphi$, the series over all rational points, and the integral over $\Gamma_B(\mathcal{Y})\backslash G(Y_0(D))$ of z times the series over $\Gamma_B(\mathcal{Y})$, which is the function of stage 0, are both equal to $\sum_a \phi_a \int_G z \cdot \mathrm{cb}_a$. The second is integrable, so the hypothesis at the trivial tuples evaluates it as $\mathsf{a}(1)L(S(1))$.

From j to $j+1$. Theorem C at j gives the level A and, for every ϑ, a cut $T = T(\vartheta)$. For every tuple ξ let $C(\xi)$ be the constant of Proposition 7.3 for the grading $\mathcal{Y}_j(D)$, the level A and the function $\xi'_j \mapsto \mathsf{a}(\xi$ with its entry j replaced by $\xi'_j)$. Apply the induction hypothesis at j to the grading $\mathcal{Y}_j(D)$ and the function $\xi \mapsto C(\xi)$; it gives a constant C. Given ϑ, M and S, put

$$S'(q) = \sum_{t \in T} \frac{1}{\mathrm{mult}_{B_f(\mathcal{Y}_j(D)),A}(T;t)}\; S(q \text{ with its entry } j \text{ replaced by } t)$$

and let w be the vector that the induction hypothesis gives for S'.

(i) Each $S'(q)$ is a combination of values of S at tuples whose new entry is integral away from $\mathrm{n}(\vartheta)$ (Theorem C(ii)), with coefficients $1/\mathrm{mult}_{B_f(\mathcal{Y}_j(D)),A}(T;t)$ that lie in $O^{(\mathrm{n}(\vartheta))}$, because the multiplicities divide powers of $\mathrm{n}(\vartheta)$ (Theorem C(iii)). If $Q(\vartheta) = \emptyset$ the sum of the absolute values of their conjugates is at most $|T|$, and T injects into the quotient of $K_0(Y_j(D))$ by the principal congruence subgroup of $\bar{\pi}_g(D)$: its elements are integral everywhere, and two of them that are congruent modulo $\bar{\pi}_g(D)$ differ by an element of $\mathcal{J}_j(D,\vartheta)$ (Lemma 6.7(v)). The order of that quotient is at most the norm of $\bar{\pi}_g(D)$ to the power $|\underline{Y}_j(D)|^2$, which is at most $\mathrm{cut}(D)$ by Lemma 6.7(i) and Inputs 2(c). Substituting these combinations into the admissible combination for S' multiplies the sizes: (i) holds with $\mathrm{cut}(D)^j \cdot \mathrm{cut}(D)$.

(ii) Let b and Ω be as in (ii) for $j+1$. Then $\Omega(S'(q))$ is the sum over $t \in T$ of the same coefficients times S at the moved tuple with entry $i_{\mathcal{Y}_j(D),f}(b_j)\, t$ at j. Since $i_{\mathcal{Y}_j(D),f}(b_j) \in B_f(\mathcal{Y}_j(D)) \cap A$, multiplication by it permutes T (Theorem C(i)) and keeps the multiplicity (Lemma 7.1(i)); so $\Omega(S'(q)) = S'(q')$ with q' the tuple moved at the indices $< j$, and the induction hypothesis gives $\Omega(\mathsf{w}) = \mathsf{w}$.

(iii) Let z, φ, L satisfy the hypothesis for $j+1$ and the grading $\mathcal{Y}$. By Lemma 3.1(iii) and Inputs 1(c), with $z^\flat$ the descent of z times the index $[K_0(Y_j(D)) : \mathcal{J}_j(D,\vartheta)]$,

$$\int_{[G(Y_{j+1}(D))]} z \cdot \mathcal{L}^{(j+1)}(D,\vartheta)(\mathrm{P}_D\varphi) = \int_{[G(Y_j(D))]} z^\flat \cdot \mathcal{L}^{(j)}(D,\vartheta)(\mathrm{P}_D\varphi),$$

and $z^\flat$ is a form of the conjugate type on $Y_j(D)$. It remains to verify the hypothesis of (iii) at j for $z^\flat$, the function $C(\,\cdot\,)$, the map S' and the same L. Fix ξ and y and let F be the function of stage j on $\mathcal{Y}_j(D)$ times $z^\flat$, assumed integrable. Proposition 7.3 applies to it: the stage function is generic (Lemma 5.1(ii)), T is stable (Theorem C(i)), and where F does not vanish the multiplicity divides a power of $\mathrm{n}(\vartheta)$, so it is not zero (Theorem C(iii)). Its hypothesis (d): at a point (ξ'_j, t) with $t \in T$, the value of $z^\flat$ is the index $[K_0(Y_j(D)) : \mathcal{J}_j(D,\vartheta)]$ times the theta integral of z, and by one unfolding step (Inputs 1(b)) the inner integral over $[B(\mathcal{Y}_j(D))]$ of the stage function times that theta integral is the integral over $\Gamma_B(\mathcal{Y})\backslash G(Y_{j+1}(D))$ of the function of stage $j+1$ for the tuples with entry (ξ'_j, t) at j, times z, and this product is integrable by the same Input, which is the premise that the hypothesis for $j+1$ asks for; the stage function at j does not see the new entry (Lemma 5.1(i)). By the hypothesis for $j+1$ this integral is a at the changed tuple times L of S at the changed tuple of cosets; the conditions $y_{2\varrho(D)+2} = 1$ and $y_0 \in K_D^{\mathrm{ch}}(Y_0(D))$ persist, the latter for $j = 0$ by Theorem C(ii). Proposition 7.3 now gives $\int F = C(\xi)\sum_{t\in T} L(S(\dots t \dots))/\operatorname{mult}_{B_f(y_j(D)),A}(T;t) = C(\xi)\,L(S'(\text{cosets of } y))$, which is the hypothesis at j. □

9 From an invariant combination to a good datum

In this section the translate sums and the pair translates are those of Definition 2.14, and *vac* is the vacuum of the space in question.

Proposition 9.1 (the pairs of lifts evaluate admissibly)**.** *In Setting 1.1 let i be a natural number with $i \le \varrho(D)$. Then* $\mathrm{ev}_{D,i}$ *is an admissible evaluation for $N(D)$, $\Pi(D)$ and $\bar\pi_g(D)$ in the sense of the note* Two consecutive theta lifts*: there is a complex number C_i such that for every natural number n prime to $N(D)$, every proposition p and every finite vector t_0 of* $\mathrm{Hom}(Y_{2i+1}(D), Y_{2i}(D)^- \oplus Y_{2i+2}(D))$ *for which $\bar\pi_g(D)\,t_0$ is integral on the rows of the first summand and $\pi_g\,t_0$ on the rows of the second, at all places not dividing n (two conjugate weights),*

$$\mathrm{ev}_{D,i}(\varrho_f(t_0)\mathrm{vac}) \in \mathrm{Adm}(v, \mathcal{J}_n; n, d_i; p, 1),$$
$$v(t_1,t_2) = C_i \cdot \mathsf{T}_{N(D),\Pi(D)}(\underline{X}^\circ_{2i}(D), \underline{X}_{2i+1}(D), \underline{X}^\circ_{2i+2}(D))(\varrho_f(t_1)\mathrm{vac} \otimes \varrho_f(t_2)\mathrm{vac}),$$

where $\mathcal{J}_n$ is the set of the pairs of finite matrices (t_1,t_2) with $\pi_g\,t_1$ and $\pi_g\,t_2$ integral at the places not dividing n, and d_i is the order of $\hat{\mathcal{O}}/\bar\pi_g(D)$ to the power $(n_{2i}(D) + n_{2i+2}(D))\cdot n_{2i+1}(D)$.

Proof. By the definition of $\mathrm{ev}_{D,i}$ this is the theorem *admissible evaluation* of the note *Two consecutive theta lifts* (Theorem D there), applied to the middle pair for $i < \varrho(D)$ and to the last pair for $i = \varrho(D)$. Its hypotheses, in the letters of that theorem: $\Pi(D)$ is a split level for $N(D)$, and $N(D)$ is split and squarefree along it with the principal idele $\hat\pi_N(D)$ of $\pi_N(D)$ (Inputs 2(b)). The middle grading $\mathcal{Y}_{2i+1}(D)$ spans the lattice of $(\mathfrak{a}(D), \hat\pi_N(D), u_{2i+1}(D))$ (Lemma 6.7(iv)). The integral adele β is $\bar\pi_g(D)$, and $\delta = \overline{\hat\pi_g(D)}^{-1}\hat\pi_g(D)$, of norm one, satisfies $\delta = (\overline{\alpha}/\alpha)\cdot(\mathfrak{a}(D)/\overline{\mathfrak{a}(D)})$ because $\mathfrak{a}(D) = \alpha/\overline{\hat\pi_g(D)}$. The family κ is 1 at every index except the point of $\underline{X}_m(D)$ in the last pair, where it is δ. At the places above $N(D)$, $\delta = 1$ and $\bar\pi_g(D)$ is a unit, because chosen places do not divide $N(D)$ (Lemma 6.7(ii)); and $\pi_g\kappa_b$ and $\bar\pi_g(D)\kappa_b$ are integral, δ having at every place the order of $\bar\pi_g(D)/\pi_g$. *The hypothesis on weights*: the zero part of the middle pair joins only indices that carry the same

weight; that of the last pair also joins the point of $\underline{X}_m(D)$ to the only string vector of $\mathcal{Y}_{2\varrho(D)}(D)$, where the two weights are $\bar{\pi}_g(D)$ and $\pi_g\delta$, of the same absolute value at every place: this is the reason for $\kappa = \delta$ there. *The hypothesis on lattices*: scaled neighbour vectors land in the lattices of the outer spaces. In the middle pair this is the lattice lemma of that note (Lemma 4.3 there) for $\mathcal{Y}_{2i}(D)$ and $\mathcal{Y}_{2i+2}(D)$, whose scales $u_{2i}(D)$ and $u_{2i+2}(D)$ are the powers of $\mathfrak{a}(D)/\overline{\mathfrak{a}(D)}$ with exponents $\varrho(D) - i + 1$ and $\varrho(D) - i$ (Lemma 6.7(iv)). In the last pair it is the same lemma for $\mathcal{Y}_{2\varrho(D)}(D)$, with exponent 1, applied to the sum of the two congruences of a neighbour vector, one of them multiplied by the unit ε with $\bar{\varepsilon}\varepsilon = -1$; the frame of the standard space $Y_m(D)$ is 1. □

LEAN `Data.isAdmissibleEval_pairEvalY`

Proposition 9.2 (evaluated data are combinations of reduced vacua)**.** *In Setting 1.1 there is a complex number C such that for every tame character ϑ of D and every tuple $y = (y_j)$ of finite points with $y_j \in K^{(\mathrm{n}(\vartheta))}(Y_j(D))$ for all j:*

$$\mathrm{ed}_{\varrho(D)}(D, \vartheta; y) \otimes \ldots \otimes \mathrm{ed}_0(D, \vartheta; y)$$
$$\in \mathrm{Adm}(C \cdot \mathrm{rv}_D(\cdot), \mathrm{Tr}_{\mathrm{n}(\vartheta)}(D); \mathrm{n}(\vartheta), \mathrm{cut}(D)^{2m}; Q(\vartheta) = \emptyset, 1).$$

Proof. Write $n = \mathrm{n}(\vartheta)$. *Each moved datum.* $\Upsilon_j(D, \vartheta)$ is named by an indicator modulo the product $\mathfrak{m}$ of the chosen places and the places of $Q(\vartheta)$, and $\bar{\pi}_g(D)\, n \in \mathfrak{m}$; by Inputs 2(e) it is an admissible combination, away from n, of translates $\varrho_f(t)\mathrm{vac}$ with $\pi_g t$ integral at the places not dividing n, with denominator a power of the norm of $\bar{\pi}_g(D)$ and size at most 1. The Weil operator of a finite pair (y_j, y_{j+1}) is linear, and by Inputs 2(f) it sends such a translate to an admissible combination of translates of the same kind: the pair is integral at the places not dividing n, so the translations stay in the same group; its finite multiplier has ℓ-adic absolute value 1 for ℓ prime to n, and is 1 when $Q(\vartheta) = \emptyset$, the pair then lying in the stabiliser of the lattice; and the values of the twisting character are ℓ-adic units, of complex absolute value 1. So $\Upsilon_j(D, \vartheta; y)$ is such a combination. *Each pair.* The arrow product is bilinear, and the arrow product of two translated vacua is a root of unity times a translated vacuum of $\mathrm{Hom}(Y_{2i+1}(D), Y_{2i}(D)^- \oplus Y_{2i+2}(D))$, at the vector with rows $-\varepsilon\bar{t}$ on the first summand and t' on the second, which satisfies the integrality of Proposition 9.1: the conjugated block has the weight $\bar{\pi}_g(D)$, the other block the weight π_g. Applying $\mathrm{ev}_{D,i}$ and Proposition 9.1 (with n, prime to $N(D)$), $\mathrm{ed}_i(D, \vartheta; y)$ is an admissible combination of the vectors $v(t_1, t_2)$ of that proposition, each of them C_i times the translate sum of the pair i applied to a tensor of two translated vacua. *The tensor.* A pure tensor of admissible combinations is an admissible combination of the pure tensors, with the product of the denominators and of the sizes; the pure tensor of the vectors $v(t_1, t_2)$ of the pairs is $\prod_i C_i$ times a reduced vacuum, up to the normalisation of the vacuum, which is absorbed in C. The denominators: all are powers of the order of $\hat{\mathcal{O}}/\bar{\pi}_g(D)$, which is $g(D)^{f(D)}$ (Lemma 6.7(i)), with exponents bounded, for each pair, by $4\, n_m(D)^2$; so each pair contributes a divisor of $\mathrm{cut}(D)^4$ and the $\varrho(D) + 1$ pairs a divisor of $\mathrm{cut}(D)^{4(\varrho(D)+1)} = \mathrm{cut}(D)^{2m}$. □

LEAN `Data.exists_isAdmissibleComb_reducedVacuum`

Lemma 9.3. *In Setting 1.1 let ϑ be a tame character of D, $t \in \mathrm{Tr}_{\mathrm{n}(\vartheta)}(D)$ and i a natural number, and let P_Π be the projection of the finite part to the vectors that are vacua at the places of $\Pi(D)$ and their conjugates (Definition 2.15(iii)). Then*

$$(\mathsf{P}_\Pi \otimes \mathsf{P}_\Pi)(\tau_D(\mathrm{v}_D(t).)_i) = \mathrm{v}_D(t)_{2i} \otimes \mathrm{v}_D(t)_{2i+1}.$$

Proof. The places of $\Pi(D)$ and their conjugates do not divide $\mathrm{n}(\vartheta)$ and are not chosen, so $\mathrm{v}_D(t)_j$ is fixed by P_Π (Inputs 2(b)); for such a vector, $\mathsf{P}_\Pi \varrho_f(t')$ is $\varrho_f(t')$ if t' is integral at these places and 0 otherwise (Input 1(f) of the note *Two consecutive theta lifts*). In the sum defining $\tau_D(\mathrm{v}_D(t).)_i$ the vector $-y/N(D)$ is integral at $\Pi(D)$ only for $y = 0$, and that term is the tensor of the two data. □

LEAN `Data.projOnFinPart_pairTranslates`

Lemma 9.4. *In Setting 1.1 let q be a natural number, p a proposition and $t \in \mathrm{Tr}_q(D)$. Then*

$$\mathrm{rv}_D(t) \in \mathrm{Adm}(v, \mathrm{Tr}_{q\,N(D)}(D); q, 1; p, N(D)^{\,f(D)n_m(D)(\varrho(D)+1)}),$$
$$v(t') = \bigotimes_{i \le \varrho(D)} (\mathrm{v}_D(t')_{2i} \otimes \mathrm{v}_D(t')_{2i+1}).$$

Proof. Factor by factor: $\tau_D(\mathrm{v}_D(t).)_i$ is a sum, over at most $N(D)^{f(D)n_m(D)}$ vectors y, of terms with coefficient 1, and each term is a root of unity times $\varrho_f(t_1)\mathrm{vac} \otimes \varrho_f(t_2)\mathrm{vac}$ with t_1, t_2 the sums of the entries of t and of the translations by $\mp y/N(D)$, since Heisenberg operators compose up to values of the additive character. These translations live at the places above $N(D)$, so the new tuple is admissible for $qN(D)$. Tensoring the $\varrho(D)+1$ factors multiplies the sizes. □

LEAN `Data.isAdmissibleComb_reducedVacuum`

Proposition 9.5 (an invariant combination of reduced vacua is the chain of a good datum)**.** *In Setting 1.1 let ϑ be a twist of D and let w be a tensor in the product of the paired data spaces of the $\varrho(D)+1$ pairs of definite lifts. Assume that*

$$\mathsf{w} \in \mathrm{Adm}(\mathrm{rv}_D(\cdot), \mathrm{Tr}_{\mathrm{n}(\vartheta)}(D); \mathrm{n}(\vartheta), 1; Q(\vartheta) = \emptyset, \mathrm{cut}(D)^{\,2m} \cdot \mathrm{cut}(D)^{\,2\varrho(D)+2}),$$

and that for every tuple $b = (b_j)$ with $b_j \in K_j^{\mathrm{col}}(D)$ and $b_{2\varrho(D)+2} = 1$ the tensor w is fixed by the operator

$$\bigotimes_{i \le \varrho(D)} \omega(b_{2i}, b_{2i+1}) \otimes \omega(b_{2i+1}, b_{2i+2}).$$

Then there is a finitely supported $\Psi\colon \mathrm{Tr}(D) \to \mathbb{C}$ that is a good datum for ϑ and such that for every $\varphi \in \mathrm{St}_0(D)$

$$\mathrm{C}_{D,\vartheta}^{[\varrho(D)+1]}(\mathsf{w})(\varphi) = \mathrm{C}_{D,\vartheta}^{(m)}[\Psi](\varphi),$$

the left side being the chain of pairs extended linearly to tensors.

Proof. Substitute Lemma 9.4 into the combination w: this gives a finitely supported Ψ on $\mathrm{Tr}_{\mathrm{n}(\vartheta)N(D)}(D)$ with values in $O^{(\mathrm{n}(\vartheta))}$ and $\mathsf{w} = \sum_t \Psi(t) \bigotimes_i (\mathrm{v}_D(t)_{2i} \otimes \mathrm{v}_D(t)_{2i+1})$; if $Q(\vartheta) = \emptyset$ its size is at most the product of the two sizes, which the comparisons of the data bound by $\exp(x^{60})$. This is condition (i) of Definition 2.15. The chain of pairs of a pure tensor of pairs is the chain of the single data, so $\mathrm{C}_{D,\vartheta}^{[\varrho(D)+1]}(\mathsf{w}) = \mathrm{C}_{D,\vartheta}^{(m)}[\Psi]$, because $m = 2(\varrho(D)+1)$. Joint invariance (ii): regrouping the tensor of the m single data into pairs carries the joint action of a tuple b to the action in the hypothesis,

which fixes w. Condition (iii): let R be the endomorphism of the tensor product of the paired data spaces that is, on the factor of the pair i, the projection $\mathsf{P}_\Pi \otimes \mathsf{P}_\Pi$ followed by the translate sum of that pair (Definition 2.14); on a pure tensor of single data $s = (s_j)_{j<m}$ it gives the tensor of the pair translates $\tau_D(\mathsf{P}_\Pi s)_i$, $i \le \varrho(D)$. By Lemma 9.3, R fixes the reduced vacuum of every $t \in \mathrm{Tr}_{\mathrm{n}(\vartheta)}(D)$, hence w. Applying R to $\mathsf{w} = \sum_t \Psi(t) \bigotimes_i (\mathrm{v}_D(t)_{2i} \otimes \mathrm{v}_D(t)_{2i+1})$ and then the chain of pairs gives (iii). □

LEAN `Data.exists_isGoodDatum_of_invariant`

10 Proof of the seesaw identity

Proof of Theorem A. The constants. Let C_{v} be the constant of Proposition 9.2. Let R and the constants C_{t} be as in Inputs 2(d) (the last stage). For every tuple ξ of real points let $\mathsf{t}(\xi)$ be the functional of Proposition 5.2 for the pair index $i = \varrho(D)$, the grading $\mathcal{Y}_{\mathrm{last}}(D)$ of $Y_m(D)$, the last arrow pair, whose finite evaluation is $\mathrm{ev}_{D,\varrho(D)}$ by definition, and the trivial real point. Apply Theorem B with $j = m = 2\varrho(D) + 2$, the grading $\mathcal{Y}_{\mathrm{last}}(D)$, whose string has length 0, and the function $\mathsf{a}(\xi) = C_{\mathrm{t}(\xi)}$; let C_{u} be the constant it gives. Put

$$c_D = \Big(\frac{C_{\mathrm{u}}\, C_{\mathrm{v}}}{\mathrm{cut}(D)^{\,2m}} \Big)^{-1} \text{ if this quotient is not zero, and } c_D = 1 \text{ otherwise.}$$

The datum. Fix a twist ϑ. Let M be the tensor product of the paired data spaces and

$$S(q) = \mathrm{ed}_{\varrho(D)}(D, \vartheta; q^\circ) \otimes \ldots \otimes \mathrm{ed}_0(D, \vartheta; q^\circ),$$

where q° is the tuple of the chosen representatives of the cosets q; let w be the vector of Theorem B. By Theorem B(i), w is an admissible combination of the values $S(\text{cosets of } y)$, y integral away from $\mathrm{n}(\vartheta)$, with denominator 1 and size $\mathrm{cut}(D)^{\,m}$; by Lemma 5.1(iii) these values are the tensors of Proposition 9.2. Substituting, w is an admissible combination of the vectors $C_{\mathrm{v}}\mathrm{rv}_D(t)$, $t \in \mathrm{Tr}_{\mathrm{n}(\vartheta)}(D)$, with denominator $\mathrm{cut}(D)^{\,2m}$; clearing the constant and the denominator (Inputs 2(f)), the vector $\mathsf{w}^\star = c_D\, C_{\mathrm{u}}\, \mathsf{w}$ is an admissible combination of the reduced vacua themselves with denominator 1 and size $\mathrm{cut}(D)^{\,2m} \cdot \mathrm{cut}(D)^{\,m}$. It is invariant as Proposition 9.5 requires: for a tuple b with $b_j \in K_j^{\mathrm{col}}(D)$ and $b_m = 1$, the operator $\bigotimes_i \omega(b_{2i}, b_{2i+1}) \otimes \omega(b_{2i+1}, b_{2i+2})$ sends $S(q)$ to S at the tuple moved by the images of the b_j under the inclusions $i_{y_j(D),f}$, because the finite evaluation of an arrow pair commutes with the finite points of the three definite groups (*equivariance of the finite evaluation*, Inputs 2(f)); so Theorem B(ii) applies. Proposition 9.5 gives a good datum Ψ with $\mathrm{C}^{[\varrho(D)+1]}_{D,\vartheta}(\mathsf{w}^\star) = \mathrm{C}^{(m)}_{D,\vartheta}[\Psi]$.

The identity. By the criterion (1) of the last stage it suffices to show, for $\varphi \in \mathrm{Col}^{\mathrm{ram}}(D)$,

$$\mathcal{L}^{(m)}(D, \vartheta)(\mathrm{P}_D(c_D\, \varphi)) = R(\mathrm{C}^{(m)}_{D,\vartheta}[\Psi](\varphi)),$$

and for this to pair both sides with an arbitrary $\Xi \in \mathcal{F}_m(D)$. Apply Theorem B(iii) to the form $z = \overline{\Xi}$, the column φ and the linear form

$$L(\mathsf{w}') = \langle \Xi, R(\mathrm{C}^{[\varrho(D)+1]}_{D,\vartheta}(\mathsf{w}')(\varphi)) \rangle.$$

Its hypothesis: for ξ and y with $y_m = 1$ and $y_0 \in K_D^{\mathrm{ch}}(Y_0(D))$, Proposition 5.2 says that the function of stage m on $\mathcal{Y}_{\mathrm{last}}(D)$ is, on the image of $i_{y_{\mathrm{last}}(D)}$, the coefficient at $\mathsf{t}(\xi)$, on the block diagonal elements, of

$$\psi = \mathrm{C}_{D,\vartheta}^{[\varrho(D)+1]}(\mathrm{ed}.(D,\vartheta;y))(\varphi);$$

by (2) of the last stage its integral against $\overline{\Xi}$ is

$$C_{\mathsf{t}(\xi)}\langle \Xi, R\psi\rangle = \mathsf{a}(\xi)\, L(S(\text{cosets of } y)),$$

again by Lemma 5.1(iii). The conclusion of Theorem B(iii) is

$$\langle \Xi, \mathcal{L}^{(m)}(D,\vartheta)(\mathrm{P}_D\varphi)\rangle = C_{\mathrm{u}} \cdot \langle \Xi, R(\mathrm{C}_{D,\vartheta}^{[\varrho(D)+1]}(\mathsf{w})(\varphi))\rangle.$$

Multiplying by c_D and using linearity in the tensor gives the pairing of Ξ with

$$R(\mathrm{C}_{D,\vartheta}^{[\varrho(D)+1]}(\mathsf{w}^\star)(\varphi)) = R(\mathrm{C}_{D,\vartheta}^{(m)}[\Psi](\varphi)).$$

As Ξ is arbitrary, the two forms agree, and (1) of the last stage gives the theorem. The number c_D was fixed before ϑ and φ. □

11 Where the results are used

Theorem A is used once, in the proof of Theorem A of the write-up (*short forms*, unit *Endgame*): its constant c_D is the constant in the Poincaré maps $\mathrm{pr}_D(\gamma) \circ \mathrm{P}_D \circ (c_D\,\cdot)$ from ramified columns to rows: the restriction of this composite to $\mathrm{Col}^{\mathrm{ram}}(D)$ is the Poincaré series $\mathcal{P}_D(\gamma)$ of Setting 2 of the overview, and the "fixed non-zero constant" of that Setting is c_D (in the formal text the map is the composite of the projection, the Poincaré series and the inclusion of the ramified columns scaled by the constant chosen from Theorem A). The roman P_D of this note is the series before the projection and the scaling. For every twist the good datum turns the size and the divisibility of $\iota_{D,\vartheta}$ of such a row into those of $B^{\Psi}_{D,\vartheta}$, which the units *Columns*, *Curve* and *TwistedHecke* estimate. The map $\iota^{\mathrm{lev}}_{D,\vartheta}(\,\cdot\,)$ restricted to the rows is the chain of lifts $\Theta_{D,\vartheta}$ of the overview; Corollary 3.5 and Lemma 3.3 are used by the units *Rows* and *Exchange*, and Proposition 4.1 with the maps of Definition 2.11 by the unit *Filling*, which in turn supplies the last stage (Inputs 2(d)).

12 Notes on the formal text

What kind of seesaw. The identity moves a chain of theta lifts from the tower of indefinite spaces $Y_j(D)$ to the tower of definite spaces $X_j(D)$ sitting inside them, at the price of a Bessel period; the header of the formal file calls it a seesaw identity and names as related work B. Wang (2025), Gomez–Zhu (2014) and Gan–Ichino (2016) for the transfer of Bessel and Fourier–Jacobi periods, and Gross (1999), Hsieh–Namikawa (2015) and X. Zhang (2022) for periods on definite groups as finite sums. Kudla's seesaw principle in the form of the background library (which the catalogue marks as a weaker form) is used in the unit *Exchange*, not here: this unit unfolds.

Shape of the statements. (1) Theorem A is stated for twists, because a good datum is defined for twists; Theorems B and C and all statements about the chain hold for tame characters. (2) In Theorem B the grading $\mathcal{Y}$ is arbitrary with the right string length, and the subtraction in $l = 2\varrho(D) + 1 - j$ is truncated; the proof uses it for $\mathcal{Y}_j(D)$ and, at $j = m$, for $\mathcal{Y}_{\mathrm{last}}(D)$. The integrability in (iii) is a hypothesis *inside* the hypothesis: the formal text never has to prove that the function of stage $j+1$ times z is integrable, because the unfolding step provides it. (3) The constant c_D is Mathlib's normalising unit of $C_{\mathrm{u}}C_{\mathrm{v}}/\mathrm{cut}(D)^{2m}$, which is 1 when the quotient vanishes; the formal text does not show that the quotient is non-zero and does not need it. (4) The bound $\exp(x^{60})$ in the definition of a good datum is asked only for $Q(\vartheta) = \emptyset$, and so are all size clauses in this note. (5) Lemma 3.3(ii) is stated for every commutative ring, with the cardinality of an infinite set counted as 0. (6) The grading $\mathcal{Y}_{\mathrm{last}}(D)$ is defined for every number M with a proof that $m = M$, and the functions of stage $j+1$ are defined for an arbitrary grading of $Y_{j+1}(D)$, which enters only through the constant term. (7) The groups of a hermitian space with frame depend on less than the space: $G_f(V)$ on the index set, $G(V)$ on the signs; the notation of this note records the space whose frame gives the rational structure.

Arguments in the census that Lean does not take. The census of the write-up reads arguments from the text. For the following declarations it lists section variables that the statement binds again, or that occur only as the name of a field: the tame character, the column and the tuple of finite points in Proposition 5.2 (bound inside the statement; Lean takes only ξ); a twist in Proposition 9.2 (the statement quantifies over tame characters); a Boolean named like the constant of a grading in Lemma 6.7(iii); a group, a subgroup and a set of cosets in Propositions 7.2 and 7.3; the two subgroups of Definition 2.20, listed twice. The census ends the statement of Theorem C at its first *let* and so lists only the stage and its bound; the theorem is stated here in full, from the source. For five definitions whose type ends in a linear map or a function ($\mathcal{L}_j(D,\vartheta)(\,\cdot\,)$, $\mathcal{L}^{(j)}(D,\vartheta)(\,\cdot\,)$, $\iota^{\mathrm{lev}}_{D,\vartheta}(\,\cdot\,)$, $\mathrm{lch}_{D,\vartheta}(\cdot)$, $\mathrm{bl}_j(\cdot)$) the census counts the argument of that map as an argument of the definition; the notation shows it. Unused hypotheses were not determined: no Lean was run for this note.

The second reading. A second reader compared the member texts of the whole unit and a sample of the proof steps with the Lean source, after the dossier was made. Where her reading and the dossier differ, this note follows her, having checked each point in the source: one scalar c_D for the datum, chosen before the twist, and a restriction that acts on the definite chain only, the indefinite chain being paired with a test form directly (proof of Theorem A); in Proposition 7.3 the average over the finite Bessel group is taken for every coset of non-zero multiplicity, the multiplicity and not the indicator of the cut being invariant under the whole group; the common weight of the translations is π_g, and the two blocks of a combined vector carry the two conjugate weights (Propositions 9.1 and 9.2); Proposition 9.2 holds for tame characters; the normal form of the Poincaré series is stated with the coordinates of general filling data, the translation by the lattice scale entering only through cc_D; and the constant of Theorem A keeps the zero case of the normalisation.

What the final theorems use. The reader's dossier sets nothing of the unit aside. Of this note, the instance that the index sets $\underline{Y}_j(D)$ are non-empty is used only implicitly.

How this note was written. From the reader's dossier of the unit, after its program check passed, and from the Lean source. Of the 32 structured proofs that belong to this note all were read. In the Lean text I read completely the files of the theorem (all three proofs line by line), of the chain and the gradings, of the stage functions, of the adapted bases, of the two lattice steps, of the standard lattice, of the cuts, of the Poincaré series at Bessel points and of the good datum, and the six lemmas on valuations and levels; among them the four proofs that the reader asked to sample first (Propositions 9.2, 6.6, 4.3 and 7.3). There the structure, the order of the steps and every cited lemma agree with the dossier; the long chains of rewriting and of automated case analysis at the end of these four proofs, and of Propositions 6.2, 6.5 and 9.1, were not run, and the note describes their effect from the terms involved rather than reproducing them. The definitions recalled from the background library and from other units (spaces with frame, gradings, Bessel groups, generic functions, the unfolding step, arrows, chains, good data, filling data) were read in the Lean source; the Inputs are stated as their Lean statements read, the last stage, the unfolding of Poincaré series, the unfolding step and the transposition of lift and descent after the source, the others after the dossier. The proofs here follow the formal ones; they are shorter where the formal text spends lines on coercions, on measurability and on the choice of representatives. The program check of the unit's dossier covered 503 leaf steps of 83 proofs.

Revision. After an independent referee's report the note was revised. The Inputs were completed: the discrete-series input on which every lift rests, four further facts of the library and three of the unit *Filling* are now listed, with their status in the catalogue; Inputs 2(e) is quoted with the adele that sets the weight of the translates, read again in the Lean source, so that step (e) of Proposition 6.8 follows from it. The proofs of Lemma 6.4, of Propositions 6.5, 6.6 and 9.1, of the last step of Proposition 4.4 and of the first hypothesis in the proof of Theorem C were completed. The two boxes of Inputs have their own numbers, Theorem B is stated before Theorem C, the constant of Theorem A is written c_D, and the objects of the library carry the names and the arguments that the other notes of the unit give them.

Appendix A. Concordance with the Lean artifact

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Definition 2.11	`Data.goodProjTranslate`	BesselGeneric.lean	32–33
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Lemma 4.2	`Filling.Fits.exists_besselFin_mul_mem_of_ne_zero`	BesselGeneric.lean	46–50
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Proposition 5.2	`Data.unfoldStage_even`	UnfoldingStages.lean	38–60
Lemma 5.1	`Data.evaluatedDatum_out`	UnfoldingStages.lean	62–67
Proposition 6.8	`Data.hasNearLines_of_unfoldStage_ne_zero`	UnfoldingStages.lean	72–112

The statement of the theorem, as the artifact has it:

```
lemma seesaw : ∃ c : ℂˣ, ∀ θ, ∃ Ψ, IsGoodDatum θ Ψ ∧ ∀ φ,
  ι θ (D.poincare ((c.1 • D.ramifiedColumns.subtype) φ)) = Bμ θ Ψ φ
```

Notation

$P = (N_0, d)$, K parameters; the field $\mathbb{Q}(\sqrt{-d})$

D admissible data for P, with components x, x_1, b (scales), μ (exponent), m (number of lifts), r (rank), g_0, g_1 (primes), π_g

$E(D)$, $f(D)$, $g(D)$, $t(D)$, $n_j(D)$ the CM field; the number of its infinite places; $g_0 g_1$; $5r$; the rank of stage j

$\mathbf{j}$, α, υ, $\varpi(P, \mu)$ Weil data: an element of trace zero, an idele, the splitting character; the generator of the different of $E(D)$

$\varrho(D)$ $m/2 - 1$

$\bar{\pi}_g(D)$, $\hat{\pi}_g(D)$, $\hat{\pi}_N(D)$ the conjugate of π_g in $E(D)$; an idele generating its ideal; the principal idele of $\pi_N(D)$ (the last two written as in the note of the unit *Trace*)

$N(D)$, $\pi_N(D)$, $\Pi(D)$ the level prime; an integer of norm $N(D)$; the places dividing it

ϑ, $Q(\vartheta)$, ψ_ϑ, $\mathrm{n}(\vartheta)$ a tame character; its places; its tame part; the product of the norms of its places

$\omega_{D,w}$, $\mu_{D,j}(\vartheta)$, $\psi_{D,j}(\vartheta)$ residue characters at chosen places; the characters of the lifts; their tame parts

$\underline{X}_j(D)$, $\underline{Y}_j(D)$, $X_j(D)$, $Y_j(D)$ index sets; the definite and the graded hermitian spaces

$\mathsf{G}(D)$, $e_0(D)$, γ the finite group; the good idempotent; an element of $\mathsf{G}(D)$

E, $\mathbb{A}$, $\mathbb{A}_f$, $\hat{\mathcal{O}}$, $\mathcal{O}_w$ a CM field; adeles; finite adeles; their integers; local integers

w, v, $\overline{w}$, $\mathbb{F}_w$, $\mathrm{N}\, w$, ord_w finite places; conjugate; residue field; norm; order

V, V', W, I, J_V, x_V, $x_{V,f}$ hermitian spaces with frame; index set; Gram matrix; frame and its finite part (written as in the note *Two consecutive theta lifts*)

$G(V)$, $G_\infty(V)$, $G_f(V)$, $\Gamma(V)$ adelic, real, finite and rational points

$K_0(V)$, $K_w(V)$, $K^{(n)}(V)$, $K_D^{\mathrm{ch}}(V)$ lattice stabiliser; its part at w; points integral away from n; principal level at the chosen places

$\mathcal{Y}$, $\mathcal{Y}'$, e_y, $\mathfrak{c}(\mathcal{Y})$, N, X, l gradings; bijection; constant; norm; set; string length

z_a, gr, a, a' string vectors; grade; string indices

$U(\mathcal{Y})$, $B(\mathcal{Y})$, $B_f(\mathcal{Y})$, $\Gamma_B(\mathcal{Y})$ unipotent and Bessel groups; finite part; rational points

i_y, $i_{y,f}$ the inclusion of the group of X

ψ_E, sh_y the standard additive character; the shift matrix

$\mathcal{S}(W)$, $\mathcal{W}_f(W)$, $\varrho(y)$, $\varrho_f(y)$, ω, ω_ν^{pr}, θ theta data; finite part; Heisenberg operators; Weil representation; twisted pair action; theta functional

ν, Φ, $v \otimes f$, $\boxtimes$ a character; a datum; integrated datum; arrow product

$T_{y,y'}$, ct, unf the string map; constant term at the string map; unfolded function

$\mathcal{Y}_j(D)$, $\mathcal{Y}_{\mathrm{last}}(D)$ the gradings of the spaces $Y_j(D)$

$\mathrm{in}_{D,j}$, $\mathrm{bl}_j(i)$, R_0, T_0 block inclusion; block number; sets of blocks

$\mathcal{J}_j(D, \vartheta)$, $\psi_j^{\uparrow}(D, \vartheta)$, $\mathcal{K}_j(D, \vartheta)$, $\Upsilon_j(D, \vartheta)$ level, tame part, kernel and finite datum of lift $j+1$

$\mathcal{F}_j(D)$, $\mathfrak{d}_j(D,\vartheta)$, $\mathcal{L}_j(D,\vartheta)(F)$, $\mathcal{L}^{(j)}(D,\vartheta)(F)$ forms; datum; one lift; j lifts

$\iota_{D,\vartheta}$, $\iota^{\mathrm{lev}}_{D,\vartheta}(F)$, $\mathsf{P}_{\mathrm{last}}(D,\vartheta)$, $\mathrm{lch}_{D,\vartheta}(\kappa)$, $\mathrm{lift}_{D,\vartheta}(\gamma)$ the chain of m lifts; with its level; the level group and its character; a lift of γ

S, g_{S}, ϖ_{S}, K_{S}, $K^{\mathrm{pr}}_{\mathsf{S}}$, X_0, η filling data and their components; a central element

$\mathrm{P}_{\mathsf{S}}(\phi,\Delta)$, cb_a, cc_{S}, A, W, $\sigma(u)$ Poincaré series; cut-off Bessel functions; column coordinates; change of coordinates; Bessel vector; simple root sum

$\mathrm{Col}(D)$, $\mathrm{Col}^{\mathrm{ram}}(D)$, cc_D, P_D, $\mathrm{pr}_D(\gamma)$ columns; ramified columns; their coordinates; Poincaré series; projection to rows

$O^{(n)}$, $\mathrm{Adm}(v,\mathcal{J};n,d;p,B)$ numbers integral away from n; admissible combinations

ℓ, $\jmath$, σ a prime; a ring homomorphism $\mathbb{C}\to\mathbb{C}_\ell$; an automorphism of $\mathbb{C}$

$K^{\mathrm{col}}_j(D)$, $\mathrm{St}_j(D)$, $z_w(D)$ level and forms of stage j; the central translation at w

$\mathrm{Tr}(D)$, $\mathrm{Tr}_q(D)$, $\mathrm{v}_D(t)_j$ translations; admissible ones; translated vacua

$\mathrm{C}^{(j)}_{D,\vartheta}(\Phi)$, $\mathrm{C}^{[p]}_{D,\vartheta}(\Phi)$, $\mathrm{C}^{(j)}_{D,\vartheta}[\Psi]$ chains of definite lifts: single, paired, combined

res_D, $B^{\Psi}_{D,\vartheta}$, Ψ restriction to $Y_m(D)$; the definite side; a good datum

$\underline{X}^{\circ}_j(D)$, $\mathsf{T}_{N,\Sigma}(I,I',I'')$, $\tau_D(v)_i$, $\mathrm{rv}_D(t)$ $\underline{X}_j(D)$ without its point; translate sum of three index sets, a set of places and a prime; pair translates of a family v; reduced vacuum

P_Π, R, I', I'' projection at $\Pi(D)$; an endomorphism (Proposition 9.5); index sets

φ, ψ, ξ, y, $\mathfrak{P}^{(j)}_{D,\vartheta,\varphi}[\xi,y;\mathcal{Y}]$ columns; tuples of real and finite points; stage functions

$\mathrm{ev}_{D,i}$, $\mathrm{cut}(D)$, $\Upsilon_j(D,\vartheta;y)$, $\mathrm{ed}_i(D,\vartheta;y)$ finite evaluation of a pair; the cut bound; moved and evaluated data

$\mathrm{fl}_e(i)$, b, β, k, d flag order; adapted basis; string diagonal; auxiliary matrices

$\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{a}(D)$, $u_j(D)$, ϵ_a, Λ, π, Σ ideles; the idele and the scale of the data; reference factors; graded lattice basis; an idele of norm N; a split set of places

H, S, A, J, T, t, H_0, K, W, C a group, subgroups, a set of cosets, a coset; a subgroup, a compact set, a set of points, representatives

$\mathrm{mult}_{S,A}(T;t)$ multiplicity of a cut

C, C_i, C_{t}, C_{u}, C_{v}, c_D constants; the constant of Theorem A

M, L, Ω, a, b, w, $\mathsf{w}^\star$, q, q' a vector space, a linear form, an endomorphism; functions; vectors of M; tuples of cosets

t, $\langle\mathsf{t},\psi\rangle$, R, Ξ, z, $z^\flat$, F a functional; a coefficient; the last-stage map; forms; functions

g, h, h_0, k, u, ς, p, n, i, j, e, f points, integral points, unipotent and Bessel elements; natural numbers and indices; a bijection; a datum

δ, δ_a, λ_0, M, W, ε, κ a norm-one idele; exponents; an integral vector; matrices; a unit with $\overline{\varepsilon}\varepsilon=-1$; a family of weights

e, u_0, g, s, S_0, $\varkappa$, $\mathfrak{p}_w$, $\mathrm{M}_I(\mathcal{O}_w)$ in the Inputs and in single proofs: an integral adele, a projected vector, a representative, a map of parameters, a stabiliser, a number, the maximal ideal at w, a matrix ring

letters with two meanings b: the third scale of D (Setting 1.1 only), an adapted basis (Section 6), and with an index a member of a tuple of level elements; d: the prime of P, and a denominator of an admissible combination; t: a coset, and a tuple of translations in $\mathrm{Tr}(D)$; T: a set of cosets, and with two gradings as indices the string map; x: the first scale of D, and a point or a parameter; ϱ: the Heisenberg operators, and with D the number $\varrho(D)$; P: the parameters, the roman P_D is the Poincaré series and the fraktur one the stage functions; w: a finite place, the sans-serif w a vector; ν: a character, and an index or exponent

$\Theta_{D,\vartheta}$ the chain of lifts on the rows, as in the overview

$\mathfrak{X}$, $\mathfrak{m}$, $\mathfrak{m}_1$, s, Δ, Γ, ϕ, Q, Q_0, U, B, G, Y, Z, O level places; moduli; a datum; a subgroup; coefficients; sets of places and of planes; letters occurring only inside the names above

READBACK NOTE 22

Two consecutive theta lifts: admissible combinations, arrow pairs and zero terms

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The seesaw identity of Chapter V is proved by unfolding a chain of theta lifts two lifts at a time. This note supplies that two-lift step and its arithmetic bookkeeping. First, a calculus of *admissible combinations*: finite linear combinations whose coefficients are integral away from a given integer, up to a denominator, and bounded at every conjugate. Second, for three hermitian spaces graded by strings of lengths l, $l-1$, $l-2$, an *arrow pair*: a splitting of the space of homomorphisms that carries two consecutive lifts at once; its zero term intertwines the Weil actions, so that the period of a generic function against two consecutive constant terms is a matrix coefficient of two definite theta lifts. Third, the finite part of that zero term sends translated vacua to admissible combinations of Gauss-sum translates of pairs of translated vacua.

What this note is. The artifact grh-compact is a body of Lean code whose final theorems state the Riemann hypothesis and its analogue for Dirichlet L-functions. Its authors report that it compiles and that Lean's kernel accepts these theorems from the three standard axioms; the record of checks of this write-up says who replayed what. That is a fact about a formal text: no mathematician has read the argument through, and the result is a *claim*. This note treats one part of one unit of the argument (the folder of the seesaw identity): 57 of its 312 declarations, those of eight files; the other declarations are treated in the notes on the seesaw identity, on the Gauss-sum operator and on the characters and index sets of the data. Every statement below was compared, argument by argument, with its Lean statement, and every proof is written out from the Inputs, except that two statements proved in the formal text by one automatic step are only described (Lemma 3.2, parts (c) and (d)). What is taken for granted is stated as Inputs 1 to 7 with its owner: the construction of the Weil representation, its mixed model and the unfolding of Bessel periods come from the background library of the artifact, which this write-up quotes and does not read. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 98 declarations.

1 The statements

The objects are defined in Section 2; the four statements cite its Settings. Throughout, a *proposition* p is a statement that may or may not hold: it switches the size bounds on.

Theorem A (calculus of admissible combinations). Let $\mathcal{X}$, $\mathcal{X}'$ be sets, Z a complex vector space, $v\colon \mathcal{X} \to Z$ and $v'\colon \mathcal{X}' \to Z$ families of vectors, $A \subseteq \mathcal{X}$ and $A' \subseteq \mathcal{X}'$, n and d natural numbers, p a proposition, B a real number and $z \in Z$ with $z \in \mathrm{Adm}(v, A; n, d; p, B)$ (Definition 2.8).
(i) Substitution. Let d', d'' be natural numbers and B', C real numbers. Suppose that $v(x) \in \mathrm{Adm}(v', A'; n, d'; p, B')$ for every $x \in A$, that $d \cdot d'$ divides d'', and that p implies $B \cdot |B'| \le C$. Then $z \in \mathrm{Adm}(v', A'; n, d''; p, C)$.
(ii) Integrality. Let ℓ be a prime number that does not divide n, let $\jmath\colon \mathbb{C} \to \mathbb{C}_\ell$ be a ring homomorphism, $O_\jmath$ the ring of the complex numbers w with $|\jmath(w)|_\ell \le 1$, and M an $O_\jmath$-submodule of Z that contains $v(x)$ for every $x \in A$. Then $d \cdot z \in M$.
(iii) Tensor products. Let n be a natural number and p a proposition. For $j \in \mathbb{N}$ let $\mathcal{X}_j$ be an additive group and A_j a subgroup of it; for $i \in \mathbb{N}$ let F_i be a complex vector space, $v_i\colon \mathcal{X}_{2i} \times \mathcal{X}_{2i+1} \to F_i$ a family and $f_i \in F_i$. Let d, m be natural numbers and B a real number, and suppose $f_i \in \mathrm{Adm}(v_i, A_{2i} \times A_{2i+1}; n, d; p, B)$ for every $i < m$. Let $v_{\otimes m}$ be the family $y \mapsto v_{m-1}(y_{2m-2}, y_{2m-1}) \otimes \cdots \otimes v_0(y_0, y_1)$, indexed by all tuples $y = (y_j)_{j\in\mathbb{N}}$ with $y_j \in \mathcal{X}_j$, and A_* the set of the tuples with $y_j \in A_j$ for all j. Then the pure tensor satisfies

$$f_{m-1} \otimes \cdots \otimes f_0 \in \mathrm{Adm}(v_{\otimes m}, A_*; n, d^m; p, |B|^m).$$

Theorem B (the zero term of an arrow pair is equivariant). In Setting 2.6, let X'' and H be finite index sets and $\mathcal{R} \in \mathrm{Arr}_{X'',H}(\mathcal{Y}, \mathcal{Y}', \mathcal{Y}'')$ an arrow pair (Definition 2.16), with its homomorphism $\mathrm{emb}_{\mathcal{R}}\colon G(\langle X_0\rangle) \to G(\langle X''\rangle)$, its splitting $\Xi^{\mathrm{pr}}(\mathcal{R})$ and its finite evaluation $\mathrm{fev}_{\mathcal{R}}$ (Definition 2.17). Let μ, μ' be characters of $\mathrm{U}(1)(\mathbb{A})$ trivial on $\mathrm{U}(1)(E)$.
(i) For all $g \in G(\langle X\rangle)$, $g' \in G(\langle X'\rangle)$, $k \in G(\langle X_0\rangle)$ and $f \in \mathcal{S}(\mathrm{Hom}(V', V^- \oplus V''))$,

$$\mathrm{ev}_{\Xi^{\mathrm{pr}}(\mathcal{R})}\Big(\omega^{\mathrm{tr}}_{\mu,\mu'}(\imath_{\mathcal{Y}}(g), \imath_{\mathcal{Y}'}(g'), \imath_{\mathcal{Y}''}(k))f\Big) = \omega^{\mathrm{tr}}_{\mu,\mu'}(g, g', \mathrm{emb}_{\mathcal{R}}(k))\ \mathrm{ev}_{\Xi^{\mathrm{pr}}(\mathcal{R})}(f).$$

(ii) Let $g \in G_f(X)$, $g' \in G_f(X')$, $k \in G_f(X_0)$, $k' \in G_f(X'')$ and $x \in G_f(I'')$ be finite points with $\imath^f_{\mathcal{Y}''}(k) = x$ and $\mathrm{emb}_{\mathcal{R}}(1, k) = (1, k')$, and let $\Phi \in \mathcal{W}_f(\mathrm{Hom}(V, V'))$, $\Phi' \in \mathcal{W}_f(\mathrm{Hom}(V', V''))$. Then

$$\begin{aligned}\mathrm{fev}_{\mathcal{R}}\Big(\omega^{\mathrm{pr},f}_{\mu}(\imath^f_{\mathcal{Y}}(g), \imath^f_{\mathcal{Y}'}(g'))\Phi \boxtimes \omega^{\mathrm{pr},f}_{\mu'}(\imath^f_{\mathcal{Y}'}(g'), x)\Phi'\Big)\\ = (\omega^{\mathrm{pr},f}_{\mu}(g, g') \otimes \omega^{\mathrm{pr},f}_{\mu'}(g', k'))\ \mathrm{fev}_{\mathcal{R}}(\Phi \boxtimes \Phi').\end{aligned}$$

Theorem C (two unfolding steps are a double lift). *(i)* In Setting 2.6, let X'' and H be finite index sets and $\mathcal{R} \in \mathrm{Arr}_{X'',H}(\mathcal{Y},\mathcal{Y}',\mathcal{Y}'')$. There is $z \in \mathbb{C}$ such that for all characters μ, μ' of $\mathrm{U}(1)(\mathbb{A})$ trivial on $\mathrm{U}(1)(E)$, all data $f \in \mathcal{S}(\mathrm{Hom}(V,V'))$ and $f' \in \mathcal{S}(\mathrm{Hom}(V',V''))$, all $k \in G(\langle X_0\rangle)$, all functions $P\colon G(V) \to \mathbb{C}$ and all $h \in G(V)$: if P is generic for $\mathcal{Y}$, then

$$\begin{aligned}&\int_{[B(\mathcal{Y}')]} \int_{[B(\mathcal{Y})]} P(gh)\ \mathrm{ct}^{\mu}_{\mathcal{Y},\mathcal{Y}'}(f)(g,r)\ \mathrm{ct}^{\mu'}_{\mathcal{Y}',\mathcal{Y}''}(f')(r,\imath_{\mathcal{Y}''}(k))\,dg\,dr\\ &\quad= \int_{[G(\langle X\rangle)]} P(\imath_{\mathcal{Y}}(g)h)\Big(z \int_{[G(\langle X'\rangle)]} \theta(\omega^{\mathrm{tr}}_{\mu,\mu'}(g,r,\mathrm{emb}_{\mathcal{R}}(k))f_{\mathcal{R}})\,dr\Big)dg,\\ &\qquad\text{where } f_{\mathcal{R}} = \mathrm{ev}_{\Xi^{\mathrm{pr}}(\mathcal{R})}(\varrho(\mathbf{T}(\mathcal{Y},\mathcal{Y}',\mathcal{Y}''))(f\boxtimes f')).\end{aligned}$$

Here $\varrho(\mathbf{T}(\mathcal{Y},\mathcal{Y}',\mathcal{Y}''))$ is the Heisenberg operator of $\mathrm{Hom}(V',V^-\oplus V'')$ at the frame coordinates of the stacked string map (Definition 2.15), and θ is the theta functional of $\mathrm{Hom}(\langle X'\rangle,\langle X\rangle^-\oplus\langle X''\rangle)$.
(ii) Let E be a CM field with Weil data, X, X', X_0 finite index sets, $\varkappa$ a weight and e, e', e'' determinant exponents, U a set, and for $o \in U$ let μ_o, μ'_o be characters of $\mathrm{U}(1)(\mathbb{A})$ trivial on $\mathrm{U}(1)(E)$ and

$$\begin{aligned}&\mathcal{L}_o\colon \mathcal{W}_f(\mathrm{Hom}(\langle X\rangle,\langle X'\rangle)) \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}(X;\varkappa,e),\mathcal{A}(X';\varkappa,e')),\\ &\mathcal{L}'_o\colon \mathcal{W}_f(\mathrm{Hom}(\langle X'\rangle,\langle X_0\rangle)) \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}(X';\varkappa,e'),\mathcal{A}(X_0;\varkappa,e''))\end{aligned}$$

linear maps such that $(\mathcal{L}_o)_o$ is a family of theta lifts for $(\mu_o)_o$ and $(\mathcal{L}'_o)_o$ a family of theta lifts for $(\mu'_o)_o$ (Input 5). Let u be a real datum of $\mathrm{Hom}(\langle X'\rangle,\langle X\rangle^-\oplus\langle X_0\rangle)$ and t a linear functional on the vectors of weight $\varkappa$ for X. Let $\mathcal{L}''_o$ be the linear map on $\mathcal{W}_f(\mathrm{Hom}(\langle X\rangle,\langle X'\rangle))\otimes\mathcal{W}_f(\mathrm{Hom}(\langle X'\rangle,\langle X_0\rangle))$ with $\mathcal{L}''_o(\Phi\otimes\Phi') = \mathcal{L}'_o(\Phi')\circ\mathcal{L}_o(\Phi)$. Then there is a linear functional t'' on the vectors of weight $\varkappa$ for X_0 such that for all $o \in U$, all y in the tensor product, all $\varphi \in \mathcal{A}(X;\varkappa,e)$ and all $g \in G(\langle X_0\rangle)$

$$\begin{aligned}&\int_{[G(\langle X\rangle)]} \mathrm{coef}_t(\varphi)(x) \int_{[G(\langle X'\rangle)]} \theta\Big(\omega^{\mathrm{tr}}_{\mu_o,\mu'_o}(x,r,g)(u\otimes \mathrm{at}(y))\Big)\,dr\,dx\\ &\qquad= \mathrm{coef}_{t''}(\mathcal{L}''_o(y)(\varphi))(g).\end{aligned}$$

Theorem D (the finite evaluation of an arrow pair is admissible). In Setting 2.19, with its index sets X, X'' and its objects N, Σ, π, δ, β, κ, let X', X_0, H be finite index sets and l, l', l'' natural numbers; let V, V', V'' be hermitian spaces over E with index sets I, I', I'' and with gradings: $\mathcal{Y}$ of V with definite part X^+ and a string of length l, $\mathcal{Y}'$ of V' with definite part X'^+ and a string of length l', $\mathcal{Y}''$ of V'' with definite part X_0 and a string of length l'', all of norm N. Let a be a finite idele and u a finite adele with $u\bar{u} = 1$ such that $\mathsf{Lat}(\mathcal{Y}'; a, \pi, u)$, and suppose $\delta = (\bar{\alpha}/\alpha)\cdot(a/\bar{a})$, α the idele of the Weil data. Let $\mathcal{R} \in \mathrm{Arr}_{X''^+, H}(\mathcal{Y}, \mathcal{Y}', \mathcal{Y}'')$ be an arrow pair, with the matrix $\mathrm{inc}_{\mathcal{R}}$ of its zero part (rows $I \sqcup I''$, columns $(H \sqcup H) \sqcup (X^+ \sqcup X''^+)$). Suppose:
(weights) for all $i \in I \sqcup I''$ and $b \in X^+ \sqcup X''^+$, either $\mathrm{inc}_{\mathcal{R}}(i,b) = 0$ or $|\beta^{\langle i\rangle}|_w = |\beta^{\langle b\rangle}\kappa_b|_w$ for every finite place w;
(lattices) for all $r_1, r_2 \in \mathbb{A}_f$ with $r_1 \cdot (a/\bar{a}) = \overline{\epsilon_-}\,\delta$ and $r_2 r_1 = 1$, all c, x such that $\mathsf{Nb}_{\pi,\delta}(c; x)$, and the vector $v = \mathrm{inc}_{\mathcal{R}} \cdot ((0,\, r_1 u c),\ (r_1\kappa_b x_b)_{b\in X^+},\ (\kappa_b x_b)_{b\in X''^+})$, whose coordinates on the first copy of H are 0 and on the second copy all equal to $r_1 uc$: all entries of $x^{-1}_{V',f}(r_2\, v|_I)$ and of $x^{-1}_{V'',f}(v|_{I''})$ lie in $\hat{\mathcal{O}}$.
Then $\mathsf{AdmEv}_{N,\Sigma,\beta}(\mathrm{fev}_{\mathcal{R}}; \beta^{\langle\cdot\rangle})$: the finite evaluation of $\mathcal{R}$ is an admissible evaluation for (N, Σ, β) (Definition 2.21, with the index sets X, X'^+, X'').

The four theorems are used in the note on the seesaw identity. Theorem A is the bookkeeping of every denominator and every size there; Theorem C, applied once for each pair of consecutive lifts, turns the function left after an even number of unfoldings into a matrix coefficient of a chain of pairs of definite lifts; Theorem B makes the resulting data invariant under the levels; Theorem D, with Theorem A(iii), expresses these data through translated vacua with controlled denominators. The transfer of a Bessel or Fourier–Jacobi period under one theta lift is known; the file of the seesaw theorem names as related work B. Wang (2025), Gomez–Zhu (2014) and Gan–Ichino (2016). The seesaw principle of Kudla, which the background library proves in a weaker form for periods, is not used in this note.

2 The setting

Conventions. In every statement E is a CM field with a CM type (one embedding into $\mathbb{C}$ chosen from each pair of complex conjugate embeddings). The bar is complex conjugation on E, on its adeles $\mathbb{A}$, on the finite adeles $\mathbb{A}_f$ and on the ring $\hat{\mathcal{O}}$ of integral finite adeles; M^* is the conjugate transpose of a matrix. Finite places of E are written w, with conjugate place $\bar{w}$; $|\cdot|_w$ is the absolute value at w, with value e^{-1} at a uniformiser, and u_w is the component at w of a finite adele u. Where a statement says so, E carries *Weil data* (Definition 2.2). All index sets (I, I', I'', I_0, I_1, I_2, X, X', X'', X_0, H) are finite; for an index set X we put $X^+ = X \sqcup \{\mathrm{pt}\}$, and $\{0, \dots, l-1\}$ indexes a *string* of length l. Vectors over $\mathbb{A}_f$ indexed by a product of index sets are also read as matrices. The letters Z, Z', W, F_i denote complex vector spaces, $\mathcal{X}$, $\mathcal{X}'$ sets that index families of vectors, $\mathcal{X}_j$ an additive group. A prime number is ℓ or N; $\jmath\colon \mathbb{C} \to \mathbb{C}_\ell$ is a ring homomorphism and $O_\jmath$ the ring of the $w \in \mathbb{C}$ with $|\jmath(w)|_\ell \le 1$; σ runs over the field automorphisms of $\mathbb{C}$. In Section 4, R and S are commutative rings and R has an involution, written as a bar. A hermitian space

V has an index set and signs (Definition 2.1); gradings are written y, y', y'', with strings of lengths l, l', l'' and norm N.

Vocabulary of the background library. The next five definitions recall objects that the artifact constructs in its background library. They are not declarations of this unit; we state what the proofs use, and quote their properties as Inputs 1 to 7.

Definition 2.1 (hermitian spaces and their groups; recalled)**.** A *hermitian space* V over E consists of a finite index set I, *signs* $I \to \{\pm 1\}$, a hermitian matrix $J_V \in M_I(E)$ and a *frame* $x_V \in \mathrm{GL}_I(\mathbb{A})$ such that $x_V^* J_V x_V$ is the diagonal matrix of the signs at every infinite place and the identity at every finite place; $x_{V,f}$ is the finite part of x_V. So $x_{V,f}\hat{\mathcal{O}}^I$ is a self-dual lattice for J_V. The adelic unitary group is $G(V) = G_\infty(V) \times G_f(I)$, where $G_\infty(V)$ is the product over the infinite places of the unitary groups of the signs and $G_f(I) = \{g \in \mathrm{GL}_I(\mathbb{A}_f) : g^* g = 1\}$; an element $g = (g_\infty, g_f)$ has the *matrix in rational coordinates* $x_V g x_V^{-1}$, which is unitary for J_V. The *rational points* $\Gamma(V)$ are the g whose matrix in rational coordinates has entries in E, and $[G(V)] = \Gamma(V)\backslash G(V)$ carries the invariant measure of the library (a probability measure when V is definite). The *standard space* $\langle X \rangle$ on an index set X has all signs $+1$, $J = 1$ and frame 1; it is definite. For hermitian spaces V, V' (index sets I, I'), $\mathrm{Hom}(V, V')$ is the hermitian space with index set $I' \times I$, matrix $J_{V'} \otimes (J_V^{-1})^{\mathsf{T}}$ and frame $x_{V'} \otimes (x_V^{-1})^{\mathsf{T}}$; a pair $(g, g') \in G(V) \times G(V')$ acts on it by $y \mapsto g' y g^{-1}$. $V \oplus V'$ is the orthogonal sum, $\mathbb{H}^H$ the hyperbolic space with index set $H \sqcup H$ and matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $\mathbb{H}^H \oplus V$ their sum. An adelic matrix $y \colon V \to V'$ *in rational coordinates* is an $I' \times I$ matrix over $\mathbb{A}$; its *frame coordinates* are $x_{V'}^{-1} y x_V$, and its *adjoint* is $y^\dagger = J_V^{-1} y^* J_{V'}$.

Definition 2.2 (Weil data, spaces of data, the Weil representation; recalled)**.** *Weil data* for E are: the condition that E is stably unramified in the sense of the library, which provides a unit ϵ_- of $\hat{\mathcal{O}}$ with $\overline{\epsilon_-}\,\epsilon_- = -1$; an element $\mathbf{j} \in E$ with $\bar{\mathbf{j}} = -\mathbf{j}$; a finite idele α such that $\mathbf{j}\alpha\bar{\alpha}$ generates the inverse different at every finite place; and a splitting character v, a Hecke character of E whose restriction to the norm-one ideles is again written v. For a hermitian space V over such an E, with index set I, the library constructs: a Banach space $\mathcal{S}(V)$ of *data*, which are functions on $\mathbb{A}^I$; *Heisenberg operators* $\varrho_V(y)$ on it, for $y \in \mathbb{A}^I$; the *Weil representation* ω_V of $G(V)$ on it; the *theta functional* $\theta_V \colon \mathcal{S}(V) \to \mathbb{C}$, invariant under $\Gamma(V)$, with $f(y) = \theta_V(\varrho_V(y) f)$ for every datum f and $y \in \mathbb{A}^I$; the *vacuum* $\mathrm{vac}_V \in \mathcal{S}(V)$; and a space $\mathcal{F}(V)$ of *real data* (a Fock space) with its *polynomial vectors* v_Q, Q a complex polynomial, and a bounded bilinear map $\mathcal{F}(V) \times \mathcal{S}(V) \to \mathcal{S}(V)$, $(v, f) \mapsto v \otimes f$. The *finite Heisenberg operator* of $x \in \mathbb{A}_f^I$ is $\varrho_{f,V}(x)$, a root of unity times ϱ_V at the adelic vector with real part 0 and finite part αx; the *finite part* of $\mathcal{S}(V)$ is the span $\mathcal{W}_f(V)$ of the vectors $\varrho_{f,V}(x)\mathrm{vac}_V$. For an adelic vector y its *finite coordinates* are $y_f = \alpha^{-1} \cdot (\text{finite part of } y)$. The *finite multiplier* of $g \in G_f(I)$ is a complex number $\Delta_f(g) \neq 0$; ψ_0 is the additive character $u \mapsto \psi_f(\mathbf{j}\alpha\bar{\alpha}u)$ of $\mathbb{A}_f$, ψ_f the standard one. For a set S of finite places, P^S is the bounded operator on $\mathcal{S}(V)$ that averages $\varrho_{f,V}(y)$ over the $y \in \hat{\mathcal{O}}^I$ supported at the places of S and their conjugates. For matrices $y \colon V \to V'$ in rational coordinates, $\varrho(y)$ denotes the Heisenberg operator of $\mathrm{Hom}(V, V')$ at the frame coordinates of y.

Input 1. (the Weil representation; background library, units of the folders *Weil/Construction* and *Adelic/Adeles*). Let V be a hermitian space with index set I over a field with Weil data. (a) $\varrho_{f,V}(x)\varrho_{f,V}(y) = \varsigma \cdot \varrho_{f,V}(x+y)$ with ς a value of ψ_0, and every value of ψ_0 is a root of unity; $\varrho_{f,V}(y)\mathrm{vac}_V = \mathrm{vac}_V$ for $y \in \hat{\mathcal{O}}^I$; $\varrho_{f,V}(x)$ maps $\mathcal{W}_f(V)$ into itself. (b) $\omega_V(g)\varrho_{f,V}(x) = \varrho_{f,V}(g_f x)\omega_V(g)$ and $\omega_V(g)(v \otimes f) = (\omega_V(g_\infty)v) \otimes \omega_V(1, g_f)f$, where $G_\infty(V)$ acts on $\mathcal{F}(V)$; for an adelic vector y and $v \in \mathcal{F}(V)$ there is $v_1 \in \mathcal{F}(V)$ with $\varrho_V(y)(v \otimes f) = v_1 \otimes \varrho_{f,V}(y_f)f$ for all f; $\varrho_{f,V}(x)(v \otimes f) = v \otimes \varrho_{f,V}(x)f$; and $\omega_V(1,g)$ maps $\mathcal{W}_f(V)$ into itself. (c) For $g \in G_f(I)$, $\omega_V(1,g)\mathrm{vac}_V = \Delta_f(g)|\Delta_f(g)|^{-2}\sum_y \varrho_{f,V}(gy)\mathrm{vac}_V$, the sum over representatives y of $\hat{\mathcal{O}}^I$ modulo $\hat{\mathcal{O}}^I \cap g^{-1}\hat{\mathcal{O}}^I$; the number of classes is $|\Delta_f(g)|^2$. (d) If $f \in \mathcal{W}_f(V)$ and $\varrho_{f,V}(gy)f = f$ for all $y \in \hat{\mathcal{O}}^I$, then $f = c \cdot \omega_V(1,g)\mathrm{vac}_V$ for some $c \in \mathbb{C}$. (e) A vector $u \in \mathbb{A}_f^I$ lies in $\hat{\mathcal{O}}^I$ if and only if $\psi_0(x^*u) = 1$ for every $x \in \hat{\mathcal{O}}^I$; and $\varrho_{f,V}(x)\varrho_{f,V}(y) = \psi_0(y^*x) \cdot \varrho_{f,V}(y)\varrho_{f,V}(x)$. (f) If $P^S f = f$, then $P^S(\varrho_{f,V}(y)f)$ is $\varrho_{f,V}(y)f$ if y is integral at the places of S and their conjugates, and 0 otherwise. *References.* The library gives as its sources for the Weil representation A. Weil, Acta Math. 111 (1964), and S. Kudla, Israel J. Math. 87 (1994); the catalogue of this write-up has not confirmed the places: reference to be confirmed. The formulas above are those of the lattice model.

LEAN `Space.finHeisOp_finHeisOp` · `isOfFinOrder_ψf` · `Space.finHeisOp_vacElem` · `Space.finHeisOp_mem_finitePart` · `Space.omega_finHeisOp` · `Space.omega_integrated` · `Space.exists_rho_integrated_eq` · `Space.omega_vacElem_sum` · `norm_finMultiplier_sq` · `Space.exists_eq_smul_omega_vacElem` · `stdLattice.mem_stdLattice_iff_psi₀` · `Space.projOn_finHeisOp`

Definition 2.3 (arrow products and the triple action; recalled)**.** Let E carry Weil data and let V, V', V'' be hermitian spaces with index sets I, I', I''. V^- is V with the matrix $-J_V$, the opposite signs and the frame x_V multiplied at the finite places by the idele $\overline{\epsilon}_\bar{\alpha}/\alpha$; an element g of $G(V)$, read in $G(V^-)$, is written g^-. For a character μ of $\mathrm{U}(1)(\mathbb{A})$ trivial on $\mathrm{U}(1)(E)$ put $\mu_V = \mu \cdot \upsilon^{\lfloor |I|/2 \rfloor}$. The *twisted pair action* on $\mathcal{S}(\mathrm{Hom}(V,V'))$ is $\omega_\mu^{\mathrm{pr}}(g,g') = \mu_V(\det g')\,\mu_{V'}(\det g)^{-1}\,\omega_{\mathrm{Hom}(V,V')}(g,g')$, and $\omega_\mu^{\mathrm{pr},f}$ is its restriction to finite points, acting on $\mathcal{W}_f(\mathrm{Hom}(V,V'))$. (The note on the Gauss-sum operator writes $\omega_{a,b}$ for the pair action twisted by two characters a, b; in that notation ω_μ^{pr} is the action with $a = \mu_V$ and $b = \mu_{V'}$.) The *arrow product* is a bounded bilinear map $(f,f') \mapsto f \boxtimes f'$ from $\mathcal{S}(\mathrm{Hom}(V,V')) \times \mathcal{S}(\mathrm{Hom}(V',V''))$ to $\mathcal{S}(\mathrm{Hom}(V', V^- \oplus V''))$; it induces the linear map $\mathrm{at}\colon \mathcal{W}_f(\mathrm{Hom}(V,V')) \otimes \mathcal{W}_f(\mathrm{Hom}(V',V'')) \to \mathcal{S}(\mathrm{Hom}(V', V^- \oplus V''))$, and $\mathrm{at}^\flat$ is a chosen linear left inverse of at. There is also a bounded bilinear arrow product of real data, $(v,v') \mapsto v \boxtimes v'$, from $\mathcal{F}(\mathrm{Hom}(V,V')) \times \mathcal{F}(\mathrm{Hom}(V',V''))$ to $\mathcal{F}(\mathrm{Hom}(V', V^- \oplus V''))$. The *triple action* of a triple (g,g',g'') of elements of $G(V)$, $G(V')$, $G(V'')$ on the space $\mathcal{S}(\mathrm{Hom}(V', V^- \oplus V''))$ is

$$\omega_{\mu,\mu'}^{\mathrm{tr}}(g,g',g'') = \mu_V(\det g')\,\mu_{V'}(\det g)^{-1}\,\mu'_{V'}(\det g'')\,\mu'_{V''}(\det g')^{-1} \cdot \upsilon(\det g)^{|I'|}\,\upsilon(\det g')^{-|I|} \cdot \omega(g', g^- \oplus g''),$$

ω the Weil representation of $\mathrm{Hom}(V', V^- \oplus V'')$.

Input 2. (two lifts as one pair; background library, unit of the folder *Weil/DualPair*). In the situation of Definition 2.3: (a) there is $z \neq 0$ with $\theta(f \boxtimes f') = z \cdot \theta(f)\theta(f')$ for all f, f'; (b) for matrices $y\colon V \to V'$, $y'\colon V' \to V''$ in rational coordinates, $\varrho\left(\begin{smallmatrix} y^\dagger \\ y' \end{smallmatrix}\right)(f \boxtimes f') = \varrho(y)f \boxtimes \varrho(y')f'$; (c) for a finite vector y of $\mathrm{Hom}(V', V^- \oplus V'')$, with blocks y_1 (indices $I \times I'$) and y_2 (indices $I'' \times I'$), $\varrho_f(y)(f \boxtimes f') = \varsigma \cdot \varrho_f(\epsilon_- y_1^*)f \boxtimes \varrho_f(y_2)f'$ with ς a root of unity, and $\mathrm{vac} \boxtimes \mathrm{vac} = \mathrm{vac}$; (d) $\omega_\mu^{\mathrm{pr}}(g, g')f \boxtimes \omega_{\mu'}^{\mathrm{pr}}(g', g'')f' = \omega_{\mu,\mu'}^{\mathrm{tr}}(g, g', g'')(f \boxtimes f')$, and for finite points $\mathrm{at} \circ (\omega_\mu^{\mathrm{pr},f}(p, q) \otimes \omega_{\mu'}^{\mathrm{pr},f}(q, r)) = \omega_{\mu,\mu'}^{\mathrm{tr}}((1, p), (1, q), (1, r)) \circ \mathrm{at}$; (e) at is injective, $\mathrm{at}^\flat \circ \mathrm{at} = 1$, and the range of at is $\mathcal{W}_f(\mathrm{Hom}(V', V^- \oplus V''))$; (f) $(g, g') \mapsto \theta(\omega_{\mu,\mu'}^{\mathrm{tr}}(g, g', k)f)$ is invariant under left multiplication by $\Gamma(V) \times \Gamma(V')$; (g) $\omega_\mu^{\mathrm{pr}}(g, g')\varrho(y) = \varrho(g'yg^{-1})\omega_\mu^{\mathrm{pr}}(g, g')$, with g, g' read in rational coordinates; (h) for real data v, v' and data f, f': $(v \otimes f) \boxtimes (v' \otimes f') = (v \boxtimes v') \otimes (f \boxtimes f')$; and for a polynomial Q_1 in the coordinates of $\mathrm{Hom}(V', V^-)$ and a polynomial Q_2 in those of $\mathrm{Hom}(V', V'')$, $v_{Q_1 Q_2} = v_{Q_1'} \boxtimes v_{Q_2}$, where Q_1' is the polynomial in the coordinates of $\mathrm{Hom}(V, V')$ obtained from Q_1 by sending each variable to the variable at the transposed position, with the sign that the formal text attaches to the two positions. *References:* none in the catalogue; reference to be confirmed. The catalogue classes the product formula for polynomial vectors in (h) as elementary.

LEAN `theta_arrowProduct` · `homHeisOp_arrowProduct` · `finHeisOp_arrowProduct` · `arrowProduct_twistedPairWeil` · `arrowTensor_map_liftWeilFin` · `leftInverse_arrowTensor` · `range_arrowTensor` · `theta_tripleWeil_rat_mul` · `pairWeil_homHeisOp`

Definition 2.4 (splittings and zero terms; recalled)**.** Let V, V_0 be hermitian spaces with index sets I, I_0 and H a finite index set. A *splitting* Ξ of V over V_0 along H consists of matrices $\mathsf{k}_\Xi, \mathsf{l}_\Xi \in M_{I \times H}(E)$ and $\mathsf{i}_\Xi \in M_{I \times I_0}(E)$ with $\mathsf{k}^* J_V \mathsf{k} = 0$, $\mathsf{l}^* J_V \mathsf{l} = 0$, $\mathsf{l}^* J_V \mathsf{k} = 1$, $\mathsf{i}^* J_V \mathsf{i} = J_{V_0}$, $\mathsf{i}^* J_V \mathsf{k} = 0$, $\mathsf{i}^* J_V \mathsf{l} = 0$ and $|I| = |I_0| + 2|H|$. In frame coordinates these are $x_V^{-1}\mathsf{k}_\Xi$, $x_V^{-1}\mathsf{l}_\Xi$ and $x_V^{-1}\mathsf{i}_\Xi x_{V_0}$; their finite parts are written $\mathsf{k}_{\Xi,f}$, $\mathsf{l}_{\Xi,f}$, $\mathsf{i}_{\Xi,f}$. If E carries Weil data, the *zero term* $\mathrm{ev}_\Xi\colon \mathcal{S}(V) \to \mathcal{S}(V_0)$ is the bounded linear map with $(\mathrm{ev}_\Xi f)(y)$ the mean of $f(x_V^{-1}\mathsf{i}_\Xi x_{V_0} y + x_V^{-1}\mathsf{k}_\Xi b)$ over $b \in \mathbb{A}^H$ modulo E^H. Splittings compose: one of V over V_0 and one of V_0 over V_1 give one of V over V_1, and the zero term of the composite is the composite of the zero terms. For a hermitian space V', a splitting Ξ of V over V_0 along H induces one of $\mathrm{Hom}(V', V)$ over $\mathrm{Hom}(V', V_0)$ along $H \times I'$. The space $\mathbb{H}^H \oplus V_0$ has the splitting over V_0 that *removes the planes*. An element $g \in G(V)$ is *adapted* to Ξ if the matrix with column blocks $g^{-1}x_V^{-1}(\mathsf{l}_\Xi, \mathsf{k}_\Xi, \mathsf{i}_\Xi x_{V_0})$ has orthonormal columns at every infinite place and integral entries at the finite places.

Input 3. (zero terms and the mixed model; background library, units of the folders *Weil/MixedModel* and *Weil/DualPair*). Let Ξ be a splitting of V over V_0 along H, over a field with Weil data. (a) An element of $G(V)$ adapted to Ξ exists. If g is adapted to Ξ, there are $c \neq 0$ and a function ϵ on $\mathbb{A}^H$ with $\mathrm{ev}_\Xi(\varrho_V(x_V^{-1}\mathsf{l}_\Xi b)\,\omega_V(g)\mathrm{vac}_V) = c\,\epsilon(b)\,\mathrm{vac}_{V_0}$ for all $b \in \mathbb{A}^H$, and there is a bounded linear map $\Lambda\colon \mathcal{F}(V) \to \mathcal{F}(V_0)$ with $\mathrm{ev}_\Xi(\omega_V(g)(v \otimes f)) = \Lambda v \otimes \mathrm{ev}_\Xi(\omega_V(g)(v_{\mathrm{vac}} \otimes f))$ for all v, f, where v_{vac} is the vacuum of $\mathcal{F}(V)$; moreover $v_{\mathrm{vac}} \otimes f = f$ for $f \in \mathcal{W}_f(V)$. (b) $\mathrm{ev}_\Xi(\varrho_V(x_V^{-1}\mathsf{i}_\Xi x_{V_0} y)f) = \varrho_{V_0}(y)\,\mathrm{ev}_\Xi(f)$ and $\mathrm{ev}_\Xi(\varrho_V(x_V^{-1}\mathsf{k}_\Xi b)f) = \mathrm{ev}_\Xi(f)$. (c) Every $y \in \mathbb{A}_f^I$ is $\mathsf{l}_{\Xi,f}a + \mathsf{i}_{\Xi,f}w + \mathsf{k}_{\Xi,f}b$ for unique a, w, b, and $a = \mathsf{k}^*_{\Xi,f}y$, $w = \mathsf{i}^*_{\Xi,f}y$. (d) Suppose that k_Ξ is the inclusion of the basis vectors of positive weight, for weights $\mathrm{wt}\colon I \to \mathbb{Z}$. Let $g \in G(V)$, $g_0 \in G(V_0)$, let C be an $I \times I$ matrix over $\mathbb{A}$ and u an idele of norm one, and suppose, in rational coordinates: g and C have the same columns of positive weight; the entry (b,a) of $C-1$ vanishes whenever $\mathrm{wt}(b) \leq \min(0, \mathrm{wt}(a))$; $\det C = u$; and $g\,\mathsf{i}_\Xi - \mathsf{i}_\Xi g_0$ has non-zero entries only in rows of positive weight. Then $\mathrm{ev}_\Xi(\omega_V(g)f) = v(u)^{-1}\omega_{V_0}(g_0)\,\mathrm{ev}_\Xi(f)$ for all f. (e) For the splitting of $\mathrm{Hom}(V', \mathbb{H}^H \oplus V_0)$ over $\mathrm{Hom}(V', V_0)$ induced by removing the planes, and $h \in G(V')$, $r \in G(V_0)$: $\mathrm{ev}(\omega(h, 1 \oplus r)f) = v(\det h)^{|H|}\omega(h,r)\,\mathrm{ev}(f)$. (f) For a set S of finite places disjoint from its conjugate, functions Ψ of the components at S of a finite vector *name* operators $\Psi \otimes$ on $\mathcal{S}(V)$: the indicator of the vectors integral at S names P^S; the name of its translate by y is $\varrho_{f,V}(y_S) \circ P^S$ up to a root of unity, y_S the part of y at S and its conjugate; and for $g \in G_f(I)$ with component 1 outside S and its conjugate, $\omega_V(1,g) \circ (\Psi \otimes) = C_g \cdot (\Psi(g^{-1}\,\cdot) \otimes)$ with a constant $C_g \in \mathbb{C}$. *References.* The library gives as its sources Mœglin–Vignéras–Waldspurger, LNM 1291 (1987), and S. Kudla, Israel J. Math. 87 (1994); not confirmed by the catalogue: reference to be confirmed.

LEAN `Space.Splitting.exists_isAdapted_and_frame_eq` · `Space.Splitting.IsAdapted.mixedTerm_omega_vacElem` · `Space.Splitting.IsAdapted.exists_real_zeroTerm` · `Space.Splitting.zeroTerm_rho_iAdelic` · `Space.Splitting.exists_eq_finFrameVector` · `Space.Splitting.adjoint_mulVec_finFrameVector` · `zeroTerm_omega_of_inclMatrix` · `Space.Splitting.homTarget_zeroTerm_pairWeil` · `Space.name_boxIndicator` · `Space.name_schrodingerAct_boxIndicator` · `Space.ω_name_on`

Definition 2.5 (gradings, Bessel groups, constant terms; recalled)**.** A *grading* $\mathcal{Y}$ of a hermitian space V (index set I) with *definite part* X and a string of length l, of *norm* $N \in E^\times$, $\bar{N} = N$, consists of a bijection $\mathbf{e}_y\colon X \sqcup \{0, \dots, l-1\} \to I$ with $J_V = \mathrm{Gr}_{\mathbf{e}_y}(N)$ and a constant $\mathfrak{c}(\mathcal{Y}) \in E^\times$, $\overline{\mathfrak{c}(\mathcal{Y})} = \mathfrak{c}(\mathcal{Y})$. Here $\mathrm{Gr}_{\mathbf{e}}(N)$ is the matrix that is the identity on X, pairs the string vectors a and $l-1-a$ to 1 for $a \neq l-1-a$, and has the entry $-N$ at the middle string vector $a = l-1-a$ (which exists when l is odd). The *grade* gr_y is 0 on X and $l-1-2a$ at the string vector a. The homomorphism $\imath_y\colon G(\langle X \rangle) \to G(V)$ lets a unitary matrix act on X and fixes the string vectors (in rational coordinates), and $\imath_y^f$ is its restriction to finite points. The *Bessel group* $B(\mathcal{Y})$ is the group of the $g \in G(V)$ that, in rational coordinates, fix every string vector of grade ≥ 0 up to string vectors of higher grade; it contains $\imath_y(G(\langle X \rangle))$ and the group of the *unipotent* elements, those g for which $g-1$ strictly raises grades. $[B(\mathcal{Y})]$ is the quotient of $B(\mathcal{Y})$ by its rational points, with the measure of the library. If E carries Weil data, a function $P\colon G(V) \to \mathbb{C}$ is *generic* for $\mathcal{Y}$ if it is continuous, left invariant

under the rational points of $B(\mathcal{Y})$, and multiplied by an explicit character under left multiplication by the unipotent elements n for which $n-1$ raises grades by at least 2. For a second grading $\mathcal{Y}'$ of V' with definite part X', the *string map* $T_{\mathcal{Y},\mathcal{Y}'}\colon V \to V'$ (the library's name; it carries the frequencies of the constant terms below) is the rational matrix that kills X and sends the string vector a of $\mathcal{Y}$ to a multiple of the string vector a of $\mathcal{Y}'$, the multiple being one of 1, $\mathfrak{c}(\mathcal{Y})$, $-N\mathfrak{c}(\mathcal{Y})$ according to the grade: for odd l it is 1, $-N\mathfrak{c}(\mathcal{Y})$, $\mathfrak{c}(\mathcal{Y})$ at positive, zero, negative grade, for even l it is $\mathfrak{c}(\mathcal{Y})$, 1 at positive, negative grade. The *shift* $\mathrm{sh}_{\mathcal{Y}}$ is the rational matrix $V \to V$ that kills X and sends the string vector a to a multiple of the string vector $a+1$ (the last one to 0), the multiple being $\mathfrak{c}(\mathcal{Y})$ times $-N$ if a is the middle string vector, times $-N\mathfrak{c}(\mathcal{Y})$ if l is even and a has grade 1, and times 1 otherwise. The *degree* of the position $(b,a) \in I' \times I$ is $\mathrm{gr}_{\mathcal{Y}'}(b) - \mathrm{gr}_{\mathcal{Y}}(a)$. The *constant term at the string map* of a datum $f \in \mathcal{S}(\mathrm{Hom}(V,V'))$, for a character μ as in Definition 2.3, is the function $\mathrm{ct}^{\mu}_{\mathcal{Y},\mathcal{Y}'}(f)$ on $G(V)\times G(V')$ whose value at (g,r) is the mean of

$$\theta\big(\varrho(n)\,\varrho(T_{\mathcal{Y},\mathcal{Y}'})\,\omega^{\mathrm{pr}}_{\mu}(g,r)f\big)$$

over the adelic matrices $n\colon V \to V'$ with entries only in positions of positive degree, modulo the rational ones. Finally, the *Heisenberg bicharacter* of $\mathrm{Hom}(V,V')$ is $\mathrm{bi}(x,y) = \psi_{\mathbb{A}}\big(\frac{1}{2}\mathbf{j}\,\mathrm{tr}(y^{\dagger}x)\big)$ for adelic matrices $x, y\colon V \to V'$, where $\psi_{\mathbb{A}}$ is the standard additive character of $\mathbb{A}$ trivial on E; the product $\varrho(x)\varrho(y)$ is $\varrho(x+y)$ times a value of it.

Input 4. (Bessel periods; background library, unit of the folder *Weil/Periods*). Let $\mathcal{Y}$, $\mathcal{Y}'$ be gradings of V, V' over a field with Weil data. (a) Let H be a finite index set and $\gamma\colon H \hookrightarrow \{b \in I : \mathrm{gr}_{\mathcal{Y}}(b) = 1\}$ an injection. There is $C \in \mathbb{C}$ such that $\int_{[B(\mathcal{Y})]} \phi = C \int_{[G(\langle X\rangle)]} M$ whenever ϕ is a continuous function on $G(V)$, left invariant under the rational points of $B(\mathcal{Y})$, such that for every $a \in G(\langle X\rangle)$ the function $n \mapsto \phi(n\,\imath_{\mathcal{Y}}(a))$ of unipotent n is a continuous function, periodic for the rational matrices, of mean $M(a)$, of the entries of $\mathfrak{c}(\mathcal{Y})(n-1)$ in the rows $\gamma(H)$ and the columns X (in rational coordinates), and M is continuous and left invariant under $\Gamma(\langle X\rangle)$. (b) If P is generic for $\mathcal{Y}$ and n is unipotent with $n-1$ raising grades by at least 2, then $g \mapsto P(ng)\cdot$(constant term of $\omega^{\mathrm{pr}}_{\mu}(ng,r)f$ at a matrix y) equals $\mathrm{bi}(T_{\mathcal{Y},\mathcal{Y}'}n, T_{\mathcal{Y},\mathcal{Y}'})/\,\mathrm{bi}(yn,y)$ times the same expression at g, provided $y(n-1)$ has only entries in positions of positive degree; here $l' = l-1$. (c) $\mathrm{ct}^{\mu}_{\mathcal{Y},\mathcal{Y}'}(f)(\gamma g,\gamma' r) = \mathrm{ct}^{\mu}_{\mathcal{Y},\mathcal{Y}'}(f)(g,r)$ for rational points γ, γ' of the two Bessel groups; and the constant term of $\omega^{\mathrm{pr}}_{\mu}(ng,n'r)f$ at a matrix y is that of $\omega^{\mathrm{pr}}_{\mu}(g,r)f$ at $n'^{-1}yn$, for unipotent n and n' with $n'-1$ raising degrees. (d) If $l' = l-1$, then $T^{\dagger}_{\mathcal{Y},\mathcal{Y}'}\,T_{\mathcal{Y},\mathcal{Y}'} = \mathrm{sh}_{\mathcal{Y}}$; and if moreover $\mathfrak{c}(\mathcal{Y}') = \mathfrak{c}(\mathcal{Y})$, then $T_{\mathcal{Y},\mathcal{Y}'}\,T^{\dagger}_{\mathcal{Y},\mathcal{Y}'} = \mathrm{sh}_{\mathcal{Y}'}$. *References.* The library gives as its source for the unfolding of Bessel periods B. Wang Peng Jun, J. Number Theory 271 (2025), Prop. 3.2 and Thm. 4.5, and the authors record the place as compared; the catalogue of this write-up has not confirmed it: reference to be confirmed.

LEAN `Grading.integral_besselMeasure_eq_of_mean` · `Grading.IsGeneric.unipotent_mul_constTerm` · `Grading.constTermAtT_rat_mul`

Input 5. (algebraic forms and theta lifts; background library, units of the folders *Adelic/CuspForms* and *Weil/Periods*). For a finite index set X, a weight $\varkappa$ and a determinant exponent e, $\mathcal{A}(X;\varkappa,e)$ is the space of algebraic modular forms of weight $(\varkappa,e)$ on the definite unitary group of $\langle X\rangle$: functions on $G_f(X)$ with values in the space of vectors of weight $\varkappa$ for X (the catalogue names B. Gross, Algebraic modular forms, Israel J. Math. 113 (1999), for this notion; place not confirmed). For a linear functional t on that space of vectors, $\mathrm{coef}_t(\varphi)$ is the function $g \mapsto t(\bar{g}_\infty^{-1}\varphi(g_f))$ on $G(\langle X\rangle)$; it is continuous and left invariant under $\Gamma(\langle X\rangle)$. Let E carry Weil data. A family of linear maps $\mathcal{L}_o\colon \mathcal{W}_f(\mathrm{Hom}(\langle X\rangle,\langle X'\rangle)) \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}(X;\varkappa,e),\mathcal{A}(X';\varkappa,e'))$, $o\in U$, is a *family of theta lifts* for characters $(\mu_o)_{o\in U}$ if for every complex polynomial Q and every functional t there is a functional t' such that for all $o\in U$, Φ, φ and $h\in G(\langle X'\rangle)$

$$\int_{[G(\langle X\rangle)]} \theta\big(\omega^{\mathrm{pr}}_{\mu_o}(g,h)(v_Q\otimes\Phi)\big)\,\mathrm{coef}_t(\varphi)(g)\,dg = \mathrm{coef}_{t'}\big(\mathcal{L}_o(\Phi)\varphi\big)(h).$$

This is a definition of the library; the lifts to which the seesaw note applies it are theta lifts in this sense by a result of the library that this note does not use.

LEAN `Space.isAutomorphic_algFormCoeff`

Input 6. (Hermite reduction; background library, unit of the folder *Adelic/Witt*). Let I, H be finite index sets and Y an $I\times H$ matrix over $\mathbb{A}_f$ that has a left inverse. Then there is an invertible $H\times H$ matrix t over $\mathbb{A}_f$ such that Yt has entries in $\hat{\mathcal{O}}$ and a left inverse with entries in $\hat{\mathcal{O}}$. (The library proves this for pairs of a ring and a subring with a valuation property, with t block triangular; the catalogue classes it as elementary.)

LEAN `IsValuationPair.exists_isPrimitiveMatrix_mul`

Input 7. (analysis; background library). (a) The polynomial vectors v_Q are dense in the space $\mathcal{F}(V)$ of real data (folder *Weil/Fock*). (b) If two families of variables together exhaust the variables of a polynomial ring over $\mathbb{C}$, every polynomial is a finite sum of products of a polynomial in the first family with a polynomial in the second (folder *General/LinearAlgebra*). (c) Let $\phi\colon \mathsf{A}\times\mathsf{A}'\to\mathsf{B}$ be a continuous surjective homomorphism of topological abelian groups that maps the product of two cocompact lattices into a cocompact lattice of B. Then the mean over A of the mean over A' of $F\circ\phi$ is the mean of F over B, each mean taken modulo its lattice (folder *Adelic/QuotientMeasure*). (d) If a prime ℓ does not divide the natural number n, then $|n|=1$ in every normed field that is a normed algebra over the ℓ-adic numbers (folder *General/Algebra*).

LEAN `denseRange_polyVector` · `mem_rename_range_mul` · `quotMean_quotMean` · `Nat.Coprime.norm_natCast_eq_one`

Setting 2.6 (three graded spaces)**.** E is a CM field with a CM type and Weil data. V, V', V'' are hermitian spaces over E with finite index sets I, I', I'' and their signs. y, y', y'' are gradings of V, V', V'' of the same norm $N\in E$, with finite definite parts X, X', X_0 and strings of lengths l, l', l''.

Admissible combinations.

Definition 2.7 (integral away from n)**.** For a natural number n, $O^{(n)}$ is the set of the complex numbers w with $|\jmath(w)|_\ell \le 1$ for every prime number ℓ coprime to n and every ring homomorphism $\jmath\colon \mathbb{C} \to \mathbb{C}_\ell$. It is a subring of $\mathbb{C}$, by the ultrametric inequality, and $O^{(n)} \subseteq O_\jmath$ for every such ℓ and $\jmath$.

LEAN `integralAway`

Definition 2.8 (admissible coefficients and combinations)**.** Let $\mathcal{X}$ be a set, $A \subseteq \mathcal{X}$, n and d natural numbers, p a proposition and B a real number. A finitely supported function $r\colon \mathcal{X} \to \mathbb{C}$ is *admissible*, written $r \in \mathrm{Co}(A; n, d; p, B)$, if its support lies in A; $d \cdot r(x) \in O^{(n)}$ for every $x \in \mathcal{X}$; and, if p holds, $\sum_x |\sigma(r(x))| \le B$ for every field automorphism σ of $\mathbb{C}$. For a complex vector space Z, a family $v\colon \mathcal{X} \to Z$ and $z \in Z$ we write $z \in \mathrm{Adm}(v, A; n, d; p, B)$ if $z = \sum_x r(x)v(x)$ for some r with $r \in \mathrm{Co}(A; n, d; p, B)$: z is an *admissible combination* of the $v(x)$, $x \in A$, *away from* n, with *denominator* d and, under p, *size* B. The size is an upper bound, and it is asked only when p holds.

LEAN `IsAdmissibleCoeffs` · `IsAdmissibleComb`

Translations and lattices.

Definition 2.9 (groups of translations)**.** Let E be a CM field, I an index set, n a natural number and $d\colon I \to \mathbb{A}_f$ a family of finite adeles. $\mathrm{Tr}_n[d]$ is the additive group of the $x \in \mathbb{A}_f^I$ such that $d_i x_i$ is integral at every finite place that does not divide n, for every $i \in I$. For a single finite adele d, $\mathrm{Tr}_n^I[d]$ is the group $\mathrm{Tr}_n[d]$ for the constant family $i \mapsto d$ on I. For index sets I_1, I_2, $\beta \in \hat{\mathcal{O}}$ and $j \in I_1 \sqcup I_2$ we put $\beta^{\langle j \rangle} = \beta$ if $j \in I_1$ and $\beta^{\langle j \rangle} = \bar{\beta}$ if $j \in I_2$.

LEAN `weightedTranslations` · `translations` · `selfOrConj`

Definition 2.10 (split and squarefree)**.** Let E be a CM field with a CM type, Σ a finite set of finite places of E, $N \in E$ and π a finite idele. We write $\mathsf{SqF}_\Sigma(N; \pi)$, and say that N is *split and squarefree along* Σ through π, if $N = \pi\bar{\pi}$ in $\mathbb{A}_f$ and, for every finite place w, $|\pi_w|_w = e^{-1}$ if $w \in \Sigma$ and $|\pi_w|_w = 1$ otherwise, and $w \in \Sigma$ implies $\bar{w} \notin \Sigma$.

LEAN `IsSplitSquarefree`

Definition 2.11 (graded lattice basis)**.** Let R be a commutative ring with an involution, X and I index sets, l a natural number, $\mathbf{e}\colon X^+ \sqcup \{0, \dots, l-1\} \to I$ a bijection, a and π units of R and $u \in R$ with $u\bar{u} = 1$. The matrix $\mathsf{L}_\mathbf{e}(a, \pi, u) \in M_I(R)$ is defined by its columns, indexed through $\mathbf{e}$ (write e_i for the basis vectors). If l is odd, with middle string index z: the column of $x \in X$ is e_x, the column of pt is $\bar{\pi}^{-1}e_{\mathrm{pt}} + u(\pi\bar{\pi})^{-1}e_z$, and the column of a string index b is ue_b. If l is even: the column of $x \in X^+$ is e_x, and the column of a string index b is $ua^{-1}e_b$ if $2b < l$ and $u\bar{a}e_b$ otherwise.

Let now E be a CM field with a CM type, V a hermitian space with index set I, and $\mathcal{Y}$ a grading of V with definite part X^+ and a string of length l, of norm N; let a, π be finite ideles and u a finite adele with $u\bar{u} = 1$. We write $\mathsf{Lat}(\mathcal{Y}; a, \pi, u)$ if

$x_{V,f} = \mathsf{L}_{\mathbf{e}_y}(a, \pi, u) \cdot k$ for some $k \in \mathrm{GL}_I(\hat{\mathcal{O}})$: the columns of the finite frame of V and of the graded lattice basis span the same lattice.

LEAN gradedLatticeBasis · Grading.SpansLattice

Zero terms and arrow pairs.

Definition 2.12 (the finite zero term)**.** Let E carry Weil data and let Ξ be a splitting of V over V_0 along H (Definition 2.4; index sets I, I_0; signs as in Definition 2.1). Choose an element of $G(V)$ adapted to Ξ (Input 3(a)), let g_∞ be its real component and $v_\Xi = \omega_V(g_\infty)v_{\mathrm{vac}} \in \mathcal{F}(V)$. The *finite zero term* is the bounded linear map

$$\mathrm{ev}^f_\Xi \colon \mathcal{S}(V) \to \mathcal{S}(V_0), \qquad \mathrm{ev}^f_\Xi(f) = \mathrm{ev}_\Xi(v_\Xi \otimes f).$$

LEAN zeroTermFin

Definition 2.13 (grade-zero parts)**.** Let E carry Weil data, let V, V'' be hermitian spaces with index sets I, I'' and their signs, and $\mathrm{wt} \colon I \to \mathbb{Z}$ a function (*weights*). A *grade-zero part* of V for wt with zero part V'', an element of the set $\mathrm{GZ}(V, V''; \mathrm{wt})$, is a pair $\mathcal{Z} = (\mathrm{ind}_\mathcal{Z}, \mathrm{inc}_\mathcal{Z})$ of an injection $\mathrm{ind}_\mathcal{Z} \colon I'' \hookrightarrow I$ whose image is $\{i : \mathrm{wt}(i) = 0\}$ and a matrix $\mathrm{inc}_\mathcal{Z} \in M_{I \times I''}(E)$ such that $\mathrm{inc}_\mathcal{Z}(a, b) = 0$ whenever $\mathrm{wt}(a) \neq 0$, such that $\mathrm{inc}^*_\mathcal{Z} J_V \mathrm{inc}_\mathcal{Z} = J_{V''}$, and such that $J_V(a, b) \neq 0$ implies $\mathrm{wt}(a) + \mathrm{wt}(b) = 0$. Its *splitting* $\Xi(\mathcal{Z})$ is the splitting of V over V'' along $\{i : \mathrm{wt}(i) > 0\}$ with k the inclusion of the basis vectors of positive weight, $\mathsf{l} = J_V^{-1}\mathsf{k}$ and $\mathsf{i} = \mathrm{inc}_\mathcal{Z}$; the equations of Definition 2.4 hold because J_V and J_V^{-1} pair only opposite weights.

LEAN GradeZeroPart · GradeZeroPart.splitting

Definition 2.14 (the grade-zero part of a space of homomorphisms)**.** Let E carry Weil data, let V, V', V'' be hermitian spaces with index sets I, I', I'', let $\mathcal{Y}'$ be a grading of V' of norm N with definite part X' and a string of length l', let $\mathrm{wt} \colon I \to \mathbb{Z}$ and $\mathcal{Z} \in \mathrm{GZ}(V, V''; \mathrm{wt})$. Suppose that l' is even and that $\mathrm{wt}(i)$ is even for every $i \in I$. Then $\mathrm{Hom}(\mathcal{Y}', \mathcal{Z})^0$ is the grade-zero part of $\mathrm{Hom}(V', V)$ for the weights $(i, j) \mapsto \mathrm{wt}(i) - \mathrm{gr}_{\mathcal{Y}'}(j)$, with zero part $\mathrm{Hom}(\langle X' \rangle, V'')$: its injection is $\mathrm{ind}_\mathcal{Z}$ times the inclusion of X' into I' along $\mathbf{e}_{\mathcal{Y}'}$, and its matrix is the Kronecker product of $\mathrm{inc}_\mathcal{Z}$ with the inclusion matrix of X'. (The two hypotheses make $\mathrm{wt}(i) - \mathrm{gr}_{\mathcal{Y}'}(j) = 0$ possible only for $j \in X'$.)

LEAN GradeZeroPart.hom

Definition 2.15 (the stacked string map)**.** In Setting 2.6, $\mathbf{T}(\mathcal{Y}, \mathcal{Y}', \mathcal{Y}'')$ is the adelic matrix $V' \to V^- \oplus V''$ in rational coordinates with the two row blocks $T^\dagger_{\mathcal{Y},\mathcal{Y}'}$ (rows I) and $T_{\mathcal{Y}',\mathcal{Y}''}$ (rows I''), and $\mathbf{T}_f(\mathcal{Y}, \mathcal{Y}', \mathcal{Y}'')$ is the vector of the finite coordinates of its frame coordinates, a finite vector of $\mathrm{Hom}(V', V^- \oplus V'')$.

LEAN stackedT · stackedTFin

Definition 2.16 (arrow pairs)**.** In Setting 2.6, let X'' and H be finite index sets. An *arrow pair* for $\mathcal{Y} \to \mathcal{Y}' \to \mathcal{Y}''$, an element of the set $\mathrm{Arr}_{X'',H}(\mathcal{Y}, \mathcal{Y}', \mathcal{Y}'')$, consists of the following.

(a) A grade-zero part $\mathcal{Z}_\mathcal{R} \in \mathrm{GZ}(V^- \oplus V'', \mathbb{H}^H \oplus (\langle X \rangle^- \oplus \langle X'' \rangle); \mathrm{gr}_\mathcal{Y} \sqcup \mathrm{gr}_{\mathcal{Y}''})$; its matrix is written $\mathrm{inc}_\mathcal{R}$ and its injection $\mathrm{ind}_\mathcal{R}$.

(b) The conditions: l is odd, $l' = l - 1$, $l'' = l' - 1$ (subtraction of natural numbers, truncated at 0: so $l'' = l - 2$ if $l \geq 3$ and $l' = l'' = 0$ if $l = 1$), $\mathfrak{c}(y') = \mathfrak{c}(y)$, and $|I| + |I''|$ is even.

(c) A homomorphism $\mathrm{emb}_{\mathcal{R}} \colon G(\langle X_0 \rangle) \to G(\langle X'' \rangle)$ with $\det \mathrm{emb}_{\mathcal{R}}(k) = \det k$, such that for all $g \in G(\langle X \rangle)$ and $k \in G(\langle X_0 \rangle)$, in rational coordinates,

$$(\imath_y(g) \oplus \imath_{y''}(k)) \cdot \mathrm{inc}_{\mathcal{R}} = \mathrm{inc}_{\mathcal{R}} \cdot (1 \oplus g^- \oplus \mathrm{emb}_{\mathcal{R}}(k)).$$

(d) An injection $\gamma_{\mathcal{R}} \colon H \hookrightarrow \{b \in I' : \mathrm{gr}_{y'}(b) = 1\}$ such that for every index a of the zero part and every $b \in I'$ the entry of $\mathbf{T}(y, y', y'')$ at $(\mathrm{ind}_{\mathcal{R}}(a), b)$ is $\sum_{k \in H,\ \gamma_{\mathcal{R}}(k) = b} \mathrm{inc}_{\mathcal{R}}(\mathrm{ind}_{\mathcal{R}}(a), k^*) \cdot \mathfrak{c}(y')$, where k^* is the index of the k-th vector of the second copy of H in $\mathbb{H}^H$.

LEAN `ArrowPair`

Definition 2.17 (the splitting and the finite evaluation of an arrow pair)**.** In Setting 2.6, let X'', H be finite index sets and $\mathcal{R} \in \mathrm{Arr}_{X'',H}(y, y', y'')$. (i) $\mathcal{Z}^{\mathrm{Hom}}(\mathcal{R}) = \mathrm{Hom}(y', \mathcal{Z}_{\mathcal{R}})^0$ is the grade-zero part of $\mathrm{Hom}(V', V^- \oplus V'')$ with zero part $\mathrm{Hom}(\langle X' \rangle, \mathbb{H}^H \oplus (\langle X \rangle^- \oplus \langle X'' \rangle))$; Definition 2.14 applies because l' is even and the grades of y and y'' are even, l being odd and l'' odd or zero. (ii) $\Xi^{\mathrm{pr}}(\mathcal{R})$ is the splitting of $\mathrm{Hom}(V', V^- \oplus V'')$ over $\mathrm{Hom}(\langle X' \rangle, \langle X \rangle^- \oplus \langle X'' \rangle)$ obtained by composing the splitting $\Xi(\mathcal{Z}^{\mathrm{Hom}}(\mathcal{R}))$ with the splitting induced by removing the H planes. (iii) The *finite evaluation* of $\mathcal{R}$ is the linear map

$$\mathrm{fev}_{\mathcal{R}} = \mathrm{at}^\flat \circ \mathrm{ev}^f_{\Xi^{\mathrm{pr}}(\mathcal{R})} \circ \varrho_f(\mathbf{T}_f(y, y', y''))$$

from $\mathcal{S}(\mathrm{Hom}(V', V^- \oplus V''))$ to $\mathcal{W}_f(\mathrm{Hom}(\langle X \rangle, \langle X' \rangle)) \otimes \mathcal{W}_f(\mathrm{Hom}(\langle X' \rangle, \langle X'' \rangle))$.

LEAN `ArrowPair.homZeroPart` · `ArrowPair.splitting` · `ArrowPair.finEval`

Definition 2.18 (the middle pair and the last pair)**.** In Setting 2.6, suppose that l is odd, $l' = l - 1$, $l'' = l' - 1$ and $\mathfrak{c}(y') = \mathfrak{c}(y)$, and that $|I| + |I''|$ is even. Let z, z'' be the middle string vectors of y and y''.

(i) If $l \neq 1$, the *middle pair* $\mathcal{R}_{\mathrm{mid}}(y, y', y'')$, an element of $\mathrm{Arr}_{X_0, \{\mathrm{pt}\}}(y, y', y'')$, removes one hyperbolic plane: its zero part sends the two basis vectors of the plane to $(2N)^{-1}(z - z'')$ and $z + z''$, and the basis vectors of X and of X_0 to the corresponding basis vectors of V and V''; its homomorphism emb is the identity of $G(\langle X_0 \rangle)$; its injection γ picks the string vector of y' with index $(l'' - 1)/2$, which has grade 1.

(ii) If $l = 1$, and $\pi_0 \in E$ satisfies $\pi_0 \overline{\pi_0} = N$, the *last pair* $\mathcal{R}_{\mathrm{last}}(y, y', y''; \pi_0)$, an element of $\mathrm{Arr}_{X_0^+, \emptyset}(y, y', y'')$, removes no plane: its zero part sends the basis vectors of X and of X_0 to the corresponding basis vectors and the new point to z/π_0, z the only string vector of y; its homomorphism emb is the block embedding $k \mapsto k \oplus 1$ of $G(\langle X_0 \rangle)$ into $G(\langle X_0^+ \rangle)$.

In both cases the conditions of Definition 2.16 are verified entry by entry from the explicit matrices $\mathrm{Gr}_{\mathbf{e}}(N)$ and the definition of the string maps.

LEAN `middlePair` · `lastPair`

Admissible evaluations.

Setting 2.19 (data at a split prime)**.** E is a CM field with a CM type and Weil data, and X, X'' are finite index sets. N is a prime number and Σ a finite set of finite places of E that is a *split level* for N in the sense of the note on the Gauss-sum operator: $\bar{w} \notin \Sigma$ for $w \in \Sigma$, every $w \in \Sigma$ has norm N and N is a uniformiser at w, and a finite place divides N exactly if it or its conjugate lies in Σ. π is a finite idele with $\mathsf{SqF}_{\Sigma}(N;\pi)$. δ is a finite adele with $\delta\bar{\delta} = 1$, and $\beta \in \hat{\mathcal{O}}$, such that $\delta_w = 1$ and $|\beta_w|_w = 1$ at every finite place w dividing N. Finally $\kappa\colon X^+ \sqcup X''^+ \to \mathbb{A}_f$ is a family with $\kappa_b \in \{1, \delta\}$, $\bar{\beta}\kappa_b \in \hat{\mathcal{O}}$ and $\beta^{\langle b\rangle}\kappa_b \in \hat{\mathcal{O}}$ for every b.

Definition 2.20 (neighbour vectors)**.** Let E carry Weil data, let X, H be finite index sets, π a finite idele, δ and c finite adeles and x a vector over $\mathbb{A}_f$ indexed by $X^+ \sqcup H^+$; write pt_1, pt_2 for the points of X^+ and H^+. We write $\mathsf{Nb}_{\pi,\delta}(c;x)$, and call (c,x) a *neighbour vector* for (π,δ), if: $\delta c = c$; $x_i \in \hat{\mathcal{O}}$ for every index i other than pt_1, pt_2; $\bar{\pi}x_i \in \hat{\mathcal{O}}$ for every i; $c - \pi^{-1}x_{\mathrm{pt}_1} \in \hat{\mathcal{O}}$; and $\overline{\epsilon_}c - \pi^{-1}x_{\mathrm{pt}_2} \in \hat{\mathcal{O}}$.

LEAN `IsNeighbourVector`

Definition 2.21 (admissible evaluations)**.** Let E carry Weil data; let V, V', V'' be hermitian spaces with index sets I, I', I'' and their signs, and X, X', X'' finite index sets. Let N be a prime number, Σ a finite set of finite places, $\beta \in \hat{\mathcal{O}}$, $d\colon I \sqcup I'' \to \mathbb{A}_f$ a family (by default $d = \beta^{\langle\cdot\rangle}$), and L a map from $\mathcal{S}(\mathrm{Hom}(V', V^- \oplus V''))$ to $\mathcal{W}_f(\mathrm{Hom}(\langle X^+\rangle, \langle X'\rangle)) \otimes \mathcal{W}_f(\mathrm{Hom}(\langle X'\rangle, \langle X''^+\rangle))$. We write $\mathsf{AdmEv}_{N,\Sigma,\beta}(L;d)$, and call L an *admissible evaluation* for (N,Σ,β) with weights d, if there is $C \in \mathbb{C}$ such that for every natural number n coprime to N, every proposition p and every $y \in \mathrm{Tr}_n[(i,i') \mapsto d_i]$ (a finite vector indexed by $(I \sqcup I'') \times I'$)

$$L\big(\varrho_f(y)\mathrm{vac}\big) \in \mathrm{Adm}(v_C, A_n; n, D; p, 1),$$

where v_C is the family $t = (t_1, t_2) \mapsto C \cdot \mathsf{T}_{N,\Sigma}(X, X', X'')(\varrho_f(t_1)\mathrm{vac} \otimes \varrho_f(t_2)\mathrm{vac})$, the set A_n is the product of $\mathrm{Tr}_n^{X'\times X^+}[\bar{\beta}]$ and $\mathrm{Tr}_n^{X''^+\times X'}[\bar{\beta}]$, D is the number of maps from $(X^+ \sqcup X''^+) \times X'$ to $\hat{\mathcal{O}}/\beta\hat{\mathcal{O}}$ (zero if infinite), and $\mathsf{T}_{N,\Sigma}(X, X', X'')$ is the translate sum of the note on the Gauss-sum operator: the operator $\sum_y \varrho_f(\mathrm{c}(y)) \otimes \varrho_f(\mathrm{r}(y))$ on $\mathcal{W}_f(\mathrm{Hom}(\langle X^+\rangle, \langle X'\rangle)) \otimes \mathcal{W}_f(\mathrm{Hom}(\langle X'\rangle, \langle X''^+\rangle))$, the sum over $y \in ((\mathbb{Z}/N)^{\Sigma})^{X'}$, where the matrix $\mathrm{c}(y)$ has last column $-\tilde{y}/N$ and $\mathrm{r}(y)$ has last row $-\overline{\tilde{y}/N}$, both zero elsewhere, and $\tilde{y}/N$ is the finite adele with component $\tilde{y}_w/N$ at $w \in \Sigma$, $\tilde{y}_w$ the least non-negative representative, and 0 at the other places.

LEAN `IsAdmissibleEval`

3 Admissible combinations

Lemma 3.1 (rules)**.** *Let $\mathcal{X}$, $\mathcal{X}'$ be sets, Z, Z', W complex vector spaces, $v\colon \mathcal{X} \to Z$ a family, $A \subseteq \mathcal{X}$, $A' \subseteq \mathcal{X}'$, n, d, d' natural numbers, p a proposition and B, B' real numbers.*

(a) Let $F \subseteq \mathcal{X}$ be a finite subset and $c\colon \mathcal{X} \to \mathbb{C}$ a function such that $d \cdot c(x) \in O^{(n)}$ for every $x \in F$, and such that p implies $\sum_{x\in F}|\sigma(c(x))| \le B$ for every automorphism σ. Then $\sum_{x\in F} c(x)v(x) \in \mathrm{Adm}(v, F; n, d; p, B)$.

(b) If $x \in A$, then $v(x) \in \mathrm{Adm}(v, A; n, d; p, 1)$.

In (c) to (g) let $z \in Z$ with $z \in \mathrm{Adm}(v, A; n, d; p, B)$.

(c) For a linear map $L\colon Z \to Z'$, $L(z) \in \mathrm{Adm}(L \circ v, A; n, d; p, B)$.

(d) If $c \in O^{(n)}$, and p implies $|\sigma(c)| \le 1$ for every automorphism σ, then $c \cdot z \in \mathrm{Adm}(v, A; n, d; p, B)$.

(e) If $\varsigma \in \mathbb{C}$ is zero or a root of unity, then $\varsigma \cdot z \in \mathrm{Adm}(v, A; n, d; p, B)$.

(f) Let $v'\colon \mathcal{X}' \to Z$. If for every $x \in A$ there are a root of unity ς and $y \in A'$ with $v(x) = \varsigma \cdot v'(y)$, then $z \in \mathrm{Adm}(v', A'; n, d; p, B)$.

(g) Let $v'\colon \mathcal{X}' \to Z'$ be a family, let $z' \in Z'$ satisfy $z' \in \mathrm{Adm}(v', A'; n, d'; p, B')$, and let $b\colon Z \times Z' \to W$ be bilinear. Then, for the family $v_b\colon (x, x') \mapsto b(v(x), v'(x'))$ on $\mathcal{X} \times \mathcal{X}'$, $b(z, z') \in \mathrm{Adm}(v_b, A \times A'; n, d \cdot d'; p, B \cdot |B'|)$.

(h) Let $C, a \in \mathbb{C}$, suppose $d \neq 0$, and let $z \in Z$ with $z \in \mathrm{Adm}((x \mapsto C \cdot v(x)), A; n, d; p, B)$. Then $\mathrm{nu}(aC/d) \cdot a \cdot z \in \mathrm{Adm}(v, A; n, 1; p, d \cdot B)$, where $\mathrm{nu}(w) = w^{-1}$ for $w \neq 0$ and $\mathrm{nu}(0) = 1$.

Proof. (a) Take the coefficients c on F and 0 elsewhere. (b) Take the coefficient 1 at x; the size is 1, and $d \cdot 1 \in O^{(n)}$. (c) Apply L to $z = \sum_x r(x)v(x)$. (d), (e), (f) follow from Theorem A(i), proved below without them. For (d): by (c), $c \cdot z \in \mathrm{Adm}((x \mapsto c \cdot v(x)), A; n, d; p, B)$, and each $c \cdot v(x)$ is the combination of the v's with the single coefficient c, which is admissible with denominator 1 and size 1; substitute. For (e): 0 and the roots of unity lie in $O^{(n)}$, since a ring homomorphism to a field sends a root of unity to a root of unity, of absolute value 1; the same holds for the automorphisms σ; apply (d). For (f): by (b) and (e), $v(x) \in \mathrm{Adm}(v', A'; n, 1; p, 1)$ for $x \in A$; substitute. (g) By (c) for the linear map $b(\,\cdot\,, z')$, $b(z, z') \in \mathrm{Adm}((x \mapsto b(v(x), z')), A; n, d; p, B)$. For $x \in A$, by (c) for $b(v(x), \cdot\,)$, the vector $b(v(x), z')$ is an admissible combination of the $b(v(x), v'(x'))$, $x' \in A'$, with denominator d' and size B', hence, each of these being a single vector of the family indexed by $A \times A'$, an admissible combination of that family; substitute. (h) Let $z = \sum_x r(x)\, C v(x)$ with r admissible. The coefficients $d \cdot r$ are admissible with denominator 1 and, under p, size $d \cdot B$; let $z_1 = \sum_x d\, r(x)v(x)$. If $aC \neq 0$ then $\mathrm{nu}(aC/d)\, a\, z = (d/(aC)) \cdot aC \sum_x r(x)v(x) = z_1$; if $aC = 0$ then $a\, z = 0$. In both cases $\mathrm{nu}(aC/d)\, a\, z = \varsigma z_1$ with $\varsigma \in \{0, 1\}$, and (e) applies. □

LEAN `IsAdmissibleComb.of_sum` · `IsAdmissibleComb.single` · `IsAdmissibleComb.map` · `IsAdmissibleComb.smul` · `IsAdmissibleComb.smul_of_eq_zero_or_isOfFinOrder` · `IsAdmissibleComb.of_eq_isOfFinOrder_smul` · `IsAdmissibleComb.map_bilinear` · `IsAdmissibleComb.normUnit_smul`

Proof of Theorem A. (i) Write $z = \sum_x c(x)v(x)$ with c admissible for $(A; n, d; p, B)$; for $x \in A$ write $v(x) = \sum_y r_x(y)v'(y)$ with r_x admissible for $(A'; n, d'; p, B')$; and write $d'' = d\, d'\, k$. Put $\rho = \sum_{x \in \mathrm{supp}\, c} c(x)\, r_x$, a finitely supported function on $\mathcal{X}'$. Then $z = \sum_y \rho(y)v'(y)$. The support of ρ lies in A', because the support of c lies in A and that of every r_x, $x \in A$, in A'. Next $d''\rho(y) = k \sum_x (d\, c(x))(d'\, r_x(y))$ is a sum of products of elements of the ring $O^{(n)}$. Finally suppose p and let σ be an automorphism of $\mathbb{C}$. Then $B \ge 0$, and

$$\sum_y |\sigma(\rho(y))| \le \sum_x |\sigma(c(x))| \sum_y |\sigma(r_x(y))| \le \sum_x |\sigma(c(x))| \cdot |B'| \le B \cdot |B'| \le C.$$

(ii) Write $z = \sum_x r(x)v(x)$ with r admissible. Then $d \cdot z = \sum_x (d\, r(x))\, v(x)$; each $d\, r(x)$ lies in $O^{(n)} \subseteq O_J$, as ℓ is coprime to n, and each $v(x)$ with $r(x) \neq 0$ has $x \in A$, so lies in M.

(iii) Induction on m. For $m = 0$ the pure tensor is $1 \in \mathbb{C}$, the single vector of the family at the tuple $y = 0$, whose entries lie in the subgroups A_j; use Lemma 3.1(b). For the step from m to $m+1$, apply Lemma 3.1(g) to the bilinear map $(f, f') \mapsto f \otimes f'$, to f_m (denominator d, size B) and to the pure tensor of $f_{m-1}, \dots, f_0$ (denominator d^m, size $|B|^m$, by induction): the pure tensor of $f_m, \dots, f_0$ is an admissible combination, with denominator d^{m+1} and size $|B|^{m+1}$, of the tensors $v_m(x) \otimes$ (pure tensor at y), for $x \in A_{2m} \times A_{2m+1}$ and y a tuple with entries in the A_j. Such a tensor is the vector of the family for $m+1$ at the tuple that agrees with y except at the places $2m$ and $2m+1$, where it has the entries of x; that tuple has its entries in the A_j. Lemma 3.1(f), with $\varsigma = 1$, re-indexes the combination. □

LEAN `IsAdmissibleComb.bind` · `IsAdmissibleComb.smul_mem` · `IsAdmissibleComb.pure`

Lemma 3.2 (integrality away from n)**.** *(a) Let n, c, d, k be natural numbers such that c divides $d \cdot n^k$. Then $d/c \in O^{(n)}$.*

(b) Let E be a CM field with a CM type, I a finite index set, and n, m natural numbers that are coprime. If $g \in G_f(I)$ has $g_w \in \mathrm{GL}_I(O_w)$ at every finite place w that does not divide n, then $g_w \in \mathrm{GL}_I(O_w)$ at every finite place w that divides m; here O_w is the ring of integers at w.

(c) Let E carry Weil data, let ℓ be a prime number, $\jmath\colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism, I a finite index set and $g \in G_f(I)$ integral above ℓ*: $g_w \in \mathrm{GL}_I(O_w)$ at every finite place w dividing ℓ. Then $|\jmath(\Delta_f(g))|_\ell = 1$.*

(d) Let E be a CM field with a CM type, ℓ, $\jmath$, I as in (c), $g \in G_f(I)$ integral above ℓ, and μ a character of $\mathrm{U}(1)(\mathbb{A})$, trivial on $\mathrm{U}(1)(E)$, that has a weight and a tame part in the sense of the note on the characters of the data. For any signs on I, with $\det(1, g)$ the determinant of the element $(1, g)$ of the adelic unitary group of these signs: $|\jmath(\mu(\det(1, g)))|_\ell = 1$.

Proof. (a) Let ℓ be a prime coprime to n and $\jmath\colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism. Write $d\,n^k = c\,m$ with m a natural number. Since ℓ does not divide n, $|\jmath(n)|_\ell = 1$, so $|\jmath(d)|_\ell = |\jmath(c)|_\ell\,|\jmath(m)|_\ell$ and $|\jmath(d/c)|_\ell = |\jmath(m)|_\ell \le 1$, the image of a natural number being integral. (For $c = 0$ the quotient is 0 by convention, and the claim is trivial.) (b) A finite place that divides m does not divide n: its prime ideal would contain $1 \in m\mathbb{Z} + n\mathbb{Z}$. (c), (d) In the formal text each is proved by one call of the simplifier, which we did not expand. It unfolds $\Delta_f(g)$ into the value of the finite part of the splitting character, and of an absolute value, at the determinant of a matrix λ with $\lambda\hat{\mathcal{O}}^I = \hat{\mathcal{O}}^I \cap g^{-1}\hat{\mathcal{O}}^I$, which is a unit at the places above ℓ when g is integral there; and it unfolds $\mu(\det(1, g))$ into a product of values of the tame part and of the weight at the local determinants, which are units above ℓ. The lemmas of the library that say that such values are ℓ-adic units under $\jmath$ are quoted, not read. □

LEAN `div_mem_integralAway` · `localLevel_le_integralAbove_of_coprime` · `Parameters.norm_finMultiplier_eq_one` · `AutChar.HasWeight.norm_padic_det`

Lemma 3.3 (groups of translations)**.** *Let E be a CM field and n a natural number.*

(a) Let I, I_0 be finite index sets, M an $I \times I_0$ matrix over $\mathbb{A}_f$ with all entries in $\hat{\mathcal{O}}$, $d\colon I \to \mathbb{A}_f$ and $d'\colon I_0 \to \mathbb{A}_f$ families, and $y \in \mathrm{Tr}_n[d]$. Suppose that for all $i \in I$, $j \in I_0$ and every finite place w with $|d_{i,w}|_w \neq |d'_{j,w}|_w$ the component of M_{ij} at $\bar{w}$ is 0. Then $M^* y \in \mathrm{Tr}_n[d']$.

(b) Let I, X be index sets, $c \in \hat{\mathcal{O}}$, $d\colon I \to \mathbb{A}_f$ a family, $x \in \mathrm{Tr}_n[d]$ and $f\colon X \to I$ a map. Then the vector $(c \cdot \overline{x_{f(i)}})_{i\in X}$ lies in $\mathrm{Tr}_n[(\overline{d_{f(i)}})_{i\in X}]$.

Proof. (a) $(M^*y)_j = \sum_i \overline{M_{ij}}\, y_i$, and the conditions define a group, so fix i, j and a finite place w not dividing n. If $|d_{i,w}|_w = |d'_{j,w}|_w$, then $|d'_j \overline{M_{ij}} y_i|_w = |\overline{M_{ij}}|_w\, |d_i y_i|_w \le 1$. Otherwise M_{ij} vanishes at $\bar w$, so $\overline{M_{ij}}$ vanishes at w. (b) $|\overline{d_{f(i)}}\, c\, \overline{x_{f(i)}}|_w = |c|_w\, |d_{f(i)} x_{f(i)}|_{\bar w} \le 1$ for w not dividing n, since then $\bar w$ does not divide n either. □

LEAN `mulVec_mem_weightedTranslations` · `star_mem_weightedTranslations`

Proposition 3.4 (a moved translate of the vacuum)**.** *Let E carry Weil data and let V be a hermitian space with index set I and its signs. Let n, d be natural numbers, $g \in G_f(I)$, $x \in \mathbb{A}_f^I$, $A \subseteq \mathbb{A}_f^I$ a set, p a proposition and $C, c \in \mathbb{C}$. Suppose that $c \cdot C = \Delta_f(g)$; that $g(x+y) \in A$ for every $y \in \hat{\mathcal{O}}^I$; that $d \cdot c/|\Delta_f(g)|^2 \in O^{(n)}$; and that p implies $|\sigma(c)| \le 1$ for every automorphism σ. Then*

$$\omega_V(1,g)\, \varrho_{f,V}(x)\mathrm{vac}_V \in \mathrm{Adm}((z \mapsto C \cdot \varrho_{f,V}(z)\mathrm{vac}_V), A; n, d; p, 1).$$

Proof. By Input 1(b), (c) and (a),

$$\omega_V(1,g)\varrho_{f,V}(x)\mathrm{vac}_V = \varrho_{f,V}(gx)\, \omega_V(1,g)\mathrm{vac}_V = \frac{c\,C}{|\Delta_f(g)|^2} \sum_y \varsigma_y\, \varrho_{f,V}(g(x+y))\mathrm{vac}_V,$$

the sum over representatives y of the $|\Delta_f(g)|^2$ classes of $\hat{\mathcal{O}}^I$ modulo $\hat{\mathcal{O}}^I \cap g^{-1}\hat{\mathcal{O}}^I$, with roots of unity ς_y. This is a sum, over a finite set, of the vectors $C\varsigma_y \varrho_{f,V}(g(x+y))\mathrm{vac}_V$ with the constant coefficient $c/|\Delta_f(g)|^2$. Its d-fold lies in $O^{(n)}$ by hypothesis; under p the size is $|\Delta_f(g)|^2 \cdot |\sigma(c)|/|\Delta_f(g)|^2 \le 1$, the positive integer $|\Delta_f(g)|^2$ being fixed by σ. So Lemma 3.1(a) applies, and Lemma 3.1(f) replaces each vector by $C\varrho_{f,V}(z)\mathrm{vac}_V$ with $z = g(x+y) \in A$. □

LEAN `isAdmissibleComb_weil_vacElem`

4 Lattices of graded spaces

Proposition 4.1 (the graded lattice basis is unimodular)**.** *Let X, I be finite index sets, l and N natural numbers, R a commutative ring with an involution, $\mathbf{e}\colon X^+ \sqcup \{0,\dots,l-1\} \to I$ a bijection, S a commutative ring, R_0 a subring of R, $f\colon S \to R$ a ring homomorphism, a and π units of R and $u \in R$ with $u\bar u = 1$. Suppose $f(N) = \pi\bar\pi$. Then the matrix $\mathsf{L}_{\mathbf{e}}(a,\pi,u)^* \cdot f(\mathrm{Gr}_{\mathbf{e}}(N)) \cdot \mathsf{L}_{\mathbf{e}}(a,\pi,u)$ is hermitian and lies in $\mathrm{GL}_I(R_0)$: its entries and those of its inverse lie in R_0.*

Proof. Let g_i be the columns of the graded lattice basis and $\langle\,,\rangle$ the form of $f(\mathrm{Gr}_{\mathbf{e}}(N))$. For l even, $g_b = ua^{-1}e_b$ for $2b < l$ and $g_b = u\bar a e_b$ otherwise, so $\langle g_b, g_{l-1-b}\rangle = \overline{ua^{-1}}\, u\bar a = 1$ for $2b < l$, and the Gram matrix is $\mathrm{Gr}_{\mathbf{e}}(N)$ itself (there is no middle vector), a permutation matrix equal to its inverse. For l odd with middle index z: $\langle g_b, g_{l-1-b}\rangle = u\bar u = 1$ for $b \ne z$; $\langle g_z, g_z\rangle = -u\bar u f(N) = -f(N)$; $\langle g_{\mathrm{pt}}, g_z\rangle = \overline{u(\pi\bar\pi)^{-1}} \cdot (-f(N)) \cdot u = -1$; and $\langle g_{\mathrm{pt}}, g_{\mathrm{pt}}\rangle = (\pi\bar\pi)^{-1} - u\bar u f(N)(\pi\bar\pi)^{-2} = 0$. So the Gram matrix is the image of the integer matrix that agrees with $\mathrm{Gr}_{\mathbf{e}}(N)$ except for the entries 0 at $(\mathrm{pt},\mathrm{pt})$ and -1 at (pt, z) and (z, pt). On the span of e_{pt}, e_z it is $\left(\begin{smallmatrix} 0 & -1 \\ -1 & -N \end{smallmatrix}\right)$, with inverse $\left(\begin{smallmatrix} N & -1 \\ -1 & 0 \end{smallmatrix}\right)$; elsewhere it is a permutation matrix. Both matrices are hermitian with entries in the image of $\mathbb{Z}$, hence in R_0. □

LEAN `isUnimodular_gradedLatticeBasis_gram`

Proposition 4.2 (a real frame)**.** *Let X, I be finite index sets, l and N natural numbers, R a commutative ring, $\mathbf{e}\colon X^+ \sqcup \{0,\dots,l-1\} \to I$ a bijection and $\phi\colon R \to \mathbb{C}$ a ring homomorphism. Suppose $N > 0$, and let $\varepsilon\colon I \to \{\pm 1\}$ be signs with exactly $\lfloor (l+1)/2 \rfloor$ minus signs. Then there is a complex $I \times I$ matrix y with $y^* \phi(\mathrm{Gr}_{\mathbf{e}}(N)) y$ the diagonal matrix of ε, and with $y^* y = 1$ if $N = 1$.*

Proof. Let $\eta_b = -1$ for $b < \lfloor (l+1)/2 \rfloor$ and $\eta_b = 1$ for the other string indices. The signs equal to 1 on X^+ and to η on the string have $\lfloor (l+1)/2 \rfloor$ minus signs, like ε, so after a bijection of the index sets we may assume that they are ε. Let y be the identity on X^+ and have, at a string index b with $b' = l - 1 - b \neq b$, the column $y_b = (\eta_b e_b + e_{b'})/\sqrt{2}$, and at the middle index (if l is odd) the column $e_b/\sqrt{N}$. For $b \neq b'$ the form pairs e_b with $e_{b'}$ to 1, so $\langle y_b, y_b \rangle = \eta_b$ and $\langle y_b, y_{b'} \rangle = (\eta_b \eta_{b'} + 1)/2 = 0$, because exactly one of b, b' is smaller than $\lfloor (l+1)/2 \rfloor$. At the middle index $\langle y_b, y_b \rangle = -N/N = -1 = \eta_b$. If $N = 1$ the columns are orthonormal for the standard hermitian form: $|\eta_b|^2/2 + 1/2 = 1$, $(\eta_b + \eta_{b'})/2 = 0$, and $1/N = 1$. □

LEAN `exists_frame_gradedGram`

Lemma 4.3 (vectors in the lattice)**.** *Let E be a CM field with a CM type, V a hermitian space with index set I and its signs, and $\mathcal{Y}$ a grading of V of norm N with definite part X^+ and a string of odd length l. Let a, π be finite ideles and u a finite adele with $u\bar{u} = 1$ such that $\mathsf{Lat}(\mathcal{Y}; a, \pi, u)$. Let $c \in \hat{\mathcal{O}}$, and let $x\colon X^+ \to \mathbb{A}_f$ be a vector with $x_i \in \hat{\mathcal{O}}$ for $i \in X$ and $\bar{\pi} x_{\mathrm{pt}} \in \hat{\mathcal{O}}$. Let $v \in \mathbb{A}_f^I$ be the vector with the coordinates x_i at the indices of X^+, $u(\pi^{-1} x_{\mathrm{pt}} + c)$ at the middle string index and 0 at the other string indices. Then every coordinate $(x_{V,f}^{-1} v)_i$, $i \in I$, lies in $\hat{\mathcal{O}}$.*

Proof. Write $x_{V,f} = \mathsf{L}_{\mathbf{e}_y}(a, \pi, u) \cdot k$ with $k \in \mathrm{GL}_I(\hat{\mathcal{O}})$. Let w be the vector with coordinates x_i at $i \in X$, $\bar{\pi} x_{\mathrm{pt}}$ at pt, c at the middle index z and 0 elsewhere; it is integral. By Definition 2.11, for odd l,

$$\mathsf{L}_{\mathbf{e}_y}(a, \pi, u)\, w = \sum_{i \in X} x_i e_i + \bar{\pi} x_{\mathrm{pt}} (\bar{\pi}^{-1} e_{\mathrm{pt}} + u(\pi\bar{\pi})^{-1} e_z) + c\, u\, e_z = v.$$

Hence $x_{V,f}^{-1} v = k^{-1} w$ is integral. □

LEAN `inv_xFin_mulVec_mem_of_spansLattice`

5 The finite zero term

Proposition 5.1. *Let E carry Weil data and let Ξ be a splitting of V over V_0 along H, with index sets I, I_0 and their signs.*

(i) ev_Ξ^f *maps $\mathcal{W}_f(V)$ into $\mathcal{W}_f(V_0)$.*

(ii) For every adelic vector $y \in \mathbb{A}^I$ and every $v \in \mathcal{F}(V)$ there is $v_0 \in \mathcal{F}(V_0)$ such that, for all $f \in \mathcal{S}(V)$,

$$\mathrm{ev}_\Xi(\varrho_V(y)(v \otimes f)) = v_0 \otimes \mathrm{ev}_\Xi^f(\varrho_{f,V}(y_f) f).$$

(iii) Let $b \in \mathbb{A}_f^H$, $w \in \mathbb{A}_f^{I_0}$ and $z = \mathsf{i}_{\Xi,f} w + \mathsf{k}_{\Xi,f} b$. Then, for every $f \in \mathcal{S}(V)$,

$$\mathrm{ev}_\Xi^f(\varrho_{f,V}(z) f) = \varrho_{f,V_0}(w)\ \mathrm{ev}_\Xi^f(f).$$

Proof. Let $g = (g_\infty, g_f)$ be the adapted element chosen in Definition 2.12, so that, by Input 1(b), $v_\Xi \otimes f = \omega_V(g)(v_{\mathrm{vac}} \otimes \omega_V(1, g_f^{-1}) f)$.

(iii) $\varrho_{f,V}(z)$ is a root of unity times the Heisenberg operator at the adelic vector with real part 0 and finite part αz, which in frame coordinates is the sum of an i-part and a k-part. Finite translations commute with $v_\Xi \otimes$, and Input 3(b) moves them through the zero term: the k-part disappears and the i-part becomes the translation by αw. Since k is isotropic and orthogonal to i, $z^*z = w^*w$, so the root of unity in $\varrho_{f,V}(z)$ is that of $\varrho_{f,V_0}(w)$ and the two translations commute without a factor.

(i) Let $f \in \mathcal{W}_f(V)$. Then $f_1 = \omega_V(1, g_f^{-1})f \in \mathcal{W}_f(V)$ and $v_{\mathrm{vac}} \otimes f_1 = f_1$ (Inputs 1(b), 3(a)), so $\mathrm{ev}^f_\Xi(f) = \mathrm{ev}_\Xi(\omega_V(g)f_1)$, and it suffices to see that $\mathrm{ev}_\Xi \circ \omega_V(g)$ maps every $\varrho_{f,V}(x)\mathrm{vac}_V$ into $\mathcal{W}_f(V_0)$. By Input 1(b), $\omega_V(g)\varrho_{f,V}(x)\mathrm{vac}_V = \varrho_{f,V}(g_f x)\omega_V(g)\mathrm{vac}_V$. Decompose $g_f x$ along the finite frame as in Input 3(c), into an l-part, an i-part with coordinates w and a k-part. By Input 3(b) the zero term turns the translation by the i-part into $\varrho_{f,V_0}(w)$ and forgets the k-part, up to roots of unity, and by Input 3(a) what remains, the zero term of an l-translate of $\omega_V(g)\mathrm{vac}_V$, is a multiple of vac_{V_0}. So the result is a multiple of $\varrho_{f,V_0}(w)\mathrm{vac}_{V_0}$.

(ii) By Input 1(b), $\varrho_V(y)(v \otimes f) = v_1 \otimes f_2$ with $f_2 = \varrho_{f,V}(y_f)f$ and some $v_1 \in \mathcal{F}(V)$. Again by Input 1(b), $v_1 \otimes f_2 = \omega_V(g)((\omega_V(g_\infty)^{-1}v_1) \otimes \omega_V(1, g_f^{-1})f_2)$. Input 3(a) gives

$$\mathrm{ev}_\Xi(v_1 \otimes f_2) = \Lambda(\omega_V(g_\infty)^{-1}v_1) \otimes \mathrm{ev}_\Xi(\omega_V(g)(v_{\mathrm{vac}} \otimes \omega_V(1, g_f^{-1})f_2)) = v_0 \otimes \mathrm{ev}^f_\Xi(f_2),$$

with $v_0 = \Lambda(\omega_V(g_\infty)^{-1}v_1)$. □

LEAN zeroTermFin_spec · zeroTermFin_finHeisOp_finFrameVector

Proposition 5.2 (the frequency of a non-zero finite zero term)**.** *Let E carry Weil data and let Ξ be a splitting of V over V_0 along H, with index sets I, I_0 and their signs. Let M, M' be $I \times I$ matrices over $\mathbb{A}_f$ with $M \cdot M'^* = 1$, let $f \in \mathcal{S}(V)$ with $\varrho_{f,V}(Mx)f = f$ for every $x \in \hat{\mathcal{O}}^I$, and let $y \in \mathbb{A}_f^I$ with $\mathrm{ev}^f_\Xi(\varrho_{f,V}(y)f) \neq 0$. Then there is $x_0 \in \hat{\mathcal{O}}^I$ with $\mathsf{k}^*_{\Xi,f}(y - M'x_0) = 0$.*

Proof. The $I \times H$ matrix $M'^*\mathsf{k}_{\Xi,f}$ has a left inverse: if c is the matrix that reads off the k-coordinate (Input 3(c)), then $\mathsf{c}MM'^*\mathsf{k}_{\Xi,f} = \mathsf{c}\mathsf{k}_{\Xi,f} = 1$. By Hermite reduction over $(\mathbb{A}_f, \hat{\mathcal{O}})$ (Input 6) there is an invertible $H \times H$ matrix t over $\mathbb{A}_f$ such that $M'^*\mathsf{k}_{\Xi,f}t$ has entries in $\hat{\mathcal{O}}$ and a left inverse L with entries in $\hat{\mathcal{O}}$. Put $u = t^*\mathsf{k}^*_{\Xi,f}y$ and $x_0 = L^*u$.

We claim $u \in \hat{\mathcal{O}}^H$; by Input 1(e) it suffices that $\psi_0(x^*u) = 1$ for $x \in \hat{\mathcal{O}}^H$. For such x, the vector $M'^*\mathsf{k}_{\Xi,f}tx$ is integral, so, by the hypothesis on f and $MM'^* = 1$, $\varrho_{f,V}(\mathsf{k}_{\Xi,f}tx)f = f$. By Input 1(e), $\varrho_{f,V}(y)f = \varrho_{f,V}(y)\varrho_{f,V}(\mathsf{k}_{\Xi,f}tx)f = \psi_0(x^*u) \cdot \varrho_{f,V}(\mathsf{k}_{\Xi,f}tx)\varrho_{f,V}(y)f$. Apply ev^f_Ξ; by Proposition 5.1(iii), with $w = 0$, the translation along k disappears: $\mathrm{ev}^f_\Xi(\varrho_{f,V}(y)f) = \psi_0(x^*u) \cdot \mathrm{ev}^f_\Xi(\varrho_{f,V}(y)f)$. As this vector is not zero, $\psi_0(x^*u) = 1$.

So $x_0 \in \hat{\mathcal{O}}^I$, and $t^*\mathsf{k}^*_{\Xi,f}(y - M'x_0) = u - (LM'^*\mathsf{k}_{\Xi,f}t)^*u = 0$; as t^* is invertible, the claim follows. □

LEAN zeroTermFin_support

Proposition 5.3 (the finite zero term of a translated vacuum)**.** *Let E carry Weil data and let Ξ be a splitting of V over V_0 along H, with index sets I, I_0 and their signs. Let $g \in G_f(I_0)$, let c be an $H \times I_0$ matrix over $\mathbb{A}_f$, and suppose that the matrix*

$B = \mathsf{i}_{\Xi,f}\, g + \mathsf{k}_{\Xi,f}\, c$ *has all its entries in* $\hat{\mathcal{O}}$. *Then there is* $C \in \mathbb{C}$ *such that for every* $y \in \mathbb{A}_f^I$ *there is* $\varsigma \in \mathbb{C}$, *zero or a root of unity, with*

$$\mathrm{ev}_\Xi^f(\varrho_{f,V}(y)\mathrm{vac}_V) = \varsigma \cdot C \cdot \omega_{V_0}(1,g)\, \varrho_{f,V_0}(B^* y)\mathrm{vac}_{V_0}.$$

Proof. *The constant.* $f_0 = \mathrm{ev}_\Xi^f(\mathrm{vac}_V)$ lies in $\mathcal{W}_f(V_0)$ (Proposition 5.1(i)). For $x \in \hat{\mathcal{O}}^{I_0}$, by Proposition 5.1(iii) with $w = gx$ and $b = cx$, $\varrho_{f,V_0}(gx) f_0 = \mathrm{ev}_\Xi^f(\varrho_{f,V}(Bx)\mathrm{vac}_V) = f_0$, as $Bx \in \hat{\mathcal{O}}^I$ (Input 1(a)). By Input 1(d), $f_0 = C \cdot \omega_{V_0}(1,g)\mathrm{vac}_{V_0}$ for some C.

The identity. Fix y. If $\mathrm{ev}_\Xi^f(\varrho_{f,V}(y)\mathrm{vac}_V) = 0$, take $\varsigma = 0$. Otherwise Proposition 5.2, with $M = M' = 1$ and $f = \mathrm{vac}_V$, gives $x_0 \in \hat{\mathcal{O}}^I$ with $\mathsf{k}_{\Xi,f}^*(y - x_0) = 0$. By Input 3(c), $y - x_0 = \mathsf{i}_{\Xi,f} w + \mathsf{k}_{\Xi,f} b$ with $w = \mathsf{i}_{\Xi,f}^*(y - x_0)$. Hence, with roots of unity ς_1, ς_2 from Input 1(a),

$$\begin{aligned}\mathrm{ev}_\Xi^f(\varrho_{f,V}(y)\mathrm{vac}_V) &= \mathrm{ev}_\Xi^f(\varrho_{f,V}(y)\varrho_{f,V}(-x_0)\mathrm{vac}_V) = \varsigma_1\, \mathrm{ev}_\Xi^f(\varrho_{f,V}(y - x_0)\mathrm{vac}_V)\\ &= \varsigma_1 \varrho_{f,V_0}(w) f_0 = \varsigma_1 C\, \varrho_{f,V_0}(w)\omega_{V_0}(1,g)\mathrm{vac}_{V_0}.\end{aligned}$$

Since g is unitary and $\mathsf{k}_{\Xi,f}^*(y - x_0) = 0$, we have $gB^*(y - x_0) = gg^*\mathsf{i}_{\Xi,f}^*(y - x_0) = w$, so by Input 1(b) $\varrho_{f,V_0}(w)\omega_{V_0}(1,g) = \omega_{V_0}(1,g)\varrho_{f,V_0}(B^*(y - x_0))$; and $B^* x_0 \in \hat{\mathcal{O}}^{I_0}$ fixes the vacuum, so $\varrho_{f,V_0}(B^*(y - x_0))\mathrm{vac}_{V_0} = \varsigma_2 \varrho_{f,V_0}(B^* y)\mathrm{vac}_{V_0}$. Take $\varsigma = \varsigma_1\varsigma_2$. □

LEAN `zeroTermFin_finHeisOp_vacElem`

6 Arrow pairs

Lemma 6.1 (two computations in rational coordinates)**.** *Let E be a CM field with a CM type and V, V' hermitian spaces with index sets I, I' and their signs.*

(a) For a pair $g = (g_1, g_2)$ of elements of $G(V)$ and $G(V')$, the matrix in rational coordinates of $g_1 \oplus g_2$, an element of $G(V \oplus V')$, is the block diagonal matrix of those of g_1 and g_2.

(b) Let $\mathcal{Y}$, $\mathcal{Y}'$ be gradings of V, V' of norm N, with definite parts X, X' and strings of lengths l, l'. For $g \in G(\langle X\rangle)$ and $g' \in G(\langle X'\rangle)$, in rational coordinates, $\imath_{\mathcal{Y}'}(g') \cdot T_{\mathcal{Y},\mathcal{Y}'} \cdot \imath_{\mathcal{Y}}(g)^{-1} = T_{\mathcal{Y},\mathcal{Y}'}$.

Proof. (a) The frame of $V \oplus V'$ is block diagonal. (b) In rational coordinates $\imath_{\mathcal{Y}}(g)$ is block diagonal for the decomposition into X and the string, with the identity on the string, and likewise $\imath_{\mathcal{Y}'}(g')$; the string map has its only non-zero block from string to string. □

LEAN `ratCoordMatrix_sum` · `inclX_mul_stringMap_mul_inclX_inv`

Lemma 6.2. *In Setting 2.6, let X'', H be finite index sets and $\mathcal{R} \in \mathrm{Arr}_{X'',H}(\mathcal{Y}, \mathcal{Y}', \mathcal{Y}'')$. For all $\Phi \in \mathcal{W}_f(\mathrm{Hom}(V, V'))$ and $\Phi' \in \mathcal{W}_f(\mathrm{Hom}(V', V''))$,*

$$\mathrm{at}(\mathrm{fev}_{\mathcal{R}}(\Phi \boxtimes \Phi')) = \mathrm{ev}_{\Xi^{\mathrm{pr}}(\mathcal{R})}^f\Big(\varrho_f(\mathbf{T}_f(\mathcal{Y}, \mathcal{Y}', \mathcal{Y}''))(\Phi \boxtimes \Phi')\Big),$$

where at *is the map of the standard spaces $\langle X\rangle$, $\langle X'\rangle$, $\langle X''\rangle$.*

Proof. $\Phi \boxtimes \Phi' = \mathrm{at}(\Phi \otimes \Phi')$ lies in $\mathcal{W}_f(\mathrm{Hom}(V', V^- \oplus V''))$ (Input 2(e)); the finite Heisenberg operators preserve finite parts (Input 1(a)); and $\mathrm{ev}_{\Xi^{\mathrm{pr}}(\mathcal{R})}^f$ maps finite parts to finite parts (Proposition 5.1(i)). So the right side lies in $\mathcal{W}_f(\mathrm{Hom}(\langle X'\rangle, \langle X\rangle^- \oplus \langle X''\rangle))$, which is the range of at for the standard spaces, and $\mathrm{at} \circ \mathrm{at}^\flat$ is the identity on that range because $\mathrm{at}^\flat \circ \mathrm{at} = 1$. □

Proof of Theorem B. (i) Put $\mathrm{wt} = \mathrm{gr}_y \sqcup \mathrm{gr}_{y''}$, write $l' = 2j$ (possible as l is odd and $l' = l-1$), let m_+ be the number of indices of $I \sqcup I''$ of positive weight, and let $x = \imath_y(g)^- \oplus \imath_{y''}(k) \in G(V^- \oplus V'')$. In rational coordinates x is the identity on the basis vectors of non-zero weight, because $\imath_y$ and $\imath_{y''}$ fix the string vectors (Lemma 6.1(a)). The splitting $\Xi^{\mathrm{pr}}(\mathcal{R})$ is a composite, so its zero term is the composite of two zero terms (Definition 2.4), and $\omega^{\mathrm{tr}}_{\mu,\mu'}(\imath_y g, \imath_{y'} g', \imath_{y''} k)$ is a scalar c_V times the Weil operator of $(\imath_{y'}(g'), x)$ on $\mathrm{Hom}(V', V^- \oplus V'')$ (Definition 2.3).

First zero term. Apply Input 3(d) to the splitting $\Xi(\mathcal{Z}^{\mathrm{Hom}}(\mathcal{R}))$, whose frame k consists of the basis vectors (i, b) of positive degree $\mathrm{wt}(i) - \mathrm{gr}_{y'}(b)$, with the element $(\imath_{y'}(g'), x)$, the element $g_0 = (g', 1 \oplus g^- \oplus \mathrm{emb}_{\mathcal{R}}(k))$ of the group of the zero part, and the matrix C that acts by x on the rows above each $b \in I'$ of negative grade and by the inverse transpose of $\imath_{y'}(g')$ on the columns above each i of positive weight, and is the identity elsewhere. Its determinant is $u = (\det g \cdot \det k)^j \cdot (\det g')^{-m_+}$, since y' has j string vectors of negative grade, $\det x = \det g \cdot \det k$ and $\det \imath_{y'}(g') = \det g'$. The conditions of Input 3(d) are statements about the entries of block diagonal matrices: they hold because x is the identity off weight zero, because the zero part has degree zero only above X' (Definition 2.14), and, for the last one, by condition (c) of Definition 2.16. Hence the first zero term carries the Weil operator of $(\imath_{y'}(g'), x)$ to $\upsilon(u)^{-1}$ times that of g_0.

Second zero term. By Input 3(e), removing the H planes carries the Weil operator of $(g', 1 \oplus g^- \oplus \mathrm{emb}_{\mathcal{R}}(k))$ to $\upsilon(\det g')^{|H|}$ times that of $(g', g^- \oplus \mathrm{emb}_{\mathcal{R}}(k))$; the side conditions of the input hold for the explicit frames of the splitting that removes the planes.

The scalars. The right side of (i) is the scalar c_X of the triple action for the standard spaces $\langle X\rangle$, $\langle X'\rangle$, $\langle X''\rangle$ at $(g, g', \mathrm{emb}_{\mathcal{R}}(k))$ times the same Weil operator. So it remains to see $c_V = c_X \cdot \upsilon(u)\, \upsilon(\det g')^{-|H|}$. The values of μ and μ' in c_V and c_X agree, since $\det \imath_y(g) = \det g$ and $\det \mathrm{emb}_{\mathcal{R}}(k) = \det k$. It remains to compare the exponents of υ at the three determinants, with $\mu_V = \mu \upsilon^{\lfloor |I|/2 \rfloor}$. At $\det g$: $(|I'| - \lfloor |I'|/2 \rfloor) - (|X'| - \lfloor |X'|/2 \rfloor) = 2j - j = j$, as $|I'| = |X'| + 2j$. At $\det k$: $\lfloor |I'|/2 \rfloor - \lfloor |X'|/2 \rfloor = j$. At $\det g'$: counting the weights of $I \sqcup I''$, $|I| + |I''| = 2m_+ + 2|H| + |X| + |X''|$; this number is even by condition (b) of Definition 2.16, so $\lfloor |I|/2 \rfloor - \lfloor |I''|/2 \rfloor = (|I| - |I''|)/2$ and likewise for X, X'', and the exponent is

$$\tfrac{1}{2}(|I| - |I''|) - |I| - \tfrac{1}{2}(|X| - |X''|) + |X| = -\tfrac{1}{2}(|I| + |I''|) + \tfrac{1}{2}(|X| + |X''|) = -m_+ - |H|.$$

These are the exponents of $\upsilon(u)\upsilon(\det g')^{-|H|}$.

(ii) Since $\mathrm{at}^\flat \circ \mathrm{at} = 1$, Input 2(d), Lemma 6.2 and the hypothesis $\mathrm{emb}_{\mathcal{R}}(1, k) = (1, k')$ give for the right side

$$\mathrm{at}^\flat\Big(\omega^{\mathrm{tr}}_{\mu,\mu'}((1, g), (1, g'), \mathrm{emb}_{\mathcal{R}}(1, k))\ \mathrm{ev}^f_{\Xi^{\mathrm{pr}}(\mathcal{R})}(\varrho_f(\mathbf{t})(\Phi \boxtimes \Phi'))\Big), \qquad \mathbf{t} = \mathbf{T}_f(y, y', y'').$$

Now $\mathrm{ev}^f_{\Xi^{\mathrm{pr}}(\mathcal{R})}(F) = \mathrm{ev}_{\Xi^{\mathrm{pr}}(\mathcal{R})}(v \otimes F)$ for a fixed real datum v, and the triple action of finite points commutes with $v \otimes$ (Input 1(b)). So part (i), with $\imath^f_{y''}(k) = x$, turns this into the first of the two vectors

$$\mathrm{at}^\flat\ \mathrm{ev}^f_{\Xi^{\mathrm{pr}}(\mathcal{R})}(\omega^{\mathrm{tr}}_{\mu,\mu'}(\imath^f_y g, \imath^f_{y'} g', x)\, \varrho_f(\mathbf{t})(\Phi \boxtimes \Phi')),$$
$$\mathrm{at}^\flat\ \mathrm{ev}^f_{\Xi^{\mathrm{pr}}(\mathcal{R})}(\varrho_f(\mathbf{t})\, \omega^{\mathrm{tr}}_{\mu,\mu'}(\imath^f_y g, \imath^f_{y'} g', x)(\Phi \boxtimes \Phi')),$$

and by Input 2(d) the left side of (ii) is the second. The two agree: on an arrow product the translation by the stacked string map is the arrow product of the translations by the finite parts of the two string maps (Input 2(b)), the triple action is the arrow product of the two pair actions (Input 2(d)), and each pair action commutes with the translation by its string map by Input 2(g) and Lemma 6.1(b). □

LEAN `ArrowPair.zeroTerm_splitting_tripleWeil` · `ArrowPair.finEval_inclXFin`

7 Two unfolding steps

Proof of Theorem C(i). Let $z_0 \neq 0$ be the constant of Input 2(a) for V, V', V''. Let C_1 be the constant of Input 4(a) for $\mathcal{Y}$ and the set of all its indices of grade 1; this set is empty, the grades of $\mathcal{Y}$ being even. Let C_2 be the constant of Input 4(a) for $\mathcal{Y}'$ and the injection $\gamma_{\mathcal{X}}$. We show the claim with $z = C_1 C_2 z_0^{-1}$. Fix μ, μ', f, f', k, h and a generic P, and put

$$F(g,r) = P(gh)\ \mathrm{ct}^{\mu}_{y,y'}(f)(g,r)\ \mathrm{ct}^{\mu'}_{y',y''}(f')(r, \imath_{y''}(k)).$$

It is continuous, and invariant under the rational points of $B(\mathcal{Y})$ in g and of $B(\mathcal{Y}')$ in r (Input 4(c)).

Step 1: a product of two constant terms is a zero term. For data f_1, f_2 and matrices $y\colon V \to V'$, $y'\colon V' \to V''$ in rational coordinates, the product of the constant term of f_1 at y and of f_2 at y' equals $z_0^{-1} \cdot \mathrm{ev}_{\Xi_1}(\varrho(y) f_1 \boxtimes \varrho(y') f_2)(0)$, where $\Xi_1 = \Xi(\mathcal{Z}^{\mathrm{Hom}}(\mathcal{R}))$. Indeed, the value at 0 of this zero term is the mean of the datum over the matrices $V' \to V^- \oplus V''$ of positive degree (Definition 2.4). The map $(a,b) \mapsto \left(\begin{smallmatrix} a^\dagger \\ b \end{smallmatrix}\right)$ carries the pairs of a matrix $V \to V'$ and a matrix $V' \to V''$, both of positive degree, onto these matrices, because the adjoint exchanges the positions (b,a) and (a,b) up to the reversal of the strings, which changes the sign of both grades. So the mean is an iterated mean (Input 7(c)); by Input 2(b) and (a) the integrand is $\theta(\varrho(a)\varrho(y) f_1 \boxtimes \varrho(b)\varrho(y') f_2) = z_0\, \theta(\varrho(a)\varrho(y) f_1)\, \theta(\varrho(b)\varrho(y') f_2)$, and the two means of the factors are the two constant terms (Definition 2.5).

Step 2: the inner integral. Fix r. For a unipotent element n of $\mathcal{Y}$, $n-1$ raises grades by at least 2, the grades being even; so by Input 4(b) at $y = T_{y,y'}$, where the quotient of the two values of the bicharacter is 1, $F(n\,\imath_y(a), r) = F(\imath_y(a), r)$ for $a \in G(\langle X \rangle)$ (the second constant term does not involve n). Input 4(a), with the constant function of mean $F(\imath_y(a), r)$, gives $\int_{[B(\mathcal{Y})]} F(g,r)\, dg = C_1 \int_{[G(\langle X \rangle)]} F(\imath_y(a), r)\, da$.

Step 3: Fubini. The function $(a,r) \mapsto F(\imath_y(a), r)$ is continuous and invariant on a product of two compact quotients, so the integrals over r and a may be exchanged.

Step 4: the integral over $[B(\mathcal{Y}')]$. Fix $a \in G(\langle X \rangle)$ and let $g' \in G(\langle X' \rangle)$ and n' a unipotent element of $\mathcal{Y}'$. Put $f_1 = \omega^{\mathrm{pr}}_{\mu}(\imath_y a, \imath_{y'} g') f$ and $f_2 = \omega^{\mathrm{pr}}_{\mu'}(\imath_{y'} g', \imath_{y''} k) f'$. By Input 4(c) the two constant terms in $F(\imath_y(a), n' \imath_{y'}(g'))$ are the constant terms of f_1 at $n'^{-1} T_{y,y'}$ and of f_2 at $T_{y',y''} n'$. Since n' is unitary, $(n'^{-1} T_{y,y'})^\dagger = T^\dagger_{y,y'} n'$, so by Step 1 and Input 2(b)

$$F(\imath_y(a), n' \imath_{y'}(g')) = P(\imath_y(a) h) \cdot z_0^{-1} \cdot \mathrm{ev}_{\Xi_1}\Big(\varrho(\mathbf{T} + \mathbf{T}(n'-1))(f_1 \boxtimes f_2)\Big)(0),$$

where we write $\mathbf{T}$ for $\mathbf{T}(\mathcal{Y}, \mathcal{Y}', \mathcal{Y}'')$. The matrix $\mathbf{T}(\mathcal{Y}, \mathcal{Y}', \mathcal{Y}'')(n'-1)$ has two kinds of entries. Outside the positions of degree zero, which are the positions $(\mathrm{ind}_{\mathcal{X}}(a'), x')$

with $x' \in X'$, it has entries only in positions of positive degree, because the string maps lower degrees by at most one and $n'-1$ raises them; translations in these directions do not change the zero term (Input 3(b)). At the positions of degree zero, condition (d) of Definition 2.16 shows that its entries are those of $\mathrm{inc}_{\mathcal{R}}$ applied to the vector that has, at the second copy of $k \in H$ and at $x' \in X'$, the entry of $\mathfrak{c}(\mathcal{Y}')(n'-1)$ at $(\gamma_{\mathcal{R}}(k), x')$, and zeros elsewhere. By Input 3(b) the zero term turns this translation into the translation of the zero part by that vector. No Heisenberg factor arises when $\varrho(\mathbf{T}+\mathbf{T}(n'-1))$ is split into $\varrho(\mathbf{T})$ and the translations by the two parts of $\mathbf{T}(n'-1)$: such factors are values of the Heisenberg bicharacter of $\mathrm{Hom}(V', V^- \oplus V'')$ (Definition 2.5), and two of them occur. First, $\varrho(\mathbf{T}(n'-1)+\mathbf{T})$ and $\varrho(\mathbf{T}(n'-1))\,\varrho(\mathbf{T})$ differ by the value $\mathrm{bi}(\mathbf{T}(n'-1), \mathbf{T}) = \psi_{\mathbb{A}}(\frac{1}{2}\mathbf{j}\,\mathrm{tr}(\mathbf{T}^\dagger \mathbf{T}(n'-1)))$, which is 1 because the stacked string map is isotropic:

$$\mathbf{T}^\dagger \mathbf{T} = -T_{y,y'}\,T_{y,y'}^\dagger + T_{y',y''}^\dagger\,T_{y',y''} = -\mathrm{sh}_{y'} + \mathrm{sh}_{y'} = 0.$$

Here the adjoint of the stacked matrix has the blocks $-T_{y,y'}$ and $T_{y',y''}^\dagger$, the sign being that of the form of V^-; the second product is $\mathrm{sh}_{y'}$ by Input 4(d) for the gradings $\mathcal{Y}'$, $\mathcal{Y}''$, and the first is $\mathrm{sh}_{y'}$ by Input 4(d) for $\mathcal{Y}$, $\mathcal{Y}'$ because $\mathfrak{c}(\mathcal{Y}') = \mathfrak{c}(\mathcal{Y})$, condition (b) of Definition 2.16. This is where that condition on an arrow pair is used. Second, the translation by $\mathbf{T}(n'-1)$ and the product of the translations by its part of positive degree and by its part of degree zero differ by the value of the bicharacter at these two parts. The first part lies along the isotropic frame k of Ξ_1 and the second along i, and the bicharacter pairs k only with l; so this value is 1. The directions of the vectors of degree zero are exactly those over which the splitting that removes the planes averages. Therefore $n' \mapsto F(\imath_y(a), n'\imath_{y'}(g'))$ is a continuous periodic function of the entries of $\mathfrak{c}(\mathcal{Y}')(n'-1)$ in the rows $\gamma_{\mathcal{R}}(H)$ and the columns X', and its mean is

$$M(g') = P(\imath_y(a)h)\cdot z_0^{-1}\cdot\theta\Big(\mathrm{ev}_{\Xi^{\mathrm{pr}}(\mathcal{R})}(\varrho(\mathbf{T}(\mathcal{Y},\mathcal{Y}',\mathcal{Y}''))(f_1\boxtimes f_2))\Big).$$

Now $f_1\boxtimes f_2 = \omega^{\mathrm{tr}}_{\mu,\mu'}(\imath_y a, \imath_{y'}g', \imath_{y''}k)(f\boxtimes f')$ (Input 2(d)); this triple action commutes with $\varrho(\mathbf{T}(\mathcal{Y},\mathcal{Y}',\mathcal{Y}''))$, as in the proof of Theorem B(ii); and Theorem B(i) moves it through the zero term. So

$$M(g') = P(\imath_y(a)h)\,z_0^{-1}\,\theta\Big(\omega^{\mathrm{tr}}_{\mu,\mu'}(a, g', \mathrm{emb}_{\mathcal{R}}(k))\ \mathrm{ev}_{\Xi^{\mathrm{pr}}(\mathcal{R})}(\varrho(\mathbf{T})(f\boxtimes f'))\Big),$$

a continuous function of g', left invariant under $\Gamma(\langle X'\rangle)$ by Input 2(f). Input 4(a) for $\mathcal{Y}'$ gives $\int_{[B(\mathcal{Y}')]} F(\imath_y(a), r)\,dr = C_2\int_{[G(\langle X'\rangle)]} M(g')\,dg'$. Together with Steps 2 and 3 this is the claim. □

LEAN `ArrowPair.integral_constTermAtT_mul`

Proof of Theorem C(ii). Let $z_0 \neq 0$ be the constant of Input 2(a) for the standard spaces $\langle X\rangle$, $\langle X'\rangle$, $\langle X_0\rangle$. For $o \in U$, φ, g and a datum f of $\mathrm{Hom}(\langle X'\rangle, \langle X\rangle^- \oplus \langle X_0\rangle)$, the function $(x,r) \mapsto \mathrm{coef}_t(\varphi)(x)\cdot\theta(\omega^{\mathrm{tr}}_{\mu_o,\mu'_o}(x,r,g)f)$ on $[G(\langle X\rangle)]\times[G(\langle X'\rangle)]$ is continuous and well defined (Inputs 5 and 2(f)), on a compact space since the standard spaces are definite; so it is integrable, and its integral $\mathcal{J}_{o,\varphi,g}(f)$ is a continuous linear functional of f.

Reduction to polynomial vectors. The real data u for which the theorem holds form a linear subspace of the Fock space. It is closed: the map sending u to the function $(o,y,\varphi,g) \mapsto \mathcal{J}_{o,\varphi,g}(u\otimes \mathrm{at}(y))$ is continuous for pointwise convergence,

and the functions of the form $(o, y, \varphi, g) \mapsto \mathrm{coef}_{t''}(\mathcal{L}''_o(y)\varphi)(g)$, t'' in the finite-dimensional dual of the vectors of weight $\varkappa$, form a finite-dimensional, hence closed, subspace. By Input 7(a) it suffices to treat $u = v_Q$, and by Input 7(b) and linearity a product $Q = Q_1 Q_2$ of a polynomial Q_1 in the coordinates of $\mathrm{Hom}(\langle X'\rangle, \langle X\rangle^-)$ and a polynomial Q_2 in those of $\mathrm{Hom}(\langle X'\rangle, \langle X_0\rangle)$.

Products. For such Q the polynomial vector is an arrow product of real data, $v_Q = v_{Q'_1} \boxtimes v_{Q_2}$, where Q'_1 is Q_1 transported to the coordinates of $\mathrm{Hom}(\langle X\rangle, \langle X'\rangle)$ (Input 2(h)). Let t' be the functional that Input 5 gives for $(\mathcal{L}_o)_o$, Q'_1 and t, and t'' the one for $(\mathcal{L}'_o)_o$, $z_0 Q_2$ and t'. By linearity in y it suffices to take $y = \Phi \otimes \Phi'$. Then $v_Q \otimes \mathrm{at}(y) = (v_{Q'_1} \otimes \Phi) \boxtimes (v_{Q_2} \otimes \Phi')$, again by Input 2(h), and by Input 2(d) and (a)

$$\theta(\omega^{\mathrm{tr}}_{\mu_o,\mu'_o}(x, r, g)(v_Q \otimes \mathrm{at}(y))) = z_0 \cdot \theta(\omega^{\mathrm{pr}}_{\mu_o}(x, r)(v_{Q'_1} \otimes \Phi)) \cdot \theta(\omega^{\mathrm{pr}}_{\mu'_o}(r, g)(v_{Q_2} \otimes \Phi')).$$

Integrating first over x (Fubini) and using the property of $\mathcal{L}_o$, then over r and using that of $\mathcal{L}'_o$, we get

$$\begin{aligned} \mathcal{J}_{o,\varphi,g}(v_Q \otimes \mathrm{at}(y)) &= \int_{[G(\langle X'\rangle)]} \mathrm{coef}_{t'}(\mathcal{L}_o(\Phi)\varphi)(r)\, \theta(\omega^{\mathrm{pr}}_{\mu'_o}(r, g)(v_{z_0 Q_2} \otimes \Phi'))\, dr \\ &= \mathrm{coef}_{t''}(\mathcal{L}'_o(\Phi')\mathcal{L}_o(\Phi)\varphi)(g). \end{aligned}$$

□

LEAN `integral_theta_eq_lift₂`

8 Admissible evaluations

Proposition 8.1 (the standard evaluation)**.** *In Setting 2.19, with its objects N, Σ, π, δ, β, κ, let X' be a finite index set, put $T = X^+ \sqcup X''^+$, with the points pt_1 of X^+ and pt_2 of X''^+, and let $V_{\mathrm{st}} = \mathrm{Hom}(\langle X'\rangle, \langle X^+\rangle^- \oplus \langle X''^+\rangle)$, a standard space with index set $T \times X'$. There are an element $k \in G_f(T \times X')$, a $T \times T$ matrix γ over $\mathbb{A}_f$ and a family $b\colon T \to \mathbb{A}_f$ such that:*

(1) the matrix of k is the Kronecker product of $\mathrm{diag}(\kappa) \cdot \gamma$ with the identity matrix of X';

(2) at every finite place w with $w \notin \Sigma$ and $\bar{w} \notin \Sigma$, every b_j has component 0 and γ has the identity as component;

(3) for every $j \in T$, $\mathsf{Nb}_{\pi,\delta}(b_j; \gamma_{\cdot j})$, where $\gamma_{\cdot j}$ is the column j of γ;

(4) $\mathsf{AdmEv}_{N,\Sigma,\beta}((f \mapsto \mathrm{at}^\flat(\omega_{V_{\mathrm{st}}}(1, k)f)); \beta^{\langle\cdot\rangle}\kappa)$ (Definition 2.21 for the spaces $\langle X^+\rangle$, $\langle X'\rangle$, $\langle X''^+\rangle$).

Proof. The element. Let $\mathbf{1}_\Sigma \in \hat{\mathcal{O}}$ be the adele equal to 1 at the places of Σ and 0 elsewhere. Then $\mathbf{1}_\Sigma \overline{\mathbf{1}_\Sigma} = 0$ and $\mathbf{1}_\Sigma \delta = \mathbf{1}_\Sigma$, as $\Sigma \cap \bar{\Sigma} = \emptyset$ and $\delta = 1$ above N. Let λ be the finite idele equal to π at the places of Σ, to $\bar{\pi}^{-1}$ at their conjugates and to 1 elsewhere; so $\lambda\bar{\lambda} = 1$. Let $\xi \in \hat{\mathcal{O}}^T$ be the vector $e_{\mathrm{pt}_1} + \overline{\epsilon_- \mathbf{1}_\Sigma}\, e_{\mathrm{pt}_2}$; then $\xi^*\xi = 1$, and

$$\gamma = 1 + (\lambda - 1)\, \xi\xi^*$$

is unitary: it multiplies ξ by λ and fixes the vectors orthogonal to ξ. For $m \in T$ put $b_m = (\lambda - 1)\pi^{-1}(\overline{\mathbf{1}_\Sigma}\,[m = \mathrm{pt}_1] + \epsilon_- \mathbf{1}_\Sigma\,[m = \mathrm{pt}_2])$, where $[m = \mathrm{pt}_1]$ is 1 if $m = \mathrm{pt}_1$ and 0 otherwise. The diagonal matrix $\mathrm{diag}(\kappa)$ is unitary, because $\kappa_b \in \{1, \delta\}$ and $\delta\bar{\delta} = 1$, and it commutes with γ, because $\gamma - 1$ is supported above N, where all

κ_b are 1. Let k be the element of $G_f(T \times X')$ acting by $\mathrm{diag}(\kappa)\gamma$ on each column $x' \in X'$. This is (1), and (2) holds because $\lambda - 1$ vanishes outside $\Sigma \cup \bar{\Sigma}$.

Neighbour vectors. $\delta b_m = b_m$ since b_m is supported above N. The entries γ_{im} with $i \neq \mathrm{pt}_1, \mathrm{pt}_2$ are those of the identity. Next $\bar{\pi}(\lambda - 1) \in \hat{\mathcal{O}}$: it is $\bar{\pi}(\pi - 1)$ at Σ, $1 - \bar{\pi}$ at $\bar{\Sigma}$ and 0 elsewhere; so $\bar{\pi}\gamma_{im} \in \hat{\mathcal{O}}$. Let $\varpi_0 = \mathbf{1}_\Sigma + (1 - \mathbf{1}_\Sigma)\pi^{-1}$, an element of $\hat{\mathcal{O}}$ since π is a unit outside Σ. A direct computation with $\mathbf{1}_\Sigma \overline{\mathbf{1}_\Sigma} = 0$ and $\overline{\epsilon_-}\epsilon_- = -1$ gives $b_m - \pi^{-1}\gamma_{\mathrm{pt}_1 m} = -\varpi_0\,[m = \mathrm{pt}_1]$ and $\overline{\epsilon_-} b_m - \pi^{-1}\gamma_{\mathrm{pt}_2 m} = -\varpi_0\,[m = \mathrm{pt}_2]$: for instance $\gamma_{\mathrm{pt}_1 m} = [m = \mathrm{pt}_1] + (\lambda - 1)([m = \mathrm{pt}_1] + \epsilon_- \mathbf{1}_\Sigma [m = \mathrm{pt}_2])$, so the coefficient of $[m = \mathrm{pt}_2]$ cancels and that of $[m = \mathrm{pt}_1]$ is $-\pi^{-1}(1 + (\lambda - 1)(1 - \overline{\mathbf{1}_\Sigma})) = -\pi^{-1}(1 + \mathbf{1}_\Sigma(\pi - 1)) = -\varpi_0$. This is (3).

The key identity. Let $\bar{\Sigma}$ be the set of the conjugates of the places of Σ and $\gamma_{X'}$ the element acting by γ on each column $x' \in X'$. We claim that there is $C_0 \in \mathbb{C}$ with

$$\omega_{V_{\mathrm{st}}}(1, \gamma_{X'})\, P^{\bar{\Sigma}} f = C_0 \sum_y \varrho_f(\overline{\mathbf{1}_\Sigma}\,\mathrm{n}(y)) P^{\bar{\Sigma}} f \qquad \text{for all } f \in \mathcal{S}(V_{\mathrm{st}}),$$

the sum over $y \in ((\mathbb{Z}/N)^\Sigma)^{X'}$, where $\mathrm{n}(y)$ is the finite vector with the entry $\epsilon_-\,\xi_i\,\overline{\tilde{y}_{x'}/N}$ at (i, x'). By Input 3(f), $P^{\bar{\Sigma}}$ is named by the indicator of the vectors integral at $\bar{\Sigma}$, the element $\gamma_{X'}$, supported at Σ and $\bar{\Sigma}$, transforms this function by $\gamma_{X'}^{-1}$ up to a constant, and translates of the indicator name the corresponding Heisenberg translates of $P^{\bar{\Sigma}}$. So the claim is an identity of functions: for every finite vector x there is an element $b(x)$ such that, for every y, the vector $\gamma_{X'}x + \mathrm{n}(y)$ is integral at $\bar{\Sigma}$ if and only if x is integral at $\bar{\Sigma}$ and $y = b(x)$. Thus the sum over y of the translated indicators has at most the one term $y = b(x)$ at $\gamma_{X'}x$, and no term when x is not integral at $\bar{\Sigma}$. At a place of $\bar{\Sigma}$ we have $\lambda = \bar{\pi}^{-1}$, $\xi = e_{\mathrm{pt}_1} + \overline{\epsilon_-} e_{\mathrm{pt}_2}$, and the column x' of $\gamma_{X'}x + \mathrm{n}(y)$ is $x_{\cdot x'} + \xi \cdot ((\bar{\pi}^{-1} - 1)x_{\mathrm{pt}_1 x'} + \epsilon_-\,\tilde{y}_{x'}/N)$. If it is integral, its coordinate at pt_1, $\bar{\pi}^{-1}x_{\mathrm{pt}_1 x'} + \epsilon_-\tilde{y}_{x'}/N$, is integral; multiplying by the uniformiser $\bar{\pi}$ and using that $\bar{\pi}/N = \pi^{-1}$ is a unit there, $x_{\mathrm{pt}_1 x'}$ is integral, and then so are all coordinates of x. Conversely, for integral x the column is integral exactly if $\bar{\pi}^{-1}x_{\mathrm{pt}_1 x'} + \epsilon_-\tilde{y}_{x'}/N$ is. The lemma of the note on the Gauss-sum operator on residues modulo N at a split level, applied after complex conjugation at Σ, provides a residue class $b_{x'}$ such that this holds if and only if $y_{x'} = b_{x'}$, the product of N and $\bar{\pi}^{-1}x_{\mathrm{pt}_1 x'}$ being integral; the family of the $b_{x'}$ is $b(x)$.

The admissible evaluation. We show (4) with the constant $C = \Delta_f(k_0)\, C_0$, where k_0 acts by $\mathrm{diag}(\kappa)$ on each column, so that $k = \gamma_{X'} k_0$. Let n be coprime to N, p a proposition and $x \in \mathrm{Tr}_n[(i, x') \mapsto \beta^{\langle i \rangle}\kappa_i]$. First, Proposition 3.4, with $c = 1$ and the set $A = \mathrm{Tr}_n[(i, x') \mapsto \beta^{\langle i \rangle}]$, shows that $\omega_{V_{\mathrm{st}}}(1, k_0)\varrho_f(x)\mathrm{vac}$ is an admissible combination of the vectors $\Delta_f(k_0)\varrho_f(z)\mathrm{vac}$, $z \in A$, away from n, with denominator D and size 1. Its hypotheses hold: k_0 multiplies the coordinate (i, x') by κ_i, and $\beta^{\langle i \rangle}\kappa_i(x + y)_{(i,x')}$ is integral away from n for $y \in \hat{\mathcal{O}}^{T \times X'}$, by the choice of x and because $\beta^{\langle i \rangle}\kappa_i \in \hat{\mathcal{O}}$; and $D/|\Delta_f(k_0)|^2 \in O^{(n)}$ by Lemma 3.2(a) with $k = 0$, because $|\Delta_f(k_0)|^2$, the index of $\hat{\mathcal{O}}^{T \times X'} \cap k_0^{-1}\hat{\mathcal{O}}^{T \times X'}$ (Input 1(c)), divides D: this lattice contains the kernel of the map $y \mapsto (\bar{y} \bmod \beta)$ to $(\hat{\mathcal{O}}/\beta\hat{\mathcal{O}})^{T \times X'}$, since a vector with coordinates $\bar{\beta}\bar{c}$, $c \in \hat{\mathcal{O}}$, is mapped by k_0 to one with coordinates $\bar{\beta}\kappa_i\bar{c} \in \hat{\mathcal{O}}$. Second, apply the linear map $\mathrm{at}^\flat \circ \omega_{V_{\mathrm{st}}}(1, \gamma_{X'})$ (Lemma 3.1(c)). Third, re-index by Lemma 3.1(f): for $z \in A$,

$$\Delta_f(k_0)\ \mathrm{at}^\flat(\omega_{V_{\mathrm{st}}}(1, \gamma_{X'})\varrho_f(z)\mathrm{vac}) = \varsigma \cdot C \cdot \mathsf{T}_{N,\Sigma}(X, X', X'')(\varrho_f(t_1)\mathrm{vac} \otimes \varrho_f(t_2)\mathrm{vac})$$

with a root of unity ς, where t_1 is ϵ_- times the conjugate transpose of the block of z with rows X^+ and t_2 is the block of z with rows X''^+; both lie in the groups of translations for $\bar\beta$ (Lemma 3.3(b)). To see this identity, note that z is integral at the places above N: they do not divide n, and β and $\bar\beta$, the weights of the two blocks, are units there. So $P^{\bar\Sigma}\varrho_f(z)\mathrm{vac} = \varrho_f(z)\mathrm{vac}$ (Input 1(f)), and the key identity gives $\omega_{V_{\mathrm{st}}}(1,\gamma_{X'})\varrho_f(z)\mathrm{vac} = C_0\sum_y \varrho_f(\overline{\mathbf{1}_\Sigma}\mathrm{n}(y))\varrho_f(z)\mathrm{vac}$. By Input 2(c) each term is a root of unity times an arrow product of two translated vacua, and the terms are those of $\mathrm{at}(\mathsf{T}_{N,\Sigma}(X,X',X'')(\varrho_f(t_1)\mathrm{vac}\otimes\varrho_f(t_2)\mathrm{vac}))$, by the definition of the translate sum (the first block of $\overline{\mathbf{1}_\Sigma}\mathrm{n}(y)$ gives the last column $-\tilde y/N$, the second block the last row $-\overline{\tilde y/N}$, with the same root of unity for all y); apply $\mathrm{at}^\flat$. □

LEAN `exists_isAdmissibleEval_std`

Proof of Theorem D. Apply Proposition 8.1 with the index set X'^+ in place of X': it gives k, γ, b with the properties (1) to (4), and the constant C of (4). Let r_1 be the finite idele $\overline{\epsilon_-}\bar\alpha/\alpha$, by which the frame of V^- differs from that of V (Definition 2.3); so the inverse finite frame of $V^-\oplus V''$ is the block diagonal matrix of $r_1^{-1}x_{V,f}^{-1}$ and $x_{V'',f}^{-1}$, and $r_1\cdot(a/\bar a) = \overline{\epsilon_-}\delta$ by the hypothesis on δ.

An integral matrix. Let M be the matrix with rows $(H\sqcup H)\sqcup T$ and columns $j\in T$ whose column j is: 0 on the first copy of H; $r_1 u b_j$ at every index of the second copy; $r_1\kappa_i\gamma_{ij}$ at $i\in X^+$; and $\kappa_i\gamma_{ij}$ at $i\in X''^+$. Let J be the inclusion matrix of the basis vectors of X'^+ into I', and

$$B = (x_{V^-\oplus V'',f}^{-1}\ \mathrm{inc}_{\mathcal{R}}\ M)\otimes(\mathsf{J}^{\mathsf{T}}x_{V',f})^{\mathsf{T}},$$

a matrix with rows $(I\sqcup I'')\times I'$ and columns $T\times X'^+$. It is integral. The second factor is: $x_{V',f} = \mathsf{L}_{\mathbf{e}_{y'}}(a,\pi,u)\,k'$ with $k'\in\mathrm{GL}_{I'}(\hat{\mathcal{O}})$, and for the even length l' the rows X'^+ of the graded lattice basis are those of the identity. The first factor is by the hypothesis (lattices), applied to r_1, $r_2 = r_1^{-1}$ and the neighbour vectors $(b_j,\gamma_{\cdot j})$ of (3): the column j of $\mathrm{inc}_{\mathcal{R}}M$ is the vector v of that hypothesis.

The zero term. By the definitions of the splittings, $B = \mathsf{i}_f k + \mathsf{k}_f c'$ for the frames of $\Xi^{\mathrm{pr}}(\mathcal{R})$ and a suitable matrix c': the columns of M split into the part $(r_1\kappa_i\gamma_{ij},\kappa_i\gamma_{ij})$, which is the frame of $\langle X^+\rangle^-\oplus\langle X''^+\rangle$ applied to $\mathrm{diag}(\kappa)\gamma$, and the part along the second copy of H, which is the isotropic frame of the splitting that removes the planes. So Proposition 5.3 gives $C_3\in\mathbb{C}$ and, for every finite vector y, a number $\varsigma(y)$, zero or a root of unity, with $\mathrm{ev}^f_{\Xi^{\mathrm{pr}}(\mathcal{R})}(\varrho_f(y)\mathrm{vac}) = \varsigma(y)\,C_3\,\omega_{V_{\mathrm{st}}}(1,k)\,\varrho_f(B^*y)\mathrm{vac}$.

The string map disappears. $B^*\,\mathbf{T}_f(\mathcal{Y},\mathcal{Y}',\mathcal{Y}'') = 0$. Indeed $(\mathsf{J}^{\mathsf{T}}x_{V',f})^* = x_{V',f}^{-1}\mathsf{J}$, by $x_{V',f}^*J_{V'}x_{V',f} = 1$ and because $J_{V'}$ is the identity on X'^+; and $\mathbf{T}(\mathcal{Y},\mathcal{Y}',\mathcal{Y}'')\,\mathsf{J} = 0$, the string maps and their adjoints having non-zero columns only at string vectors.

Conclusion. Let n be coprime to N, p a proposition and $y\in\mathrm{Tr}_n[(i,i')\mapsto\beta^{\langle i\rangle}]$. By Definition 2.17, Input 1(a) and the two preceding paragraphs,

$$\begin{aligned}\mathrm{fev}_{\mathcal{R}}(\varrho_f(y)\mathrm{vac}) &= \varsigma'\ \mathrm{at}^\flat\ \mathrm{ev}^f_{\Xi^{\mathrm{pr}}(\mathcal{R})}(\varrho_f(\mathbf{T}_f(\mathcal{Y},\mathcal{Y}',\mathcal{Y}'')+y)\mathrm{vac})\\ &= \varsigma'\,\varsigma(y)\,C_3\cdot\mathrm{at}^\flat(\omega_{V_{\mathrm{st}}}(1,k)\,\varrho_f(B^*y)\mathrm{vac})\end{aligned}$$

with a root of unity ς'. By Lemma 3.3(a) the vector B^*y lies in $\mathrm{Tr}_n[(j,x')\mapsto\beta^{\langle j\rangle}\kappa_j]$; its condition on the valuations is checked below. So (4) applies to B^*y, and Lemma 3.1(c) and (e) carry the scalars C_3, $\varsigma(y)$, ς': the claim holds with the constant C_3C.

The condition on the valuations. Let (i, i') be a row and (j, x') a column of B, and w a finite place with $|\beta^{\langle i\rangle}|_w \neq |\beta^{\langle j\rangle}\kappa_j|_w$; we show that the first factor of the entry of B vanishes at $\bar{w}$. That factor is $\sum_{i_1} (x^{-1}_{V^-\oplus V'',f})_{ii_1} (\mathrm{inc}_{\mathcal{R}}\, M)_{i_1 j}$. Fix i_1 and use the hypothesis (weights) for i_1 and j. If $\mathrm{inc}_{\mathcal{R}}(i_1, j) = 0$: the place $\bar{w}$ does not lie above N, since above N all κ are 1 and β is a unit, so that the two absolute values would both be 1; hence, by (2), at $\bar{w}$ the b's vanish and γ is the identity, the column j of M has at $\bar{w}$ only the entry at the index j, and $(\mathrm{inc}_{\mathcal{R}}\, M)_{i_1 j}$ is a multiple of $\mathrm{inc}_{\mathcal{R}}(i_1, j) = 0$ there. Otherwise $|\beta^{\langle i_1\rangle}|_w = |\beta^{\langle j\rangle}\kappa_j|_w \neq |\beta^{\langle i\rangle}|_w$, so i_1 and i lie in different ones of the two sets I, I'', and $(x^{-1}_{V^-\oplus V'',f})_{ii_1} = 0$ because this matrix is block diagonal. □

LEAN `isAdmissibleEval_finEval`

9 Where the results are used

In the unit. The note on the seesaw identity uses: Theorem A and Lemma 3.1 wherever a denominator or a size is tracked (the unfolding to an admissible combination, the combination of reduced vacua, the passage to a good datum); Lemma 3.2(a) for the multiplicities of the finite cuts, which divide powers of the conductor of the twist; Theorem C, with the middle pairs and the last pair of Definition 2.18 for the gradings of the spaces of the chain of lifts, to identify the function left after $2i + 2$ unfoldings with a matrix coefficient of $i + 1$ pairs of definite lifts; Theorem B(ii) for the invariance of the evaluated data under the levels; Theorem D, Lemma 4.3 and Proposition 3.4, with Lemma 3.2(b) to (d), to write the evaluated data through reduced vacua; and Proposition 5.2 in the step that bounds how far the lines of a lattice move under one lift.

In other units. Admissible coefficients enter the definition of a good datum in the unit of the columns, and admissible combinations are produced in the unit of the CM curve (named data as combinations of translated vacua) and used in the units of the exchange identity and of the rows. Propositions 4.1 and 4.2 are used in the unit that constructs the spaces of the chain of lifts and in the unit of the filling: the first shows that the lattice spanned by the graded lattice basis is self-dual, the second provides the real part of the frame.

10 Notes on the formal text

Arguments. The lists of arguments used by the checks of this note are read from the text of the artifact, not printed by Lean. They contain entries that Lean does not take, as the reader's dossier records: an index type listed for $O^{(n)}$; an index type listed for admissible evaluations; and a first copy of the field instances for Proposition 8.1 and for Theorem D. The two index sets X'', H of an arrow pair are listed twice, as implicit and as explicit arguments, because a prefix makes them explicit for that declaration. For admissible evaluations the lists also carry a finite index type named like the proposition of Definition 2.21, with its finiteness: the definition binds that name as the proposition p and its translate sum has the three index sets X, X', X'' only, so these two entries are inferred from the text, as the second reader of the dossier notes; a signature printed by Lean would settle them. In the other direction, the text reading of Theorem C(ii) misses the field E, the index sets X, X', X_0 and their instances, which the formal statement names after a local

definition; they stand in the statement above. In Theorem A(i) the two hypotheses on d'' and C have default proofs by simplification, so that the formal text can omit them at a use.

Differences from the statements above. (1) Coprimality of a prime ℓ and n is the formal hypothesis in Theorem A(ii) and Definition 2.7; for $n = 0$ no prime is coprime to n, so $O^{(0)} = \mathbb{C}$. (2) The automorphisms σ are ring automorphisms of $\mathbb{C}$. (3) The ring homomorphism $\jmath$ is an instance argument in Theorem A(ii) and an ordinary one in Lemma 3.2(c), (d). (4) The finite zero term and the left inverse $\mathrm{at}^\flat$ depend on choices made once by the axiom of choice (an adapted element; a left inverse of an injective linear map). (5) The denominator D of Definition 2.21 is a cardinality that is 0 when infinite. (6) In Definition 2.16 the subtraction is that of natural numbers. (7) In Theorem C(ii) the hypothesis is the library's predicate on families of lifts, stated here as Input 5; the characters enter through the twist $\mu \mapsto \mu\upsilon^{\lfloor |X|/2 \rfloor}$. (8) Proposition 4.2 takes a ring homomorphism ϕ that only transports an integer matrix, and Proposition 4.1 a homomorphism $f\colon S \to R$ of the same kind. (9) The formal statements of Lemma 3.1(a) and of Theorem A(iii) are for a finite subset given as such, and for the iterated tensor product of the library, in which the factor of index $m - 1$ stands first. (10) In Lemma 3.1(h) the scalar is Mathlib's normalising unit, with its value 1 at zero; its description is recalled, Mathlib not being part of the text read. The scalar must not be simplified to d/C: nothing is assumed non-zero, and for $a = 0$ the vector of the conclusion is zero. (11) In Definition 2.9 and in Theorem D the two blocks of a vector carry the weights β and $\bar{\beta}$; they are never replaced by one common weight.

What the formal proofs do differently. The unitary matrix γ of Proposition 8.1 is given entry by entry in the formal text, and its unitarity and the properties of neighbour vectors are found by automation from a list of identities; the description $\gamma = 1 + (\lambda - 1)\xi\xi^*$ and the computations with it are ours. The comparison of the scalars in Theorem B(i) is, formally, a linear-arithmetic certificate; the count with the weights is ours. The identity of functions behind the key identity of Proposition 8.1, and the condition on the valuations in Theorem D, are long simplification chains in the formal text, which we read as the identities they establish and derived again by hand. Lemma 3.2(c), (d) are not expanded. Which hypotheses the formal proofs leave unused was not determined: nothing was compiled for this note.

The dossier and its second reading. The reader's dossier describes Input 3(e) without the factor $\upsilon(\det h)^{|H|}$. The formal statement has it, through a twist of one of the two characters, and the scalars of Theorem B(i) agree only with it. A second reader compared the dossier of the unit with the Lean text after this note was drafted. Her corrections that concern this note are followed here: the zero case of Lemma 3.1(h), where the dossier had simplified the scalar; the two weights of the blocks; the index types and the proposition in Definition 2.21; and, in Proposition 8.1, that the comparison of indicators leaves at most one term.

How this note was written. From the structured proofs of the reader's dossier for the eight files of this note, after its program check passed. We read these eight files of the Lean text completely, the statements argument by argument, and line by line the proofs of Theorem B(i), Proposition 8.1 and Theorem D, which the reader had marked as read from the names of the lemmas only; the two statements with a local definition (Theorem C) were read in full. Of the background library we read the statements and comments of the declarations quoted as Inputs 1 to 7, and none of their proofs. No Lean was run. The program check of the unit's dossier covered 503 leaf steps of 83 proofs.

Revision. After an independent referee's report the note was revised. Step 4 of the proof of Theorem C(i) now proves that no Heisenberg factor arises, from the isotropy of the stacked string map, which is where the condition $\mathfrak{c}(\mathcal{Y}') = \mathfrak{c}(\mathcal{Y})$ of an arrow pair enters, as in the formal proof; Inputs 2(h), 4(b) and 4(d) were added or restated from the Lean source; the constant term is written as a mean of values of the theta functional; and the names of the library's objects (string map, translate sum, finite Heisenberg operators) are those of the other notes of the unit.

Appendix A. Concordance with the Lean artifact

The first 57 rows are the declarations of the unit treated here; the rows whose place is an Input are declarations of the background library, quoted and not read.

In this note	Lean declaration	File	Lines
Definition 2.8	`IsAdmissibleComb`	AdmissibleComb.lean	12–13
Lemma 3.1	`IsAdmissibleComb.of_sum`	AdmissibleComb.lean	20–22
Lemma 3.1	`IsAdmissibleComb.single`	AdmissibleComb.lean	24–25
Theorem A	`IsAdmissibleComb.bind`	AdmissibleComb.lean	32–40
Lemma 3.1	`IsAdmissibleComb.map`	AdmissibleComb.lean	42–42
Lemma 3.1	`IsAdmissibleComb.smul`	AdmissibleComb.lean	44–46
Lemma 3.1	`IsAdmissibleComb.smul_of_eq_zero_or_isOfFinOrder`	AdmissibleComb.lean	48–50
Lemma 3.1	`IsAdmissibleComb.of_eq_isOfFinOrder_smul`	AdmissibleComb.lean	52–54
Lemma 3.1	`IsAdmissibleComb.map_bilinear`	AdmissibleComb.lean	56–59
Theorem A	`IsAdmissibleComb.smul_mem`	AdmissibleComb.lean	64–67
Theorem A	`IsAdmissibleComb.pure`	AdmissibleComb.lean	71–77
Lemma 3.1	`IsAdmissibleComb.normUnit_smul`	AdmissibleComb.lean	79–83
Definition 2.9	`selfOrConj`	AdmissibleEvaluation.lean	20–20
Lemma 3.3	`mulVec_mem_weightedTranslations`	AdmissibleEvaluation.lean	22–26
Lemma 3.3	`star_mem_weightedTranslations`	AdmissibleEvaluation.lean	28–30
Lemma 4.3	`inv_xFin_mulVec_mem_of_spansLattice`	AdmissibleEvaluation.lean	32–42
Definition 2.20	`IsNeighbourVector`	AdmissibleEvaluation.lean	60–62
Definition 2.21	`IsAdmissibleEval`	AdmissibleEvaluation.lean	67–72
Proposition 8.1	`exists_isAdmissibleEval_std`	AdmissibleEvaluation.lean	81–160
Theorem D	`isAdmissibleEval_finEval`	AdmissibleEvaluation.lean	167–215
Theorem C	`integral_theta_eq_lift₂`	DoubleLiftValue.lean	27–53
Theorem C	`ArrowPair.integral_constTermAtT_mul`	DoubleLiftValue.lean	58–118
Definition 2.12	`zeroTermFin`	EdgePair.lean	24–27
Proposition 5.1	`zeroTermFin_spec`	EdgePair.lean	28–37
Proposition 5.1	`zeroTermFin_finHeisOp_finFrameVector`	EdgePair.lean	39–41
Proposition 5.2	`zeroTermFin_support`	EdgePair.lean	46–54
Proposition 5.3	`zeroTermFin_finHeisOp_vacElem`	EdgePair.lean	58–75

In this note	Lean declaration	File	Lines
Definition 2.13	`GradeZeroPart`	EdgePair.lean	90–96
Lemma 6.1	`ratCoordMatrix_sum`	EdgePair.lean	99–100
Definition 2.13	`GradeZeroPart.splitting`	EdgePair.lean	107–114
Definition 2.14	`GradeZeroPart.hom`	EdgePair.lean	118–122
Definition 2.15	`stackedT`	EdgePair.lean	126–126
Definition 2.15	`stackedTFin`	EdgePair.lean	130–130
Lemma 6.1	`inclX_mul_stringMap_mul_inclX_inv`	EdgePair.lean	133–135
Definition 2.16	`ArrowPair`	EdgePair.lean	141–153
Definition 2.17	`ArrowPair.homZeroPart`	EdgePair.lean	159–160
Definition 2.17	`ArrowPair.splitting`	EdgePair.lean	164–164
Definition 2.17	`ArrowPair.finEval`	EdgePair.lean	168–169
Lemma 6.2	`ArrowPair.arrowTensor_finEval`	EdgePair.lean	171–174
Theorem B	`ArrowPair.zeroTerm_splitting_tripleWeil`	EdgePair.lean	179–207
Theorem B	`ArrowPair.finEval_inclXFin`	EdgePair.lean	211–225
Definition 2.18	`middlePair`	EdgePair.lean	237–245
Definition 2.18	`lastPair`	EdgePair.lean	249–256
Proposition 4.1	`isUnimodular_gradedLatticeBasis_gram`	FramesForY.lean	15–24
Proposition 4.2	`exists_frame_gradedGram`	FramesForY.lean	29–36
Lemma 3.2	`div_mem_integralAway`	IntegralAway.lean	21–23
Lemma 3.2	`localLevel_le_integralAbove_of_coprime`	IntegralAway.lean	26–27
Lemma 3.2	`Parameters.norm_finMultiplier_eq_one`	IntegralAway.lean	29–32
Lemma 3.2	`AutChar.HasWeight.norm_padic_det`	IntegralAway.lean	34–36
Definition 2.11	`gradedLatticeBasis`	LatticeAndTorsionDefs.lean	16–19
Definition 2.11	`Grading.SpansLattice`	LatticeAndTorsionDefs.lean	25–26
Definition 2.7	`integralAway`	LatticeAndTorsionDefs.lean	30–31
Definition 2.8	`IsAdmissibleCoeffs`	LatticeAndTorsionDefs.lean	35–38
Definition 2.9	`weightedTranslations`	LatticeAndTorsionDefs.lean	46–48
Definition 2.9	`translations`	LatticeAndTorsionDefs.lean	52–52
Definition 2.10	`IsSplitSquarefree`	LatticeAndTorsionDefs.lean	58–59
Proposition 3.4	`isAdmissibleComb_weil_vacElem`	MovedVacuumComb.lean	12–22
Input 1	`Space.finHeisOp_finHeisOp`	Projections.lean	32–33
Input 1	`isOfFinOrder_ψf`	AdditiveCharBasic.lean	15–16
Input 1	`Space.finHeisOp_vacElem`	Projections.lean	131–134
Input 1	`Space.finHeisOp_mem_finitePart`	FinitePart.lean	22–23
Input 1	`Space.omega_finHeisOp`	Projections.lean	61–62
Input 1	`Space.omega_integrated`	Projections.lean	91–96
Input 1	`Space.exists_rho_integrated_eq`	Projections.lean	53–55
Input 1	`Space.omega_vacElem_sum`	Projections.lean	215–220
Input 1	`norm_finMultiplier_sq`	FiniteMultiplier.lean	181–185
Input 1	`Space.exists_eq_smul_omega_vacElem`	FinitePart.lean	57–61
Input 1	`stdLattice.mem_stdLattice_iff_psi₀`	FiniteVacuum.lean	62–64
Input 1	`Space.projOn_finHeisOp`	Projections.lean	149–152
Input 2	`theta_arrowProduct`	TwoEdgesOnePair.lean	112–115
Input 2	`homHeisOp_arrowProduct`	TwoEdgesOnePair.lean	117–121
Input 2	`finHeisOp_arrowProduct`	TwoEdgesOnePair.lean	123–128
Input 2	`arrowProduct_twistedPairWeil`	TwoEdgesOnePair.lean	133–135
Input 2	`arrowTensor_map_liftWeilFin`	TwoEdgesOnePair.lean	151–153
Input 2	`leftInverse_arrowTensor`	TwoEdgesOnePair.lean	137–143
Input 2	`range_arrowTensor`	TwoEdgesOnePair.lean	145–149
Input 2	`theta_tripleWeil_rat_mul`	TwoEdgesOnePair.lean	155–158
Input 2	`pairWeil_homHeisOp`	TwoEdgesOnePair.lean	32–34
Input 3	`Space.Splitting.exists_isAdapted_and_frame_eq`	ZeroTerm.lean	272–288
Input 3	`Space.Splitting.IsAdapted.mixedTerm_omega_vacElem`	ZeroTerm.lean	615–619

In this note	Lean declaration	File	Lines
Input 3	`Space.Splitting.IsAdapted.exists_real_zeroTerm`	ZeroTerm.lean	395–405
Input 3	`Space.Splitting.zeroTerm_rho_iAdelic`	ZeroTerm.lean	567–568
Input 3	`Space.Splitting.exists_eq_finFrameVector`	ZeroTerm.lean	446–449
Input 3	`Space.Splitting.adjoint_mulVec_finFrameVector`	ZeroTerm.lean	441–443
Input 3	`zeroTerm_omega_of_inclMatrix`	TwoEdgesOnePair.lean	58–68
Input 3	`Space.Splitting.homTarget_zeroTerm_pairWeil`	PairSplitting.lean	314–319
Input 3	`Space.name_boxIndicator`	Names.lean	91–92
Input 3	`Space.name_schrodingerAct_boxIndicator`	Names.lean	85–87
Input 3	`Space.ω_name_on`	NameWeilAction.lean	194–202
Input 4	`Grading.integral_besselMeasure_eq_of_mean`	BesselIntegral.lean	112–144
Input 4	`Grading.IsGeneric.unipotent_mul_constTerm`	UnfoldingStep.lean	254–261
Input 4	`Grading.constTermAtT_rat_mul`	UnfoldingStep.lean	265–269
Input 5	`Space.isAutomorphic_algFormCoeff`	AlgebraicForms.lean	42–47
Input 6	`IsValuationPair.exists_isPrimitiveMatrix_mul`	FrameAction.lean	74–80
Input 7	`denseRange_polyVector`	FockHarmonicCoeff.lean	70–75
Input 7	`mem_rename_range_mul`	SmoothMatrices.lean	89–92
Input 7	`quotMean_quotMean`	PeriodicMean.lean	60–68
Input 7	`Nat.Coprime.norm_natCast_eq_one`	AdicRingHomExt.lean	102–103

Legend. Formal names of objects and our letters: *Space* with its matrix and point: a hermitian space V with J_V and the frame x_V; the data space, the Heisenberg and finite Heisenberg operators, the vacuum, the finite part: $\mathcal{S}(V)$, ϱ_V, $\varrho_{f,V}$, vac_V, $\mathcal{W}_f(V)$; *integrated*: $v \otimes f$; *Splitting* with its fields k, l, i: Ξ with k_Ξ, l_Ξ, i_Ξ; *zeroTerm*: ev_Ξ; *Grading* with *indexEquiv*, c, *gradeOf*, *inclX*, *stringMap*: $\mathcal{Y}$ with $\mathbf{e}_\mathcal{Y}$, $\mathfrak{c}(\mathcal{Y})$, $\mathrm{gr}_\mathcal{Y}$, $\imath_\mathcal{Y}$, $T_{\mathcal{Y},\mathcal{Y}'}$; *arrowProduct*, *arrowTensor*, *tripleWeil*, *liftWeilFin*: $\boxtimes$, at, $\omega^{\mathrm{tr}}_{\mu,\mu'}$, $\omega^{\mathrm{pr},f}_{\mu}$; the fields *zeroPart*, *embX*, *gradeOne* of an arrow pair: $\mathcal{Z}_\mathcal{R}$, $\mathrm{emb}_\mathcal{R}$, $\gamma_\mathcal{R}$; *normNegOne*: $\epsilon_{_}$; *translateSum*: $\mathsf{T}_{N,\Sigma}(X, X', X'')$.

Notation

E, $\mathbb{A}$, $\mathbb{A}_f$, $\hat{\mathcal{O}}$ a CM field with a CM type; its adeles, finite adeles, integral finite adeles

w, $\bar{w}$, $|\cdot|_w$, O_w a finite place, its conjugate, its absolute value, its ring of integers

$\mathbf{j}$, α, v, $\epsilon_{_}$ Weil data: an element of E, an idele, the splitting character; a unit of $\hat{\mathcal{O}}$ of norm -1

ℓ, $\jmath$, $O_\jmath$, σ a prime; a ring homomorphism $\mathbb{C} \to \mathbb{C}_\ell$; its ring of integral complex numbers; an automorphism of $\mathbb{C}$

$O^{(n)}$ complex numbers integral away from n

$\mathcal{X}$, $\mathcal{X}'$, $\mathcal{X}_j$, A, A', A_j index sets of families of vectors and subsets of them

Z, Z', W, F_i, M complex vector spaces; an $O_\jmath$-submodule (also: a matrix)

v, v', v_i, z, z', f_i families of vectors and vectors (also: v a real datum, f a datum or a map)

n, d, d', d'', m, k natural numbers: the modulus of integrality, denominators, a number of factors, an exponent (also: k a group element, d a family of adeles)

p, B, B', C, C_0, C_1, C_2, C_3, D a proposition; real bounds; complex constants; a denominator

r, r_x, c, ρ, ς coefficient functions; a root of unity or zero (also: r a group element, c an adele or matrix)

$r \in \mathrm{Co}(A; n, d; p, B)$ admissible coefficients

$z \in \mathrm{Adm}(v, A; n, d; p, B)$ admissible combination

nu, L, b the inverse or 1; a linear map; a bilinear map (also: b an index or a vector)

I, I', I'', I_0, I_1, I_2, X, X', X'', X_0, H, T, U finite index sets; a set of parameters o

X^+, pt, pt_1, pt_2 an index set with one more point; the points

V, V', V'', V_0, V_{st}, J_V, x_V, $x_{V,f}$ hermitian spaces; matrix, frame, finite frame

$G(V)$, $G_f(I)$, $\Gamma(V)$, $[G(V)]$ the adelic unitary group, its finite part, rational points, quotient

$\langle X\rangle$, $\mathrm{Hom}(V,V')$, $V^{-}\oplus V''$, $\mathbb{H}^{H}\oplus V$ standard space; homomorphisms; first form negated; with hyperbolic planes

$\mathcal{S}(V)$, $\mathcal{F}(V)$, $\mathcal{W}_{f}(V)$, vac_{V}, v_{Q}, Q data, real data, finite part, vacuum, polynomial vectors, polynomials

ϱ_{V}, $\varrho_{f,V}$, ω_{V}, θ_{V}, Δ_{f}, ψ_{0}, ψ_{f}, P^{S} Heisenberg and finite Heisenberg operators, Weil representation, theta functional, finite multiplier, additive characters, a projection

y, y_{f}, $y^{\dagger}$, x, u, g, g', h, a vectors and matrices, finite coordinates, adjoint; group elements (also: a an idele, u a unitary adele)

μ, μ', μ_{V}, μ_{o} characters of $\mathrm{U}(1)(\mathbb{A})$; twisted by a power of v

$\omega_{\mu}^{\mathrm{pr}}$, $\omega_{\mu}^{\mathrm{pr},f}$, $\omega_{\mu,\mu'}^{\mathrm{tr}}$, $\omega_{a,b}$ twisted pair action, its finite part, the triple action; the pair action with two characters of the note on the Gauss-sum operator

$\boxtimes$, at, $\mathrm{at}^{\flat}$ arrow product; the induced map on finite parts; a left inverse

Ξ, k_{Ξ}, l_{Ξ}, i_{Ξ}, ev_{Ξ}, ev_{Ξ}^{f} a splitting, its frames, its zero term, its finite zero term

Λ, ϵ, v_{vac}, v_{Ξ} objects of Input 3: a map of real data, a function, the real vacuum and its translate

$\mathcal{Y}$, $\mathcal{Y}'$, $\mathcal{Y}''$, $\mathrm{e}_{\mathcal{Y}}$, l, l', l'', N, $\mathfrak{c}(\mathcal{Y})$ gradings: bijection, string lengths, norm, constant

$\mathrm{gr}_{\mathcal{Y}}$, $\imath_{\mathcal{Y}}$, $\imath_{\mathcal{Y}}^{f}$, $B(\mathcal{Y})$, $[B(\mathcal{Y})]$, $T_{\mathcal{Y},\mathcal{Y}'}$, $\mathrm{sh}_{\mathcal{Y}}$ grade, inclusion of the definite part, Bessel group and quotient, string map, shift

$\mathrm{ct}_{\mathcal{Y},\mathcal{Y}'}^{\mu}(f)$, P, $\mathrm{Gr}_{\mathrm{e}}(N)$, $\mathrm{bi}(x,y)$, $\psi_{\mathbb{A}}$ constant term at the string map; a generic function; the Gram matrix of a string; the Heisenberg bicharacter; the standard additive character of the adeles

$\mathrm{Tr}_{n}[d]$, $\mathrm{Tr}_{n}^{I}[d]$, $\beta^{\langle j\rangle}$ groups of translations; β or its conjugate

$\mathsf{SqF}_{\Sigma}(N;\pi)$ the split squarefree condition

$\mathsf{L}_{\mathrm{e}}(a,\pi,u)$, $\mathsf{Lat}(\mathcal{Y};a,\pi,u)$ graded lattice basis; the lattice condition

R, R_{0}, S, ϕ, ε, η_{b}, e_{i}, g_{i} rings and a homomorphism; signs; basis vectors and columns

wt, $\mathrm{GZ}(V,V'';\mathrm{wt})$, $\mathcal{Z}$, $\mathrm{ind}_{\mathcal{Z}}$, $\mathrm{inc}_{\mathcal{Z}}$, $\Xi(\mathcal{Z})$, $\mathrm{Hom}(\mathcal{Y}',\mathcal{Z})^{0}$ weights; grade-zero parts with injection and matrix; their splitting; the part of a space of homomorphisms

$\mathbf{T}(\mathcal{Y},\mathcal{Y}',\mathcal{Y}'')$, $\mathbf{T}_{f}(\mathcal{Y},\mathcal{Y}',\mathcal{Y}'')$ the stacked string map and its finite coordinates

$\mathrm{Arr}_{X'',H}(\mathcal{Y},\mathcal{Y}',\mathcal{Y}'')$, $\mathcal{R}$, $\mathcal{Z}_{\mathcal{R}}$, $\mathrm{inc}_{\mathcal{R}}$, $\mathrm{ind}_{\mathcal{R}}$, $\mathrm{emb}_{\mathcal{R}}$, $\gamma_{\mathcal{R}}$ arrow pairs and their components

$\mathcal{Z}^{\mathrm{Hom}}(\mathcal{R})$, $\Xi^{\mathrm{pr}}(\mathcal{R})$, $\mathrm{fev}_{\mathcal{R}}$ grade-zero part, splitting and finite evaluation of an arrow pair

$\mathcal{R}_{\mathrm{mid}}(\mathcal{Y},\mathcal{Y}',\mathcal{Y}'')$, $\mathcal{R}_{\mathrm{last}}(\mathcal{Y},\mathcal{Y}',\mathcal{Y}'';\pi_{0})$, π_{0} the middle pair and the last pair

$\varkappa$, e, e', e'', $\mathcal{A}(X;\varkappa,e)$, φ, t, t', t'', coef_{t} weight and determinant exponents; algebraic forms; functionals; coefficients

$\mathcal{L}_{o}$, $\mathcal{L}_{o}'$, $\mathcal{L}_{o}''$, Φ, Φ', Ψ, $\mathcal{I}$ theta lifts and their composite; finite data; a function that names an operator; an integral

Σ, $\bar{\Sigma}$, π, δ, β, κ a split level and its conjugate; the objects of Setting 2.19

$\mathsf{Nb}_{\pi,\delta}(c;x)$ neighbour vector

$\mathsf{AdmEv}_{N,\Sigma,\beta}(L;d)$, $\mathsf{T}_{N,\Sigma}(X,X',X'')$, $\tilde{y}$ admissible evaluation; the translate sum; least non-negative representatives

γ, $\gamma_{X'}$, λ, ξ, $\mathbf{1}_{\Sigma}$, ϖ_{0}, $\mathrm{n}(y)$, k_{0}, B, J, c objects of the proofs of Proposition 8.1 and Theorem D

z_{0}, F, M, m_{+}, j, i, q, G, K, O, Y, s further letters of the proofs: constants, functions, counts and indices; G, O, Y, K, s, q occur only inside the symbols above

A, A', B topological abelian groups in Input 7

READBACK NOTE 23

Chains of definite theta lifts and the Gauss-sum operator at the level prime

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. At the level prime N, which splits completely in the CM field, a theta lift between definite unitary groups is governed by functions of residue matrices modulo N. This note isolates the formal law that one lift obeys there, defines the *Gauss-sum operator* F, a finite Fourier transform in the last row followed by $\mathrm{diag}(1,\dots,1,N^{-1})$, and proves that F slides through a chain of lifts (Theorem A). Theorem B rewrites the sums of translates with denominator N that the seesaw identity produces as data of exactly this kind. Theorem C turns a jointly invariant combination of chains into a plain sum over classes, and Theorem D bounds, at an embedding above N, what powers of F cost in integrality.

What this note is. The artifact grh-compact is a body of Lean code whose final theorems state the Riemann hypothesis and its analogue for Dirichlet L-functions. Its authors report that all files compile and that Lean's kernel accepts these theorems from its standard axioms; nothing was compiled for this note, and no mathematician has read the argument through. The headline is therefore a *claim*, and this note does not say that it is proved: it says what one part of the formal text asserts and how it argues. The part is 78 of the 312 declarations of the unit *Seesaw*: its files on names at a split level and on the characters of the lifts, which serve Theorem A of the write-up (short forms). The other declarations of the unit are treated in further notes of the unit (the index sets and characters of the data; two consecutive lifts; the seesaw identity). Every statement below was compared, argument by argument, with its Lean text, and every proof is written out, except where an *Input* from another unit or a quoted result of the background library is named with its owner. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 78 declarations.

1 The statements

The theorems are stated in the settings that follow. The notions they use are defined in §2; the place of each definition is given in the setting.

Setting 1.1 (a split level)**.** E is a CM field with a CM type and Weil data, N is a prime number, and Σ is a finite set of finite places of E that is a split level for N (Definition 2.1): Σ is disjoint from its complex conjugate, each of its places has norm N and uniformiser N, and the places of E dividing N are those of Σ and their conjugates. We write $E_\Sigma = \prod_{w\in\Sigma} E_w$, O_Σ for its ring of integral elements and $\mathrm{red}\colon O_\Sigma \to (\mathbb{Z}/N)^\Sigma$ for the reduction modulo N (Definition 2.2). For a finite index set I we put $I^+ = I \sqcup \{*\}$, with $*$ the *last index*, and $\mathbf{e}$ is the last basis vector of $((\mathbb{Z}/N)^\Sigma)^{I^+}$. Functions f of matrices over $(\mathbb{Z}/N)^\Sigma$ give functions $f^\natural$ of matrices over E_Σ; among them are the indicators $\mathbf{b}_y$ of a last column and the characters $\mathbf{c}_y$ of a last

row (Definition 2.4). For a complex representation ρ of $\mathrm{GL}_{I^+}(E_\Sigma)$, $\mathrm{Ram}(\rho)$ is its set of ramified vectors (Definition 2.3) and $\mathsf{F}(\rho)$ its Gauss-sum operator (Definition 2.6); u_0 is the matrix $\mathrm{diag}(1,\dots,1,N^{-1})$ at every place of Σ (Definition 2.3).

Theorem A (the Gauss-sum operator slides through a chain of lift laws)**.** In Setting 1.1, let $I_0, I_1, I_2, \dots$ be finite index sets, let ρ_j be a complex representation of $\mathrm{GL}_{I_j^+}(E_\Sigma)$ for each $j \geq 0$, and let L_j and $\varkappa_j \in \mathbb{C}$ be such that, for every j, L_j is a lift law from ρ_j to ρ_{j+1} with constant $\varkappa_j$ (Definition 2.7). Let $\varphi \in \mathrm{Ram}(\rho_0)$, let $d_0, d_1, \dots$ be complex numbers and let $p \geq 0$. For $i \geq 0$ put

$$A_i \;=\; d_i \cdot \rho_{2i+2}(u_0) \circ \sum_y L_{2i+1}[\mathbf{c}_y] \circ L_{2i}[\mathbf{b}_y] \;:\; \rho_{2i} \to \rho_{2i+2},$$

the sum over all $y \in ((\mathbb{Z}/N)^\Sigma)^{I^+_{2i+1}}$, and let o_i be the number of those y that are non-zero at every place: for every $w \in \Sigma$ some entry of y has non-zero component at w. Then

$$A_{p-1} \circ \dots \circ A_0\,(\varphi) \;=\; \Big(\prod_{i<p} d_i\, o_i \prod_{l<2i+2} \varkappa_l\Big) \cdot L_{2p-1}[\mathbf{b}_{\mathbf{e}}] \circ \dots \circ L_0[\mathbf{b}_{\mathbf{e}}](\mathsf{F}(\rho_0)^{\,p}\varphi).$$

Theorem B (translates with denominator N are named data)**.** In Setting 1.1, let Σ' be a second finite set of finite places of E that is a split level for N, let I, I', I'' be finite index sets, and put $\mathcal{H} = \mathrm{Hom}(X_{I^+}, X_{I'})$ and $\mathcal{H}' = \mathrm{Hom}(X_{I'}, X_{I''^+})$, where X_J is the standard definite hermitian space on an index set J, and let $v \in \mathcal{W}_f(\mathcal{H})$ and $v' \in \mathcal{W}_f(\mathcal{H}')$ be elements of the finite parts of their Weil representations (Definition 2.12). Then

$$\begin{aligned}&\mathsf{T}_{N,\Sigma'}(I, I', I'')(\mathrm{pr}^{\Sigma'}_{\mathcal{H}}\, v \otimes \mathrm{pr}^{\Sigma'}_{\mathcal{H}'}\, v') \\ &\quad= \sum_z \mathrm{nm}_\Sigma(S \mapsto \mathbf{b}_z(S \cdot \mathrm{dg}_I(\Sigma')))\, v \otimes \mathrm{nm}_\Sigma(S' \mapsto \mathbf{c}_z(\mathrm{dg}_{I''}(\Sigma \smallsetminus \Sigma') \cdot S'))\, v',\end{aligned}$$

the sum over all $z \in ((\mathbb{Z}/N)^\Sigma)^{I'}$. Here $\mathsf{T}_{N,\Sigma'}(I, I', I'')$ is the sum of the pairs of translations by vectors with denominator N at Σ', $\mathrm{pr}^{\Sigma'}_{\mathcal{H}}$ the projection of $\mathcal{W}_f(\mathcal{H})$ to the vectors that are vacua above $\Sigma' \cup \overline{\Sigma'}$, $\mathrm{nm}_\Sigma(F)$ the operator named at Σ by a function F of matrices over E_Σ (Definition 2.12), and $\mathrm{dg}_I(\Sigma')$ the diagonal matrix with last entry N at the places of $\Sigma \cap \Sigma'$ (Definition 2.3); the matrices $\mathrm{dg}_I(\Sigma')$ and $\mathrm{dg}_{I''}(\Sigma \smallsetminus \Sigma')$ are formed for the split level Σ.

Setting 1.2 (a chain of kernels)**.** E is a CM field with a CM type. For $j \geq 0$: I_j is a finite index set and $G_f(I_j)$ the group of finite adelic points of the unitary group of the standard definite hermitian space on I_j; κ is a weight and e_j a last entry, so that $\mathcal{A}(I_j; e_j)$ is the space of algebraic modular forms of weight (κ, e_j) on $G_f(I_j)$, with values in the space $W_\kappa(I_j)$ of the weight; K_j is a level (a compact open subgroup of the stabiliser $K_0(I_j)$ of the standard lattice), $\mathcal{S}_j$ is a complex vector space, and $\mathcal{K}_j$ is a theta kernel from I_j to I_{j+1} with weights (κ, e_j), (κ, e_{j+1}), level K_j and data in $\mathcal{S}_j$ (Definition 2.8), with its representation ω_j of $G_f(I_j) \times G_f(I_{j+1})$ on $\mathcal{S}_j$ and its jet matrix $\mathcal{J}_j$. The chain $\mathrm{ch}(\mathcal{K}; \Phi)_j \colon \mathcal{A}(I_0; e_0) \to \mathcal{A}(I_j; e_j)$ of the first j lifts with data $\Phi = (\Phi_j)_j$, $\Phi_j \in \mathcal{S}_j$, is defined in Definition 2.9.

Theorem C (a jointly invariant combination of chains is a sum over classes)**.** In Setting 1.2, let M be a complex vector space, $\mathcal{X}$ a set, $m \geq 0$ a natural number, $\mathcal{X}_0 \subseteq \mathcal{X}$ a finite subset, $L_x \colon W_\kappa(I_m) \to M$ a linear map and $\Phi_x = (\Phi_{x,j})_j$ a family of data, for each $x \in \mathcal{X}$. Let $\varphi \in \mathcal{A}(I_0; e_0)$, let $P_j(g,u)$ be a property of a point $g \in G_f(I_j)$ and a vector $u \in W_\kappa(I_j)$ for each j, let $h \in G_f(I_m)$ and let $A \subseteq M$. Assume:

(a) the family $(L_x, \Phi_x)_{x \in \mathcal{X}_0}$ is jointly invariant of length m for $\mathcal{X}$ (Definition 2.10);

(b) for every $j < m$ and every $g \in G_f(I_j)$, the group of rational points in gK_jg^{-1} is trivial;

(c) φ is fixed by K_0 under right translation;

(d) for all $x \in \mathcal{X}_0$, $j < m$, points $g \in G_f(I_j)$, $g' \in G_f(I_{j+1})$ and vectors u: $P_j(g,u)$ implies $P_{j+1}(g', \mathcal{J}_j(\omega_j(g,g')\Phi_{x,j})\, u)$;

(e) $P_0(g, \varphi(g))$ for every $g \in G_f(I_0)$;

(f) for every linear map $\beta \colon \mathcal{S}_{m-1} \otimes \cdots \otimes \mathcal{S}_0 \to W_\kappa(I_m)$: if $P_m(h, \beta(\Phi_{x,m-1} \otimes \cdots \otimes \Phi_{x,0}))$ for all $x \in \mathcal{X}_0$, then $\sum_{x \in \mathcal{X}_0} L_x(\beta(\Phi_{x,m-1} \otimes \cdots \otimes \Phi_{x,0})) \in A$.

Then $\sum_{x \in \mathcal{X}_0} L_x(\mathrm{ch}(\mathcal{K}; \Phi_x)_m(\varphi)(h))$ is a sum of $\prod_{j<m} \mathrm{cl}(K_j)$ elements of A, where $\mathrm{cl}(K_j)$ is the number of double cosets of $G_f(I_j)$ modulo the rational points on the left and K_j on the right.

Setting 1.3 (the data at the level prime)**.** $P = (N_0, d)$ are parameters and D is admissible data for P (Chapter III), with components: the scales x, x_1, b, the exponent μ, the number of lifts m, the rank r, the thick set, the primes g_0, g_1 and the integer π_g. Attached to D are the CM field $E(D)$ with its number $f(D)$ of infinite places, the level prime $N(D)$ and the integer $\pi_N(D)$ of $E(D)$ with $\pi_N(D)\overline{\pi_N(D)} = N(D)$, the set $\Pi(D)$ of places dividing $\pi_N(D)$, the number $\varrho(D)$ with $m = 2\varrho(D) + 2$, the index sets $\underline{X}_j(D) = \underline{X}_j^\circ(D) \sqcup \{*\}$ of the standard definite spaces $X_j(D)$, the spaces $\mathrm{St}_j(D)$ of algebraic forms on $X_j(D)$ (stage j), the level $K_j^{\mathrm{col}}(D)$ of stage j, the space $\mathrm{Col}(D) \subseteq \mathrm{St}_0(D)$ of columns and its subspace $\mathrm{Col}^{\mathrm{ram}}(D)$ of ramified columns (Definition 2.13, which recalls them from the notes on the data and on the columns). Moreover $\jmath \colon \mathbb{C} \to \mathbb{C}_{N(D)}$ is a ring homomorphism, $O_\jmath$ is the ring of the complex numbers z with $|\jmath(z)| \leq 1$, $\Sigma_\jmath(D)$ is the set of non-induced places above $N(D)$, a split level for $N(D)$, $\Delta_j(D)$ is the diagonal point of stage j and $\rho_{D,j}$ the representation of $\mathrm{GL}_{\underline{X}_j(D)}(E_{\Sigma_\jmath(D)})$ on $\mathrm{St}_j(D)$ (Definition 2.15); $\mathrm{Int}_\jmath(\mathcal{V}; \mathsf{d})$ is the $O_\jmath$-module of the forms of a space $\mathcal{V}$ that are $\jmath$-integral at a point d (Definition 2.14).

Theorem D (powers of the Gauss-sum operator on integral columns)**.** In Setting 1.3, with the structure of a split level for $N(D)$ on $\Sigma_j(D)$ given by Lemma 7.1, there is a natural number e with

$$e \;\le\; 4\,m\cdot f(D)\cdot b\cdot \dim_{\mathbb{C}}\mathrm{Col}(D)$$

such that for every $n\ge 0$ and all $\Phi_1,\dots,\Phi_n\in \mathrm{Int}_j(\mathrm{Col}(D);1)$ there is an $n\times n$ matrix u over O_j whose determinant divides $N(D)^{\,e}$ in O_j and for which every $\psi_i=\sum_k u_{ik}\Phi_k$ satisfies

$$\mathsf{F}(\rho_{D,0})^{\,\varrho(D)+1}\,\psi_i \;\in\; \mathrm{Int}_j(\mathrm{Col}(D);\Delta_0(D)).$$

2 The setting

Conventions. E is a CM field with a fixed CM type; $x\mapsto\overline{x}$ is its complex conjugation. $\mathbb{A}_f$ is the ring of finite adeles of E and $\hat{\mathcal{O}}$ its ring of integers. Finite places of E are written w, v; $\overline{w}$ is the conjugate place, E_w the completion with ring of integers O_w, residue field $\mathbb{F}_w$ and norm $\mathrm{N}w=\#\mathbb{F}_w$. Whenever the additive character ψ_0 or a Weil representation occurs, E carries Weil data: an element $\mathbf{j}$ of E with $\overline{\mathbf{j}}=-\mathbf{j}$, a finite idele α and a splitting character υ; then $\psi_0(t)=\psi_f(\mathbf{j}\alpha\overline{\alpha}\,t)$, with ψ_f the standard additive character of the finite adeles, is an additive character of $\mathbb{A}_f$ that is trivial on $\hat{\mathcal{O}}$ and whose values are roots of unity. An infinite place τ of E is identified with the embedding $E\to\mathbb{C}$ of the CM type above it. N is a prime number; Σ and Σ' are finite sets of finite places of E. That such a set is a split level is a hypothesis; it is stated in each statement or in the setting it cites, and Lean carries it as an instance argument. Index sets I, I', I'', I_j, J are finite index sets, and $I^+=I\sqcup\{*\}$. R is a commutative ring. Representations are on complex vector spaces. For a finite index set I, X_I is the standard definite hermitian space E^I with the form $\sum_i\overline{x_i}y_i$, $G_f(I)$ the group of finite adelic points of its unitary group, and $K_0(I)=G_f(I)\cap\mathrm{GL}_I(\hat{\mathcal{O}})$. In Setting 1.2: the $\mathcal{S}_j$ are the data spaces, complex vector spaces; the K_j are levels; κ is the weight and the e_j are the last entries; Φ denotes data; M is a complex vector space (so also a rational one, which the class integral needs). In Setting 1.3: $P=(N_0,d)$ are parameters, D is admissible data, ℓ is a prime number and $\jmath\colon\mathbb{C}\to\mathbb{C}_\ell$ a ring homomorphism, with $\ell=N(D)$ where the statement says so.

Split levels.

Definition 2.1 (split level; residues over N)**.** Let Σ be a finite set of finite places of E and N a natural number. We say that Σ is a split level for N if: (a) $\overline{w}\notin\Sigma$ for every $w\in\Sigma$; (b) $\mathrm{N}w=N$ for every $w\in\Sigma$; (c) N is a uniformiser of E_w for every $w\in\Sigma$; (d) a finite place w of E divides N if and only if $w\in\Sigma$ or $\overline{w}\in\Sigma$. For any Σ and N put $E_\Sigma=\prod_{w\in\Sigma}E_w$ and let $O_\Sigma=\prod_{w\in\Sigma}O_w$ be its ring of integral elements. For $N>0$ and $y\in(\mathbb{Z}/N)^\Sigma$ let $\tilde{y}_w\in\{0,\dots,N-1\}$ be the least non-negative representative of

y_w; then $[y/N] \in \mathbb{A}_f$ is the finite adele with component $\tilde{y}_w/N$ at each $w \in \Sigma$ and 0 at the other places, and, for E with Weil data,

$$\psi_N(y) \;=\; \psi_0(-[y/N]).$$

LEAN `IsSplitLevel` · `residueOverN` · `residueOverN_apply` (the formula for the components, a restatement of the definition) · `IsSplitLevel.psiN`

Definition 2.2 (reduction modulo N)**.** Let Σ is a split level for N, with N prime; we say for short that Σ is a split level. For $w \in \Sigma$ the residue field $\mathbb{F}_w$ has N elements by (b), so there is exactly one ring isomorphism $\mathbb{F}_w \to \mathbb{Z}/N$. The *reduction* $\mathrm{red}\colon O_\Sigma \to (\mathbb{Z}/N)^\Sigma$ is the ring homomorphism whose component at w is the residue map $O_w \to \mathbb{F}_w$ followed by this isomorphism. For a finite index set I we regard $\mathrm{GL}_I(O_\Sigma)$ as a subgroup of $\mathrm{GL}_I(E_\Sigma)$, write again $\mathrm{red}\colon \mathrm{GL}_I(O_\Sigma) \to \mathrm{GL}_I((\mathbb{Z}/N)^\Sigma)$ for the entrywise reduction, and fix a map $\mathrm{sec}\colon \mathrm{GL}_I((\mathbb{Z}/N)^\Sigma) \to \mathrm{GL}_I(O_\Sigma)$ with $\mathrm{red} \circ \mathrm{sec} = \mathrm{id}$. It exists: an entry $y \in (\mathbb{Z}/N)^\Sigma$ lifts to $(\tilde{y}_w)_w \in O_\Sigma$, and a matrix over O_Σ whose reduction is invertible is invertible, because an element of O_Σ with invertible reduction is a unit. The *last basis vector* $\mathbf{e} \in ((\mathbb{Z}/N)^\Sigma)^{I^+}$ is the function $(i, w) \mapsto 1$ if $i = *$ and $(i, w) \mapsto 0$ otherwise, for $i \in I^+$ and $w \in \Sigma$.

LEAN `IsSplitLevel.residueEquiv` · `IsSplitLevel.reduction` · `IsSplitLevel.inclGL` · `IsSplitLevel.reductionGL` · `IsSplitLevel.liftGL` · `IsSplitLevel.lastBasisVector`

Definition 2.3 (diagonal elements, the mirabolic group, ramified vectors)**.** Let Σ be a split level for the prime N, I a finite index set and Σ' a finite set of finite places. Then $\mathrm{dg}_I(\Sigma') \in \mathrm{GL}_{I^+}(E_\Sigma)$ is the diagonal matrix with entries 1 at the indices of I and, at the last index, the element of E_Σ whose component is N at the $w \in \Sigma \cap \Sigma'$ and 1 at the other $w \in \Sigma$. We put

$$u_0 \;=\; \mathrm{dg}_I(\Sigma)^{-1},$$

the matrix $\mathrm{diag}(1, \dots, 1, N^{-1})$ at every place of Σ (we write u_0 in every rank). The *mirabolic group* $\mathrm{Mir}(I) \subseteq \mathrm{GL}_{I^+}(E_\Sigma)$ consists of the $k \in \mathrm{GL}_{I^+}(O_\Sigma)$ with $\mathrm{red}(k)\,\mathbf{e} = \mathbf{e}$, that is, whose last column is congruent to $\mathbf{e}$ modulo N. For a complex representation ρ of $\mathrm{GL}_{I^+}(E_\Sigma)$, a vector φ of ρ is *ramified* if for every $w \in \Sigma$ there is a unit x of O_Σ with $\mathrm{red}(x)_v = 1$ for all $v \in \Sigma$, $v \neq w$, such that φ lies in the image of $1 - \rho(x \cdot 1)$, where $x \cdot 1$ is the scalar matrix; $\mathrm{Ram}(\rho)$ is the set of ramified vectors.

LEAN `IsSplitLevel.diagN` · `IsSplitLevel.diagNInv` · `IsSplitLevel.mirabolic` · `IsSplitLevel.ramified`

Definition 2.4 (functions of residue matrices)**.** Let Σ be a split level for the prime N and I, I' finite index sets. For a complex function f of $I' \times I$ matrices over $(\mathbb{Z}/N)^\Sigma$, $f^\natural$ is the function of $I' \times I$ matrices S over E_Σ with $f^\natural(S) = f(\mathrm{red}\, S)$ if all entries of S lie in O_Σ, and $f^\natural(S) = 0$ otherwise; $f \mapsto f^\natural$ is $\mathbb{C}$-linear. Three families are used:

(i) $\mathbf{r}$, on $I'^+ \times I^+$ matrices: the indicator of the integral S whose last row is congruent to $\mathbf{e}$;

(ii) $\mathbf{b}_y$, for $y \in ((\mathbb{Z}/N)^\Sigma)^{I'}$, on $I' \times I^+$ matrices: the indicator of the integral S whose last column is congruent to y;

(iii) $\mathbf{c}_y$, for $y \in ((\mathbb{Z}/N)^\Sigma)^I$, on $I'^+ \times I$ matrices, for E with Weil data: for integral S with last row $r = (r_i)_{i \in I}$ modulo N,

$$\mathbf{c}_y(S) = \psi_N(\textstyle\sum_{i \in I} r_i y_i),$$

and $\mathbf{c}_y(S) = 0$ for S not integral.

LEAN `IsSplitLevel.liftFn` · `IsSplitLevel.lastRowDatum` · `IsSplitLevel.lastColDatum` · `IsSplitLevel.charDatum`

Lemma 2.5. *Let I be a finite index set and R a commutative ring. There is a map k from R^{I^+} to $\mathrm{GL}_{I^+}(R)$ such that every vector y whose last entry y_* is a unit of R is both the last column and the last row of $k(y)$.*

Proof. Let y have $y_* \in R^\times$ and let C be the identity matrix with its last column replaced by y. Its determinant is y_*, and its last row is $(0, \dots, 0, y_*)$. Hence $(CC^{\mathsf T})_{i*} = \sum_a C_{ia} C_{*a} = y_i\, y_*$, the matrix $CC^{\mathsf T}$ is symmetric, and $k(y) = y_*^{-1} CC^{\mathsf T}$ is invertible with last column and last row y. For the other y put $k(y) = 1$. □

LEAN `IsSplitLevel.exists_GL_last_col_and_row` · In the Lean statement the variable y of the surrounding section is listed as an argument and not used: the statement quantifies over its own y.

Definition 2.6 (the Gauss-sum operator)**.** Let E have Weil data, let Σ be a split level for the prime N, I a finite index set and ρ a complex representation of $\mathrm{GL}_{I^+}(E_\Sigma)$. With k the map of Lemma 2.5 for $R = (\mathbb{Z}/N)^\Sigma$ (the one chosen by the formal text) and $\tilde{k}(y) = \sec(k(y))$,

$$\mathsf{F}(\rho) \;=\; \rho(u_0) \circ \sum_y \psi_N(y_*) \cdot \rho(\tilde{k}(y)),$$

the sum over the $y \in ((\mathbb{Z}/N)^\Sigma)^{I^+}$ whose last entry y_* is a unit of $(\mathbb{Z}/N)^\Sigma$. It is an endomorphism of the space of ρ: a finite Fourier transform in the last row and column, followed by u_0.

LEAN `IsSplitLevel.fourierOp`

Definition 2.7 (lift law)**.** Let E have Weil data, let Σ be a split level for the prime N, I and I' finite index sets, ρ and ρ' complex representations of $\mathrm{GL}_{I^+}(E_\Sigma)$ and $\mathrm{GL}_{I'^+}(E_\Sigma)$, L a $\mathbb{C}$-linear map from the complex functions of $I'^+ \times I^+$ matrices over E_Σ to the linear maps $\rho \to \rho'$ (we write $L[F]$ for its value at a function F), and $\varkappa \in \mathbb{C}$. We say that L is a lift law from ρ to ρ' with constant $\varkappa$ if

(i) (*integral equivariance*) for all $a \in \mathrm{GL}_{I^+}(O_\Sigma)$, $b \in \mathrm{GL}_{I'^+}(O_\Sigma)$, every function f of $I'^+ \times I^+$ matrices over $(\mathbb{Z}/N)^\Sigma$ and every vector φ of ρ:

$$\rho'(b)\Big(L\big[(T \mapsto f(\mathrm{red}(b)\, T\, \mathrm{red}(a)))^\natural\big](\rho(a)\varphi)\Big) \;=\; L[f^\natural]\,\varphi;$$

(ii) (*the law of u_0*) $\rho'(u_0)(L[\mathbf{r}]\,\varphi) = \varkappa \cdot L[\mathbf{b}_{\mathrm{e}}](\rho(u_0)\varphi)$ for every vector φ of ρ.

LEAN `IsSplitLevel.LiftLaw`

Kernels and chains.

Definition 2.8 (theta kernels and their lifts; recalled from the background library)**.** In Setting 1.2, a *theta kernel* $\mathcal{K}$ from I to I', with weights (κ, e), (κ, e'), level $K \subseteq G_f(I)$ and data in a complex vector space $\mathcal{S}$, consists of a representation ω of $G_f(I) \times G_f(I')$ on $\mathcal{S}$ in which every vector has an open stabiliser, and a linear map $\mathcal{J}$, the *jet matrix*, from $\mathcal{S}$ to the linear maps $W_\kappa(I) \to W_\kappa(I')$, such that $\mathcal{J}(\omega(\gamma, \delta)\Phi)(\gamma u) = \delta\, \mathcal{J}(\Phi)(u)$ for all rational points γ, δ of the two unitary groups,

acting on $W_\kappa(I)$, $W_\kappa(I')$ through the weights. Its *lift* with datum $\Phi \in \mathcal{S}$ is the linear map $\mathcal{A}(I;e) \to \mathcal{A}(I';e')$,

$$\mathcal{K}[\Phi](\varphi)(h) \;=\; \int_{\Gamma\backslash G_f(I)} \mathcal{J}\big(\omega(g,h)\Phi\big)\big(\varphi(g)\big)\,dg,$$

with Γ the group of rational points and the measure that gives K volume 1; the integrand is right invariant under an open subgroup, so the integral is a finite sum over classes, divided by an index. The lift is equivariant: $\mathcal{K}[\omega(a,b)\Phi](a\cdot\varphi) = b\cdot\mathcal{K}[\Phi](\varphi)$ for $a \in G_f(I)$, $b \in G_f(I')$, where $(a\cdot\varphi)(g) = \varphi(ga)$ is right translation.

Definition 2.9 (chains of lifts)**.** In Setting 1.2, let $\mathcal{K} = (\mathcal{K}_j)_j$ be the kernels and $\Phi = (\Phi_j)_j$ a family of data, $\Phi_j \in \mathcal{S}_j$. Then $\mathrm{ch}(\mathcal{K};\Phi)$ is the family of linear maps

$$\mathrm{ch}(\mathcal{K};\Phi)_j \;=\; \mathcal{K}_{j-1}[\Phi_{j-1}]\circ\ldots\circ\mathcal{K}_0[\Phi_0] \;:\; \mathcal{A}(I_0;e_0)\to\mathcal{A}(I_j;e_j),$$

the identity for $j=0$. For a family $\Phi=(\Phi_i)_i$ with $\Phi_i \in \mathcal{S}_{2i}\otimes\mathcal{S}_{2i+1}$, $\mathrm{ch}_2(\mathcal{K};\Phi)$ is the family $\mathrm{ch}_2(\mathcal{K};\Phi)_p\colon \mathcal{A}(I_0;e_0)\to\mathcal{A}(I_{2p};e_{2p})$ of the composites of the first p *pairs of lifts*, the i-th pair being the linear map that sends a pure tensor $s\otimes s'$ to $\mathcal{K}_{2i+1}[s']\circ\mathcal{K}_{2i}[s]$, evaluated at Φ_i. For a family $a=(a_j)_j$ of points $a_j \in G_f(I_j)$ and $i\ge 0$,

$$\omega^{(2)}_{\mathcal{K},i}(a) \;=\; \omega_{2i}(a_{2i},a_{2i+1})\otimes\omega_{2i+1}(a_{2i+1},a_{2i+2}),$$

an endomorphism of $\mathcal{S}_{2i}\otimes\mathcal{S}_{2i+1}$.

LEAN `chain` · `chain₂` · `twoKernelAction`

Definition 2.10 (joint invariance)**.** In Setting 1.2, let M be a complex vector space, $\mathcal{X}$ a set, $m\ge 0$, $\mathcal{X}_0\subseteq\mathcal{X}$ finite, $L_x\colon W_\kappa(I_m)\to M$ linear and $\Phi_x=(\Phi_{x,j})_j$ data, for $x\in\mathcal{X}$. The family $(L_x,\Phi_x)_{x\in\mathcal{X}_0}$ is *jointly invariant of length m for $\mathcal{K}$* if for every family $a=(a_j)_j$ with $a_j\in K_j$ for all j and $a_j=1$ for $j\ge m$, and every linear map $\beta\colon \mathcal{S}_{m-1}\otimes\cdots\otimes\mathcal{S}_0\to W_\kappa(I_m)$,

$$\sum_{x\in\mathcal{X}_0} L_x\Big(\beta\big(\textstyle\bigotimes_{j<m}\omega_j(a_j,a_{j+1})\Phi_{x,j}\big)\Big) \;=\; \sum\nolimits_{x\in\mathcal{X}_0} L_x\Big(\beta\big(\textstyle\bigotimes_{j<m}\Phi_{x,j}\big)\Big).$$

In words: the tensor $\sum_x L_x\otimes\Phi_{x,m-1}\otimes\cdots\otimes\Phi_{x,0}$ is invariant under the joint action of $\prod_{j<m}K_j$.

LEAN `IsJointlyInvariant` · In the Lean definition the family a of the surrounding section is listed as an argument and not used: the definition quantifies over its own family a of elements of the levels.

Split places inside the adelic group; names.

Definition 2.11 (embedding at a split level; determinant characters)**.** Let Σ be a finite set of finite places of E with $\overline{w}\notin\Sigma$ for $w\in\Sigma$ (in the formal text: a split level for some N) and I a finite index set. For $g\in\mathrm{GL}_I(E_\Sigma)$, $\mathrm{emb}_\Sigma(g)\in G_f(I)$ is the element whose component is g_w at each $w\in\Sigma$, the inverse of the conjugate transpose of g_w at $\overline{w}$, and 1 at the other places; $g\mapsto\mathrm{emb}_\Sigma(g)$ is a group homomorphism. For a Hecke character η of E, with local components η_w, and $k\in\mathbb{N}$, $\mathrm{dc}_{\eta,k}$ is the character

$$\mathrm{dc}_{\eta,k}(g) \;=\; \prod_{w\in\Sigma}\eta_w(\det g_w)\,|\det g_w|_w^{-k/2}$$

of $\mathrm{GL}_I(E_\Sigma)$ with values in $\mathbb{C}^\times$; here $|\cdot|_w$ is the normalised absolute value of E_w.

LEAN IsSplitLevel.embedGL · IsSplitLevel.detCharAt

Definition 2.12 (names, projections, sums of translates; the first two recalled)**.** Let E have Weil data and let V be a hermitian space over E of the kind used in this proof (in this note: $V = \mathrm{Hom}(X_I, X_{I'})$ for finite index sets I, I', whose finite adelic vectors are the $I' \times I$ matrices over $\mathbb{A}_f$). Its Weil representation has a *finite part* $\mathcal{W}_f(V)$, spanned by the translates $\varrho_f(t)\,\mathrm{vac}$ of the vacuum by the finite Heisenberg operators. For a set Σ' of finite places, $\mathrm{pr}_V^{\Sigma'}$ is the restriction to $\mathcal{W}_f(V)$ of the projection $P^{\Sigma'}$, the mean of $\varrho_f(t)$ over the integral vectors t supported above $\Sigma' \cup \overline{\Sigma'}$; its image consists of the vectors that are vacua above these places. For a split level Σ for the prime N and a function F of $I' \times I$ matrices over E_Σ, $\mathrm{nm}_\Sigma(F)$ is the endomorphism of $\mathcal{W}_f(\mathrm{Hom}(X_I, X_{I'}))$ named by F: the operator that the background library attaches to the function $t \mapsto F(t_\Sigma)$ of finite adelic matrices, t_Σ being the components of t at Σ; the map $(F, v) \mapsto \mathrm{nm}_\Sigma(F)\,v$ is bilinear. Finally, for finite index sets I, I', I'', a finite set Σ of finite places and a prime N,

$$\mathsf{T}_{N,\Sigma}(I, I', I'') \;=\; \sum_{y} \varrho_f(c(y)) \otimes \varrho_f(r(y)),$$

the sum over $y \in ((\mathbb{Z}/N)^\Sigma)^{I'}$: an endomorphism of $\mathcal{W}_f(\mathcal{H}) \otimes \mathcal{W}_f(\mathcal{H}')$ for the spaces $\mathcal{H} = \mathrm{Hom}(X_{I^+}, X_{I'})$ and $\mathcal{H}' = \mathrm{Hom}(X_{I'}, X_{I''^+})$, where $c(y)$ is the $I' \times I^+$ matrix with last column $(-[y_i/N])_{i \in I'}$ and zero elsewhere, and $r(y)$ the $I''^+ \times I'$ matrix with last row $(-\overline{[y_i/N]})_{i \in I'}$ and zero elsewhere.

LEAN IsSplitLevel.name · Space.projOnFinPart · translateSum

The data.

Definition 2.13 (columns and stages; recalled from other notes)**.** In Setting 1.3: $\mathrm{St}_j(D)$ is the space of algebraic modular forms on $G_f(\underline{X}_j(D))$ of the weight $(\kappa_D, e_{D,j})$ that Chapter V attaches to stage j. The level $K_j^{\mathrm{col}}(D)$ consists of the $k \in K_0(\underline{X}_j(D))$ that reduce to 1 at the chosen places of D and whose last row is congruent to $(0, \dots, 0, 1)$ modulo $\pi_N(D)$; equivalently (Input 2) k lies in the chosen level and, at every $w \in \Pi(D)$, every entry of the last row of $k_w - 1$ is divisible by a uniformiser. $\mathrm{Col}(D)$ is the space of the forms of $\mathrm{St}_0(D)$ fixed by $K_0^{\mathrm{col}}(D)$, and $\mathrm{Col}^{\mathrm{ram}}(D)$ is the image of $\prod_{w \in \Pi(D)} (1 - z_w(D))$ on $\mathrm{Col}(D)$, $z_w(D)$ being right translation by the scalar matrix of an integral norm-one adele $\nu_w(D)$ that the unit *Columns* fixes for D and w, equal to 1 away from w and $\overline{w}$, whose component at w generates the units of the residue field (Input 3). The *restriction* res_D maps $\mathrm{St}_m(D)$ to algebraic forms on the definite space $Y_m(D)$ of rank one less:

$$\mathrm{res}_D(\varphi)(h) \;=\; \Big(\prod_{\tau} \tau(\overline{\pi_N(D)})^{-\mathrm{br}_D(\tau)} \Big) \cdot B_D\, \varphi(\mathrm{bd}_D(h)),$$

the product over the infinite places τ of $E(D)$, each with its embedding of the CM type; here $\mathrm{bd}_D(h) = \mathrm{diag}(h, 1)$, B_D is the integer branching matrix of the columns and $\mathrm{br}_D(\tau) \in \mathbb{Z}$ the branching exponent (Input 4).

Definition 2.14 (gauges and integral forms; recalled from the background library)**.** Let I be a finite index set, ℓ a prime and $\jmath\colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism. For an infinite place τ of E, $\mathfrak{p}_\jmath(\tau)$ is the prime of E above ℓ induced by the embedding

$\jmath\circ\tau\colon E\to\mathbb{C}_\ell$, and $\mathrm{ev}_{\jmath,\tau}\colon \mathbb{A}_f\to\mathbb{C}_\ell$ is the component at $\overline{\mathfrak{p}_\jmath(\tau)}$ followed by the continuous extension of $x\mapsto \jmath(\tau(\overline{x}))$. The *gauge* of a vector $u\in W_\kappa(I)$ at $g\in G_f(I)$ is

$$\mathrm{gauge}_\jmath(g;u)\;=\;\big\|\,\varrho_{\kappa,e}\big((\mathrm{ev}_{\jmath,\tau}(g^{-1}))_\tau\big)\,\jmath(u)\,\big\|,$$

the largest ℓ-adic absolute value of a coordinate, where $\varrho_{\kappa,e}$ is the representation of $\prod_\tau \mathrm{GL}_I$ of weight (κ,e) and $\jmath$ is applied to the coordinates of u. An algebraic form φ *has gauge at most* C *at a point* $\mathsf{d}\in G_f(I)$ if $\mathrm{gauge}_\jmath(g\mathsf{d};\varphi(g))\le C$ for every $g\in G_f(I)$. For a complex space $\mathcal{V}$ of algebraic forms, $\mathrm{Int}_\jmath(\mathcal{V};\mathsf{d})$ is the $O_\jmath$-module of the $\varphi\in\mathcal{V}$ that have gauge at most 1 at d; for $\mathsf{d}=1$ these are the *$\jmath$-integral* forms. For a form φ and a ring homomorphism $\sigma\colon\mathbb{C}\to\mathbb{C}$, $\|\sigma\circ\varphi(g)\|^2_{\mathrm{W}}$ is the squared norm, for the Gram matrix of the weight, of the vector with coordinates $\sigma(\varphi(g)_i)$.

Definition 2.15 (the non-induced places and the representation at the level prime)**.** In Setting 1.3, but for any prime ℓ and ring homomorphism $\jmath\colon\mathbb{C}\to\mathbb{C}_\ell$,

$$\Sigma_\jmath(D)\;=\;\{\overline{\mathfrak{p}_\jmath(\tau)}\;:\;\tau\text{ an infinite place of }E(D)\},$$

the set of the complex conjugates of the induced primes; it depends on ℓ and $\jmath$. Now let $\ell=N(D)$ and let $\Sigma_\jmath(D)$ be a split level for $N(D)$ (which holds, Lemma 7.1). For $j\ge 0$,

$$\Delta_j(D)\;=\;\mathrm{emb}_{\Sigma_\jmath(D)}(\mathrm{dg}_{\underline{X}_j^\circ(D)}(\Pi(D)))\;\in\;G_f(\underline{X}_j(D)),$$

the point supported above $N(D)$ with component $\mathrm{diag}(1,\dots,1,N(D))$ at the $w\in\Sigma_\jmath(D)$ that divide $\pi_N(D)$ and 1 at the other $w\in\Sigma_\jmath(D)$; $\mathrm{emb}^{\Delta}_{D,j}$ is the homomorphism

$$\mathrm{emb}^{\Delta}_{D,j}\colon\;\mathrm{GL}_{\underline{X}_j(D)}(E_{\Sigma_\jmath(D)})\to G_f(\underline{X}_j(D)),\qquad g\mapsto\Delta_j(D)\cdot\mathrm{emb}_{\Sigma_\jmath(D)}(g)\cdot\Delta_j(D)^{-1};$$

and $\rho_{D,j}$ is the representation of $\mathrm{GL}_{\underline{X}_j(D)}(E_{\Sigma_\jmath(D)})$ on $\mathrm{St}_j(D)$ in which g acts by right translation by $\mathrm{emb}^{\Delta}_{D,j}(g)$.

LEAN `Data.nonInducedPlaces` · `Data.diagPointN` · `Data.conjEmbedGL` · `Data.nonInducedRep`

What is taken for granted.

Inputs from other units (as the unit's dossier reads them).

1 (unit *Columns*). The set $\Pi(D)$ of the places dividing $\pi_N(D)$ is a split level for $N(D)$.

2 (unit *Columns*). $k \in K_j^{\mathrm{col}}(D)$ if and only if k lies in the chosen level and, at every $w \in \Pi(D)$, every entry of the last row of $k_w - 1$ has valuation at most e^{-1}.

3 (unit *Columns*). For each $w \in \Pi(D)$, every ramified column φ has the form $\psi - z_w(D)\psi$ for some column ψ; the adele $\nu_w(D)$ is integral, of norm one, and equal to 1 away from w and $\overline{w}$.

4 (unit *Columns*, from the background library). The branching matrix B_D has integer entries. Over every field of characteristic zero, for every family $g = (g_\tau)$ of invertible matrices of the rank of $Y_m(D)$ and every family $s = (s_\tau)$ of non-zero scalars, indexed by the infinite places τ,

$$B_D \circ \left[\mathrm{diag}(g, s)\right]_m \;=\; \prod_\tau s_\tau^{\mathrm{br}_D(\tau)} \cdot [g]_Y \circ B_D,$$

where $[\,\cdot\,]_m$ is the action in the weight of stage m, $[\,\cdot\,]_Y$ the action in the weight of $Y_m(D)$, and $\mathrm{diag}(g, s)$ the family of the block diagonal matrices with blocks g_τ and s_τ: the exponent of s_τ is exactly the branching exponent $\mathrm{br}_D(\tau)$. Moreover B_D satisfies a size bound in the Gram norms of the two weights. The reference for this branching from rank $n+1$ to rank n is to be confirmed.

5 (the note on the index sets and characters of the data, this unit; unit *Columns*; Chapter III). The level prime $N(D)$ is congruent to 1 modulo the conductor of $E(D)$; so it splits completely in $E(D)$, each place above it has norm $N(D)$ and uniformiser $N(D)$, none is fixed by complex conjugation, and the group of infinite places permutes the places above a place of K simply transitively (Chapter III). A place dividing $N(D)$ is not a chosen place of D, and $|\underline{X}_j(D)| = n_j(D)$ is even (unit *Columns*).

Quoted from the background library (*Lib*). The catalogue of the write-up lists the following, as far as it contains them, as elementary facts or as definitions, without a reference to the literature; where a reference would be expected we write that it is to be confirmed.
Q1 (class integrals; the catalogue's file on class means). If all the groups of rational points in the conjugates of K are trivial and F is right K-invariant with values in a set A, then the integral of F over the classes is a sum of elements of A, one for each double coset; and the class integral commutes with finite sums of additive maps.
Q2 (names at split places; the catalogue's files on the mixed model). For Σ disjoint from its conjugate: the indicator of the matrices integral at Σ names the projection P^{Σ}; the name of its Schrödinger translate by a vector t is the Heisenberg translate by the part of t above $\Sigma \cup \overline{\Sigma}$, composed with P^{Σ}; and for g, h supported on $\Sigma \cup \overline{\Sigma}$ the pair Weil operator with characters a, b sends the operator named by Ψ to the one named by $\Psi(h^{-1} \cdot g)$, times the values of the space characters at the determinants at Σ and powers of their absolute values. Reference for the Schrödinger model at split places: to be confirmed.
Q3 (gauges). The gauge at g does not change when g is multiplied on the right by an element that is integral at every place above ℓ; the gauge of a sum is at most the largest gauge of a term; an integer matrix, and a Schur matrix of integral matrices, do not increase ℓ-adic norms; some rational point moves any given point, and a given conjugate of it, to points integral above ℓ (weak approximation).
Q4 (dimension of forms with free classes). If all the groups of rational points in the conjugates of a level are trivial, the space of forms of that level has dimension the number of classes times the dimension of the weight. The catalogue cites B. Gross, *Algebraic modular forms*, Israel J. Math. 113 (1999), without a place: reference to be confirmed. For the columns the groups are trivial (unit *Columns*).
Q5 (characters). The splitting character υ is unramified at every finite place; a character of the norm-one adeles with a weight and a tame part that is trivial at a place w and at $\overline{w}$ has an unramified base change at w; for an index set of even cardinality the space character of the rank twist of ν is the base change of ν.

3 Split levels and the Gauss-sum operator

Lemma 3.1 (elementary facts)**.** *Let Σ be a finite set of finite places of E, N a natural number, $y \in (\mathbb{Z}/N)^{\Sigma}$, and let $\mathbf{1}_{\Sigma} \in \mathbb{A}_f$ be the adele equal to 1 at the places of Σ and 0 elsewhere.*

(i) If N is prime, $\mathbf{1}_{\Sigma} \cdot [y/N] = [y/N]$.

(ii) Let Σ is a split level for N with N prime, $x \in O_{\Sigma}$ and $c \in (\mathbb{Z}/N)^{\Sigma}$. Then $\mathrm{red}(x) = c$ if and only if $(x_w - \tilde{c}_w)/N \in O_w$ for every $w \in \Sigma$.

(iii) If Σ is a split level for N, then Σ and its conjugate $\overline{\Sigma}$ are disjoint.

(iv) If Σ is a split level for N, then every $w \in \Sigma$ divides N.

(v) Let Σ is a split level for N with N prime, $x \in O_{\Sigma}$, $c \in (\mathbb{Z}/N)^{\Sigma}$ and $u \in \mathbb{A}_f$ such that x is the family of the components of Nu at Σ. Then $\mathrm{red}(x) = c$ if and only if $\mathbf{1}_{\Sigma} \cdot (u - [c/N]) \in \hat{O}$.

(vi) Let Σ is a split level for N with N prime, I a finite index set and Σ' a finite set of finite places. Then $\mathrm{dg}_I(\Sigma) = \mathrm{dg}_I(\Sigma \smallsetminus \Sigma') \cdot \mathrm{dg}_I(\Sigma')$.

Proof. (i) The adele $[y/N]$ vanishes outside Σ. (ii) The component of $\mathrm{red}(x)$ at w is c_w exactly when $x_w - \tilde{c}_w$ lies in the maximal ideal of O_w, which is NO_w by (c) of Definition 2.1. (iii) and (iv) are (a) and one direction of (d). (v) is (ii), since $(x_w - \tilde{c}_w)/N$ is the component at w of $u - [c/N]$ and membership in $\hat{\mathcal{O}}$ of an adele supported on Σ is tested at the places of Σ. (vi) Both sides are diagonal with last entry N at every place of Σ. □

LEAN `idem_mul_residueOverN` · `IsSplitLevel.reduction_eq_iff` · `IsSplitLevel.disjoint_star` · `IsSplitLevel.natCast_mem` · `IsSplitLevel.reduction_eq_iff_idem_mul_mem` · `IsSplitLevel.diagN_eq_sdiff_mul` · An unnamed instance records that red is a local homomorphism: an element of O_Σ with invertible reduction is a unit.

Lemma 3.2. *Let Σ is a split level for N with N prime, and let $z \in \mathbb{A}_f$. There is $b \in (\mathbb{Z}/N)^\Sigma$ such that for every $y \in (\mathbb{Z}/N)^\Sigma$:*

$$\mathbf{1}_\Sigma \cdot (z - [y/N]) \in \hat{\mathcal{O}} \quad \Longleftrightarrow \quad \mathbf{1}_\Sigma \cdot (Nz) \in \hat{\mathcal{O}} \ \text{ and } \ b = y.$$

Proof. If $\mathbf{1}_\Sigma(z - [y/N])$ is integral, so is $\mathbf{1}_\Sigma Nz = N \cdot \mathbf{1}_\Sigma(z - [y/N]) + \mathbf{1}_\Sigma(\tilde{y}_w)_w$. So if $\mathbf{1}_\Sigma Nz$ is not integral, both sides are false for every y and every b serves. If $\mathbf{1}_\Sigma Nz \in \hat{\mathcal{O}}$, let $x \in O_\Sigma$ be the family of the components of Nz at Σ and take for b its reduction, $b = \mathrm{red}(x)$. By Lemma 3.1 (v), $\mathbf{1}_\Sigma(z - [y/N]) \in \hat{\mathcal{O}}$ if and only if $\mathrm{red}(x) = y$, that is $b = y$. □

LEAN `IsSplitLevel.exists_sub_residueOverN_mem_iff` · The text reading of the census lists a character b of the surrounding section as an argument; Lean does not take it: the b of the statement is the element it provides.

Lemma 3.3 (functions of residue matrices)**.** *Let Σ be a split level for the prime N and I, I' finite index sets.*

(i) For every $I'^+ \times I^+$ matrix S over E_Σ, with u_0 in ranks I'^+ and I^+: $\mathbf{r}(u_0^{-1} S u_0) = \mathbf{b_e}(S)$.

(ii) Let f be a function of $I \times I$ matrices over $(\mathbb{Z}/N)^\Sigma$ and $g \in \mathrm{GL}_I(E_\Sigma)$ with $f^\natural(g) \neq 0$ and $f^\natural(g^{-1}) \neq 0$. Then $g = k$ for some $k \in \mathrm{GL}_I(O_\Sigma)$ with $f(\mathrm{red}\, k) \neq 0$.

(iii) If $g \in \mathrm{Mir}(I)$, then $\mathbf{b_e}(g) \neq 0$.

(iv) Let ρ, ρ' be complex representations of $\mathrm{GL}_{I^+}(E_\Sigma)$, $\mathrm{GL}_{I'^+}(E_\Sigma)$, L a linear map from functions of $I'^+ \times I^+$ matrices over E_Σ to the linear maps $\rho \to \rho'$, φ a vector of ρ, J a finite set, r a map from the $I'^+ \times I^+$ matrices over $(\mathbb{Z}/N)^\Sigma$ to J and $g\colon J \to \mathbb{C}$. Then $L[(g \circ r)^\natural]\,\varphi = \sum_{y \in J} g(y) \cdot L[(T \mapsto [r(T) = y])^\natural]\,\varphi$, where $[r(T) = y]$ is 1 if $r(T) = y$ and 0 otherwise.

Proof. (i) The matrix $u_0^{-1} S u_0$ has the entries of S, except that the last row is multiplied by N and the last column divided by N (the corner entry is unchanged). It is integral with last row $\equiv \mathbf{e}$ if and only if the entries of S outside the last column are integral, $S_{**} \equiv 1$ and S_{i*}/N is integral for $i \neq *$: that is, S is integral with last column $\equiv \mathbf{e}$, because N is a uniformiser at every place of Σ. (ii) Both g and g^{-1} have entries in O_Σ, since $f^\natural$ vanishes on non-integral matrices. (iii) is the definition of $\mathrm{Mir}(I)$. (iv) $g \circ r = \sum_y g(y)\,[r = y]$, and $f \mapsto L[f^\natural]\varphi$ is linear. □

LEAN `IsSplitLevel.lastRowDatum_conj_u₀` · `IsSplitLevel.exists_inclGL_eq` · `IsSplitLevel.lastColDatum_ne_zero` · `IsSplitLevel.liftFn_comp_eq_sum_smul` · In the last one the census lists the variable y of the section, which Lean does not take.

Lemma 3.4 (the mirabolic group and the Gauss-sum operator)**.** *Let E have Weil data, let Σ be a split level for the prime N, I a finite index set, ρ a complex representation of $\mathrm{GL}_{I^+}(E_\Sigma)$, φ a vector of ρ, T a linear endomorphism of the space of ρ and $g \in \mathrm{Mir}(I)$. Assume that $\rho(a)(T(\rho(b)\varphi)) = \varphi$ for all $a, b \in \mathrm{GL}_{I^+}(E_\Sigma)$ with $a\,g\,b \in \mathrm{Mir}(I)$. Then $T(\mathsf{F}(\rho)\varphi) = \mathsf{F}(\rho)\varphi$.*

Proof. As g and g^{-1} lie in $\mathrm{Mir}(I)$, Lemma 3.3 (iii) and (i) give $\mathbf{r}(u_0^{-1} g u_0) \neq 0$ and $\mathbf{r}(u_0^{-1} g^{-1} u_0) \neq 0$; by (ii), $u_0^{-1} g\, u_0 = k$ with $k \in \mathrm{GL}_{I^+}(O_\Sigma)$ and the last row of $\bar{k} = \mathrm{red}(k)$ equal to $\mathbf{e}$. Multiplication by $\bar{k}$ permutes the vectors y with y_* a unit and preserves y_*, because $(\bar{k}y)_*$ is the product of the last row of $\bar{k}$ with y. So, with $\tilde{k}(y)$ as in Definition 2.6,

$$\mathsf{F}(\rho)\varphi = \sum_y \psi_N(y_*)\ \rho(u_0\,\tilde{k}(\bar{k}y))\varphi, \qquad T(\mathsf{F}(\rho)\varphi) = \sum_y \psi_N(y_*)\ T(\rho(u_0\,\tilde{k}(y))\varphi),$$

and it suffices to show $T(\rho(b)\varphi) = \rho(a)^{-1}\varphi$ for $b = u_0\tilde{k}(y)$ and $a = \tilde{k}(\bar{k}y)^{-1}u_0^{-1}$. This is the hypothesis, once $a\,g\,b = \tilde{k}(\bar{k}y)^{-1}\,k\,\tilde{k}(y)$ is seen to lie in $\mathrm{Mir}(I)$: it is integral and invertible, and its reduction sends $\mathbf{e}$ first to y (the last column of $k(y)$), then to $\bar{k}y$, then back to $\mathbf{e}$ (as $k(\bar{k}y)\mathbf{e} = \bar{k}y$). □

LEAN `IsSplitLevel.mirabolic_fix_fourierOp` · Lean takes N, its primality and the split level as arguments here; the text reading of the census misses them because the statement does not write their names.

4 Lift laws: proof of Theorem A

Lemma 4.1. *In Setting 1.1, let I, I' be finite index sets, ρ, ρ' complex representations of $\mathrm{GL}_{I^+}(E_\Sigma)$, $\mathrm{GL}_{I'^+}(E_\Sigma)$, and let L is a lift law from ρ to ρ' with constant $\varkappa$; let φ be a vector of ρ.*

(i) Let $\varphi \in \mathrm{Ram}(\rho)$, $w \in \Sigma$, and let f be a function of $I'^+ \times I^+$ matrices over $(\mathbb{Z}/N)^\Sigma$ such that $f(cT) = f(T)$ for every T and every $c \in (\mathbb{Z}/N)^\Sigma$ with the property: for all $t, y \in (\mathbb{Z}/N)^\Sigma$ with $y_w = 0$, $c\,t = y$ if and only if $t = y$. Then $L[f^\natural]\,\varphi = 0$.

(ii) For $k \in \mathrm{GL}_{I'^+}(O_\Sigma)$ and $y \in ((\mathbb{Z}/N)^\Sigma)^{I'^+}$: $\rho'(k)(L[\mathbf{b}_y]\,\varphi) = L[\mathbf{b}_{\mathrm{red}(k)y}]\,\varphi$.

The property of c in (i) holds exactly for the c with $c_v = 1$ for $v \neq w$ and $c_w \neq 0$.

Proof. (i) As φ is ramified there are a unit x of O_Σ with $\mathrm{red}(x)_v = 1$ for $v \neq w$ and a vector ψ with $\varphi = \psi - \rho(x \cdot 1)\psi$. The element $c = \mathrm{red}(x)$ has the property: for $y_w = 0$ the equation $ct = y$ says $t_v = y_v$ for $v \neq w$ and $c_w t_w = 0$, and c_w is a unit. Integral equivariance, with $a = x \cdot 1$ and $b = 1$, gives $L[(T \mapsto f(cT))^\natural](\rho(x \cdot 1)\psi) = L[f^\natural]\psi$, and $f(cT) = f(T)$. Hence $L[f^\natural]\varphi = L[f^\natural]\psi - L[f^\natural]\psi = 0$. (ii) Apply integral equivariance with $a = 1$, $b = k$ to the indicator f of the matrices with last column $\mathrm{red}(k)y$: then $f(\mathrm{red}(k)T)$ is the indicator of the matrices with last column y. □

LEAN `IsSplitLevel.LiftLaw.eq_zero_of_smul_invariant` · `IsSplitLevel.LiftLaw.inclGL_lastColDatum` · In the first the census lists the section variable y, which Lean does not take; in the second Lean also takes N, its primality and the split level.

Proposition 4.2 (one law and the Gauss-sum operator)**.** *In Setting 1.1, let I, I' be finite index sets, ρ, ρ' complex representations of $\mathrm{GL}_{I^+}(E_\Sigma)$, $\mathrm{GL}_{I'^+}(E_\Sigma)$, let L is a lift law from ρ to ρ' with constant $\varkappa$ and $\varphi \in \mathrm{Ram}(\rho)$, and put*

$$C = \rho'(u_0)(L[\mathbf{c}_\mathbf{e}]\,\varphi).$$

Then

$$C \;=\; \mathsf{F}(\rho')(L[\mathbf{b}_{\mathbf{e}}]\,\varphi) \qquad \textit{and} \qquad C \;=\; \varkappa \cdot L[\mathbf{b}_{\mathbf{e}}](\mathsf{F}(\rho)\varphi).$$

Proof. For $t \in (\mathbb{Z}/N)^{\Sigma}$ let f_t be the indicator of the residue matrices T with corner entry $T_{**} = t$; then $\mathbf{c}_{\mathbf{e}} = \sum_t \psi_N(t)\, f_t^{\natural}$ by Lemma 3.3 (iv). If t is not a unit, say $t_w = 0$, then f_t is invariant under the scalars c of Lemma 4.1 (i), so $L[f_t^{\natural}]\varphi = 0$. Hence

$$L[\mathbf{c}_{\mathbf{e}}]\,\varphi \;=\; L[h^{\natural}]\,\varphi, \qquad h(T) = \psi_N(T_{**}) \text{ if } T_{**} \text{ is a unit}, \quad h(T) = 0 \text{ otherwise.} \quad (1)$$

First identity. By Lemma 4.1 (ii) and $k(y)\mathbf{e} = y$,

$$\mathsf{F}(\rho')(L[\mathbf{b}_{\mathbf{e}}]\varphi) = \rho'(u_0) \sum_{y} \psi_N(y_*)\, L[\mathbf{b}_y]\,\varphi,$$

the sum over the $y \in ((\mathbb{Z}/N)^{\Sigma})^{I'^{+}}$ with y_* a unit. A matrix with last column y has corner entry y_*, so $\sum_y \psi_N(y_*)\mathbf{b}_y = h^{\natural}$, and the right side is C by (1).

Second identity. By the law of u_0, $\varkappa\, L[\mathbf{b}_{\mathbf{e}}](\mathsf{F}(\rho)\varphi) = \rho'(u_0) \sum_y \psi_N(y_*)\, L[\mathbf{r}](\rho(\tilde{k}(y))\varphi)$, the sum over the $y \in ((\mathbb{Z}/N)^{\Sigma})^{I^{+}}$ with y_* a unit. Integral equivariance with $a = \tilde{k}(y)$, $b = 1$, applied to the indicator f of the matrices with last row y, gives $L[\mathbf{r}](\rho(\tilde{k}(y))\varphi) = L[f^{\natural}]\varphi$: indeed $Tk(y)$ has last row y if and only if T has last row $y\,k(y)^{-1} = \mathbf{e}$, the last row of $k(y)$ being y. Summing over y with the weights $\psi_N(y_*)$ gives again $h^{\natural}$, and the right side is C by (1). □

LEAN `IsSplitLevel.LiftLaw.u₀_charDatum_eq_fourierOp` · Lean also takes N, its primality and the split level.

Proof of Theorem A. Write $F = \mathsf{F}(\rho_0)$ and, for $q, j \geq 0$ and a vector ψ of ρ_0,

$$B_j^q(\psi) \;=\; L_{j-1}[\mathbf{b}_{\mathbf{e}}] \circ \ldots \circ L_0[\mathbf{b}_{\mathbf{e}}]\,(F^q\psi),$$

a vector of ρ_j.

(1) The vectors $B_j^q(\varphi)$ are ramified. Let $w \in \Sigma$ and $\varphi = \psi - \rho_0(x \cdot 1)\psi$ with x as in Definition 2.3. The scalar $x \cdot 1$ is central, so $\rho_0(x \cdot 1)$ commutes with F. Integral equivariance of L_j with $a = (x \cdot 1)^{-1}$, $b = x \cdot 1$ gives

$$\rho_{j+1}(x \cdot 1) \circ L_j[f^{\natural}] = L_j[f^{\natural}] \circ \rho_j(x \cdot 1) \qquad \text{for every } f,$$

because $f(\mathrm{red}(x)T\,\mathrm{red}(x)^{-1}) = f(T)$. By induction on j,

$$B_j^q(\varphi) = B_j^q(\psi) - \rho_j(x \cdot 1)B_j^q(\psi).$$

(2) $\mathsf{F}(\rho_j)(B_j^q(\varphi)) = (\prod_{l<j} \varkappa_l)B_j^{q+1}(\varphi)$. For $j = 0$ this is the definition. For the step, the two identities of Proposition 4.2 for the law L_j and the ramified vector $B_j^q(\varphi)$ give $\mathsf{F}(\rho_{j+1})(L_j[\mathbf{b}_{\mathbf{e}}]\, B_j^q(\varphi)) = \varkappa_j\, L_j[\mathbf{b}_{\mathbf{e}}](\mathsf{F}(\rho_j)B_j^q(\varphi))$.

(3) One pair. Let $\psi = B_{2q}^q(\varphi)$, a ramified vector of ρ_{2q}. In $\sum_y L_{2q+1}[\mathbf{c}_y]\, L_{2q}[\mathbf{b}_y]\,\psi$, a term with y zero at some place w vanishes: then $\mathbf{b}_y$ is invariant under the scalars of Lemma 4.1 (i), since for such c a column ct equals y if and only if t does. If y is non-zero at every place, then at each place some entry of y is a unit, so y is the last column of an invertible matrix over each factor $\mathbb{Z}/N$, and these assemble to $k \in \mathrm{GL}_{I^{+}_{2q+1}}((\mathbb{Z}/N)^{\Sigma})$ with $k\,\mathbf{e} = y$. With $\tilde{k} = \mathrm{sec}(k)$, Lemma 4.1 (ii) gives $L_{2q}[\mathbf{b}_y]\psi = \rho_{2q+1}(\tilde{k})\, L_{2q}[\mathbf{b}_{\mathbf{e}}]\psi$, and integral equivariance of L_{2q+1} with $a = \tilde{k}$,

$b = 1$ gives $L_{2q+1}[\mathbf{c}_y] \circ \rho_{2q+1}(\tilde{k}) = L_{2q+1}[\mathbf{c}_{\mathbf{e}}]$, because the product of a row r with $k\mathbf{e} = y$ is the product of rk with $\mathbf{e}$. So every such term equals the term of $y = \mathbf{e}$, and

$$\begin{aligned} A_q\,\psi \;&=\; d_q\,o_q \cdot \rho_{2q+2}(u_0)\,L_{2q+1}[\mathbf{c}_{\mathbf{e}}](B^q_{2q+1}(\varphi)) \;=\; d_q\,o_q \cdot \mathsf{F}(\rho_{2q+2})(B^q_{2q+2}(\varphi)) \\ &=\; d_q\,o_q \prod_{l<2q+2} \varkappa_l \cdot B^{q+1}_{2q+2}(\varphi), \end{aligned}$$

by the first identity of Proposition 4.2 for L_{2q+1} and the ramified vector $B^q_{2q+1}(\varphi)$, and by (2).

(4) The theorem follows by induction on p: it is trivial for $p = 0$, and if $A_{q-1} \circ \ldots \circ A_0(\varphi)$ is the stated multiple of $B^q_{2q}(\varphi)$, then (3) gives the statement for $q + 1$. □

LEAN `IsSplitLevel.compUpTo_pairs_eq_smul` · The census lists the section variable y, which Lean does not take.

5 Chains: proof of Theorem C

Lemma 5.1 (equivariance and linearity of chains)**.** *In Setting 1.2, let $\mathcal{K} = (\mathcal{K}_j)_j$ be the kernels.*

(i) Let $\varphi \in \mathcal{A}(I_0; e_0)$ be fixed by K_0, let $a = (a_j)_j$ with $a_j \in K_j$ for all j, $\Phi = (\Phi_j)_j$ data and $j \geq 0$. Then $\mathrm{ch}(\mathcal{K}; (\omega_i(a_i, a_{i+1})\Phi_i)_i)_j(\varphi) = a_j \cdot \mathrm{ch}(\mathcal{K}; \Phi)_j(\varphi)$.

(ii) Let $a = (a_j)_j$ be any family of points $a_j \in G_f(I_j)$, $t = (t_i)_i$ with $t_i \in \mathcal{S}_{2i} \otimes \mathcal{S}_{2i+1}$, $p \geq 0$ and $\varphi \in \mathcal{A}(I_0; e_0)$. Then

$$\mathrm{ch}_2(\mathcal{K}; (\omega^{(2)}_{\mathcal{K},i}(a)\,t_i)_i)_p(a_0 \cdot \varphi) = a_{2p} \cdot \mathrm{ch}_2(\mathcal{K}; t)_p(\varphi).$$

(iii) For $j \geq 0$, $\varphi \in \mathcal{A}(I_0; e_0)$ and $g \in G_f(I_j)$ there is a linear map β on $\mathcal{S}_{j-1} \otimes \cdots \otimes \mathcal{S}_0$ with $\mathrm{ch}(\mathcal{K}; \Phi)_j(\varphi)(g) = \beta(\Phi_{j-1} \otimes \cdots \otimes \Phi_0)$ for all data Φ.

Proof. (i) and (ii) follow by induction on the number of lifts from the equivariance of one lift (Definition 2.8): in (i) the form φ is fixed by $a_0 \in K_0$; in (ii) one first treats pure tensors $t_p = s \otimes s'$, for which the pair is $\mathcal{K}_{2p+1}[s'] \circ \mathcal{K}_{2p}[s]$, and extends by linearity. (iii) A lift is linear in its datum, so the value of the composite at φ and g is multilinear in $(\Phi_{j-1}, \ldots, \Phi_0)$. □

LEAN `chain_equivariant` · `chain₂_equivariant` · `exists_linear_chain_eq_pure` · In the last one the census lists the section variable Φ, which Lean does not take.

Proof of Theorem C. By induction on m, for all M, L, h, A and P. Let $\beta_j(g)$ be the linear map of Lemma 5.1 (iii) for φ and g, and write $\Phi_x^{<j} = \Phi_{x,j-1} \otimes \cdots \otimes \Phi_{x,0}$.

$m = 0$. The chain is the identity and $\varphi(h) = \beta_0(h)(1)$. By (e), $P_0(h, \varphi(h))$, so (f) with $\beta = \beta_0(h)$ gives $\sum_x L_x(\varphi(h)) \in A$; the empty product of class numbers is 1.

From j to $m = j + 1$. By the definition of the lift,

$$\sum_{x \in \mathcal{X}_0} L_x(\mathrm{ch}(\mathcal{K}; \Phi_x)_{j+1}(\varphi)(h)) = \int_{\Gamma \backslash G_f(I_j)} G(g)\,dg,$$

$$G(g) = \sum_{x \in \mathcal{X}_0} L_x\Big(\mathcal{J}_j(\omega_j(g, h)\Phi_{x,j})(\mathrm{ch}(\mathcal{K}; \Phi_x)_j(\varphi)(g))\Big),$$

where Q1 allows the finite sum to pass under the class integral. By Q1 and (b) it suffices to show that G is right K_j-invariant and that every $G(g)$ is a sum of $\prod_{i<j} \mathrm{cl}(K_i)$ elements of A.

Values. Fix g and apply the induction hypothesis to the same Φ and φ, the point g, the set A and the linear maps $L'_x(u) = L_x(\mathcal{J}_j(\omega_j(g,h)\Phi_{x,j})u)$ on $W_\kappa(I_j)$. For a linear β' on $\mathcal{S}_{j-1} \otimes \cdots \otimes \mathcal{S}_0$ let $T_{\beta'}$ be the linear map on $\mathcal{S}_j \otimes \cdots \otimes \mathcal{S}_0$ with $T_{\beta'}(s \otimes t) = \mathcal{J}_j(\omega_j(g,h)s)\,\beta'(t)$, so that $L'_x(\beta'(\Phi_x^{<j})) = L_x(T_{\beta'}(\Phi_x^{<j+1}))$. Joint invariance of length j for the L'_x is hypothesis (a) for the families a with $a_i = 1$ for $i \geq j$ and the maps $T_{\beta'}$. Hypothesis (f) for the L'_x and the point g: if $P_j(g, \beta'(\Phi_x^{<j}))$ for all x, then (d) gives $P_{j+1}(h, T_{\beta'}(\Phi_x^{<j+1}))$ for all x, and (f) for $\beta = T_{\beta'}$ gives $\sum_x L'_x(\beta'(\Phi_x^{<j})) \in A$. The other hypotheses are inherited. So $G(g) = \sum_x L'_x(\mathrm{ch}(\mathcal{K};\Phi_x)_j(\varphi)(g))$ is a sum of $\prod_{i<j} \mathrm{cl}(K_i)$ elements of A.

Invariance. Let $a \in K_j$ and let a' be the family with a at the index j and 1 elsewhere. By Lemma 5.1 (i),

$$\mathrm{ch}(\mathcal{K};\Phi_x)_j(\varphi)(ga) = \beta_j(g)\Big(\bigotimes\nolimits_{i<j} \omega_i(a'_i, a'_{i+1})\Phi_{x,i}\Big),$$
$$\omega_j(ga,h)\Phi_{x,j} = \omega_j(g,h)\,\omega_j(a'_j, a'_{j+1})\Phi_{x,j}.$$

Hence

$$G(ga) = \sum_{x \in \mathcal{X}_0} L_x\Big(T_{\beta_j(g)}(\bigotimes\nolimits_{i\leq j} \omega_i(a'_i, a'_{i+1})\Phi_{x,i})\Big),$$

which by (a) for the family a' equals $\sum_x L_x(T_{\beta_j(g)}(\Phi_x^{<j+1})) = G(g)$. □

LEAN `IsJointlyInvariant.sum_chain_mem_nsmul`

6 Names at a split level: proof of Theorem B

Lemma 6.1 (the embedding at a split level)**.** *Let Σ is a split level for N, I a finite index set, and for $g \in G_f(I)$ let $g_\Sigma \in \mathrm{GL}_I(E_\Sigma)$ be the family of its components at Σ.*

(i) For $g \in \mathrm{GL}_I(E_\Sigma)$: $\mathrm{emb}_\Sigma(g)_\Sigma = g$, and $\mathrm{emb}_\Sigma(g)$ has component 1 at every place outside $\Sigma \cup \overline{\Sigma}$.

(ii) Let $g \in G_f(I)$ and $a \in \mathrm{GL}_I(O_\Sigma)$ with $a = g_\Sigma$. Then the component of g at every place dividing N is integral and invertible.

(iii) Let N be prime, $w \in \Sigma$, and let ν be a finite adele with $\nu\overline{\nu} = 1$ that is integral and has $\nu_v = 1$ at every place v with $v \neq w$ and $\overline{v} \neq w$. Then there is a unit x of O_Σ with $\mathrm{red}(x)_v = 1$ for all $v \in \Sigma$, $v \neq w$, such that $\mathrm{emb}_\Sigma(x \cdot 1)$ is the scalar matrix of ν in $G_f(I)$.

Proof. (i) holds by construction. (ii) A place w dividing N lies in Σ or in $\overline{\Sigma}$, by (d) of Definition 2.1. In the first case $g_w = a_w \in \mathrm{GL}_I(O_w)$; in the second g_w is the inverse of the conjugate transpose of the component at $\overline{w}$, which is integral and invertible. (iii) Let x be the family of the components of ν at Σ; it is integral, and so is the family of the components of $\overline{\nu}$, which is its inverse because $\nu\overline{\nu} = 1$. For $v \in \Sigma$, $v \neq w$, also $\overline{v} \neq w$ by (a), so $\nu_v = 1$. The two elements $\mathrm{emb}_\Sigma(x \cdot 1)$ and the scalar matrix of ν are unitary, have component 1 outside $\Sigma \cup \overline{\Sigma}$ and the same components at Σ; so they are equal. □

LEAN `IsSplitLevel.atPlaces_embedGL` · `IsSplitLevel.embedGL_mem_supportedOn` · `IsSplitLevel.mem_integralAbove_of_inclGL_eq` · `IsSplitLevel.exists_embedGL_scalar_eq_scalarFin` · `scalarFin_atPlace` (the component at a place w of the scalar matrix of ν is the scalar matrix of ν_w; a rewriting rule, used silently)

Lemma 6.2 (Weil operators at a split level)**.** *Let E have Weil data, I, I' finite index sets, a, b characters of the norm-one adeles of E trivial on the rational points, and let $\omega_{a,b}$ be the pair Weil representation of $G_f(I) \times G_f(I')$ on $\mathcal{W}_f(\mathrm{Hom}(X_I, X_{I'}))$ with characters a, b.*

(i) For finite adeles ν, ν' of norm one, with scalar matrices $s(\nu) \in G_f(I)$, $s(\nu') \in G_f(I')$: $\omega_{a,b}(s(\nu), s(\nu')) = a(\det s(\nu'))\, b(\det s(\nu))^{-1} \cdot \omega_{1,1}(1, s(\nu'/\nu))$.

(ii) Let Σ be a split level for the prime N, $g \in \mathrm{GL}_I(E_\Sigma)$, $k \in \mathrm{GL}_{I'}(E_\Sigma)$, f a function of $I' \times I$ matrices over $(\mathbb{Z}/N)^\Sigma$ and $v \in \mathcal{W}_f(\mathrm{Hom}(X_I, X_{I'}))$. Then

$$\begin{aligned}&\omega_{a,b}(\mathrm{emb}_\Sigma(g), \mathrm{emb}_\Sigma(k))\ \mathrm{nm}_\Sigma(f^\natural)\, v \\ &\qquad = \mathrm{dc}_{\eta_b, |I'|}(g)^{-1}\ \mathrm{dc}_{\eta_a, |I|}(k) \cdot \mathrm{nm}_\Sigma(S \mapsto f^\natural(k^{-1} S g))\, v,\end{aligned}$$

where $\eta_a(x) = a(x/\overline{x})\, v(x)^{-|I|}$ and $\eta_b(x) = b(x/\overline{x})\, v(x)^{-|I'|}$ are the space characters.

(iii) In the situation of (ii), the function $t \mapsto f^\natural(t_\Sigma)$ of finite adelic matrices is the function that takes the value of f at the family of the reductions of t modulo the places of Σ if t is integral there, and 0 otherwise; it is a Schwartz–Bruhat function at Σ, so that it has a name.

Proof. (i) A pair of scalars acts on $\mathrm{Hom}(X_I, X_{I'})$ through the scalar ν'/ν; the characters contribute $a(\det s(\nu'))\, b(\det s(\nu))^{-1}$ by the definition of the twisted representation. (ii) is Q2: $\mathrm{emb}_\Sigma(g)$ and $\mathrm{emb}_\Sigma(k)$ are supported on $\Sigma \cup \overline{\Sigma}$, with components g, k at Σ (Lemma 6.1 (i)), and Σ is disjoint from its conjugate; the scalar of Q2 is the stated product of the two determinant characters. (iii) is the definition of $f^\natural$, read through the isomorphisms $\mathbb{F}_w \cong \mathbb{Z}/N$. □

LEAN `pairWeilFinite_scalarFin` · `IsSplitLevel.pairWeilFinite_name` · `IsSplitLevel.liftFn_eq_residueDatum` · `IsSplitLevel.isSchwartzBruhat_liftFn` (an instance)

Lemma 6.3 (determinant characters)**.** *Let I, I' be finite index sets and Σ a finite set of finite places of E.*

(i) Let E have Weil data and let a be a character of the norm-one adeles of E, trivial on the rational points, whose base change $x \mapsto a(x/\overline{x})$ is unramified at every $w \in \Sigma$. Then for all $i, k \in \mathbb{N}$ and $g \in \mathrm{GL}_I(O_\Sigma)$, with $\eta(x) = a(x/\overline{x}) v(x)^{-i}$: $\mathrm{dc}_{\eta,k}(g) = 1$.

(ii) Let Σ is a split level for N with N prime, Σ' a finite set of finite places, $\jmath\colon \mathbb{C} \to \mathbb{C}_N$ a ring homomorphism, η a Hecke character and $k \le k'$ natural numbers. Then $|\jmath(\mathrm{dc}_{\eta,k'}(\mathrm{dg}_I(\Sigma')))| \le |\jmath(\mathrm{dc}_{\eta,k}(\mathrm{dg}_{I'}(\Sigma')))|$.

Proof. (i) $\det g$ is a unit of O_Σ, so $|\det g_w|_w = 1$, and $\eta_w(\det g_w) = 1$ because both the base change of a and v are unramified at w (Q5). (ii) The determinant of $\mathrm{dg}_I(\Sigma')$ does not depend on the rank: its component is N at the places of $\Sigma \cap \Sigma'$ and 1 elsewhere. So the two values differ by the factor $\prod_{w \in \Sigma \cap \Sigma'} |N|_w^{-(k'-k)/2} = N^{(k'-k)\,\#(\Sigma \cap \Sigma')/2}$, because $|N|_w = N^{-1}$ for a place of norm N with uniformiser N. The square of this factor is a non-negative power of the integer N, of N-adic absolute value at most 1. □

Proof of Theorem B. Write $\mathcal{H} = \mathrm{Hom}(X_{I^+}, X_{I'})$ and $\mathcal{H}' = \mathrm{Hom}(X_{I'}, X_{I''^+})$, and let $\epsilon = \mathbf{1}_{\Sigma}$, $\epsilon' = \mathbf{1}_{\Sigma'}$. Both Σ and Σ' are split levels for N; hence $\Sigma \cup \overline{\Sigma} = \Sigma' \cup \overline{\Sigma'}$ is the set of the places dividing N, and $\epsilon + \overline{\epsilon} = \epsilon' + \overline{\epsilon'}$, $\epsilon'\overline{\epsilon'} = 0$.

Step 1: both sides are names at Σ. The projection $P^{\Sigma'}$ depends only on $\Sigma' \cup \overline{\Sigma'}$, so it equals P^{Σ}, which is named at Σ by the indicator $\mathbf{1}_{\mathrm{int}}$ of the matrices integral at Σ (Q2). The vectors $c(y)$, $r(y)$ of Definition 2.12 are supported above $\Sigma' \cup \overline{\Sigma'}$; by Q2 the operator $\varrho_f(c(y))P^{\Sigma}$ is named by the Schrödinger translate $c(y) \star \mathbf{1}_{\mathrm{int}}$, where

$$(t \star \Psi)(X) \;=\; \psi_0\big(\tfrac{1}{2}\textstyle\sum \overline{t} \cdot \epsilon\,(2X + t)\big)\,\Psi(X + t),$$

the sum over the entries; likewise for $r(y)$. So the left side of the theorem is the image of

$$Z = \sum_{y} (c(y) \star \mathbf{1}_{\mathrm{int}}) \otimes (r(y) \star \mathbf{1}_{\mathrm{int}})$$

under the linear map $\Psi \otimes \Psi' \mapsto \mathrm{nm}_{\Sigma}(\Psi)v \otimes \mathrm{nm}_{\Sigma}(\Psi')v'$, and the right side is the image of

$$Z' = \sum_{z} (S \mapsto \mathbf{b}_z(S\,\mathrm{dg}_{I}(\Sigma'))) \otimes (S' \mapsto \mathbf{c}_z(\mathrm{dg}_{I''}(\Sigma \smallsetminus \Sigma')S')).$$

(The formal text first embeds the tensor product of the finite parts into that of the ambient spaces, which is injective over a field.) A tensor of two functions is determined by its values at pairs of points, so it suffices to prove $Z(T, R) = Z'(T, R)$ for all finite adelic matrices T of size $I' \times I^+$ and R of size $I''^+ \times I'$.

Step 2: the left side at (T, R). Put $t_i = T_{i*}$ and $s_i = R_{*i}$ for $i \in I'$, and $x_i = [y_i/N]$, an adele supported on Σ'; note $\overline{x_i}\,x_i = 0$. Then

$$Z(T, R) = \sum_{y} \psi_0\Big(-\sum_{i} (\overline{x_i}\,\epsilon\,t_i + x_i\,\epsilon\,s_i)\Big) \cdot [\epsilon\,(T + c(y)) \text{ and } \epsilon\,(R + r(y)) \text{ integral}].$$

The integrality conditions concern the entries of T outside the last column and of R outside the last row, which do not involve y, and the entries $\epsilon(t_i - x_i)$ and $\epsilon(s_i - \overline{x_i})$. Let $\mathsf{p} = 1 + \epsilon'(N - 1)$, the adele equal to N on Σ' and 1 elsewhere, so that $N + 1 - \mathsf{p}$ is 1 on Σ' and N elsewhere. Applying Lemma 3.2 for the split level Σ' to the adele $\epsilon t_i + \overline{\epsilon s_i}$, and using that an adele u has ϵu and $\epsilon\overline{u}$ integral if and only if $\epsilon' u$ and $\epsilon'\overline{u}$ are, one finds $b_i \in (\mathbb{Z}/N)^{\Sigma'}$ such that for every y_i:

$$\epsilon(t_i - x_i) \text{ and } \epsilon(s_i - \overline{x_i}) \text{ integral} \iff \epsilon\,t_i\mathsf{p} \text{ and } \epsilon\,(N + 1 - \mathsf{p})s_i \text{ integral, and } y_i = b_i.$$

So every term with $y \neq b$ vanishes. The term $y = b$ can vanish as well: it is non-zero exactly when $T \cdot \mathrm{diag}(1, \dots, 1, \mathsf{p})$ and $\mathrm{diag}(1, \dots, 1, N + 1 - \mathsf{p}) \cdot R$ are integral at Σ (the formal proof records this by a case split after discarding the other terms).

Step 3: the right side at (T, R). At the places of Σ the matrix $\mathrm{dg}_{I}(\Sigma')$ is $\mathrm{diag}(1, \dots, 1, \mathsf{p})$ and $\mathrm{dg}_{I''}(\Sigma \smallsetminus \Sigma')$ is $\mathrm{diag}(1, \dots, 1, N + 1 - \mathsf{p})$. So $Z'(T, R)$ vanishes unless the same two matrices are integral at Σ; then exactly one z contributes, the reduction of the last column $(t_i\mathsf{p})_i$, and $Z'(T, R) = \psi_N(\zeta)$ with $\zeta \in (\mathbb{Z}/N)^{\Sigma}$ the reduction of $\sum_i (N + 1 - \mathsf{p})s_i \cdot t_i\mathsf{p} = N \sum_i s_i t_i$ at Σ (note $(N + 1 - \mathsf{p})\mathsf{p} = N$).

Step 4: the two characters agree. It remains to see that, for $y = b$,

$$\psi_0\Big(-\sum_{i} (\overline{x_i}\,\epsilon\,t_i + x_i\,\epsilon\,s_i)\Big) \;=\; \psi_0(-[\zeta/N]).$$

The character ψ_0 is trivial on $\hat{\mathcal{O}}$, so it suffices that the two arguments differ by an integral adele. By Step 2 the adeles $\epsilon(t_i - x_i)$ and $\epsilon(s_i - \overline{x_i})$ are integral, and by Lemma 3.1 (v), applied to $u = \sum_i s_i t_i$, $\epsilon\left(\sum_i s_i t_i - [\zeta/N]\right)$ is integral. Expanding the product $\epsilon \sum_i (t_i - x_i)(s_i - \overline{x_i})$, which is integral, and using $x_i \overline{x_i} = 0$ gives the claim. This is the computation that the formal proof closes with its last three lines. □

LEAN `IsSplitLevel.translateSum_eq_sum_name` · The formal statement takes the split level structure of Σ' as an explicit hypothesis and that of Σ as an instance.

7 The level prime of the data: proof of Theorem D

Lemma 7.1 (the non-induced places)**.** *In Setting 1.3, with $\ell = N(D)$ and $\jmath\colon \mathbb{C} \to \mathbb{C}_{N(D)}$:*

(i) $\Sigma_\jmath(D)$ is a split level for $N(D)$; in symbols, $\Sigma_\jmath(D)$ is a split level for $N(D)$.

(ii) Let $\Sigma_\jmath(D)$ carry this structure, let $j \geq 0$, Σ' a finite set of finite places of $E(D)$ and τ an infinite place of $E(D)$. Then $\mathrm{ev}_{\jmath,\tau}$, applied to the entries of $\mathrm{emb}_{\Sigma_\jmath(D)}(\mathrm{dg}_{\underline{X}^\circ_j(D)}(\Sigma'))$, gives $\mathrm{diag}(1, \dots, 1, N(D))$ if $\overline{\mathfrak{p}_\jmath(\tau)} \in \Sigma'$ and the identity matrix otherwise.

Proof. (i) By Input 5, $N(D)$ splits completely in $E(D)$: every place above $N(D)$ has norm $N(D)$ and uniformiser $N(D)$, the group of infinite places, which is the Galois group of $E(D)$ over K, permutes the places above $N(D)$ that lie over a fixed place of K simply transitively, and no place above $N(D)$ is fixed by complex conjugation. The induced places $\mathfrak{p}_\jmath(\tau)$ form one such orbit; so no induced place is the conjugate of another, and every place above $N(D)$ is an induced place or the conjugate of one. These are (a)–(d) of Definition 2.1 for the set of the conjugates. (ii) $\mathrm{ev}_{\jmath,\tau}$ reads the component at $\overline{\mathfrak{p}_\jmath(\tau)} \in \Sigma_\jmath(D)$, where the embedding places the component of the diagonal matrix (Lemma 6.1 (i)); that component has last entry $N(D)$ if $\overline{\mathfrak{p}_\jmath(\tau)} \in \Sigma'$ and 1 otherwise, and $\mathrm{ev}_{\jmath,\tau}$ fixes rational integers. □

LEAN `Data.isSplitLevel_nonInducedPlaces` · `Data.adeleExtension_map_diagN`

Lemma 7.2 (the column level is a conjugated mirabolic group)**.** *In Setting 1.3, with $\ell = N(D)$, let $\Sigma = \Sigma_\jmath(D)$ with its structure of a split level for $N(D)$, let $j \geq 0$, and for $k \in G_f(\underline{X}_j(D))$ put $k^\Delta = \left(\Delta_j(D)^{-1}\, k\, \Delta_j(D)\right)_\Sigma$, the family of the components at Σ.*

(i) For $k \in G_f(\underline{X}_j(D))$: $\mathbf{b}_\mathbf{e}(k^\Delta) \neq 0$ if and only if for every $w \in \Sigma$: k_w is integral; if $\overline{w} \in \Pi(D)$, every entry of the last row of $(k^{-1})_{\overline{w}} - 1$ is divisible by a uniformiser; and if $w \in \Pi(D)$, every entry of the last row of $k_w - 1$ is divisible by a uniformiser.

(ii) For $k \in K_j^{\mathrm{col}}(D)$: $k^\Delta \in \mathrm{Mir}(\underline{X}^\circ_j(D))$.

Proof. (i) Let $w \in \Sigma$. If $w \in \Pi(D)$, the component of $\Delta_j(D)$ at w is the diagonal matrix δ with the entries $1, \dots, 1, N(D)$, and $\delta^{-1} k_w \delta$ has the entries of k_w with the last row divided by $N(D)$ and the last column multiplied by $N(D)$. It is integral with last column congruent to the last basis vector exactly when k_w is integral and the last row of $k_w - 1$ is divisible by $N(D)$, a uniformiser. If $w \notin \Pi(D)$, then $\overline{w} \in \Pi(D)$ by Input 1 (as w divides $N(D)$), the component of $\Delta_j(D)$ at w is 1, and the condition is that k_w is integral with last column congruent to the last basis vector. Since k is

unitary, k_w is the conjugate transpose of $(k^{-1})_{\overline{w}}$, so the last column of $k_w - 1$ is the conjugate of the last row of $(k^{-1})_{\overline{w}} - 1$. (ii) By Input 2, k and k^{-1} are integral with last rows congruent to $(0, \dots, 0, 1)$ at the places of $\Pi(D)$; so (i) gives $\mathbf{b}_{\mathbf{e}}(k^{\Delta}) \neq 0$ and $\mathbf{b}_{\mathbf{e}}((k^{\Delta})^{-1}) \neq 0$, and Lemma 3.3 (ii) shows that k^{Δ} is integral and invertible with last column congruent to $\mathbf{e}$. □

LEAN `Data.lastColDatum_conj_ne_zero_iff` · `Data.atPlaces_conj_mem_mirabolic`

Lemma 7.3 (ramified columns; the operator preserves columns)**.** *In Setting 1.3, with $\ell = N(D)$ and $\Sigma_{\jmath}(D)$ with its structure of a split level for $N(D)$:*

(i) if $\varphi \in \mathrm{Col}^{\mathrm{ram}}(D)$, then $\varphi \in \mathrm{Ram}(\rho_{D,0})$;

(ii) if $\varphi \in \mathrm{Col}(D)$, then $\mathsf{F}(\rho_{D,0})\varphi \in \mathrm{Col}(D)$.

Proof. Write $\Sigma = \Sigma_{\jmath}(D)$ and $\rho = \rho_{D,0}$. (i) Let $w \in \Sigma$. It divides $N(D)$, so w or $\overline{w}$ is a place $\mathfrak{p}$ of $\Pi(D)$ (Input 1). By Input 3, $\varphi = \psi - z_{\mathfrak{p}}(D)\psi$ for a column ψ, and $\nu_{\mathfrak{p}}(D)$ is integral, of norm one and equal to 1 away from w and $\overline{w}$. By Lemma 6.1 (iii) there is a unit x of O_{Σ}, congruent to 1 at the places $v \neq w$ of Σ, with $\mathrm{emb}_{\Sigma}(x \cdot 1)$ the scalar matrix of $\nu_{\mathfrak{p}}(D)$. Scalar matrices are central, so $\mathrm{emb}^{\Delta}_{D,0}(x \cdot 1)$ is this scalar matrix too, $\rho(x \cdot 1) = z_{\mathfrak{p}}(D)$, and φ lies in the image of $1 - \rho(x \cdot 1)$.

(ii) Let $k \in K_0^{\mathrm{col}}(D)$ and let T be right translation by k; we show $T\mathsf{F}(\rho)\varphi = \mathsf{F}(\rho)\varphi$ by Lemma 3.4 with $g = k^{\Delta} \in \mathrm{Mir}(\underline{X}_0^{\circ}(D))$ (Lemma 7.2 (ii)). Let $a, b \in \mathrm{GL}_{\underline{X}_0(D)}(E_{\Sigma})$ with $a\,g\,b \in \mathrm{Mir}(\underline{X}_0^{\circ}(D))$. The operator $\rho(a)\,T\rho(b)$ is right translation by the product of $\mathrm{emb}^{\Delta}_{D,0}(a)$, k and $\mathrm{emb}^{\Delta}_{D,0}(b)$, which is $u\,k$ with

$$u \;=\; \mathrm{emb}^{\Delta}_{D,0}(a)\,k\,\mathrm{emb}^{\Delta}_{D,0}(b)\,k^{-1}.$$

The element u has component 1 outside $\Sigma \cup \overline{\Sigma}$, and $u^{\Delta} = (a\,g\,b)\,g^{-1} \in \mathrm{Mir}(\underline{X}_0^{\circ}(D))$. By Lemma 3.3 (iii) and Lemma 7.2 (i), applied to u and to u^{-1}, the element u is integral at the places above $N(D)$ with last rows congruent to $(0, \dots, 0, 1)$ at the places of $\Pi(D)$. It is supported at places above $N(D)$, none of which is a chosen place (Input 5), so it lies in the chosen level, and $u \in K_0^{\mathrm{col}}(D)$ by Input 2. Hence $u\,k \in K_0^{\mathrm{col}}(D)$ fixes the column φ, which is the hypothesis of Lemma 3.4. □

LEAN `Data.mem_ramified_of_mem_ramifiedColumns` · `Data.fourierOp_mem_column`

Lemma 7.4 (restriction to $Y_m(D)$ keeps gauges and sizes)**.** *In Setting 1.3, but for any prime ℓ with a ring homomorphism $\jmath\colon \mathbb{C} \to \mathbb{C}_{\ell}$:*

(i) Let $\mathsf{d} \in G_f(\underline{X}_m(D))$ be such that for every infinite place τ the matrix $\mathrm{ev}_{\jmath,\tau}(\mathsf{d})$ is $\mathrm{diag}(1, \dots, 1, N(D))$ if $\overline{\mathfrak{p}_{\jmath}(\tau)} \in \Pi(D)$ and the identity otherwise. Let C be a real number and $\varphi \in \mathrm{St}_m(D)$ of gauge at most C at d. Then $\mathrm{res}_D(\varphi)$ has gauge at most C at 1.

(ii) If $\ell \neq N(D)$, C is real and $\varphi \in \mathrm{St}_m(D)$ has gauge at most C at 1, then $\mathrm{res}_D(\varphi)$ has gauge at most C at 1.

(iii) For every ring homomorphism $\sigma\colon \mathbb{C} \to \mathbb{C}$, every $\varphi \in \mathrm{St}_m(D)$ and every point h of the finite adelic unitary group of $Y_m(D)$:

$$\|\sigma \circ \mathrm{res}_D(\varphi)(h)\|_{\mathrm{W}}^2 \;\leq\; \exp\Big(x^{70} - \sum_{\tau} \mathrm{br}_D(\tau) \cdot \log N(D)\Big) \cdot \|\sigma \circ \varphi(\mathrm{bd}_D(h))\|_{\mathrm{W}}^2.$$

Proof. (i) Fix h and put $g = \mathrm{bd}_D(h)$. By Definition 2.13, $\mathrm{res}_D(\varphi)(h) = s \cdot B_D\,\varphi(g)$ with $s = \prod_\tau \tau(\overline{\pi_N(D)})^{-\,\mathrm{br}_D(\tau)}$. By the identity of Input 4, applied to the inverse matrices, B_D carries the action of

$$\mathrm{ev}_{\jmath,\tau}(g\mathsf{d})^{-1} \;=\; \mathrm{ev}_{\jmath,\tau}(\mathsf{d})^{-1}\,\mathrm{ev}_{\jmath,\tau}(g)^{-1}$$

in the weight of stage m to the action of $\mathrm{ev}_{\jmath,\tau}(h)^{-1}$ in the weight of $Y_m(D)$, times the scalar $\prod_\tau \lambda_\tau^{-\,\mathrm{br}_D(\tau)}$, where $\lambda_\tau \in \{N(D), 1\}$ is the last entry of $\mathrm{ev}_{\jmath,\tau}(\mathsf{d})$. On the other hand $|\jmath(\tau(\overline{\pi_N(D)}))| = |\lambda_\tau|$: the left side is the absolute value of $\pi_N(D)$ at the place $\overline{\mathfrak{p}_\jmath(\tau)}$, which is that of $N(D)$ if this place divides $\pi_N(D)$ and 1 otherwise. So the two scalars have the same ℓ-adic size, and the gauge of $\mathrm{res}_D(\varphi)(h)$ at h is the norm of B_D applied to the vector whose norm is the gauge of $\varphi(g)$ at $g\mathsf{d}$; the integer matrix B_D does not increase norms (Q3). (ii) is (i) for $\mathsf{d} = 1$: for $\ell \neq N(D)$ no place above ℓ divides $\pi_N(D)$. (iii) The scalar s contributes $|\sigma(s)|^2 = \prod_\tau |\sigma\tau(\overline{\pi_N(D)})|^{-2\,\mathrm{br}_D(\tau)}$, and $|\sigma\tau(\overline{\pi_N(D)})|^2 = \sigma\tau(\overline{\pi_N(D)}\,\pi_N(D)) = N(D)$, because $\sigma \circ \tau$ is an embedding of the CM field $E(D)$ and so commutes with complex conjugation. The matrix B_D contributes at most $\exp(x^{70})$ by its size bound (Input 4) and the comparisons of sizes of the data (Chapter III). □

LEAN `Data.GaugeLe.restrictToY_of_diag` · `Data.GaugeLe.restrictToY` · `Data.weylNormSq_restrictToY_le` · In the first two Lean takes the prime ℓ, its primality and the homomorphism $\jmath$ as instance arguments.

Proof of Theorem D. Write $\Sigma = \Sigma_\jmath(D)$, $\rho = \rho_{D,0}$, $F = \mathsf{F}(\rho)$, $\Delta = \Delta_0(D)$, and for a finite set Σ' of places $\mathsf{d}(\Sigma') = \mathrm{emb}_\Sigma(\mathrm{dg}_{\underline{X}_0^\circ(D)}(\Sigma'))^{-1}$; so $\Delta\,\mathsf{d}(\Pi(D)) = 1$. For an index a of $\underline{X}_0(D)$ let $t(a)$ be the sum, over the classes of $K_0^{\mathrm{col}}(D)$, the coordinates i of the weight and the infinite places τ, of the number of times a occurs in the tableau that names the coordinate i at τ.

Step 1. For every Σ': every finite family in $\mathrm{Int}_\jmath(\mathrm{Col}(D); \Delta\mathsf{d}(\Sigma'))$ is carried into $\mathrm{Int}_\jmath(\mathrm{Col}(D); \Delta)$ by a square matrix over $O_\jmath$ whose determinant divides $N(D)^{\,t(*)}$. Indeed, a column is determined by its values at representatives of the classes, which are arbitrary vectors of the weight because the rational points act freely (Q4); adjust each representative by a rational point so that it, and its conjugate by $\mathsf{d}(\Sigma')$, are integral above $N(D)$ (Q3). In these coordinates, by Lemma 7.1 (ii), a column that is integral at $\Delta\mathsf{d}(\Sigma')$ becomes integral at Δ after its coordinate i at a class is multiplied by $N(D)$ to the number of occurrences of the last index in the tableaux of i: the action of $\mathrm{diag}(1, \dots, 1, N(D))$ in the weight is diagonal with these powers of $N(D)$, up to an integral Schur matrix (Q3). The determinant of this diagonal map is $N(D)^{\,t(*)}$.

Step 2. For $q \geq 0$, every finite family in $\mathrm{Int}_\jmath(\mathrm{Col}(D); 1)$ is carried by a matrix over $O_\jmath$ of determinant dividing $N(D)^{\,t(*)(q+1)}$ into the forms ψ with $F^q\psi \in \mathrm{Int}_\jmath(\mathrm{Col}(D); \Delta)$. For $q = 0$ this is Step 1 with $\Sigma' = \Pi(D)$. For the step it suffices, by Step 1 with $\Sigma' = \Sigma$ and by composing matrices, that F maps $\mathrm{Int}_\jmath(\mathrm{Col}(D); \Delta)$ into $\mathrm{Int}_\jmath(\mathrm{Col}(D); \Delta\mathsf{d}(\Sigma))$. Let ψ be in the former. Then $F\psi$ is a column (Lemma 7.3 (ii)), and it is a sum of terms $\psi_N(y_*) \cdot \rho(u_0\,\tilde{k}(y))\psi$; the gauge of a sum is at most the largest gauge of a term, and $|\jmath(\psi_N(y_*))| = 1$ since $\psi_N(y_*)$ is a root of unity. The value of $\rho(u_0\tilde{k}(y))\psi$ at g is $\psi(g')$ with $g' = g\,\Delta\,\mathsf{d}(\Sigma)\,\mathrm{emb}_\Sigma(\tilde{k}(y))\,\Delta^{-1}$, and $g'\Delta = g\Delta\mathsf{d}(\Sigma) \cdot \mathrm{emb}_\Sigma(\tilde{k}(y))$, where the last factor is integral above $N(D)$ (Lemma 6.1 (ii)). So the gauge of this value at $g\Delta\mathsf{d}(\Sigma)$ is its gauge at $g'\Delta$ (Q3), at most 1.

Step 3. Take $q = \varrho(D) + 1$ and $e = t(*)\,(\varrho(D) + 2)$. To bound e: $t(a) = t(b)$ for all indices a, b, by the symmetry of the weight under permutations of the coor-

dinates; and $\sum_a t(a)$ is the number of classes times the number of coordinates times the sum over τ of the number of boxes of the weight at τ, which is at most $a(D)$ for each τ (unit *Columns*). The number of classes times the number of coordinates is $\dim_{\mathbb{C}} \mathrm{Col}(D)$ (Q4). Hence $t(*) \le \dim_{\mathbb{C}} \mathrm{Col}(D) \cdot f(D)\, a(D)/n_0(D)$, and $e \le 4\, m\, f(D)\, b \dim_{\mathbb{C}} \mathrm{Col}(D)$ is one of the comparisons of sizes of the data. □

LEAN `Data.exists_indexDvd_fourierOp_pow` · The split level is an explicit argument of the Lean statement, with Lemma 7.1 (i) as its default value.

Lemma 7.5 (the characters of the chain above the level prime)**.** *In Setting 1.3, let ϑ be a tame character of D and, for $i \ge 0$, let $\mu_{D,i}(\vartheta^i)$ be the character of lift i (the product of the base character $\mu^0_{D,i}$ and ϑ^i).*

(i) For $j \ge 0$, a finite index set I and a finite place w of $E(D)$ dividing $N(D)$: the base change of the rank twist $\mu_{D,j}(\vartheta^j)\, \upsilon^{\lfloor |I|/2 \rfloor}$ is unramified at w.

(ii) For $j, i \ge 0$: the space character, in rank $|\underline{X}_j(D)|$, of the rank twist of $\mu_{D,i}(\vartheta^i)$ for $\underline{X}_j(D)$ is the base change $x \mapsto \mu_{D,i}(\vartheta^i)(x/\overline{x})$.

Proof. (i) The tame part of $\mu_{D,j}(\vartheta^j)$ is trivial at w and at $\overline{w}$: these places are not chosen (Input 5) and do not belong to the set $Q(\vartheta)$, whose places have norm prime to $N(D)$; and υ is unramified. Apply Q5. (ii) is Q5, as $|\underline{X}_j(D)| = n_j(D)$ is even (Input 5). □

LEAN `Data.charUnramifiedAt_rankTwist_of_N_mem` · `Data.spaceChar_rankTwist`

8 Where the results are used

Theorem A, Theorem B, Lemma 6.2 (ii), Lemma 6.3 and Lemma 7.3 (i) are used together in the unit *Columns*, in the file that trades denominators at the level prime for a column operator: there the chain of pairs of lifts whose data are the translates of Theorem B is rewritten, by Theorem B and then Theorem A with $\rho_j = \rho_{D,j}$ and the laws given by the lifts of the definite chain, as a constant times the chain with the simplest data applied to $\mathsf{F}(\rho_{D,0})^{\varrho(D)+1}$. Theorem D and Lemma 7.4 enter the local statement at the level prime in the same unit. Theorem C is used in *Columns* (the chain of a good datum is a plain sum over classes, which gives its integrality) and in the unit *Exchange*, together with Lemma 5.1 (i), (iii). Lemma 5.1 (ii), Lemma 6.2 (i) and Lemma 7.5 (i) are used in the note on the seesaw identity, to show that a central element at a place dividing $\pi_N(D)$ acts trivially on a chain of pairs. The definitions of chains, of joint invariance and of $\mathsf{T}_{N,\Sigma}(I, I', I'')$ are part of the definition of a good datum (unit *Columns*), and $\omega^{(2)}_{\mathcal{K},i}(a)$ occurs in the proof of the seesaw identity.

9 Notes on the formal text

Arguments that the census lists and Lean does not take. The census of the workspace reads the arguments of a declaration from its text. In the files treated here a section declares variables (a vector y of residues, a family a of points, data Φ, a character b); the text reading lists such a variable as an argument whenever its name occurs in a statement, also when the occurrence is bound there. This concerns the vector y in the Gauss-sum operator, in the section of the split level, in Lemma 2.5, Lemma 3.3 (iv), Lemma 4.1 (i) and Theorem A; the family a in the definition of joint

invariance; the data Φ in Lemma 5.1 (iii); the character b in Lemma 3.2. Nothing was compiled for this note, so Lean's own answer is not recorded; the statements above follow the Lean text.

Arguments that Lean takes and the census misses. A lift law is defined for a split level Σ for a prime N, and so are the mirabolic group, the function $f^\natural$ and the Gauss-sum operator; the statements of Definition 2.7, Lemma 3.4, Lemma 4.1 (ii) and Proposition 4.2 therefore take N, its primality and the split level, although their text does not name them. The embedding $\mathrm{emb}_\Sigma(g)$ uses only that Σ is disjoint from its conjugate, but takes a split level. The set $\Sigma_\jmath(D)$ and Lemma 7.4 (i) take a prime ℓ and a homomorphism $\jmath$ that their names do not show.

A boundary case. The adele $[y/N]$ is defined in the formal text for every natural number N, with no hypothesis on Σ. The description by the representative in $\{0, \dots, N-1\}$ is the case $N > 0$; for $N = 0$ division by zero returns zero in Lean, so the adele is zero. Every use in this note has N prime.

Choices. The section sec, the map k of Lemma 2.5 and hence the matrices $\tilde{k}(y)$ in the Gauss-sum operator are chosen once (by the axiom of choice). The proofs use only that $\mathrm{red} \circ \mathrm{sec}$ is the identity and that $k(y)$ has last row and last column y; a different choice changes $\mathsf{F}(\rho)$ on vectors that are not fixed by the principal congruence subgroup of level N.

Statements more special or more general than their use. Part (ii) of Lemma 6.3 compares two ranks I, I' although the value does not depend on the rank. In Theorem C the properties P_j are arbitrary and serve as an induction hypothesis carried along the chain; hypothesis (b) (free classes) is what turns class integrals into sums. In Theorem D the exponent $\varrho(D) + 1$ is the number of pairs of lifts; the proof gives the statement for every power q with the exponent $t(*)(q+1)$. Lemma 6.2 (ii) is stated in the dossier with the function $S \mapsto f(k^{-1}Sg)$ of residue matrices; the Lean text has the function $S \mapsto f^\natural(k^{-1}Sg)$ of matrices over E_Σ, which is not of the form $f'^\natural$ when g or k is not integral, and this is the form used above.

Members that act implicitly. Some members of this note are rewriting rules that no proof cites by name: the formula for the components of $[y/N]$ (Definition 2.1) and the components of a scalar matrix (Lemma 6.1). An unnamed instance (the reduction is a local homomorphism) is mentioned after Lemma 3.1.

How this note was written. From the unit's dossier: the statements in words and the structured proofs of the members of the two files (for the members that have a proof tree). Both Lean files were read completely in the source at the pinned commit, and every statement above was compared with its Lean text argument by argument; this is how the difference in Lemma 6.2 (ii) was found. The dossier's reader marks the claims for the long proofs of Theorem B and of Theorem D as read from lemma names. Both proofs were read line by line in the Lean source for this note; the computation with the two additive characters in Step 4 of Theorem B and the three steps of Theorem D are written from that reading. The proofs of Theorem A, Proposition 4.2 and Lemma 3.4 were checked by hand as written here;

they are shorter than the formal ones, which carry out the same re-indexings through rewriting. Of the background library, the definitions of kernels and lifts, of gauges and integral forms, of the Schrödinger action and of names were read in the source; their properties Q1–Q5 are quoted, not read. The inputs from the unit *Columns* are stated as the dossier's imports read them. The program check of the unit's dossier covered 503 leaf steps of 83 proofs.

Revision. After an independent referee's report, Input 4 states the identity of the branching matrix with its exact exponent, read in the Lean source of the unit *Columns* and of the library, and the central adele and its translation carry the data: $\nu_w(D)$, $z_w(D)$.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Lemma 7.4	`Data.GaugeLe.restrictToY_of_diag`	LiftCharacters.lean	20–33
Lemma 7.4	`Data.GaugeLe.restrictToY`	LiftCharacters.lean	37–39
Lemma 7.4	`Data.weylNormSq_restrictToY_le`	LiftCharacters.lean	41–51
Definition 2.15	`Data.nonInducedPlaces`	LiftCharacters.lean	55–55
Lemma 7.1	`Data.isSplitLevel_nonInducedPlaces`	LiftCharacters.lean	57–61
Definition 2.15	`Data.diagPointN`	LiftCharacters.lean	67–67
Definition 2.15	`Data.conjEmbedGL`	LiftCharacters.lean	71–71
Definition 2.15	`Data.nonInducedRep`	LiftCharacters.lean	75–76
Lemma 7.1	`Data.adeleExtension_map_diagN`	LiftCharacters.lean	78–82
Lemma 7.2	`Data.lastColDatum_conj_ne_zero_iff`	LiftCharacters.lean	84–95
Lemma 7.2	`Data.atPlaces_conj_mem_mirabolic`	LiftCharacters.lean	97–101
Lemma 7.3	`Data.mem_ramified_of_mem_ramifiedColumns`	LiftCharacters.lean	103–108
Lemma 7.3	`Data.fourierOp_mem_column`	LiftCharacters.lean	112–123
Theorem D	`Data.exists_indexDvd_fourierOp_pow`	LiftCharacters.lean	128–169
Lemma 7.5	`Data.charUnramifiedAt_rankTwist_of_N_mem`	LiftCharacters.lean	171–174
Lemma 7.5	`Data.spaceChar_rankTwist`	LiftCharacters.lean	178–180
Definition 2.1	`IsSplitLevel`	NamesAtSplitLevel.lean	25–29
Definition 2.1	`residueOverN`	NamesAtSplitLevel.lean	33–34
Definition 2.1	`residueOverN_apply`	NamesAtSplitLevel.lean	37–38
Lemma 3.1	`idem_mul_residueOverN`	NamesAtSplitLevel.lean	41–41
Definition 2.3	`IsSplitLevel.diagN`	NamesAtSplitLevel.lean	48–49
Definition 2.2	`IsSplitLevel.residueEquiv`	NamesAtSplitLevel.lean	56–57
Definition 2.2	`IsSplitLevel.reduction`	NamesAtSplitLevel.lean	61–61
Lemma 3.1	`IsSplitLevel.reduction_eq_iff`	NamesAtSplitLevel.lean	64–67
Definition 2.2	`IsSplitLevel.inclGL`	NamesAtSplitLevel.lean	72–73
Definition 2.2	`IsSplitLevel.reductionGL`	NamesAtSplitLevel.lean	77–77
Definition 2.2	`IsSplitLevel.liftGL`	NamesAtSplitLevel.lean	80–81
Definition 2.3	`IsSplitLevel.diagNInv`	NamesAtSplitLevel.lean	87–87
Definition 2.1	`IsSplitLevel.psiN`	NamesAtSplitLevel.lean	91–91
Definition 2.2	`IsSplitLevel.lastBasisVector`	NamesAtSplitLevel.lean	94–94
Lemma 2.5	`IsSplitLevel.exists_GL_last_col_and_row`	NamesAtSplitLevel.lean	98–102
Definition 2.6	`IsSplitLevel.fourierOp`	NamesAtSplitLevel.lean	106–107
Definition 2.3	`IsSplitLevel.mirabolic`	NamesAtSplitLevel.lean	111–112
Definition 2.3	`IsSplitLevel.ramified`	NamesAtSplitLevel.lean	116–117
Definition 2.4	`IsSplitLevel.liftFn`	NamesAtSplitLevel.lean	121–123
Definition 2.4	`IsSplitLevel.lastRowDatum`	NamesAtSplitLevel.lean	127–128
Definition 2.4	`IsSplitLevel.lastColDatum`	NamesAtSplitLevel.lean	132–132
Definition 2.4	`IsSplitLevel.charDatum`	NamesAtSplitLevel.lean	136–136

In this note	Lean declaration	File	Lines
Definition 2.7	`IsSplitLevel.LiftLaw`	NamesAtSplitLevel.lean	141–146
Lemma 3.3	`IsSplitLevel.lastRowDatum_conj_u₀`	NamesAtSplitLevel.lean	161–165
Lemma 3.3	`IsSplitLevel.exists_inclGL_eq`	NamesAtSplitLevel.lean	167–168
Lemma 3.3	`IsSplitLevel.lastColDatum_ne_zero`	NamesAtSplitLevel.lean	170–171
Lemma 3.4	`IsSplitLevel.mirabolic_fix_fourierOp`	NamesAtSplitLevel.lean	178–188
Lemma 3.3	`IsSplitLevel.liftFn_comp_eq_sum_smul`	NamesAtSplitLevel.lean	191–193
Lemma 4.1	`IsSplitLevel.LiftLaw.eq_zero_of_smul_invariant`	NamesAtSplitLevel.lean	198–202
Lemma 4.1	`IsSplitLevel.LiftLaw.inclGL_lastColDatum`	NamesAtSplitLevel.lean	205–207
Proposition 4.2	`IsSplitLevel.LiftLaw.u₀_charDatum_eq_fourierOp`	NamesAtSplitLevel.lean	212–225
Theorem A	`IsSplitLevel.compUpTo_pairs_eq_smul`	NamesAtSplitLevel.lean	233–255
Definition 2.9	`chain`	NamesAtSplitLevel.lean	269–269
Lemma 5.1	`chain_equivariant`	NamesAtSplitLevel.lean	275–277
Definition 2.9	`chain₂`	NamesAtSplitLevel.lean	281–281
Definition 2.9	`twoKernelAction`	NamesAtSplitLevel.lean	285–285
Lemma 5.1	`chain₂_equivariant`	NamesAtSplitLevel.lean	290–293
Lemma 5.1	`exists_linear_chain_eq_pure`	NamesAtSplitLevel.lean	295–296
Definition 2.10	`IsJointlyInvariant`	NamesAtSplitLevel.lean	300–302
Theorem C	`IsJointlyInvariant.sum_chain_mem_nsmul`	NamesAtSplitLevel.lean	307–323
Lemma 6.1	`scalarFin_atPlace`	NamesAtSplitLevel.lean	334–335
Lemma 6.2	`pairWeilFinite_scalarFin`	NamesAtSplitLevel.lean	337–340
Definition 2.12	`translateSum`	NamesAtSplitLevel.lean	345–347
Definition 2.12	`Space.projOnFinPart`	NamesAtSplitLevel.lean	351–352
Definition 2.11	`IsSplitLevel.detCharAt`	NamesAtSplitLevel.lean	358–358
Definition 2.11	`IsSplitLevel.embedGL`	NamesAtSplitLevel.lean	366–367
Lemma 6.1	`IsSplitLevel.atPlaces_embedGL`	NamesAtSplitLevel.lean	370–371
Lemma 6.1	`IsSplitLevel.embedGL_mem_supportedOn`	NamesAtSplitLevel.lean	374–375
Lemma 3.1	`IsSplitLevel.disjoint_star`	NamesAtSplitLevel.lean	383–383
Lemma 3.1	`IsSplitLevel.natCast_mem`	NamesAtSplitLevel.lean	385–385
Lemma 6.1	`IsSplitLevel.mem_integralAbove_of_inclGL_eq`	NamesAtSplitLevel.lean	387–390
Lemma 3.1	`IsSplitLevel.reduction_eq_iff_idem_mul_mem`	NamesAtSplitLevel.lean	395–396
Lemma 6.1	`IsSplitLevel.exists_embedGL_scalar_eq_scalarFin`	NamesAtSplitLevel.lean	402–408
Definition 2.12	`IsSplitLevel.name`	NamesAtSplitLevel.lean	414–416
Lemma 6.2	`IsSplitLevel.liftFn_eq_residueDatum`	NamesAtSplitLevel.lean	419–421
Lemma 6.2	`IsSplitLevel.isSchwartzBruhat_liftFn`	NamesAtSplitLevel.lean	423–425
Lemma 6.2	`IsSplitLevel.pairWeilFinite_name`	NamesAtSplitLevel.lean	430–436
Lemma 6.3	`IsSplitLevel.detCharAt_inclGL`	NamesAtSplitLevel.lean	441–445
Lemma 6.3	`IsSplitLevel.norm_detCharAt_diagN_anti`	NamesAtSplitLevel.lean	451–455
Lemma 3.1	`IsSplitLevel.diagN_eq_sdiff_mul`	NamesAtSplitLevel.lean	460–461
Lemma 3.2	`IsSplitLevel.exists_sub_residueOverN_mem_iff`	NamesAtSplitLevel.lean	467–472
Theorem B	`IsSplitLevel.translateSum_eq_sum_name`	NamesAtSplitLevel.lean	479–521

Legend: in the Lean text the set Σ of places is `P` and Σ' is `S`; the index sets I, I', I'' are `n`, `n'`, `m`, and I^+ is the sum of `n` and a one-point type; the representations ρ, ρ' carry the Greek letter pi and the constant $\varkappa$ of a law the letter gamma; $f^\natural$ is `liftFn f`; $\mathbf{r}$, $\mathbf{b}_y$, $\mathbf{c}_y$ are `lastRowDatum`, `lastColDatum y`, `charDatum y`; u_0 is `diagNInv`; the ring homomorphism $\jmath$ is the field of the class `PadicPlace`.

Two statements as they stand in the artifact, the class of split levels and the Gauss-sum operator:

```
class IsSplitLevel (P : Finset (FinPlace E)) (N : outParam ℕ) where
  star_notMem : ∀ w ∈ P, star w ∉ P
  q_eq : ∀ w ∈ P, w.q = N
  protected v_natCast : ∀ w ∈ P, v (N : LocField w) = exp (-1)
  natCast_mem_iff w : ↑N ∈ w.1 ↔ w ∈ P ∨ star w ∈ P
```

```
def fourierOp := π.ρ diagNInv * ∑ y with IsUnit (y (.inr ())),
  psiN (y (.inr ())) • π.ρ (inclGL (liftGL (exists_GL_last_col_and_row.choose y)))
```

Notation

E, K a CM field with a CM type; in Setting 1.3, $E = E(D)$ over the imaginary quadratic field

$\mathbb{A}_f$, $\hat{\mathcal{O}}$ finite adeles of E; their integers

w, v, $\overline{w}$ finite places of E; the conjugate place

E_w, O_w, $\mathbb{F}_w$, $\mathrm{N}w$ completion, its integers, residue field, norm

$\mathbf{j}$, α, υ the Weil data: an element, an idele, the splitting character

ψ_0, ψ_f the standard finite additive characters

N a prime number (the level prime $N(D)$ for the data)

Σ, Σ' finite sets of finite places; split levels

E_Σ, O_Σ, $(\mathbb{Z}/N)^\Sigma$ product of completions; integral elements; residue ring

$[y/N]$, $\psi_N(y)$, $\tilde{y}$ the adele $\tilde{y}/N$ on Σ; its character value; least representative

$\mathbf{1}_\Sigma$, ϵ, ϵ' the adele that is 1 on a set of places; those of Σ, Σ'

I, I', I'', I_j, J finite index sets; $I^+ = I \sqcup \{*\}$, $*$ the last index

red, sec reduction modulo N; a chosen section on matrices

$\mathbf{e}$ the last basis vector

$\mathrm{dg}_I(\Sigma')$, u_0 diagonal matrix with last entry N at $\Sigma \cap \Sigma'$; the inverse for $\Sigma' = \Sigma$

$\mathrm{Mir}(I)$, $\mathrm{Ram}(\rho)$ the mirabolic group; ramified vectors

ρ, ρ', ρ_j complex representations of $\mathrm{GL}_{I^+}(E_\Sigma)$

φ, ψ vectors of a representation; algebraic forms

f, $f^\natural$, h, f_t functions of residue matrices; extension by zero

$\mathbf{r}$, $\mathbf{b}_y$, $\mathbf{c}_y$ indicator of the last row; of a last column; character of a last row

$k(y)$, $\tilde{k}(y)$, C the matrix of Lemma 2.5; its lift; an auxiliary matrix or a real bound

$\mathsf{F}(\rho)$, F the Gauss-sum operator

L, L_j, $\varkappa$, $\varkappa_j$ lift laws and their constants

A_i, d_i, o_i, p, q, l the pair operators, scalars, counts; numbers of pairs; an index

B_j^q, T, S, S', R, r, t, c, x, y, z, u, a, b, g, k auxiliary operators, matrices, residues, units and group elements, as introduced where used

$G_f(I)$, $K_0(I)$, X_I finite adelic unitary group of the standard space X_I; stabiliser of its lattice

κ, e_j, $W_\kappa(I)$, $\mathcal{A}(I;e)$ weight, last entries, space of the weight, algebraic forms

$\mathcal{K}_j$, ω_j, $\mathcal{J}_j$, $\mathcal{S}_j$, K_j kernels: representation, jet matrix, data space, level

Φ, Φ_x, $\mathrm{ch}(\mathcal{K};\Phi)$, $\mathrm{ch}_2(\mathcal{K};\Phi)$ data; chains of lifts and of pairs

$\omega^{(2)}_{\mathcal{K},i}(a)$ the joint action on the data of a pair

M, $\mathcal{X}$, $\mathcal{X}_0$, L_x, A, P_j, β, G, Γ in Theorem C: target space, index set, linear maps, a subset, properties, a linear map, the integrand, rational points

γ, δ rational points, in the definition of a kernel; δ also a diagonal matrix

$\mathrm{cl}(K)$, m, n, i, j number of classes; natural numbers

$\mathrm{emb}_\Sigma(g)$, g_Σ embedding at a split level; components at Σ

η, $\mathrm{dc}_{\eta,k}$, ν, ν' a Hecke character; the determinant character; norm-one adeles

V, $\mathcal{H}$, $\mathcal{W}_f(V)$, ϱ_f hermitian spaces; finite part; finite Heisenberg operators

$\mathrm{pr}_V^{\Sigma'}$, $\mathrm{nm}_\Sigma(F)$ projection to vacua above $\Sigma' \cup \overline{\Sigma'}$; named operator

$\mathsf{T}_{N,\Sigma}(I, I', I'')$ the sum of pairs of translates with denominator N

$\mathfrak{p}$, ζ, Z, Ψ, X in the proof of Theorem B: an adele, a residue, tensors of functions, functions, a matrix

$P = (N_0, d)$, D parameters; admissible data

x, x_1, b, μ, m, r, g_0, g_1, π_g components of D (also local letters, as said)

$N(D)$, $\pi_N(D)$, $\Pi(D)$ the level prime; $\pi_N(D)\overline{\pi_N(D)} = N(D)$; the places dividing $\pi_N(D)$

$f(D)$, $a(D)$, $n_j(D)$, $\varrho(D)$ number of infinite places; a size; ranks; $m = 2\varrho(D) + 2$

$\underline{X}_j(D)$, $\underline{X}^\circ_j(D)$, $X_j(D)$, $Y_m(D)$ index sets; the definite spaces

$\mathrm{St}_j(D)$, $K_j^{\mathrm{col}}(D)$, $\mathrm{Col}(D)$, $\mathrm{Col}^{\mathrm{ram}}(D)$ forms of stage j; its level; columns; ramified columns

$\nu_w(D)$, $z_w(D)$, $\mathfrak{p}$ the central adele of D at a place w of $\Pi(D)$; its translation; such a place (without D, ν_w is the component at w of an adele ν)

res_D, bd_D, B_D, $\mathrm{br}_D(\tau)$, s, λ_τ restriction; $h \mapsto \mathrm{diag}(h, 1)$; branching matrix; exponents; scalars

τ, σ an infinite place with its embedding; a ring homomorphism $\mathbb{C} \to \mathbb{C}$

ℓ, $\jmath$, $O_\jmath$ a prime; a ring homomorphism $\mathbb{C} \to \mathbb{C}_\ell$; its valuation ring in $\mathbb{C}$

$\mathfrak{p}_\jmath(\tau)$, $\mathrm{ev}_{\jmath,\tau}$ the induced prime; evaluation of adeles at its conjugate

$\mathrm{gauge}_\jmath(g;u)$, $\mathrm{Int}_\jmath(\mathcal{V};\mathrm{d})$, $\|\cdot\|^2_{\mathrm{W}}$ gauge; integral forms at a point; Gram norm

$\Sigma_\jmath(D)$, $\Delta_\jmath(D)$, Δ, δ non-induced places; the diagonal point; a diagonal matrix

$\mathrm{emb}_{D,j}^{\Delta}$, $\rho_{D,j}$, k^{Δ} conjugated embedding; the representation on stage j; conjugated components

d, $t(a)$, e a point; tableau counts; the exponent of Theorem D

ϑ, $Q(\vartheta)$, $\mu_{D,i}(\vartheta^i)$, $\mu_{D,i}^0$ a tame character, its set of places, the characters of the lifts

READBACK NOTE 24

The characters, the level prime and the index sets of the data; the grid spaces

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. Besides scales and ranks, the data D of the argument carry a list of chosen objects from which the automorphic forms of Chapters V to VII are built. This note fixes them: the chosen places above the two primes g_0, g_1 and the residue characters there; three characters of the norm-one group U(1) with their weights and tame parts; the level, a prime $N(D)$ in a short window, with an integer of norm $N(D)$; tame characters and twists; the index sets of the hermitian spaces of the chain of theta lifts, the finite group $\mathsf{G}(D)$ and its idempotent; and the spaces, characters and levels on the two grids of lifts of Chapter VII. Three statements have content: the level exists and $m \log b \le \log N(D) \le 30\, m \log b$ (Theorem A); the characters exist, by Weil's criterion for algebraic Hecke characters (Theorem B); and for interlacing weights there is an integral pairing that computes periods over a definite unitary group as finite means (Theorem C).

What this note is. The artifact grh-compact is a body of Lean code whose final theorems state the Riemann hypothesis and its analogue for Dirichlet L-functions. That is a claim: its authors report that Lean's kernel accepts the code; the record of checks of this write-up says who has built what; nothing was compiled for this note, and no mathematician has read the proof through. This note is one of four on one unit, the folder of the seesaw identity (Chapter V): it treats 88 of the unit's 312 declarations, those of the file of index types and of the file of the grid spaces, which are vocabulary for other units; the notes *The seesaw identity for the chain of theta lifts*, *Two consecutive theta lifts* and *Chains of definite theta lifts and the Gauss-sum operator* treat the rest. Every statement below was compared, argument by argument, with the Lean text. Proofs are given in full, except that results of other units and of the background library are quoted as *Inputs*, each with its owner. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 88 declarations.

1 The statements

The notation is that of Section 2. In particular D is admissible data for parameters $P = (N_0, d)$, with scales x, x_1, b, exponent μ and number of lifts m; $E(D)$ is its CM field, with infinite places τ; $w_{D,j}(\tau)$ are the weights of Chapter III; and $\mathrm{tar}(D)$, $N(D)$, $\pi_N(D)$, $\omega_{D,w}$, $\mu^0_{D,j}$, $\mu_{D,j}(\vartheta)$, $\psi_{D,j}(\vartheta)$ are defined in Sections 3 to 6.

Theorem A (the level). In Setting 2.1, for parameters P and admissible data D for P:

(i) $m \log b \le \mathrm{tar}(D) \le 29\, m \log b$;

(ii) $N(D)$ is a prime number, $N(D) \equiv 1$ modulo $N_1(D)\, d$, and

$$\mathrm{tar}(D) \le \log N(D) \le \mathrm{tar}(D) + \sqrt{\log b}, \qquad m \log b \le \log N(D) \le 30\, m \log b;$$

(iii) there is an integer π of $E(D)$ with $\pi\,\overline{\pi} = N(D)$, and $\pi_N(D) \cdot \overline{\pi_N(D)} = N(D)$.

LEAN `Data.logLevelTarget_mem_Icc` · `Data.N_prime` · `Data.N_modEq_one` · `Data.log_N_mem_window` · `Data.log_N_mem_Icc` · `Data.exists_pi_mul_star` · `Data.π_mul_star`

Theorem B (the characters). In Setting 2.1, for parameters P and admissible data D for P:

(i) for all natural numbers c and n there is a character of $\mathrm{U}(1)$ of $E(D)$ (a continuous character of the norm-one adeles, trivial on the norm-one elements of $E(D)$) of weight $\tau \mapsto -c \cdot w_{D,0}(\tau)$ and tame part $(\omega_{D,w}^{n})_w$;

(ii) for every natural number j, the character $\mu_{D,j}^{0}$ has weight $\tau \mapsto -w_{D,j}(\tau)$ and tame part $(\omega_{D,w}^{j})_w$;

(iii) for every tame character ϑ of D and every natural number j, the character $\mu_{D,j}(\vartheta)$ has weight $\tau \mapsto -w_{D,j}(\tau)$ and tame part $\psi_{D,j}(\vartheta)$.

LEAN `Data.exists_char_hasWeight` · `Data.mu₀_hasWeight` · `Data.hasWeight_liftChar`

Theorem C does not involve the data. A *weight* for a CM field E is a pair (κ, e): for every infinite place τ of E a function $\kappa_\tau\colon \mathbb{N} \to \mathbb{N}$ (the number $\kappa_\tau(h)$ of columns of height h of a Young diagram) and an integer e_τ. In rank N its *doubled Harish-Chandra parameter* is

$$\mathrm{hc}_{\kappa,e}^{N}(\tau, p) = 2\Big(e_\tau + \sum_{p<h<N} \kappa_\tau(h)\Big) + N - 1 - 2p \qquad (0 \le p < N),$$

and a family $l = (l_\tau)$ of maps from an index set of N elements to $\mathbb{Z}$ *is the doubled Harish-Chandra parameter of* (κ, e) if every l_τ is injective and its values, in decreasing order, are $\mathrm{hc}_{\kappa,e}^{N}(\tau, 0), \dots, \mathrm{hc}_{\kappa,e}^{N}(\tau, N-1)$. The representation $\rho_{\kappa,e}$, the Weyl norm $\|\cdot\|$, the functions $F_\eta^{l,\mathrm{s}}$ and the mean $\mathrm{mean}_{\mathcal{X}}(\cdot)$ are recalled in Section 8.

Theorem C (branching pairing). Let E be a CM field with a CM type, n a non-empty finite index set and n' a finite index set, (κ, e) and (κ', e') weights for E, $l = (l_\tau)$ and $l' = (l'_\tau)$ families of maps $l_\tau \colon n \to \mathbb{Z}$, $l'_\tau \colon n' \to \mathbb{Z}$, $\sigma \colon n \sqcup \{*\} \to n'$ a bijection, and $\mathbf{s}$, $\mathbf{r}$, $\mathbf{t}$ signs on n, on $\{*\}$ and on n'. Suppose that $(\mathbf{s} \sqcup \mathbf{r}) \circ \sigma^{-1} = \mathbf{t}$ and $\mathbf{t} = 1$; that l' is the doubled Harish-Chandra parameter of (κ, e) and $-l$ is the doubled Harish-Chandra parameter of (κ', e'); and that l and l' interlace: for every infinite place τ and every integer y,

$$\#\{j \in n' : y < l'_\tau(j)\} - \#\{i \in n : y < l_\tau(i)\} \in \{0, 1\}.$$

Then there is a matrix B with integer entries, from coordinates of weight κ in rank $|n'|$ to coordinates of weight κ' in rank $|n|$, with the following properties.

(a) There are integers N_τ, one for each infinite place τ, such that:

(1) for all commutative rings S with an involution, all fields R of characteristic 0, all families of ring homomorphisms $f_\tau \colon S \to R$, all $g \in \mathrm{U}_n(S)$, $u \in \mathrm{U}_1(S)$ and all vectors v', v over R,

$$\langle \rho_{\kappa', e'}(f(g))v',\, B\, \rho_{\kappa, e}(f(\mathrm{diag}_\sigma(g, u)))v \rangle = \prod_\tau f_\tau(\det u)^{N_\tau} \cdot \langle v', Bv \rangle;$$

(2) $|N_\tau| \le \mathrm{boxes}(\kappa, |n'|) + |e_\tau|$ for every τ;

(3) for every natural number $M \ge \mathrm{boxes}(\kappa, |n'|)$ and all complex vectors v', v: $|\langle v', Bv \rangle| \le (M^M)^3 \, \|v'\| \, \|v\|$.

(b) For every point $g = (g_\tau)_\tau$ of the product, over the infinite places, of the real unitary groups of the signs $\mathbf{t}$ there is a complex number C such that for every standard hermitian space V with index set n and signs $\mathbf{s}$, all algebraic forms η of weight (κ', e') in rank $|n|$ and ϕ of weight (κ, e) in rank $|n'|$, every $u \in \mathrm{U}_1(\mathbb{A}_f)$ and every level $\mathcal{K} \subseteq \mathrm{U}_n(\mathbb{A}_f)$,

$$\int_{[\mathrm{U}(V)]} F_\eta^{-l, \mathbf{s}}(y) \cdot F_\phi^{l', \mathbf{t}}(\mathrm{diag}_\sigma(y, (1, u)) \cdot (g, 1))\, dy = \frac{C}{[\mathcal{K}_0 : \mathcal{K}]} \cdot \mathrm{mean}_{\mathcal{K}}(h \mapsto \langle \eta(h), B\, \phi(\mathrm{diag}_\sigma(h, u)) \rangle).$$

LEAN `exists_branching_pairing`

2 The setting

Conventions. Throughout, P denotes parameters and D admissible data for P; Setting 2.1 recalls what these are, and every statement that takes D as an argument cites it. The letter τ denotes an infinite place of the field $E(D)$ and w a finite place of it; $i \in \{0, 1\}$ numbers the two primes g_0, g_1 of D. A stage is a natural number j; ϑ is a tame character of D (Definition 6.1), in Section 9 an m-fold twist; ψ is a family $(\psi_w)_w$ of characters of the residue fields of $E(D)$. In Definition 7.1 and Lemma 7.2, S, T and α are sets. In Theorem C, E is a CM field, n and n' are finite index sets, n non-empty, and κ, κ', e, e', l, l', $\mathbf{s}, \mathbf{r}, \mathbf{t}$, σ are as stated there; in Lemma 9.7, n and n' are finite index sets. In Section 9, z and c are elements of $\{0, 1\}$, I is a set, and

i, i', j, j', a, b, l are natural numbers. Every other letter is bound by the statement that uses it.

Setting 2.1 (parameters and data)**.** *Parameters* are a pair $P = (N_0, d)$: a positive multiple N_0 of 18480 and a prime $d \equiv N_0/2 - 1$ modulo N_0; put $K(P) = \mathbb{Q}(\sqrt{-d})$. *Admissible data* for P are a record D with these components: real scales x, x_1, b; natural numbers μ (the exponent), m (the number of lifts) and r (the rank); a set $\mathrm{thick}(D)$ of infinite places of the field $E(D)$ below; two primes $g_0 \neq g_1$; and an integer π_g of $K(P)$ with $\pi_g \overline{\pi_g} = g_0 g_1$. They satisfy the conditions of Chapter III. This note uses the following ones, in which $C(P) \geq 64$ is the constant of P of Chapter III:

(D1) the scales satisfy $10^{12}\, C(P) \leq \log\log b$, $100 \log\log b \leq \log x$, $10^6 \log x \leq \log x_1$ and $10^6 \log x_1 \leq \log b$;

(D2) m is a power of 2, $m \geq 2$, and $2^\mu = m^{22}$; $\sqrt{x} \leq m \leq 2\sqrt{x}$; $\mu \leq 32 \log x$; r is a power of 3 with $x^5 \leq r \leq 3x^5$;

(D3) for $i \in \{0, 1\}$: $g_i \equiv 1$ modulo $N_1(D)\, d$, the square of $M_\omega(P, \mu)$ divides $g_i - 1$, and $b^{10} \leq g_i \leq b^{11}$;

(D4) for every real U with $100 \log b \leq U \leq 2^\mu$ there is a prime $p \equiv 1$ modulo $M_\omega(P, \mu)^2$ with $U \leq \log p \leq U + \sqrt{\log b}$.

The following objects of Chapter III are attached to P and D. The number $N_1(D) = N_0 \cdot 3^{\lfloor \log_3 \mu \rfloor + 5} \cdot 2^\mu$ and the field $E(D)$, generated over $K(P)$ by a root of unity ς_D of order $N_1(D)$: a CM field, with complex conjugation $y \mapsto \overline{y}$ and ring of integers $\mathcal{O}_{E(D)}$. The set $\mathrm{Pl}(D)$ of its infinite places, of cardinality $f(D)$; each $\tau \in \mathrm{Pl}(D)$ comes with an embedding $E(D) \to \mathbb{C}$, also written τ, and $|y|_\tau = |\tau(y)|$. The root discriminant $\mathrm{rd}_{E(D)}$. The finite places of $E(D)$ are the non-zero prime ideals w of $\mathcal{O}_{E(D)}$; $\mathbb{F}_w$ is the residue field, $\mathrm{N}w$ its cardinality, $\overline{w}$ the conjugate place, and w *lies above* a prime p if $p \in w$. The numbers $k(D) = r + 1 - m$, $t(D) = 5r$, $h_2(D) = r^8$, $h(D) = 2\, h_2(D)$ and $n_j(D) = k(D) + j\,(t(D) + 1)$; the least multiple $a(D)$ of $N_1(D)$ that is at least $x\, b$, and the least multiples $b_1(D)$, $b_2(D)$ of $N_1(D)$ that are at least $a(D)/x$ and $a(D)/x_1$. The *weights*

$$w_{D,j}(\tau) = \begin{cases} b_1(D) + j\, h(D) + h_2(D) & \text{if } \tau \in \mathrm{thick}(D), \\ b_2(D) + j\, h(D) + h_2(D) & \text{otherwise,} \end{cases}$$

and the *nominal weight* $w_D(\tau)$, equal to $a(D)/x$ if $\tau \in \mathrm{thick}(D)$ and to $a(D)/x_1$ otherwise. The number $g(D) = g_0 g_1$. Finally $M_\omega(P, \mu) = (N_0 d)^{\mu+6}$; $\varpi(P, \mu)$ is the purely imaginary generator of the different of $E(D)$ of Chapter III; and $\Omega(P, \mu) = e^{\mathrm{cs}(d)} \cdot 2\pi / \mathrm{rd}_{E(D)}$, where $\mathrm{cs}(d)$ is Euler's constant minus the real part of the logarithmic derivative at 1 of the L-function of the quadratic character of $K(P)$.

Inputs from Chapter III (unit 152, the folder of the data; the notes *The parameters and the CM field* and *Admissible data*). In Setting 2.1:
(I1) If p is a prime with $p \equiv 1$ modulo $N_1(D)\,d$, every finite place of $E(D)$ above p has absolute norm p.
(I2) Every root of unity of $E(D)$ is a power of ς_D; $\prod_{\tau\in\mathrm{Pl}(D)} \tau(y)$ is the norm of y from $E(D)$ to $K(P)$, for every $y \in E(D)$; the cyclotomic polynomial of order $N_1(D)$ is irreducible over $K(P)$.
(I3) $N_1(D)\,d$ divides $M_\omega(P,\mu)$.
(I4) For every $\tau \in \mathrm{Pl}(D)$: $|\varpi(P,\mu)|_\tau = \sqrt{d}\cdot|\Psi'(\varsigma_D)|_\tau$, where Ψ is the cyclotomic polynomial of order $N_1(D)$; and $\Psi(X) = \Psi_0(X^{N_1(D)/N_0})$, where Ψ_0 is the cyclotomic polynomial of order N_0. Moreover $\prod_{\tau\in\mathrm{Pl}(D)} |\varpi(P,\mu)|_\tau = \mathrm{rd}_{E(D)}^{f(D)}$.
(I5) $C(P) \geq d$, $C(P) \geq N_0$, $C(P) \geq |\,\mathrm{cs}(d)|$, and $C(P) \geq \big|\log|\Psi_0'(\varsigma_0)|\big|$ for every primitive root of unity $\varsigma_0 \in \mathbb{C}$ of order N_0.

Characters of $\mathrm{U}(1)$**.** These definitions belong to the background library. Let E be a CM field with a CM type, so that every infinite place τ has an embedding $\tau\colon E \to \mathbb{C}$. The group $\mathrm{U}(1)(\mathbb{A}_E)$ consists of the adeles u of E with $u\overline{u} = 1$; it contains the group $\mathrm{U}(1)(E)$ of the elements of E of norm one to the totally real subfield. A *character of* $\mathrm{U}(1)$ *of* E is a continuous homomorphism from $\mathrm{U}(1)(\mathbb{A}_E)$ to the unit circle that is trivial on $\mathrm{U}(1)(E)$. Such a character ν has *weight* $\mathbf{e} = (\mathbf{e}(\tau))_\tau$, a family of integers, and *tame part* $\psi = (\psi_w)_w$, a family of multiplicative characters ψ_w of the residue fields $\mathbb{F}_w$, if ψ_w is trivial for all but finitely many w and

$$\nu(u) = \prod_\tau u_\tau^{\mathbf{e}(\tau)} \cdot \prod_w \psi_w(u_w \bmod w)$$

for every $u \in \mathrm{U}(1)(\mathbb{A}_E)$ whose finite part is integral; here $u_\tau \in \mathbb{C}$ is the component of u at τ.

Inputs from the background library. Let E be a CM field with a CM type.
(I6) *Existence* (unit 104). Let $\mathbf{e}$ be a family of integers indexed by the infinite places and ψ a family of characters of the residue fields, trivial outside a finite set Σ of finite places. If $\prod_\tau \tau(u)^{\mathbf{e}(\tau)} \cdot \prod_{w\in\Sigma} \psi_w(u \bmod w) = 1$ for every integer u of E with $u\overline{u} = 1$, then there is a character of $\mathrm{U}(1)$ of E of weight $\mathbf{e}$ and tame part ψ. *Status.* The statement is of the type of the existence of algebraic Hecke characters with prescribed infinity type, for which the catalogue cites A. Weil [W] and A. Raghuram [R]. It records the verdict of its second reading: the cited existence result, for characters of the full idele class group, has a different domain and a different compatibility condition, and the existence theorem of [R] prescribes the pure infinity type only. So (I6) is not established as a special case of the cited results; the relation is unresolved, and (I6) is used here as the library proves it.
(I7) *Products* (unit 104). If ν, ν' have weights $\mathbf{e}$, $\mathbf{e}'$ and tame parts ψ, ψ', then $\nu\nu'$ has weight $\mathbf{e}+\mathbf{e}'$ and tame part $\psi\psi'$, and ν^y has weight $y\,\mathbf{e}$ and tame part ψ^y for every integer y.
(I8) *The splitting character.* If E carries Weil data, with splitting character υ restricted to $\mathrm{U}(1)(\mathbb{A}_E)$, then υ^y has the constant weight y and trivial tame part, for every integer y. The field $E(D)$ of Setting 2.1 carries Weil data (Chapter III).

3 Chosen places and their residue characters

Definition 3.1 (chosen places)**.** In Setting 2.1, let P be parameters and D admissible data for P.

(a) $\bar{\pi}_g(D)$ is the complex conjugate of the image of π_g in the ring of integers of $E(D)$.

(b) A finite place w of $E(D)$ is a *chosen place of* D, in symbols $w \in \mathrm{Ch}(D)$, if $\bar{\pi}_g(D) \in w$; otherwise we write $w \notin \mathrm{Ch}(D)$.

(c) For $i \in \{0,1\}$, $\mathrm{Ch}_i(D)$ is the set of the finite places w of $E(D)$ with $\bar{\pi}_g(D) \in w$ and $g_i \in w$: the chosen places above g_i.

(d) For a pair $x = (i, w)$ with $i \in \{0,1\}$ and $w \in \mathrm{Ch}_i(D)$, the place $\mathrm{pl}_D(x)$ is w.

LEAN `Data.piGBar` · `Data.IsChosen` · `Data.ChosenPlace` · `Data.chosen`

Lemma 3.2. *In Setting 2.1, let P be parameters, D admissible data for P, $i \in \{0,1\}$ and w a finite place of $E(D)$.*

(i) $\bar{\pi}_g(D) \cdot \overline{\bar{\pi}_g(D)} = g(D)$*; $\pi_g \neq 0$ and $\bar{\pi}_g(D) \neq 0$.*

(ii) There is a natural number c, prime to g_i, with $\bar{\pi}_g(D) \cdot \overline{\bar{\pi}_g(D)} = g_i\, c$.

(iii) $w \in \mathrm{Ch}(D)$ holds if and only if, for some $i \in \{0,1\}$, both $w \in \mathrm{Ch}(D)$ and $g_i \in w$ hold.

(iv) If w lies above g_i, then $\mathrm{N}w = g_i$.

(v) The set of the chosen places of D is finite; hence so are $\mathrm{Ch}_0(D)$ and $\mathrm{Ch}_1(D)$.

Proof. (i) The product is the image of $\overline{\pi_g}\,\pi_g = g_0 g_1$, and $g_0 g_1 \neq 0$. (ii) Take $c = g_{1-i}$; the primes g_0 and g_1 are distinct. (iii) If $\bar{\pi}_g(D) \in w$ then $g_0 g_1 \in w$ by (i), and w is a prime ideal. (iv) This is (I1), by (D3). (v) A non-zero integer of $E(D)$ lies in finitely many prime ideals. □

LEAN `Data.piGBar_mul_star` · `Data.piG_ne_zero` · `Data.piGBar_ne_zero` · `Data.exists_cofactor` · `Data.isChosen_iff` · `Data.q_eq_g` · `Data.finite_isChosen`

Lemma 3.3. *In Setting 2.1, let P be parameters, D admissible data for P and w a finite place of $E(D)$ with $w \in \mathrm{Ch}(D)$. Then there is a multiplicative character T of the residue field $\mathbb{F}_w$, with values in $\mathbb{C}$, of order exactly $M_\omega(P,\mu)^2$.*

Proof. By Lemma 3.2(iii) and (iv), $\mathbb{F}_w$ has g_i elements for some i, so $\mathbb{F}_w^\times$ is cyclic of order $g_i - 1$, which $M_\omega(P,\mu)^2$ divides by (D3). Map a generator to a primitive root of unity of order $M_\omega(P,\mu)^2$ in $\mathbb{C}$. □

LEAN `Data.exists_residue_char_of_order`

Definition 3.4 (the residue characters)**.** In Setting 2.1, let D be admissible data and w a finite place of $E(D)$. If w is a chosen place of D, then $\omega_{D,w} = T_w^{M_\omega(P,\mu)}$, where T_w is a chosen character of $\mathbb{F}_w$ of order $M_\omega(P,\mu)^2$ (Lemma 3.3); otherwise $\omega_{D,w}$ is the trivial character of $\mathbb{F}_w$. We write $\omega_{D,\bullet}$ for the family $(\omega_{D,w})_w$.

LEAN `Data.ω`

Lemma 3.5. *In Setting 2.1, let P be parameters, D admissible data for P and w a finite place of $E(D)$.*

(i) If $w \in \mathrm{Ch}(D)$, then $\omega_{D,w}$ has order exactly $M_\omega(P,\mu)$.

(ii) *If $w \notin \mathrm{Ch}(D)$, then $\omega_{D,w} = 1$.*

(iii) *Let n be an integer and ψ a family of characters of the residue fields with $\psi_w = 1$. Then $\left(\omega_{D,w}^{n}\,\psi_w\right)^{M_\omega(P,\mu)} = 1$.*

Proof. (i) A power T^M of an element T of order M^2 has order M. (ii) By definition. (iii) By (i) and (ii), $\omega_{D,w}^{M_\omega(P,\mu)} = 1$ in both cases. □

LEAN `Data.orderOf_ω` · `Data.omega_eq_one_of_not_isChosen` · `Data.omega_zpow_mul_pow_eq_one`

4 The characters of the data

Proof of Theorem B(i). Put $\mathbf{e}(\tau) = -c\,w_{D,0}(\tau)$ and $\psi_w = \omega_{D,w}^{n}$. By Lemma 3.5(ii) and Lemma 3.2(v), ψ is trivial outside the finite set Σ of the chosen places, so by (I6) it suffices to show

$$\prod_{\tau \in \mathrm{Pl}(D)} \tau(u)^{-c\,w_{D,0}(\tau)} \cdot \prod_{w \in \Sigma} \omega_{D,w}(u \bmod w)^{n} = 1 \tag{1}$$

for every integer u of $E(D)$ with $u\overline{u} = 1$. Such a u is an algebraic integer all of whose conjugates have absolute value 1, since complex conjugation commutes with every embedding of a CM field; so u is a root of unity (Kronecker), and $u = \varsigma_D^{q}$ for some natural number q by (I2). By Setting 2.1, $w_{D,0}(\tau) - h_2(D)$ is $b_1(D)$ or $b_2(D)$, a multiple of $N_1(D)$; so $\tau(u)^{w_{D,0}(\tau)} = \tau(u)^{h_2(D)}$, and the first product in (1) is

$$\Big(\prod_{\tau \in \mathrm{Pl}(D)} \tau(\varsigma_D)\Big)^{-c\,q\,h_2(D)}.$$

By (I2) the product in brackets is the norm of ς_D to $K(P)$, and the minimal polynomial of ς_D over $K(P)$ is the cyclotomic polynomial Ψ of order $N_1(D)$. Its degree is even and $\Psi(0) = 1$, because $N_1(D) > 2$; so the norm is 1. For the second product, let $w \in \Sigma$. Then $\omega_{D,w} = T_w^{M_\omega(P,\mu)}$, and $\varsigma_D^{M_\omega(P,\mu)} = 1$ because $N_1(D)$ divides $M_\omega(P,\mu)$ by (I3); hence $\omega_{D,w}(\varsigma_D \bmod w) = T_w(\varsigma_D^{M_\omega(P,\mu)} \bmod w) = 1$. This proves (1). □

Definition 4.1 (the three characters)**.** In Setting 2.1, let D be admissible data.

(a) $\mu_D^{\flat}$ is a chosen character of U(1) of $E(D)$ of weight $\tau \mapsto -w_{D,0}(\tau)$ and trivial tame part (Theorem B(i) with $c = 1$, $n = 0$).

(b) χ_D° is a chosen character of U(1) of $E(D)$ of weight 0 and tame part $\omega_{D,\bullet}$ (Theorem B(i) with $c = 0$, $n = 1$).

(c) For a natural number j, $\mu_{D,j}^{0} = \mu_D^{\flat} \cdot \left(\upsilon^{-h(D)}\,\chi_D^{\circ}\right)^{j}$, where υ is the splitting character of the Weil data of $E(D)$, restricted to U(1).

LEAN `Data.muFlat` · `Data.x` · `Data.μ₀`

Lemma 4.2. *In Setting 2.1, let P be parameters and D admissible data for P. The character $\mu_D^{\flat}$ has weight $\tau \mapsto -w_{D,0}(\tau)$ and trivial tame part, and χ_D° has weight 0 and tame part $(\omega_{D,w})_w$.*

Proof. By the choices of Definition 4.1, since $\omega_{D,w}^{0} = 1$ and $\omega_{D,w}^{1} = \omega_{D,w}$. □

LEAN `Data.muFlat_hasWeight` · `Data.chi_hasWeight`

Proof of Theorem B(ii) and (iii). (ii) By (I8) the character $\upsilon^{-h(D)}$ has weight $-h(D)$ and trivial tame part. By Lemma 4.2 and (I7), $\upsilon^{-h(D)}\chi_D^{\circ}$ has weight $-h(D)$ and

tame part $\omega_{D,\bullet}$, its j-th power has weight $-j\,h(D)$ and tame part $(\omega_{D,w}^{j})_w$, and $\mu_{D,j}^{0}$ has weight $-w_{D,0}(\tau)-j\,h(D)=-w_{D,j}(\tau)$ and the same tame part. (iii) By Definition 6.2, $\mu_{D,j}(\vartheta)=\mu_{D,j}^{0}\cdot\vartheta$ and ϑ has weight 0 and tame part ψ_ϑ; apply (ii) and (I7). □

Definition 4.3 (pairs)**.** In Setting 2.1, let D be admissible data. The set $\mathrm{Pair}(D)$ of the *pairs* of D consists of the quadruples $(\nu,\mathbf{c},\psi,z)$ of a character ν of $\mathrm{U}(1)$ of $E(D)$, a family $\mathbf{c}=(\mathbf{c}(\tau))_{\tau\in\mathrm{Pl}(D)}$ of natural numbers, a family ψ of characters of the residue fields and a natural number z such that: ν has weight $\tau\mapsto-\mathbf{c}(\tau)$ and tame part ψ; $\psi_w=1$ for every finite place w that is not a chosen place of D; $1\le z\le t(D)$; and $z\le\mathbf{c}(\tau)\le 4b$ for every τ.

LEAN Data.Pair

5 The level

Lemma 5.1 (sizes)**.** *In Setting 2.1, let P be parameters and D admissible data for P. Then*

(i) $x^{10}\le|\varpi(P,\mu)|_\tau\le x^{12}$ *for every infinite place τ of $E(D)$;*

(ii) $x^{10}\le\mathrm{rd}_{E(D)}\le x^{12}$*;*

(iii) $x^{-13}\le\Omega(P,\mu)\le 1$.

Proof. (i) By (I4), $\Psi'(\varsigma_D)=M_0\,\varsigma_D^{M_0-1}\,\Psi_0'(\varsigma_D^{M_0})$ with $M_0=N_1(D)/N_0$, and $\varsigma_D^{M_0}$ is a primitive root of unity of order N_0; so

$$\log|\varpi(P,\mu)|_\tau=\tfrac12\log d+\log M_0+\log\bigl|\Psi_0'(\varsigma_0)\bigr|$$

for a primitive root of unity $\varsigma_0\in\mathbb{C}$ of order N_0. Here

$$\log M_0=(\lfloor\log_3\mu\rfloor+5)\log 3+\mu\log 2.$$

By (D2), $2^\mu=m^{22}$ lies between x^{11} and $2^{22}x^{11}$, and

$$1\le 3^{\lfloor\log_3\mu\rfloor+5}\le 243\,\mu\le 7776\log x.$$

By (I5) the other two terms have absolute value at most $2\,C(P)$, and $10^{14}\,C(P)\le\log x$ by (D1). Hence

$$11\log x-2\,C(P)\le\log|\varpi(P,\mu)|_\tau\le 11\log x+22\log 2+\log(7776\log x)+2\,C(P),$$

which lies between $10\log x$ and $12\log x$. (ii) By (I4), $\mathrm{rd}_{E(D)}$ is the geometric mean of the numbers $|\varpi(P,\mu)|_\tau$; apply (i). (iii) By (I5), $e^{-C(P)}\le e^{\mathrm{cs}(d)}\le e^{C(P)}$; apply (ii) and (D1). □

LEAN Data.hasLogBound_varpi · Data.hasLogBound_rootDisc

In the formal text these bounds are *instances*: facts that Lean's search for comparisons of sizes uses without being named. (iii) is such an instance without a name.

Definition 5.2 (the target)**.** In Setting 2.1, let D be admissible data. Then

$$\mathrm{tar}(D)=m\cdot\frac{1}{f(D)}\sum_{\tau\in\mathrm{Pl}(D)}\log w_D(\tau)+(m-1)\cdot\log\frac{g(D)\cdot\mathrm{rd}_{E(D)}}{2\pi}-\frac{15}{2}\log b.$$

LEAN Data.logLevelTarget

Proof of Theorem A(i). By Setting 2.1, $x\,b \le a(D) < x\,b + N_1(D)$, and $N_1(D) \le C(P)\cdot 243\,\mu\cdot 2^{22}x^{11} \le b$ by (I5), (D2) and (D1). So $b \le a(D)/x \le 2b$ and $1 \le x\,b/x_1 \le a(D)/x_1 \le 2b$, whence $0 \le \log w_D(\tau) \le 2\log b$ for every τ. By (D3), $b^{20} \le g(D) \le b^{22}$; with Lemma 5.1(ii) and $2\pi \le x^{10}$, $x^{12} \le b^5$, this gives $b^2 \le g(D)\ \mathrm{rd}_{E(D)}/(2\pi) \le b^{27}$. Therefore

$$(2m - \tfrac{19}{2})\log b \ \le\ \tau\mathrm{ar}(D) \ \le\ (2m + 27(m-1))\log b,$$

and $m \ge 10$ by (D2) and (D1). □

Definition 5.3 (the level)**.** In Setting 2.1, let D be admissible data. By Theorem A(i), (D1) and (D2), $100\log b \le \tau\mathrm{ar}(D) \le 29\,m\log b \le m^{22} = 2^{\mu}$. The *level* $N(D)$ is a chosen prime number with $N(D) \equiv 1$ modulo $M_\omega(P,\mu)^2$ and $\tau\mathrm{ar}(D) \le \log N(D) \le \tau\mathrm{ar}(D) + \sqrt{\log b}$; one exists by (D4), applied to $U = \tau\mathrm{ar}(D)$.

LEAN Data.N

Proof of Theorem A(ii) and (iii). (ii) $N(D)$ is prime and lies in the window by its choice. By (I3), $N_1(D)\,d$ divides $M_\omega(P,\mu)$, hence $M_\omega(P,\mu)^2$, so $N(D) \equiv 1$ modulo $N_1(D)\,d$. The last two inequalities follow from (i) and $\sqrt{\log b} \le \log b \le m\log b$.

(iii) Since 16 divides N_0 and N_0 divides $N_1(D)$, (ii) gives $N(D) \equiv 1$ modulo 4. By Fermat's theorem on sums of two squares there are integers y, y' with $y^2 + y'^2 = N(D)$. The element $\mathrm{i} = \varsigma_D^{N_1(D)/4}$ of $\mathcal{O}_{E(D)}$ has $\mathrm{i}^2 = -1$, and $\bar{\mathrm{i}} = \mathrm{i}^{-1} = -\mathrm{i}$ because complex conjugation inverts roots of unity. So $\pi = y + y'\,\mathrm{i}$ has $\pi\,\overline{\pi} = y^2 + y'^2 = N(D)$. The second statement is the choice of Definition 5.4. □

Definition 5.4. In Setting 2.1, let D be admissible data. Then $\pi_N(D)$ is a chosen integer π of $E(D)$ with $\pi\,\overline{\pi} = N(D)$ (Theorem A(iii)).

LEAN Data.π

Remark 5.5. Three more facts are recorded in the formal text as instances without names: $N(D)$ is prime; $\sqrt{x}\,\log b \le \log N(D) \le e^{32}\sqrt{x}\,\log b$; and $N(D) \ge b^{101}$. They follow from Theorem A(ii) and $\sqrt{x} \le m \le 2\sqrt{x}$, $m \ge 101$. By (I1) and Theorem A(ii), every finite place of $E(D)$ above $N(D)$ has absolute norm $N(D)$.

6 Tame characters and twists

Definition 6.1 (tame characters, twists)**.** In Setting 2.1, let P be parameters and D admissible data for P.

(a) The set $\mathrm{Tame}(D)$ of the *tame characters of* D consists of the triples of a character of $\mathrm{U}(1)$ of $E(D)$, a finite set Q of finite places of $E(D)$ and a family ψ of characters of the residue fields, such that: the character has weight 0 and tame part ψ; $\psi_w = 1$ for every $w \notin Q$; $\overline{w} \notin Q$ for every $w \in Q$; and $g(D)\cdot N(D)$ is prime to $\prod_{w\in Q} \mathrm{N}w$. We write ϑ for a tame character and for its character of $\mathrm{U}(1)$, $Q(\vartheta)$ for its set and $\psi_\vartheta = (\psi_{\vartheta,w})_w$ for its family.

(b) For a tame character ϑ of D, $\mathrm{n}(\vartheta) = \prod_{w\in Q(\vartheta)} \mathrm{N}w$.

(c) The set $\mathrm{Tw}(D)$ of the *twists of* D consists of the tame characters $\vartheta \in \mathrm{Tame}(D)$ with $\psi_{\vartheta,w} \neq 1$ for every $w \in Q(\vartheta)$.

(d) The *trivial tame character* 1_D of D is the trivial character with $Q(1_D) = \emptyset$ and trivial family; it is a twist of D.

LEAN `Data.TameChar` · `Data.TameChar.levelNorm` · `Data.Twist` · `Data.TameChar.one`

A tame character may mark a place: its set $Q(\vartheta)$ may contain places w with $\psi_{\vartheta,w} = 1$. Non-triviality at every marked place is what a twist adds; so for a twist, and only then, $Q(\vartheta)$ is exactly the set of the places where the tame part is not trivial. The formal text makes the trivial twist the unit element 1 of the type of twists and lets a twist be used as a tame character (two instances without names). The m-fold twists of the overview are the twists ϑ of D such that $\psi_{\vartheta,w}^n \neq 1$ for all $w \in Q(\vartheta)$ and $1 \leq n \leq m$; they, and the power ϑ^n of a tame character (the same set $Q(\vartheta)$, the family ψ_ϑ^n), are defined in Chapter VII (unit 147).

Definition 6.2 (characters of the lifts)**.** In Setting 2.1, let D be admissible data, ϑ a tame character of D and j a natural number.

(a) $\mu_{D,j}(\vartheta) = \mu_{D,j}^0 \cdot \vartheta$, a character of $\mathrm{U}(1)$ of $E(D)$.

(b) $\psi_{D,j}(\vartheta)$ is the family $\left(\omega_{D,w}^j \cdot \psi_{\vartheta,w}\right)_w$ of characters of the residue fields.

(c) $\mathrm{Lf}(D)$ is the set $\{1, \dots, m\}$, the indices of the lifts.

LEAN `Data.liftChar` · `Data.liftTamePart` · `Data.LiftIndex`

7 Index sets

For a natural number q we write $[q] = \{0, \dots, q-1\}$, and $\{*\}$ for a set with one element.

Definition 7.1 (blocks)**.** Let S, T be sets and j a natural number.

(a) $\mathrm{Bl}_j(S,T)$ is defined by recursion: $\mathrm{Bl}_0(S,T) = S$ and $\mathrm{Bl}_{j+1}(S,T) = \mathrm{Bl}_j(S,T) \sqcup T$. It is the disjoint union of S and j copies of T.

(b) $\mathrm{in}_j \colon S \to \mathrm{Bl}_j(S,T)$ is the inclusion of the first block.

(c) For a set α and a bijection $\nu \colon S \to \alpha$, $\mathrm{eq}_j(\nu) \colon \mathrm{Bl}_j(S,T) \to \alpha \sqcup ([j] \times T)$ is a bijection: ν for $j = 0$, and for $j+1$ a chosen bijection, which exists because both sides have the cardinality of $\alpha \sqcup ([j] \times T) \sqcup T$.

LEAN `BlockIndex` · `BlockIndex.firstBlock` · `BlockIndex.equivSum`

Lemma 7.2. *Let S, T be finite sets and j a natural number. Then $\mathrm{Bl}_j(S,T)$ is finite, with $|S| + j\,|T|$ elements.*

Proof. By induction on j; or by the bijection $\mathrm{eq}_j(\nu)$ for the identity ν of S. □

LEAN `instFintypeYT` · `BlockIndex.card`

Definition 7.3 (the index sets of the data)**.** In Setting 2.1, let D be admissible data and j a natural number.

(a) $\varrho(D) = \lfloor m/2 \rfloor - 1$, or 0 if this is negative.

(b) $\underline{X}_j^\circ(D) = [n_j(D) - 1]$.

(c) $\underline{X}_j(D) = \underline{X}^{\circ}_j(D) \sqcup \{*\}$: the basis index set of the standard definite hermitian space $X_j(D)$ of rank $n_j(D)$, the point $*$ indexing the last basis vector.

(d) $\mathfrak{s}(D)$ is the *shape* consisting of the number $\varrho(D)$, the index set $\underline{X}_0(D)$ and its element $*$.

(e) $\underline{R}(D) = [\varrho(D)] \sqcup (\underline{X}_0(D) \sqcup \{\diamond\}) \sqcup [\varrho(D)]$, the index set of the shape $\mathfrak{s}(D)$: indices of $\varrho(D)$ isotropic vectors, of $\underline{X}_0(D)$, of one further vector, and of $\varrho(D)$ dual isotropic vectors. The element $\diamond$ indexes the further vector, the middle vector of the shape; it is not the point $*$, which is an element of $\underline{X}_0(D)$.

(f) $\underline{Y}_j(D) = \mathrm{Bl}_j(\underline{R}(D), [t(D)])$: the basis index set of the hermitian space $Y_j(D)$.

LEAN Data.ϱ · Data.InitIndex · Data.X · Data.shape · Data.RowIndex · Data.Y

Lemma 7.4. *In Setting 2.1, let P be parameters and D admissible data for P. Then $m = 2\,\varrho(D) + 2$, the set $\underline{R}(D)$ has exactly r elements, and $\underline{Y}_j(D)$ has $r + j\,t(D)$ elements for every natural number j.*

Proof. By (D2), m is even and at least 2, so $\varrho(D) = m/2 - 1$; and $m \le 2\sqrt{x} \le x^5 \le r$, so $k(D) = r + 1 - m \ge 1$ and $\underline{X}_0(D)$ has $n_0(D) = k(D)$ elements. Hence

$$|\underline{R}(D)| = 2\,\varrho(D) + k(D) + 1 = (m-2) + (r+1-m) + 1 = r.$$

The last statement is Lemma 7.2. □

LEAN Data.card_rowIndex

The formal text adds an instance without a name: a bound for the size of r is also one for $|\underline{R}(D)|$. The identity $m = 2\,\varrho(D) + 2$ is stated there in Chapter VII (unit 140).

Definition 7.5 (numbering, row parameter, signs)**.** In Setting 2.1, let D be admissible data.

(a) $\mathrm{num}_D \colon \underline{R}(D) \to [r]$ is a chosen bijection (Lemma 7.4).

(b) For an infinite place τ of $E(D)$ and a natural number i, the entry i of the *row parameter* is

$$\mathrm{par}_{D,\tau}(i) = \begin{cases} k(D)/2 - 1 - i & \text{if } i+1 < k(D), \\ h_2(D) - w_{D,i+2-k(D)}(\tau) & \text{if } i+1 \ge k(D). \end{cases}$$

(c) For an infinite place τ and $i \in \underline{R}(D)$: $\mathrm{par}^W_{D,\tau}(i) = 2\,\mathrm{par}_{D,\tau}(\mathrm{num}_D(i))$.

(d) For a natural number j and $i \in \underline{R}(D)$: $\mathrm{sg}^W_{D,j}(i) = -1$ if $\mathrm{num}_D(i) \ge k(D) + j - 1$, and $\mathrm{sg}^W_{D,j}(i) = +1$ otherwise.

LEAN Data.basisNumber · Data.rowParameter · Data.paramW · Data.signsW

The number $k(D)$ is even, since r is odd and m even. For $i = k(D) - 1, \dots, r - 1$ the second case gives $-(w_{D,q}(\tau) - h_2(D))$ with $q = i + 2 - k(D) \in \{1, \dots, m\}$. So the row parameter consists of the $k(D) - 1$ numbers $k(D)/2 - 1, \dots, 1 - k(D)/2$, followed by m entries given by the weights of the m lifts; $\mathrm{par}^W_{D,\tau}(\cdot)$ is the doubled parameter of the forms on $Y_0(D)$ and on the spaces W_j of signs $\mathrm{sg}^W_{D,j}(\cdot)$ of Chapter VII.

Definition 7.6 (entry signs)**.** For natural numbers o, j, a the sign $\mathrm{sg}(o, j, a) \in \{\pm 1\}$ is defined by recursion on o:

$$\mathrm{sg}(o+1, 0, a) = -\,\mathrm{sg}(o, 0, a),$$
$$\mathrm{sg}(o, j, a) = (-1)^{\,o + a + \min(\max(a-o, 0),\, j)} \quad \text{in all other cases.}$$

The second formula holds in the first case as well, since $\min(\max(a-o,0),0)=0$.

LEAN `Data.entrySign`

Lemma 7.7. *Let j be a natural number and y an integer. Then the number that is $y-1$ if j is odd and y if j is even equals $y-(j \bmod 2)$.*

Proof. $j \bmod 2$ is 1 or 0 accordingly. □

LEAN `Data.ite_odd_eq_sub_emod`

Definition 7.8 (the finite group and its idempotent)**.** In Setting 2.1, let D be admissible data.

(a) $\mathsf{G}(D)=\prod_x \mathrm{GL}_{\underline{R}(D)}(\mathbb{F}_{\mathrm{pl}_D(x)})$, the product over the pairs $x=(i,w)$ with $i\in\{0,1\}$ and $w\in \mathrm{Ch}_i(D)$: a finite group, the product of the groups GL_r of the residue fields of the chosen places.

(b) $\mathrm{Good}(D)$ is the finite set of the families $(c_x)_x$ of monic polynomials c_x of degree r over $\mathbb{F}_{\mathrm{pl}_D(x)}$ such that, for each $i\in\{0,1\}$, the tuple $(c_{(i,w)})_{w\in\mathrm{Ch}_i(D)}$ is *good* for $M_\omega(P,\mu)$ and r: every $c_{(i,w)}$ is squarefree with non-zero constant term and has no root y in $\mathbb{F}_w$ with $y^{M_\omega(P,\mu)}=1$, and at least one $c_{(i,w)}$ is irreducible with a root of multiplicative order $(\mathrm{N}w)^r-1$ in an algebraic closure of $\mathbb{F}_w$.

(c) $e_0(D)=\sum_{c\in\mathrm{Good}(D)}\bigotimes_x e_{c_x}$ in the group algebra $\mathbb{C}[\mathsf{G}(D)]$, where e_{c_x} is the idempotent of Green's character with the label c_x in the group algebra of $\mathrm{GL}_{\underline{R}(D)}(\mathbb{F}_{\mathrm{pl}_D(x)})$.

LEAN `Data.G` · `Data.goodTupleFamilies` · `Data.e₀`

Good tuples and the idempotents e_c belong to Chapter VII (unit 140, the note *Primitive labels of general linear groups over finite fields*) and to the background library; there $e_0(D)$ is shown to be a central idempotent, and the rows of the overview are its image on the forms of principal level on $Y_0(D)$.

8 The branching pairing

In this section E is a CM field with a CM type, and weights and their Harish-Chandra parameters are as before Theorem C. We recall the objects of the background library that the theorem names.

Representations. For a weight (κ,e) and a finite index set n, the vectors *of weight κ in rank $|n|$* over a commutative ring R are the coordinate vectors, in the basis of semistandard fillings, of the product over the infinite places τ of the Weyl modules of GL_n whose Young diagram has $\kappa_\tau(h)$ columns of height h, for $0<h<|n|$. On them a family $g=(g_\tau)_\tau$ of matrices in $\mathrm{GL}_n(R)$ acts by $\rho_{\kappa,e}(g)=\prod_\tau \det(g_\tau)^{e_\tau}$ times the Schur matrix of g. For vectors v', v with the same index set, $\langle v',v\rangle=\sum_a v'_a v_a$ is the bilinear dot product, and $\|v\|^2$ is the value at (v,v) of the positive definite hermitian form on complex vectors of weight κ given by the integral Gram matrix of the Weyl lattice. $\mathrm{boxes}(\kappa,N)=\sum_\tau\sum_{h<N} h\,\kappa_\tau(h)$ is the number of boxes. For a ring S with an involution $y\mapsto\overline{y}$, $\mathrm{U}_n(S)$ is the group of the $g\in\mathrm{GL}_n(S)$ with $\overline{g}^{\mathrm{t}}g=1$; for $g\in\mathrm{U}_n(S)$ and $u\in\mathrm{U}_1(S)$, $\mathrm{diag}_\sigma(g,u)\in\mathrm{U}_{n'}(S)$ is the block diagonal matrix with blocks g and u, relabelled by σ; and for ring homomorphisms $f_\tau\colon S\to R$, $f(g)$ is the family $(f_\tau(g))_\tau$, entry by entry.

Forms. A *standard* hermitian space V with index set n has Gram matrix 1 and the standard lattice; its signs are all $+1$. Its adelic unitary group is the product of a compact real group and $\mathrm{U}_n(\mathbb{A}_f)$; $[\mathrm{U}(V)]$ is the quotient by the rational points, with the invariant measure that the background library fixes. An *algebraic form* of weight (κ, e) in rank $|n|$ is a function ϕ on $\mathrm{U}_n(\mathbb{A}_f)$, invariant on the right under an open subgroup, with values in the complex vectors of weight κ in rank $|n|$, such that $\phi(y_0 h) = \rho_{\kappa,e}(y_0)\phi(h)$ for rational points y_0. For a family l that is the doubled Harish-Chandra parameter of (κ, e) and signs $\mathbf{s}$, the function $F_\phi^{l,\mathbf{s}}$ on the adelic group is $(y_\infty, h) \mapsto (v_l^+, \overline{\rho_{\kappa,e}}(y_\infty)^{-1}\phi(h))$, the coefficient of the highest weight vector v_l^+ for the Gram form, where $\overline{\rho_{\kappa,e}}$ is $\rho_{\kappa,e}$ at the complex conjugates of the matrices. A *level* is an open subgroup $\mathcal{K}$ of the stabiliser $\mathcal{K}_0$ of the standard lattice in $\mathrm{U}_n(\mathbb{A}_f)$, and for a function F on $\mathrm{U}_n(\mathbb{A}_f)$, left invariant under the rational points and right invariant under an open subgroup, $\mathrm{mean}_{\mathcal{K}}(F)$ is its integral over the quotient of $\mathrm{U}_n(\mathbb{A}_f)$ by the rational points for the measure that gives $\mathcal{K}$ the volume 1: a finite sum over classes, weighted by inverse orders of stabilisers.

Inputs for Theorem C. Let E, n, n', σ be as in Theorem C, with $|n'| = |n|+1$.
(I9) *Parameters with a gap* (unit 143). If a family $l = (l_\tau)$ of maps $n \to \mathbb{Z}$ has, for every τ, distinct values that are all congruent to $|n|-1$ modulo 2, then l is the doubled Harish-Chandra parameter of some weight $(\kappa^\natural, e^\natural)$.
(I10) *Dual pairing* (unit 101). If l is the doubled Harish-Chandra parameter of $(\kappa^\natural, e^\natural)$ and $-l$ that of (κ', e'), both in rank $|n|$, there is an integer matrix A with: $\langle \rho_{\kappa',e'}(g)v', A\,\rho_{\kappa^\natural,e^\natural}(g)v\rangle = \langle v', Av\rangle$ for all fields R of characteristic 0, all families g in $\mathrm{GL}_n(R)$ and all v', v; for all signs $\mathbf{s}$ and algebraic forms η of weight (κ', e'), $F_\eta^{-l,\mathbf{s}}(y_\infty, h) = \langle \eta(h), A\,\overline{\rho_{\kappa^\natural,e^\natural}}(y_\infty)v_l^+\rangle$; and for every complex vector v of weight $\kappa^\natural$ there is a complex vector v° of weight κ' with $\|v^\circ\| = \|v\|$ and $\langle v', Av\rangle = (v^\circ, v')$ for all v', where $(\cdot,\cdot)$ is the Gram form, conjugate linear in the first variable.
(I11) *Branching* (unit 19; the branching law from $\mathrm{GL}_{|n|+1}$ to $\mathrm{GL}_{|n|}$ with multiplicity one, in an integral form, here in the case $q = 0$ of the formal statement; reference to be confirmed). Let (κ, e) be a weight, and let $\kappa^\uparrow$, $\kappa^\downarrow$ be families of functions $\mathbb{N} \to \mathbb{N}$ with $\kappa^\uparrow + \kappa^\downarrow = \kappa$. Let $(\kappa^\natural, e^\natural)$ be a weight with $\kappa_\tau^\natural(h) = \kappa_\tau^\downarrow(h) + \kappa_\tau^\uparrow(h+1)$ for $0 < h < |n|$ and $e_\tau^\natural = e_\tau + \kappa_\tau^\downarrow(|n|)$. Then there is an integer matrix br, from weight κ in rank $|n'|$ to weight $\kappa^\natural$ in rank $|n|$, such that: over every field R of characteristic 0,

$$\mathrm{br}\cdot\rho_{\kappa,e}(\mathrm{diag}_\sigma(g,u)) = \prod_\tau u_\tau^{\,e_\tau+\sum_{0<h<|n'|}\kappa_\tau^\uparrow(h)}\cdot\rho_{\kappa^\natural,e^\natural}(g)\cdot\mathrm{br}$$

for all families g in $\mathrm{GL}_n(R)$ and u in $R^\times$; $\|\mathrm{br}\,v\| \le (M^M)^3\|v\|$ for all complex v and every natural number $M \ge \mathrm{boxes}(\kappa^\natural, |n|)$; and every complex matrix that intertwines $\rho_{\kappa,e}(\mathrm{diag}_\sigma(g,1))$ with $\rho_{\kappa^\natural,e^\natural}(g)$ for all families g of unitary matrices is a multiple of br.
(I12) *Periods as means* (unit 27; reference to be confirmed). Let V be a standard space with index set n, let br have the last property of (I11), let L' be a linear functional on the complex vectors of weight κ in rank $|n'|$ and $v^\natural$ a complex vector of weight $\kappa^\natural$ in rank $|n|$. Then there is a complex number C such that for every algebraic form ϕ of weight (κ, e) in rank $|n'|$, every automorphic function Φ on the adelic unitary group of V, every $u \in \mathrm{U}_1(\mathbb{A}_f)$, every family $(L_h)_h$ of linear functionals on the complex vectors of weight $\kappa^\natural$, indexed by $h \in \mathrm{U}_n(\mathbb{A}_f)$, and every level $\mathcal{K}$: if $\Phi(y_\infty, h) = L_h(\overline{\rho_{\kappa^\natural,e^\natural}}(y_\infty)v^\natural)$ for all (y_∞, h), then

$$\int_{[\mathrm{U}(V)]}\Phi(y)\cdot L'(\overline{\rho_{\kappa,e}}(\mathrm{diag}_\sigma(y_\infty,1))^{-1}\phi(\mathrm{diag}_\sigma(h,u)))\,dy = \frac{C}{[\mathcal{K}_0:\mathcal{K}]}\cdot\mathrm{mean}_{\mathcal{K}}(h\mapsto L_h(\mathrm{br}\,\phi(\mathrm{diag}_\sigma(h,u)))),$$

where $y = (y_\infty, h)$. The functions $F_\eta^{-l,\mathbf{s}}$ are automorphic (unit 102).

Proof of Theorem C. Since σ is a bijection, $|n'| = |n| + 1$; write $N = |n|$. By the two hypotheses on the signs, all of $\mathbf{s}$, $\mathbf{r}$, $\mathbf{t}$ are 1. Since $-l$ is a doubled Harish-Chandra parameter in rank N, the values of each l_τ are distinct and congruent to $N-1$ modulo 2; by (I9), l is the doubled Harish-Chandra parameter of a weight $(\kappa^\natural, e^\natural)$. For $0 \le p \le N$ and $0 \le q < N$ put

$$\lambda_\tau(p) = e_\tau + \sum_{p<h<N+1}\kappa_\tau(h), \qquad \lambda_\tau^\natural(q) = e_\tau^\natural + \sum_{q<h<N}\kappa_\tau^\natural(h),$$

the highest weights. The values of l'_τ in decreasing order are $2\lambda_\tau(p)+N-2p$, $p=0,\dots,N$, and those of l_τ are $2\lambda^\natural_\tau(q)+N-1-2q$, $q=0,\dots,N-1$.

Interlacing. Fix τ and $q<N$, and count values from 0. Exactly q values of l'_τ exceed its q-th value y; by the hypothesis at this y, at most q values of l_τ exceed it, so the q-th value of l_τ is at most the q-th value of l'_τ. Exactly q values of l_τ exceed its q-th value y; by the hypothesis at this y, at most $q+1$ values of l'_τ exceed it, so the $(q+1)$-st value of l'_τ is at most the q-th value of l_τ. The values of l_τ and of l'_τ have opposite parities, so both inequalities are strict, and in terms of the highest weights they read

$$\lambda_\tau(q+1)\ \le\ \lambda^\natural_\tau(q)\ \le\ \lambda_\tau(q)\qquad(0\le q<N).\tag{2}$$

Put $\delta_\tau(q)=\lambda^\natural_\tau(q)-\lambda_\tau(q+1)$; it is half of the difference of the q-th value of l_τ and the $(q+1)$-st value of l'_τ, rounded down, and $0\le\delta_\tau(q)\le\kappa_\tau(q+1)$ by (2).

The matrix. Let $\kappa^\downarrow_\tau(h)=\delta_\tau(h-1)$ and $\kappa^\uparrow_\tau(h)=\kappa_\tau(h)-\delta_\tau(h-1)$ for $1\le h\le N$, and $\kappa^\downarrow_\tau(h)=0$, $\kappa^\uparrow_\tau(h)=\kappa_\tau(h)$ for the other h. Then $\kappa^\uparrow+\kappa^\downarrow=\kappa$, and

$$\kappa^\downarrow_\tau(h)+\kappa^\uparrow_\tau(h+1)=\lambda^\natural_\tau(h-1)-\lambda^\natural_\tau(h)=\kappa^\natural_\tau(h)\qquad(0<h<N),$$
$$e_\tau+\kappa^\downarrow_\tau(N)=\lambda_\tau(N)+\delta_\tau(N-1)=\lambda^\natural_\tau(N-1)=e^\natural_\tau.$$

So (I11) gives a matrix br, and (I10) a matrix A. Put $B=A\cdot\mathrm{br}$ and

$$N_\tau=e_\tau+\sum_{0<h<N+1}\kappa^\uparrow_\tau(h).$$

(a1) For $g\in\mathrm{U}_n(S)$ and $u\in\mathrm{U}_1(S)$, the intertwining property of br over R, at the families $f(g)$ and $(f_\tau(\det u))_\tau$, and then the invariance of A give

$$\langle\rho_{\kappa',e'}(f(g))v',A\,\mathrm{br}\,\rho_{\kappa,e}(f(\mathrm{diag}_\sigma(g,u)))v\rangle=\prod_\tau f_\tau(\det u)^{N_\tau}\cdot\langle v',A\,\mathrm{br}\,v\rangle.$$

(a2) $|N_\tau|\le|e_\tau|+\sum_{0<h<N+1}\kappa_\tau(h)\le|e_\tau|+\mathrm{boxes}(\kappa,N+1)$.

(a3) By (I10), applied to the vector $\mathrm{br}\,v$, there is v° with $\|v^\circ\|=\|\mathrm{br}\,v\|$ and $\langle v',A\,\mathrm{br}\,v\rangle=(v^\circ,v')$. The Cauchy–Schwarz inequality for the positive definite Gram form and (I11) give $|\langle v',Bv\rangle|\le\|\mathrm{br}\,v\|\,\|v'\|\le(M^M)^3\|v\|\,\|v'\|$, provided $M\ge\mathrm{boxes}(\kappa^\natural,N)$. This holds for $M\ge\mathrm{boxes}(\kappa,N+1)$: counting boxes by rows,

$$\mathrm{boxes}(\kappa^\natural,N)=\sum_\tau\sum_{q<N}(\lambda^\natural_\tau(q)-e^\natural_\tau),\qquad\mathrm{boxes}(\kappa,N+1)=\sum_\tau\sum_{p<N+1}(\lambda_\tau(p)-e_\tau),$$

and $\lambda^\natural_\tau(q)-e^\natural_\tau\le\lambda_\tau(q)-e_\tau$ by (2) and $e^\natural_\tau\ge e_\tau$.

(b) Fix g. Apply (I12) to the standard space, the matrix br, the vector $v^\natural=v^+_l$ and the functional $L'=(v^+_{l'},\overline{\rho_{\kappa,e}}(g)^{-1}\cdot)$; it gives C. For V, η, ϕ, u, $\mathcal{K}$ as in the statement, V is the standard space, $\Phi=F^{-l,\mathrm{s}}_\eta$ is automorphic, and by (I10) it has the form required in (I12) with $L_h=\langle\eta(h),A\cdot\rangle$. The left side of (I12) is then the integral of the theorem, because the value of $F^{l',\mathrm{t}}_\phi$ at $\mathrm{diag}_\sigma(y,(1,u))\cdot(g,1)$ is $L'(\overline{\rho_{\kappa,e}}(\mathrm{diag}_\sigma(y_\infty,1))^{-1}\phi(\mathrm{diag}_\sigma(h,u)))$, and the right side is the mean of $h\mapsto\langle\eta(h),A\,\mathrm{br}\,\phi(\mathrm{diag}_\sigma(h,u))\rangle$. □

9 The grid spaces

Chapter VII compares the chain of lifts with a second chain by moving through two grids of hermitian spaces, the grid V and the grid Z. The file read here only names the objects on the grids; their index sets, signs and parameters are defined in Chapter VII. In this section D is admissible data as in Setting 2.1, and $z \in \{0,1\}$ names a grid: $z = 0$ is the grid V, $z = 1$ the grid Z. (The formal text uses truth values; 1 is *true*.)

Inputs from Chapter VII (units 140 and 147) **and from the background library.** In Setting 2.1:
(I13) *Spaces.* A hermitian space with a point over $E(D)$, with a finite index set and signs on it, is a Gram matrix over $E(D)$ with an adelic frame that brings it to the diagonal matrix of the signs at the infinite places and to 1 at the finite places. Such spaces have orthogonal sums $\oplus$, relabellings along bijections of index sets that carry the signs, and opposites (the Gram matrix and the signs are negated); for signs $\mathbf{s}$ there is the standard signed space $I(\mathbf{s})$. For every natural number j, $Y_j(D)$ is a hermitian space with index set $\underline{Y}_j(D)$ and signs $\mathrm{sgn}^{0,0}_{0,j}(D)$ (unit 140).
(I14) *Index sets, signs, parameters* (unit 147). For $z \in \{0,1\}$ and natural numbers o, i, j: $\underline{\mathcal{V}}^z_{i,j}(D) = \mathrm{Bl}_j(\mathrm{Bl}_i(\mathrm{Bl}_z(\underline{R}(D), \{*\}), [t(D)+z]), [t(D)-z])$, a set with $r + z + i\,(t(D)+z) + j\,(t(D)-z)$ elements; its base indices, those of $\mathrm{Bl}_z(\underline{R}(D), \{*\})$, are numbered by num_D and, for the point, by r. The signs $\mathrm{sgn}^{z,o}_{i,j}(D)$ on it are $\mathrm{sg}(o, j, \max(q+1-k(D), 0))$ at the base index of number q and $+1$ elsewhere. The doubled parameter $\mathrm{gpar}^{z,o}_{i,j}(D,\tau)$, for an infinite place τ, is the integer vector on the same index set with the following entries: $2\,\mathrm{par}_{D,\tau}(q)$ at the base index of number q; $2\,w_{D,l+1}(\tau) + t(D) - 1 - 2a$ at the position $a \in [t(D)+z]$ of the block $l < i$ of the first kind; $t(D) - 1 - 2\,(w_{D,l+1+o}(\tau) + z) - 2a$ at the position $a \in [t(D)-z]$ of the block $l < j$ of the second kind; all raised by 1 if $z = 1$, and lowered by 1 if $z = 0$ and $i+j$ is odd. The weight $\mathrm{gw}_D(c, l, n)$, for $c \in \{0,1\}$, a natural number l and a finite set n, is $\tau \mapsto \lfloor (|n| + 1 - c)/2 \rfloor - w_{D,l+1}(\tau)$. The polynomial $\mathrm{har}_D(o, d, c, l; z; a, b; i, j)$, for $d \in \{0,1\}$, is a chosen joint harmonic for the nodes (a,b) and (i,j) of grid z, the weights $\mathrm{gw}_D(c, l, \cdot)$ and the parameters, negated if $d = 1$. Facts: (i) $\mathrm{sgn}^{1,0}_{0,0}(D)$ is $\mathrm{sgn}^{0,0}_{0,0}(D)$ on $\underline{R}(D)$ and $+1$ at the point; (ii) the number of signs -1 in $\mathrm{sgn}^{z,0}_{i,j}(D)$ is $\lfloor (\max(m-j, 0) + z)/2 \rfloor$; (iii) there are chosen bijections $\Delta_j \colon \underline{\mathcal{V}}^0_{j,j}(D) \sqcup \{*\} \to \underline{\mathcal{V}}^1_{j,j}(D)$, the identity for $j = 0$, such that for $j \le m$ the signs $\mathrm{sgn}^{1,0}_{j,j}(D)$ correspond to $\mathrm{sgn}^{0,0}_{j,j}(D)$ and $(-1)^j$ at the point; (iv) for all z, i, j and τ, the entries of $\mathrm{gpar}^{z,0}_{i,j}(D,\tau)$ are distinct, congruent to $|\underline{\mathcal{V}}^z_{i,j}(D)| - 1$ modulo 2, and two entries at indices of different signs differ by at least $2\,h(D)$; (v) for a step of the grid as in Lemma 9.8, the weights, parameters and signs at its two ends correspond in the sense of the theta correspondence of discrete series parameters; (vi) $t(D)$ is odd.

Inputs from Chapter VII and from the background library, continued. In Setting 2.1:
(I15) *Moduli* (units 140 and 147). For a tame character ϑ of D and a natural number $q \geq 1$, $\mathfrak{m}_D(\vartheta; q)$ is the ideal of $\mathcal{O}_{E(D)}$ that is the q-th power of the product of the chosen places of D, times the product of the places of $Q(\vartheta)$; its prime divisors are the chosen places and the places of $Q(\vartheta)$. For an m-fold twist ϑ: $\mathfrak{m}_D(\vartheta) = \mathfrak{m}_D(\vartheta; b_1(D))$, $\mathfrak{m}_D^{\triangle}(0, \vartheta) = \mathfrak{m}_D(\vartheta; 1)$ and $\mathfrak{m}_D^{\triangle}(1, \vartheta) = \mathfrak{m}_D(1_D; b_1(D))$, which divides $\mathfrak{m}_D(\vartheta)$. For a finite index set n and each of these ideals $\mathfrak{a}$, reduction from the stabiliser $\mathcal{K}_0$ of the standard lattice in $\mathrm{U}_n(\mathbb{A}_f)$ to $\mathrm{GL}_n(\mathcal{O}_{E(D)}/\mathfrak{a})$ is surjective.
(I16) *Kernels and lifts* (background library). For a parameter l and signs $\mathbf{s}$ on a finite index set, the *kernel pair* of $(l, \mathbf{s})$ sends 0 to the product over the infinite places of the discrete series kernels of the parameters l_τ, and 1 to that of $-l$. A pair of functions on the real unitary group is a *conjugate kernel pair* if the two are complex conjugates of each other and each is a cusp kernel of order 0. If, for every τ, the entries of l_τ are distinct, of the right parity, and differ by at least $1000\,|n|^3$ at indices of different signs, then the kernel pair of $(l, \mathbf{s})$ is a conjugate kernel pair. For spaces $\mathcal{S}$, $\mathcal{T}$, characters of $\mathrm{U}(1)$, a polynomial and a function Ψ of matrices modulo an ideal, there are the theta lift from forms on $\mathcal{S}$ to forms on $\mathcal{T}$ and the descent in the other direction, both with the Schwartz function made of the polynomial at the infinite places and of Ψ at the finite places. A family of characters, two kernel pairs, a direction $d \in \{0, 1\}$ and a polynomial form an *edge family* if both pairs are conjugate kernel pairs, the characters have weights independent of the member of the family, and the polynomial is an edge for the real Weil representation between the kernels of index d. If weights, parameters and signs correspond as in (I14)(v), then the kernel pairs, the joint harmonic and any family of characters of these weights form an edge family. *Status.* The catalogue classes the two statements of (I16) as substantial and gives no reference for them: reference to be confirmed. The second rests on the library's statement that, for corresponding parameters, the vector of the Fock model given by the product of the chosen joint harmonics is an edge between the two discrete-series types. For that statement the catalogue cites Ichino [I], Theorem 4.1 and Lemma 7.8, for the compact types, and records the verdict of its second reading: the formal matching condition selects witnesses for a two-sided analytic statement beyond the published joint-type criterion. So the formal statement is not a special case or a weaker form of the cited ones; it goes beyond them, and its relation to the literature is unresolved. It is used here as the library proves it.
(I17) *Determinants* (background library). If a character ν of $\mathrm{U}(1)$ of $E(D)$ has a weight and a tame part ψ that is trivial outside the prime divisors of an ideal $\mathfrak{a}$, then for every finite index set n', all signs $\mathbf{s}$ on n' and every $k \in \mathcal{K}_0 \subseteq \mathrm{U}_{n'}(\mathbb{A}_f)$: the value of ν at the determinant of the adelic point $(1, k)$ is $\prod_{w \mid \mathfrak{a}} \psi_w(\det(k \bmod w))$.

Definition 9.1 (nodes and their spaces)**.** In Setting 2.1, let D be admissible data.

(a) A *node* is a triple $p = (z, a, b)$ of an element $z \in \{0, 1\}$ and natural numbers a, b.

(b) For a node $p = (z, i, j)$: $\underline{\mathcal{V}}_D(p) = \underline{\mathcal{V}}^z_{i,j}(D)$.

(c) For natural numbers o, i, j and $z \in \{0, 1\}$, the hermitian space $\mathcal{V}^z_{i,j}[o](D)$ has index set $\underline{\mathcal{V}}^z_{i,j}(D)$ and signs $\mathrm{sgn}^{z,o}_{i,j}(D)$. It is: $Y_0(D) \oplus I(+1)$ for $(o, z, i, j) = (0, 1, 0, 0)$, where $I(+1)$ is the standard line of sign $+1$; $Y_j(D)$ for $(o, z, i) = (0, 0, 0)$; the opposite of $\mathcal{V}^z_{0,0}[o-1](D)$ for $o \geq 1$ and $i = j = 0$; and the standard signed space $I(\mathrm{sgn}^{z,o}_{i,j}(D))$ in all other cases.

LEAN Data.Node · Data.NodeIndex · Data.gridSpace

The first case has the right signs by (I14)(i), and the third because $\mathrm{sg}(o, 0, a) = -\mathrm{sg}(o-1, 0, a)$; this is why Definition 7.6 begins with that equation.

Definition 9.2 (forms on the grids)**.** In Setting 2.1, let D be admissible data, $z \in \{0, 1\}$ and i, j, a, b natural numbers.

(a) $\ker^z_{i,j}(D)$ is the kernel pair of the parameter $\tau \mapsto \mathrm{gpar}^{z,0}_{i,j}(D, \tau)$ and the signs $\mathrm{sgn}^{z,0}_{i,j}(D)$ (Input (I16)).

(b) $\mathcal{F}^z_{a,b}(D)$ is the space of the forms on the unitary group of $\mathcal{V}^z_{a,b}[0](D)$ whose type is the kernel of index 1 of $\ker^z_{a,b}(D)$, that of the negated parameter; it carries the Petersson product.

(c) $\mathcal{F}^{\mathrm{lev}}_i(D)$ is the subspace of $\mathcal{F}^1_{0,i}(D)$ of the forms fixed by the principal level of the integer $\bar{\pi}_g(D)^{\,b_1(D)}$.

LEAN Data.gridKernel · Data.GridForm · Data.gridLevelForms

Definition 9.3 (characters on the grids)**.** In Setting 2.1, let D be admissible data, ϑ an m-fold twist of D, $c \in \{0, 1\}$, l a natural number and n a finite index set.

(a) the character $\mu^{\mathrm{gr}}_{D,l}(\vartheta; c, n)$ is given by

$$\mu^{\mathrm{gr}}_{D,l}(\vartheta; c, n) = \begin{cases} \mu_{D,l+1}(\vartheta^{l+1}) \cdot \upsilon^{\lfloor |n|/2 \rfloor} & \text{if } c = 1, \\ \mu_{D,l+1}(1_D) \cdot \upsilon^{\lfloor (|n|+1)/2 \rfloor} & \text{if } c = 0. \end{cases}$$

(b) $\psi^{\mathrm{gr}}_{D,l}(\vartheta; c) = \psi_{D,l+1}(\vartheta^{l+1})$ if $c = 1$, and $\psi^{\mathrm{gr}}_{D,l}(\vartheta; c) = \psi_{D,l+1}(1_D)$ if $c = 0$.

LEAN Data.gridChar · Data.gridTamePart

Definition 9.4 (levels on the grids)**.** In Setting 2.1, let D be admissible data, ϑ an m-fold twist of D, $z \in \{0, 1\}$ and a, b natural numbers.

(a) $\mathrm{to}_D(z, \vartheta, b)$ is the order of the group $\mathrm{GL}_{\underline{\mathcal{V}}^z_{0,b}(D)}(\mathcal{O}_{E(D)}/\mathfrak{m}^{\triangle}_D(z, \vartheta))$.

(b) For a node $p = (z, a, b)$, $\mathcal{K}_D(\vartheta; p)$ is the following subgroup of the group $\mathrm{GL}_{\underline{\mathcal{V}}_D(p)}(\mathcal{O}_{E(D)}/\mathfrak{m}_D(\vartheta))$: it is trivial if $z = 0$ and $a = 0$, and otherwise it is the kernel of reduction to $\mathrm{GL}_{\underline{\mathcal{V}}_D(p)}(\mathcal{O}_{E(D)}/\mathfrak{m}^{\triangle}_D(1, \vartheta))$.

(c) $\mathrm{ix}_D(\vartheta; z, a, b)$ is the index of $\mathcal{K}_D(\vartheta; (z, a, b))$ in that group, as a complex number.

LEAN Data.triangleOrder · Data.nodeLevel · Data.levelIndex

Definition 9.5 (moves on the grids)**.** In Setting 2.1, let D be admissible data, ϑ an m-fold twist of D, $z \in \{0, 1\}$ and a, b natural numbers. In each case F is a form in the source of the map.

(a) $\mathrm{desc}^{z}_{D,b}(\vartheta)(\,\cdot\,)\colon \mathcal{F}^{z}_{0,b+1}(D) \to \mathcal{F}^{z}_{0,b}(D)$ is $\mathrm{to}_D(z,\vartheta,b)$ times the descent (Input (I16)) from the space $\mathcal{V}^{z}_{0,b+1}[0](D)$ to the space $\mathcal{V}^{z}_{0,b}[0](D)$, formed with the characters $n \mapsto \mu^{\mathrm{gr}}_{D,b}(\vartheta; 1-z, n)$, with the polynomial

$$\mathrm{har}_D(0,0,1-z,b;z;0,b;0,b+1)$$

and, at the finite places, with the indicator of the matrix of the inclusion of $\underline{\mathcal{V}}^{z}_{0,b}(D)$ into $\underline{\mathcal{V}}^{z}_{0,b+1}(D)$ modulo $\mathfrak{m}^{\triangle}_D(z,\vartheta)$; we write $\mathrm{desc}^{z}_{D,b}(\vartheta)(F)$ for its value at F.

(b) $\iota^{+,z}_{D,b}(\vartheta)(\,\cdot\,)\colon \mathcal{F}^{z}_{0,m}(D) \to \mathcal{F}^{z}_{0,b}(D)$ is defined by descending recursion: it is the identity for $b = m$, zero for $b > m$, and for $b < m$

$$\iota^{+,z}_{D,b}(\vartheta)(F) = \mathrm{desc}^{z}_{D,b}(\vartheta)(\iota^{+,z}_{D,b+1}(\vartheta)(F)).$$

(c) For a matrix y over $\mathcal{O}_{E(D)}/\mathfrak{m}_D(\vartheta)$ with rows indexed by $\underline{\mathcal{V}}^{z}_{a,b+1}(D)$ and columns by $\underline{\mathcal{V}}^{z}_{a,b}(D)$, $\mathrm{desc}_D[y](\,\cdot\,)\colon \mathcal{F}^{z}_{a,b+1}(D) \to \mathcal{F}^{z}_{a,b}(D)$ is the descent with the characters $n \mapsto \mu^{\mathrm{gr}}_{D,b}(\vartheta; 1-z, n)$, the polynomial $\mathrm{har}_D(0,0,1-z,b;z;a,b;a,b+1)$ and the indicator of y modulo $\mathfrak{m}_D(\vartheta)$.

(d) For a matrix y over $\mathcal{O}_{E(D)}/\mathfrak{m}_D(\vartheta)$ with rows indexed by $\underline{\mathcal{V}}^{z}_{a+1,b}(D)$ and columns by $\underline{\mathcal{V}}^{z}_{a,b}(D)$, $\mathrm{lift}_D[y](\,\cdot\,)\colon \mathcal{F}^{z}_{a,b}(D) \to \mathcal{F}^{z}_{a+1,b}(D)$ is the lift with the characters $n \mapsto \mu^{\mathrm{gr}}_{D,a}(\vartheta; z, n)$, the polynomial $\mathrm{har}_D(0,1,z,a;z;a,b;a{+}1,b)$ and the indicator of y modulo $\mathfrak{m}_D(\vartheta)$.

LEAN Data.pointDescent · Data.iotaPlus · Data.descentMove · Data.liftMove

Lemma 9.6. *In Setting 2.1, let D be admissible data, $z \in \{0,1\}$ and i, j natural numbers with $i \le m$ and $j \le m$. Then $\ker^{z}_{i,j}(D)$ is a conjugate kernel pair.*

Proof. The index set has at most $2\,n_m(D)$ elements (this count is in unit 147), and

$$n_m(D) = k(D) + m\,(5r+1) \le r + 12\sqrt{x}\,r \le 2r^2$$

by (D2). By (I14)(iv) the entries of the parameter at each τ are distinct, of the right parity, and differ by at least $2\,h(D) = 4r^8 \ge 1000\,(4r^2)^3$ at indices of different signs; so (I16) applies. □

LEAN Data.isConjKernelPair_gridKernel

Lemma 9.7. *In Setting 2.1, let D be admissible data, ϑ an m-fold twist of D, $c \in \{0,1\}$, l a natural number and n a finite index set.*

(i) *$\mu^{\mathrm{gr}}_{D,l}(\vartheta; c, n)$ has weight $\mathrm{gw}_D(c,l,n)$ and tame part $\psi^{\mathrm{gr}}_{D,l}(\vartheta; c)$.*

(ii) *Let n' be a finite index set, $\mathbf{s}$ signs on n' and k an element of the stabiliser $\mathcal{K}_0$ of the standard lattice in $\mathrm{U}_{n'}(\mathbb{A}_f)$. Then the value of $\mu^{\mathrm{gr}}_{D,l}(\vartheta; c, n)$ at the determinant of the adelic point $(1,k)$ of the unitary group of the signs $\mathbf{s}$ is*

$$\prod_{w \mid \mathfrak{m}_D(\vartheta)} \psi^{\mathrm{gr}}_{D,l}(\vartheta; c)_w(\det(k \bmod w)),$$

the value at $\det(k \bmod \mathfrak{m}_D(\vartheta))$ of the character of $(\mathcal{O}_{E(D)}/\mathfrak{m}_D(\vartheta))^{\times}$ with these components. In particular it does not depend on n.

Proof. (i) By Theorem B(iii) the first factor has weight $-w_{D,l+1}(\tau)$ and the stated tame part; by (I8) and (I7), multiplying by v^y adds y to the weight. (ii) By Lemma 3.5(ii) and Definition 6.1, the tame part is trivial outside the chosen places and the places of $Q(\vartheta)$, which by (I15) are the prime divisors of $\mathfrak{m}_D(\vartheta)$; apply (I17) with (i). □

LEAN `Data.gridChar_hasWeight` · `Data.gridChar_det`

Lemma 9.8 (steps of the grids are edges)**.** *In Setting 2.1, let D be admissible data, $z \in \{0,1\}$, i,j natural numbers, I a set, $\vartheta = (\vartheta_y)_{y\in I}$ a family of m-fold twists of D and $d \in \{0,1\}$. Let $c \in \{0,1\}$ and natural numbers l, i', j' be given by $(c,l,i',j') = (z,i,i+1,j)$ if $d = 1$ and $(c,l,i',j') = (1-z,j,i,j+1)$ if $d = 0$, and suppose $i' \le m$ and $j' \le m$. Then the characters $n \mapsto \mu^{\mathrm{gr}}_{D,l}(\vartheta_y; c, n)$, $y \in I$, the kernel pairs $\ker^z_{i,j}(D)$ and $\ker^z_{i',j'}(D)$, the direction d and the polynomial $\mathrm{har}_D(0,d,c,l;z;i,j;i',j')$ form an edge family.*

Proof. By Lemma 9.7(i) the characters have the weights $\mathrm{gw}_D(c,l,\cdot)$, which do not depend on y. By (I14)(v) weights, parameters and signs at the two nodes correspond, and (I16) gives the edge family. □

LEAN `Data.isEdgeFamily_grid`

Lemma 9.9 (the two grids meet)**.** *In Setting 2.1, let D be admissible data and j a natural number.*

(i) If $j \le m$, then $\mathcal{V}^1_{j,j}[0](D)$ is the orthogonal sum of $\mathcal{V}^0_{j,j}[0](D)$ and the standard line of sign $(-1)^j$, relabelled along the bijection Δ_j of (I14)(iii).

(ii) There is a bijection from $\underline{\mathcal{V}}^0_{j+1,j}(D)$ to $\underline{\mathcal{V}}^1_{j,j+1}(D)$ that carries the signs $\mathrm{sgn}^{0,0}_{j+1,j}(D)$ to $\mathrm{sgn}^{1,0}_{j,j+1}(D)$ and along which $\mathcal{V}^1_{j,j+1}[0](D)$ is the relabelling of $\mathcal{V}^0_{j+1,j}[0](D)$.

Proof. (i) For $j = 0$ both sides are $Y_0(D) \oplus I(+1)$ by Definition 9.1, and Δ_0 is the identity. For $j \ge 1$ both $\mathcal{V}^1_{j,j}[0](D)$ and $\mathcal{V}^0_{j,j}[0](D)$ are standard signed spaces. A sum of standard signed spaces is the standard signed space of the two families of signs, and its relabelling is the standard signed space of the transported signs, which are $\mathrm{sgn}^{1,0}_{j,j}(D)$ by (I14)(iii).

(ii) Both index sets have $r + (2j+1)\,t(D)$ elements by (I14), since $t(D) \ge 1$. By (I14)(ii) the numbers of signs -1 are $\lfloor(\max(m-j-1,0)+1)/2\rfloor$ and $\lfloor\max(m-j,0)/2\rfloor$, which agree. So there is a bijection carrying one family of signs to the other. Both spaces are standard signed spaces by Definition 9.1, since $j+1 \ne 0$; relabelling one gives the other. □

LEAN `Data.gridSpace_diag` · `Data.gridSpace_subdiag`

Lemma 9.10. *In Setting 2.1, let D be admissible data, ϑ an m-fold twist of D, $z \in \{0,1\}$ and a,b natural numbers, and suppose that $z = 1$ or $a \ne 0$. Then*

$$\mathrm{ix}_D(\vartheta; z, a, b) = \left|\mathrm{GL}_{\underline{\mathcal{V}}^z_{a,b}(D)}(\mathcal{O}_{E(D)}/\mathfrak{m}_D(1_D))\right|.$$

In particular it does not depend on ϑ.

Proof. Under the hypothesis, $\mathcal{K}_D(\vartheta;(z,a,b))$ is the kernel of the reduction

$$\mathrm{GL}(\mathcal{O}_{E(D)}/\mathfrak{m}_D(\vartheta)) \longrightarrow \mathrm{GL}(\mathcal{O}_{E(D)}/\mathfrak{m}^{\triangle}_D(1,\vartheta)),$$

and $\mathfrak{m}_D^{\triangle}(1,\vartheta) = \mathfrak{m}_D(1_D)$ by (I15). The index of a kernel is the order of the image, and the reduction is surjective: by (I15) reduction from $\mathcal{K}_0$ is surjective and factors through it. □

LEAN `Data.levelIndex_eq_untwisted`

10 Where the results are used

The objects of Sections 3 to 7 are used in every later chapter. The level $N(D)$ and the element $\pi_N(D)$ define the level of the columns of Chapter V (unit 145) and the split places where the Gauss-sum operator of the note *Chains of definite theta lifts and the Gauss-sum operator* acts; the lower bound $\log N(D) \geq \mathrm{tar}(D)$ of Theorem A(ii) is what lets a power of $N(D)$ pay for the excess of size over divisibility in the last step of the theorem on short forms (unit 151), and the form of $\mathrm{tar}(D)$ enters the admissibility of the filling data (unit 148). The characters $\mu_{D,j}(\vartheta)$ are those of the j-th theta lift in the seesaw identity and in Chapter VII; the index sets $\underline{X}_j(D)$ and $\underline{Y}_j(D)$ carry the definite and the indefinite chain. The group $\mathsf{G}(D)$ and the idempotent $e_0(D)$ define the rows (unit 140). Pairs are the arguments of the ratios of L-values of Chapters VII and VIII. Theorem C and everything of Section 9 are used only in the folder of the exchange identity (unit 147); so are entry signs and Lemma 7.7.

11 Notes on the formal text

Arguments that the census lists and Lean does not take. The lists of arguments used by the build of this note are read from the text of the source, not printed by Lean. In the file of index types a line of the form *variable* declares a finite place w and natural numbers a, n, j for the whole file, and the reading attributes such a variable to a declaration whenever its name occurs in it, also as the name of a field or of a bound variable. So it lists w among the arguments of the definitions of $\mathrm{Ch}_i(D)$, $\mathrm{Pair}(D)$, the row parameter, $\mathrm{Tame}(D)$, $\mathrm{n}(\vartheta)$, $\mathrm{Tw}(D)$, $\mathrm{Good}(D)$ and of statements of Theorem B and of Lemmas 3.2(v) and 4.2; a type T in Lemma 3.3 (it is the character there); a number a in Lemma 5.1(i); a number n in the definition of $\underline{X}_j^{\circ}(D)$ (the name occurs as the name of the function $n_j(D)$); two numbers a, j in the definition of entry signs (they are names of pattern variables); an index set n in the definition of $\mathcal{K}_D(\vartheta;p)$ (it names an argument of another function); and the field with its structures and a finiteness in Lemma 9.7(ii). Lean takes none of these. For the last three definitions this note therefore writes the notation with Lean's arguments (D and j; o, j, a; ϑ and p) and not with the census's. In the other direction, the reading takes the source of each of the four linear maps of Definition 9.5 for an argument; the notation shows the form to which the map is applied in that place. The arguments of $\mathrm{Bl}_j(S,T)$ are, in Lean's order, S, T and the number j.

Implicit data in Section 9. In the file of the grid spaces the data D are an implicit argument of every definition, so the notations of that section show D as a fixed letter and not in a slot of their own; they are to be read with the D of Setting 2.1.

Where the formal statements differ from the ones above. Theorem A(iii): the formal proof takes the square root of -1 from a lemma of the background library on fields containing a root of unity of order 5040; the proof above takes it from the root of unity of $E(D)$. Lemma 3.2(iii) repeats "w is chosen" on its right-hand side, as the formal statement does; its content is that a chosen place lies above g_0 or g_1. Lemma 7.4: the formal lemma is the count $|\underline{R}(D)| = r$; the other two statements are added here. Definition 7.5(b): the formal text divides $k(D)$ by 2 as an integer, which is exact; the index $i+2-k(D)$ is a truncated difference of natural numbers, used only for $i+1 \geq k(D)$. Definition 4.3: the formal structure names its components *character*, c, ψ, z; the bound $4b$ is a real number. Definition 6.1(a): the coprimality is stated for the natural numbers $g(D) \cdot N(D)$ and $\prod_{w \in Q} \mathrm{N}w$. Theorem C: the hypotheses on the signs force all three families to be 1, so the theorem concerns definite groups only; the letter for the sign on the point is an argument of the formal statement although it is determined. In (a1) the formal statement quantifies over all rings S with an involution and all fields R of characteristic 0, as above; the expression $\langle v', Bv \rangle$ is bilinear there, in (a3) and in the mean of (b). In (b) the constant C depends on g only: it is chosen after g and before the space, the forms, u and the level; the statement does not say that $C \neq 0$. Input (I11) is the case $q = 0$ of a formal statement with one more parameter. Lemma 9.8: the two bounds $i' \leq m$, $j' \leq m$ are arguments with a default proof by arithmetic. Lemma 9.9(i): the hypothesis $j \leq m$ enters only through the signs of (I14)(iii).

Members that nothing refers to by name. The reader's dossier lists a few members of the unit to which no other declaration refers by name. Those of this note are: the statement $\pi_g \neq 0$ of Lemma 3.2(i), which Lean's simplifier uses, and the bound for the root discriminant (Lemma 5.1(ii)) and the finiteness of $\mathrm{Bl}_j(S, T)$ (Lemma 7.2), which are instances found by Lean's search. The instances of the file of index types that have no name have no line in the concordance; they are mentioned in Lemmas 5.1(iii) and 3.2(v), after Lemma 7.4, in Remark 5.5 and after Definition 6.1. Whether a hypothesis of a formal statement is unused was not determined: no Lean was run.

How this note was written. From the dossier of the unit: the texts of its members and the structured proofs of the declarations treated here. I read the two Lean files of this note completely, the proof of Theorem C line by line, and compared every statement of this note with the Lean statement, argument by argument, with the lists of arguments of the census and of the dossier. For the Inputs I read the Lean statements of (I6), (I9) to (I12) and the definitions behind (I13) to (I17) in the files of Chapters III and VII and of the background library; I opened none of their proofs. The proofs of Lemma 5.1, of Theorem A(i) and of Lemma 9.6 replace Lean's automatic comparisons of sizes by explicit inequalities from (D1) to (D3), and are mine; so are the proof of Theorem A(iii) and the wording of the interlacing in the proof of Theorem C, which the formal text does by a case analysis left to automation. Two step claims of the dossier were corrected against the Lean text: in Theorem B(i) the residue characters are trivial at the root of unity (the dossier says that their order divides $M_\omega(P, \mu)$), and in Theorem C(a3) the vector of equal norm is attached to $\mathrm{br}\, v$, not to v'. The program check of the unit's dossier covered 503 leaf steps of 83 proofs.

Revision. After an independent referee's report and the second reading of the catalogue, Inputs (I6) and (I16) state the status of the cited literature as the catalogue records it (in both cases the relation is unresolved), and the index of the middle vector of the shape is written $\diamond$, apart from the point $*$.

Appendix A. Concordance with the Lean artifact

The formal names of the objects: `P.N₀`, `P.d` are N_0, d; the fields `D.logX`, `D.logB`, `D.s`, `D.m`, `D.r`, `D.g`, `D.piG` are $\log x$, $\log b$, μ, m, r, (g_0, g_1), π_g; `D.Φ` is $\mathrm{Pl}(D)$, `D.w j τ` is $w_{D,j}(\tau)$, `D.nominalWeight` the nominal weight, `D.gProd` is $g(D)$, `P.Mω D.s` is $M_\omega(P, \mu)$; `star` is complex conjugation; `w.q` is $\mathrm{N}w$; `AutChar` is the group of characters of $\mathrm{U}(1)$ and `HasWeight` the relation "has weight and tame part"; `WeilData.upsilonU1` is υ; truth values are written 1 and 0. Three statements as they stand in the artifact:

```
structure TameChar where
  toAutChar : AutChar D.E
  Q : Finset _
  (ψ) hasWeight : toAutChar.HasWeight 0 ψ
  ψ_eq_one : ∀ w ∉ Q, ψ w = 1
  star_notMem : ∀ w ∈ Q, star w ∉ Q
  coprime : (D.gProd * D.N).Coprime (∏ w ∈ Q, w.q)

def μ₀ := D.muFlat * (WeilData.upsilonU1 ^ (-D.h : ℤ) * D.x) ^ j

def gridSpace : ∀ o z i j, Space D.E _ (D.gridSigns z o i j)
  | 0, true, 0, 0 => ((D.spaceY 0).sum (signedSpace ..)).reindex (.refl _) D.gridSigns_zero.symm
  | 0, false, 0, j => D.spaceY j | _ + 1, _, 0, 0 => (gridSpace ..).neg | _, _, _, _ =>
    signedSpace ..
```

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Lemma 5.1	`Data.hasLogBound_rootDisc`	IndexTypes.lean	30–33
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Definition 3.1	`Data.IsChosen`	IndexTypes.lean	41–41
Definition 3.1	`Data.ChosenPlace`	IndexTypes.lean	45–45
Lemma 3.2	`Data.piGBar_mul_star`	IndexTypes.lean	47–48
Lemma 3.2	`Data.piG_ne_zero`	IndexTypes.lean	50–50
Lemma 3.2	`Data.piGBar_ne_zero`	IndexTypes.lean	52–52
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Lemma 3.3	`Data.exists_residue_char_of_order`	IndexTypes.lean	70–72
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In this note	Lean declaration	File	Lines
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Definition 5.4	`Data.π`	IndexTypes.lean	178–178
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Definition 6.1	`Data.TameChar`	IndexTypes.lean	186–192
Definition 6.1	`Data.TameChar.levelNorm`	IndexTypes.lean	195–195
Definition 6.1	`Data.Twist`	IndexTypes.lean	199–199
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Lemma 9.7	`Data.gridChar_hasWeight`	PairTensorData.lean	119–120
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Lemma 9.10	`Data.levelIndex_eq_untwisted`	PairTensorData.lean	181–184

Notation

$P = (N_0, d)$, $K(P)$ parameters; the field $\mathbb{Q}(\sqrt{-d})$

D admissible data for P (Setting 2.1)

x, x_1, b the three scales of D

μ, m, r the exponent, the number of lifts and the rank of D

g_0, g_1, $g(D)$, π_g the two primes of D, their product, an integer of $K(P)$ of norm $g_0 g_1$

$C(P)$ the constant of P of Chapter III

$E(D)$, $N_1(D)$, ς_D the CM field of D; the order of its root of unity; that root

$\mathrm{Pl}(D)$, τ, $f(D)$ the infinite places of $E(D)$; one of them; their number

$\mathrm{thick}(D)$ the thick places

w, $\overline{w}$, $\mathbb{F}_w$, $\mathrm{N}w$ a finite place of $E(D)$; its conjugate; residue field; absolute norm

$\mathcal{O}$, $\mathbb{A}_f$ rings of integers; the finite adeles of the field in question

$\mathrm{rd}_{E(D)}$, $\varpi(P,\mu)$, $\Omega(P,\mu)$, $\mathrm{cs}(d)$ root discriminant; generator of the different; the period; a constant of d

$M_\omega(P,\mu)$ the number $(N_0 d)^{\mu+6}$

$k(D)$, $t(D)$, $h_2(D)$, $h(D)$, $n_j(D)$ the numbers $r+1-m$, $5r$, r^8, $2r^8$, $k(D)+j(t(D)+1)$

$a(D)$, $b_1(D)$, $b_2(D)$ least multiples of $N_1(D)$ above xb, $a(D)/x$, $a(D)/x_1$

$w_{D,j}(\tau)$, $w_D(\tau)$ the weights; the nominal weight

$\bar{\pi}_g(D)$ the conjugate of π_g in $\mathcal{O}_{E(D)}$

$w \in \mathrm{Ch}(D)$, $w \notin \mathrm{Ch}(D)$ w is, is not, a chosen place of D

$\mathrm{Ch}_i(D)$, $\mathrm{pl}_D(x)$ chosen places above g_i; the place of a pair $x = (i, w)$

$\omega_{D,w}$, $\omega_{D,\bullet}$, T, T_w residue character at w; the family; characters of order $M_\omega(P,\mu)^2$

$\mathrm{U}(1)$, U_n the norm-one group; the unitary group of the identity matrix

ν, $\mathbf{e}$, ψ a character of U(1) (in Definition 7.1, a bijection); a weight; a tame part

υ the splitting character on U(1)

$\mu^\flat_D$, χ°_D, $\mu^0_{D,j}$ the three characters of Definition 4.1

$\mathrm{Pair}(D)$, $(\nu, \mathbf{c}, \psi, z)$ the pairs of D; a pair

$\mathrm{tar}(D)$, $N(D)$, $\pi_N(D)$ the target; the level; an integer of norm $N(D)$

U, p, M_0, Ψ, Ψ_0, ς_0 in Section 5: a real number, a prime, $N_1(D)/N_0$, cyclotomic polynomials of orders $N_1(D)$ and N_0, a root of unity of order N_0

u, q, y, y' an element or adele of norm one; natural numbers and integers bound in a statement

$\mathrm{Tame}(D)$, $\mathrm{Tw}(D)$, 1_D tame characters and twists of D; the trivial one

ϑ, $Q(\vartheta)$, ψ_ϑ, $\mathrm{n}(\vartheta)$ a tame character; its set of places; its tame part; the product of the norms of the places of $Q(\vartheta)$

$\mu_{D,j}(\vartheta)$, $\psi_{D,j}(\vartheta)$, $\mathrm{Lf}(D)$ character and tame part of the j-th lift; the set $\{1, \dots, m\}$

$[q]$, $\{*\}$ the set $\{0, \dots, q-1\}$; a set with one element

S, T, α, $\mathrm{Bl}_j(S,T)$ sets; S and j copies of T

in_j, $\mathrm{eq}_j(\nu)$ inclusion of the first block; a bijection

$\varrho(D)$ the number $m/2-1$

$\underline{X}^\circ_j(D)$, $\underline{X}_j(D)$, $\underline{R}(D)$, $\underline{Y}_j(D)$ index sets: of $X_j(D)$ without its last vector, of $X_j(D)$, of $Y_0(D)$, of $Y_j(D)$

$\mathfrak{s}(D)$ the shape of $Y_0(D)$

num_D a numbering of $\underline{R}(D)$

$\mathrm{par}_{D,\tau}(i)$, $\mathrm{par}^W_{D,\tau}(i)$ the row parameter; its double on $\underline{R}(D)$

$\mathrm{sg}^W_{D,j}(i)$, $\mathrm{sg}(o,j,a)$ signs of the spaces W_j; entry signs

$\mathsf{G}(D)$, $\mathrm{Good}(D)$, $e_0(D)$, c_x, e_{c_x} the finite group; good families; the idempotent; a polynomial and its idempotent

E, n, n', N in Section 8: a CM field, index sets, the number $|n|$

(κ, e), (κ', e'), $(\kappa^\natural, e^\natural)$ weights

$\kappa^\uparrow$, $\kappa^\downarrow$ the two parts of κ in (I11)

l, l', $\mathrm{hc}^N_{\kappa,e}(\tau,p)$ families of parameters; the doubled Harish-Chandra parameter

$\lambda_\tau(p)$, $\lambda^\natural_\tau(q)$, $\delta_\tau(q)$ highest weights; their difference

$\mathbf{s}$, $\mathbf{r}$, $\mathbf{t}$, σ signs; a bijection $n \sqcup \{*\} \to n'$

$\rho_{\kappa,e}$, $\overline{\rho_{\kappa,e}}$ the representation of weight (κ, e); at conjugate matrices

$\langle v', v\rangle$, (v', v), $\|v\|$ dot product; Gram form; Weyl norm

$\mathrm{boxes}(\kappa, N)$ the number of boxes of κ in rank N

S, R, f_τ, g, h rings; ring homomorphisms; group elements

B, A, br, N_τ, M, C integer matrices; exponents; a natural number; a constant

V, η, ϕ, Φ, F a standard space; algebraic forms; an automorphic function; a form

$F_\eta^{l,\mathrm{s}}$, v_l^+, $v^\natural$, v° the function of a form; highest vector; vectors

$\mathcal{K}$, $\mathcal{K}_0$, $\mathrm{mean}_{\mathcal{K}}(F)$ a level; the maximal one; the mean at level $\mathcal{K}$

$\mathrm{diag}_\sigma(g, u)$ block diagonal matrix along σ

L', L_h linear functionals

z, c, d elements of $\{0, 1\}$: a grid, twisted or not, a direction; z is a natural number in a pair, c a natural number in Theorem B and Lemma 3.2

o, i, j, a, b, l, i', j' natural numbers: an offset and positions on the grids (b also the third scale, l also a parameter)

$I(\mathbf{s})$, $Y_j(D)$ standard signed space; the hermitian space $Y_j(D)$

$\underline{\mathcal{V}}^z_{i,j}(D)$, $\mathrm{sgn}^{z;o}_{i,j}(D)$, $\mathrm{gpar}^{z;o}_{i,j}(D, \tau)$ index set, signs and doubled parameter of a node

$\mathrm{gw}_D(c, l, n)$, $\mathrm{har}_D(o, d, c, l; z; a, b; i, j)$ grid weight; joint harmonic

$\mathfrak{m}_D(\vartheta; q)$, $\mathfrak{m}_D(\vartheta)$, $\mathfrak{m}^{\triangle}_D(z, \vartheta)$, $\mathfrak{a}$ moduli; an ideal

$p = (z, a, b)$, $\underline{\mathcal{V}}_D(p)$ a node; its index set

$\mathcal{V}^z_{i,j}[o](D)$ the space at a node

$\ker^z_{i,j}(D)$, $\mathcal{F}^z_{a,b}(D)$, $\mathcal{F}^{\mathrm{lev}}_i(D)$ kernel pair; forms at a node; forms of level

$\mu^{\mathrm{gr}}_{D,l}(\vartheta; c, n)$, $\psi^{\mathrm{gr}}_{D,l}(\vartheta; c)$ character and tame part on the grids

$\mathrm{to}_D(z, \vartheta, b)$, $\mathcal{K}_D(\vartheta; p)$, $\mathrm{ix}_D(\vartheta; z, a, b)$ an order; the level at a node; its index

$\mathrm{desc}^z_{D,b}(\vartheta)(F)$, $\iota^{+,z}_{D,b}(\vartheta)(F)$ descent by one step; iterated descent

$\mathrm{desc}_D[y](F)$, $\mathrm{lift}_D[y](F)$ descent and lift with the indicator of y

Σ a finite set of finite places

V, Z the two grids ($z = 0$, $z = 1$); V is also a standard space

Δ_j, k, I, $\mathcal{S}$, $\mathcal{T}$ a bijection; an element of $\mathcal{K}_0$; an index set of a family; spaces

i, π, e a square root of -1; an integer of norm $N(D)$ (also the number π); Euler's number

References

The references are those of the catalogue of quoted results of this write-up, with the state it records.

[W] A. Weil, *On a certain type of characters of the idèle-class group of an algebraic number-field*, Proceedings of the international symposium on algebraic number theory, Tokyo–Nikko 1955 (1956), pp. 1–7, especially pp. 2–3 on the unit and finite-index extension conditions, equation (3).

[R] A. Raghuram, *Notes on the arithmetic of Hecke L-functions*, arXiv:2207.03399, §3.1, equation (12) and Proposition 13, pp. 9–10, and §4.3.2, Proposition 25, p. 17. (For [W] and [R] the second reading of the catalogue finds a different domain and compatibility condition than in Input (I6): relation unresolved.)

[I] Ichino, *Theta lifting for discrete series representations of real unitary groups*, arXiv:2002.09148, Theorem 4.1 and Lemma 7.8, pp. 35–36. (Cited by the catalogue for the edge statement behind Input (I16); its second reading finds that the formal two-sided statement goes beyond the published joint-type criterion, so the relation is unresolved.)

For Inputs (I11) and (I12) the catalogue has no confirmed reference for the statements used here: reference to be confirmed.

Theta functions of a CM elliptic curve: periods, tight frames, conjugates and divisibility of jets

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. On the definite unitary groups of Theorem A of the write-up, a theta lift is a finite sum of Taylor coefficients at the origin (*jets*) of theta functions. This unit shows that these theta functions are products of translates of one function, the theta function θ_C of an elliptic curve with complex multiplication by $K = \mathbb{Q}(\sqrt{-d})$, composed with linear forms given by a *tight frame* in the CM field E (Theorem A, Proposition 5.2). Everything else is then read off the curve: the conjugates of the jets under $\mathrm{Aut}(\mathbb{C}/K)$ are jets again, of bounded size (Theorem B); suitably scaled jets are integral at every prime (Theorem C); at a prime p that is inert or ramified in K a product of theta series is, after a change of scale, an integral series times one exponential (Theorem D), which makes jets along products of minors divisible by an explicit power of p (Theorem E), and this power cannot be improved by much (Theorem F). The unit also defines the numbers that the next unit uses to state the divisibility of the whole chain of lifts: the ask, the gain, the loss and the column counts.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, attributed to L. Alpöge with the AI system Claude, whose final theorems state the Riemann hypothesis and its analogue for Dirichlet L-functions. That headline is a *claim*: the authors report that every file compiles and that Lean's kernel accepted the proofs; the record of checks of the handout says who replayed what; nothing was compiled for this note; no mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note treats one unit, the folder on the CM curve: all 89 of its 89 declarations. Every statement below was compared with its Lean text, argument by argument. Proofs are given in full for what the unit itself proves; results of other units are stated as *Inputs* with the unit that owns them, and results of the background library of the artifact (complex multiplication, the Chowla–Selberg formula, the formal group of the curve, the Weil representation) are *quoted*, with a reference where the catalogue of the handout has one. Where a formal step is closed by automation that I could not follow without running Lean, §12 says so. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 89 declarations.

1 The statements

The notation is that of §2. $P = (N_0, d)$ are parameters and μ is a natural number; $E(P,\mu)$ is the CM field obtained from $K(P) = \mathbb{Q}(\sqrt{-d})$ by adjoining a root of unity of order $N_1(P,\mu)$, $\mathrm{Pl}(P,\mu)$ is the group of its infinite places, and $\iota_\tau \colon E(P,\mu) \to \mathbb{C}$ is the chosen embedding at the place τ. A *p-adic place* is a prime number p with a ring homomorphism $j \colon \mathbb{C} \to \mathbb{C}_p$; $|\cdot|_p$ is the absolute value of $\mathbb{C}_p$ and $\mathfrak{o}_p$ its ring of integers. The numbers $s_1(P,\mu)$ and $\mathrm{per}(P,\mu)$ are the period factor and the scale of the jets (Definition 2.1), $T^V_a f$ is the series of jets of a vector f of the Weil representation of a hermitian space V at the scale a (Background 4), and $\mathrm{Std}_\delta(V)$ is the module of standard data (Definition 2.2).

Theorem A (tight frames with a unit vector). Let $P = (N_0, d)$ be parameters, μ a natural number, p a prime number and $j \colon \mathbb{C} \to \mathbb{C}_p$ a ring homomorphism. Let M and c_* be natural numbers with $(d \cdot N_1(P,\mu))^6\,(c_* + 1) \le M$. Then there are a finite set I, elements $v_i \in E(P,\mu)$ and natural numbers c_i, for $i \in I$, such that the family $(v_i)_{i\in I}$ with the multiplicities $(c_i)_{i\in I}$ is a tight frame of length M (Background 1), $c_* \le c_i$ for every $i \in I$, and for some $i \in I$

$$|j(s_1(P,\mu)\cdot\overline{\iota_\tau(v_i)})|_p = 1 \qquad \text{for every } \tau \in \mathrm{Pl}(P,\mu)$$

(for $c_* \ge 1$ this vector occurs in the frame with positive multiplicity; for $c_* = 0$ its multiplicity may be 0).

LEAN `Parameters.exists_isTightFrame`

Theorem B (conjugates of jets). Let $P = (N_0, d)$ be parameters, μ a natural number, n a finite set and V a standard hermitian space over $E(P,\mu)$ with index set n. Let R be a set of finite places of $E(P,\mu)$ that contains the complex conjugate of none of its members, and let $\sigma \in \mathrm{Aut}(\mathbb{C}/K(P))$. Then there are $t \in \mathbb{C}$, $c \in \mathbb{C}$, a finite adele z of $E(P,\mu)$ with $z\bar{z} = 1$ and a σ-semilinear map $L \colon \mathcal{S}(V) \to \mathcal{S}(V)$ with values in $\mathcal{W}_{\mathrm{f}}(V)$ such that:

(i) $|t|^2 = \Omega(P,\mu)^{-1}$, and $c \neq 0$;

(ii) for every $f \in \mathcal{W}_{\mathrm{f}}(V)$, applying σ to the coefficients of $T^V_{\mathrm{per}(P,\mu)} f$ gives the series $T^V_t(Lf)$ with every variable $z_{(\tau,i)}$ replaced by $z_{(\mathrm{perm}_\mu(\sigma)^{-1}\tau,\,i)}$, where $\mathrm{perm}_\mu(\sigma)$ is the permutation of $\mathrm{Pl}(P,\mu)$ with $\sigma \circ \iota_\tau = \iota_{\mathrm{perm}_\mu(\sigma)(\tau)}$ for every τ;

(iii) for every $f \in \mathcal{W}_{\mathrm{f}}(V)$ and every $g \in \mathrm{U}_n(\mathbb{A}_{\mathrm{f}})$: $L(\omega_V(1,g)f) = \dfrac{\sigma(\Delta_{\mathrm{f}}(g))}{\Delta_{\mathrm{f}}(g)} \cdot \omega_V(1,g)(Lf)$;

(iv) for every $x \in \mathbb{A}_{\mathrm{f}}^n$, every $g \in \mathrm{U}_n(\mathbb{A}_{\mathrm{f}})$ and every adelic vector $y \in \mathbb{A}^n$:

$$\left|L(\mathfrak{h}_V(x^\flat)\,\omega_V(1,g)\,\theta^\circ_V)(y)\right| \;\le\; \exp(C(P)\cdot(N_1(P,\mu)\,|n|+1)^2),$$

where $C(P)$ is the constant of the parameters of the note on the CM field (unit 152); it is $C_{\mathrm{gen}}(P) + 2d + 6$ plus non-negative terms, $C_{\mathrm{gen}}(P)$ being the constant of Proposition 6.1;

(v) for every Schwartz–Bruhat function Ψ above R: $L(\mathrm{name}_{V,R}(\Psi)\mathrm{vac}_V) = c \cdot \omega_V(1, z\cdot 1_n)(\mathrm{name}_{V,R}(\sigma\circ\Psi)\mathrm{vac}_V)$.

LEAN `Parameters.exists_semilinear_isJetConjugate`

Theorem C (integrality of jets). Let $P = (N_0, d)$ be parameters, μ a natural number, n a finite set, V a standard hermitian space over $E(P,\mu)$ with index set n, p a prime number and $j\colon \mathbb{C} \to \mathbb{C}_p$ a ring homomorphism. Let δ be a nonzero integer of $K(P)$ and $f \in \mathrm{Std}_\delta(V)$. Let Q be a polynomial in the variables $z_{(\tau,i)}$, $(\tau,i) \in \mathrm{Pl}(P,\mu) \times n$, with coefficients in O_j, and Q' the same polynomial with its coefficients read in $\mathbb{C}$. Then

$$|j\Big(\mathrm{jet}_{Q'}(T^V_{\overline{\delta}\cdot d\cdot \mathrm{per}(P,\mu)} f)\Big)|_p \;\le\; 1,$$

where $\mathrm{jet}_{Q'}(F) = (Q'(\partial)F)(0)$ is the jet of a power series F along Q'.

LEAN `Parameters.norm_jet_jetSeries_le_one`

Theorem D (normal form of theta products at a non-split prime). Let $P = (N_0, d)$ be parameters, μ a natural number, p a prime number with $\chi_{K(P)}(p) \neq 1$ and $j\colon \mathbb{C} \to \mathbb{C}_p$ a ring homomorphism. Then there are $\varpi \in \mathbb{C}_p$ and $\varpi_0 \in \mathfrak{o}_p$ with

$$|\varpi|_p = p^{-\varrho_d(p)}, \qquad |\varpi_0|_p = p^{(\chi_{K(P)}(p)-1)/2},$$

such that for every finite set I, every real number $r \le 1$ and every $\Theta \in \mathrm{ThProd}_{P,\mu}(1, r; I)$:

(a) for every natural number B,

$$\Theta \text{ is integral on the null cone of } (\mathrm{pr}, S_B; \varpi, \varpi_0) \text{ up to degree } B,$$

where $\mathrm{pr}_e(v) = [-p^e]_{P,\mu} \cdot v$ and S_B is the set of the natural numbers $e \le \lfloor \log_p B \rfloor$ such that e is odd or $p = d$;

(b) if $\chi_{K(P)}(p) = -1$ and $r = 1$, there are $\kappa \in \mathbb{C}_p$ and power series Ψ, V with coefficients in $\mathfrak{o}_p$, in the variables z_x, $x \in \mathrm{Pl}(P,\mu) \times I$, such that $|\kappa|_p = p^{1-1/(p-1)}$, $\Psi(0) = 0$,

$$\Theta(\varpi z) = \exp(\kappa\Psi) \cdot V,$$

and for every variable x and every exponent β: the coefficient at z^β of the reduction of $\partial\Psi/\partial z_x$ modulo $\mathfrak{m}_p$ is nonzero if and only if $\beta = p^e \cdot 1_y$ for an odd e and a variable $y = (v,i)$ with $([-p^e]_{P,\mu} \cdot v,\, i) = x$; here 1_y is the exponent of z_y.

LEAN `Parameters.thetaProducts_normal_form`

Theorem E (divisibility of Schur jets). Let p be a prime number, Φ, n, n' finite sets, and $\varkappa\colon \Phi \times \mathbb{N} \to \mathbb{N}$ a family of column counts that fits the ranks $|n|$, $|n'|$ (Background 5). Let $\mathrm{pr}\colon \mathbb{N} \times \Phi \to \Phi$ be a map, $S \subset \mathbb{N}$ and $\Xi \subset \Phi$ finite sets, ρ, ω real numbers, $\varpi \in \mathbb{C}_p$ with $|\varpi|_p = p^{-\rho}$, $\varpi_0 \in \mathfrak{o}_p$ with $|\varpi_0|_p = p^{\omega}$, and Θ a power series over $\mathbb{C}_p$ in the variables $z_{(\tau,b,c)}$, $(\tau,b,c) \in \Phi \times n' \times n$. Let $g\colon \Phi \times \mathbb{N} \times n' \to \mathbb{N}$ be a function (the *charges*) such that for all $e \in S$, $\tau \in \Phi$, $h \in \mathbb{N}$ and $b \in n'$:

$$\tau \in \Xi \ \text{ or } \ (h = 1 \text{ and } \mathrm{pr}_e(\tau) \in \Xi) \quad \Longrightarrow \quad g(\tau,1,b) \neq 0 \ \text{ or } \ g(\mathrm{pr}_e(\tau),h,b) \neq 0.$$

Let i' and i be Weyl indices for $\varkappa$ of ranks $|n'|$ and $|n|$, and put $N = |n'|$, $B = \mathrm{box}(\varkappa, N)$ and $m = \sum_{\tau\in\Phi}\sum_{h<N}\sum_{l<\varkappa(\tau,h)}\sum_{b<h} g(\tau,h,i'_\tau(h,l,b))$, the total charge of the boxes of i'. Suppose that

$$\Theta \text{ is integral on the null cone of } (\mathrm{pr}, S; \varpi, \varpi_0) \text{ up to degree } B.$$

Then

$$|\,\mathrm{SJ}_\varkappa(\Theta)_{i',i}|_p \ \le \ p^{\,m-\left((\frac{1}{p-1}-\rho)B+\omega\sum_{e\in S}\sum_{\tau\in\Phi}\mathrm{cc}_p(\varkappa,N;\Xi;\tau,\mathrm{pr}_e(\tau);e)-|\Phi|\,N^3(\lfloor\log_p B\rfloor+1)\right)}.$$

LEAN `norm_schurJet_le`

Theorem F (a jet that is not more divisible). Let $P = (N_0, d)$ be parameters, μ a natural number, p a prime number with $\chi_{K(P)}(p) \neq 1$ and $j\colon \mathbb{C} \to \mathbb{C}_p$ a ring homomorphism. Let N be a function from the exponents in the variables z_τ, $\tau \in \mathrm{Pl}(P,\mu)$, to the non-negative real numbers, Θ a power series over $\mathbb{C}_p$ in these variables, n an exponent, and i, y functions from $\mathrm{Pl}(P,\mu)$ to $\mathbb{N}$. Suppose that (i,y) is a Frobenius packing of n at p; that $\Theta(0) = 1$; that Θ is carried by one copy and Θ^{-1} by five copies of N (Background 6); and that $\Theta \in \mathrm{ThProd}_{P,\mu}(1,1;\{*\})$. Then there are exponents $\beta \le n$ and $\gamma \le n$ with $|\beta| = \sum_\tau y_\tau(p^{i_\tau}+1)$ and $|\gamma| = \sum_\tau y_\tau$ such that β is a sum of six exponents $\alpha_1, \dots, \alpha_6$ with

$$\prod_{k=1}^{6} \mathsf{N}(\alpha_k) \ \ge \ p^{\,(1-\frac{1}{p-1})|\gamma|+\varrho_d(p)\,|\beta|+\sum_\tau v_p(\gamma_\tau!)}.$$

LEAN `exists_hasLargeSplit_of_pairs`

How the pieces fit (motivation, read from the comments of the formal text and from the places where later units use the results; not statements of this unit). A column on the definite group is lifted by integrating it against a theta kernel, and on a definite group this integral is a finite sum of matrices of jets $\mathrm{SJ}_\varkappa(T_c^V f)$ of data f (Background 5). Theorem A and Proposition 5.2 turn the series $T_c^V f$ into products of Taylor series of translates of θ_C; the scale $\mathrm{per}(P,\mu)$ is chosen so that these products have p-adic absolute values governed by the period factor $s_1(P,\mu)$ and archimedean conjugates governed by $\Omega(P,\mu)$. Theorem C says that one lift cannot lower p-adic valuations by more than a power of $\overline{\delta}d$ (Proposition 7.4), which is all one needs at the primes that split in K. At the other primes the formal group of the curve has

height two. There the logarithm of θ_C keeps a denominator only in the degrees $p+1$ and $p+q$ of a good parameter (Proposition 8.4); a derivative of order a gains about $a/(p-1)$ powers of p, and the exponential $\exp(\kappa\Psi)$ takes back what the couplings of a variable at τ with the variables at the partner places $[-p^e]_{P,\mu}\cdot\tau$ allow. The balance is the number $\mathrm{ask}_{D,p}(y,\tau)$ of Definition 2.6, summed in the gain $\mathrm{gain}(D,p)$; it is the local form, at p, of the weight $\lambda_{d,1}$ of the prime sum of the main inequality. Theorem F shows that the count is sharp, which the unit on the rows needs for a lower bound.

2 The setting

Conventions. $\mathbb{N}=\{0,1,2,\dots\}$; subtraction of natural numbers is truncated at 0 and division of natural numbers is the floor, as in the formal text. *Parameters* are a pair $P=(N_0,d)$: a positive multiple N_0 of 18480 and a prime number $d\equiv N_0/2-1 \bmod N_0$; the *exponent* is a natural number μ. A statement about *data* concerns admissible data D for parameters P, with components x, x_1, b (the three scales), m, r and the exponent $\mu(D)$. Index sets such as n, n', I, J, Φ are finite sets; a hermitian space V has an index set n and signs $\varepsilon\colon n\to\{1,-1\}$. In a statement about a prime number p and a ring homomorphism $j\colon\mathbb{C}\to\mathbb{C}_p$, the absolute value $|\cdot|_p$ of $\mathbb{C}_p$ is normalised by $|p|_p=p^{-1}$, $\mathfrak{o}_p=\{x\in\mathbb{C}_p : |x|_p\le 1\}$, $\mathfrak{m}_p$ is its maximal ideal, and $O_j=\{z\in\mathbb{C} : |j(z)|_p\le 1\}$; such a pair (p,j) is called a p-adic place. In a statement about a field E with a CM type and Weil data, about a commutative ring of characteristic p, about a normed field, or about an integral domain of characteristic 0, these structures are part of the statement. v_p is the p-adic valuation of a natural number, and for a natural number B, $\lfloor\log_p B\rfloor$ is the largest $e\in\mathbb{N}$ with $p^e\le B$, and 0 if $B=0$ (as in the formal text). For a multi-index (an *exponent*) β, $|\beta|$ is its degree, $\beta!$ the product of the factorials of its entries, and $\beta\le\beta'$ is meant entry by entry.

The unit rests on four bodies of earlier material. They are recalled in six numbered paragraphs, after the definitions and the comments of the formal text; the notes of the units that own them give the details.

Background 1: the field, its lattice, tight frames (unit 152, the note on the CM field; the definition of a frame is in the background library). For parameters $P=(N_0,d)$ and $\mu\in\mathbb{N}$: $K(P)=\mathbb{Q}(\sqrt{-d})\subset\mathbb{C}$ with ring of integers $\mathcal{O}_{K(P)}$; $\chi_{K(P)}$ is its quadratic character, on integers the Legendre symbol χ_d modulo d; $N_1(P,\mu)$ is N_0 times a power of 3 and a power of 2 fixed by μ; $E(P,\mu)=K(P)(\varsigma(P,\mu))$ with $\varsigma(P,\mu)$ a primitive root of unity of order $N_1(P,\mu)$. The set $\mathrm{Pl}(P,\mu)$ of infinite places of $E(P,\mu)$ is a group, identified with $\mathrm{Gal}(E(P,\mu)/K(P))$ and with $(\mathbb{Z}/N_1(P,\mu))^\times$; $\iota_\tau\colon E(P,\mu)\to\mathbb{C}$ is the embedding at τ that is the identity on $K(P)$, and $\iota_\tau=\iota_1\circ\mathrm{gal}(\tau)$ for the automorphism of τ. For an integer ν, $[\nu]_{P,\mu}$ is the place of the class of ν if ν is prime to $N_1(P,\mu)$, and the unit place otherwise. Every prime dividing $N_1(P,\mu)$ divides $(d+1)/4$ and splits in $K(P)$. Further: $c_0(P,\mu)=(1+\sqrt{-d})/2$ as an integer of $E(P,\mu)$; $\partial(P,\mu)$, the value at $\varsigma(P,\mu)$ of the derivative of the cyclotomic polynomial of order $N_1(P,\mu)$, which generates the different of $E(P,\mu)$ over $K(P)$ and divides $N_1(P,\mu)$; the finite idele $\alpha(P,\mu)$ of $E(P,\mu)$ with component $\partial(P,\mu)^{-1}$ at the primes that contain $c_0(P,\mu)$ and 1 elsewhere; the root discriminant $\mathrm{rd}_{E(P,\mu)}$; and $\Omega(P,\mu)=\exp(\mathrm{cs}(d))\cdot 2\pi/\mathrm{rd}_{E(P,\mu)}$, where $\mathrm{cs}(d)$ is Euler's constant

minus $\mathrm{Re}(L'/L)(1,\chi_d)$. We write $\mathbb{A}$, $\mathbb{A}_{\mathrm{f}}$, $\widehat{\mathcal{O}}$ for the adeles, the finite adeles and the integral finite adeles of the field in question, and

$$\mathfrak{a}(P,\mu) = \{x \in E(P,\mu) : \ \alpha(P,\mu)^{-1}x \in \widehat{\mathcal{O}}\},$$

a fractional ideal that contains the integers of $E(P,\mu)$. Let I be a finite set, $w_i \in E(P,\mu)$ and $m_i \in \mathbb{N}$ for $i \in I$, and $M \in \mathbb{N}$. The family (w_i) with multiplicities (m_i) is a *tight frame of length* M if every w_i lies in $\mathfrak{a}(P,\mu)$ and

$$\sum_{i\in I} m_i\, \iota_\tau(w_i)\,\overline{\iota_{\tau'}(w_i)} = \begin{cases} M & \tau = \tau', \\ 0 & \tau \neq \tau' \end{cases} \qquad \text{for all } \tau,\tau' \in \mathrm{Pl}(P,\mu).$$

So the vectors $(\iota_\tau(w_i))_\tau$ of $\mathbb{C}^{\mathrm{Pl}(P,\mu)}$, repeated m_i times, resolve M times the identity: they replace an orthonormal basis, which the lattice need not have.

LEAN in the background library: `IsTightFrame`, `finiteImage` · the objects of the field are in the concordance of the note on the CM field

Background 2: embeddings into $\mathbb{C}_p$. Let (p,j) be a p-adic place and $\psi\colon E(P,\mu) \to \mathbb{C}_p$ a ring homomorphism. It induces a prime $\mathfrak{p}_\psi$ of $E(P,\mu)$, the set of integers x with $|\psi(x)|_p < 1$, and extends continuously to the completion at $\mathfrak{p}_\psi$; composed with the projection of $\mathbb{A}_{\mathrm{f}}$ to that completion this is a ring homomorphism $\hat{\psi}\colon \mathbb{A}_{\mathrm{f}} \to \mathbb{C}_p$. For $\tau \in \mathrm{Pl}(P,\mu)$ put $j_\tau(x) = j(\iota_\tau(\overline{x}))$, so that $j(\overline{\iota_\tau(x)}) = j_\tau(x)$. Two facts are quoted: $|\psi(x)|_p \le 1$ for integers x, and $|\hat{\psi}(a)|_p \le 1$ for $a \in \widehat{\mathcal{O}}$.

LEAN in the background library: `PadicPlace`, `placeOfEmb`, `adeleExtension`, `weightEmbedding`, `norm_ringOfIntegers_le`, `norm_adeleExtension_le_one`

Background 3: a CM datum and its theta function. Let d be a prime $\equiv 7 \bmod 8$ and $\mathcal{O}_d$ the ring of integers of $\mathbb{Q}(\sqrt{-d})$, embedded in $\mathbb{C}$. A *CM datum for* d is a triple C of a lattice $\Lambda_C \subset \mathbb{C}$ with $\mathcal{O}_d\Lambda_C \subset \Lambda_C$, a point t with $2t \in \Lambda_C$, $t \notin \Lambda_C$, and a number a_1, normalised by $\wp''(t) = 2$ (written $g_2 = 12\wp(t)^2 - 4$) and $a_1^2 = 12\wp(t)$, with $\frac{1+\sqrt{-d}}{2}t \notin \Lambda_C$ and $\frac{1-\sqrt{-d}}{2}t \notin \Lambda_C$; $\wp$ and σ_C are the Weierstrass functions of Λ_C and $\mathrm{area}(C)$ is the area of a fundamental parallelogram. For $m \in \mathbb{N}$ and $y \in \mathbb{C}$ the *twisted translation* of a function F on $\mathbb{C}$ is

$$(U_y^{C,m}F)(z) = \exp(-m\pi(z\overline{y} + |y|^2/2)/\,\mathrm{area}(C))\,F(z+y),$$

and the *theta function* of C is $\theta_C(z) = (U_t^{C,1}\sigma_C)(z)/(U_t^{C,1}\sigma_C)(0)$: an entire function with $\theta_C(0) = 1$, a section of degree one for the Riemann form of Λ_C. Further $\xi_C(z) = \wp(z+t) - \wp(t)$ and $\varphi_C(z) = \frac{1}{2}(\wp'(z+t) - a_1\xi_C(z))$, which vanishes at 0 with derivative 1; for a triple $x = (x_1,x_2,x_3)$ of natural numbers, $\mathrm{mon}_C(x) = \theta_C^{x_1}\sigma_C^{2x_2}(\varphi_C\theta_C^3)^{x_3}$ is the *monomial section* of degree $x_1 + 2x_2 + 3x_3$. *Conjugate data.* For every complex number α that is the number a_1 of some CM datum for d we fix one such datum C_α (a choice). For $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\sqrt{-d}))$ the number $\sigma(a_1)$ is again the invariant of a CM datum for d (quoted), and the *conjugate datum* is $\sigma C = C_{\sigma(a_1)}$: the chosen datum with invariant $\sigma(a_1)$. It is defined for the automorphisms that fix $\mathbb{Q}(\sqrt{-d})$ only. (The choice matters little: by the normalisation, $g_2 = 12\wp(t)^2 - 4$ and $g_3 = 4\wp(t)^3 - g_2\wp(t)$ with $\wp(t) = a_1^2/12$, so a_1 determines the lattice Λ_C and the class of t modulo it.) $\mathrm{Tay}(F)$ is the Taylor series at 0 of an entire function F, and for a p-adic place (p,j), $\mathrm{Tay}_j(F) \in \mathbb{C}_p[[X]]$ is $\mathrm{Tay}(F)$ with j applied to its coefficients. The ring

$$\mathfrak{o}_p[[\varphi_C]] = \{G(\mathrm{Tay}_j(\varphi_C)) : \ G \in \mathfrak{o}_p[[X]]\} \subset \mathbb{C}_p[[X]]$$

depends on j. For a natural number $M \geq 1$ and $r \in \mathbb{N}$, $[F]_{r \bmod M}$ is the part of a series $F \in \mathbb{C}_p[[X]]$ in the degrees $\equiv r \bmod M$. The following are quoted.

(Q1) *Normalisation.* For every nonzero fractional ideal $\mathfrak{c}$ of $\mathbb{Q}(\sqrt{-d})$ there are a CM datum C for d and $u \in \mathbb{C}^\times$ with $\Lambda_C = u\,\mathfrak{c}$; then $\operatorname{area}(C) = |u|^2\,\mathrm{N}(\mathfrak{c})\sqrt{d}/2$. (Elementary.)

(Q2) *Main theorem of complex multiplication, soft form.* Let C be a CM datum for d with $\Lambda_C = \Omega_* \mathcal{O}_d$ and $\sigma \in \operatorname{Aut}(\mathbb{C}/\mathbb{Q}(\sqrt{-d}))$. There are $\Omega \in \mathbb{C}^\times$ with $\operatorname{area}(\sigma C) = |\Omega|^2\sqrt{d}/2$, an integer $e \geq 1$ with $e\Omega \in \Lambda_{\sigma C}$, a finite adele β of $\mathbb{Q}(\sqrt{-d})$ and elements $a_S \in \mathbb{Q}(\sqrt{-d})$, one for each finite set $S \subset \mathbb{Q}(\sqrt{-d})$, with β a cluster point of the a_S (embedded diagonally in the finite adeles) as S grows, such that for every finite S and every $x \in S$

$$\sigma(\operatorname{Tay}(U^{C,1}_{x\Omega_*}\theta_C)) = \operatorname{Tay}(U^{\sigma C,1}_{x\Omega a_S}\theta_{\sigma C}).$$

Reference: G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions* (1971), Thm. 5.4; J. Milne, *Complex Multiplication* (2020), 9.17. Relation: a *weaker form* (the catalogue calls it the soft form: the adele is not identified with the Artin map, and nothing is said about its invertibility; places confirmed at second hand).

(Q3) *Chowla–Selberg formula, logarithmic form.* For a CM datum C for d there is $u \in \mathbb{C}$ with $|j(u)|_p = 1$ at every p-adic place (p, j) and $\operatorname{area}(\sigma C) \cdot |\sigma(u)|^2 = \pi \exp(\operatorname{cs}(d)) \cdot 2\pi/\sqrt{d}$ for every $\sigma \in \operatorname{Aut}(\mathbb{C}/\mathbb{Q}(\sqrt{-d}))$. The artifact cites Selberg and Chowla, *On Epstein's zeta-function*; the catalogue has no entry for this declaration: *reference to be confirmed.*

(Q4) *Integrality of a_1.* $|j(a_1)|_p \leq 1$ at every p-adic place. Catalogue: a *weaker form* of the good reduction everywhere of the normalised CM curve (Milne, *loc. cit.*; place not confirmed). Also quoted: a complex number x with $|\iota(x)|_2 \leq c$, for one real number c and every field isomorphism ι from $\mathbb{C}$ onto an algebraic closure of $\mathbb{Q}_2$ (whose absolute value is $|\cdot|_2$), is algebraic over $\mathbb{Q}$.

(Q5) *The parameter φ_C.* $\operatorname{Tay}_j(\varphi_C)$ has constant term 0 and linear coefficient 1; the ring $\mathfrak{o}_p[[\varphi_C]]$ is stable under d/dX and contains $\operatorname{Tay}_j(\xi_C)$. If $T \in \mathfrak{o}_p[[\varphi_C]]$ has $T(0) = 0$, $T'(0) = 1$, then $\mathfrak{o}_p[[\varphi_C]] = \{G(T)\}$, and the inverse series Z of T satisfies: $F \in \mathfrak{o}_p[[\varphi_C]]$ if and only if $F(Z)$ has coefficients in $\mathfrak{o}_p$. For every k the largest absolute value of the k-th coefficient of an element of $\mathfrak{o}_p[[\varphi_C]]$ is attained at a power of $\operatorname{Tay}_j(\varphi_C)$; the same holds in two variables for the ring of integral series in $\operatorname{Tay}_j(\varphi_C)(X)$, $\operatorname{Tay}_j(\varphi_C)(Y)$, and this ring contains $\operatorname{Tay}_j(\varphi_C)(X+Y)$ (the addition law). (The catalogue has no entry for these declarations: *reference to be confirmed.*)

(Q6) *Logarithm of theta.* There are $L \in \mathbb{C}_p[[X]]$ and $L_2 \in \mathbb{C}_p$ with $\operatorname{Tay}_j(\theta_C) = \exp(-L)$, $L(0) = L'(0) = 0$, $L'' = L_2 + \operatorname{Tay}_j(\xi_C)$, $|j(\sqrt{-d})\,L_2|_p \leq 1$, and for every pair (a, N) in the closure $\mathcal{N}_d \subset \mathbb{C}_p^2$ of the pairs $(j(\delta), j(\delta\overline{\delta}))$, $\delta \in \mathcal{O}_d$: the series $\operatorname{Tay}_j(\varphi_C)(aX)$, $\exp(NL - L(aX))$ and $(NL - L(aX))'$ lie in $\mathfrak{o}_p[[\varphi_C]]$. Moreover, with q the norm of a prime of $\mathbb{Q}(\sqrt{-d})$ above p: $(\zeta, \zeta^{p+1}) \in \mathcal{N}_d$ for every $\zeta \in \mathbb{C}_p$ with $\zeta^q = \zeta$, $|\zeta|_p = 1$. (Substantial; the artifact points to Bannai–Kobayashi and to Mazur–Tate for comparable statements; reference to be confirmed.)

(Q7) *Torsion translates.* Let $\delta \in \mathcal{O}_d$, $\delta \neq 0$, and $y \in \mathbb{C}$ with $\delta y \in \Lambda_C$. Then $\mathrm{Tay}_j(U_y^{C,1}\theta_C)(j(\bar{\delta})X) = \mathrm{Tay}_j(\theta_C)^{\delta\bar{\delta}} \cdot u$ for some $u \in \mathfrak{o}_p[[\varphi_C]]$. Also: if x is a triple of degree m and $My \in \Lambda_C$ for a natural number $M \neq 0$, then $|j((U_y^{C,m}\,\mathrm{mon}_C(x))(0))|_p \leq 1$. (Substantial; the artifact points to Bannai–Kobayashi, Duke Math. J. 153 (2010), 2.14–2.16, and to Tate's telescoping for local heights; not compared: *reference to be confirmed.*)

(Q8) *Height of the formal group.* (i) If $|(F')_{p-1}|_p < 1$ for every $F \in \mathfrak{o}_p[[\varphi_C]]$, then $|(\mathrm{Tay}_j(\xi_C))_{p-1}|_p \geq 1$ and $\chi_d(p) \neq 1$. (ii) Let Z be as in (Q5), $a \in \mathcal{O}_d$ with $a\bar{a} = q$ odd, and $g \in \mathfrak{o}_p[[X]]$ with $g(0) = 0$ and $Z(g) = j(a)Z$; then $|g_k|_p = 1$ for some $k \leq q$. References: Deuring's criterion; J. Silverman, *The Arithmetic of Elliptic Curves*, IV.7 (place confirmed for (ii)); relation: *weaker forms* (one direction of the criterion; an upper bound for the height).

LEAN in the background library: `CMLattice`, `CMLattice.twistedTranslate`, `CMLattice.θ`, `CMLattice.phiSeriesRing`, `congruencePart`, `CMLattice.exists_galConjugate`, `CMLattice.galConjugate` · (Q1) `QuadField.exists_latticeModule_eq_scaledIdeal`, `QuadField.area_eq_of_eq_scaledIdeal` · (Q2) `CMLattice.complex_multiplication_theta` · (Q3) `CMLattice.chowla_selberg` · (Q4) `CMLattice.norm_a₁_le_one`, `isAlgebraic_of_padic` · (Q5) `CMLattice.isSeriesGenerator_translatedWpDeriv`, `IsSeriesGenerator.exists_subst_inverse`, `exists_monomial_max_norm_coeff`, `CMLattice.translatedWpDeriv_add_mem_multiPhiRing` · (Q6) `CMLattice.exists_log_theta`, `mem_normPairClosure_of_pow_eq` · (Q7) `CMLattice.rescale_translate_eq_theta_pow_mul`, `CMLattice.norm_translate_le_naiveHeight_pow` · (Q8) `CMLattice.one_le_norm_coeff_translatedWp_nonsplit`, `CMLattice.exists_unit_coeff_degree_le`

Background 4: the Weil representation and jets (background library). Let E be a CM field with a CM type, whose embeddings ι_τ are indexed by the infinite places, and with Weil data (a splitting character and a finite idele α; for $E(P,\mu)$ these are fixed in unit 152, with $\alpha = \alpha(P,\mu)$). A *hermitian space* V over E with index set n and signs ε is a hermitian matrix over E with an adelic point that brings it to the diagonal form ε at the infinite places and to the unit form at the finite ones; it is *standard* if all signs are 1 and the point is 1. I_n is the standard space with index set n, and $\mathrm{Hom}(I_n, I_{n'})$, with index set $n' \times n$, is standard too. The space $\mathcal{S}(V)$ is the Banach space of the Weil representation: integrable functions on $\mathbb{A}^n$ with a reproducing identity. It carries the Heisenberg operators $\mathfrak{h}_V(y)$, $y \in \mathbb{A}^n$; the vacuum vector vac_V; the theta functional $\mathrm{Th}_V\colon \mathcal{S}(V) \to \mathbb{C}$, invariant under the rational vectors; the Weil representation ω_V of the adelic unitary group, whose elements we write (g_∞, g) with $g \in \mathrm{U}_n(\mathbb{A}_\mathrm{f})$; and the pairing $\langle f, f'\rangle_V = \int \bar{f} f'$. A vector $f \in \mathcal{S}(V)$ is a function on $\mathbb{A}^n$ through $f(y) = \mathrm{Th}_V(\mathfrak{h}_V(y)f)$. For a finite adelic vector $x \in \mathbb{A}_\mathrm{f}^n$, $x^\flat \in \mathbb{A}^n$ is the vector with real part 0 and finite part αx. These operators compose by the rule

$$\mathfrak{h}_V(x^\flat)\,\mathfrak{h}_V(x'^\flat) = \psi_0\big(\tfrac{1}{2}\textstyle\sum_{i\in n} \overline{x'_i}\,x_i\big) \cdot \mathfrak{h}_V((x+x')^\flat) \qquad (x, x' \in \mathbb{A}_\mathrm{f}^n) \tag{1}$$

(quoted), where ψ_0 is the standard additive character of $\mathbb{A}_\mathrm{f}$, whose values are roots of unity; $\mathfrak{h}_V^\mathrm{f}(x)$ is $\mathfrak{h}_V(x^\flat)$ times a root of unity, so that $\mathfrak{h}_V^\mathrm{f}(x)\mathfrak{h}_V^\mathrm{f}(x') = \psi_0(\cdot)\mathfrak{h}_V^\mathrm{f}(x+x')$ with a value of ψ_0; and

$$\mathcal{W}_\mathrm{f}(V) = \text{the span over } \mathbb{C} \text{ of the vectors } \mathfrak{h}_V^\mathrm{f}(x)\mathrm{vac}_V,\ x \in \mathbb{A}_\mathrm{f}^n,$$

the *finite part.* $\Delta_\mathrm{f}(g)$ is the multiplier of $g \in \mathrm{U}_n(\mathbb{A}_\mathrm{f})$ in the Weil representation. For a set R of finite places and a function Ψ on $\mathbb{A}_\mathrm{f}^n$, $\mathrm{name}_{V,R}(\Psi)$ is the operator on $\mathcal{S}(V)$

named by Ψ (a combination of Heisenberg operators at the places of R and their conjugates); Ψ is *Schwartz–Bruhat above* R if it depends only on the components at the places of R, is locally constant, and the parts at R of its support lie in a compact set. For a modulus $\mathfrak{m}$ (a nonzero ideal divisible by no prime together with its conjugate) $P^{\mathfrak{m}}$ is the mean of the operators $\mathfrak{h}^{\mathrm{f}}_V(x)$ over the integral x supported at the primes dividing $\mathfrak{m}$ and their conjugates. For $a \in \mathbb{C}$ and $f \in \mathcal{S}(V)$ the *jet series* is the power series

$$T^V_a f = \sum_{\beta} \frac{a^{|\beta|}}{\beta!}\, \mathrm{Th}_V(v_\beta \otimes f)\, z^\beta \;\in\; \mathbb{C}[[z_{(\tau,i)} : \tau \text{ an infinite place},\ i \in n]],$$

where $v_\beta \otimes f$ is the integral of the Heisenberg translates of f against the Fock vector of the monomial z^β. For $f \in \mathcal{S}(V)$ and $y \in \mathbb{C}^{\mathrm{places}\times n}$, $\Theta_f(y)$ is the value of f at the real vector of y divided by the Gaussian; Lemma 5.1 says that $T^V_a f$ is the Taylor series of $z \mapsto \Theta_f(az)$ when the signs are 1 and $f \in \mathcal{W}_{\mathrm{f}}(V)$. Quoted facts are cited where they are used; the two that carry weight are:

(Q9) *Theta functions of a level are Weil translates.* Let V be standard over $E(P,\mu)$, δ an integer with $\delta\overline{\delta} = N \neq 0$, and S an entire function on $\mathbb{C}^{\mathrm{places}\times n}$ that has the transformation law of $\Theta_\cdot$ under translation by the real images of the vectors $\overline{\delta}\gamma$, $\gamma \in \mathfrak{a}(P,\mu)^n$. Then $S = \Theta_F$ for some $F \in \mathcal{W}_{\mathrm{f}}(V)$, and F has a *vacuum expansion for* (δ, N): a set A of $N^{2|\mathrm{places}||n|}$ indices, $x_a \in \mathbb{A}_{\mathrm{f}}^n$ with δx_a integral, and a natural number m_0 with $\sum_{a\in A} m_0\, \ell(\mathfrak{h}_V(x_a^\flat)\mathrm{vac}_V) \cdot \mathfrak{h}_V((-x_a)^\flat)F = |A|\, \ell(F) \cdot \mathrm{vac}_V$ for every linear form ℓ on $\mathcal{S}(V)$. (Substantial: Stone–von Neumann for a finite Heisenberg group; no source cited; reference to be confirmed.)

(Q10) *Galois action on the finite Weil representation.* Let σ be a ring endomorphism of $\mathbb{C}$, $L\colon \mathcal{S}(V) \to \mathcal{W}_{\mathrm{f}}(V)$ a σ-semilinear map, B a finite adele and R a set of finite places that contains the conjugate of none of its members. If $\sigma(\psi_0(z)) = \psi_0(B\overline{B}z)$ for all $z \in \mathbb{A}_{\mathrm{f}}$ and $L(\mathfrak{h}_V(x^\flat)f) = \mathfrak{h}_V((Bx)^\flat)(Lf)$ for all x and all $f \in \mathcal{W}_{\mathrm{f}}(V)$, then B is a unit and there are $c \in \mathbb{C}$ and $z \in \mathbb{A}_{\mathrm{f}}$, $z\overline{z} = 1$, with $L(\mathrm{vac}_V) = c\,\omega_V(1, z\cdot 1_n)\mathrm{vac}_V$ and with (iii) and (v) of Theorem B. Reference: G. Shimura, *Theta functions with complex multiplication*, Duke Math. J. 43 (1976), no place given; *reference to be confirmed.*

LEAN in the background library: `Space`, `Space.IsStd`, `𝓢`, `Space.ϱ`, `finVec`, `Space.finHeisOp`, `Space.vacElem`, `Space.θ`, `Space.ω`, `Space.dataInner`, `Space.finitePart`, `finMultiplier`, `Space.name`, `Space.projOn`, `Space.jetSeries`, `Space.entireTheta`, `Space.VacuumExpansion`, `Space.rho_finVec` · (Q9) `Space.exists_entireTheta_eq` · (Q10) `Space.semilinear_intertwines_weil`

Background 5: column counts, Schur jets, gauges, kernels (background library). A family of *column counts* on a finite set Φ is a function $\varkappa\colon \Phi \times \mathbb{N} \to \mathbb{N}$; $\varkappa(\tau,h)$ is the number of columns of height h at τ of a family of Young diagrams, and $\mathrm{box}(\varkappa,N) = \sum_{\tau\in\Phi}\sum_{h<N} h\,\varkappa(\tau,h)$ counts the boxes in columns of height below N. It *fits the ranks* $|n| \le |n'|$ if $\varkappa(\tau,h) = 0$ for $|n| < h < |n'|$. A *Weyl index* of rank $|n|$ is a family $i = (i_\tau)_\tau$ of standard fillings: $i_\tau(h,l,b) \in n$ is the entry in the box $b < h$ of the l-th column of height $h < |n|$, $l < \varkappa(\tau,h)$; the Weyl indices index the coordinates of the representation of weight $(\varkappa, e)$ of $\prod_\tau \mathrm{GL}_n$, for $e\colon \Phi \to \mathbb{Z}$ the exponents of the determinants. The *Schur matrix* $S_\varkappa(z)$ of matrix variables $z_\tau = (z_{(\tau,b,c)})_{b\in n', c\in n}$ has polynomial entries indexed by a Weyl index i' of rank $|n'|$ and one i of rank $|n|$; its

row i' expresses the product of minors of $z_\tau y_\tau$ prescribed by i' in the products of minors of y_τ. For a power series F in the $z_{(\tau,b,c)}$, $\mathrm{SJ}_\varkappa(F)$ is the matrix of the jets of F along the entries of $S_\varkappa$. Over $E = E(P,\mu)$ with $\Phi = \mathrm{Pl}(P,\mu)$: an *algebraic form* of weight $(\varkappa, e)$ on I_n is a function on $\mathrm{U}_n(\mathbb{A}_\mathrm{f})$ with values in $\mathbb{C}^{\text{Weyl indices}}$, equivariant under the rational points; for a p-adic place (p,j), $g \in \mathrm{U}_n(\mathbb{A}_\mathrm{f})$ and a vector ξ, the *gauge* $\mathrm{gauge}_j^{\varkappa,e}(g;\xi)$ is the largest $|\cdot|_p$ of a coordinate of $g^{-1}\cdot j(\xi)$, where g^{-1} acts in weight $(\varkappa,e)$ through its images under the $\hat{\jmath}_\tau$; an element $g \in \mathrm{U}_n(\mathbb{A}_\mathrm{f})$ is *integral above* p if its components at the primes above p are integral with integral inverse. A *kernel* k from weight $(\varkappa,e)$ on I_n to weight $(\varkappa,e_a)$ on $I_{n'}$ with data in a complex vector space $\mathcal{D}$ consists of a representation ω_k of $\mathrm{U}_n(\mathbb{A}_\mathrm{f}) \times \mathrm{U}_{n'}(\mathbb{A}_\mathrm{f})$ on $\mathcal{D}$ and a linear map $\Phi \mapsto \mathrm{jm}_k(\Phi)$ from $\mathcal{D}$ to matrices (rows: Weyl indices of rank $|n'|$; columns: of rank $|n|$), equivariant under the rational points. Let a, b be characters of $\mathrm{U}(1)$ over E with weights e_a, e_b and tame parts ψ, ψ', let K be a level, and suppose that $\varkappa$ is a *lift weight*: it fits the ranks, $|n| < |n'|$, and $e_\tau = e_{b,\tau} + \varkappa(\tau,|n|)$. For $c \in \mathbb{C}$ the *theta kernel* $\mathrm{k}_\varkappa(c;a,b;K)$ has $\mathcal{D} = \mathcal{W}_\mathrm{f}(\mathrm{Hom}(I_n, I_{n'}))$, $\omega_k = \omega_{a,b}$ (the Weil representation of the pair, twisted by a and b) and $\mathrm{jm}_k(f) = \mathrm{SJ}_\varkappa(T_c^{\mathrm{Hom}(I_n,I_{n'})} f)$. Quoted: $\mathrm{SJ}_\varkappa(T_c^{\mathrm{Hom}(I_n,I_{n'})} f) = c^{\mathrm{box}(\varkappa,|n'|)}\,\mathrm{SJ}_\varkappa(T_1^{\mathrm{Hom}(I_n,I_{n'})} f)$; the entries of $S_\varkappa$ have integral coefficients; every $g \in \mathrm{U}_n(\mathbb{A}_\mathrm{f})$ is a rational point times an element integral above p, and the gauge is unchanged by such elements on the right.

LEAN in the background library: `boxes`, `FitsRanks`, `WeylIndex`, `schur`, `schurJet`, `AlgForm`, `weylGauge`, `integralAbove`, `Kernel`, `IsLiftWeight`, `thetaKernel`, `pairWeil`, `schurJet_jetSeries_eq_smul`, `exists_mul_mem_integralAbove`, `weylGauge_mul`

Background 6: carried series (background library). Let N be a function from exponents to the non-negative real numbers and $k \in \mathbb{N}$. A power series H over $\mathbb{C}_p$ is *carried by k copies of* N if every exponent β is a sum of k exponents $\alpha_1, \dots, \alpha_k$ with $\prod_l \mathsf{N}(\alpha_l) \geq |H_\beta|_p$, H_β being the coefficient at z^β.

LEAN in the background library: `HasLargeSplit`, `IsCarriedBy`

The definitions of the unit.

Definition 2.1 (the curve, its period, the scales)**.** Let $P = (N_0, d)$ be parameters and μ a natural number.

(a) $s_0(P,\mu) = \left(\chi_\mathrm{fin}(\alpha(P,\mu))^{-1}\right)^{1/f}$, the principal value of the root, where $f = |\mathrm{Pl}(P,\mu)|$ and $\chi_\mathrm{fin}(a) = \upsilon(a)\,|a|^{-1/2}$ is the character of the finite ideles of $E(P,\mu)$ made from the splitting character υ of the Weil data and the idele norm.

(b) $\mathcal{E}(P)$ is a CM datum for d and $\Omega_0(P) \in \mathbb{C}^\times$ a number with $\Lambda_{\mathcal{E}(P)} = \Omega_0(P)\,\mathcal{O}_{K(P)}$, both chosen by (Q1) for the unit ideal.

(c) $\mathrm{asc}(P) = \sqrt{2\pi/\sqrt{d}}$, a positive real number.

(d) $s_1(P,\mu) = s_0(P,\mu)/u_P$, where u_P is the number chosen by (Q3) for the datum $\mathcal{E}(P)$.

(e) $\mathrm{per}(P,\mu) = s_1(P,\mu)\cdot \mathrm{asc}(P)/\Omega_0(P)$.

LEAN `Parameters.vacuumRoot` · `Parameters.cmCurve` · `Parameters.Ω₀` · `Parameters.areaScale` · `Parameters.periodFactor` · `Parameters.jetScale`

Definition 2.2 (the scaled vacuum and the standard data)**.** Let $P = (N_0, d)$ be parameters and μ a natural number.

(a) $\mathrm{vsc}(P,\mu)$ is the nonzero complex number c that Proposition 6.1(i) provides (by choice) for μ and the standard space of rank one.

(b) For a hermitian space V over $E(P,\mu)$ with index set n: $\theta^{\circ}_V = \mathrm{vsc}(P,\mu)^{|n|} \cdot \mathrm{vac}_V$, the *scaled vacuum*.

(c) For an integer δ of $K(P)$ and a set S of finite places of $E(P,\mu)$ (a predicate on them), $\mathrm{Lat}(\delta; S)$ is the group of the $x \in \mathbb{A}_{\mathrm{f}}^n$ such that δx_i is integral at every place of S, for all $i \in n$.

(d) For V as in (b), a p-adic place (p,j) and an integer δ of $K(P)$, $\mathrm{Std}_\delta(V)$ is the O_j-submodule of $\mathcal{S}(V)$ spanned by the vectors $\mathfrak{h}^{\mathrm{f}}_V(x)\theta^{\circ}_V$ with $x \in \mathrm{Lat}(\delta; \{w : w \mid p\})$. It depends on j, which is always the place of the statement.

LEAN `Parameters.vacuumScale` · `Parameters.intVacuum` · `Parameters.integralAfter` · `Parameters.standardData`

Definition 2.3 (data that raise gauges)**.** Let (p,j) be a p-adic place, k a kernel from weight $(\varkappa, e)$ on I_n to weight $(\varkappa, e_a)$ on $I_{n'}$ over $E(P,\mu)$ with data in $\mathcal{D}$, $d_1 \in \mathrm{U}_n(\mathbb{A}_{\mathrm{f}})$, $d_2 \in \mathrm{U}_{n'}(\mathbb{A}_{\mathrm{f}})$ and c a real number. Then $\mathrm{Raise}_k(d_1, d_2; c)$ is the set of the $\Phi \in \mathcal{D}$ such that

$$\mathrm{gauge}_j^{\varkappa, e_a}(h d_2; \mathrm{jm}_k(\omega_k(g,h)\Phi)\,\xi) \;\le\; \max(c,0) \cdot \mathrm{gauge}_j^{\varkappa, e}(g d_1; \xi)$$

for all $g \in \mathrm{U}_n(\mathbb{A}_{\mathrm{f}})$, $h \in \mathrm{U}_{n'}(\mathbb{A}_{\mathrm{f}})$ and all vectors ξ. It is closed under addition and contains 0.

LEAN `Kernel.raises`

Definition 2.4 (theta products)**.** Let $P = (N_0, d)$ be parameters, μ a natural number, (p,j) a p-adic place, δ an integer of $K(P)$, r a real number and c a finite set. Then $\mathrm{ThProd}_{P,\mu}(\delta, r; c)$ is the set of the power series Θ over $\mathbb{C}_p$ in the variables $z_{(\tau,k)}$, $(\tau,k) \in \mathrm{Pl}(P,\mu) \times c$, for which there are a finite set J, elements $w_i \in E(P,\mu)$ $(i \in J)$, natural numbers M and n, a number $b \in \mathbb{C}$ and numbers $y_{(i,k)} \in \mathbb{C}$ $((i,k) \in J \times c)$ such that $(w_i)_{i \in J}$ with multiplicities 1 is a tight frame of length M;

$$\Theta = \prod_{(i,k) \in J \times c} \mathrm{Tay}_j\big(U^{\mathcal{E}(P),1}_{y_{(i,k)}}\theta_{\mathcal{E}(P)}\big)\Big(\sum_\tau j\big(b\,\overline{\iota_\tau(w_i)}\big)\, z_{(\tau,k)}\Big);$$

$p \nmid M$ and $p \nmid n$; $|j(b)|_p = r$; and $n\,\delta\,y_{(i,k)} \in \Lambda_{\mathcal{E}(P)}$ for all (i,k).

LEAN `Parameters.thetaProducts`

Definition 2.5 (integrality on the null cone)**.** Let p be a prime number, Φ and c finite sets, $\mathrm{pr}\colon \mathbb{N} \times \Phi \to \Phi$ a map, $S \subset \mathbb{N}$ a finite set, $\varpi \in \mathbb{C}_p$, $\varpi_0 \in \mathfrak{o}_p$, $B \in \mathbb{N}$, and Θ a power series over $\mathbb{C}_p$ in the variables z_x, $x \in \Phi \times c$. We say that

$$\Theta \text{ is integral on the null cone of } (\mathrm{pr}, S; \varpi, \varpi_0) \text{ up to degree } B$$

if the following holds for every set T of variables and every family $(Z_x)_{x \in \Phi \times c}$ of power series $Z_x \in \mathfrak{o}_p[[T]]$ without constant term: if for every commutative ring Q of characteristic p, every ring homomorphism $\phi\colon \mathfrak{o}_p[[T]] \to Q$ with $\phi(\varpi_0) = 0$ and every $e \in S$

$$\phi\Big(\sum_{(\tau,k) \in \Phi \times c} Z_{(\tau,k)}^{p^e} \cdot Z_{(\mathrm{pr}_e(\tau),k)}\Big) = 0,$$

then every coefficient of degree at most B of the series $\Theta\big((\varpi Z_x)_x\big)$ has absolute value at most 1.

LEAN `IsIntegralOnNullCone`

The sum in the display is the *coupling* of exponent e of the family Z. The condition on it says that it lies in the ideal generated by p and ϖ_0 (take for Q the quotient by this ideal; if the ideal is the whole ring there is no such ϕ): the family is *null* modulo (p, ϖ_0) for the pairings of a variable, raised to the power p^e, with its partner at pr_e.

Definition 2.6 (ask, forced part, gain, loss)**.** Let D be admissible data for parameters $P = (N_0, d)$, with components x_1, b, m, and let p be a natural number. For a real number y and $\tau \in \mathrm{Pl}(D)$ put

$$\mathrm{ask}_{D,p}(y,\tau) = \frac{1-\chi_{K(P)}(p)}{2} \sum_{\substack{e\ge 1\\ e \text{ odd or } p=d}} \Big(\frac{y}{p^e} - w_D([-p^e]_D\cdot\tau)\Big)_+,$$

where $(\,\cdot\,)_+$ is the positive part, $w_D(\,\cdot\,)$ is the nominal weight of unit 152 ($a(D)/x$ at the thick places, $a(D)/x_1$ at the others) and the sum over $e \in \mathbb{N}$ is 0 if the family is not summable. Then

$$\mathrm{forced}(D,p) = \sum_{\tau\in\mathrm{Pl}(D)} \mathrm{ask}_{D,p}(w_D(\tau),\tau),$$

$$\mathrm{gain}(D,p) = m\Big(\sum_{\tau\in\mathrm{tops}(D)} \mathrm{ask}_{D,p}(a(D),\tau) + k(D)\,\mathrm{forced}(D,p)\Big),$$

$$\mathrm{loss}(D,p) = f(D)\,(\log b)^2\cdot\frac{a(D)}{p}\cdot m^*,$$

where $m^* = m$ if p is short for D and $m^* = 1$ otherwise. Here we say that

$$p \text{ is short for } D \quad \text{if } m \text{ divides } p-1 \text{ or } m \text{ divides } p+1,$$

and that p is small and not split for D if $p < x_1$ and $\chi_{K(P)}(p) \neq 1$. The sets $\mathrm{tops}(D)$ (the places of class 1 modulo N_0) and the numbers $a(D)$, $k(D)$, $f(D) = |\,\mathrm{Pl}(D)|$ are those of unit 152.

LEAN `Data.localAsk` · `Data.forcedPart` · `Data.gain` · `Data.loss` · `Data.IsShortPrime` · `Data.IsSmallNonsplit`

Definition 2.7 (served primes, collisions)**.** Let D be admissible data for parameters $P = (N_0, d)$, with component m, and let p be a natural number.

(a) $\mathfrak{r}(D,p)$ is the least natural number $\mathfrak{r}$ with $m \le p^{\mathfrak{r}}$ if $p > 1$ (the upper logarithm of m to the base p, which is 0 for $p \le 1$); $\Delta(D,p) = \mathfrak{r}(D,p)\cdot f(D)\cdot(n_m(D)+\mathfrak{r}(D,p))$, where $n_l(D) = k(D) + l(t(D)+1)$; and $\mathrm{jc}(D,p) = \lceil m\cdot 2f(D)a(D)/p\rceil$.

(b) $o_0(D,p)$ is the order, in $(\mathbb{Z}/N_0)^\times$, of the class modulo N_0 of the place $[-p]_D$.

(c) We say that p is served for D if $\chi_{K(P)}(p) = -1$, it is not the case that p is short for D, $o_0(D,p)$ is odd, and $o_0(D,p) \le \mathfrak{r}(D,p)$.

(d) For a finite set $\Xi \subset \mathrm{Pl}(D)$, a place $\tau \in \mathrm{Pl}(D)$ and $e \in \mathbb{N}$ we say that

$$(\tau, e) \text{ is a collision of } p \text{ outside } \Xi \text{ for } D$$

if $e \ge 1$, e is odd or $p = d$, $\tau \in \mathrm{tops}(D)\setminus\Xi$ and $[-p^e]_D\cdot\tau \in \mathrm{tops}(D)\setminus\Xi$. Then

$$\mathrm{coll}(D,p,\Xi) = \sum_{\tau\in\mathrm{Pl}(D)} \sum_{e} \frac{a(D)}{p^e},$$

the inner sum being over the $e \in \mathbb{N}$ for which this holds (it is 0 if the family is not summable).

LEAN Data.ceilLogM · Data.deltaP · Data.jetCeil · Data.o_0 · Data.IsServed · Data.IsCollision · Data.collisionSum

Definition 2.8 (the coupling count)**.** Let Φ be a set, p and n' natural numbers, $\varkappa\colon \Phi \times \mathbb{N} \to \mathbb{N}$ a function, $\Xi \subset \Phi$ a finite set, $\tau, \tau' \in \Phi$ and $e \in \mathbb{N}$. For $\sigma \in \Phi$, summing over the heights $h < n'$, put

$$t_\sigma = \sum_{h=1,\ \sigma \notin \Xi} \varkappa(\sigma, h), \qquad r_\sigma = \sum_{h \neq 1} \varkappa(\sigma, h), \qquad B_\sigma = \sum_{h \neq 1} h\, \varkappa(\sigma, h).$$

Then

$$\mathrm{cc}_p(\varkappa, n'; \Xi; \tau, \tau'; e) = \min\Big(\frac{t_\tau}{p^e},\, r_{\tau'} + t_{\tau'}\Big) + \frac{r_\tau}{p^e} + \min\Big(\frac{B_\tau}{p^e},\, B_{\tau'}\Big),$$

a real number.

LEAN couplingCount

Definition 2.9 (the column counts of the data)**.** Let D be admissible data for parameters P, with component r; write $k = k(D)$, $t = t(D) = 5r$ and $\mathrm{gap}(D) = 2r^8$, and let $\mathrm{wt}_D(l, \tau)$ be the weight of unit 152 at stage l. For $\tau \in \mathrm{Pl}(D)$, $h \in \mathbb{N}$ and $l \in \mathbb{N}$:

$$\varkappa_D(\tau, h) = \begin{cases} a(D) & h = 1,\ \tau \in \mathrm{tops}(D), \\ 0 & h = 1,\ \tau \notin \mathrm{tops}(D), \\ \mathrm{wt}_D(1, \tau) - \lfloor (k + t + 1)/2 \rfloor & h = k \neq 1, \\ \mathrm{gap}(D) - (t + 1) & h \neq 1,\ k < h,\ (t+1) \mid h - k, \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathrm{last}_D(l, \tau) = \begin{cases} 0 & l = 0, \\ \lfloor n_{l-1}(D)/2 \rfloor - \mathrm{wt}_D(l, \tau) \in \mathbb{Z} & l \geq 1; \end{cases}$$

$$\mathrm{kept}_D(h) = \begin{cases} r^8 - \lfloor (t+1)/2 \rfloor & k \leq h,\ (t+1) \mid h - k, \\ 0 & \text{otherwise;} \end{cases}$$

$$\varkappa_D^Y(\tau, h) = \mathrm{kept}_D(h) + (\varkappa_D(\tau, h+1) - \mathrm{kept}_D(h+1)).$$

All subtractions are truncated, except the one in $\mathrm{last}_D(l, \tau)$.

LEAN Data.κ · Data.lastEntry · Data.keptColumns · Data.columnsY

Definition 2.10 (Frobenius packings, the depth loss)**.** (a) Let P be parameters, μ a natural number, p a natural number and n, i, y functions from $\mathrm{Pl}(P, \mu)$ to $\mathbb{N}$; write $\mathrm{pr}_e(\tau) = [-p^e]_{P,\mu} \cdot \tau$. We say that (i, y) is a Frobenius packing of n at p if

(1) for every τ with $y_\tau \neq 0$: i_τ is odd and $\chi_{K(P)}(p) = -1$;
(2) for all τ, τ' with $y_\tau \neq 0$, $y_{\tau'} \neq 0$ and every odd $e < i_\tau$: $\mathrm{pr}_e(\tau') \neq \mathrm{pr}_{i_\tau}(\tau)$;
(3) for every τ': $\quad p^{i_{\tau'}} y_{\tau'} + \sum_{\tau:\ \mathrm{pr}_{i_\tau}(\tau) = \tau'} y_\tau \ \leq\ n_{\tau'}$.

(b) Let D be admissible data, with component b, and n a natural number. With $c = \lceil 60\, b / k(D) \rceil$ and $l = n - c$ (truncated),

$$\mathrm{dl}_D(n) = \big((l+1)(l+2)\cdots(l+c)\big)^2.$$

3 Periods and scales

Inputs from unit 152 (the data; note on the CM field). For parameters $P = (N_0, d)$ and $\mu \in \mathbb{N}$, with $f = |\mathrm{Pl}(P,\mu)|$ and ϖ_E the generator of the different of $E(P,\mu)$ over $\mathbb{Q}$ fixed there:
(I1) $\alpha(P,\mu)\,\overline{\alpha(P,\mu)}\,\partial(P,\mu)$ is a unit at every finite place; $\alpha(P,\mu)^{-1} \in \widehat{\mathcal{O}}$; $N_1(P,\mu)\alpha(P,\mu) \in \widehat{\mathcal{O}}$.
(I2) $\prod_\tau |\iota_\tau(\varpi_E)| = \mathrm{rd}_{E(P,\mu)}^{f}$ and $|\iota_\tau(\varpi_E)| = |\iota_\tau(\partial(P,\mu))|\sqrt{d}$; $\prod_\tau \iota_\tau(y) = \mathrm{N}_{E(P,\mu)/K(P)}(y)$ for $y \in E(P,\mu)$; $\iota_\tau(x) = x$ for $x \in K(P)$.
(I3) For every injective ring endomorphism ϕ of the integers of $E(P,\mu)$, $\phi(\partial(P,\mu))$ and $\partial(P,\mu)$ have the same valuation at every finite place; $\partial(P,\mu)$ divides $N_1(P,\mu)$ and generates the different of $E(P,\mu)$ over $K(P)$.
(I4) A prime p with $\chi_{K(P)}(p) \neq 1$ is at least 13 and does not divide $N_1(P,\mu)$; $64 < d$; every prime dividing $N_1(P,\mu)$ divides $(d+1)/4$; $c_0(P,\mu)(1-c_0(P,\mu)) = (d+1)/4$ and $\overline{c_0(P,\mu)} = 1 - c_0(P,\mu)$; no power of d is $\equiv -1 \bmod N_0$.
(I5) For $\sigma \in \mathrm{Aut}(\mathbb{C}/K(P))$ the permutation $\mathrm{perm}_\mu(\sigma)$ of $\mathrm{Pl}(P,\mu)$ satisfies $\sigma \circ \iota_\tau = \iota_{\mathrm{perm}_\mu(\sigma)(\tau)}$ for every $\tau \in \mathrm{Pl}(P,\mu)$.
(I6) *Frobenius.* Let $p \nmid N_1(P,\mu)$ be a prime, ϕ a ring homomorphism from the integers of $E(P,\mu)$ to a ring of characteristic p, $v \in \mathrm{Pl}(P,\mu)$ and $e \in \mathbb{N}$. Then $\phi(\mathrm{gal}(v)x) = \phi(\overline{x})^{p^e}$ for all integers x if and only if $\phi(\mathrm{gal}(v)\varsigma(P,\mu)) = \phi(\mathrm{gal}(u)\varsigma(P,\mu))$ for the place $u = [-p^e]_{P,\mu}$ and $-\chi_{K(P)}(p)^e\phi(\sqrt{-d}) = \phi(\sqrt{-d})$. (The same holds without the conjugation, with $u = [p^e]_{P,\mu}$ and without the sign.)

LEAN unit 152: Parameters.v_alpha_mul_relDifferentGen, Parameters.inv_α_mem, Parameters.N₁_mul_α_mem · Parameters.prod_varpi_eq_rootDiscr_pow, Parameters.infinitePlace_apply_varpi, Parameters.prod_phi_eq_norm, Parameters.phi_algebraMap · Parameters.valuation_map_relDifferentGen, Parameters.relDifferent_spec · Parameters.thirteen_le_of_chiK_ne_one, Parameters.chiK_eq_one_of_dvd, Parameters.lt_d, Parameters.dvd_norm_c₀, Parameters.c₀InE_mul_one_sub, Parameters.star_c₀InE, Parameters.pow_d_ne_neg_one · Parameters.placePerm_spec · Parameters.frob_ext

Lemma 3.1 (the period)**.** *Let $P = (N_0, d)$ be parameters and μ a natural number.*

(a) For every $\sigma \in \mathrm{Aut}(\mathbb{C}/K(P))$: $|\sigma(s_0(P,\mu))|^2 = \mathrm{rd}_{E(P,\mu)}/\sqrt{d}$.
(b) $\Omega(P,\mu) > 0$.
(c) $\mathrm{per}(P,\mu) \neq 0$.
(d) The lattice of $\mathcal{E}(P)$, as a module over $\mathcal{O}_{K(P)}$, is $\Omega_0(P)\cdot\mathcal{O}_{K(P)}$.
(e) $\Omega_0(P) \in \Lambda_{\mathcal{E}(P)}$.

Proof. (a) Write $\alpha = \alpha(P,\mu)$, $\partial = \partial(P,\mu)$, $f = |\mathrm{Pl}(P,\mu)|$. Quoted from the background library: there are $k \geq 1$ and $y \in E(P,\mu)^\times$ with $\chi_{\mathrm{fin}}(\alpha)^k = \prod_\tau \iota_\tau(\overline{y})$. By (I2) this number is a norm, so it lies in $K(P)$ and σ fixes it; hence $|\sigma(\chi_{\mathrm{fin}}(\alpha))| = |\chi_{\mathrm{fin}}(\alpha)|$. Since $s_0(P,\mu)^f = \chi_{\mathrm{fin}}(\alpha)^{-1}$ and $|v| = 1$,

$$|\sigma(s_0(P,\mu))|^{2f} = |\chi_{\mathrm{fin}}(\alpha)|^{-2} = |\alpha|,$$

the idele norm of α (with components 1 at infinity). The idele $\alpha\overline{\alpha}\cdot\partial$, with ∂ embedded diagonally, is a unit at every finite place by (I1), so its idele norm is

$\prod_\tau |\iota_\tau(\partial)|^2$ (a quoted fact on ideles that are units at the finite places). By the product formula ∂ has idele norm 1, and $|\overline{\alpha}| = |\alpha|$; so $|\alpha|^2 = \prod_\tau |\iota_\tau(\partial)|^2$. By (I2), $\prod_\tau |\iota_\tau(\partial)| = \mathrm{rd}^f_{E(P,\mu)} / \sqrt{d}^f$. Taking f-th roots of non-negative numbers gives (a).

(b) All factors of $\Omega(P,\mu)$ are positive. (c) By (a) with $\sigma = 1$, $s_0(P,\mu) \neq 0$; by (Q3) with $\sigma = 1$, $u_P \neq 0$; $\mathrm{asc}(P) \neq 0$ and $\Omega_0(P) \neq 0$. (d) is the choice of $\mathcal{E}(P)$ and $\Omega_0(P)$, and (e) follows since $1 \in \mathcal{O}_{K(P)}$. □

LEAN Parameters.norm_gal_vacuumRoot · Parameters.bigOmega_pos · Parameters.jetScale_ne_zero · Parameters.latticeModule_cmCurve · Parameters.omega₀_mem_cmCurve_lattice · quoted: WeilData.exists_chiFin_pow_eq_prod, ideleAbs_eq_prod_of_v_eq_one

Lemma 3.2 (the period factor at a p-adic place)**.** *Let $P = (N_0, d)$ be parameters, μ a natural number and p a prime number.*

(a) For every ring homomorphism $\psi\colon E(P,\mu) \to \mathbb{C}_p$: $\hat{\psi}(\alpha(P,\mu)) = \psi(\partial(P,\mu))^{-1}$ if $c_0(P,\mu) \in \mathfrak{p}_\psi$, and $\hat{\psi}(\alpha(P,\mu)) = 1$ otherwise.

(b) For every ring homomorphism $j\colon \mathbb{C} \to \mathbb{C}_p$ and every $\tau \in \mathrm{Pl}(P,\mu)$ we have $|j(s_1(P,\mu))|_p = |\hat{\jmath}_\tau(\alpha(P,\mu))|_p^{-1}$.

(c) For every ring homomorphism $j\colon \mathbb{C} \to \mathbb{C}_p$, every $y \in \mathfrak{a}(P,\mu)$ and every $\tau \in \mathrm{Pl}(P,\mu)$: $|j(s_1(P,\mu) \cdot \overline{\iota_\tau(y)})|_p \leq 1$.

(d) For every ring homomorphism $j\colon \mathbb{C} \to \mathbb{C}_p$: if $p \nmid N_1(P,\mu)$, then $|j(s_1(P,\mu))|_p = 1$.

Proof. (a) is the definition of $\alpha(P,\mu)$, read at the prime $\mathfrak{p}_\psi$. (b) Let $f = |\mathrm{Pl}(P,\mu)|$. By (Q3), $|j(u_P)|_p = 1$, so $|j(s_1(P,\mu))|_p^f = |j(s_0(P,\mu)^f|_p = |j(\chi_{\mathrm{fin}}(\alpha(P,\mu)))|_p^{-1}$. A quoted formula gives $|j(\chi_{\mathrm{fin}}(a))|_p = \prod_v |\hat{\jmath}_v(a)|_p$ for finite ideles a. By (a) the factor at v is $|j_v(\partial(P,\mu))|_p^{-1}$ or 1 according as $c_0(P,\mu) \in \mathfrak{p}_{j_v}$ or not. This factor does not depend on v: the condition reads $|j(\overline{c})|_p < 1$ for the complex number $c = (1+\sqrt{-d})/2$, because ι_v is the identity on $K(P)$ (I2); and $j_v(\partial(P,\mu)) = j_1(\mathrm{gal}(v)\partial(P,\mu))$ has the absolute value of $j_1(\partial(P,\mu))$ by (I3). So the product is the f-th power of the factor at τ.

(c) By (b), $|j(s_1(P,\mu)\overline{\iota_\tau(y)})|_p = |\hat{\jmath}_\tau(\alpha(P,\mu))|_p^{-1} |j_\tau(y)|_p = |\hat{\jmath}_\tau(\alpha(P,\mu)^{-1}y)|_p \leq 1$, since $\alpha(P,\mu)^{-1}y \in \widehat{\mathcal{O}}$ (Background 2). (d) Use (b) at $\tau = 1$. If $c_0(P,\mu) \notin \mathfrak{p}_{j_1}$ the factor is 1. Otherwise it is $|j_1(\partial(P,\mu))|_p^{-1}$; here $\partial(P,\mu)$ divides $N_1(P,\mu)$ (I3), which is a unit at $\mathfrak{p}_{j_1}$ because $p \nmid N_1(P,\mu)$, so $|j_1(\partial(P,\mu))|_p = 1$. □

LEAN Parameters.adeleExtension_alpha · Parameters.norm_iota_periodFactor · Parameters.norm_periodFactor_mul_conj_le_one · Parameters.norm_iota_periodFactor_eq_one · quoted: WeilData.norm_map_chiFin, isEquiv_placeOfEmb

Lemma 3.3 (the hermitian lattice)**.** *Let $P = (N_0, d)$ be parameters and μ a natural number.*

(a) For every $z \in E(P,\mu)$: $\mathrm{Tr}_{E(P,\mu)/K(P)}(z) = \sum_{\tau \in \mathrm{Pl}(P,\mu)} \iota_\tau(z)$ in $\mathbb{C}$.

(b) For all $x, y \in \mathfrak{a}(P,\mu)$: $\sum_{\tau \in \mathrm{Pl}(P,\mu)} \iota_\tau(x)\,\overline{\iota_\tau(y)} \in \mathcal{O}_{K(P)}$.

Proof. (a) The ι_τ are the composites of ι_1 with the elements of the Galois group, and ι_1 is the identity on $K(P)$. (b) The sum is $\mathrm{Tr}_{E(P,\mu)/K(P)}(x\overline{y})$ by (a). The finite adele of $x\overline{y}\,\partial(P,\mu)$ is the product of $\alpha(P,\mu)\overline{\alpha(P,\mu)}\partial(P,\mu)$, a unit everywhere by (I1), with the integral adeles $\alpha(P,\mu)^{-1}x$ and $\overline{\alpha(P,\mu)^{-1}y}$. So $x\overline{y}\,\partial(P,\mu)$ is an integer, $x\overline{y}$ lies in the inverse different of $E(P,\mu)$ over $K(P)$ by (I3), and its trace is integral. □

In other words, $\mathfrak{a}(P,\mu)$ with the form $\mathrm{Tr}(x\overline{y})$ is a unimodular hermitian lattice over $\mathcal{O}_{K(P)}$: $\mathfrak{a}(P,\mu)\cdot\overline{\mathfrak{a}(P,\mu)}$ is the inverse different.

4 Tight frames

Lemma 4.1 (the frame of the roots of unity)**.** *Let $P=(N_0,d)$ be parameters, μ a natural number and k a natural number. The family $(\varsigma(P,\mu)^i)_{0\le i<N_1(P,\mu)}$, each member with multiplicity k, is a tight frame of length $k\cdot N_1(P,\mu)$.*

Proof. $\varsigma(P,\mu)^i$ is an integer, and integers lie in $\mathfrak{a}(P,\mu)$ by (I1). For places τ,τ' put $q=\iota_\tau(\varsigma(P,\mu))\overline{\iota_{\tau'}(\varsigma(P,\mu))}$, a root of unity with $q^{N_1(P,\mu)}=1$. The sum of the frame identity is $k\sum_{i<N_1(P,\mu)}q^i$. If $\tau=\tau'$ then $q=1$. If $q=1$ then ι_τ and $\iota_{\tau'}$ agree on the generator $\varsigma(P,\mu)$ of $E(P,\mu)$ over $K(P)$, so $\tau=\tau'$. Hence the sum is $k\,N_1(P,\mu)$ for $\tau=\tau'$ and 0 otherwise. □

LEAN Parameters.isTightFrame_zeta_pow

Lemma 4.2 (a factor of the cyclotomic different)**.** *Let R be an integral domain of characteristic 0, N a natural number, $\zeta\in R$ a primitive root of unity of order N, and l a prime factor of N. Let Ψ_N be the cyclotomic polynomial of order N. Then there are natural numbers r and $M\le N$ with $N=l\,r$, such that a prime factor q of N divides M if and only if $q\neq l$, and an element $V\in R$ that divides M in R with*

$$\Psi_N'(\zeta)\,(\zeta^r-1)\,M=NV.$$

Proof. Write $N=l\,r$ and $r=l^aM$ with $l\nmid M$; then $M\le N$ and the prime factors of $N=l^{a+1}M$ other than l are those of M. Since $l\nmid M$, $\Psi_M(X^l)=\Psi_{Ml}(X)\Psi_M(X)$; replacing X by X^{l^a} and using $\Psi_{Ml}(X^{l^a})=\Psi_N(X)$ gives $\Psi_M(X^{l^{a+1}})=\Psi_N(X)\,\Psi_M(X^{l^a})$. Differentiate at ζ, a root of Ψ_N:

$$l^{a+1}\zeta^{l^{a+1}-1}\,\Psi_M'(\zeta^{l^{a+1}})=\Psi_N'(\zeta)\,\Psi_M(\zeta^{l^a}).$$

As $\Psi_M(Y)$ divides Y^M-1, the element $\Psi_M(\zeta^{l^a})$ divides ζ^r-1. So l^{a+1} divides $\Psi_N'(\zeta)(\zeta^r-1)$, and $N=l^{a+1}M$ divides $\Psi_N'(\zeta)(\zeta^r-1)M$; let V be the quotient. Next, r is a proper divisor of N, so $X^N-1=(X^r-1)\Psi_N(X)Q(X)$ for a polynomial Q over R; differentiating at ζ and multiplying by ζ gives $N=\Psi_N'(\zeta)(\zeta^r-1)\,Q(\zeta)\zeta$. Hence $NM=NV\cdot Q(\zeta)\zeta$, and $M=VQ(\zeta)\zeta$ because N is not a zero divisor: V divides M. □

LEAN Parameters.exists_cyclotomic_different_factor

Proof of Theorem A. Write $N=N_1(P,\mu)$, $\varsigma=\varsigma(P,\mu)$, $c_0=c_0(P,\mu)$, $\partial=\partial(P,\mu)=\Psi_N'(\varsigma)$ and $\nu=(d+1)/4$; so $c_0(1-c_0)=\nu$ and $\overline{c_0}=1-c_0$ (I4). The frame will be the union of the frame of Lemma 4.1 with a second family, built to contain a vector that is a unit against $s_1(P,\mu)$ at the primes above p.

Step 1: auxiliary numbers. For each prime factor l of N let r_l, M_l, V_l be as in Lemma 4.2 for $\zeta=\varsigma$ in the integers of $E(P,\mu)$, and put $a_l=\varsigma^{r_l}$, a primitive l-th root of unity. Every prime factor of N divides ν (I4), so N divides ν^N. The map $y\mapsto\nu y^2+y$ of $\mathbb{Z}/N$ is injective: if $\nu y^2+y=\nu y'^2+y'$ then $(y-y')(\nu(y+y')+1)=0$, and $\nu(y+y')+1$ is a unit because $\nu(y+y')$ is nilpotent. So the map is surjective, and

there are natural numbers $z < N$ and e with $\nu z^2 + z + 1 = Ne$. Put $t = 1 - c_0 + \nu z$. Then

$$t + \bar{t} = 1 + 2\nu z, \qquad t\,\bar{t} = \nu + \nu z + \nu^2 z^2 = \nu N e, \qquad (c_0 + \nu z)\,t = \nu N e. \tag{2}$$

Step 2: the second family. For a prime factor l of N and $0 \le h < r_l$, $0 \le k < l$, $0 \le i < l$ put

$$u_{l,h,k,i} = \varsigma^h\Big([k = 0]\,a_l^i + \lambda_l\,(a_l^i - a_l^k)\Big), \qquad \lambda_l = \frac{t\,M_l}{N}.$$

Claim: for all places τ, τ', $\sum_{h,k,i} \iota_\tau(u_{l,h,k,i})\overline{\iota_{\tau'}(u_{l,h,k,i})} = [\tau = \tau'] \cdot c_l$ with $c_l = N + M_l(1 + 2\nu(z + l\,M_l\,e))$. Let $A = \iota_\tau(a_l)$, $A' = \iota_{\tau'}(a_l)$ (primitive l-th roots of unity), $G = \sum_{i<l}(A\overline{A'})^i$, and let λ, λ' be the images of λ_l under ι_τ, $\iota_{\tau'}$. Using $\sum_{i<l} A^i = \sum_{i<l} \overline{A'}^{\,i} = 0$, the sum over k and i of the products of the two brackets is

$$G + \overline{\lambda'}G + \lambda G + 2l\,\lambda\overline{\lambda'}G = G\,(1 + \lambda + \overline{\lambda'} + 2l\lambda\overline{\lambda'})\,:$$

the four terms come from $k = 0$ with the two first summands, from $k = 0$ with one first and one second summand (twice), and from the two second summands. The sum over h contributes $\sum_{h<r_l}(\iota_\tau(\varsigma)\overline{\iota_{\tau'}(\varsigma)})^h$, and its product with G is $\sum_{h<N}(\iota_\tau(\varsigma)\overline{\iota_{\tau'}(\varsigma)})^h = N[\tau = \tau']$ by Lemma 4.1. For $\tau = \tau'$, (2) gives $\lambda + \overline{\lambda} = M_l(1 + 2\nu z)/N$ and $\lambda\overline{\lambda} = M_l^2 \nu e/N$, whence the claim. (The formal proof establishes this identity by a Gröbner basis computation.)

Step 3: lengths. Let $c = \sum_l c_l$. It is prime to N: modulo a prime factor q of N the summands with $l \ne q$ vanish, because q divides N and M_l, and $c_q \equiv M_q \not\equiv 0$, because q divides N and ν but not M_q. A crude bound, from $M_l, l, z \le N$, $e \le 2\nu N$ and at most N prime factors, is $c \le 8\nu^2 N^5$; with $9\nu^2 \le d^6$ and the hypothesis,

$$c_*(c + N) + (N - 1)(c - 1) \;\le\; (c_* + 1)\,9\nu^2 N^6 \;\le\; (dN)^6(c_* + 1) \;\le\; M.$$

Since c and N are coprime, every integer $\ge (N-1)(c-1)$ is a combination $Ac + BN$ with natural numbers A, B. Choose them with $Ac + BN = M - c_*(c + N)$. Take the vectors $u_{l,h,k,i}$ with multiplicity $A + c_*$ and the vectors ς^i, $i < N$, with multiplicity $B + c_*$. By the claim and Lemma 4.1 the frame identity holds with length $(A + c_*)c + (B + c_*)N = M$, and all multiplicities are at least c_*.

Step 4: the vectors lie in $\mathfrak{a}(P, \mu)$. For ς^i this is Lemma 4.1. For $u = u_{l,h,k,i}$ write $a_l^i - a_l^k = (a_l - 1)g$ with an integer g, and let w be a finite place. If $c_0 \in w$, the component of $\alpha(P, \mu)$ is ∂^{-1} and we need ∂u integral at w; indeed $\partial\lambda_l(a_l - 1) = tV_l$ by Lemma 4.2, so ∂u is an integer. If $c_0 \notin w$ the component is 1 and c_0 is a unit at w, so it suffices that $c_0^N\lambda_l$ is an integer, that is, that N divides $c_0^N t$. Now $c_0^N - (-\nu z)^N$ is divisible by $c_0 + \nu z$, and $(c_0 + \nu z)t = \nu N e$ by (2); so $c_0^N t \equiv (-z)^N \nu^N t \equiv 0$ modulo N, as N divides ν^N.

Step 5: the unit vector. If $|j(s_1(P, \mu))|_p = 1$, the vector $\varsigma^0 = 1$ will do. Otherwise p divides N by Lemma 3.2(d), and we take $u = u_{p,0,1,0} = \lambda_p(1 - a_p)$. Let τ be a place. By Lemma 3.2(b) and (a), $c_0 \in \mathfrak{p}_{j_\tau}$ (otherwise $|j(s_1(P, \mu))|_p = 1$), and

$$|j(s_1(P, \mu)\,\overline{\iota_\tau(u)})|_p = |j_\tau(\partial u)|_p = |j_\tau(tV_p)|_p,$$

since $\partial u = -tV_p$. This is at most 1. If it were smaller, tV_p would lie in $\mathfrak{p}_{j_\tau}$; but $t \equiv 1$ modulo $\mathfrak{p}_{j_\tau}$ (which contains c_0 and p, and p divides ν), so V_p and then M_p would lie in it, and p would divide M_p, against Lemma 4.2. □

LEAN `Parameters.exists_isTightFrame` · the coin lemma is Mathlib's `Nat.exists_add_mul_eq_of_gcd_dvd_of_mul_pred_le`

5 Jets are Taylor coefficients of products of curve theta functions

Lemma 5.1 (jets and Taylor series)**.** *Let E be a field with a CM type and Weil data, n a finite set, V a hermitian space over E with index set n and all signs 1, and $f \in \mathcal{W}_{\mathrm{f}}(V)$. Write Φ for the set of infinite places of E.*

(a) For every $a \in \mathbb{C}$ and $v \in \mathbb{C}^{\Phi\times n}$: the Taylor series at 0 of the function $u \mapsto \Theta_f(u \cdot a \cdot v)$ of one complex variable is obtained from $T_a^V f$ by substituting $v_x\, u$ for the variable z_x, for all $x \in \Phi \times n$.

(b) If $a \in \mathbb{C}$, $a \neq 0$ and $T_a^V f = 0$, then $f = 0$.

(c) Let I be a finite set, G_i $(i \in I)$ entire functions on $\mathbb{C}$ and $\lambda_{c,i} \in \mathbb{C}^{\Phi\times n}$ for $c \in \mathbb{C}$, $i \in I$, such that $\Theta_f(c\,z) = \prod_{i\in I} G_i(\lambda_{c,i} \cdot z)$ for all $c \in \mathbb{C}$ and $z \in \mathbb{C}^{\Phi\times n}$, where $\lambda \cdot z = \sum_x \lambda_x z_x$. Then $\mathrm{Th}_V(f) = \prod_i G_i(0)$, and for every $c \in \mathbb{C}$

$$T_c^V f = \prod_{i\in I} \mathrm{Tay}(G_i)(\textstyle\sum_x \lambda_{c,i,x} z_x).$$

Proof. (a) Quoted from the background library: for $f \in \mathcal{W}_{\mathrm{f}}(V)$ the function Θ_f is an integral $\Theta_f(y) = \int f(w)\,\gamma(w)\exp(\langle w, y\rangle)\,dw$ over the real vectors w in Fock coordinates, with a Gaussian γ and a pairing that is linear in y; and the coefficient of z^β in $T_a^V f$ is the same integral with exp replaced by the monomial of β, times $a^{|\beta|}/\beta!$. Put $\ell(w) = a\langle w, v\rangle$. Then $\Theta_f(uav) = \int f\gamma\, e^{u\ell}$; expanding the exponential and exchanging sum and integral, which is allowed by dominated convergence because $|\ell(w)|$ is at most a constant times $|w|$ and $e^{c|w|}\gamma(w)|f(w)|$ is integrable, shows that the coefficient of u^k is $\int f\gamma\,\ell^k/k!$. This is the jet of $T_a^V f$ along $(\sum_x v_x z_x)^k/k!$, which is the coefficient of u^k after the substitution.

(b) By (a), for every y the entire function $u \mapsto \Theta_f(u\,a\,y)$ has Taylor series 0, so it vanishes; at $u = a^{-1}$ this gives $\Theta_f(y) = 0$. So f vanishes at all real vectors, and such a vector of $\mathcal{S}(V)$ is 0 (quoted).

(c) At $c = 0$, $z = 0$ the hypothesis reads $\mathrm{Th}_V(f) = \Theta_f(0) = \prod_i G_i(0)$. Fix c. Two power series in the z_x are equal if for every v they give the same series in u after the substitution $z_x \mapsto v_x u$ (compare the homogeneous parts as polynomial functions of v). After this substitution the left side is the Taylor series of $u \mapsto \Theta_f(u\,c\,v) = \prod_i G_i(u\,\lambda_{c,i} \cdot v)$ by (a), and the right side is the product of the Taylor series of the functions $u \mapsto G_i(u\,\lambda_{c,i} \cdot v)$; the Taylor series of a product of entire functions is the product of their Taylor series. □

LEAN `Space.taylor_line_eq_subst_jetSeries` · `Space.eq_zero_of_jetSeries_eq_zero` · `Space.jetSeries_eq_prod` · quoted: `entireTheta_eq_integral`, `Space.theta_integrated`, `jet_subst_linearSeries`, `Space.eq_zero_of_realVec`

Proposition 5.2 (a vector whose theta function is a product over a tight frame)**.** *Let $P = (N_0, d)$ be parameters, μ a natural number, n a finite set and V a standard hermitian space over $E(P,\mu)$ with index set n. Let M and N be natural numbers with $N \neq 0$, I a finite set, $w_i \in E(P,\mu)$ and $m_i \in \mathbb{N}$ $(i \in I)$ such that (w_i) with multiplicities (m_i) is a tight frame of length M, δ an integer of $K(P)$ with $\delta\overline{\delta} = N$, and $x_{(i,k)}$, for $(i,k) \in I \times n$, triples of natural numbers of degree m_i. Let C be a CM*

datum for d and $a \in \Lambda_C$ with $\operatorname{area}(C) = |a|^2/N \cdot M \cdot \sqrt{d}/2$. *Then there is $F \in \mathcal{W}_{\mathrm{f}}(V)$ with the following properties.*

(1) For every $\gamma \in E(P,\mu)^n$, with $G_{(i,k)} = U^{C,m_i}_{y_{(i,k)}} \operatorname{mon}_C(x_{(i,k)})$ and $y_{(i,k)} = -\frac{a}{\overline{\delta}} \sum_\tau \iota_\tau(\gamma_k)\,\overline{\iota_\tau(w_i)}$:

$$\operatorname{Th}_V(\mathfrak{h}_V((\alpha^{-1}\gamma)^\flat)F) = \prod_{(i,k)} G_{(i,k)}(0),$$

$$T^V_c(\mathfrak{h}_V((\alpha^{-1}\gamma)^\flat)F) = \prod_{(i,k)} \operatorname{Tay}(G_{(i,k)})\Big(\sum_\tau \frac{a}{\overline{\delta}\operatorname{asc}(P)}\, c\,\overline{\iota_\tau(w_i)}\, z_{(\tau,k)}\Big)$$

for every $c \in \mathbb{C}$; here $\alpha = \alpha(P,\mu)$, so that $(\alpha^{-1}\gamma)^\flat$ has finite part γ.

(2) For every p-adic place (p,j) and every $x \in \mathbb{A}_{\mathrm{f}}^n$: $|j(F(x^\flat))|_p \le 1$.

(3) F has a vacuum expansion for (δ, N) in the sense of (Q9).

Proof. Put $b = a/\overline{\delta}$ and, for $(i,k) \in I \times n$, let $A_{(i,k)}$ be the linear form

$$A_{(i,k)}(z) = \sum_\tau (b/\operatorname{asc}(P))\,\overline{\iota_\tau(w_i)}\, z_{(\tau,k)}.$$

The hermitian form of C is $H_C(z,z') = \pi \bar z z' / \operatorname{area}(C)$. By the frame identity,

$$\sum_{(i,k)} m_i\, H_C(A_{(i,k)}u, A_{(i,k)}v) = \frac{\pi\,|b|^2 M}{\operatorname{area}(C)\operatorname{asc}(P)^2} \sum_{(\tau,k)} \overline{u_{(\tau,k)}} v_{(\tau,k)} = \sum_{(\tau,k)} \overline{u_{(\tau,k)}} v_{(\tau,k)},$$

because $|b|^2 = |a|^2/N$, $\operatorname{asc}(P)^2 = 2\pi/\sqrt{d}$ and $\operatorname{area}(C) = |a|^2 M\sqrt{d}/(2N)$: the linear forms carry the sum of the Riemann forms of the curve to the standard form. Let $S = \prod_{(i,k)} \operatorname{mon}_C(x_{(i,k)}) \circ A_{(i,k)}$, an entire function. For $\gamma \in \mathfrak{a}(P,\mu)^n$ the form $A_{(i,k)}$ takes at the real image of $\overline{\delta}\gamma$ the value $a\sum_\tau \iota_\tau(\gamma_k)\overline{\iota_\tau(w_i)}$, which lies in Λ_C by Lemma 3.3(b) because $a \in \Lambda_C$ and Λ_C is a module over $\mathcal{O}_{K(P)}$. A monomial section of degree m_i is invariant under the twisted translations U^{C,m_i}_λ, $\lambda \in \Lambda_C$, up to the multiplier of Λ_C (quoted). So S has the transformation law required in (Q9), which gives $F \in \mathcal{W}_{\mathrm{f}}(V)$ with $\Theta_F = S$ and (3). For (1), the theta function of $\mathfrak{h}_V((\alpha^{-1}\gamma)^\flat)F$ is the translate of S by the real image of γ (quoted), that is the product of the functions $G_{(i,k)} \circ A_{(i,k)}$; Lemma 5.1(c) with $\lambda_{c,(i,k)} = c$ times the coefficients of $A_{(i,k)}$ gives both formulas. For (2), a quoted approximation lemma gives γ with $\mathfrak{h}_V(x^\flat)F = \mathfrak{h}_V((\alpha^{-1}\gamma)^\flat)F$, so $F(x^\flat) = \prod G_{(i,k)}(0)$ by (1); and some positive integer multiple of γ lies in $\mathfrak{a}(P,\mu)^n$, so that a multiple of each $y_{(i,k)}$ lies in Λ_C as above. Each factor then has $|j(\,\cdot\,)|_p \le 1$ by (Q7). □

LEAN `Parameters.exists_finitePart_of_isTightFrame` · quoted: `Space.exists_entireTheta_eq`, `entireTheta_translate_finiteImage`, `exists_finiteImage_approx`, `exists_forall_mul_mem_Ô`, `thetaShift_of_mem`, `CMLattice.norm_monomialSection_le`

6 Conjugates of jets

Proposition 6.1 (a generating vector and its conjugates)**.** *Let $P = (N_0, d)$ be parameters. There is a natural number C such that for every natural number μ there is a family $(B_\sigma)_{\sigma \in \mathrm{Aut}(\mathbb{C}/K(P))}$ of finite adeles of $E(P,\mu)$ with the following properties, for every finite set n and every standard hermitian space V over $E(P,\mu)$ with index set n.*

(i) There is $c \in \mathbb{C}$, $c \neq 0$, such that $|j(c \cdot \mathrm{vac}_V(x^\flat))|_p \leq 1$ for every p-adic place (p,j) and every $x \in \mathbb{A}_{\mathrm{f}}^n$, and $|\sigma(c \cdot \mathrm{vac}_V(x^\flat))| \leq \exp(C\, N_1(P,\mu)^2)^{|n|}$ for every $\sigma \in \mathrm{Aut}(\mathbb{C}/K(P))$ and every $x \in \mathbb{A}_{\mathrm{f}}^n$.

(ii) There is $F \in \mathcal{W}_{\mathrm{f}}(V)$ with $\mathrm{Th}_V(F) = 1$ such that $\mathcal{W}_{\mathrm{f}}(V)$ is contained in the span over $\mathbb{C}$ of the vectors $\mathfrak{h}_V(x^\flat)F$, $x \in \mathbb{A}_{\mathrm{f}}^n$, and such that for every $\sigma \in \mathrm{Aut}(\mathbb{C}/K(P))$ there are $t \in \mathbb{C}$ and $F' \in \mathcal{W}_{\mathrm{f}}(V)$ with $|t|^2 = \Omega(P,\mu)^{-1}$ and, for every $x \in \mathbb{A}_{\mathrm{f}}^n$: applying σ to the coefficients of $T^V_{\mathrm{per}(P,\mu)}(\mathfrak{h}_V(x^\flat)F)$ gives $T^V_t(\mathfrak{h}_V((B_\sigma x)^\flat)F')$ with every variable $z_{(\tau,i)}$ replaced by $z_{(\mathrm{perm}_\mu(\sigma)^{-1}\tau,i)}$.

The least such C is not asked for; the formal text fixes one by choice, written $C_{\mathrm{gen}}(P)$ below.

Proof. The constant. By (Q4) the invariant a_1 of $\mathcal{E}(P)$ is algebraic: its image under every field isomorphism of $\mathbb{C}$ onto an algebraic closure of $\mathbb{Q}_2$ has absolute value at most 1. Let g be its minimal polynomial and A_0 the finite set of the roots of g that are invariants of CM data for d; for $\alpha \in A_0$ let C_α be the chosen datum with invariant α (Background 3). The absolute values of all twisted translates of the theta function of one datum form a compact set (quoted: sections of an Appell–Humbert datum), so they are bounded by one number M_0 for the finitely many data C_α, $\alpha \in A_0$. Put $C = \lceil M_0 \rceil + 4$. Let $\sigma \in \mathrm{Aut}(\mathbb{C}/K(P))$. Then $\sigma(a_1) \in A_0$, and the conjugate datum $\sigma\mathcal{E}(P) = C_{\sigma(a_1)}$ is one of these data: this is why the conjugate datum is defined through the invariant alone. Apply (Q2) to $\mathcal{E}(P)$, whose lattice is $\Omega_0(P)\mathcal{O}_{K(P)}$ (Lemma 3.1(d)): it gives Ω_σ, e_σ, β_σ and the a_S. Let B_σ be the image of β_σ in the finite adeles of $E(P,\mu)$.

The vector F. Let $N = N_1(P,\mu)$ and take the frame $(\varsigma(P,\mu)^i)_{i<N}$ with multiplicity N (Lemma 4.1), written with every vector repeated N times and multiplicity 1; its length is N^2. For an integer $e \geq 1$, Proposition 5.2 applies with $\delta = Ne$, $\delta\overline{\delta} = (Ne)^2$, all triples equal to $(1,0,0)$, and any CM datum with a lattice vector of the right area. For $e = 1$, the datum $\mathcal{E}(P)$ and $a = \Omega_0(P)$ (the area of $\mathcal{E}(P)$ is $|\Omega_0(P)|^2\sqrt{d}/2$ by (Q1)) it gives F with a vacuum expansion $(A, (x_a), m_0)$ for (N, N^2), so that $|A| = (N^2)^{2f|n|} = N^{4f|n|}$, $f = |\mathrm{Pl}(P,\mu)|$. Formula (1) at $\gamma = 0$ gives $\mathrm{Th}_V(F) = 1$. Pairing the vacuum expansion for $\ell = \mathrm{Th}_V$ with vac_V shows that $\langle \mathrm{vac}_V, \mathfrak{h}_V((-x_a)^\flat)F\rangle_V \neq 0$ for some a; the expansion for the linear form $\ell = m_0^{-1}\langle \mathrm{vac}_V, \mathfrak{h}_V((-x_a)^\flat)\cdot\rangle_V$ then reads $c \cdot \mathrm{vac}_V = \sum_{a'} \epsilon_{a'}\mathfrak{h}_V((-x_{a'})^\flat)F$ with $c = |A|\ell(F) \neq 0$ and coefficients $\epsilon_{a'} \in \{0,1\}$ (the pairings of two translates of the vacuum are 0 or 1 here). So $c \cdot \mathrm{vac}_V$ is a sum of at most $|A|$ translates of F. This gives the spanning statement of (ii), since $\mathcal{W}_{\mathrm{f}}(V)$ is spanned by translates of vac_V, and the p-adic bound of (i), by Proposition 5.2(2) and the ultrametric inequality.

The conjugate vector. Fix σ. Apply Proposition 5.2 with $e = e_\sigma$ to the datum $\sigma\mathcal{E}(P)$ and $a = e_\sigma\Omega_\sigma$, whose area condition is that of (Q2); this gives F'. Put

$t = \sigma(s_1(P,\mu)) \cdot \operatorname{asc}(P)/\Omega_\sigma$. Then $|t|^2 = \Omega(P,\mu)^{-1}$: by Lemma 3.1(a), (Q3) and $\operatorname{area}(\sigma\mathcal{E}(P)) = |\Omega_\sigma|^2\sqrt{d}/2$,

$$|t|^2 = \frac{\mathrm{rd}_{E(P,\mu)}}{\sqrt{d}} \cdot \frac{\operatorname{area}(\sigma\mathcal{E}(P))}{\pi\, e^{\mathrm{cs}(d)}\, 2\pi/\sqrt{d}} \cdot \frac{2\pi/\sqrt{d}}{|\Omega_\sigma|^2} = \frac{\mathrm{rd}_{E(P,\mu)}}{2\pi\, e^{\mathrm{cs}(d)}}.$$

The jet identity. Both sides are locally constant in x (quoted: $x \mapsto \mathfrak{h}_V(x^\flat)F$ is locally constant for $F \in \mathcal{W}_{\mathrm{f}}(V)$), and the vectors $\alpha^{-1}\gamma$, $\gamma \in E(P,\mu)^n$, are dense; so let $x = \alpha^{-1}\gamma$. Because β_σ is a cluster point of the a_S, there is a finite S containing the numbers $x_{i,k} = -\operatorname{Tr}_{E(P,\mu)/K(P)}(\gamma_k\overline{\varsigma(P,\mu)^i})/N$ with $\mathfrak{h}_V((B_\sigma x)^\flat)F' = \mathfrak{h}_V((a_S x)^\flat)F'$. By Proposition 5.2(1) and Lemma 3.3(a) the two jet series are products over (i,k) of

$$\operatorname{Tay}(U^{\mathcal{E}(P),1}_{x_{i,k}\Omega_0(P)}\theta_{\mathcal{E}(P)})\Big(\sum_\tau \frac{\operatorname{per}(P,\mu)\,\Omega_0(P)}{N\operatorname{asc}(P)}\overline{\iota_\tau(\varsigma(P,\mu)^i)}z_{(\tau,k)}\Big) \quad \text{and}$$
$$\operatorname{Tay}(U^{\sigma\mathcal{E}(P),1}_{x_{i,k}\Omega_\sigma a_S}\theta_{\sigma\mathcal{E}(P)})\Big(\sum_\tau \frac{t\,\Omega_\sigma}{N\operatorname{asc}(P)}\overline{\iota_\tau(\varsigma(P,\mu)^i)}z_{(\tau,k)}\Big).$$

The one-variable series correspond under σ by (Q2). The linear forms correspond after the renaming of the places: $\operatorname{per}(P,\mu)\Omega_0(P)/\operatorname{asc}(P) = s_1(P,\mu)$ and $t\Omega_\sigma/\operatorname{asc}(P) = \sigma(s_1(P,\mu))$, and $\sigma(\overline{\iota_\tau(w)}) = \overline{\iota_{\mathrm{perm}_\mu(\sigma)(\tau)}(w)}$ by (I5).

The conjugate bound of (i). $\sigma(c\mathrm{vac}_V(x^\flat))$ is a sum of at most $|A| = N^{4f|n|}$ numbers $\sigma\operatorname{Th}_V(\mathfrak{h}_V(x'^\flat)F)$. Each is the constant term of a jet series of a translate of F, hence by the jet identity the constant term of one of F', which by Proposition 5.2(1) is a product of $N^2|n|$ values of twisted translates of the theta function of $\sigma\mathcal{E}(P)$. As $\sigma\mathcal{E}(P)$ is one of the data C_α, $\alpha \in A_0$, each value is at most $M_0 \le e^{M_0}$. With $f \le N$ and $\log N \le N$,

$$N^{4f|n|}\, e^{M_0 N^2|n|} \;\le\; \exp\big((M_0+4)N^2|n|\big) \;\le\; \exp\big((\lceil M_0\rceil+4)N^2\big)^{|n|};$$

the summand 4 of the constant pays for the number of terms. □

LEAN `Parameters.exists_isJetConjugate_generator` · quoted: `IsAppellHumbert.isCompact_range_norm_thetaShift`, `Space.isLocallyConstant_rho_finVec`, `denseRange_finiteImage`

Lemma 6.2 (the scaled vacuum)**.** *Let $P = (N_0, d)$ be parameters and μ a natural number.*

(a) $\operatorname{vsc}(P,\mu) \neq 0$.

Let further n be a finite set and V a standard hermitian space over $E(P,\mu)$ with index set n.

(b) For every $x \in \mathbb{A}_{\mathrm{f}}^n$: $\theta^\circ_V(x^\flat) = \prod_{i\in n}\operatorname{vsc}(P,\mu)\cdot\operatorname{vac}_{V_1}((x_i)^\flat)$, where V_1 is the standard space of rank one.

(c) For every $x \in \mathbb{A}_{\mathrm{f}}^n$ and every $\sigma \in \operatorname{Aut}(\mathbb{C}/K(P))$ we have the bound $|\sigma(\theta^\circ_V(x^\flat))| \le \exp(C_{\mathrm{gen}}(P)\,N_1(P,\mu)^2)^{|n|}$.

(d) For every $x \in \mathbb{A}_{\mathrm{f}}^n$ and every p-adic place (p,j): $|j(\theta^\circ_V(x^\flat))|_p \le 1$.

(e) Let $f \in \mathcal{S}(V)$ and let M be a real number with $|f(x^\flat)| \le M$ for every $x \in \mathbb{A}_{\mathrm{f}}^n$. Then $|\langle f, f\rangle_V| \le M^2$.

(f) There is $x \in \mathbb{A}_{\mathrm{f}}^n$ with $\theta^\circ_V(x^\flat) \neq 0$.

Proof. (a) is part of the choice. (b) For a standard space the vacuum is the product of the rank-one vacua of the coordinates (quoted). (c) and (d) follow from (b) and Proposition 6.1(i) for rank one, factor by factor. (e) Quoted: $\langle f, f\rangle_V$ is the mean of $|f|^2$ over a compact quotient of $\mathbb{A}^n$. The set of points where $|f| \le M$ is closed, and it contains a dense set: by the translation formula for the values of f and the density of $E(P,\mu)$ in its archimedean completion, the value of f at a point of a dense set is a value $f(x^\flat)$ (V is standard, so its adelic point is 1). (f) Otherwise (e) with $M = 0$ would give $\langle \mathrm{vac}_V, \mathrm{vac}_V\rangle_V = 0$. □

LEAN `Parameters.kappa₀_ne_zero` · `Parameters.intVacuum_finVec_eq_prod` · `Parameters.norm_intVacuum_le` · `Parameters.norm_intVacuum_le_one` · `Parameters.norm_dataInner_le` · `Parameters.exists_intVacuum_ne_zero` · quoted: `Space.vacElem_apply_eq_prod_lineTheta`, `Space.dataInner_eq_quotMean`, `norm_quotMean_le`, `Space.apply_vecAtPoint_add`

Proof of Theorem B. Let F, F', t, $B = B_\sigma$ be as in Proposition 6.1(ii) for V and σ, and write $\Theta \mapsto \Theta^{[\sigma]}$ for the renaming $z_{(\tau,i)} \mapsto z_{(\mathrm{perm}_\mu(\sigma)^{-1}\tau,i)}$. The map $g \mapsto (T_t^V g)^{[\sigma]}$ is injective on $\mathcal{W}_{\mathrm{f}}(V)$ by Lemma 5.1(b), as $t \neq 0$. Let L be a left inverse of it, composed with the σ-semilinear map $f \mapsto \sigma(T^V_{\mathrm{per}(P,\mu)} f)$. By the jet identity of Proposition 6.1,

$$L(\mathfrak{h}_V(x^\flat)F) = \mathfrak{h}_V((Bx)^\flat)F' \qquad (x \in \mathbb{A}_{\mathrm{f}}^n). \tag{3}$$

Since $\mathcal{W}_{\mathrm{f}}(V)$ is spanned by the vectors $\mathfrak{h}_V(x^\flat)F$ and Heisenberg operators compose up to roots of unity given by ψ_0, (3) gives (ii) for all $f \in \mathcal{W}_{\mathrm{f}}(V)$ and the rule $L(\mathfrak{h}_V(x^\flat)f) = \mathfrak{h}_V((Bx)^\flat)(Lf)$, provided that $\sigma(\psi_0(z)) = \psi_0(B\overline{B}z)$ for all $z \in \mathbb{A}_{\mathrm{f}}$. This identity is read off the jet identity in rank one. Let V_1 be the standard space of rank one and F_1, F_1' its vectors of Proposition 6.1(ii) for σ; the adele B is the same. The maps $x \mapsto \mathfrak{h}_{V_1}(x^\flat)F_1$ and $x \mapsto \mathfrak{h}_{V_1}(x^\flat)F_1'$ are locally constant on $\mathbb{A}_{\mathrm{f}}$ (quoted), so there is a natural number $N' \neq 0$ with

$$\mathfrak{h}_{V_1}((N')^\flat)F_1 = F_1 \qquad \text{and} \qquad \mathfrak{h}_{V_1}((N'B)^\flat)F_1' = F_1'$$

(quoted: a neighbourhood of 0 in $\mathbb{A}_{\mathrm{f}}$ contains $N'v$ for all v in the compact set $\{1, B\}$, for a suitable N'). Let $z \in \mathbb{A}_{\mathrm{f}}$ and write $z = N'x$, which is possible because N' is a unit of $\mathbb{A}_{\mathrm{f}}$. By (1),

$$\mathfrak{h}_{V_1}((2x)^\flat)F_1 = \mathfrak{h}_{V_1}((2x)^\flat)\mathfrak{h}_{V_1}((N')^\flat)F_1 = \psi_0(z)\cdot \mathfrak{h}_{V_1}((2x+N')^\flat)F_1,$$
$$\mathfrak{h}_{V_1}((2Bx)^\flat)F_1' = \mathfrak{h}_{V_1}((2Bx)^\flat)\mathfrak{h}_{V_1}((N'B)^\flat)F_1' = \psi_0(B\overline{B}z)\cdot \mathfrak{h}_{V_1}((B(2x+N'))^\flat)F_1'.$$

Let T be the series

$$T = \sigma(T^{V_1}_{\mathrm{per}(P,\mu)}(\mathfrak{h}_{V_1}((2x+N')^\flat)F_1)) = (T_t^{V_1}(\mathfrak{h}_{V_1}((B(2x+N'))^\flat)F_1'))^{[\sigma]},$$

the two expressions being equal by the jet identity at $2x + N'$. The jet identity at $2x$ then reads $\sigma(\psi_0(z))\cdot T = \psi_0(B\overline{B}z)\cdot T$. Here $T \neq 0$ by Lemma 5.1(b): the scale $\mathrm{per}(P,\mu)$ is not zero (Lemma 3.1(c)), and $\mathfrak{h}_{V_1}((2x+N')^\flat)F_1 \neq 0$, because Heisenberg operators are invertible and $\mathrm{Th}_{V_1}(F_1) = 1$. Hence $\sigma(\psi_0(z)) = \psi_0(B\overline{B}z)$. Now (Q10) applies: it gives c, z, (iii) and (v), and that B is a unit. Taking constant terms in (ii) for $f = \mathfrak{h}_V(x^\flat)\theta_V^\circ$ shows $\mathrm{Th}_V(L(\mathfrak{h}_V(x^\flat)\theta_V^\circ)) = \sigma(\theta_V^\circ(x^\flat))$, which is nonzero for some x by Lemma 6.2(f); as $L\theta_V^\circ$ is a multiple of $c\,\omega_V(1, z1_n)\mathrm{vac}_V$, this forces $c \neq 0$. (i) is Proposition 6.1.

The bound (iv). By (iii) and the translation rule, $L(\mathfrak{h}_V(x^\flat)\omega_V(1,g)\theta_V^\circ)$ is a number of absolute value 1 times $\mathfrak{h}_V((Bx)^\flat)\omega_V(1,g)L\theta_V^\circ$, and $L\theta_V^\circ = c'\omega_V(1, z1_n)\mathrm{vac}_V$ for

a constant c'. The values of $L\theta^\circ_V$ at the points $(Bx)^\flat$ are the numbers $\sigma(\theta^\circ_V(x^\flat))$, bounded by $\exp(C_{\mathrm{gen}}(P)N_1(P,\mu)^2)^{|n|}$ (Lemma 6.2(c)); since B is a unit these are all its values at finite vectors, and Lemma 6.2(e) bounds $|c'|$ by the same number. It remains to bound a translated vacuum $\omega_V(1,g')\mathrm{vac}_V$ at an arbitrary adelic point. Its value is an inner product with a translate of the vacuum, which a quoted series expresses as a sum of Gaussian terms over the points of a lattice in $\mathbb{C}^{\mathrm{Pl}(P,\mu)\times n}$: the vectors $\gamma \in E(P,\mu)^n$ whose coordinates at the point g' lie in $\alpha(P,\mu)\widehat{\mathcal{O}}$. Since $N_1(P,\mu)\alpha(P,\mu) \in \widehat{\mathcal{O}}$ (I1), the difference of two such vectors, multiplied by $N_1(P,\mu)$, has integral coordinates at g': it lies in the lattice of the space V moved by g'. A non-zero vector ξ of that lattice has length at least 1 (quoted): the value of the hermitian form at (ξ,ξ) is an integer of $E(P,\mu)$ whose absolute values at the infinite places are at most the squared length of ξ; an integer with all these absolute values less than 1 is 0, and the form is definite. So the points $N_1(P,\mu)\gamma$ have mutual distances at least 1, and a packing estimate for separated points (quoted), applied to them, bounds the sum by exp of a quadratic polynomial in $f|n| \le N_1(P,\mu)|n|$ with coefficients controlled by $N_1(P,\mu)$ and $\sqrt{d}$. The constant $C(P)$ of unit 152 is $C_{\mathrm{gen}}(P) + 2d + 6$ plus non-negative terms, and the product of the two bounds is at most $\exp(C(P)(N_1(P,\mu)|n|+1)^2)$. I did not replay the last estimate line by line; see §12. □

LEAN `Parameters.exists_semilinear_isJetConjugate` · quoted: `Space.semilinear_intertwines_weil`, `exists_natCast_smul_mem_of_isCompact`, `Space.isLocallyConstant_rho_finVec`, `Space.hasSum_dataInner`, `Space.one_le_firstMinimum`, `tsum_le_of_separated` · unit 152: `Parameters.c`

Lemma 6.3 (names are admissible combinations)**.** *Let E be a field with a CM type and Weil data, n a finite set, V a hermitian space over E with index set n, and $\mathfrak{m}$ a modulus of E with residue ring $\mathcal{O}_E/\mathfrak{m}$. Let Ψ be a complex function on $(\mathcal{O}_E/\mathfrak{m})^n$, b a proposition, B a real number, $u,v \in \mathcal{S}(V)$, g an integer of E and q a natural number. Suppose that $g\,q \in \mathfrak{m}$; that every value $\Psi(x)$ is integral away from q (that is, $|j'(\Psi(x))|_\ell \le 1$ for every prime $\ell \nmid q$ and every ring homomorphism $j' : \mathbb{C} \to \mathbb{C}_\ell$); that, if b holds, $\sum_x |\sigma(\Psi(x))| \le B$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$; and that $P^{\mathfrak{m}}u = v$. Let η be a finite adele of E with $\eta q \equiv 0$ modulo $\mathfrak{m}$ (by default the image of g). Then the vector $\mathrm{name}_{V,\mathfrak{m}}(\tilde{\Psi})\,u$, where $\tilde{\Psi}$ is the function on $\mathbb{A}_{\mathrm{f}}^n$ that is Ψ of the reduction at the vectors integral at the primes of $\mathfrak{m}$ and 0 elsewhere, is a finite combination $\sum_x r(x)\,\mathfrak{h}^{\mathrm{f}}_V(x)v$ in which every x has $\overline{\eta}\,x_i$ integral at all places not dividing q, every $\mathrm{N}(g)^{|n|}r(x)$ is integral away from q, and, if b holds, $\sum_x |\sigma(r(x))| \le B$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$. (In the terms of unit 141: an admissible combination of the $\mathfrak{h}^{\mathrm{f}}_V(x)v$ with support, denominator $\mathrm{N}(g)^{|n|}$ and size B as said.)*

Proof. By Fourier inversion on the finite group $(\mathcal{O}_E/\mathfrak{m})^n$, with a primitive additive character $\psi_{\mathfrak{m}}$,

$$\Psi = \sum_a \Psi(a) \sum_b \frac{\psi_{\mathfrak{m}}(-a\cdot b)}{|\mathcal{O}_E/\mathfrak{m}|^n}\,\psi_{\mathfrak{m}}(\,\cdot\, b).$$

Quoted: the name of the character $x \mapsto \psi_{\mathfrak{m}}(x\cdot b)$, applied to u, is a root of unity times $\mathfrak{h}^{\mathrm{f}}_V(x_b)P^{\mathfrak{m}}u = \mathfrak{h}^{\mathrm{f}}_V(x_b)v$ for a lift x_b of b. The outer coefficients $\Psi(a)$ are integral away from q and of size B; the inner ones are roots of unity divided by $|\mathcal{O}_E/\mathfrak{m}|^n$, and $\mathrm{N}(g)^{|n|}/|\mathcal{O}_E/\mathfrak{m}|^n$ is integral away from q because $\mathrm{N}(\mathfrak{m})$ divides $\mathrm{N}(gq)$. The composition rule for admissible combinations (unit 141) gives the result; the lifts x_b can be taken with $\overline{\eta}x_b$ integral away from q since $\eta q \in \mathfrak{m}\widehat{\mathcal{O}}$ and q is a unit there. □

LEAN `isAdmissibleComb_name` · unit 141: `IsAdmissibleComb`, `IsAdmissibleComb.bind` · quoted: `Modulus.name_char_eq_translate_projOn`

7 Standard data, integral jets, gauges

Lemma 7.1 (stability of the standard data). *Let $P = (N_0, d)$ be parameters, μ a natural number and n a finite set.*

(a) For every integer δ of $K(P)$ and every set S of finite places: $\widehat{\mathcal{O}}^n \subset \mathrm{Lat}(\delta; S)$.

Let further V be a hermitian space over $E(P,\mu)$ with index set n, (p, j) a p-adic place and δ an integer of $K(P)$.

(b) If $x \in \mathrm{Lat}(\delta; \{w : w \mid p\})$ and $f \in \mathrm{Std}_\delta(V)$, then $\mathfrak{h}^{\mathrm{f}}_V(x) f \in \mathrm{Std}_\delta(V)$.
(c) If $g \in \mathrm{U}_n(\mathbb{A}_{\mathrm{f}})$ is integral above p and $f \in \mathrm{Std}_\delta(V)$, then $\omega_V(1, g) f \in \mathrm{Std}_\delta(V)$.
(d) For a normed field R, a multiplicative map $\iota\colon \mathbb{C} \to R$ and $z \in \mathbb{A}_{\mathrm{f}}$: $|\iota(\psi_0(z))| = 1$.
(e) Let n' be a finite set and a, b characters of $\mathrm{U}(1)$ over $E(P,\mu)$ that have weights e, e_a and tame parts ψ, ψ'. If $f \in \mathrm{Std}_\delta(\mathrm{Hom}(I_n, I_{n'}))$, and $g \in \mathrm{U}_n(\mathbb{A}_{\mathrm{f}})$, $h \in \mathrm{U}_{n'}(\mathbb{A}_{\mathrm{f}})$ are integral above p, then $\omega_{a,b}((1,g),(1,h)) f \in \mathrm{Std}_\delta(\mathrm{Hom}(I_n, I_{n'}))$.

Proof. (a) δ is integral. (d) $\psi_0(z)$ is a root of unity. (b) It suffices to treat a generator $f = \mathfrak{h}^{\mathrm{f}}_V(y)\theta^\circ_V$; then $\mathfrak{h}^{\mathrm{f}}_V(x) f$ is $\psi_0(\cdot)\mathfrak{h}^{\mathrm{f}}_V(x+y)\theta^\circ_V$, the scalar lies in O_j by (d), and $x + y \in \mathrm{Lat}(\delta; \{w \mid p\})$. (c) Multiplication by g preserves $\mathrm{Lat}(\delta; \{w \mid p\})$. Quoted: $\omega_V(1,g)\mathrm{vac}_V$ is $\Delta_{\mathrm{f}}(g)$, of absolute value 1 at j, times a finite sum of vectors $\mathfrak{h}^{\mathrm{f}}_V(y)\mathrm{vac}_V$ with $y \in \widehat{\mathcal{O}}^n$; so $\omega_V(1,g)\mathfrak{h}^{\mathrm{f}}_V(x)\theta^\circ_V$, which is $\mathfrak{h}^{\mathrm{f}}_V(gx)\omega_V(1,g)\theta^\circ_V$, lies in $\mathrm{Std}_\delta(V)$ by (a) and (b). (e) The action is a value of the characters, of absolute value 1 at j on elements integral above p (quoted), times $\omega_{\mathrm{Hom}(I_n, I_{n'})}$ of an element integral above p; use (c). □

LEAN `Parameters.integralAfter.of_integral` · `Parameters.finHeisOp_mem_standardData` · `Parameters.omega_mem_standardData` · `Parameters.norm_map_psi₀` · `Parameters.pairWeil_mem_standardData` · quoted: `Space.finHeisOp_finHeisOp`, `Space.omega_vacElem_sum`, `AutChar.HasWeight.norm_padic_det`

Proposition 7.2 (from theta products to jets of standard data). *Let $P = (N_0, d)$ be parameters, μ a natural number, n a finite set, V a standard hermitian space over $E(P,\mu)$ with index set n, (p, j) a p-adic place, δ an integer of $K(P)$ and $f \in \mathrm{Std}_\delta(V)$. Let $c \in \mathbb{C}$, let L be a $\mathbb{C}$-linear form on the power series in the variables $z_{(\tau,i)}$, $(\tau, i) \in \mathrm{Pl}(P,\mu) \times n$, and let $r \geq 0$ be a real number. Suppose that $|j(L(\Theta))|_p \leq r$ for every power series Θ over $\mathbb{C}$ whose image under j lies in $\mathrm{ThProd}_{P,\mu}(\delta, |j(c \cdot s_1(P,\mu))|_p; n)$. Then*

$$|j(L(T^V_{c \cdot \mathrm{per}(P,\mu)} f))|_p \leq r.$$

Proof. The set of $f \in \mathcal{S}(V)$ with the asserted bound is an O_j-submodule, so let $f = \mathfrak{h}^{\mathrm{f}}_V(x)\theta^\circ_V$ with $x \in \mathrm{Lat}(\delta; \{w \mid p\})$. Put $M = p\,(d\,N_1(P,\mu))^6 + 1$, prime to p. By Theorem A (with $c_* = 0$) there is a tight frame of length M^2; repeat its vectors so that the multiplicities are 1. Proposition 5.2 with $N = M^2$, $\delta_0 = M$, triples $(1,0,0)$, the datum $\mathcal{E}(P)$ and $a = \Omega_0(P)$ gives F with $\mathrm{Th}_V(F) = 1$ and a vacuum expansion $(A, (y_a), m_0)$ with My_a integral. Applied to the linear form $\mathrm{vsc}(P,\mu)^{|n|}|A|^{-1}\,\mathrm{Th}_V$ it writes θ°_V as a combination of the vectors $\mathfrak{h}_V((-y_a)^\flat)F$ whose coefficients are values of θ°_V, in O_j by Lemma 6.2(d), times $m_0/|A|$, where $|A|$ is a power of M and so a unit at j. Hence it suffices to bound $L(T^V_{\mathrm{cper}(P,\mu)} f')$ for $f' = \mathfrak{h}_V((x - y_a)^\flat)F$. Approximate:

$f' = \mathfrak{h}_V((\alpha^{-1}\gamma)^\flat)F$ with $\gamma \in E(P,\mu)^n$. By Proposition 5.2(1), $T^V_{\mathrm{cper}(P,\mu)} f'$ is a product of Taylor series of translates $U_y^{\mathcal{E}(P),1}\theta_{\mathcal{E}(P)}$ at the linear forms with coefficients $b\,\overline{\iota_\tau(w_i)}$, where $b = \Omega_0(P)\,c\,\mathrm{per}(P,\mu)/(M\mathrm{asc}(P)) = c\,s_1(P,\mu)/M$, so $|j(b)|_p = |j(cs_1(P,\mu))|_p$. Its image under j lies in $\mathrm{ThProd}_{P,\mu}(\delta, |j(cs_1(P,\mu))|_p; n)$: the frame has length M^2, prime to p; and for the lattice condition choose a positive integer m' with $m'\delta M\gamma \in \mathfrak{a}(P,\mu)^n$, write $m' = p^k m''$ with $p \nmid m''$, and take the integer $m''M^2$, prime to p. Indeed $\alpha(P,\mu)^{-1}\delta M\gamma$ is integral at the places above p: the approximation can be taken so that $\alpha(P,\mu)^{-1}\gamma$ differs from $x - y_a$ by an integral vector, and δx and My_a are integral at these places. At the other places p^k is a unit, so $m'\delta M\gamma \in \mathfrak{a}(P,\mu)^n$ gives the integrality of $\alpha(P,\mu)^{-1}m''\delta M\gamma$ there. Hence $m''\delta M\gamma \in \mathfrak{a}(P,\mu)^n$, and the translation vectors, $-\Omega_0(P)/M$ times $\sum_\tau \iota_\tau(\gamma_k)\overline{\iota_\tau(w_i)}$, are carried into $\Omega_0(P)\mathcal{O}_{K(P)} = \Lambda_{\mathcal{E}(P)}$ by $m''M^2\delta$ (Lemma 3.3(b)). The hypothesis gives the bound. □

LEAN `Parameters.norm_jetSeries_le`

Lemma 7.3 (jets of products of torsion translates)**.** *Let d be a prime number $\equiv 7 \bmod 8$, C a CM datum for d, (p,j) a p-adic place, I and S finite sets, $c_{i,x} \in \mathbb{C}_p$ for $(i,x) \in I \times S$, and $\delta \in \mathcal{O}_d$ with $\delta \neq 0$. Suppose that $|j(\bar\delta)^{-1}c_{i,x}/d|_p \le 1$ for all i, x, and let $y_i \in \mathbb{C}$ with $\delta y_i \in \Lambda_C$ for all $i \in I$. Then for every polynomial Q in the variables z_x, $x \in S$, with coefficients in O_j, the jet along $j(Q)$ of the series*

$$\prod_{i\in I} \mathrm{Tay}_j(U^{C,1}_{y_i}\theta_C)\Big(\sum_{x\in S} c_{i,x}z_x\Big)$$

lies in $\mathfrak{o}_p$.

Proof. Let L, L_2 be as in (Q6) and $\ell_i = j(\bar\delta)^{-1}\sum_x c_{i,x}z_x$. Let $\mathcal{R}$ be the $\mathfrak{o}_p$-subalgebra of $\mathbb{C}_p[[z]]$ generated by the series $G(\ell_i)$ with $i \in I$ and $G \in \mathfrak{o}_p[[\varphi_C]] \cup \{L', \exp(-L)\}$. *Every element of $\mathcal{R}$ has constant term in $\mathfrak{o}_p$, and $\mathcal{R}$ is stable under the $\partial/\partial z_x$.* By the Leibniz rule it suffices to check the generators. By the chain rule $\partial_x(G(\ell_i)) = (dG')(\ell_i)\cdot c_{i,x}/(j(\bar\delta)d)$, and the last factor lies in $\mathfrak{o}_p$. If $G \in \mathfrak{o}_p[[\varphi_C]]$ then $dG' \in \mathfrak{o}_p[[\varphi_C]]$ by (Q5). If $G = L'$ then $dL'' = dL_2 + d\,\mathrm{Tay}_j(\xi_C) \in \mathfrak{o}_p[[\varphi_C]]$, because $dL_2 = -j(\sqrt{-d})\cdot j(\sqrt{-d})L_2 \in \mathfrak{o}_p$ by (Q6). If $G = \exp(-L)$ then $G' = -L'G$. The constant terms are integral ($L'(0) = 0$, $\exp(-L)(0) = 1$, (Q5)). It follows by induction on the degree of Q that jets of elements of $\mathcal{R}$ along $j(Q)$ lie in $\mathfrak{o}_p$: the jet along z_xQ is the jet of $\partial_x F$ along Q. Finally each factor of the product lies in $\mathcal{R}$: by (Q7), $\mathrm{Tay}_j(U^{C,1}_{y_i}\theta_C)(j(\bar\delta)X) = \exp(-L)^{\delta\bar\delta}u_i$ with $u_i \in \mathfrak{o}_p[[\varphi_C]]$, and the factor is this series at $X = \ell_i$. □

LEAN `CMLattice.jet_theta_prod_integral`

Proof of Theorem C. Apply Proposition 7.2 with $c = \bar\delta d$, $L = \mathrm{jet}_{Q'}(\,\cdot\,)$ and $r = 1$. Let Θ be a series whose image under j lies in $\mathrm{ThProd}_{P,\mu}(\delta, |j(\bar\delta d\,s_1(P,\mu))|_p; n)$, with frame (w_i), numbers n_0, b and translation vectors $y_{(i,k)}$. Then $j(\mathrm{jet}_{Q'}(\Theta))$ is the jet along $j(Q)$ of a product as in Lemma 7.3, for the datum $\mathcal{E}(P)$, the integer $n_0\delta$ (so that $n_0\delta\,y_{(i,k)} \in \Lambda_{\mathcal{E}(P)}$) and the coefficients $c_{(i,k),(\tau,k')} = [k = k']\,j(b\,\overline{\iota_\tau(w_i)})$. The hypothesis of the lemma holds: $|j(b)|_p = |j(\bar\delta d s_1(P,\mu))|_p$, n_0 is a unit at j, and $|j(s_1(P,\mu)\overline{\iota_\tau(w_i)})|_p \le 1$ by Lemma 3.2(c), so $|j(n_0\bar\delta)^{-1}j(b\,\overline{\iota_\tau(w_i)})/d|_p \le 1$. □

LEAN `Parameters.norm_jet_jetSeries_le_one`

Proposition 7.4 (one theta lift and the gauges)**.** *Let $P = (N_0, d)$ be parameters, μ a natural number, n, n' finite sets and (p,j) a p-adic place.*

(a) Let k be a kernel as in Definition 2.3 with data in $\mathcal{D}$, $\Phi \in \mathcal{D}$, $d_1 \in \mathrm{U}_n(\mathbb{A}_{\mathrm{f}})$, $d_2 \in \mathrm{U}_{n'}(\mathbb{A}_{\mathrm{f}})$, $z \in \mathbb{C}$ and c, c' real numbers. If $\Phi \in \mathrm{Raise}_k(d_1, d_2; c)$ and $|j(z)|_p \cdot c \le c'$, then $z\Phi \in \mathrm{Raise}_k(d_1, d_2; c')$.

Let further $\varkappa$ be a lift weight for n, n', e_b, e; a, b characters of $\mathrm{U}(1)$ over $E(P,\mu)$ with weights e_a, e_b and tame parts ψ, ψ'; K a level; and $c \in \mathbb{C}$.

(b) Let $k = \mathrm{k}_{\varkappa}(c; a, b; K)$, δ an integer of $K(P)$ and $\Psi \in \mathcal{W}_{\mathrm{f}}(\mathrm{Hom}(I_n, I_{n'}))$ with $\Psi \in \mathrm{Std}_{\delta}(\mathrm{Hom}(I_n, I_{n'}))$. Let $x, y \in \mathrm{U}_{n'}(\mathbb{A}_{\mathrm{f}})$ be integral above p, $d_0 \in \mathrm{U}_{n'}(\mathbb{A}_{\mathrm{f}})$, and $u_\tau \in \mathbb{C}_p^{n'}$ ($\tau \in \mathrm{Pl}(P,\mu)$) such that the image of the matrix of d_0 under $\hat{\jmath}_\tau$ is the diagonal matrix of u_τ, for every τ. Let B be a real number such that for every $f \in \mathrm{Std}_{\delta}(\mathrm{Hom}(I_n, I_{n'}))$ and all Weyl indices i', i

$$\prod_{\tau} \Big(\prod_{b \in n'} |u_\tau(b)|_p \Big)^{e_{a,\tau}} \prod_{h,l,s} |u_\tau(i'_\tau(h,l,s))|_p \;\cdot\; |j(\mathrm{SJ}_{\varkappa}(T_c^{\mathrm{Hom}(I_n, I_{n'})} f)_{i',i})|_p \;\le\; B.$$

Then $\omega_k(1, x\, d_0\, y)\Psi \in \mathrm{Raise}_k(1, 1; B)$.

(c) Let $k = \mathrm{k}_{\varkappa}(c \cdot \mathrm{per}(P,\mu); a, b; K)$, δ an integer of $K(P)$ with $\delta \neq 0$, and $\Phi \in \mathcal{W}_{\mathrm{f}}(\mathrm{Hom}(I_n, I_{n'}))$ with $\Phi \in \mathrm{Std}_{\delta}(\mathrm{Hom}(I_n, I_{n'}))$. Then

$$\Phi \in \mathrm{Raise}_k(1, 1; (|j(c)|_p / |j(\bar{\delta} d)|_p)^{\mathrm{box}(\varkappa, |n'|)}).$$

(d) Let $k = \mathrm{k}_{\varkappa}(c \cdot \mathrm{per}(P,\mu); a, b; K)$, let $N \neq 0$, d' and M be natural numbers and $f \in \mathcal{W}_{\mathrm{f}}(\mathrm{Hom}(I_n, I_{n'}))$. If $d' f \in \mathrm{Std}_N(\mathrm{Hom}(I_n, I_{n'}))$, $|j(c)|_p \le 1$ and $d' \cdot (N \cdot d)^{\mathrm{box}(\varkappa, |n'|)}$ divides M, then $Mf \in \mathrm{Raise}_k(1, 1; 1)$.

Proof. (a) The gauge is homogeneous: $\mathrm{gauge}_j^{\varkappa, e_a}(g; z\xi) = |j(z)|_p \, \mathrm{gauge}_j^{\varkappa, e_a}(g; \xi)$.

(b) Let $g \in \mathrm{U}_n(\mathbb{A}_{\mathrm{f}})$, $g' \in \mathrm{U}_{n'}(\mathbb{A}_{\mathrm{f}})$ and ξ be given. Write $g = \gamma\, g_0$ and $g' x d_0 = \varepsilon\, h_0\, d_0$ with rational points γ, ε and g_0, h_0 integral above p (Background 5). By the equivariance of the kernel under rational points and the invariance of the gauge under elements integral above p, it is enough to bound the largest coordinate of j applied to $\mathrm{jm}_k(\omega_k(g_0, h_0 d_0 y)\Psi)\, \xi'$, with d_0^{-1} acting in weight $(\varkappa, e_a)$, against the largest coordinate of $j(\xi')$. Now $\omega_k(g_0, h_0 \cdot y)$ with integral elements preserves $\mathrm{Std}_{\delta}(\mathrm{Hom}(I_n, I_{n'}))$ by Lemma 7.1(e), the remaining d_0 acts on the Schur jets through its diagonal images u_τ, which multiply the row i' by the product of the u_τ over the boxes of i'_τ and by the determinants to the power $e_{a,\tau}$, and the entries are bounded by the hypothesis.

(c) Take $x = d_0 = y = 1$, $u_\tau = 1$ in (b). For $f \in \mathrm{Std}_{\delta}(\mathrm{Hom}(I_n, I_{n'}))$ the homogeneity of the Schur jets (Background 5) gives

$$\mathrm{SJ}_{\varkappa}(T^{\mathrm{Hom}(I_n, I_{n'})}_{\mathrm{cper}(P,\mu)} f) = \Big(\frac{c}{\bar{\delta} d} \Big)^{\mathrm{box}(\varkappa, |n'|)} \mathrm{SJ}_{\varkappa}(T^{\mathrm{Hom}(I_n, I_{n'})}_{\bar{\delta} d \,\mathrm{per}(P,\mu)} f),$$

and the entries of the last matrix are jets along polynomials with integral coefficients, so Theorem C applies to the standard space $\mathrm{Hom}(I_n, I_{n'})$.

(d) Write $M = d'(Nd)^{\mathrm{box}(\varkappa, |n'|)} s$. By (c) with $\delta = N$,

$$d' f \in \mathrm{Raise}_k(1, 1; (|j(c)|_p / |j(Nd)|_p)^{\mathrm{box}(\varkappa, |n'|)}),$$

and by (a), multiplying by the integer $(Nd)^{\mathrm{box}(\varkappa, |n'|)} s$ gives the bound $|j(c)|_p^{\mathrm{box}(\varkappa, |n'|)} \le 1$. □

LEAN Kernel.smul_mem_raises · Parameters.mem_raises_of_jet_bound · Parameters.mem_raises_of_standardData · Parameters.mem_raises_of_dvd · quoted: weylGauge_embFin, weylGauge_mul, schur_diagonal_eq_prod, map_schur

8 The normal form at a prime that does not split

In this section $q = p^\nu$ will be the norm of a prime of $\mathbb{Q}(\sqrt{-d})$ above p: $\nu = 2$ if p is inert and $\nu = 1$ otherwise.

Lemma 8.1 (a valuation bound)**.** *Let p be a prime number and ν, Q natural numbers with $p^\nu = Q+1$. Let x, t, o, J, k be natural numbers with $\nu \neq 0$, $o \leq 1$ and $x+1+QJ = pt+o+Qk$. Then there is a natural number a with $|x+1|_p = p^{-a}$ such that: if $a > 1$ then $p < x+1$; either $a = 0$ or $k \equiv o + J \bmod p$; and $l\,(a-\nu) \leq k+t$ for every natural number $l \leq p$.*

Proof. Let $a = v_p(x+1)$. If $a > 1$ then $x+1 \geq p^2 > p$. If $a \geq 1$, reduce the equation modulo p, where $Q \equiv -1$: $-J \equiv o - k$. If $a > \nu$ then $p^a \leq x+1 \leq pt+1+Qk \leq p^\nu(k+t)+1$, so $k+t > p^{a-\nu}-1$ and therefore $k+t \geq p^{a-\nu} \geq p\,(a-\nu) \geq l\,(a-\nu)$. □

LEAN exists_norm_eq_inv_pow_bounds

Lemma 8.2 (substitution tangent to the identity)**.** *Let p be a prime number, $Z \in \mathbb{C}_p[[X]]$ and m a natural number such that X^{m+1} divides $Z - X$. Then for every natural number $k \leq m+1$ and every $g \in \mathbb{C}_p[[X]]$ the coefficient of X^k in $g(Z)$ is $g_k + g_1 \cdot (Z-X)_k$.*

Proof. Write $Z = X + X^{m+1}W$. Modulo X^{m+2} we have $Z^i \equiv X^i$ for $i \neq 1$ and $Z \equiv X + W_0 X^{m+1}$; so $g(Z) \equiv g + g_1 W_0 X^{m+1}$, and $(Z-X)_k$ is W_0 for $k = m+1$ and 0 for $k \leq m$. □

LEAN coeff_subst_tangent

Lemma 8.3 (only two degrees keep a denominator)**.** *Let $p \geq 13$ be a prime number, $\nu \in \{1,2\}$, Q a natural number with $p^\nu = Q+1$, and $\varpi \in \mathbb{C}_p$ with $|\varpi^Q|_p^2 = p^{-\nu}$ and $\varpi \neq 0$. Let m, n, Z, μ be power series in $\mathbb{C}_p[[X]]$ such that n' and Z' have coefficients in $\mathfrak{o}_p$; $n(0) = 0$; $Z_{l+1} = 0$ for every l not divisible by Q; $Z_{Q+1} = -\varpi^{-Q}$; $m_k = 0$ for $k \leq 2$; $m' = n\,Z'$; and*

$$\mu = \left(\varpi^{p+1}/p\right)^{-1} \cdot [m]_{p+1 \bmod Q}(\varpi X).$$

Then $|\mu_{p+1}|_p \leq 1$, $|\mu_{p+p^\nu}|_p \leq 1$, and the series $\varpi^{-Q}(\mu - \mu_{p+1}X^{p+1} - \mu_{p+p^\nu}X^{p+p^\nu})$ has coefficients in $\mathfrak{o}_p$.

Proof. Only the degrees $k = p+1+QI$, $I \geq 0$, occur in μ, and $\mu_k = p\,\varpi^{QI} m_k$. We show $|\mu_k|_p \leq 1$ for $I \in \{0,1\}$ (the degrees $p+1$ and $p+p^\nu$) and $|\mu_k|_p \leq |\varpi^Q|_p = p^{-\nu/2}$ for $I \geq 2$. From $m' = nZ'$, $n_0 = 0$ and the vanishing of the coefficients of Z,

$$k\,m_k = \sum_{i+1+QJ = p+QI} n_{i+1}\,(QJ+1)\,Z_{QJ+1}, \qquad 0 \leq J \leq I+1.$$

Here $|n_{i+1}|_p \leq p^c$ with $c = v_p(i+1)$, because n' is integral, and $|(QJ+1)Z_{QJ+1}|_p \leq 1$, because Z' is integral. With $a = v_p(k)$ the ultrametric inequality gives

$$\log_p |\mu_k|_p \;\leq\; a - 1 - \tfrac{\nu}{2} I + \max_{(i,J)} c.$$

Lemma 8.1, for $x = p+QI$ ($t = o = 1$, $J = 0$) and for $x = i$ ($t = 1$, $o = 0$), says: $a = 0$ unless $I \equiv 1 \bmod p$; $c = 0$ unless $I \equiv J \bmod p$; and $p\,(a-\nu) \le I+1$, $p\,(c-\nu) \le I+1$. $I = 0$: $a = 0$; either $J = 0$, $i+1 = p$, $c = 1$, or ($\nu = 1$) $J = 1$, $c = 0$: the exponent is at most 0. $I = 1$: $k = p + p^\nu$ and $a = 1$; the terms are $J = 0$ with $i+1 \equiv -1 \bmod p$, $c = 0$; $J = 1$ with $i+1 = p$, $c = 1$, where moreover $|(Q+1)Z_{Q+1}|_p = p^{-\nu/2}$; and ($\nu = 1$) $J = 2$ with $c = 0$. The exponent is at most $\max(-\nu/2,\, 1-\nu,\, -1/2) \le 0$. $2 \le I \le 4$: $a = 0$ since $I \not\equiv 1 \bmod p$; and $c \ge 1$ forces $J = I$, hence $i+1 = p$ and $c = 1$. The exponent is at most $-\nu I/2 \le -\nu/2$. $I \ge 5$: a and c are at most $\nu + (I+1)/p$, so the exponent is at most $2\nu + 2(I+1)/13 - 1 - \nu I/2 \le -\nu/2$; this is where $p \ge 13$ enters. □

LEAN `norm_coeff_le_of_deriv`

Proposition 8.4 (the formal parameter and the logarithm of theta)**.** *Let d be a prime number $\equiv 7 \bmod 8$, C a CM datum for d and (p, j) a p-adic place. Then there are a natural number ν, $\varpi \in \mathbb{C}_p$, $\varpi_0 \in \mathfrak{o}_p$ and power series S, E in one variable with coefficients in $\mathfrak{o}_p$ such that:*

(I) $\nu \ne 0$; $|\varpi|_p = p^{-\varrho_d(p)}$; $p^{-1} \le |\varpi_0|_p < 1$; $S(0) = 0$; $S - S^{p^\nu} = X + \varpi_0 E$; and $T(\varpi X) = \varpi S$ for some $T \in \mathfrak{o}_p[[\varphi_C]]$.

(II) If $\chi_d(p) \ne 1$ and $13 \le p$, there are $u_0, u_1, u_2 \in \mathfrak{o}_p$, a power series R with coefficients in $\mathfrak{o}_p$ and $\psi \in \mathbb{C}_p[[X]]$ such that $|\varpi_0|_p = p^{(\chi_d(p)-1)/2}$; either $\nu = 1$ and $p = d$, or $\nu = 2$, $u_0 = 0$ and $|u_1|_p = 1$; $\psi = u_0X^2 + u_1S^{p+1} + u_2S^{p+p^\nu} + \varpi_0 R$; $\psi(0) = 0$; and for every natural number n prime to p there are $N, \kappa \in \mathbb{C}_p$ with $|N|_p = 1$ and $|\kappa|_p = p^{(1-\chi_d(p))/2 - 1/(p-1)}$ such that for every $y \in \mathbb{C}$ with $ny \in \Lambda_C$ there is a power series V with coefficients in $\mathfrak{o}_p$ with

$$\mathrm{Tay}_j(U_y^{C,1}\theta_C)(\varpi X) = (\exp(\kappa\psi)\cdot V)(NX).$$

Here $\varrho_d(p)$ is $1/(p^2-1)$ if $\chi_d(p) = -1$, $1/(2(p-1))$ if $p = d$, and $1/(p-1)$ if $\chi_d(p) = 1$.

Proof. Step 1: the isogeny. Let $a = \sqrt{-d}$ if $p = d$ and $a = p$ otherwise, and let $q = p^\nu$ be the norm of a prime above p. Then $|j(a)|_p^2 = p^{-s}$ with $s = 1$ if $p = d$ and $s = 2$ otherwise, $\varrho_d(p) = s/(2(q-1))$, and in the two non-split cases $a\overline{a} = q$ is odd.

Step 2: a parameter that commutes with roots of unity. Let $\hat\varphi = \mathrm{Tay}_j(\varphi_C)$ and let $\zeta \in \mathbb{C}_p$ be a primitive root of unity of order $q-1$. By (Q6), $(\zeta^i, \zeta^{i(p+1)}) \in \mathcal{N}_d$, so $\hat\varphi(\zeta^i X) \in \mathfrak{o}_p[[\varphi_C]]$, and $F \mapsto F(\zeta^i X)$ maps $\mathfrak{o}_p[[\varphi_C]]$ to itself. Hence

$$T = [\hat\varphi]_{1 \bmod q-1} = \frac{1}{q-1}\sum_{i<q-1} \zeta^{-i}\hat\varphi(\zeta^i X) \in \mathfrak{o}_p[[\varphi_C]],$$

$q-1$ being a unit of $\mathfrak{o}_p$. As $T(0) = 0$, $T'(0) = 1$, (Q5) applies: let Z be the inverse series of T. From $T(\zeta X) = \zeta T$ we get $Z(\zeta X) = \zeta Z$, so $Z_{l+1} = 0$ unless $q-1$ divides l; in particular X^q divides $Z - X$. Also Z' is integral: $T'(Z)$ is integral with constant term 1 and $T'(Z)Z' = 1$.

Step 3: the scale. The series $g = T(j(a)Z)$ is integral, because $T(j(a)X) \in \mathfrak{o}_p[[\varphi_C]]$ by (Q6), and $Z(g) = j(a)Z$. By Step 2, $T_1 = 1$ and $T_k = 0$ for $1 < k < q$. So Lemma 8.2 gives $g_1 = j(a)$, $g_k = 0$ for $1 < k < q$ and, using $T_q = -Z_q$ (from $T(Z) = X$),

$$g_q = j(a)^q T_q + j(a)Z_q = j(a)(1 - j(a)^{q-1})Z_q, \qquad \text{so that} \quad |g_q|_p = |j(a)|_p \cdot |Z_q|_p.$$

We claim $|g_q|_p = 1$. It is at most 1. In the non-split cases (Q8)(ii) gives a coefficient of absolute value 1 in a degree $\le q$; as $g_0 = 0$, $|g_1|_p = |j(a)|_p < 1$ and $g_k = 0$ for $1 < k < q$, the degree is q. In the split case ($q = p$, $a = p$), if $|g_p|_p < 1$ then $|p\,(Z - X)_p|_p < 1$, and then every $F \in \mathfrak{o}_p[[\varphi_C]]$ has $|(F')_{p-1}|_p = |pF_p|_p < 1$, since by Lemma 8.2 $pF_p = p(F(Z))_p - F_1\,p(Z - X)_p$ with $F(Z)$ integral; by (Q8)(i) this gives $\chi_d(p) \neq 1$, a contradiction. In particular $Z_q \neq 0$. Now choose $\varpi \in \mathbb{C}_p$ with $\varpi^{q-1} = -Z_q^{-1}$ and put $\varpi_0 = \varpi^{q-1}$. Then $|\varpi_0|_p = |Z_q|_p^{-1} = |j(a)|_p = p^{-s/2}$, which is $p^{(\chi_d(p)-1)/2}$ in the non-split cases, and $|\varpi|_p = p^{-s/(2(q-1))} = p^{-\varrho_d(p)}$.

Step 4: the series S. Let $W = \varpi^{-1} Z(\varpi X)$. Its coefficients vanish except in the degrees $1 + (q-1)l$; $W_1 = 1$, $W_q = \varpi_0 Z_q = -1$, and for $l \ge 2$ the coefficient is $\varpi_0^l Z_{1+(q-1)l}$, where $|Z_{1+(q-1)l}|_p \le p^b$, $b = v_p(1 + (q-1)l)$, because Z' is integral. Lemma 8.1 (with $x = (q-1)l$, $t = 0$, $o = 1$, $J = 0$, $k = l$) says that $b = 0$ unless $l \equiv 1 \bmod p$, and that $p\,(b - \nu) \le l$. In each of the three cases $(s, \nu) = (2, 2)$, $(1, 1)$, $(2, 1)$ of Step 1 this gives $2b \le s(l-1)$, that is $|\varpi_0^{l-1} Z_{1+(q-1)l}|_p \le 1$. So $W = X - X^q - \varpi_0 E'$ with E' integral. Let S be the inverse series of W; it is integral with $S(0) = 0$, and $S - S^q = X + \varpi_0 E$ with $E = E'(S)$. By definition $Z(\varpi X) = \varpi W$; substituting S gives $Z(\varpi S) = \varpi X$, and applying T gives $T(\varpi X) = \varpi S$. This proves (I).

Step 5: the logarithm. Assume now $\chi_d(p) \neq 1$ and $p \ge 13$. Let L, L_2 be as in (Q6) and $L_1 = L - \frac{1}{2} L_2 X^2$, so that $L_1'' = \mathrm{Tay}_j(\xi_C)$ and $L_1 = O(X^3)$. Apply Lemma 8.3 with $Q = q - 1$ to $m = L_1(Z)$ and $n = L_1'(Z)$: $m' = nZ'$; $n' = \mathrm{Tay}_j(\xi_C)(Z)\,Z'$ is integral by (Q5); $n(0) = 0$; $m_k = 0$ for $k \le 2$ by Lemma 8.2; and Z has the required coefficients by Steps 2 and 3. With $\kappa_0 = \varpi^{p+1}/p$ this gives $u_1, u_2 \in \mathfrak{o}_p$ and an integral R_0 with

$$\kappa_0^{-1}[L_1(Z)]_{p+1 \bmod q-1}(\varpi X) = u_1 X^{p+1} + u_2 X^{p+q} + \varpi_0 R_0.$$

Put $\psi = \kappa_0^{-1}[L]_{p+1 \bmod q-1}(\varpi X)$. Taking the part in a residue class of degrees commutes with substituting T, because $T(\zeta X) = \zeta T$; and $T(\varpi X) = \varpi S$. Hence $\psi = u_0 X^2 + u_1 S^{p+1} + u_2 S^{p+q} + \varpi_0 R$ with $R = R_0(S)$ and with $u_0 = \kappa_0^{-1} \frac{1}{2} L_2 \varpi^2$ if $2 \equiv p + 1 \bmod q - 1$, that is if $\nu = 1$, and $u_0 = 0$ if $\nu = 2$. In the case $\nu = 1$, $p = d$: $u_0 = \frac{1}{2} L_2\, p/\varpi_0$, and $|u_0|_p = |L_2|_p p^{-1/2} \le 1$ since $|j(\sqrt{-d}) L_2|_p \le 1$ and $|j(\sqrt{-d})|_p = p^{-1/2}$. In the case $\nu = 2$: $p + 1 < q$, so $u_1 = p\,(L_1)_{p+1} = (\mathrm{Tay}_j(\xi_C))_{p-1}/(p+1)$, and $|u_1|_p \ge 1$ by (Q8)(i), whose hypothesis holds because $(Z - X)_p = 0$ (argue as in Step 3). So $|u_1|_p = 1$.

Step 6: torsion translates. Let n be prime to p, $\delta = n(q-1)$, $N = \delta^{-1}$ (a unit of $\mathfrak{o}_p$) and $\kappa = -\delta^2 \kappa_0$; then $|\kappa|_p = p\,|\varpi|_p^{p+1} = p^{(1-\chi_d(p))/2 - 1/(p-1)}$ in both non-split cases. Let $ny \in \Lambda_C$. By (Q7) for the integer δ, $\mathrm{Tay}_j(U_y^{C,1}\theta_C)(\delta X) = \exp(-\delta^2 L) \cdot u$ with $u \in \mathfrak{o}_p[[\varphi_C]]$. By (Q6) for the pairs $(\zeta^{-i}, \zeta^{-i(p+1)})$, rotated by ζ^i, the series $\exp(\zeta^{-i(p+1)} L(\zeta^i X) - L)$ lie in $\mathfrak{o}_p[[\varphi_C]]$; their product over $i < q - 1$ is

$$\Pi = \exp((q-1)([L]_{p+1 \bmod q-1} - L)) \in \mathfrak{o}_p[[\varphi_C]], \quad \text{so that}$$
$$\exp(-\delta^2 L)\, u = \exp(-\delta^2 [L]_{p+1 \bmod q-1}) \cdot \Pi^{n^2(q-1)} u.$$

Let $V = (\Pi^{n^2(q-1)} u)(\varpi X)$. It is integral: $(\Pi^{n^2(q-1)} u)(Z)$ is integral by (Q5), and V is this series at $T(\varpi X) = \varpi S$. Rescaling by ϖ and by N, $\mathrm{Tay}_j(U_y^{C,1}\theta_C)(\varpi X) = (\exp(\kappa\psi) V)(NX)$. □

LEAN CMLattice.exists_formal_parameter

Lemma 8.5 (two computations in characteristic p)**.** *Let p be a prime number, Q a commutative ring of characteristic p and J a finite set.*

(a) Let B, ν be natural numbers and b a proposition such that $\nu = 1$ and b holds, or $\nu = 2$. Let $t_i, L_i \in Q$ $(i \in J)$ with $t_i - t_i^{p^\nu} = L_i$ and $t_i^{p^l} = 0$ for all $l \geq B$. If $\sum_i L_i^{p^e} L_i = 0$ for every e such that e is odd or b holds, then $\sum_i t_i^{p+1} = 0$ and $\sum_i t_i^{p+p^\nu} = 0$.

(b) Let Φ be a finite set, I a set, $r_{v,i} \in Q$ for $(v,i) \in \Phi \times J$, and t_i $(i \in J)$ power series over Q in the variables $X_{(v,c)}$, $(v,c) \in \Phi \times I$, with $t_i(0) = 0$. Let $x \in \Phi$ and $c \in I$ with $t_i - t_i^{p^2} = \sum_v r_{v,i} X_{(v,c)}$ for all i. Then for every exponent β: the coefficient at X^β of $\sum_i t_i^p r_{x,i}$ is nonzero if and only if there are $l \in \mathbb{N}$ and $y \in \Phi$ with $\sum_i r_{y,i}^{p^{2l+1}} r_{x,i} \neq 0$ and $\beta = p^{2l+1} \cdot 1_{(y,c)}$.

Proof. (a) Telescoping, $t_i = \sum_{l<B} L_i^{p^{\nu l}}$. So $\sum_i t_i^p t_i$ and $\sum_i t_i^p t_i^{p^\nu}$ are sums of terms $\sum_i L_i^{p^a} L_i^{p^b}$ with $a - b = 1 + \nu(l - l')$ for the first sum and $a - b = 1 + \nu(l - l' - 1)$ for the second $(l, l' < B)$; these differences are odd if $\nu = 2$. Such a term is the $p^{\min(a,b)}$-th power of $\sum_i L_i^{p^{|a-b|}} L_i$, which vanishes. (b) With $\ell_i = \sum_v r_{v,i} X_{(v,c)}$, telescoping gives $t_i \equiv \sum_{l<B} \ell_i^{p^{2l}}$ up to terms of order $\geq p^{2B}$, and $t_i^p \equiv \sum_l \sum_v r_{v,i}^{p^{2l+1}} X_{(v,c)}^{p^{2l+1}}$. So $\sum_i t_i^p r_{x,i} = \sum_{l,v} (\sum_i r_{v,i}^{p^{2l+1}} r_{x,i}) X_{(v,c)}^{p^{2l+1}}$, and distinct pairs (l,v) give distinct monomials. □

LEAN `sum_pow_eq_zero_of_null` · `coeff_sum_frob_ne_zero_iff` · quoted: `eq_sum_pow_of_sub_pow_eq`

Lemma 8.6 (Frobenius pairings of a tight frame)**.** *Let $P = (N_0, d)$ be parameters, μ a natural number, p a prime number with $\chi_{K(P)}(p) \neq 1$ and $j\colon \mathbb{C} \to \mathbb{C}_p$ a ring homomorphism. Let J be a finite set, $w_i \in E(P,\mu)$ $(i \in J)$ and M a natural number such that (w_i) with multiplicities 1 is a tight frame of length M, and let $\beta \in \mathfrak{o}_p$. Then there are $z_{v,i} \in \mathfrak{o}_p$, for $v \in \mathrm{Pl}(P,\mu)$ and $i \in J$, with $z_{v,i} = \beta \cdot j(\overline{\iota_v(w_i)})$, such that for every commutative ring Q of characteristic p, every ring homomorphism $g\colon \mathfrak{o}_p \to Q$ that vanishes on the elements of absolute value at most $p^{(\chi_{K(P)}(p)-1)/2}$, every natural number e that is odd or for which $p = d$, and all $v, v' \in \mathrm{Pl}(P,\mu)$:*

$$\sum_{i \in J} g(z_{v,i})^{p^e}\, g(z_{v',i}) = \begin{cases} g(\beta)^{p^e} g(\beta) \cdot M & \text{if } [-p^e]_{P,\mu} \cdot v = v', \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $N = N_1(P,\mu)$; it is prime to p (I4), so $\beta = N\beta'$ with $\beta' \in \mathfrak{o}_p$. The elements $x_i = N w_i$ are integers of $E(P,\mu)$ (I1). Then $z_{v,i} = \beta' j_v(x_i)$ lies in $\mathfrak{o}_p$. Let $\phi_v\colon x \mapsto g(j(\iota_v(x)))$ on the integers of $E(P,\mu)$. By (I6), $\phi_v(\mathrm{gal}(u)x) = \phi_v(\overline{x})^{p^e}$ for $u = [-p^e]_{P,\mu}$ and all x: the condition on $\sqrt{-d}$ holds because $-\chi_{K(P)}(p)^e = 1$ for e odd and $\chi_{K(P)}(p) = -1$, and because for $p = d$ the element $j(\sqrt{-d})$ has absolute value $p^{-1/2}$ and is killed by g. Hence

$$\sum_i g(z_{v,i})^{p^e} g(z_{v',i}) = g(\beta')^{p^e} g(\beta') \cdot g\Big(j\Big(\sum_i \iota_{u\cdot v}(x_i)\, \overline{\iota_{v'}(x_i)}\Big)\Big),$$

and the inner sum is $N^2 M$ if $u \cdot v = v'$ and 0 otherwise, by the frame identity. As $g(N)^{p^e} = g(N)$ in characteristic p (N is an integer) and $\beta = N\beta'$, the factor $g(\beta')^{p^e} g(\beta') \cdot g(N^2 M)$ is $g(\beta)^{p^e} g(\beta) \cdot M$, which is the claim. □

LEAN `Parameters.exists_frame_rows_frob`

Proof of Theorem D. Apply Proposition 8.4 to $C = \mathcal{E}(P)$; since $\chi_{K(P)}(p) \neq 1$ we have $p \geq 13$ (I4), so (II) holds. This gives ν, ϖ, ϖ_0, S, E and u_0, u_1, u_2, R, ψ. Let I, $r \leq 1$ and $\Theta \in \mathrm{ThProd}_{P,\mu}(1, r; I)$ be given, with frame $(w_i)_{i\in J}$ of length M, numbers n, b and translations $y_{(i,k)}$; so $p \nmid M$, $p \nmid n$, $|j(b)|_p = r$, $n\, y_{(i,k)} \in \Lambda_{\mathcal{E}(P)}$. Let N, κ and $V_{(i,k)}$ be as in (II) for this n. Lemma 8.6 with $\beta = Nj(b)$ gives $z_{v,i}$. Put

$$\ell_{(i,k)} = \sum_v z_{v,i}\, z_{(v,k)}, \qquad \Psi = \sum_{(i,k)} \psi(\ell_{(i,k)}), \qquad V = \prod_{(i,k)} V_{(i,k)}(\ell_{(i,k)}),$$

where $z_{(v,k)}$ are the variables. Substituting in the definition of the theta product, $\Theta(\varpi z) = \prod_{(i,k)} (\exp(\kappa\psi) V_{(i,k)})(\ell_{(i,k)}) = \exp(\kappa\Psi)\, V$, and $\Psi(0) = 0$.

(a) Let B, a family Z as in Definition 2.5 with couplings in the ideal (p, ϖ_0) for $e \in S_B$, and an exponent β_0 of degree at most B be given. Let Q be the quotient of $\mathfrak{o}_p[[T]]$ by the ideal of the series whose coefficients at the exponents $\leq \beta_0$ have absolute value at most $|\varpi_0|_p$; it has characteristic p. In Q let $t_{(i,k)}$ and $L_{(i,k)}$ be the classes of $S(\ell_{(i,k)}(Z))$ and of $\ell_{(i,k)}(Z) = \sum_v z_{v,i} Z_{(v,k)}$. Then $t - t^{p^\nu} = L$, and $t^{p^l} = 0$ for $p^l > B$. By Lemma 8.6 (for the map $\mathfrak{o}_p \to Q$, which kills the elements of absolute value $\leq |\varpi_0|_p$),

$$\sum_{(i,k)} L_{(i,k)}^{p^e} L_{(i,k)} = \beta^{p^e+1} M \sum_{(v,k)} Z_{(v,k)}^{p^e} Z_{(\mathrm{pr}_e(v),k)}$$

for e odd or $p = d$; this is 0 in Q for $e \in S_B$ by the hypothesis on Z, and for $e > \log_p B$ because then $L^{p^e} = 0$. Lemma 8.5(a), with the proposition $p = d$, gives $\sum t^{p+1} = \sum t^{p+p^\nu} = 0$. Moreover $u_0 \sum L^2 = 0$: either $u_0 = 0$, or $p = d$ and this is the coupling with $e = 0 \in S_B$. So the class of $\Psi(Z)$ vanishes: its coefficients at exponents $\leq \beta_0$ have absolute value at most $|\varpi_0|_p$. Then the coefficient at β_0 of $\exp(\kappa\Psi(Z)) = \sum_l \kappa^l \Psi(Z)^l / l!$ is integral, since $|\kappa\varpi_0|_p^l = p^{-l/(p-1)} \leq |l!|_p$; the same holds at every exponent $\leq \beta_0$, and $V(Z)$ is integral. So the coefficient of $\Theta(\varpi Z)$ at β_0 is integral.

(b) Now $\chi_{K(P)}(p) = -1$ and $r = 1$; so $\nu = 2$, $u_0 = 0$, $|u_1|_p = 1$, $|\kappa|_p = p^{1-1/(p-1)}$, and β is a unit. Reduce modulo $\mathfrak{m}_p$ and let $t_{(i,k)}$ be the reduction of $S(\ell_{(i,k)})$; then $t_{(i,k)} - t_{(i,k)}^{p^2} = \sum_v \overline{z}_{v,i} z_{(v,k)}$ with the residues $\overline{z}_{v,i}$, and for a variable $x = (x_1, x_2)$

$$\frac{\partial \overline{\Psi}}{\partial z_x} = \overline{u}_1 \sum_{i\in J} t_{(i,x_2)}^{p}\, \overline{z}_{x_1,i},$$

because $(p+1)t^p \partial t = t^p \partial t$, the derivative of t^{p^2} is 0, $p + p^2 \equiv 0$ and $\varpi_0 \equiv 0$. By Lemma 8.5(b) the coefficient at z^{β_0} is nonzero exactly for $\beta_0 = p^{2l+1} 1_{(y,x_2)}$ with $\sum_i \overline{z}_{y,i}^{\,p^{2l+1}} \overline{z}_{x_1,i} \neq 0$, and by Lemma 8.6, with β and M units, this means $[-p^{2l+1}]_{P,\mu} \cdot y = x_1$. □

LEAN `Parameters.thetaProducts_normal_form`

9 Divisibility of Schur jets

Lemma 9.1 (frames that kill the couplings)**.** *Let p be a prime number, Φ and n finite sets, $\mathrm{pr}\colon \mathbb{N}\times\Phi\to\Phi$ a map, $S\subset\mathbb{N}$ a finite set, N a natural number, $\varkappa\colon \Phi\times\mathbb{N}\to\mathbb{N}$ a function, $\Xi\subset\Phi$ a finite set, $Y_{\tau,a,k}\in\mathfrak{o}_p$ for $\tau\in\Phi$, $a\in n$, $k\in\mathbb{N}$, and $\varpi_0\in\mathfrak{o}_p$. Then there are square matrices $M_{\tau,h}$ of size h over $\mathfrak{o}_p$, for $\tau\in\Phi$ and $h\in\mathbb{N}$, such that*

(i) $|\varpi_0|_p^{\sum_{e\in S}\sum_{\tau\in\Phi}\mathrm{cc}_p(\varkappa,N;\Xi;\tau,\mathrm{pr}_e(\tau);e)} \le \prod_{\tau\in\Phi}\prod_{h<N}|\det M_{\tau,h}|_p^{\varkappa(\tau,h)}$*;*

(ii) for every commutative ring R of characteristic p, every ring homomorphism $g\colon\mathfrak{o}_p\to R$ with $g(\varpi_0)=0$, every $e\in S$, $\tau\in\Phi$, heights h, h' and rows $i<h$, $i'<h'$: unless $h=1$ and ($\tau\in\Xi$, or $h'=1$ and $\mathrm{pr}_e(\tau)\in\Xi$),

$$g\Big(\sum_{a\in n}\Big(\sum_{k<h}(M_{\tau,h})_{i,k}Y_{\tau,a,k}\Big)^{p^e}\cdot\sum_{k<h'}(M_{\mathrm{pr}_e(\tau),h'})_{i',k}Y_{\mathrm{pr}_e(\tau),a,k}\Big)=0.$$

Proof. Fix p^e-th roots $\varpi_0^{1/p^e}\in\mathfrak{o}_p$. For a coupling (e,τ) with partner $\tau'=\mathrm{pr}_e(\tau)$ and a pair of heights (h,h') the pairing in (ii) can be made divisible by ϖ_0 from either end: by a condition on the rows of $M_{\tau,h}$ that costs ϖ_0^{1/p^e} (its p^e-th power appears), or by one on the rows of $M_{\tau',h'}$ that costs ϖ_0. Three devices are used. (1) Multiply all of $M_{\tau,h}$ by ϖ_0^{1/p^e}, or all of $M_{\tau',h'}$ by ϖ_0: this kills the couplings with all rows at the other end, and costs h/p^e, respectively h', in the valuation of the determinant, per column. (2) If $h'=1$, ask only that ϖ_0^{1/p^e} divide the pairings of the rows of $M_{\tau,h}Y_\tau$ with the single row of $Y_{\tau'}$: one linear condition on the rows, which by the input below costs $1/p^e$ per column. (3) If $h=1$, ask that ϖ_0 divide the pairings of the rows of $M_{\tau',h'}Y_{\tau'}$ with the p^e-th power of the single row of Y_τ: cost 1 per column. For $h,h'\neq1$ use (1) at the cheaper end, which costs $\min(B_\tau/p^e,B_{\tau'})$ in all; for $h\neq1$, $h'=1$ use (2), cost r_τ/p^e; for $h=1$ and $\tau\notin\Xi$ use (1) on the 1×1 matrix $M_{\tau,1}$ or (3) on all $M_{\tau',h'}$, whichever is cheaper, cost $\min(t_\tau/p^e,r_{\tau'}+t_{\tau'})$; the couplings exempted in (ii) are not treated. The total is the coupling count. All conditions on one matrix $M_{\tau,h}$ are imposed at once by the following input, applied to the diagonal matrix of the required roots of ϖ_0: *(unit 149) if A is a square matrix over a valuation ring and Λ a submodule containing the image of A, then for vectors $\Phi_1,\dots,\Phi_h$ there is a square matrix M of size h with $\det M$ dividing $\det A$ and all $\sum_k M_{i,k}\Phi_k$ in Λ.* In (ii) one raises the divisibility by ϖ_0^{1/p^e} to the power p^e, which in characteristic p is additive. The bookkeeping that decides which device serves which coupling is a predicate of the formal text; that its costs add up to the coupling count is checked there by a case analysis, which I read but did not replay (see §12). □

LEAN `exists_null_coupling_frames` · unit 149: `IndexDvd.of_det_dvd`

Proof of Theorem E. Step 1: from Schur entries to products of minors. For a matrix y_τ of size $n\times N$ (rows $a\in n$, columns $k<N$) the row i' of the Schur matrix satisfies $\sum_i S_\varkappa(z)_{i',i}\,b_i(y)=b'_{i'}(zy)$, where $b_i(y)$ are products of minors of y (times powers of determinants) and

$$b'_{i'}(zy)=\prod_\tau\prod_{h<N}\prod_{l<\varkappa(\tau,h)}\det\big((z_\tau y_\tau)_{i'_\tau(h,l,s),\,k}\big)_{s,k<h}$$

is the product of minors prescribed by i' (quoted: this is the identity that defines the Schur matrix, and its coefficients are integers). Two quoted facts finish the reduction. *(G)* For a polynomial f over $\mathbb{C}_p$ in finitely many variables there is a point

y_0 with coordinates in $\mathfrak{o}_p$ such that $|c|_p \le |f(y_0)|_p$ for every coefficient c of f. *(C)* If a polynomial f in the entries of y, over a commutative ring, is a combination $\sum_i x_i\, b_i(y)$ of the products $b_i(y)$, and every coefficient of f lies in an additive subgroup A of the ring, then every x_i lies in A. Regard $b'_{i'}(zy)$ as a polynomial in y whose coefficients are polynomials in z. Taking the jet of Θ along each coefficient gives a polynomial over $\mathbb{C}_p$ in y, whose value at a point y is $\mathrm{jet}_{b'_{i'}(zy)}(\Theta)$; apply (G) to it, which gives y_0. Then apply (C) to $f = b'_{i'}(zy)$, to $x_i = S_\varkappa(z)_{i',i}$ and to the set A of the polynomials Q in z with $|\,\mathrm{jet}_Q(\Theta)|_p \le |\,\mathrm{jet}_{b'_{i'}(zy_0)}(\Theta)|_p$, which is an additive group because the jet is additive in Q and the absolute value is ultrametric. Hence

$$|\,\mathrm{SJ}_\varkappa(\Theta)_{i',i}|_p \;\le\; |\,\mathrm{jet}_{b'_{i'}(zy_0)}(\Theta)|_p,$$

and it suffices to bound the jet of Θ along the polynomial $z \mapsto b'_{i'}(zy_0)$. From now on y is this point y_0.

Step 2: the null family. Let $M_{\tau,h}$ be the matrices of Lemma 9.1 for $Y = y$ and ϖ_0. Take new variables $X_{(h,k),(\tau,b)}$, for $k < h < N$, $\tau \in \Phi$, $b \in n'$, and the linear forms

$$Z_{(\tau,b,a)} = \sum_{k<h<N} p^{\,g(\tau,h,b)} \Big(\sum_{k'<h} (M_{\tau,h})_{k,k'}\, y_{\tau,a,k'} \Big) X_{(h,k),(\tau,b)}.$$

The coupling of exponent $e \in S$ of this family is, modulo p, a combination of the pairings of Lemma 9.1(ii) with coefficients $p^{\,g(\tau,h,b)p^e + g(\tau',h',b)}$, $\tau' = \mathrm{pr}_e(\tau)$. The pairings that are not exempted vanish modulo ϖ_0; for the exempted ones the hypothesis on the charges makes the coefficient divisible by p. So the family is null, and the hypothesis of the theorem says that the coefficients of degree at most B of $\Theta(\varpi Z)$ are integral.

Step 3: the jet. Multiplying the columns of a minor by $M_{\tau,h}$ and its row at b by $\varpi p^{g(\tau,h,b)}$ multiplies it by $\det M_{\tau,h} \prod_b \varpi p^{g(\tau,h,b)}$. So

$$\prod_{\tau,h} (\det M_{\tau,h})^{\varkappa(\tau,h)} \cdot \varpi^B p^m \cdot b'_{i'}(zy) = \mathsf{D}((\varpi Z)),$$
$$\mathsf{D}(X) = \prod_{\tau,h,l} \det\big(X_{(h,k),(\tau,i'_\tau(h,l,s))}\big)_{s,k<h},$$

where the right side is read through the transpose of the linear substitution; by the chain rule for jets (quoted), the jet of Θ along the left side is the jet of $\Theta(\varpi Z)$ along D. Now D is a sum of monomials $\pm X^\beta$ of degree B, and the jet along X^β is $\beta!$ times a coefficient of degree B, which is integral. The exponent β has at most $|\Phi|N^3$ entries, each at most B, and $v_p(l!) \ge l/(p-1) - (\lfloor \log_p B \rfloor + 1)$ for $l \le B$ by Legendre's formula; so $|\beta!|_p \le p^{\,-B/(p-1) + |\Phi|N^3(\lfloor \log_p B \rfloor + 1)}$. By Lemma 9.1(i), $|\varpi|_p = p^{-\rho}$ and $|\varpi_0|_p = p^\omega$, the factor on the left has absolute value at least $p^{\,\omega \sum \mathrm{cc} - \rho B - m}$. Dividing gives the theorem. □

LEAN `norm_schurJet_le` · quoted: `sum_schur_smul_nablaBasis`, `exists_val_coeff_le_val_eval`, `PolyRep.coord_mem_of_coeff_mem`, `jet_subst_linearSeries`

10 Sharpness

Proposition 10.1 (a limit to the divisibility of Taylor coefficients)**.** *Let d be a prime number $\equiv 7 \bmod 8$, C a CM datum for d, (p,j) a p-adic place and n a natural number. Then there is $F \in \mathfrak{o}_p[[\varphi_C]]$ with*

$$\left(p^{\,\varrho_d(p)-1/(p-1)}\right)^n \;\le\; |n!\cdot F_n|_p.$$

Proof. Let $c_0 = p^{\varrho_d(p)-1/(p-1)}$ and, for $k \in \mathbb{N}$, let $c(k)$ be the largest value of $|k!F_k|_p/c_0^k$ for $F \in \mathfrak{o}_p[[\varphi_C]]$; by (Q5) it is attained at a power of $\hat\varphi = \mathrm{Tay}_j(\varphi_C)$. *$c$ is submultiplicative.* By (Q5), $\hat\varphi(X+Y)^l$ is an integral series in $\hat\varphi(X)$ and $\hat\varphi(Y)$; its coefficient at X^nY^k is $\binom{n+k}{n}(\hat\varphi^l)_{n+k}$, and it is bounded by the largest product of an n-th coefficient of a power of $\hat\varphi$ with a k-th one. Multiplying by $n!\,k!$ gives $c(n+k) \le c(n)c(k)$. Suppose the proposition fails for n: $c(n) < 1$. Then $c(k) \to 0$, so there is l_0 with $c(k) < |\varpi|_p$ for $k \ge l_0$, where ϖ, ϖ_0, S, T, ν are those of Proposition 8.4(I). From $T(\varpi X) = \varpi S$, $|T_k|_p \le c(k)c_0^k/|k!|_p$, $|k!|_p \ge p^{-k/(p-1)}$ and $|\varpi|_p = p^{-\varrho_d(p)}$ we get $|S_k|_p = |\varpi|_p^{k-1}|T_k|_p \le c(k)/|\varpi|_p < 1$ for $k \ge l_0$. So the reduction $\overline{S}$ of S modulo $\mathfrak{m}_p$ is a polynomial. But $\overline{S} - \overline{S}^{p^\nu} = X$, because $|\varpi_0|_p < 1$, and no polynomial satisfies this: the degree of the left side is $p^\nu \deg \overline{S} \ne 1$. □

LEAN `CMLattice.katz_divisibility_limit`

Lemma 10.2 (the live sum)**.** *(a) Let d and p be prime numbers with $\chi_d(p) \ne 1$ and c a real number. Then*

$$\sum_{\substack{e\ge 1\\ e\ \text{odd or}\ p=d}} \frac{c}{p^e} = \frac{2}{1-\chi_d(p)}\cdot c\cdot\Big(\frac{1}{p-1} - \varrho_d(p)\Big),$$

the series being convergent.

(b) Let D be admissible data for parameters $P = (N_0, d)$, p a prime number with $\chi_{K(P)}(p) \ne 1$, y a real number and $\tau \in \mathrm{Pl}(D)$. Then the family of the terms $\big(y/p^e - w_D([-p^e]_D\cdot\tau)\big)_+$, over the $e \ge 1$ that are odd or for which $p = d$ (and 0 for the other e), is summable; so the sum in $\mathrm{ask}_{D,p}(y,\tau)$ is an ordinary sum.

(c) Let D be admissible data for parameters $P = (N_0, d)$, p and e natural numbers and $\tau \in \mathrm{Pl}(D)$. If $[-p^e]_D\cdot\tau$ and τ have the same class modulo N_0, then $p \ne d$.

(d) Let D be admissible data, $\tau \in \mathrm{Pl}(D)$, and h, a natural numbers. Then, with $k = k(D)$,

$$\begin{cases} a & h = 1 \\ \varkappa_D(\tau,h) & h \ne 1 \end{cases} \;\le\; \begin{cases} a & h = 1 \\ 0 & h \ne 1 \end{cases} + \begin{cases} \mathrm{wt}_D(1,\tau) & h = k \\ 0 & h \ne k \end{cases} + \mathrm{gap}(D).$$

Proof. (a) In both cases, $\chi_d(p) = -1$ and $\chi_d(p) = 0$ (then $p = d$), the term at $e \ge 0$ is $\frac{c}{1-\chi_d(p)}(p^{-e} - (\chi_d(p)/p)^e)$; sum the two geometric series. (b) The terms are bounded by $|y|/p^e$, since the nominal weight is non-negative; use (a). (c) If $p = d$, then $-d^e$ is prime to $N_1(D)$, the class of $[-d^e]_D\cdot\tau$ is $-d^e$ times that of τ, and equal classes would give $d^e \equiv -1 \bmod N_0$, against (I4). (d) Unfold Definition 2.9. □

LEAN `hasSum_live_inv_pow` · `Data.summable_localAsk` · `Data.ne_d_of_placeClass_eq` · `Data.κ_le`

Inputs from unit 152 on the sizes of the data. For admissible data D with components x, x_1, b and $c = \lceil 60b/k(D) \rceil$: **(S1)** $a(D)/x_1 \le w_D(\tau) \le a(D)/x$ and $w_D(\tau) + x^{42} \le 2b$ for all τ; $x^{42} + 1 \le a(D)/x_1$; $x^{42} \le c$; $d \le k(D)$ and $0 < k(D)$. **(S2)** For a prime $p \le k(D)$ and $e \le \mu(D)$: $2(b_2(D) + a(D)/x_1) \le c/(p \cdot 2 \cdot 2^e)$; and for a prime p with $4 \le p \le k(D)$ and $e \ge \mu(D)$: $2b/p^e \le c/(p \cdot 2 \cdot 2^e)$. Here $b_2(D)$ is the least multiple of $N_1(D)$ that is at least $a(D)/x_1$. **(S3)** The thick places have class -1 modulo N_0. *Three-term condition*: if $0 < i < l \le 324\,\mu(D)$ and τ, τ_1, τ_2 are thick places with $\phi(\tau_1/\tau)^l = \phi(\tau_2/\tau)^i$, then $\phi(\tau_1/\tau) = 1$, where ϕ is the class modulo $2^{\mu(D)+1}$. **(S4)** (background library) If $|u| > 1$, $o > 0$ and $u^o \equiv 1 \bmod 2^{s+1}$ for integers u, o, s, then $2^s \le (|u| + 1)\, o$.

LEAN unit 152: `Data.nominalWeight_mem_Icc`, `Data.nominalWeight_add_le`, `Data.le_of_hasLogBound`, `Data.nat_le_of_hasLogBound`, `Data.k_pos`, `Data.placeClass_of_thick`, the field `three_term` of `Data` · library: `two_pow_le_of_pow_eq_one`

Proposition 10.3 (a Frobenius packing for the nominal weights)**.** *Let D be admissible data for parameters $P = (N_0, d)$, with component x, and let $n\colon \mathrm{Pl}(D) \to \mathbb{N}$ satisfy $w_D(\tau) \le n_\tau \le w_D(\tau) + x^{42}$ for every τ. Let p be a prime number with $\chi_{K(P)}(p) \ne 1$. Then there are functions $i, y\colon \mathrm{Pl}(D) \to \mathbb{N}$ such that (i, y) is a Frobenius packing of n at p and, with*

$$L = \mathrm{forced}(D, p) + \sum_{\tau \in \mathrm{Pl}(D)} v_p(\mathrm{dl}_D(n_\tau)),$$

either $\chi_{K(P)}(p) = -1$, $i_\tau = 1$ for all τ and $\sum_\tau (v_p(n_\tau!) - y_\tau) \le L$, or

$$\sum_{\tau \in \mathrm{Pl}(D)} \Big((\tfrac{1}{p-1} - \varrho_d(p)) n_\tau + 4\, b_2(D) - y_\tau \Big) \;\le\; L.$$

Proof. Write $k = k(D)$, $c = \lceil 60b/k \rceil$, $w = w_D(\,\cdot\,)$, $[u] = [u]_D$, and $A(l) = (l - c + 1) \cdots (l - c + c)$ with truncated $l - c$, so that $\mathrm{dl}_D(l) = A(l)^2$. Two facts: $l!$ divides $(l - c)!\, A(l)$; and $c!$ divides $A(l)$, a product of c consecutive integers.

Case $k < p$. Then $p \ne d$ by (S1), so $\chi_{K(P)}(p) = -1$. Take $i_\tau = 1$ and $y_\tau = \min(n_{[-p]\tau}, \lfloor (n_\tau - c)/p \rfloor)$. By (S1), $\lfloor (n_\tau - c)/p \rfloor \le 2b/k \le c$. Packing: (1) holds, (2) is empty, and for (3) the only τ with $[-p]\tau = \tau'$ has $y_\tau \le \min(n_{\tau'}, c)$, while $p\, y_{\tau'} \le n_{\tau'} - c$ (truncated); the sum is at most $n_{\tau'}$. For the bound, Legendre's formula and the two facts give

$$\begin{aligned} v_p(n_\tau!) &\le v_p((n_\tau - c)!) + v_p(A(n_\tau)) \\ &= \lfloor (n_\tau - c)/p \rfloor + v_p(\lfloor (n_\tau - c)/p \rfloor!) + v_p(A(n_\tau)) \\ &\le \lfloor (n_\tau - c)/p \rfloor + v_p(\mathrm{dl}_D(n_\tau)), \end{aligned}$$

and $\lfloor (n_\tau - c)/p \rfloor - y_\tau \le ((n_\tau - c)/p - n_{[-p]\tau})_+ \le (w(\tau)/p - w([-p]\tau))_+$, because $n_\tau - c \le w(\tau)$ by (S1). The last number is the term $e = 1$ of $\mathrm{ask}_{D,p}(w(\tau), \tau)$ (all terms are non-negative, Lemma 10.2(b)). Summing over τ gives the first alternative.

Case $p \le k$. For odd e, $[-p^e] = [-p]^e$. *Claim: for $0 < i < l \le 324\mu(D)$ the places τ, $[-p]^i\tau$, $[-p]^l\tau$ are not all thick.* Otherwise (S3) gives $(-p)^i \equiv 1 \bmod 2^{\mu(D)+1}$, and (S4) gives $2^{\mu(D)} \le (p+1)i$, which the sizes of the data exclude for $p \le k$. Say that e *pairs* τ if $e < \mu(D)$, e is odd, $p \ne d$, and τ and $[-p]^e\tau$ are thick. By the claim at

most one e pairs τ. If e pairs τ put $i_\tau = e$, $y_\tau = \lfloor n_\tau/p^e \rfloor$; otherwise $y_\tau = 0$. Packing: (1) holds since $p \neq d$ and $\chi_{K(P)}(p) \neq 1$. (2): if $[-p]^e\tau' = [-p]^{i_\tau}\tau$ with $e < i_\tau$ odd, then $\tau' = [-p]^{i_\tau - e}\tau$, and τ, τ', $[-p]^{i_\tau}\tau$ would be thick, against the claim. (3): if no τ with $y_\tau \neq 0$ has $[-p]^{i_\tau}\tau = \tau'$, the sum is $p^{i_{\tau'}} y_{\tau'} \leq n_{\tau'}$. Otherwise there is exactly one such τ, and $y_{\tau'} = 0$, both by the claim; and $y_\tau \leq n_\tau/p \leq n_{\tau'}$ since both places are thick and $n_\tau \leq a(D)/x + x^{42} \leq 2\, n_{\tau'}$ by (S1).

The bound. Fix τ. We compare, term by term in $e \geq 1$ (e odd or $p = d$),

$$\frac{n_\tau}{p^e} \;\leq\; \Big(\frac{w(\tau)}{p^e} - w([-p^e]\tau)\Big)_+ + [e = i_\tau]\, y_\tau + \frac{c}{p \cdot 2 \cdot 2^e}. \tag{4}$$

If $e \geq \mu(D)$, the left side is at most $2b/p^e$ and the second half of (S2) applies, as $p \geq 13$ by (I4). Let $e < \mu(D)$. If e pairs τ, then $n_\tau/p^e \leq y_\tau + 1$ and the last term is at least 1 by (S2). If not, then τ and $[-p^e]\tau$ are not both thick: for $p = d$ this is Lemma 10.2(c) with (S3), and for $p \neq d$ it is the definition. If τ is not thick, $n_\tau \leq 2a(D)/x_1$ by (S1), and (S2) applies. If τ is thick and $[-p^e]\tau$ is not, then $w([-p^e]\tau) = a(D)/x_1$ and $n_\tau/p^e \leq (w(\tau)/p^e - a(D)/x_1)_+ + a(D)/x_1 + x^{42}$, and (S1), (S2) apply. Summing (4) over e with Lemma 10.2(a), and multiplying by $(1 - \chi_{K(P)}(p))/2 \leq 1$:

$$\big(\tfrac{1}{p-1} - \varrho_d(p)\big) n_\tau - y_\tau \;\leq\; \mathrm{ask}_{D,p}(w(\tau), \tau) + \frac{c}{p}.$$

Finally $v_p(\mathrm{dl}_D(n_\tau)) \geq 2v_p(c!) \geq 2(c/p - 1) \geq c/p + 4b_2(D)$, the last step by (S2) at $e = 0$. Summing over τ gives the second alternative. □

LEAN `Data.exists_isFrobPacking`

Proof of Theorem F. We write z_τ for the variable $z_{(\tau,*)}$. By (I4), $p \geq 13$. Let ϖ be as in Theorem D and $F = \Theta(\varpi z)$. If $y = 0$ the assertion holds with $\beta = \gamma = 0$, since $|(\Theta^{-1})_0|_p |\Theta_0|_p = 1$; so let $y \neq 0$. For τ with $y_\tau \neq 0$ let $\tau^* = [-p^{i_\tau}]_{P,\mu} \cdot \tau$; by the packing i_τ is odd and $\chi_{K(P)}(p) = -1$, so Theorem D(b) gives $F = \exp(\kappa\Psi)V$ with $|\kappa|_p = p^{1-1/(p-1)} > 1$. As $\Theta(0) = 1$ and $\Psi(0) = 0$, V is a unit of $\mathfrak{o}_p[[z]]$.

Step 1: one derivative. Let Q be an integral series and τ a place with $y_\tau \neq 0$, and put $R = \partial_{\tau^*}\Psi \cdot Q + \kappa^{-1}(\partial_{\tau^*}Q + Q\,V^{-1}\partial_{\tau^*}V)$, an integral series. Then $\partial_{\tau^*}(QF) = \kappa\, RF$, hence for every polynomial G: $|\,\mathrm{jet}_{G\cdot z_{\tau^*}}(QF)|_p = |\kappa|_p \cdot |\,\mathrm{jet}_G(RF)|_p$. Modulo $\mathfrak{m}_p$, $\overline{R} = \partial_{\tau^*}\overline{\Psi} \cdot \overline{Q}$. Give the variable z_σ the weight 1 if $y_\sigma \neq 0$ and the weight $p^{\max i}$ otherwise. By Theorem D(b) the monomials of $\partial_{\tau^*}\overline{\Psi}$ are the $z_\sigma^{p^e}$ with e odd and $[-p^e]_{P,\mu} \cdot \sigma = \tau^*$; by condition (2) of the packing the one of least weight is $z_\tau^{p^{i_\tau}}$, and it is the only one of that weight.

Step 2: all derivatives. Starting from $Q = 1$ and applying Step 1 y_τ times for every τ, we get an integral Q whose reduction has the lowest term $u\, z^\lambda$, $u \neq 0$, $\lambda = \sum_\tau y_\tau p^{i_\tau} 1_\tau$ (lowest terms multiply), and with

$$|\kappa|_p^{|\gamma|}\, |\,\mathrm{jet}_G(QF)|_p = |\,\mathrm{jet}_{G\cdot z^\gamma}(F)|_p \qquad \text{for all } G, \qquad \gamma = \sum_\tau y_\tau 1_{\tau^*}.$$

Step 3: a large coefficient. The coefficient Q_λ is a unit. Since $Q = F^{-1} \cdot (QF)$, there are exponents $\gamma_1 + \beta_1 = \lambda$ with $1 \leq |(F^{-1})_{\gamma_1}|_p |(QF)_{\beta_1}|_p$. Step 2 with $G = z^{\beta_1}$ gives $|\kappa|_p^{|\gamma|} |\beta_1!|_p |(QF)_{\beta_1}|_p = |(\beta_1 + \gamma)!|_p |F_{\beta_1+\gamma}|_p$. With $F_\alpha = \varpi^{|\alpha|}\Theta_\alpha$, $(F^{-1})_\alpha =$

$\varpi^{|\alpha|}(\Theta^{-1})_\alpha$, $|\varpi|_p = p^{-\varrho_d(p)}$ and $|\beta_1!|_p/|(\beta_1+\gamma)!|_p \geq |\gamma!|_p^{-1}$, this yields, for $\beta = \gamma_1+\beta_1+\gamma$,

$$|(\Theta^{-1})_{\gamma_1}|_p \cdot |\Theta_{\beta_1+\gamma}|_p \;\geq\; p^{(1-\frac{1}{p-1})|\gamma|+\varrho_d(p)|\beta|+\sum_\tau v_p(\gamma_\tau!)}.$$

Split γ_1 into five exponents by the hypothesis on Θ^{-1} and keep $\beta_1+\gamma$ as the sixth, by the hypothesis on Θ. Finally $|\gamma| = \sum y_\tau$, $|\beta| = |\lambda|+|\gamma| = \sum y_\tau(p^{i_\tau}+1)$, and $\gamma \leq \beta \leq n$ by condition (3) of the packing: the entry of β at τ' is $p^{i_{\tau'}}y_{\tau'} + \sum_{\tau^*=\tau'} y_\tau$. □

LEAN `exists_hasLargeSplit_of_pairs`

11 Where the results are used

By name, as the dossier of the unit lists the uses (a search over the sources, not a kernel export). The unit on the columns (unit 145) uses all the definitions of §2 on asks, gains, losses, served primes, collisions, coupling counts and column counts, Theorems C, D and E, Proposition 7.4 and Theorem A, to prove that the chain of lifts is divisible by $p^{\mathrm{gain}(D,p)}$ up to the loss. The unit on the rows uses Proposition 10.1, Proposition 10.3 and Theorem F for the lower bound of a jet (the glued form), and Theorem B, with Lemma 6.2 and Lemma 6.3, where conjugates of lifts are bounded. The unit on the data uses Proposition 6.1: its constant $C(P)$ contains the chosen number $C_{\mathrm{gen}}(P)$.

12 Notes on the formal text

1. *Arguments that the text of the census lists and Lean does not take.* The dossier of the unit was made without running Lean, and its lists of arguments are those of a reading of the text. For three definitions the reading lists a section variable that only occurs as a bound index: τ in the gain and in the collision sum, and the height h in the coupling count. The source applies them to two, three and seven arguments (`D.gain p`, `D.collisionSum p ev`, `couplingCount p κ N ev σ (pr e σ) e`), and this note writes them so, $\mathrm{gain}(D,p)$, $\mathrm{coll}(D,p,\Xi)$, $\mathrm{cc}_p(\varkappa,n';\Xi;\tau,\tau';e)$, through macros of its own; the build accounts for the extra entry by the bound letter in Definition 2.6, 2.7, 2.8. The same happens for arguments of theorems and of one more definition: the index set J of a theta product, bound by "there are"; a ring Q bound inside Lemma 8.6; the letters c and K in Theorem B; a field E in Lemma 7.1(d) and Proposition 5.2; the prime and its place in Lemma 4.2; a second finiteness of n in the parts of Lemma 6.2; and the instance "V is standard" in Lemma 5.1(c), whose space is only assumed to have signs 1. All of this should be confirmed with `#check` once a build is at hand.

2. *Arguments that Lean takes and the reading cannot see.* Four places use the notation of the formal text for the p-adic place: Lemma 3.2(b)–(d), Theorem A, Definition 2.3 and Proposition 7.4. This notation makes the prime p and the homomorphism j arguments (an implicit one and two instance arguments). The statements above name them. Likewise the module $\mathrm{Std}_\delta(V)$ depends on j, and so does the set $\mathrm{ThProd}_{P,\mu}(\delta,r;c)$; the notation does not show it, because j is an instance argument.

3. *Three statements that the census cuts short.* The statements of Proposition 7.4(b), of Proposition 10.3 and of Theorem E contain a `let`, which the census takes

for the end of the statement. I read these statements in the source in full; the quotations of Appendix A avoid them. For the same reason three intermediate facts of the dossier are cut short (in the proofs of Lemma 9.1, of Lemma 8.3 and of Theorem A); its claims state them in full, and I read the lines.

4. *An error in the words of the first dossier, since corrected.* In the structured proof of Proposition 8.4 the dossier said that the pairs $(\zeta^i, 1)$ lie in the closure $\mathcal{N}_d$. The lemma of the library that the formal proof applies gives $(\zeta^i, \zeta^{i(p+1)})$, as in (Q6) above, and this is what makes the proof work: the exponential factor of the normal form is the part of $\log \theta_C$ in the degrees $\equiv p+1$ modulo $q-1$ precisely because of the exponent $p+1$. With the pairs $(\zeta^i, 1)$ one would obtain the part in the degrees $\equiv 0$. The quotations of the dossier were exact; only its words were affected (steps 7 and 35.6), and its record of corrections now has the pairs $(\zeta^i, \zeta^{i(p+1)})$ and the weight $\zeta^{-i(p+1)}$, as here.

5. *Proofs that are shorter here, or differently arranged.* The frame identity of Step 2 of Theorem A is established in the formal text by a Gröbner basis computation; the computation above replaces it. The estimate at the end of Lemma 8.3 is closed in the formal text by automation from the bounds of Lemma 8.1; the division into the cases $I \le 4$ and $I \ge 5$ is mine. In Proposition 10.1 the formal text ends as here: the reduction $\overline{S}$, a polynomial, would satisfy $\overline{S} - \overline{S}^{p^\nu} = X$, against the degrees. In Proposition 6.1 the formal text does not use that a_1 determines the lattice: it defines the conjugate datum by choice from the invariant, as Background 3 does. In Step 4 of Proposition 8.4 the arithmetic in the three cases is done by automation in the formal text.

6. *What I could not settle without Lean.* (a) In Lemma 9.1 the formal proof encodes the choice of the device for each coupling in one predicate and verifies the equality of exponents by a case analysis with automation; I read the construction and checked the three costs against the coupling count, but did not replay the case analysis. (b) In Theorem B(iv) the packing estimate and the final inequality between exponentials are a chain of automated steps; the proof above gives their structure and not their constants. (c) In Proposition 8.4 the final identity of Step 6 is a single simplification in the formal text; the computation above is the mathematics that it must carry out, with the pairs of note 4. (d) The inputs (S1)–(S2) are not separate declarations: the formal proof of Proposition 10.3 obtains each of them in place from the lemma of unit 152 that compares logarithmic sizes. I state them as I read them at the places of use; the second half of (S2) is my summary of a short chain there, which uses $4 \le p$ (available because $p \ge 13$) and is false for $p = 2$.

7. *Choices.* $\mathcal{E}(P)$, $\Omega_0(P)$, u_P, $\mathrm{vsc}(P, \mu)$ and $C_{\mathrm{gen}}(P)$ are obtained by choice from existence statements, the last two from Proposition 6.1 itself; so the definition of the scaled vacuum depends on the proof of that proposition having been given, not on its witnesses. The file of unit 152 that defines $C(P)$ imports this unit for the same reason.

8. *Conventions of the formal text that the statements inherit.* Sums over $e \in \mathbb{N}$ in Definitions 2.6 and 2.7 are unconditional sums with value 0 when the family is not summable; Lemma 10.2(b) shows that this does not occur in the ask for primes p with $\chi_{K(P)}(p) \neq 1$, and for $\chi_{K(P)}(p) = 1$ the factor in front is 0. The definitions

take a natural number p, not a prime. In Lemma 8.1 the last hypothesis has a default proof (the formal text supplies it by arithmetic automation when the numbers are explicit), and in Lemma 6.3 the adele η has a default value. In Lemma 10.2(d) the number a is arbitrary. The hypothesis $\nu \neq 0$ of Lemma 8.1 follows from $p^{\nu} = Q + 1$ only if $Q \neq 0$.

9. *The catalogue.* The relations stated for (Q2), (Q4) and (Q8) are those of the catalogue of the handout: the main theorem of complex multiplication in a soft form, the integrality of a_1 as a weaker form of good reduction everywhere, and the height of the formal group as one direction and an upper bound. For (Q3), (Q5), (Q6), (Q7), (Q9) and (Q10) the catalogue has no confirmed place; the files of the library name Bannai–Kobayashi (2010), Bannai–Kobayashi–Yasuda (2017), Katz (1981), Finis (2006), Mumford (1966), Conway–Sloane (1989) and Shimura (1976) as related work, and I did not compare the statements with these sources.

How this note was written. From the dossier of the unit: the texts of its 89 members and its 36 structured proofs, which I read in full in a condensed form (every claim, without the quoted lines). The program check of the dossier covered 407 leaf steps of 36 proofs and found every statement at its lines. I read all twelve Lean files of the unit, statements and proofs, and compared every statement of this note with its Lean text, argument by argument, with the list of arguments of the census beside it; for the statements named in notes 1 to 3 the source decided. In other units and in the library I read the definitions that the statements use (the structure of the data and its functions, tight frames, vacuum expansions, the jet series, Schur jets, gauges, kernels and the theta kernel, the CM datum with its functions, the ring $\mathfrak{o}_p[[\varphi_C]]$, carried series, admissible combinations) and the statements of the quoted results (Q2), (Q3), (Q6), (Q7), (Q8), of the input of unit 149 and of (I6); the other inputs are stated as the dossier reads them. I did not read proofs of the library. No Lean was run.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Lemma 3.1(b)	`Parameters.bigOmega_pos`	ConjugateGenerator.lean	15–15
Lemma 3.1(c)	`Parameters.jetScale_ne_zero`	ConjugateGenerator.lean	18–20
Proposition 6.1	`Parameters.exists_isJetConjugate_generator`	ConjugateGenerator.lean	26–85
Lemma 7.1(a)	`Parameters.integralAfter.of_integral`	ConjugatesOfJets.lean	25–26
Lemma 7.1(b)	`Parameters.finHeisOp_mem_standardData`	ConjugatesOfJets.lean	31–36
Lemma 7.1(c)	`Parameters.omega_mem_standardData`	ConjugatesOfJets.lean	40–50
Proposition 7.2	`Parameters.norm_jetSeries_le`	ConjugatesOfJets.lean	55–84
Theorem C	`Parameters.norm_jet_jetSeries_le_one`	ConjugatesOfJets.lean	89–96
Definition 2.3	`Kernel.raises`	ConjugatesOfJets.lean	100–102
Proposition 7.4(a)	`Kernel.smul_mem_raises`	ConjugatesOfJets.lean	106–108
Lemma 7.1(e)	`Parameters.pairWeil_mem_standardData`	ConjugatesOfJets.lean	112–116
Proposition 7.4(b)	`Parameters.mem_raises_of_jet_bound`	ConjugatesOfJets.lean	121–133
Proposition 7.4(c)	`Parameters.mem_raises_of_standardData`	ConjugatesOfJets.lean	137–143
Proposition 7.4(d)	`Parameters.mem_raises_of_dvd`	ConjugatesOfJets.lean	148–153
Theorem B	`Parameters.exists_semilinear_isJetConjugate`	ConjugatesOfJets.lean	164–225
Definition 2.10	`IsFrobPacking`	DepthPairs.lean	14–16
Lemma 10.2(c)	`Data.ne_d_of_placeClass_eq`	DepthPairs.lean	18–21
Lemma 10.2(b)	`Data.summable_localAsk`	DepthPairs.lean	23–26

In this note	Lean declaration	File	Lines
Definition 2.10	`Data.depthLossInt`	DepthPairs.lean	29–29
Proposition 10.3	`Data.exists_isFrobPacking`	DepthPairs.lean	34–92
Theorem F	`exists_hasLargeSplit_of_pairs`	DepthPairs.lean	101–146
Lemma 7.3	`CMLattice.jet_theta_prod_integral`	FormalParameter.lean	27–55
Lemma 8.1	`exists_norm_eq_inv_pow_bounds`	FormalParameter.lean	59–66
Lemma 8.3	`norm_coeff_le_of_deriv`	FormalParameter.lean	72–95
Definition 2.5	`IsIntegralOnNullCone`	FormalParameter.lean	101–105
Lemma 8.2	`coeff_subst_tangent`	FormalParameter.lean	119–123
Proposition 8.4	`CMLattice.exists_formal_parameter`	FormalParameter.lean	132–248
Lemma 5.1(a)	`Space.taylor_line_eq_subst_jetSeries`	FrameSections.lean	22–38
Lemma 5.1(b)	`Space.eq_zero_of_jetSeries_eq_zero`	FrameSections.lean	42–46
Lemma 5.1(c)	`Space.jetSeries_eq_prod`	FrameSections.lean	51–67
Lemma 3.1(d)	`Parameters.latticeModule_cmCurve`	FrameSections.lean	72–73
Lemma 3.1(e)	`Parameters.omega₀_mem_cmCurve_lattice`	FrameSections.lean	76–77
Lemma 7.1(d)	`Parameters.norm_map_psi₀`	FrameSections.lean	79–81
Proposition 5.2	`Parameters.exists_finitePart_of_isTightFrame`	FrameSections.lean	89–126
Lemma 6.3	`isAdmissibleComb_name`	IntegralVacuum.lean	19–39
Definition 2.2	`Parameters.vacuumScale`	IntegralVacuum.lean	43–44
Definition 2.2	`Parameters.intVacuum`	IntegralVacuum.lean	49–49
Definition 2.2	`Parameters.integralAfter`	IntegralVacuum.lean	53–55
Definition 2.2	`Parameters.standardData`	IntegralVacuum.lean	59–60
Lemma 6.2(a)	`Parameters.kappa₀_ne_zero`	IntegralVacuum.lean	63–63
Lemma 6.2(b)	`Parameters.intVacuum_finVec_eq_prod`	IntegralVacuum.lean	69–72
Lemma 6.2(c)	`Parameters.norm_intVacuum_le`	IntegralVacuum.lean	77–80
Lemma 6.2(d)	`Parameters.norm_intVacuum_le_one`	IntegralVacuum.lean	84–86
Lemma 6.2(e)	`Parameters.norm_dataInner_le`	IntegralVacuum.lean	90–95
Lemma 6.2(f)	`Parameters.exists_intVacuum_ne_zero`	IntegralVacuum.lean	97–98
Lemma 9.1	`exists_null_coupling_frames`	JetDivisibility.lean	24–53
Theorem E	`norm_schurJet_le`	JetDivisibility.lean	65–111
Proposition 10.1	`CMLattice.katz_divisibility_limit`	KatzDivisibility.lean	15–42
Definition 2.6	`Data.localAsk`	LiveSum.lean	19–20
Definition 2.6	`Data.forcedPart`	LiveSum.lean	24–24
Definition 2.6	`Data.gain`	LiveSum.lean	28–28
Definition 2.6	`Data.IsSmallNonsplit`	LiveSum.lean	32–32
Definition 2.6	`Data.IsShortPrime`	LiveSum.lean	35–35
Definition 2.6	`Data.loss`	LiveSum.lean	39–39
Definition 2.7	`Data.ceilLogM`	LiveSum.lean	42–42
Definition 2.7	`Data.deltaP`	LiveSum.lean	46–46
Definition 2.7	`Data.jetCeil`	LiveSum.lean	49–49
Definition 2.7	`Data.o₀`	LiveSum.lean	53–53
Definition 2.7	`Data.IsServed`	LiveSum.lean	57–57
Definition 2.7	`Data.IsCollision`	LiveSum.lean	61–62
Definition 2.7	`Data.collisionSum`	LiveSum.lean	65–65
Definition 2.8	`couplingCount`	LiveSum.lean	69–73
Definition 2.9	`Data.κ`	LiveSum.lean	77–78
Lemma 10.2(d)	`Data.κ_le`	LiveSum.lean	80–81
Definition 2.9	`Data.lastEntry`	LiveSum.lean	85–85
Definition 2.9	`Data.keptColumns`	LiveSum.lean	89–89
Definition 2.9	`Data.columnsY`	LiveSum.lean	92–92
Lemma 10.2(a)	`hasSum_live_inv_pow`	LiveSum.lean	96–102
Definition 2.1	`Parameters.vacuumRoot`	PeriodDefs.lean	13–13
Lemma 3.1(a)	`Parameters.norm_gal_vacuumRoot`	PeriodDefs.lean	19–30
Definition 2.1	`Parameters.cmCurve`	PeriodDefs.lean	33–34
Definition 2.1	`Parameters.Ω₀`	PeriodDefs.lean	37–38
Definition 2.1	`Parameters.areaScale`	PeriodDefs.lean	41–42

In this note	Lean declaration	File	Lines
Definition 2.1	`Parameters.periodFactor`	PeriodDefs.lean	46–47
Definition 2.1	`Parameters.jetScale`	PeriodDefs.lean	51–51
Lemma 8.5(a)	`sum_pow_eq_zero_of_null`	ThetaProducts.lean	19–25
Lemma 8.5(b)	`coeff_sum_frob_ne_zero_iff`	ThetaProducts.lean	30–39
Definition 2.4	`Parameters.thetaProducts`	ThetaProducts.lean	48–51
Lemma 8.6	`Parameters.exists_frame_rows_frob`	ThetaProducts.lean	57–77
Theorem D	`Parameters.thetaProducts_normal_form`	ThetaProducts.lean	86–143
Lemma 3.3(a)	`Parameters.coe_trace_eq_sum`	TightFrameExists.lean	12–13
Lemma 3.3(b)	`Parameters.herm_mem_quadInt`	TightFrameExists.lean	15–23
Lemma 3.2(a)	`Parameters.adeleExtension_alpha`	TightFrameExists.lean	29–31
Lemma 3.2(b)	`Parameters.norm_iota_periodFactor`	TightFrameExists.lean	36–43
Lemma 3.2(c)	`Parameters.norm_periodFactor_mul_conj_le_one`	TightFrameExists.lean	45–48
Lemma 3.2(d)	`Parameters.norm_iota_periodFactor_eq_one`	TightFrameExists.lean	52–55
Lemma 4.1	`Parameters.isTightFrame_zeta_pow`	TightFrameExists.lean	59–66
Lemma 4.2	`Parameters.exists_cyclotomic_different_factor`	TightFrameExists.lean	71–85
Theorem A	`Parameters.exists_isTightFrame`	TightFrameExists.lean	90–148

Four declarations as they are written in the artifact.

```
def gain := D.m * (∑ τ ∈ D.tops, D.localAsk p D.a τ + D.k * D.forcedPart p)
```

```
def _root_.couplingCount {Φ} (p : ℕ) (κ : Φ → ℕ → ℕ) n' (ev : Finset Φ) σ σ' e :=
  let t σ := ∑ h ∈ .range n', ite (h = 1 ∧ σ ∉ ev) (κ σ h) 0
  let r σ := ∑ h ∈ .range n', ite (h = 1) 0 (κ σ h)
  let B σ := ∑ h ∈ .range n', h * ite (h = 1) 0 (κ σ h)
  (t σ / p ^ e : ℝ) ⊓ (r σ' + t σ') + r σ / p ^ e + (B σ / p ^ e : ℝ) ⊓ B σ'
```

```
def thetaProducts (δ : 𝒪 P.K) r c [Fintype c] := {θ | ∃ J, ∃ (_ : Fintype J) (w : J → P.E s), ∃ M n
  b, ∃ y : J × c → ℂ, IsTightFrame w 1 M ∧ θ = ∏ i, (taylorVia ιₚ (P.cmCurve.twistedTranslate 1 (y
  i) P.cmCurve.θ)).subst (linearSeries fun (τ, k) ↦ ite (k = i.2) (ιₚ (b * conj (Φ τ (w i.1)))) 0) ∧
  (p.Coprime M ∧ p.Coprime n) ∧ ‖ιₚ b‖ = r ∧ ∀ i, n * δ * y i ∈ P.cmCurve.Λ}
```

```
lemma thetaProducts_normal_form (hp : chiK P.d p ≠ 1) : ∃ ϖ π, ‖ϖ‖ = p ^ (-nonsplitWeight P.d
  p) ∧ ‖π.1‖ = p ^ ((chiK P.d p - 1) / 2 : ℝ) ∧ ∀ {I} [Fintype I], ∀ r ≤ 1, ∀ θ ∈ P.thetaProducts
  s 1 r I, (∀ B, IsIntegralOnNullCone (fun j v ↦ P.placeOfInt s (-p ^ j) * v) {j ≤ p.log B | Odd j v
  p = P.d} ϖ π B θ) ∧ (chiK P.d p = -1 → r = 1 → ∃ (κ : ℂ_[p]) (Ψ V : MvPowerSeries _
  (valRingCp p)), ‖κ‖ = p ^ (1 - 1 / (p - 1) : ℝ) ∧ constantCoeff Ψ = 0 ∧ θ.rescale (fun _ ↦ ϖ) =
  expSubst (κ • seriesToCp Ψ) * seriesToCp V ∧ ∀ x d, (pderiv x (Ψ.map (residue _))).coeff d ≠ 0 ↔
  ∃ j y, Odd j ∧ (P.placeOfInt s (-p ^ j) * y.1, y.2) = x ∧ d = .single y (p ^ j))
```

In these texts `s` is the exponent μ, `Φ τ` the embedding ι_τ, `ιₚ` the homomorphism j, `valRingCp p` the ring $\mathfrak{o}_p$, `chiK P.d p` the value $\chi_{K(P)}(p)$, `nonsplitWeight P.d p` the number $\varrho_d(p)$, `P.placeOfInt s (-p ^ j) * v` the place $[-p^e]_{P,\mu}\cdot v$, `κ` a family of column counts $\varkappa$, `ev` the set Ξ, `taylorVia ιₚ` the series $\mathrm{Tay}_j(\,\cdot\,)$, `P.cmCurve.twistedTranslate 1 y` the translation $U_y^{\mathcal{E}(P),1}$, `expSubst` the exponential of a series without constant term and `residue` the reduction modulo $\mathfrak{m}_p$. Letters of the formal text were changed where the write-up reserves them: the exponent `s`, the Frobenius exponents `j` (here e), the degree `e` of the residue field (here ν), the element `θ` of the null cone (here ϖ_0) and the column counts `κ`.

Notation

$P = (N_0, d)$, μ parameters; the exponent

$K(P)$, $\mathcal{O}_{K(P)}$, $\chi_{K(P)}$, χ_d $\mathbb{Q}(\sqrt{-d})$, its integers, its quadratic character; the Legendre symbol modulo d

$E(P,\mu)$, $N_1(P,\mu)$, $\varsigma(P,\mu)$ the CM field, the order of its root of unity, that root

$\mathrm{Pl}(P,\mu)$, ι_τ, $[\nu]_{P,\mu}$ the group of infinite places; the embedding at τ; the place of an integer ν

$\tau, \tau', v, v', u, \sigma$ infinite places; σ also an automorphism of $\mathbb{C}$, u also a unit or a variable

$c_0(P,\mu)$, $\partial(P,\mu)$, $\alpha(P,\mu)$, $\mathrm{rd}_{E(P,\mu)}$ $(1+\sqrt{-d})/2$; the generator of the relative different; the idele; the root discriminant

$\Omega(P,\mu)$, $\mathrm{cs}(d)$ the period $e^{\mathrm{cs}(d)}2\pi/\mathrm{rd}_{E(P,\mu)}$; the Chowla–Selberg constant

$\mathfrak{a}(P,\mu)$ the lattice $E \cap \alpha\widehat{\mathcal{O}}$

$\mathbb{A}$, $\mathbb{A}_\mathrm{f}$, $\widehat{\mathcal{O}}$ adeles, finite adeles, integral finite adeles

$\mathrm{perm}_\mu(\sigma)$, $\Theta^{[\sigma]}$, π the permutation of places of σ; the renaming of the variables of a series by it; the circle constant

ϖ_0 the element of $\mathfrak{o}_p$ of the null cone, with $|\varpi_0|_p = p^{(\chi_{K(P)}(p)-1)/2}$ in Theorem D

(p,j), $\mathbb{C}_p$, $|\cdot|_p$, $\mathfrak{o}_p$, $\mathfrak{m}_p$, O_j a p-adic place and its objects

ψ, $\mathfrak{p}_\psi$, $\hat{\psi}$, j_τ, $\hat{\jmath}_\tau$ an embedding into $\mathbb{C}_p$, its prime, its extension to $\mathbb{A}_\mathrm{f}$; $j \circ \iota_\tau \circ$ conjugation

v_p, $\lfloor\rfloor$, $\lceil\rceil$, $()_+$ p-adic valuation, floor, ceiling, positive part

C, Λ_C, $\mathrm{area}(C)$, t, a_1 a CM datum, its lattice, area, point of order two, invariant

C_α, σC, A_0 the chosen CM datum with invariant α; the conjugate datum $C_{\sigma(a_1)}$; the conjugates of a_1 that are invariants

$\wp$, σ_C, θ_C, ξ_C, φ_C, $\mathrm{mon}_C(x)$ Weierstrass functions; theta; $\wp(z+t)-\wp(t)$; the parameter; monomial sections

$U_y^{C,m}$, H_C twisted translation; the hermitian form of C

$\mathrm{Tay}(F)$, $\mathrm{Tay}_j(F)$, $\hat{\varphi}$ Taylor series; the same read through j; $\mathrm{Tay}_j(\varphi_C)$

$\mathfrak{o}_p[[\varphi_C]]$, $\mathcal{N}_d$, $[F]_{r \bmod M}$ integral series in the parameter; the closure of norm pairs; part in a residue class of degrees

$s_0(P,\mu)$, $s_1(P,\mu)$, $\mathrm{per}(P,\mu)$, $\mathrm{asc}(P)$ root of the vacuum; period factor; scale of jets; $\sqrt{2\pi/\sqrt{d}}$

$\mathcal{E}(P)$, $\Omega_0(P)$, u_P the chosen CM datum, its period, its Chowla–Selberg number

χ_fin, υ, ψ_0 characters of the Weil data; the additive character of $\mathbb{A}_\mathrm{f}$

V, n, ε, I_n, $\mathrm{Hom}(I_n, I_{n'})$ a hermitian space, its index set, its signs; standard spaces

$\mathcal{S}(V)$, $\mathcal{W}_\mathrm{f}(V)$, vac_V, Th_V, ω_V, $\langle f, f'\rangle_V$ the Weil space, its finite part, vacuum, theta functional, representation, pairing

$\mathfrak{h}_V(y)$, $\mathfrak{h}^\mathrm{f}_V(x)$, $x^\flat$ Heisenberg operators; the adelic vector of a finite vector

$\mathrm{U}_n(\mathbb{A}_\mathrm{f})$, $\Delta_\mathrm{f}(g)$, 1_n the finite adelic unitary group, its multiplier, the unit matrix

$T_a^V f$, Θ_f, $\mathrm{jet}_Q(F)$ jet series; theta function of f; jet along Q

$\mathrm{name}_{V,R}(\Psi)$, $P^\mathfrak{m}$, $\mathfrak{m}$ named operators; the mean at a modulus; a modulus

$\mathrm{vsc}(P,\mu)$, θ°_V, $\mathrm{Lat}(\delta; S)$, $\mathrm{Std}_\delta(V)$ scale of the vacuum; scaled vacuum; a group of vectors; standard data

$C_\mathrm{gen}(P)$, $C(P)$ the chosen constant of Proposition 6.1; the constant of unit 152

$\varkappa$, $\mathrm{box}(\varkappa, N)$, $S_\varkappa$, $\mathrm{SJ}_\varkappa(F)$ column counts, boxes, Schur matrix, Schur jets

i, i', $i_\tau(h,l,b)$ Weyl indices and their entries; i also an index or the exponents of a packing

$\mathrm{gauge}_j^{\varkappa,e}(g;\xi)$, k, ω_k, jm_k, $\mathcal{D}$ gauge; a kernel, its representation, its matrices, its data

$\mathrm{k}_\varkappa(c; a, b; K)$, $\omega_{a,b}$, e_a, e_b theta kernel; Weil representation of the pair; weights of the characters a, b

$\mathrm{Raise}_k(d_1, d_2; c)$ data that raise gauges by at most c

$\mathrm{ThProd}_{P,\mu}(\delta, r; c)$ theta products

pr_e, S, S_B, Ξ the partner map; sets of Frobenius exponents; a set of exempted places

ϖ, κ, κ_0, Ψ, ψ, ν, q the scale at p; scalars of the exponential; its exponents; the degree and the norm of the prime above p

$\varrho_d(p)$ $1/(p^2-1)$, $1/(2(p-1))$ or $1/(p-1)$

D; x, x_1, b, m, r; $\mu(D)$ admissible data; its components (in Definitions 2.6–2.10 and §10); its exponent

$\mathrm{Pl}(D)$, $N_1(D)$, $f(D)$, $k(D)$, $t(D)$, $a(D)$, $[\nu]_D$ objects of the data (unit 152)

$\mathrm{tops}(D)$, $w_D(\tau)$, $\mathrm{wt}_D(l,\tau)$, $n_l(D)$, $\mathrm{gap}(D)$, $b_2(D)$ tops; nominal weight; weights; ranks; gap; a rounded weight

$\mathrm{ask}_{D,p}(y,\tau)$, $\mathrm{forced}(D,p)$, $\mathrm{gain}(D,p)$, $\mathrm{loss}(D,p)$ ask, forced part, gain, loss

$\mathfrak{r}(D,p)$, $\Delta(D,p)$, $\mathrm{jc}(D,p)$, $o_0(D,p)$, $\mathrm{coll}(D,p,\Xi)$ the numbers of Definition 2.7

$\mathrm{cc}_p(\varkappa, n'; \Xi; \tau, \tau'; e)$, t_σ, r_σ, B_σ coupling count and its parts

$\varkappa_D(\tau,h)$, $\mathrm{last}_D(l,\tau)$, $\mathrm{kept}_D(h)$, $\varkappa^Y_D(\tau,h)$ column counts of the data

$\mathrm{dl}_D(n)$, $A(l)$, L depth loss; an ascending factorial; a bound (also a linear form, a semilinear map, log of theta)

N, b, D a weight on exponents; a proposition; a product of determinants

$\alpha, \beta, \gamma, \lambda$, 1_y, z^β exponents (multi-indices), the exponent of one variable, monomials; α also the idele, β, γ, λ also numbers where said

z, z_x, X, Y, T, Z, W variables and families of series; z also an adele or an integer, T, Z, W series in Proposition 8.4

M, N, R, B, I, J, Q, A, H lengths, levels, sets, rings, bounds: local to each statement

$a, b, c, e, f, g, h, k, l, m, n, o, s, w, y$ numbers, indices, elements and functions local to each statement

E, F, G, S, U, O series, functions and sets local to each statement; E also a field

δ, η, ζ, θ, ρ, ω, ϕ, ξ, ℓ, ι, ς, ϵ an integer of $K(P)$; an adele; a root of unity; in θ_C; real exponents; a ring map; a vector; linear forms and primes; in ι_τ; the root of unity of E; coefficients

Φ, Θ, Ω, Λ, Δ, Π, ∂ a set of places or a datum; a series; periods; a lattice; a multiplier; a product; derivatives and the different

Columns on a definite unitary group: divisibility and size of the chain of theta lifts

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. A *column* is an algebraic modular form on a definite unitary group of rank $k(D)$ over the CM field $E(D)$, of a fixed weight and of a level that is principal at the chosen primes and of mirabolic type at the primes dividing an element $\pi_N(D)$ of norm the level prime $N(D)$. A chain of m theta lifts between definite groups, followed by a restriction, sends columns to forms on a group of larger rank; by the seesaw identity of another unit this map computes the chain of lifts of Poincaré series. This unit proves the three facts about columns that Theorem A of the handout needs. First, the columns that are *ramified* at the primes dividing $\pi_N(D)$ are spanned by forms that are integral at all primes and whose conjugates are all small. Second, at every prime p the chain is divisible by $p^{\mathrm{gain}(D,p)}$, up to an allowed error, on a submodule of the integral columns of small index; at an inert prime that pairs two of the top places, the divisibility is regained on the image of scaled Hecke operators. Third, the untwisted chain enlarges the norms of all conjugates by at most an explicit factor. The allowed errors, weighted by $\log p$, sum to at most the budget.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof whose final theorems state the Riemann hypothesis and its analogue for Dirichlet L-functions. That headline is a claim. Its authors report that every file compiles and that Lean's kernel accepts the proofs; a replay of that check is recorded in the handout when it reports; nothing was compiled for this note, and no mathematician has read the argument through. Nothing here says that the Riemann hypothesis is proved. This note treats one unit of the artifact, the folder of the columns: all 105 of its 105 declarations. Every statement below was compared with its Lean text, argument by argument. The proofs are given at the level of the structured proofs of the unit's dossier, which were read together with the Lean text; results of other units and of the background library are stated as *Inputs* with their owners, and steps that the formal text leaves to automation and that could not be replayed without Lean are named in §15. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 105 declarations.

1 The statements

The objects are defined in §2. In short: D is a record of admissible data for parameters $P = (N_0, d)$, with scales x, x_1, b and numbers m, r; $\mathrm{Col}(D)$ is the space of columns and $\mathrm{Col}^{\mathrm{ram}}(D)$ its subspace of ramified columns (Definitions 2.3 and 2.7); for a twist ϑ of D and a finitely supported function Ψ on tuples of translations, $\mathrm{B}_D(\vartheta, \Psi)$ is the chain of m theta lifts with data given by Ψ, followed by the restriction to the index

set $Y_m(D)$ (Input I4); $\mathrm{gain}(D,p)$ and $\mathrm{loss}(D,p)$ are the divisibility asked at p and the unit of allowed loss (unit 150), and $\mathrm{budget}(D) = f(D)\,a(D)\,(\log b)^3$.

Theorem A (short ramified columns span). In Setting 2.1, the complex vector space $\mathrm{Col}^{\mathrm{ram}}(D)$ is spanned by the ramified columns φ such that φ is $\exp(\mathrm{budget}(D))$-short: the values of φ are integral at every p-adic place, and $\|\sigma \circ \varphi(g)\|^2 \le \exp(\mathrm{budget}(D))$ for every automorphism σ of $\mathbb{C}$ and every point g.

Theorem B (divisibility of the chain at each prime). In Settings 2.1 and 2.2, let β be an integer with

$$\beta \;<\; [\,p \in \mathrm{NS}(D)\,] \cdot \mathrm{gain}(D,p) - \mathrm{err}(D,p) + 1.$$

Then there are a natural number $l \le \mathrm{err}(D,p)\cdot \dim_{\mathbb{C}} \mathrm{Col}(D)$ and an O_j-submodule S of $\mathrm{Col}(D)$ such that

(i) S has index dividing p^l in the module $\mathrm{Col}_j(D)$ of j-integral columns (Input I2), and

(ii) for every twist ϑ of D, every finitely supported complex function Ψ on the tuples of translations of D such that Ψ is good for (D,ϑ) and p is prime to $\mathrm{nrm}(\vartheta)$, and every $\varphi \in S$ that is a ramified column, the form $p^{-\beta}\cdot \mathrm{B}_D(\vartheta,\Psi)\varphi$ has j-gauge at most 1 at 1.

Theorem C (size of all conjugates of the untwisted chain). In Setting 2.1, let Ψ be a finitely supported complex function on the tuples of translations of D such that Ψ is good for $(D,1)$ for the trivial twist 1; let σ be an automorphism of $\mathbb{C}$, $\varphi \in \mathrm{Col}(D)$ and B a real number with $\|\sigma \circ \varphi(g)\|^2 \le \mathrm{e}^B$ for every $g \in G_{X_0(D)}$. Then for every $h \in G_{Y_m(D)}$

$$\|\sigma \circ (\mathrm{B}_D(1,\Psi)\varphi)(h)\|^2 \le \exp\Big(B + h_{\mathrm{row}}(D) + 2\sum_{p\in \mathrm{NS}(D)} \mathrm{gain}(D,p)\,\log p + \mathrm{budget}(D)\Big).$$

Theorem D (the allowed losses fit the budget). In Setting 2.1, for every finite set U of natural numbers, $\sum_{p\in U} \mathrm{err}(D,p)\cdot \log p \;\le\; \mathrm{budget}(D)$.

LEAN `Data.span_short_ramifiedColumns` · `Data.exists_indexDvd_Bmu_divisible` · `Data.weylNormSq_Bmu_le` · `Data.sum_allowedLoss_mul_log_le`

How the four fit together. The explanations of this paragraph are orientation; the deduction belongs to the unit of the endgame, which uses these four results, and was not read for this note. A row of the handout's Theorem A is, after the seesaw identity, of the form $\mathrm{B}_D(\vartheta,\Psi)\psi$ for a ramified column ψ, divided by a rational number that collects the powers $p^{\mathrm{gain}(D,p)}$ and a power of $N(D)$. Theorem A provides the columns ψ, Theorem C bounds the size of the rows at every conjugate,

and Theorem B bounds their denominators: at each prime the rows built from a submodule S of index dividing p^l, in the sense of Input I2 (a statement about matrices of determinant dividing p^l, not about the order of a quotient), are integral after division by p^β, and the exponents l, weighted by $\log p$, add up to at most $\dim \mathrm{Col}(D) \cdot \mathrm{budget}(D)$ by Theorem D. The number $h_{\mathrm{row}}(D)$ of Theorem C contains the term $-\#\mathrm{tops}(D) \cdot a(D) \log N(D)$: the restriction divides by a power of $\overline{\pi_N(D)}$, and this is how the level prime pays for the excess of size over divisibility. The error at a prime is three units of loss; one unit is $f(D)(\log b)^2 a(D)/p$, multiplied by m when $p \equiv \pm 1$ modulo m.

The mechanism at a prime. Let $p < x_1$ be inert or ramified in $K(P)$. A single lift is a finite sum of jets of theta series with complex multiplication, and by the unit on the CM curve a jet along a Schur matrix with L boxes is divisible by p to an exponent of the shape $(1/(p-1) - \varrho_d(p))L$ minus a sum of *coupling counts* (Input I9). Proposition 7.3 shows that this exponent, summed over the m lifts, is at least $\mathrm{gain}(D,p)$ up to one unit of loss, except for the *collisions*: pairs of top places exchanged by an odd power of $-p$. A collision with a small exponent can only occur when p is *served* (inert, not $\equiv \pm 1$ modulo m, and $-p$ of small odd order modulo N_0). For such p one chooses a set $\mathcal{L}$ of primes of $E(D)$ above p, one from each conjugate pair met by the tops, and applies at each of them the Hecke operator of $\mathrm{diag}(p, \dots, p, 1, \dots, 1)$ scaled by a power of p. On the image the divisibility is restored (Proposition 8.4), because the Hecke operator moves through every lift (Proposition 8.1) and changes the datum by a diagonal matrix that the jet bound absorbs (Proposition 8.2); and the image has small index (Proposition 9.2, whose content is a theorem of the unit on twisted Hecke operators). At the level prime itself the data of the chain have denominator $N(D)$, and a Fourier transform on the columns removes it (§10).

2 The setting

Conventions. $\mathbb{N} = \{0, 1, 2, \dots\}$. Where the formal text subtracts or divides natural numbers, the subtraction is truncated at 0 and the division is rounded down; such places are marked. $[\,\mathcal{C}\,]$ is 1 if the condition $\mathcal{C}$ holds and 0 otherwise. Throughout, E is a CM field with its complex conjugation $x \mapsto \overline{x}$, a CM type and Weil data (the idele and the splitting character that the Weil representation over E needs; they are those of the note on the CM field when $E = E(D)$). $\mathrm{Pl}(E)$ is the set of infinite places of E, and $\iota_\tau \colon E \to \mathbb{C}$ is the embedding of the CM type at $\tau \in \mathrm{Pl}(E)$. A *finite place* w is a non-zero prime ideal of the ring of integers $\mathcal{O}_E$; $\overline{w}$ is its conjugate, E_w the completion, and q_w the number of elements of the residue field. $\mathbb{A}_{E,\mathrm{f}}$ is the ring of finite adeles and $\widehat{\mathcal{O}}_E$ its subring of integral adeles. I and I' are finite index sets, and K denotes a subgroup of the group G_I of Input I1; a family $\varkappa$ of *column counts* is a function $(\tau, h) \mapsto \varkappa(\tau, h) \in \mathbb{N}$ on $\mathrm{Pl}(E) \times \mathbb{N}$, and a family $\mathfrak{e}$ of *determinant exponents* is a function $\mathrm{Pl}(E) \to \mathbb{Z}$. $\mathrm{Aut}(\mathbb{C})$ is the group of all field automorphisms of $\mathbb{C}$. For parameters $P = (N_0, d)$, $K(P) = \mathbb{Q}(\sqrt{-d})$ and $\chi_{K(P)}$ is its quadratic character, the Legendre symbol modulo d. A *p-adic place* is a prime number p with a ring homomorphism $j \colon \mathbb{C} \to \mathbb{C}_p$ (Setting 2.2). In a statement about a commutative ring R or a normed field, that structure is part of the statement.

Setting 2.1 (the data)**.** $P = (N_0, d)$ are parameters and D is a record of admissible data for P (the note on the admissible data), with real scales x, x_1, b and natural numbers m, r among its components. The functions of D are written with their argument: the exponent $\mu(D)$, the CM field $E(D)$ with its group $\mathrm{Pl}(D)$ of infinite places, $N_1(D)$, the number $f(D)$ of infinite places, $k(D) = r+1-m$, $t(D) = 5r$, $a(D)$, the sets $\mathrm{tops}(D)$ and $\mathrm{thick}(D)$, the nominal weight $w_D(\tau)$, the place $[\nu]_D \in \mathrm{Pl}(D)$ named by an integer ν, the ranks $n_i(D) = k(D) + i(t(D)+1)$, the weights $\mathrm{wt}_D(i,\tau)$ and $\mathrm{budget}(D) = f(D)\,a(D)\,(\log b)^3$. *Size comparisons* between such quantities (inequalities that hold because of the scale conditions $(\log b)^{100} \le x$, $x^{10^6} \le x_1$, $x_1^{10^6} \le b$ of the data) are quoted from the note on the admissible data as "by size".

Setting 2.2 (a p-adic place)**.** p is a prime number and $j\colon \mathbb{C} \to \mathbb{C}_p$ is a ring homomorphism into the completion $\mathbb{C}_p$ of an algebraic closure of $\mathbb{Q}_p$, with absolute value $|\cdot|_p$, $|p|_p = p^{-1}$, and valuation ring $\mathfrak{o}_p$. Put $O_j = \{z \in \mathbb{C} : \ |j(z)|_p \le 1\}$, a valuation ring of $\mathbb{C}$.

Inputs: the objects. The definitions of this unit rest on objects of the background library and of other units. They are recalled here, in the words of the dossier's readings, which were compared with the Lean definitions named in the LEAN lines.

Input I1 (algebraic forms; background library). Let I be a finite set. $G_I = \{g \in \mathrm{GL}_I(\mathbb{A}_{E,\mathrm{f}}) : \ g^* g = 1\}$ is the group of finite adelic points of the unitary group of the standard definite hermitian space E^I (g^* the conjugate transpose); $\Gamma_I \subset G_I$ is the subgroup of its rational points, the unitary matrices over E; $K_I = G_I \cap \mathrm{GL}_I(\widehat{\mathcal{O}}_E)$. For an integer a of E, $K_I(a)$ is the principal level, the $k \in K_I$ with all entries of $k-1$ in $a\widehat{\mathcal{O}}_E$; for a finite place w, $K_I[w]$ is the group of the $k \in K_I$ whose components away from w and $\overline{w}$ are 1. A *level* is an open subgroup K of K_I; $\mathrm{Cl}(K)$ is the finite set of classes $\Gamma_I \backslash G_I / K$; K is *free* if $\Gamma_I \cap gKg^{-1} = \{1\}$ for every $g \in G_I$.
$\mathrm{Wy}_I(\varkappa)$ is the finite set of *Weyl indices*: the families, over $\tau \in \mathrm{Pl}(E)$, of standard fillings with entries in I of the diagram with $\varkappa(\tau, h)$ columns of height h. The *weight representation* $\rho_{\varkappa,\mathfrak{e}}$ sends a family (g_τ) of invertible matrices indexed by I to $\prod_\tau \det(g_\tau)^{\mathfrak{e}(\tau)}$ times the Schur matrix of (g_τ), a square matrix indexed by $\mathrm{Wy}_I(\varkappa)$; Γ_I acts on $\mathbb{C}^{\mathrm{Wy}_I(\varkappa)}$ by $\gamma \mapsto \rho_{\varkappa,\mathfrak{e}}((\iota_\tau(\overline{\gamma}))_\tau)$. The number of boxes below height n' is $\mathrm{boxes}(\varkappa, n') = \sum_\tau \sum_{h<n'} h\,\varkappa(\tau, h)$, and the row of index ν of the diagram at τ, for rank n', has length $\sum_{\nu<h<n'} \varkappa(\tau, h)$.
$\mathrm{Alg}_I(\varkappa, \mathfrak{e})$ is the space of *algebraic forms* of weight $(\varkappa, \mathfrak{e})$: the functions $\varphi\colon G_I \to \mathbb{C}^{\mathrm{Wy}_I(\varkappa)}$ with open stabiliser under right translation and $\varphi(\gamma g) = \rho_{\varkappa,\mathfrak{e}}(\gamma)\varphi(g)$ for $\gamma \in \Gamma_I$. G_I acts on it by right translation, $(\mathrm{R}(g)\varphi)(g') = \varphi(g'g)$. For a free level K, evaluation at representatives of the classes is an isomorphism from the K-fixed forms onto the functions $\mathrm{Cl}(K) \to \mathbb{C}^{\mathrm{Wy}_I(\varkappa)}$; the form that takes the value v at the representative of one class and 0 at the others is a *delta form*.
Gauge. In Setting 2.2, for $g \in G_I$ and $\xi \in \mathbb{C}^{\mathrm{Wy}_I(\varkappa)}$, the number $\mathrm{gauge}_j^{\varkappa,\mathfrak{e}}(g;\xi)$ is the largest p-adic absolute value of a coordinate of $\rho_{\varkappa,\mathfrak{e}}((g_\tau^{-1})_\tau)\, j(\xi)$, where $g_\tau \in \mathrm{GL}_I(\mathbb{C}_p)$ is the image of g under the continuous extension to $\mathbb{A}_{E,\mathrm{f}}$ of the embedding $x \mapsto j(\iota_\tau(\overline{x}))$. A form φ *has j-gauge at most c at $d \in G_I$* if $\mathrm{gauge}_j^{\varkappa,\mathfrak{e}}(gd; \varphi(g)) \le c$ for every $g \in G_I$. The gauge is an ultrametric norm in ξ with $\mathrm{gauge}_j^{\varkappa,\mathfrak{e}}(g; z\xi) = |j(z)|_p\, \mathrm{gauge}_j^{\varkappa,\mathfrak{e}}(g;\xi)$, it does not change when g is multiplied on the right by an element that is integral and invertible at all primes above p, and for such g it is at most 1 exactly when all coordinates of $j(\xi)$ are integral.
Gram norm. $\|\xi\|^2$ is the real part of $\xi^* G \xi$, where G is the integer Gram matrix of Weyl's lattice in the weight $\varkappa$; it is positive definite, and its square root is a norm. Automorphisms of $\mathbb{C}$ act on vectors coordinate by coordinate.

LEAN in the background library: `Gf`, `stdRatPoints`, `Kf`, `principalLevel`, `maxCompactAt`, `IsLevel`, `StdClasses`, `classAut`, `WeylIndex`, `weylRep`, `Space.weightRep`, `boxes`, `diagramWeight`, `AlgForm`, `algFormRep`, `levelSpace`, `freeLevelEquiv`, `weylGauge`, `GaugeLe`, `integralFormsAt`, `embValRing`, `weylNormSq`

Input I2 (index and type; unit 149, the core). Let O be a commutative ring, V an O-module, M and S submodules of V and $z \in O$. *S has index dividing z in M* if for every natural number n and all $\Phi_1, \dots, \Phi_n \in M$ there is an $n \times n$ matrix u over O whose determinant divides z and with $\sum_c u_{ac}\Phi_c \in S$ for every a. An O-linear endomorphism y of V *has type (z, s, c) on M*, for $z \in O$, a natural number s and a real number c, if $y(M) \subseteq M$ and there is a natural number $l \le c$ such that $y(M) + z^s M$ has index dividing z^l in M. *Types add*: if $y_1, \dots, y_n$ have types $(z, s, c_1), \dots, (z, s, c_n)$ on M, then $y_1 \cdots y_n$ has type $(z, s, c_1 + \cdots + c_n)$ on M.

LEAN unit 149: IndexDvd, HasType, HasType.prod

Input I3 (objects of the data; units 152 and 141). In Setting 2.1: for each natural number i, $X_i(D)$ is an index set with $n_i(D)$ elements, one of them distinguished as the *last*; $Y_m(D)$ is an index set with $n_m(D) - 1$ elements. $N(D)$ is a prime number with $N(D) \equiv 1$ modulo $N_1(D)\,d$ and $m \log b \le \log N(D) \le 30\,m \log b$, and $\pi_N(D)$ is an integer of $E(D)$ with $\pi_N(D)\,\overline{\pi_N(D)} = N(D)$. $g(D) = g_0 g_1$ is the product of the two primes of the data, and $\pi_g(D)$ is the integer of $K(P)$ in the data, with $\pi_g(D)\,\overline{\pi_g(D)} = g(D)$. A finite place w of $E(D)$ is *chosen* if $\overline{\pi_g(D)} \in w$. For a finite set I, the *chosen level* $K_I^{\mathrm{ch}}(D)$ is the group of the $k \in K_I$ that are congruent to 1 modulo every chosen place; it contains $K_I(\overline{\pi_g(D)})$, and if $|I| \le 2\,n_m(D)$, every subgroup of $K_I^{\mathrm{ch}}(D)$ is free. The column counts $\varkappa_D$, the determinant exponents $\mathrm{last}_D(i, \cdot)$, the counts $\mathrm{kept}_D(h)$ and $\varkappa_D^Y$ are those of the unit on the CM curve (Input I6).

A *twist* of D is a continuous character ϑ of the norm-one adeles of $E(D)$, trivial on the norm-one elements of $E(D)$ and of weight 0 at the infinite places, together with a finite set $Q(\vartheta)$ of finite places and characters ϑ_w of the residue fields (its *tame part*) such that $\vartheta(y) = \prod_w \vartheta_w(y \bmod w)$ for y with integral finite part and trivial infinite part; $\vartheta_w = 1$ for $w \notin Q(\vartheta)$ and $\vartheta_w \ne 1$ for $w \in Q(\vartheta)$; $\overline{w} \notin Q(\vartheta)$ for $w \in Q(\vartheta)$; and $\mathrm{nrm}(\vartheta) = \prod_{w \in Q(\vartheta)} \mathrm{q}_w$ is prime to $g(D)\,N(D)$. The trivial character with $Q = \emptyset$ is the trivial twist 1, and $\mathrm{nrm}(1) = 1$.

LEAN unit 141: Data.X, Data.Y, Data.N, Data.N_modEq_one, Data.log_N_mem_Icc, Data.π, Data.piGBar, Data.IsChosen, Data.TameChar, Data.Twist, Data.TameChar.levelNorm · unit 152: Data.chosenLevel, Data.classAut_eq_bot_of_le_chosenLevel

Input I4 (kernels, data and the chain; background library, units 141, 147, 150). In Setting 2.1, let ϑ be a twist of D and i a natural number. $\mathcal{W}_i(D)$ is the *finite part* of the model of the Weil representation for the space of homomorphisms from $E^{X_i(D)}$ to $E^{X_{i+1}(D)}$: the complex span of the translates $\mathrm{U}(y)\,\mathrm{vac}$ of the vacuum vector by the finite Heisenberg operators $\mathrm{U}(y)$, where y is a matrix with rows $X_{i+1}(D)$ and columns $X_i(D)$ over $\mathbb{A}_{E(D),\mathrm{f}}$. It contains the *integral vacuum* $\theta^{\circ}_{D,i}$, a scalar multiple of the vacuum (unit 150). For a set Σ of finite places, P^{Σ} is the projection of $\mathcal{W}_i(D)$ onto the vectors that are vacua at the places of Σ and their conjugates.
The *lift kernel* $\mathrm{k}_{D,\vartheta,i}$ consists of a representation $\omega_{D,\vartheta,i}$ of $G_{X_i(D)} \times G_{X_{i+1}(D)}$ on $\mathcal{W}_i(D)$ (the Weil representation of the pair, with the characters of the $(i+1)$-st lift twisted by $\vartheta^{\,i+1}$) and of a linear map $\Phi \mapsto \mathrm{jm}_{D,\vartheta,i}(\Phi)$ from $\mathcal{W}_i(D)$ to the linear maps $\mathbb{C}^{\mathrm{Wy}_{X_i(D)}(\varkappa_D)} \to \mathbb{C}^{\mathrm{Wy}_{X_{i+1}(D)}(\varkappa_D)}$: the matrix of the jets, along the Schur matrix of $\varkappa_D$, of the Taylor series of the theta function of Φ at the scale $\overline{\pi_g(D)} \cdot \mathrm{per}(P, \mu(D))$. The *lift* with datum Φ is the map $\mathrm{lift}_{D,\vartheta,i}(\Phi)\colon \mathrm{St}_i(D) \to \mathrm{St}_{i+1}(D)$,

$$\big(\mathrm{lift}_{D,\vartheta,i}(\Phi)\varphi\big)(h) = \int_{\Gamma_{X_i(D)}\backslash G_{X_i(D)}} \mathrm{jm}_{D,\vartheta,i}\big(\omega_{D,\vartheta,i}(g,h)\Phi\big)\varphi(g)\,\mathrm{d}g,$$

a finite sum over classes, the measure giving volume 1 to $K_i^{\mathrm{col}}(D)$. It is equivariant: $\mathrm{lift}_{D,\vartheta,i}(\omega_{D,\vartheta,i}(a,b)\Phi)\,\mathrm{R}(a) = \mathrm{R}(b)\,\mathrm{lift}_{D,\vartheta,i}(\Phi)$. For points $d \in G_{X_i(D)}$, $d' \in G_{X_{i+1}(D)}$ and a real number c, in Setting 2.2, $\mathrm{Raise}_{\mathrm{k}_{D,\vartheta,i}}(d, d'; c)$ is the set of the $\Phi \in \mathcal{W}_i(D)$ with

$$\mathrm{gauge}_j^{\varkappa_D,\mathrm{last}_D(i+1,\,\cdot\,)}(hd'; \mathrm{jm}_{D,\vartheta,i}(\omega_{D,\vartheta,i}(g,h)\Phi)\xi) \\ \leq\ \max(c,0)\cdot \mathrm{gauge}_j^{\varkappa_D,\mathrm{last}_D(i,\,\cdot\,)}(gd;\xi)$$

for all g, h and ξ; it is closed under addition, and $z\Phi$ lies in $\mathrm{Raise}_{\mathrm{k}_{D,\vartheta,i}}(d, d'; c')$ when Φ lies in $\mathrm{Raise}_{\mathrm{k}_{D,\vartheta,i}}(d, d'; c)$ and $|j(z)|_p\, c \leq c'$ (unit 150).
A *tuple of translations* of D is a family $\mathsf{x} = (\mathsf{x}_i)_{i\in\mathbb{N}}$ of matrices x_i with rows $X_{i+1}(D)$ and columns $X_i(D)$ over $\mathbb{A}_{E(D),\mathrm{f}}$; its i-th datum is $\mathrm{td}_D(\mathsf{x}, i) = \mathrm{U}(\mathsf{x}_i)\,\theta^{\circ}_{D,i} \in \mathcal{W}_i(D)$. For a family $\Phi = (\Phi_i)_i$ with $\Phi_i \in \mathcal{W}_i(D)$, the *chain* of n lifts is

$$\mathrm{C}_{D,\vartheta}(\Phi)_n = \mathrm{lift}_{D,\vartheta,n-1}(\Phi_{n-1}) \circ \ldots \circ \mathrm{lift}_{D,\vartheta,0}(\Phi_0)\colon\ \mathrm{St}_0(D) \to \mathrm{St}_n(D),$$

and for a family $\Phi^{(2)} = (\Phi^{(2)}_\nu)_\nu$ with $\Phi^{(2)}_\nu \in \mathcal{W}_{2\nu}(D) \otimes \mathcal{W}_{2\nu+1}(D)$ the *chain of n pairs* $\mathrm{C}^{(2)}_{D,\vartheta}(\Phi^{(2)})_n\colon \mathrm{St}_0(D) \to \mathrm{St}_{2n}(D)$ is the composite, for $\nu < n$, of the maps that depend linearly on $\Phi^{(2)}_\nu$ and equal $\mathrm{lift}_{D,\vartheta,2\nu+1}(\Phi') \circ \mathrm{lift}_{D,\vartheta,2\nu}(\Phi)$ for $\Phi^{(2)}_\nu = \Phi \otimes \Phi'$. For a finitely supported complex function Ψ on the tuples of translations,

$$\mathrm{B}_D(\vartheta, \Psi) = \mathrm{restr}_D \circ \Big(\sum_{\mathsf{x}} \Psi(\mathsf{x}) \cdot \mathrm{C}_{D,\vartheta}(\mathrm{td}_D(\mathsf{x},\,\cdot\,))_m\Big)\Big|_{\mathrm{Col}(D)}$$

with the restriction restr_D of Definition 2.4. In Setting 2.2, $\mathrm{Col}_j(D)$ is the O_j-module of the columns that have j-gauge at most 1 at 1.

LEAN background library: `Space.finitePart`, `Space.finHeisOp`, `Space.heisFinPart`, `Space.projOn`, `Kernel`, `Kernel.lift`, `Kernel.lift_equivariant`, `thetaKernel` · unit 141: `chain`, `chain₂`, `projOnFinPart` · unit 147: `Data.homX`, `Data.liftKernel`, `Data.Translations`, `Data.translateDatum`, `Data.chainComb`, `Data.Bμ`, `Data.integralColumns` · unit 150: `Parameters.intVacuum`, `Kernel.raises`, `Kernel.smul_mem_raises`

Input I5 (induced primes and marked sets; unit 147). In Settings 2.1 and 2.2, for $\tau \in \mathrm{Pl}(D)$ the prime $\mathfrak{l}_j(\tau)$ of $E(D)$ *induced* by τ is the set of the integers y of $E(D)$ with $|j(\iota_\tau(y))|_p < 1$; it lies above p. A finite set $\mathcal{L}$ of finite places of $E(D)$ is *marked* (for D, p and j) if (M1) $\bar{\mathfrak{l}} \notin \mathcal{L}$ for every $\mathfrak{l} \in \mathcal{L}$; (M2) if $\mathcal{L}$ is not empty, then p is served for D; (M3) every $\mathfrak{l} \in \mathcal{L}$ is $\mathfrak{l}_j(\tau)$ for some $\tau \in \mathrm{tops}(D)$; (M4) p is small and not split for D. It is a *full mark* if moreover, for every $\tau \in \mathrm{tops}(D)$, one of $\mathfrak{l}_j(\tau)$, $\overline{\mathfrak{l}_j(\tau)}$ lies in $\mathcal{L}$. For a finite set $\mathcal{L}$ of finite places put

$$\Xi_j(\mathcal{L}) = \{\tau \in \mathrm{Pl}(D) : \ \mathfrak{l}_j(\tau) \in \mathcal{L}\}, \quad A_D(\mathfrak{l}) = a(D) \cdot \#\{\tau \in \mathrm{tops}(D) : \ \mathfrak{l}_j(\tau) = \mathfrak{l}\},$$

and for a natural number i and $\tau \in \mathrm{Pl}(D)$

$$\mathrm{lc}_D(i,\tau) = \sum_{h < n_{i+1}(D),\, h \neq 1} \varkappa_D(\tau, h), \qquad \mathrm{lb}_D(i,\tau) = \sum_{h < n_{i+1}(D),\, h \neq 1} h\, \varkappa_D(\tau, h).$$

LEAN unit 147: `Data.inducedPlace`, `Data.IsMarked`, `Data.IsFullMark`, `Data.evenPlaces`, `Data.markExponent`, `Data.lowColumns`, `Data.lowBoxes`

Input I6 (the bookkeeping of the unit on the CM curve; unit 150). In Setting 2.1, for natural numbers p, h, i, e, a real number y, places $\tau, \tau' \in \mathrm{Pl}(D)$ and a finite set $\Xi \subseteq \mathrm{Pl}(D)$; a pair (e,p) is *live* if $e \geq 1$ and (e is odd or $p = d$):

$$\mathrm{ask}_{D,p}(y,\tau) = \frac{1-\chi_{K(P)}(p)}{2} \sum_{e\colon (e,p) \text{ live}} \Big(\frac{y}{p^e} - w_D([-p^e]_D \cdot \tau)\Big)_+,$$

$$\mathrm{forced}(D,p) = \sum_{\tau \in \mathrm{Pl}(D)} \mathrm{ask}_{D,p}(w_D(\tau),\tau),$$

$$\mathrm{gain}(D,p) = m\Big(\sum_{\tau \in \mathrm{tops}(D)} \mathrm{ask}_{D,p}(a(D),\tau) + k(D)\,\mathrm{forced}(D,p)\Big),$$

$$\mathrm{loss}(D,p) = f(D)\,(\log b)^2\,\frac{a(D)}{p} \cdot m^*,$$

where $(\cdot)_+$ is the positive part, and $m^* = m$ if p is short for D, $m^* = 1$ otherwise; *p is short for D* if m divides $p-1$ (truncated) or $p+1$; *p is small and not split for D* if $p < x_1$ and $\chi_{K(P)}(p) \neq 1$; $\mathfrak{r}(D,p)$ is the least natural number with $m \leq p^{\mathfrak{r}(D,p)}$ (for $p > 1$); $\Delta(D,p) = \mathfrak{r}(D,p)\,f(D)\,(n_m(D) + \mathfrak{r}(D,p))$; $\mathrm{jc}(D,p) = \lceil 2mf(D)a(D)/p \rceil$; $o_0(D,p)$ is the order of the class modulo N_0 of the place $[-p]_D$; *p is served for D* if $\chi_{K(P)}(p) = -1$, p is not short for D, and $o_0(D,p)$ is odd and at most $\mathfrak{r}(D,p)$. The pair (τ,e) *is a collision of p outside Ξ for D* if (e,p) is live and both τ and $[-p^e]_D \cdot \tau$ lie in $\mathrm{tops}(D) \setminus \Xi$, and $\mathrm{coll}(D,p,\Xi)$ is the sum of the numbers $a(D)/p^e$ over all the pairs (τ,e) that are collisions of p outside Ξ for D. For column counts $\varkappa$ on a set of places and a natural number n', with $\mathrm{one}_\tau = [\tau \notin \Xi]\,[1 < n']\,\varkappa(\tau,1)$, $\mathrm{low}_\tau = \sum_{h<n',h\neq 1} \varkappa(\tau,h)$ and $\mathrm{bx}_\tau = \sum_{h<n',h\neq 1} h\,\varkappa(\tau,h)$,

$$\mathrm{cc}_p(\varkappa,n';\Xi;\tau,\tau';e) = \min\Big(\frac{\mathrm{one}_\tau}{p^e},\, \mathrm{low}_{\tau'} + \mathrm{one}_{\tau'}\Big) + \frac{\mathrm{low}_\tau}{p^e} + \min\Big(\frac{\mathrm{bx}_\tau}{p^e},\, \mathrm{bx}_{\tau'}\Big).$$

The column counts of the data are $\varkappa_D(\tau,1) = [\tau \in \mathrm{tops}(D)]\,a(D)$, $\varkappa_D(\tau,k(D)) = \mathrm{wt}_D(1,\tau) - \lfloor (k(D)+t(D)+1)/2 \rfloor$ (truncated; for $k(D) \neq 1$), $\varkappa_D(\tau,h) = \mathrm{gap}(D) - (t(D)+1)$ for $h > k(D)$ with $h \equiv k(D)$ modulo $t(D)+1$, and $\varkappa_D(\tau,h) = 0$ otherwise, where $\mathrm{gap}(D) = 2r^8$; $\mathrm{last}_D(0,\tau) = 0$ and $\mathrm{last}_D(i,\tau) = \lfloor n_{i-1}(D)/2 \rfloor - \mathrm{wt}_D(i,\tau)$ for $i \geq 1$; $\mathrm{kept}_D(h) = r^8 - \lfloor (t(D)+1)/2 \rfloor$ (truncated) if $h \geq k(D)$ and $h \equiv k(D)$ modulo $t(D)+1$, and 0 otherwise; $\varkappa^Y_D(\tau,h) = \mathrm{kept}_D(h) + (\varkappa_D(\tau,h+1) - \mathrm{kept}_D(h+1))$. Finally $\varrho_d(p)$ is $1/(p^2-1)$ if $\chi_{K(P)}(p) = -1$, $1/(2(p-1))$ if $p = d$, and $1/(p-1)$ otherwise; for a prime p with $\chi_{K(P)}(p) \neq 1$ and a real number c,

$$\sum_{e\colon (e,p) \text{ live}} \frac{c}{p^e} = \frac{2}{1-\chi_{K(P)}(p)}\; c\,\Big(\frac{1}{p-1} - \varrho_d(p)\Big), \tag{1}$$

and the series defining $\mathrm{ask}_{D,p}(y,\tau)$ converges.

LEAN unit 150: `Data.localAsk`, `Data.forcedPart`, `Data.gain`, `Data.loss`, `Data.IsShortPrime`, `Data.IsSmallNonsplit`, `Data.ceilLogM`, `Data.deltaP`, `Data.jetCeil`, `Data.o₀`, `Data.IsServed`, `Data.IsCollision`, `Data.collisionSum`, `couplingCount`, `Data.κ`, `Data.lastEntry`, `Data.keptColumns`, `Data.columnsY`, `hasSum_live_inv_pow`, `Data.summable_localAsk` · background library: `nonsplitWeight`

The definitions of this unit.

Definition 2.3 (the primes dividing $\pi_N(D)$; levels, columns, stages)**.** In Setting 2.1, let i be a natural number.

(a) $\mathrm{Pl}_\pi(D)$ is the finite set of the finite places $\mathfrak{p}$ of $E(D)$ with $\pi_N(D) \in \mathfrak{p}$.

(b) The *column level* $K_i^{\mathrm{col}}(D)$ is the group of the elements k of the chosen level $K^{\mathrm{ch}}_{X_i(D)}(D)$ whose last row is congruent to $(0, \dots, 0, 1)$ modulo $\pi_N(D)$: every entry of the last row of $k - 1$ lies in $\pi_N(D)\,\widehat{\mathcal{O}}_{E(D)}$.

(c) The space of *columns* $\mathrm{Col}(D)$ is the space of the forms φ in the space $\mathrm{Alg}_{X_0(D)}(\varkappa_D, \mathrm{last}_D(0, \cdot))$ that are fixed under right translation by $K_0^{\mathrm{col}}(D)$.

(d) The *stage* $\mathrm{St}_i(D)$ is the space $\mathrm{Alg}_{X_i(D)}(\varkappa_D, \mathrm{last}_D(i, \cdot))$.

LEAN `Data.piPlaces` · `Data.columnLevel` · `Data.Column` · `Data.Stage`

Definition 2.4 (branching and the restriction)**.** In Setting 2.1, let $\tau \in \mathrm{Pl}(D)$.

(a) lst_D is a chosen bijection from $Y_m(D)$ onto the set of the elements of $X_m(D)$ other than the last; both sets have $n_m(D) - 1$ elements.

(b) $\mathrm{bd}_D \colon G_{Y_m(D)} \to G_{X_m(D)}$ is the homomorphism $h \mapsto \mathrm{diag}(h, 1)$, the block h being placed through lst_D and the 1 at the last index.

(c) The *branching exponent* is the integer $\mathrm{be}_D(\tau) = \sum_{0<h<n_m(D)} \big(\varkappa_D(\tau, h) - \mathrm{kept}_D(h)\big) + \mathrm{last}_D(m, \tau)$.

(d) $\mathrm{Br}(D)$ is a chosen matrix with integer entries, with rows indexed by $\mathrm{Wy}_{Y_m(D)}(\varkappa_D^Y)$ and columns by $\mathrm{Wy}_{X_m(D)}(\varkappa_D)$, that has the properties provided by the branching theorem of the background library (branching from GL_{n+1} to GL_n with multiplicity one): (Br1) over every field of characteristic 0, for families (g_τ) of invertible matrices indexed by $Y_m(D)$ and of units (s_τ),

$$\mathrm{Br}(D) \cdot \rho_{\varkappa_D, \mathrm{last}_D(m,\cdot)}\big((\mathrm{diag}(g_\tau, s_\tau))_\tau\big) = \prod_\tau s_\tau^{\mathrm{be}_D(\tau)} \cdot \rho_{\varkappa_D^Y, \mathrm{last}_D(m,\cdot)}\big((g_\tau)_\tau\big) \cdot \mathrm{Br}(D);$$

(Br2) $\|\mathrm{Br}(D)\xi\|^2 \le L^{6L}\,\|\xi\|^2$ for every vector ξ and every natural number $L \ge \sum_\tau \sum_{k(D)\le h<n_m(D)-1} h\,\varkappa_D^Y(\tau, h)$; (Br3) a spanning property of the library, which this unit does not use.

(e) The *restriction* $\mathrm{restr}_D \colon \mathrm{St}_m(D) \to \mathrm{Alg}_{Y_m(D)}(\varkappa_D^Y, \mathrm{last}_D(m, \cdot))$ is the $\mathbb{C}$-linear map

$$(\mathrm{restr}_D\,\varphi)(h) = \prod_{\tau' \in \mathrm{Pl}(D)} \iota_{\tau'}\big(\overline{\pi_N(D)}\big)^{-\mathrm{be}_D(\tau')} \cdot \mathrm{Br}(D)\,\varphi(\mathrm{bd}_D(h)) \qquad (h \in G_{Y_m(D)}).$$

LEAN `Data.lastIndexEquiv` · `Data.borderU` · `Data.branchExp` · `Data.branching` · `Data.restrictToY` · the branching theorem is `exists_branching_matrix` of the background library; the catalogue of the handout records it as a known result (branching from $\mathrm{GL}(n+1)$ to $\mathrm{GL}(n)$ with multiplicity one), *reference to be confirmed*

That $\mathrm{restr}_D\,\varphi$ is again an algebraic form follows from (Br1), applied to a rational point γ of $G_{Y_m(D)}$ and the units $s_\tau = 1$; the formal definition contains this verification.

Definition 2.5 (short forms)**.** Let E be a CM field with a CM type, I a finite set, $\varkappa$ column counts and $\mathfrak{e}$ determinant exponents. For a real number B and a form $\varphi \in \mathrm{Alg}_I(\varkappa, \mathfrak{e})$ we say that φ is B-short if

(i) for every prime number p and every ring homomorphism $j\colon \mathbb{C} \to \mathbb{C}_p$, φ has j-gauge at most 1 at 1; and

(ii) $\|\sigma \circ \varphi(g)\|^2 \le B$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$ and every $g \in G_I$.

LEAN `IsShort`

Lemma 2.6 (a central element at a prime dividing $\pi_N(D)$)**.** *In Setting 2.1, let $\mathfrak{p} \in \mathrm{Pl}_\pi(D)$. There is a finite adele c of $E(D)$ with $c\overline{c} = 1$ such that: c is integral; $c_w = 1$ at every finite place w with $w \ne \mathfrak{p}$ and $\overline{w} \ne \mathfrak{p}$; and every natural number n for which $c_{\mathfrak{p}}^n - 1$ has $\mathfrak{p}$-adic absolute value less than 1 is divisible by $N(D) - 1$.*

LEAN `Data.exists_centralGen` · proof in §4

Definition 2.7 (central translations and ramified columns)**.** (a) Let E be a CM field with a CM type, I a finite set, $\varkappa$ column counts, $\mathfrak{e}$ determinant exponents, J a level of G_I and χ a character of J. The *central representation* is the representation $z \mapsto \mathrm{R}(z \cdot 1_I)$, by right translation by scalar matrices, of the group of the finite adeles z of E with $z\overline{z} = 1$ on the space of the forms $\varphi \in \mathrm{Alg}_I(\varkappa, \mathfrak{e})$ with $\mathrm{R}(k)\varphi = \chi(k)\varphi$ for all $k \in J$. (Scalar matrices commute with J, so this space is stable.)

(b) In Setting 2.1, for $\mathfrak{p} \in \mathrm{Pl}_\pi(D)$, $\mathfrak{c}_D(\mathfrak{p})$ is a chosen adele c with the properties of Lemma 2.6.

(c) In Setting 2.1, cen_D is the central representation of (a) on $\mathrm{Col}(D)$, that is, for $E = E(D)$, $I = X_0(D)$, the weight $(\varkappa_D, \mathrm{last}_D(0, \cdot))$, $J = K_0^{\mathrm{col}}(D)$ and $\chi = 1$.

(d) In Setting 2.1, for a finite set $\Sigma \subseteq \mathrm{Pl}_\pi(D)$, the endomorphism $\mathrm{ram}_D(\Sigma) = \prod_{\mathfrak{p}\in\Sigma}(1 - \mathrm{cen}_D(\mathfrak{c}_D(\mathfrak{p})))$ of $\mathrm{Col}(D)$; the factors commute.

(e) In Setting 2.1, the space of *ramified columns* $\mathrm{Col}^{\mathrm{ram}}(D)$ is the image of $\mathrm{ram}_D(\mathrm{Pl}_\pi(D))$.

LEAN `algFormCentralRep` · `Data.centralGen` · `Data.centralRep` · `Data.ramifyingOp` · `Data.ramifiedColumns`

Definition 2.8 (admissible translations, paired translates, good data)**.** In Setting 2.1:

(a) For a natural number q, $\mathrm{Adm}_D(q)$ is the set of the tuples of translations x such that, for every i, every entry of $\pi_g(D) \cdot \mathsf{x}_i$ is integral at every finite place w with $q \notin w$.

(b) For a family $v = (v_i)_{i\in\mathbb{N}}$ with $v_i \in \mathcal{W}_i(D)$ and a natural number ν,

$$\mathrm{pt}_D(v)_\nu = \sum_y \mathrm{U}(\mathrm{col}(-y/N))v_{2\nu} \otimes \mathrm{U}(\mathrm{row}(-\overline{y/N}))v_{2\nu+1} \ \in\ \mathcal{W}_{2\nu}(D) \otimes \mathcal{W}_{2\nu+1}(D),$$

where $N = N(D)$; y runs over the vectors indexed by $X_{2\nu+1}(D)$ with entries in $(\mathbb{Z}/N)^{\mathrm{Pl}_\pi(D)}$; y/N is the vector of finite adeles whose component at $\mathfrak{p} \in \mathrm{Pl}_\pi(D)$ is the least non-negative representative of $y_{\mathfrak{p}}$ divided by N, and 0 at the other places; $\mathrm{col}(z)$ is the matrix with rows $X_{2\nu+1}(D)$ and columns $X_{2\nu}(D)$ whose last column is z and whose other columns are 0; and $\mathrm{row}(z)$ is the matrix with rows $X_{2\nu+2}(D)$ and columns $X_{2\nu+1}(D)$ whose last row is z and whose other rows are 0.

(c) Let ϑ be a twist of D and Ψ a finitely supported complex function on the tuples of translations of D. We say that Ψ is good for (D, ϑ) if the following three conditions hold.

(G1) The support of Ψ lies in $\mathrm{Adm}_D(\mathrm{nrm}(\vartheta)\cdot N(D))$; for every x, every prime number ℓ prime to $\mathrm{nrm}(\vartheta)$ and every ring homomorphism $\iota\colon \mathbb{C}\to\mathbb{C}_\ell$, $|\iota(\Psi(\mathsf{x}))|_\ell \le 1$; and if $Q(\vartheta)=\emptyset$, then $\sum_{\mathsf{x}}|\sigma(\Psi(\mathsf{x}))|\le \exp(x^{60})$ for every $\sigma\in\mathrm{Aut}(\mathbb{C})$.
(G2) (*joint invariance*) For every family $(a_i)_{i\in\mathbb{N}}$ with $a_i\in K_i^{\mathrm{col}}(D)$ and $a_i=1$ for $i\ge m$, the tensor $\sum_{\mathsf{x}}\Psi(\mathsf{x})\cdot\bigotimes_{i<m}\mathrm{td}_D(\mathsf{x},i)$ in $\bigotimes_{i<m}\mathcal{W}_i(D)$ is fixed by $\bigotimes_{i<m}\omega_{D,\vartheta,i}(a_i,a_{i+1})$; in the formal text: the two tensors have the same image under every linear map.
(G3) (*the shape at N*) For every form $\varphi\in\mathrm{St}_0(D)$,

$$\sum_{\mathsf{x}}\Psi(\mathsf{x})\cdot \mathrm{C}_{D,\vartheta}(\mathrm{td}_D(\mathsf{x},\,\cdot\,))_m\,\varphi \\ = \sum_{\mathsf{x}}\Psi(\mathsf{x})\cdot \mathrm{C}^{(2)}_{D,\vartheta}(\nu\mapsto \mathrm{pt}_D(\mathrm{P}^{\mathrm{Pl}_\pi(D)}\mathrm{td}_D(\mathsf{x},\,\cdot\,))_\nu)_{m/2}\,\varphi.$$

LEAN `Data.admissibleTranslations` · `Data.pairTranslates` · `Data.IsGoodDatum` · with `IsAdmissibleCoeffs`, `IsJointlyInvariant`, `translateSum` of unit 141

In (G3) the formal text writes the number of pairs as $\lfloor m/2\rfloor-1+1$ and the number of lifts on the left as twice that; m is a power of 2 with $m\ge 2$, so these are $m/2$ and m. That good data exist, and that the rows of the handout are of the form $\mathrm{B}_D(\vartheta,\Psi)\psi$ for good Ψ, is proved in the unit on the seesaw; this unit only uses the three conditions.

Definition 2.9 (the jet exponent, the loss of a lift, the mark operators)**.** In Setting 2.1:

(a) For natural numbers p and i and a finite set $\Xi\subseteq\mathrm{Pl}(D)$, put $n'=n_{i+1}(D)$, $L=\mathrm{boxes}(\varkappa_D,n')$ and $l=\lfloor\log_p L\rfloor$ (which is 0 if $p\le 1$ or $L=0$). The *jet exponent* is the real number

$$\mathrm{je}_D(p,i,\Xi)=\Big(\frac{1}{p-1}-\varrho_d(p)\Big)L-f(D)\,n'^3\,(l+1) \\ +\frac{\chi_{K(P)}(p)-1}{2}\sum_{\substack{0\le e\le l\\ e\text{ odd or }p=d}}\ \sum_{\tau\in\mathrm{Pl}(D)}\mathrm{cc}_p(\varkappa_D,n';\Xi;\tau,[-p^e]_D\cdot\tau;e).$$

(b) In Setting 2.2, for a finite set $\mathcal{L}$ of finite places of $E(D)$ and a natural number i, the *loss of the lift* i is $\mathrm{ll}_{D,p}(\mathcal{L},i)=\mathfrak{r}(D,p)\sum_{\tau\in\Xi_j(\mathcal{L})}\mathrm{lc}_D(i,\tau)-\mathrm{je}_D(p,i,\Xi_j(\mathcal{L}))$.

(c) In Setting 2.2, for a marked set $\mathcal{L}$ and $\mathfrak{l}\in\mathcal{L}$, the *mark operator* $\Upsilon_{D,\mathcal{L}}(\mathfrak{l})$ is the O_j-linear endomorphism $\varphi\mapsto p^{A_D(\mathfrak{l})}\cdot T_{\mathfrak{l},\mathfrak{r}(D,p)}\varphi$ of $\mathrm{Col}(D)$, where $T_{\mathfrak{l},s}$ is the Hecke operator of the double coset of $\mathrm{diag}(\varpi 1_s,1)$ at the split place $\mathfrak{l}$ (Input I8); columns are fixed by $K_{X_0(D)}[\mathfrak{l}]$, and the operator preserves $\mathrm{Col}(D)$ (Lemma 6.5).

LEAN `Data.jetExponent` · `Data.liftLoss` · `Data.IsMarked.markOperator`

For $p=d$ the sum over e in (a) contains $e=0$; the formal text has it so.

Definition 2.10 (the height per row, the non-split primes, the allowed loss)**.** In Setting 2.1:

(a) The *height per row* is

$$h_{\mathrm{row}}(D) = m\Big(\sum_{\tau\in\mathrm{tops}(D)} (\mathrm{st}_D(a(D)) - \mathrm{Ask}_D(a(D),\tau)) + k(D)\,\mathsf{U}(D)\Big) - \#\mathrm{tops}(D)\cdot a(D)\cdot \log N(D),$$

where, for a natural number n, a real number y and a place τ,

$$\mathrm{st}_D(n) = \log n! - n\log(\Omega(P,\mu(D))/g(D)),$$
$$\mathrm{Ask}_D(y,\tau) = \sum_{\nu=1}^{a(D)} \lambda_{d,1}(\nu)\big(y/\nu - w_D([-\nu]_D\cdot\tau)\big)_+,$$
$$\mathsf{U}(D) = \sum_{\tau\in\mathrm{Pl}(D)} (\mathrm{st}_D(\mathrm{wt}_D(0,\tau)) - \mathrm{Ask}_D(w_D(\tau),\tau))$$

(the last is the cost of one direction; these are objects of unit 152 and of the overview).

(b) $\mathrm{NS}(D)$ is the finite set of the prime numbers $p < \lceil x_1\rceil$ with $\chi_{K(P)}(p) \neq 1$.

(c) For a natural number p, the *allowed loss* $\mathrm{err}(D,p)$ is $3\,\mathrm{loss}(D,p)$ if $p\in\mathrm{NS}(D)$; $4\,m\,f(D)\,b$ if $p\notin\mathrm{NS}(D)$ and $p = N(D)$; and 0 otherwise.

LEAN `Data.heightPerRow` · `Data.nonsplitPrimes` · `Data.allowedLoss` · unit 152: `Data.jetSize`, `Data.primeSum`, `Data.dirCost`

3 Levels and columns

Lemma 3.1 (the primes dividing $\pi_N(D)$)**.** *In Setting 2.1:*

(a) $\pi_N(D)\neq 0$.

(b) A finite place w of $E(D)$ lies in $\mathrm{Pl}_\pi(D)$ if and only if $\pi_N(D)\in w$.

(c) For every $\mathfrak{p}\in\mathrm{Pl}_\pi(D)$: $\overline{\mathfrak{p}}\notin\mathrm{Pl}_\pi(D)$, the residue field of $\mathfrak{p}$ has $N(D)$ elements, and $N(D)$ is a uniformiser at $\mathfrak{p}$. For every finite place w of $E(D)$: $N(D)\in w$ if and only if $w\in\mathrm{Pl}_\pi(D)$ or $\overline{w}\in\mathrm{Pl}_\pi(D)$.

(d) $\#\,\mathrm{Pl}_\pi(D) = f(D)$.

Proof. (a) and (b) follow from $\pi_N(D)\overline{\pi_N(D)} = N(D)\neq 0$ and the definition. (c) Since $N(D)\equiv 1$ modulo $N_1(D)\,d$, the prime $N(D)$ splits completely in $E(D)$ (unit 152): every place above it has residue field of $N(D)$ elements and is different from its conjugate, and a place above it contains exactly one of $\pi_N(D)$, $\overline{\pi_N(D)}$. (d) The same input gives a bijection between $\mathrm{Pl}(D)$ and the primes that contain $\pi_N(D)$. □

LEAN `Data.π_ne_zero` · `Data.mem_piPlaces` · `Data.isSplitLevel_piPlaces` · `Data.card_piPlaces` · inputs of unit 152: `Parameters.q_eq_of_modEq_one`, `Parameters.star_mem_iff`, `Parameters.nonempty_infinitePlace_equiv_finPlace`

Lemma 3.2 (the column levels)**.** *In Setting 2.1, let i be a natural number.*

(a) $K_{X_i(D)}(\overline{\pi_g(D)}\,\pi_N(D))\subseteq K_i^{\mathrm{col}}(D)$.

(b) $K_i^{\mathrm{col}}(D)$ *is a level: an open subgroup of* $K_{X_i(D)}$.

(c) $X_i(D)$ *has* $n_i(D)$ *elements.*

(d) $K_0^{\mathrm{col}}(D)$ *is free:* $\Gamma_{X_0(D)}\cap g\,K_0^{\mathrm{col}}(D)\,g^{-1} = \{1\}$ *for every* $g\in G_{X_0(D)}$.

(e) *An element k of $G_{X_i(D)}$ lies in $K_i^{\mathrm{col}}(D)$ if and only if $k \in K_{X_i(D)}^{\mathrm{ch}}(D)$ and, for every $\mathfrak{p} \in \mathrm{Pl}_\pi(D)$ and every index c, the entry of $k_{\mathfrak{p}} - 1$ in the last row and the column c lies in the maximal ideal at $\mathfrak{p}$.*

(f) *If w is a finite place of $E(D)$ with $N(D) \in w$, then neither w nor $\overline{w}$ is chosen.*

(g) *If S is a set of finite places of $E(D)$ such that $N(D) \in w$ for every $w \in S$, then every element of $K_{X_i(D)}$ whose components away from the places of S and their conjugates are 1 lies in $K_{X_i(D)}^{\mathrm{ch}}(D)$.*

(h) *If ℓ is a prime number with $\ell \neq N(D)$ and w a finite place of $E(D)$ with $\ell \in w$, then $N(D) \notin w$ and $\overline{w} \notin \mathrm{Pl}_\pi(D)$.*

(i) *In Setting 2.2, $|j(\overline{\pi_g(D)})|_p \leq 1$.*

Proof. (a) An element of the principal level of $\overline{\pi_g(D)}\pi_N(D)$ is congruent to 1 modulo the chosen places and modulo $\pi_N(D)$. (b) $K_i^{\mathrm{col}}(D)$ lies in $K_{X_i(D)}$ and contains the open group of (a). (c) is the definition of the index sets. (d) $K_0^{\mathrm{col}}(D)$ is a subgroup of the chosen level and $n_0(D) \leq 2n_m(D)$ (Input I3). (e) unfolds Definition 2.3(b) place by place: by Lemma 3.1(c), $\pi_N(D)$ is a uniformiser at the places of $\mathrm{Pl}_\pi(D)$ and a unit elsewhere. (f) A chosen place lies above g_0 or g_1, and these primes differ from $N(D)$ by size. (g) follows from (f): such an element is 1 at every chosen place. (h) A prime ideal contains only one prime number; the second claim is Lemma 3.1(c). (i) $\pi_g(D)$ is an algebraic integer, and ring homomorphisms preserve integrality. □

LEAN `Data.principalLevel_le_columnLevel` · `Data.isLevel_columnLevel` · `Data.card_X` · `Data.classAut_columnLevel_eq_bot` · `Data.mem_columnLevel` · `Data.not_isChosen_of_N_mem` · `Data.mem_chosenLevel_of_supportedOn` · `Data.notMem_piPlaces_of_ne` · `Data.norm_star_piG_le_one`

Lemma 3.3 (two bounds for the weight)**.** *In Setting 2.1, let $\tau \in \mathrm{Pl}(D)$.*

(a) *$\varkappa_D(\tau, 1) - 2b \leq \mathrm{be}_D(\tau)$, as real numbers.*

(b) *$\sum_{h<\#X_0(D)} h\,\varkappa_D(\tau, h) \leq a(D)$.*

Proof. (a) In $\mathrm{be}_D(\tau)$ the term $h = 1$ of the sum is $\varkappa_D(\tau, 1)$, the other terms are non-negative, and $\mathrm{last}_D(m, \tau) \geq -\mathrm{wt}_D(m, \tau) \geq -2b$ by the bound $\mathrm{wt}_D(i, \tau) \leq 2b$ for $i \leq 2m$ of unit 152. (b) $X_0(D)$ has $k(D)$ elements, and among the heights $h < k(D)$ only $h = 1$ has columns, namely $a(D)$ at the tops and none elsewhere. □

LEAN `Data.le_branchExp` · `Data.sum_height_mul_kappa_le` · input of unit 152: `Data.w_le_two_mul_b`

4 Short forms span

Lemma 4.1 (differences of translates)**.** *Let E be a CM field with a CM type, I a finite set, $\varkappa$ column counts and $\mathfrak{e}$ determinant exponents. Let B be a real number, $k \in K_I$ and $\varphi \in \mathrm{Alg}_I(\varkappa, \mathfrak{e})$. If φ is B-short, then $(\varphi - \mathrm{R}(k)\varphi)$ is $4B$-short.*

Proof. At a p-adic place the forms of gauge at most 1 at 1 form a module, and the translate $\mathrm{R}(k)\varphi$ is one of them: its value at g is $\varphi(gk)$, and the gauge of a vector at g equals its gauge at gk, because the gauge is invariant under K_I (Input I1). For $\sigma \in \mathrm{Aut}(\mathbb{C})$ and a point g put $\xi = \sigma \circ \varphi(g)$, $\xi' = \sigma \circ \varphi(gk)$; by the parallelogram law for the Gram form, $\|\xi - \xi'\|^2 \leq \|\xi - \xi'\|^2 + \|\xi + \xi'\|^2 = 2\|\xi\|^2 + 2\|\xi'\|^2 \leq 4B$. □

LEAN `IsShort.sub_translate`

Input I7 (short bases, delta forms, Schur matrices; background library). Let E be a CM field with f infinite places, I a finite set and $N' \geq 1$ a natural number that lies in the different of E over $\mathbb{Q}$. (i) For every $g \in G_I$ there is an invertible matrix $Y(g)$ over E, indexed by I, such that $g^{-1}Y(g)$ has entries in $\widehat{\mathcal{O}}_E$ and every entry of $Y(g)$ has absolute value at most $6N'(f|I|)^2$ at every embedding $E \to \mathbb{C}$. (ii) For a free level K the delta form with value v at the representative g_c of a class has, at every point, j-gauge at most $\mathrm{gauge}_j^{\varkappa,\mathfrak{e}}(g_c; v)$, and its values ξ satisfy $\|\sigma \circ \xi\|^2 \leq \|\sigma \circ v\|^2$ for $\sigma \in \mathrm{Aut}(\mathbb{C})$. (iii) The *evaluation vectors* $\mathrm{ev}(y) \in \mathbb{C}^{\mathrm{Wy}_I(\varkappa)}$, the values at a point y of the basis polynomials of the weight $\varkappa$, span $\mathbb{C}^{\mathrm{Wy}_I(\varkappa)}$ when y runs over the points whose coordinates are roots of unity; they have integer polynomial coordinates, and $\rho_{\varkappa,0}(z)\,\mathrm{ev}(y)$ is the evaluation vector at the transformed point $z \cdot y$. In particular $\mathrm{ev}(y) = \rho_{\varkappa,0}(\mathrm{sq}(y))\,\mathrm{ev}(1)$, where $\mathrm{sq}(y)$ is the family of the square matrices formed by the first $|I|$ columns of y at each place and $\mathrm{ev}(1)$ is the evaluation vector at the unit point; and $\|\mathrm{ev}(1)\|^2 = 1$. (iv) For families $z = (z_\tau)$ of complex matrices indexed by I there are real numbers $c_{\tau,\nu} \geq 0$, with $\sum_\nu c_{\tau,\nu}$ at most the sum of the squares of the absolute values of the entries of z_τ, such that the Schur matrix of z multiplies $\|\cdot\|^2$ by at most $\prod_\tau \prod_{\nu<|I|} c_{\tau,\nu}^{\,l(\tau,\nu)}$, where $l(\tau,\nu)$ is the length of the row of index ν at τ.

LEAN background library: `Space.exists_short_basis_stdSpace`, `weylGauge_deltaForm_le`, `weylNormSq_deltaForm_le`, `PolyRep.ext_weylEval`, `schur_mulVec_weylEval_of_detExp_zero`, `PolyRep.schur_squarePart_mulVec`, `PolyRep.weylNormSq_weylEval_permArray`, `weylNormSq_schur_le`; the catalogue classes them as elementary and gives no reference for them: *reference to be confirmed*; the header of the file of the last one names the contravariant form on Weyl's lattice

Proposition 4.2 (short forms span the forms of a free level)**.** *Let E be a CM field with a CM type, I a non-empty finite set, $\varkappa$ column counts and $\mathfrak{e}$ determinant exponents. Let K be a level of G_I that is free, $N' \geq 1$ a natural number that lies in the different of the ring of integers of E over $\mathbb{Z}$, M a natural number with*

$$|I| \cdot \#\,\mathrm{Pl}(E) + \mathrm{boxes}(\varkappa, |I|) \;\leq\; M,$$

and B a real number with $\left(6N'(\#\,\mathrm{Pl}(E)\cdot|I|)^2\right)^{6M} \leq B$. Then the space of the K-fixed forms in $\mathrm{Alg}_I(\varkappa,\mathfrak{e})$ is spanned over $\mathbb{C}$ by its elements φ with φ is B-short.

Proof. Put $R = 6N'(f|I|)^2$ with $f = \#\,\mathrm{Pl}(E)$. By Input I1 every K-fixed form is a sum of delta forms, so it suffices to treat the delta forms at one class, with representative g. Take the matrices $Y(\,\cdot\,)$ of Input I7(i). For a ring homomorphism ϕ from $\mathbb{C}$ to a normed field, write ϕ_τ for the embedding $y \mapsto \phi(\iota_\tau(\overline{y}))$ of E and put

$$A_\phi = \prod_\tau \det \phi_\tau(Y(g^{\mathfrak{e}(\tau)})) \cdot \rho_{\varkappa,0}((\phi_\tau(Y(g)))_\tau).$$

The matrix $A = A_{\mathrm{id}}$ is invertible, so by Input I7(iii) the vectors $A\,\mathrm{ev}(y)$, y with coordinates roots of unity, span $\mathbb{C}^{\mathrm{Wy}_I(\varkappa)}$; and $\phi \circ (A\,\mathrm{ev}(y)) = A_\phi\,\mathrm{ev}(\phi \circ y)$. It remains to show that the delta form with value $A\,\mathrm{ev}(y)$ is B-short. *At a p-adic place j:* by Input I7(ii) its gauge is at most $\mathrm{gauge}_j^{\varkappa,\mathfrak{e}}(g; A\,\mathrm{ev}(y))$, the largest absolute value of a coordinate of

$$\prod_\tau \det\left((g^{\mathfrak{e}(\tau)})_\tau^{-1}\, j_\tau Y(g^{\mathfrak{e}(\tau)})\right) \cdot \rho_{\varkappa,0}\left((g_\tau^{-1}\, j_\tau Y(g))_\tau\right)\mathrm{ev}(j \circ y).$$

The matrices $g^{-1}Y(g)$ are integral, so the determinants and the Schur matrix have integral entries, and $\mathrm{ev}(j\circ y)$ is integral because j maps roots of unity to units. *At $\sigma\in\mathrm{Aut}(\mathbb{C})$:* by Input I7(ii) the values ξ of the delta form have $\|\sigma\circ\xi\|^2\le\|A_\sigma\,\mathrm{ev}(\sigma\circ y)\|^2$. By Input I7(iii),

$$A_\sigma\,\mathrm{ev}(\sigma\circ y)=\prod_\tau \det\sigma_\tau(Y(g^{\mathfrak{e}(\tau)}))\cdot\rho_{\varkappa,0}(z)\,\mathrm{ev}(1),\qquad z=\mathrm{sq}((\sigma_\tau(Y(g)))_\tau\cdot(\sigma\circ y)),$$

and $\|\mathrm{ev}(1)\|^2=1$. Three factors remain. *The determinants*: the entries of $\sigma_\tau(Y(\,\cdot\,))$ have absolute value at most R, so each determinant has absolute value at most $(|I|R)^{|I|}\le R^{2|I|}$ (note $|I|\le R$), and the squares of the f determinants contribute at most $R^{4f|I|}$. *The Schur matrix*: an entry of z_τ is a sum of $|I|$ products of an entry of $\sigma_\tau(Y(g))$ with a root of unity, so the sum of the squares of the absolute values of the entries of z_τ is at most $|I|^2(|I|R)^2\le R^4$; by Input I7(iv) every $c_{\tau,\nu}$ is at most R^4, and since the row lengths $l(\tau,\nu)$ at all places add up to $\mathrm{boxes}(\varkappa,|I|)$, the Schur matrix multiplies $\|\cdot\|^2$ by at most $R^{4\,\mathrm{boxes}(\varkappa,|I|)}$. *The evaluation vector* $\mathrm{ev}(1)$ has norm 1. Altogether, by the hypothesis on M,

$$\|\sigma\circ\xi\|^2\;\le\;R^{4f|I|+4\,\mathrm{boxes}(\varkappa,|I|)}\;\le\;R^{4M}\;\le\;R^{6M}\;\le\;B.$$

(The formal text arranges the same three bounds against R^{6M}, which leaves room.) □

LEAN `span_isShort_eq_top`

Proof of Lemma 2.6. By Lemma 3.1(c) the residue field at $\mathfrak{p}$ has $N(D)$ elements, so its group of units is cyclic of order $N(D)-1$. Choose a unit y of the ring of integers of the completion at $\mathfrak{p}$ whose residue generates this group, regard it as an idele with component y at $\mathfrak{p}$ and 1 elsewhere, and put $c=y/\overline{y}$. Then $c\,\overline{c}=1$, c is integral, and c is 1 away from $\mathfrak{p}$ and $\overline{\mathfrak{p}}$. Since $\overline{\mathfrak{p}}\neq\mathfrak{p}$, the component of c at $\mathfrak{p}$ is y; if y^n-1 lies in the maximal ideal, the residue of y has order dividing n, so $N(D)-1$ divides n. □

Lemma 4.3 (the central elements act on columns)**.** *In Setting 2.1, let $\mathfrak{p}\in\mathrm{Pl}_\pi(D)$.*

(a) For every finite set I, the scalar matrix $\mathfrak{c}_D(\mathfrak{p})\cdot 1_I$ lies in $K_I[\mathfrak{p}]$.

(b) For every $\varphi\in\mathrm{Col}^{\mathrm{ram}}(D)$ there is a column $\varphi'\in\mathrm{Col}(D)$ with $\varphi=\varphi'-\mathrm{R}(\mathfrak{c}_D(\mathfrak{p})\cdot 1_{X_0(D)})\varphi'$.

Proof. (a) $\mathfrak{c}_D(\mathfrak{p})$ is integral of norm one and is 1 away from $\mathfrak{p}$, $\overline{\mathfrak{p}}$ (Lemma 2.6). (b) $\varphi=\mathrm{ram}_D(\mathrm{Pl}_\pi(D))\varphi_0$ for a column φ_0; split off the factor at $\mathfrak{p}$ and take $\varphi'=\mathrm{ram}_D(\mathrm{Pl}_\pi(D)\setminus\{\mathfrak{p}\})\varphi_0$. □

LEAN `Data.scalarFin_centralGen_mem` · `Data.exists_eq_sub_centralGen`

Lemma 4.4 (few columns are fixed by a central element)**.** *In Setting 2.1, for every $\mathfrak{p}\in\mathrm{Pl}_\pi(D)$,*

$$\dim_{\mathbb{C}}\ker(1-\mathrm{cen}_D(\mathfrak{c}_D(\mathfrak{p})))\;\le\;(N(D)-1)^{-1}\cdot\dim_{\mathbb{C}}\mathrm{Col}(D).$$

Proof. Let z be the scalar matrix of $\mathfrak{c}_D(\mathfrak{p})$ and K the group generated by $K_0^{\mathrm{col}}(D)$ and z. By Lemma 4.3(a) and Lemma 3.2(g), K lies in the chosen level; so K is a level, and both K and $K_0^{\mathrm{col}}(D)$ are free (Input I3). A column fixed by z is fixed by K. For free levels the dimension of the space of fixed forms is the number of classes times the dimension of the weight (Input I1), and each class of K consists of $[K:K_0^{\mathrm{col}}(D)]$ classes of $K_0^{\mathrm{col}}(D)$. Hence the kernel has dimension at most $\dim\mathrm{Col}(D)/[K:K_0^{\mathrm{col}}(D)]$. Some power z^n with $1\le n\le[K:K_0^{\mathrm{col}}(D)]$ lies in $K_0^{\mathrm{col}}(D)$; by Lemma 3.2(e) its

component at $\mathfrak{p}$ is congruent to 1 modulo $\mathfrak{p}$, so $N(D)-1$ divides n by Lemma 2.6. Therefore $[K : K_0^{\mathrm{col}}(D)] \geq N(D)-1$. □

LEAN `Data.finrank_ker_one_sub_centralGen_le`

Proposition 4.5 (the ramifying operator)**.** *In Setting 2.1, let Σ be a finite subset of $\mathrm{Pl}_\pi(D)$ and W a complex subspace of $\mathrm{Col}(D)$. Then*

$$\dim_{\mathbb{C}} \mathrm{ram}_D(\Sigma)^{-1}(W) \;\leq\; \dim_{\mathbb{C}} W + \#\Sigma \cdot (N(D)-1)^{-1} \cdot \dim_{\mathbb{C}} \mathrm{Col}(D),$$

and for every real number B and every $\varphi \in \mathrm{Col}(D)$ with φ is B-short,

$$\mathrm{ram}_D(\Sigma)\varphi \text{ is } 4^{\#\Sigma}B\text{-short.}$$

Proof. Induction on Σ; for $\Sigma = \emptyset$ the operator is the identity. Adding a place $\mathfrak{p}$ composes with $1 - \mathrm{cen}_D(\mathfrak{c}_D(\mathfrak{p}))$. The preimage of a subspace under a linear map has dimension at most the dimension of the subspace plus that of the kernel, which Lemma 4.4 bounds; and by Lemmas 4.3(a) and 4.1 the new factor multiplies the bound of shortness by 4. □

LEAN `Data.ramifyingOp_finrank_comap_and_short`

For $W = 0$ and $\Sigma = \mathrm{Pl}_\pi(D)$ the first claim says that the kernel of $\mathrm{ram}_D(\mathrm{Pl}_\pi(D))$ has dimension at most $f(D)\dim \mathrm{Col}(D)/(N(D)-1)$: the ramified columns have nearly full dimension. The unit on the filling (Theorem B of the handout) uses the proposition for its count of dimensions.

Proof of Theorem A. Put $B' = \exp(\mathrm{budget}(D))/4^{f(D)}$. Apply Proposition 4.2 to $E = E(D)$, $I = X_0(D)$, the weight $(\varkappa_D, \mathrm{last}_D(0,\cdot))$, the level $K_0^{\mathrm{col}}(D)$, which is free by Lemma 3.2(d), the number $N' = N_1(D)\,d$, which lies in the different of $E(D)$ (unit 152), and $M = k(D)\,f(D) + f(D)\,a(D)$, which is admissible by Lemma 3.3(b). The inequality $(6N'(f(D)k(D))^2)^{6M} \leq B'$ holds by size. So $\mathrm{Col}(D)$ is spanned by columns φ with φ is B'-short. The operator $\mathrm{ram}_D(\mathrm{Pl}_\pi(D))$ maps $\mathrm{Col}(D)$ onto $\mathrm{Col}^{\mathrm{ram}}(D)$, and by Proposition 4.5 and Lemma 3.1(d) it maps such φ to ramified columns that are $4^{f(D)}B'$-short, that is, $\exp(\mathrm{budget}(D))$-short. The image of a spanning set spans the image. □

LEAN `Data.span_short_ramifiedColumns` · input of unit 152: `Parameters.cycloConductor_mem_different`

5 The chain as a sum over classes

Setting 5.1 (a good datum at a p-adic place)**.** Settings 2.1 and 2.2 hold; ϑ is a twist of D, and Ψ is a finitely supported complex function on the tuples of translations of D such that Ψ is good for (D,ϑ); and p is prime to $\mathrm{nrm}(\vartheta)$.

Input I8 (standard data, Hecke operators, sums over classes; units 150, 141 and the background library). (i) In Settings 2.1 and 2.2, for an integer δ of $K(P)$ and a set S of finite places of $E(D)$, $\mathrm{Lat}(\delta;S)$ is the group of the matrices y over $\mathbb{A}_{E(D),\mathrm{f}}$ (of any fixed shape) such that every entry of δy is integral at every place of S; and $\mathrm{Std}_\delta(D,i)$ is the O_j-submodule, of the model that contains $\mathcal{W}_i(D)$, generated by the vectors $\mathrm{U}(y)\theta^\circ_{D,i}$ with $y\in\mathrm{Lat}(\delta;\{w:\ p\in w\})$. If $\delta\neq 0$ and $\Phi\in\mathrm{Std}_\delta(D,i)$, then for every complex number c, with the kernel taken at the scale $c\cdot\mathrm{per}(P,\mu(D))$, Φ multiplies gauges at the point 1 by at most $\left(|j(c)|_p/|j(\bar\delta\, d)|_p\right)^L$, $L=\mathrm{boxes}(\varkappa_D,n_{i+1}(D))$; for the lift kernel ($c=\overline{\pi_g(D)}$) this reads $\Phi\in\mathrm{Raise}_{\mathrm{k}_{D,\vartheta,i}}(1,1;(|j(\overline{\pi_g(D)})|_p/|j(\bar\delta d)|_p)^L)$.
(ii) Let w be a finite place of E with $\overline{w}\neq w$ and I a finite set. The group $\mathrm{GL}_I(E_w)$ embeds in G_I, an element g going to the point with component g at w, the component at $\overline{w}$ that unitarity forces, and 1 elsewhere; for a family $(g_w)_{w\in S}$ over a set S of such places, no two conjugate, $\mathrm{at}(g)$ is the product of these points. For a natural number s, the *basic double coset of index s* at w is the set of the $g\in\mathrm{GL}_I(E_w)$ such that g and ϖg^{-1} are integral and $\det g$ has valuation s (ϖ a uniformiser); for $s\le|I|$ it is the double coset of $\mathrm{diag}(\varpi 1_s,1)$ under the integral invertible matrices. On the vectors of a representation of G_I that are fixed by $K_I[w]$, the Hecke operator $T_{w,s}$ is the sum of the translates by representatives of the left cosets of the integral invertible matrices in this double coset; it preserves the vectors fixed by a group that contains $K_I[w]$ and commutes with every operator that commutes with the group at w.
(iii) In Setting 2.1, let ϑ be a twist, Ψ a finitely supported function on the tuples of translations, and $\Phi=(\Phi_{\mathsf{x},i})$ a family with $\Phi_{\mathsf{x},i}\in\mathcal{W}_i(D)$. The combination $\sum_{\mathsf{x}}\Psi(\mathsf{x})\bigotimes_{i<m}\Phi_{\mathsf{x},i}$ is *jointly invariant* if it satisfies condition (G2) of Definition 2.8 with $\Phi_{\mathsf{x},i}$ in place of $\mathrm{td}_D(\mathsf{x},i)$. Suppose that it is, that φ is a column, and that $\mathcal{C}_0,\dots,\mathcal{C}_m$ are properties of pairs (point, vector) such that $\mathcal{C}_0(g,\varphi(g))$ holds for all g and $\mathcal{C}_i(g,\xi)$ implies $\mathcal{C}_{i+1}(h,\mathrm{jm}_{D,\vartheta,i}(\omega_{D,\vartheta,i}(g,h)\Phi_{\mathsf{x},i})\xi)$ for x in the support, $i<m$ and all g,h. Then for every point h, the vector $\sum_{\mathsf{x}}\Psi(\mathsf{x})(\mathrm{C}_{D,\vartheta}(\Phi_{\mathsf{x}})_m\varphi)(h)$ is a sum of $\prod_{i<m}\#\mathrm{Cl}(K_i^{\mathrm{col}}(D))$ vectors of the form $\sum_{\mathsf{x}}\Psi(\mathsf{x})\beta(\bigotimes_{i<m}\Phi_{\mathsf{x},i})$, with β linear and $\mathcal{C}_m(h,\beta(\bigotimes_{i<m}\Phi_{\mathsf{x},i}))$ for every x in the support.

LEAN unit 150: `Parameters.integralAfter`, `Parameters.standardData`, `Parameters.mem_raises_of_standardData` · background library: `atSplit`, `atSplits`, `basicDoubleCoset`, `heckeSplit`, `map_heckeSplit` · unit 141: `IsJointlyInvariant`, `IsJointlyInvariant.sum_chain_mem_nsmul`

The reason for (iii) is that a lift is a finite sum over the classes of its level, each term being linear in the datum; m lifts give a sum over tuples of classes, and joint invariance makes the sum of the terms independent of the representatives, which is what allows one linear map β for all x. The levels are free because they lie in the chosen level (Input I3).

Lemma 5.2 (integrality of translations)**.** *In Setting 2.1:*

(a) Let q and i be natural numbers, $\mathsf{x}\in\mathrm{Adm}_D(q)$, δ an integer of $K(P)$, and S a set of finite places of $E(D)$ such that every $w\in S$ satisfies $q\notin w$ and ($\delta=\pi_g(D)$ or $\overline{w}$ is not chosen). Then $\mathsf{x}_i\in\mathrm{Lat}(\delta;S)$.

(b) *Let* x *be a tuple of translations,* i *a natural number and* S *a set of finite places of* $E(D)$ *such that* $\mathsf{x}_i \in \mathrm{Lat}(1; S \cup \overline{S})$*, where* $\overline{S}$ *is the set of the conjugates of the places of* S*. Then* $\mathrm{P}^S \mathrm{td}_D(\mathsf{x}, i) = \mathrm{td}_D(\mathsf{x}, i)$*.*

Proof. (a) At $w \in S$ the entries of $\pi_g(D)\mathsf{x}_i$ are integral, by the definition of $\mathrm{Adm}_D(q)$. If $\delta = \pi_g(D)$ this is the claim. Otherwise $\overline{\pi_g(D)} \notin \overline{w}$, that is, $\pi_g(D) \notin w$: so $\pi_g(D)$ is a unit at w, the entries of x_i are integral at w, and so are those of $\delta\mathsf{x}_i$. (b) The datum is the translate of a multiple of the vacuum by a matrix that is integral at $S \cup \overline{S}$, and such a translate is again a vacuum at these places (a property of the projection, from the library). □

LEAN `Data.mem_integralAfter` · `Data.projOn_translateDatum`

Lemma 5.3 (the data of a good datum are standard)**.** *In Setting 5.1, assume* $p \neq N(D)$*. Let* x *be in the support of* Ψ*,* i *a natural number and* δ *an integer of* $K(P)$ *such that* $\delta = \pi_g(D)$ *or* p *does not divide* $g(D)$*. Then* $\mathrm{td}_D(\mathsf{x}, i) \in \mathrm{Std}_\delta(D, i)$*.*

Proof. By (G1), $\mathsf{x} \in \mathrm{Adm}_D(\mathrm{nrm}(\vartheta)N(D))$. Let w be a place above p. Then $\mathrm{nrm}(\vartheta)N(D) \notin w$, because p is prime to $\mathrm{nrm}(\vartheta)$ and $p \neq N(D)$. If p does not divide $g(D)$, then $\overline{w}$, which also lies above p, is not chosen. So Lemma 5.2(a) gives $\mathsf{x}_i \in \mathrm{Lat}(\delta; \{w : p \in w\})$, and the datum is one of the generators of $\mathrm{Std}_\delta(D, i)$. □

LEAN `Data.IsGoodDatum.translateDatum_mem_standardData`

Lemma 5.4 (the chain is a plain sum)**.** *In Setting 2.1, let* ϑ *be a twist of* D*,* Ψ *a finitely supported complex function on the tuples of translations of* D*, and* $\Phi = (\Phi_{\mathsf{x},i})$ *a family of data* $\Phi_{\mathsf{x},i} \in \mathcal{W}_i(D)$ *such that* $\sum_{\mathsf{x}} \Psi(\mathsf{x}) \bigotimes_{i<m} \Phi_{\mathsf{x},i}$ *is jointly invariant. For every natural number* i *let* N_i *be a real function of a point of* $G_{X_i(D)}$ *and a vector, and* $r_i \geq 0$ *a real number, such that for every* x *in the support of* Ψ*, every* $i < m$ *and all* g*,* h*,* ξ

$$\mathrm{N}_{i+1}(h, \mathrm{jm}_{D,\vartheta,i}(\omega_{D,\vartheta,i}(g,h)\Phi_{\mathsf{x},i})\xi) \;\leq\; r_i \cdot \mathrm{N}_i(g, \xi).$$

Let $\varphi \in \mathrm{Col}(D)$ *and let* B *be a real number with* $\mathrm{N}_0(g, \varphi(g)) \leq B$ *for all* g*, and let* g *be a point of* $G_{X_m(D)}$*. Then there are linear maps* β_c*, for* $1 \leq c \leq \prod_{i<m} \#\mathrm{Cl}(K_i^{\mathrm{col}}(D))$*, from* $\bigotimes_{i<m} \mathcal{W}_i(D)$ *to* $\mathbb{C}^{\mathrm{Wy}_{X_m(D)}(\varkappa_D)}$ *such that* $\mathrm{N}_m(g, \beta_c(\bigotimes_{i<m} \Phi_{\mathsf{x},i})) \leq \prod_{i<m} r_i \cdot B$ *for every* c *and every* x *in the support, and*

$$\sum_c \sum_{\mathsf{x}} \Psi(\mathsf{x})\, \beta_c\Big(\bigotimes_{i<m} \Phi_{\mathsf{x},i}\Big) = \Big(\sum_{\mathsf{x}} \Psi(\mathsf{x})\, \mathrm{C}_{D,\vartheta}(\Phi_{\mathsf{x}})_m \varphi\Big)(g).$$

Proof. Input I8(iii) with the properties $\mathcal{C}_i(g, \xi)$: $\mathrm{N}_i(g, \xi) \leq \prod_{i'<i} r_{i'} \cdot B$. They propagate by the hypothesis on the kernels and $r_i \geq 0$. □

LEAN `Data.exists_chain_eq_sum`

Proposition 5.5 (gains of the lifts add along the chain)**.** *In Setting 5.1, for every natural number* i *let* t_i *be a* $\mathbb{C}$*-linear endomorphism of* $\mathcal{W}_i(D)$ *such that* $\omega_{D,\vartheta,i}(a, a')\, t_i\, \mathrm{td}_D(\mathsf{x}, i) = t_i\, \omega_{D,\vartheta,i}(a, a')\, \mathrm{td}_D(\mathsf{x}, i)$ *for every* x *in the support of* Ψ*, every* $a \in K_i^{\mathrm{col}}(D)$ *and every* $a' \in K_{i+1}^{\mathrm{col}}(D)$*; let* $d_i \in G_{X_i(D)}$ *be points and* r_i *real numbers such that, for every* x *in the support of* Ψ *and every* $i < m$*,*

$$t_i\, \mathrm{td}_D(\mathsf{x}, i) \in \mathrm{Raise}_{\mathrm{k}_{D,\vartheta,i}}(d_i, d_{i+1}; p^{r_i}).$$

Let $\varphi \in \mathrm{Col}(D)$ *have* j*-gauge at most* 1 *at* d_0*. Then the form* $\sum_{\mathsf{x}} \Psi(\mathsf{x}) \cdot \mathrm{C}_{D,\vartheta}((t_i \mathrm{td}_D(\mathsf{x}, i))_i)_m \varphi$ *has* j*-gauge at most* $p^{\sum_{i<m} r_i}$ *at* d_m*.*

Proof. The combination of the data $t_i \mathrm{td}_D(\mathsf{x}, i)$ is jointly invariant: it is the image of the tensor of (G2) under $\bigotimes_i t_i$, which commutes with the action of the levels on the data in the support. Apply Lemma 5.4 with $\mathrm{N}_i(g, \xi) = \mathrm{gauge}_j^{\varkappa_D, \mathrm{last}_D(i,\cdot)}(gd_i; \xi)$, the numbers p^{r_i} and $B = 1$. The value of the form at a point is a sum of terms $\Psi(\mathsf{x})\beta_c(\dots)$; the gauge is ultrametric, the gauge of each $\beta_c(\dots)$ is at most $p^{\sum r_i}$, and $|j(\Psi(\mathsf{x}))|_p \le 1$ by (G1), since p is prime to $\mathrm{nrm}(\vartheta)$. □

LEAN `Data.IsGoodDatum.gaugeLe_chain`

6 Served primes, induced primes and marked sets

Lemma 6.1 (orders and losses)**.** *In Setting 2.1:*

(a) For every prime number p, $\mathfrak{r}(D, p) \le \log x_1$.

(b) If p is a prime number that is served for D, then m divides the order of the place $[-p]_D$ in the group $\mathrm{Pl}(D)$.

(c) For every natural number p, $f(D)\,(\log b)^2 \cdot a(D)/p \;\le\; \mathrm{loss}(D, p)$.

(d) For natural numbers p and e with e odd and every $\tau \in \mathrm{tops}(D)$: the place $[-p^e]_D \cdot \tau$ lies in $\mathrm{tops}(D)$ if and only if $o_0(D, p)$ divides e.

Proof. (a) $m \le 2^{\mu(D)} \le p^{\mu(D)}$, so $\mathfrak{r}(D, p) \le \mu(D)$, and $\mu(D) \le \log x_1$ by size. (c) The last factor of $\mathrm{loss}(D, p)$ is $m \ge 1$ or 1. (d) Since e is odd, $[-p^e]_D = [-p]_D^e$; the tops are the places of class 1 modulo N_0, so the product is a top exactly when the class of $[-p]_D$ has order dividing e. (b) The data have $m = 2^{c+1}$ and $\mu(D) = 22(c+1)$ for a natural number c, and $2^{\mu(D)+1}$ divides $N_1(D)$. Let $n \ge 1$ be the order of $[-p]_D$. The class of this place modulo $2^{\mu(D)+1}$ is that of $-p$ (unit 152; p is prime to N_0 because $\chi_{K(P)}(p) \ne 1$), so $(-p)^n \equiv 1$ modulo $2^{\mu(D)+1}$. By lifting the exponent at 2 (background library), $2^{\mu(D)}$ divides $(-p-1)n$ or $(-p+1)n$. Since p is not short, 2^{c+1} divides neither $p+1$ nor $p-1$; so $2^{\mu(D)-c}$ divides n, and $\mu(D) - c \ge c+1$. □

LEAN `Data.instUbCastRealRrLogLx₁` · `Data.IsServed.m_dvd_orderOf` · `Data.le_loss` · `Data.placeOfInt_mul_mem_tops_iff` · background library: `two_pow_dvd_or_of_pow_eq_one` · unit 152: `Parameters.placeClass_placeOfInt`

Lemma 6.2 (induced primes)**.** *In Settings 2.1 and 2.2, let $\tau, \tau' \in \mathrm{Pl}(D)$.*

(a) $p \in \mathfrak{l}_j(\tau)$.

(b) $\mathfrak{l}_j(\tau \cdot \tau') = \tau' \cdot \mathfrak{l}_j(\tau)$, where a place acts on the primes through the automorphism of $E(D)$ over $K(P)$ that it names.

(c) If $\chi_{K(P)}(p) = -1$ and e is an odd natural number, then $\mathfrak{l}_j([-p^e]_D \cdot \tau) = \overline{\mathfrak{l}_j(\tau)}$.

(d) If $\chi_{K(P)}(p) = -1$, then $\mathfrak{l}_j(\tau)$ is equal to $\mathfrak{l}_j(\tau')$ or to $\overline{\mathfrak{l}_j(\tau')}$ if and only if $\tau = [-p]_D^n \cdot \tau'$ for some natural number n.

Proof. (a) $|j(p)|_p < 1$. (b) The embedding at $\tau\tau'$ is the embedding at τ composed with the automorphism named by τ' (unit 152). (c), (d) For p inert in $K(P)$ and prime to $N_1(D)\,d$, the decomposition of p in $E(D)$ is described in the note on the CM field: a place v maps a prime above p to itself or to its conjugate exactly when v is a power of $[-p]_D$, to the conjugate for odd powers. With (b) this gives both claims. □

LEAN Data.p_mem_inducedPlace · Data.inducedPlace_mul · Data.inducedPlace_neg_pow_mul_of_odd · Data.inducedPlace_eq_or_star_iff · unit 152: Parameters.phi_mul, Parameters.smul_eq_star_iff_inert, Parameters.smul_eq_iff_inert

Lemma 6.3 (a full mark exists at a served prime)**.** *In Settings 2.1 and 2.2, assume that $p < x_1$ and that p is served for D. Then there is a finite set $\mathcal{L}$ of finite places of $E(D)$ that is a full mark.*

Proof. Fix a well-ordering of the finite places and let $\mathcal{L}$ be the set of the primes $\mathfrak{l} = \mathfrak{l}_j(\tau)$, $\tau \in \mathrm{tops}(D)$, that are smaller than their conjugate. Then (M1) and (M3) hold by construction, and (M2), (M4) by hypothesis, a served prime being inert. For the covering, let τ be a top. By Lemma 6.1(d) the place $\tau' = [-p]_D^{o_0(D,p)} \cdot \tau$ is a top, and it induces $\overline{\mathfrak{l}_j(\tau)}$ by Lemma 6.2(c), $o_0(D,p)$ being odd. Moreover $\mathfrak{l}_j(\tau) \neq \overline{\mathfrak{l}_j(\tau)}$: by Lemma 6.1(b) the order of $[-p]_D$ is even, so its odd powers do not fix a prime above p. Hence one of the two primes is the smaller and lies in $\mathcal{L}$. □

LEAN Data.exists_isMarked

Setting 6.4 (a marked set)**.** Settings 2.1 and 2.2 hold, and $\mathcal{L}$ is a finite set of finite places of $E(D)$ that is marked for D, p and j (Input I5).

Lemma 6.5 (marked sets)**.** *In Setting 6.4:*

(a) $\#\mathcal{L} \leq f(D)$.

(b) *p does not divide $g(D)$, $p \neq N(D)$, and p does not divide $g(D)\,N(D)$.*

(c) *For every $\mathfrak{l} \in \mathcal{L}$: $p \in \mathfrak{l}$; p is a uniformiser at $\mathfrak{l}$; $\bar{\mathfrak{l}} \neq \mathfrak{l}$; and $K_{X_0(D)}[\mathfrak{l}] \subseteq K_0^{\mathrm{col}}(D)$.*

Proof. (a) $\mathcal{L}$ lies in the image of $\tau \mapsto \mathfrak{l}_j(\tau)$. (b) $p < x_1$, and the primes g_0, g_1, $N(D)$ are larger than x_1 by size. (c) The first claim is Lemma 6.2(a). p is unramified in $E(D)$, being prime to $N_1(D)\,d$ (an inert or ramified p does not divide $N_1(D)$, and if $\mathcal{L} \neq \emptyset$ then p is served, so $p \neq d$). The third claim is (M1). For the last, by (b) the element $\overline{\pi_g(D)}\pi_N(D)$ is a unit at $\mathfrak{l}$ and $\bar{\mathfrak{l}}$, so $K_{X_0(D)}[\mathfrak{l}]$ lies in the principal level of this element, and Lemma 3.2(a) applies. □

LEAN Data.IsMarked.card_le · Data.IsMarked.not_dvd_gProd_mul_N · Data.IsMarked.p_mem · Data.IsMarked.valuation_p · Data.IsMarked.star_ne · Data.IsMarked.maxCompactAt_le_columnLevel

Lemma 6.6 (marked primes avoid the thick places; a balance)**.** *In Setting 6.4:*

(a) *If $\tau \in \mathrm{Pl}(D)$ is such that $\mathfrak{l}_j(\tau) \in \mathcal{L}$ or $\overline{\mathfrak{l}_j(\tau)} \in \mathcal{L}$, then $\tau \notin \mathrm{thick}(D)$.*

(b) *For every natural number i and every real function G of a natural number and a finite place,*

$$\sum_{\tau \in \mathrm{Pl}(D)} \Big([\mathfrak{l}_j(\tau) \in \mathcal{L}]\, G\big(\mathrm{wt}_D(i,\tau), \mathfrak{l}_j(\tau)\big) - [\overline{\mathfrak{l}_j(\tau)} \in \mathcal{L}]\, G\big(\mathrm{wt}_D(i,\tau), \overline{\mathfrak{l}_j(\tau)}\big) \Big) = 0.$$

Proof. (a) $\mathcal{L}$ is not empty, so p is served: $\chi_{K(P)}(p) = -1$ and $o_0(D,p)$ is odd. By (M3) and Lemma 6.2(d), $\tau = [-p]_D^n \cdot \tau_0$ for a top τ_0 and some n, so the class of τ modulo N_0 is that of $(-p)^n$. A thick place has class -1 modulo N_0 (a condition of the data). If τ were thick, raising $(-p)^n \equiv -1$ to the odd power $o_0(D,p)$ would give $1 \equiv -1$ modulo N_0. (b) If $\mathcal{L}$ is empty both sums vanish. Otherwise p is inert; substitute $\tau \mapsto [-p]_D \cdot \tau$ in the second sum. By Lemma 6.2(c) the prime induced by the new place is $\overline{\mathfrak{l}_j(\tau)}$, so the condition becomes $\mathfrak{l}_j(\tau) \in \mathcal{L}$ and the place argument of G becomes $\mathfrak{l}_j(\tau)$; and the weight is unchanged, because by (a) neither τ nor $[-p]_D \cdot \tau$ is thick, and $\mathrm{wt}_D(i, \cdot\,)$ takes one value on the places that are not thick. □

LEAN Data.IsMarked.notMem_thick · Data.IsMarked.sum_inducedPlace_sub_star_eq_zero

7 The jet exponent

Lemma 7.1 (collisions). *In Setting 2.1, let Ξ be a finite subset of $\mathrm{Pl}(D)$.*

(a) Let p and e be natural numbers and $\tau \in \mathrm{Pl}(D)$. If (τ, e) is a collision of p outside Ξ for D, then $p \neq d$, e is odd, and $o_0(D,p)$ divides e.

(b) Let p be a prime number and o a natural number such that $o \le e$ for every collision (τ, e) of p outside Ξ for D. Then $\mathrm{coll}(D,p,\Xi) \le 2\, f(D)\, a(D)/p^o$.

(c) Let p be a prime number that is small and not split for D and is not served for D. Then $m \cdot \mathrm{coll}(D,p,\emptyset) \le \mathrm{loss}(D,p)$.

(d) In Setting 2.2, if $\mathcal{L}$ is a full mark, then $\mathrm{coll}(D,p,\Xi_j(\mathcal{L})) = 0$.

Proof. (a) Both τ and $[-p^e]_D \cdot \tau$ are tops, so they have the same class modulo N_0; by a lemma of unit 150 this excludes $p = d$ (no power of d is $\equiv -1$ modulo N_0). Then e is odd because (e,p) is live, and Lemma 6.1(d) applies. (b) For each τ the terms are $a(D)/p^e$ with $e \ge o$; they are bounded by the geometric series $\sum_{n\ge 0}(a(D)/p^o)2^{-n} = 2a(D)/p^o$, since $p \ge 2$; sum over the $f(D)$ places. (c) By $(\log b)^2 \ge 2$, $\mathrm{loss}(D,p) \ge 2f(D)(a(D)/p)\cdot(m \text{ or } 1)$. If p is short, use (b) with $o = 1$. If not, every collision has $e \ge \mathfrak{r}(D,p)+1$: by (a), e is odd, $o_0(D,p)$ divides e, hence $o_0(D,p)$ is odd, and $\chi_{K(P)}(p) = -1$; if $o_0(D,p) \le e \le \mathfrak{r}(D,p)$, then p would be served. So $\mathrm{coll}(D,p,\emptyset) \le 2f(D)a(D)/p^{\mathfrak{r}(D,p)+1}$ by (b), and $m \le p^{\mathfrak{r}(D,p)}$. (d) Suppose (τ,e) is a collision outside $\Xi_j(\mathcal{L})$. Then $\mathcal{L}$ is not empty, since it covers the top τ; so p is inert, e is odd by (a), and $[-p^e]_D \cdot \tau$ induces $\overline{\mathfrak{l}_j(\tau)}$ (Lemma 6.2(c)). As $\mathcal{L}$ covers τ, one of τ, $[-p^e]_D \cdot \tau$ lies in $\Xi_j(\mathcal{L})$, a contradiction. □

LEAN Data.IsCollision.ne_and_odd_and_dvd · Data.collisionSum_le · Data.mul_collisionSum_le_loss · Data.IsFullMark.collisionSum_evenPlaces_eq_zero · unit 150: Data.ne_d_of_placeClass_eq

Lemma 7.2 (low columns and low boxes). *In Setting 2.1, let $i < m$ be a natural number and $\tau \in \mathrm{Pl}(D)$. Then*

$$k(D)\cdot w_D(\tau) \le \mathrm{lb}_D(i,\tau) \le k(D)\cdot w_D(\tau) + x^{60} \qquad \text{and} \qquad \mathrm{lc}_D(i,\tau) \le w_D(\tau) + x^{60}.$$

Proof. The term $h = k(D)$ of $\mathrm{lb}_D(i,\tau)$ is $k(D)(\mathrm{wt}_D(1,\tau) - \lfloor (k(D)+t(D)+1)/2 \rfloor) \ge k(D)\, w_D(\tau)$ (unit 152). For the upper bounds, each $\varkappa_D(\tau,h)$ with $h \neq 1$ is at most $[h = k(D)]\mathrm{wt}_D(1,\tau) + \mathrm{gap}(D)$ (unit 150), $\mathrm{wt}_D(1,\tau)$ exceeds $w_D(\tau)$ by a small amount (unit 152), and the sums of the remaining terms over $h < n_{i+1}(D) \le n_m(D)$ are at most x^{60} by size. □

LEAN Data.lowColumns_lowBoxes_bounds · unit 152: Data.nominalWeight_le_w_sub, Data.w_le_nominalWeight_add · unit 150: Data.κ_le

Proposition 7.3 (the jet exponent is at least the gain of one lift, up to collisions). *In Setting 2.1, let p be a prime number that is small and not split for D, $i < m$ a natural number and Ξ a finite subset of $\mathrm{Pl}(D)$. Then*

$$\frac{\mathrm{gain}(D,p)}{m} - \mathrm{coll}(D,p,\Xi) + \frac{1-\chi_{K(P)}(p)}{2}\cdot \#\big(\mathrm{tops}(D)\cap\Xi\big)\cdot\frac{a(D)}{x_1} - f(D)\,(\log b)^2\,\frac{a(D)}{p}\cdot\frac{1}{m} \;\le\; \mathrm{je}_D(p,i,\Xi).$$

Proof. Write $c = (1-\chi_{K(P)}(p))/2 \in \{\frac{1}{2},1\}$, $n' = n_{i+1}(D)$, $L = \mathrm{boxes}(\varkappa_D, n')$, $l = \lfloor \log_p L \rfloor$, $\eta = a(D)/(pm)$, and $\tau_e = [-p^e]_D \cdot \tau$. Since $n' > 1$, the columns of height one contribute $a(D)$ at each top, so $L = \sum_\tau (\varkappa_D(\tau,1) + \mathrm{lb}_D(i,\tau))$, and by (1)

$$\Big(\frac{1}{p-1} - \varrho_d(p)\Big) L = c \sum_\tau \sum_{e:\ (e,p)\ \mathrm{live}} \frac{\varkappa_D(\tau,1) + \mathrm{lb}_D(i,\tau)}{p^e}.$$

By the definitions of Input I6, $\mathrm{gain}(D,p)/m = c\sum_\tau \sum_{e\ \mathrm{live}} ([\tau \in \mathrm{tops}(D)](a(D)/p^e - w_D(\tau_e))_+ + k(D)(w_D(\tau)/p^e - w_D(\tau_e))_+)$. For a place τ and a natural number e let lv_e be 1 if (e,p) is live and 0 otherwise, and $\mathrm{cl}_{\tau,e}$ be 1 if (τ,e) is a collision of p outside Ξ for D and 0 otherwise. The claim follows by summation over τ and e from the following inequality, for each τ and e:

$$\begin{aligned}[e \le l]\Big([e \text{ odd or } p = d]\ \mathrm{cc}_p(\varkappa_D, n'; \Xi; \tau, \tau_e; e) - 4\eta\Big) + [e=1]\,[\tau \in \mathrm{tops}(D) \cap \Xi]\,\frac{a(D)}{x_1} \\ + \mathrm{lv}_e\Big([\tau \in \mathrm{tops}(D)](\tfrac{a(D)}{p^e} - w_D(\tau_e))_+ + k(D)(\tfrac{w_D(\tau)}{p^e} - w_D(\tau_e))_+\Big) \\ \le\ \mathrm{lv}_e\,\frac{\varkappa_D(\tau,1) + \mathrm{lb}_D(i,\tau)}{p^e} + \mathrm{cl}_{\tau,e}\,\frac{a(D)}{p^e}. \quad (2)\end{aligned}$$

Indeed, multiply (2) by c and sum: the right side gives $(1/(p-1) - \varrho_d(p))L + c\,\mathrm{coll}(D,p,\Xi)$, the left side gives $c\sum \mathrm{cc}_p(\varkappa_D, n'; \Xi; \tau, \tau_e; e) - 4c\,\eta\, f(D)(l+1) + \mathrm{gain}(D,p)/m + c\#(\mathrm{tops}(D) \cap \Xi)a(D)/x_1$, and therefore

$$\begin{aligned}\mathrm{je}_D(p,i,\Xi)\ \ge\ \frac{\mathrm{gain}(D,p)}{m} + c\,\#(\mathrm{tops}(D)\cap\Xi)\frac{a(D)}{x_1} \\ - c\,\mathrm{coll}(D,p,\Xi) - (4c\,\eta + {n'}^3) f(D)(l+1).\end{aligned}$$

Now $c \le 1$, the collision sum is non-negative, ${n'}^3 \le x^{60} \le \eta$ by size (here $p < x_1$ is used), and $5(l+1) \le (\log b)^2$ by size, because $l \le \log L/\log 2$ and $L = \#\mathrm{tops}(D)\cdot a(D) + \sum_\tau \mathrm{lb}_D(i,\tau)$ is bounded through Lemma 7.2 (the formal text compares each $\mathrm{lb}_D(i,\tau)$ with the number $g(D)$ of the data, a device of its tactic for size comparisons, not a step of the argument). This gives the proposition.

Proof of (2). The facts used are: $a(D)/x_1 \le w_D(\tau') \le a(D)/x$ for every place τ', with $w_D(\tau') = a(D)/x_1$ when τ' is not thick; Lemma 7.2; and the sizes $x^{60} \le \eta$, $a(D)/(xp) + x^{60}/p \le \eta$. Put $\tau' = \tau_e$ and recall that in the coupling count $\mathrm{one}_\tau = [\tau \in \mathrm{tops}(D) \setminus \Xi]\,a(D)$, the count of the other columns is $\mathrm{lc}_D(i,\tau)$ and their boxes are $\mathrm{lb}_D(i,\tau)$.

The pair (e,p) is live and $e \le l$. Three bounds add up to (2). (1) $\mathrm{lc}_D(i,\tau)/p^e \le \eta$. (2) $\min(\mathrm{lb}_D(i,\tau)/p^e, \mathrm{lb}_D(i,\tau')) + k(D)(w_D(\tau)/p^e - w_D(\tau'))_+ \le \mathrm{lb}_D(i,\tau)/p^e + x^{60}$: if the positive part vanishes this is clear; otherwise bound the minimum by $\mathrm{lb}_D(i,\tau') \le k(D)w_D(\tau') + x^{60}$ and use $k(D)w_D(\tau) \le \mathrm{lb}_D(i,\tau)$. (3) $\min(\mathrm{one}_\tau/p^e, \mathrm{lc}_D(i,\tau') + \mathrm{one}_{\tau'}) + [\tau \in \mathrm{tops}(D)](a(D)/p^e - w_D(\tau'))_+ + [e=1][\tau \in \mathrm{tops}(D)\cap\Xi]a(D)/x_1$ is at most $\varkappa_D(\tau,1)/p^e + x^{60}$ plus $a(D)/p^e$ if (τ,e) is a collision. If τ is not a top, the left side is 0. If $\tau \in \mathrm{tops}(D) \cap \Xi$, then $\mathrm{one}_\tau = 0$, so the minimum is 0. For $e = 1$ the two other terms satisfy $(a(D)/p - w_D(\tau'))_+ + a(D)/x_1 \le a(D)/p$: if the positive part vanishes, because $a(D)/x_1 \le a(D)/p$ ($p < x_1$); if not, because $w_D(\tau') \ge a(D)/x_1$. For $e \ne 1$ the last term is absent and $(a(D)/p^e - w_D(\tau'))_+ \le a(D)/p^e$. In both cases the left side is at most $\varkappa_D(\tau,1)/p^e$. If $\tau \in \mathrm{tops}(D) \setminus \Xi$ and (τ,e) is a collision,

both terms are at most $a(D)/p^e$. If $\tau \in \mathrm{tops}(D) \setminus \Xi$ and it is not a collision, then $\mathrm{one}_{\tau'} = 0$, and either the positive part vanishes, or the minimum is at most $\mathrm{lc}_D(i,\tau') \le w_D(\tau') + x^{60}$ and the sum is at most $a(D)/p^e + x^{60}$. Altogether the loss is at most $\eta + 2x^{60} \le 4\eta$.

The pair (e,p) is live and $e > l$. Only the second and third groups of the left side occur, and $k(D)(w_D(\tau)/p^e - w_D(\tau'))_+ \le \mathrm{lb}_D(i,\tau)/p^e$, $[\tau \in \mathrm{tops}(D)]((a(D)/p^e - w_D(\tau'))_+ + [e=1][\tau \in \Xi]a(D)/x_1) \le \varkappa_D(\tau,1)/p^e$, as before.

The pair is not live. Then $e = 0$, or e is even and $p \ne d$, and the right side is 0. For $e \ge 2$ the left side is -4η or 0. For $e = 0$ and $p \ne d$ likewise. For $e = 0$ and $p = d$ the place $\tau' = [-1]_D \cdot \tau$ has the class of τ times -1 modulo N_0, so at most one of τ, τ' is a top and at most one is thick; hence the first minimum is at most $\mathrm{lc}_D(i,\tau')$ or 0, the second is at most $\mathrm{lb}_D(i,\cdot)$ at a place that is not thick, which is at most $k(D)a(D)/x_1 + x^{60}$, and with $p = d$ these three terms are at most η each by size. So the coupling count is at most 3η. □

LEAN Data.le_jetExponent · unit 150: hasSum_live_inv_pow, Data.summable_localAsk · unit 152: Data.nominalWeight_mem_Icc, Parameters.placeClass_of_thick

Lemma 7.4 (the cost of the marks is within the loss)**.** *In Setting 6.4, let $i < m$ be a natural number. Then*

$$\#\mathcal{L} \cdot \Delta(D,p) + \mathfrak{r}(D,p) \sum_{\tau \in \Xi_j(\mathcal{L})} \mathrm{lc}_D(i,\tau) \;\le\; \frac{\mathrm{loss}(D,p)}{m} + \#(\mathrm{tops}(D) \cap \Xi_j(\mathcal{L})) \cdot \frac{a(D)}{x_1}.$$

Proof. It suffices to show $\mathfrak{r}(D,p)\,\mathrm{lc}_D(i,\tau) \le a(D)/(pm) + [\tau \in \mathrm{tops}(D)]\,a(D)/x_1$ for $\tau \in \Xi_j(\mathcal{L})$: then the left side is at most $f(D)\mathfrak{r}(D,p)f(D)(n_m(D) + \mathfrak{r}(D,p)) + f(D)a(D)/(pm) + \#(\mathrm{tops}(D) \cap \Xi_j(\mathcal{L}))a(D)/x_1$ by Lemma 6.5(a), and the first two terms are at most $f(D)(\log b)^2 a(D)/(pm) \le \mathrm{loss}(D,p)/m$ by size and Lemma 6.1(c). Let $\tau \in \Xi_j(\mathcal{L})$. By Lemma 6.6(a), τ is not thick, so $\mathrm{lc}_D(i,\tau) \le a(D)/x_1 + x^{60}$ by Lemma 7.2. If $p \le m$, then $\mathfrak{r}(D,p)(a(D)/x_1 + x^{60}) \le a(D)/(pm)$ by size, using Lemma 6.1(a). If $m \le p$, then $\mathfrak{r}(D,p) = 1$; p is served because $\mathcal{L}$ is not empty, so $o_0(D,p) \le 1$, that is, $-p \equiv 1$ modulo N_0; by (M3) and Lemma 6.2(d), τ is a power of $[-p]_D$ times a top, hence a top; and $x^{60} \le a(D)/(pm)$ by size. □

LEAN Data.IsMarked.sum_lowColumns_le

8 Hecke operators through the chain

Input I9 (divisibility of jets; unit 150, and the Hecke transfer of the background library). (i) (*Gauges from a jet bound.*) In Settings 2.1 and 2.2, let ϑ be a twist, i a natural number, δ an integer of $K(P)$ and $\Phi' \in \mathrm{Std}_{\delta}(D,i)$. Let $y, y' \in G_{X_{i+1}(D)}$ be integral and invertible at all primes above p, and let y'' be a point of $G_{X_{i+1}(D)}$ whose image under $x \mapsto j(\iota_\tau(\overline{x}))$ is, for every τ, a diagonal matrix $\mathrm{diag}(u_\tau)$. Let B be a real number such that for every $\Phi \in \mathrm{Std}_{\delta}(D,i)$ and all Weyl indices c', c the entry (c',c) of the jet matrix of Φ at the scale of the lift kernel, multiplied by $\prod_\tau (\prod_a |u_\tau(a)|_p)^{\mathrm{last}_D(i+1,\tau)}$ and by the product of $|u_\tau(a)|_p$ over the entries a of the tableau c' at τ, has p-adic absolute value at most B. Then $\omega_{D,\vartheta,i}(1, yy''y')\Phi' \in \mathrm{Raise}_{\mathrm{k}_{D,\vartheta,i}}(1,1;B)$.
(ii) (*Jets of standard data are governed by theta products.*) For $\Phi \in \mathrm{Std}_{\delta}(D,i)$, a complex number c and a $\mathbb{C}$-linear form lf on power series: if $|j(\mathrm{lf}(\Theta))|_p \le r$ for every power series Θ whose image under j lies in the set $\mathrm{ThProd}_{P,\mu(D)}(\delta, |j(c\,s_1)|_p; X_{i+1}(D) \times X_i(D))$ of theta products of the CM curve (s_1 the period factor), then $|j(\mathrm{lf}(T))|_p \le r$ for the Taylor series T of the theta function of Φ at the scale $c \cdot \mathrm{per}(P, \mu(D))$. If p does not divide $N_1(D)$, then $|j(s_1)|_p = 1$.
(iii) (*Normal form and divisibility of Schur jets.*) Let $\chi_{K(P)}(p) \neq 1$. Every theta product Θ in $\mathrm{ThProd}_{P,\mu(D)}(1, r; I)$ with $r \le 1$ is *integral on the null cone* in every degree, for the maps $\tau \mapsto [-p^e]_D \cdot \tau$, the exponents $e \le \lfloor \log_p L \rfloor$ with e odd or $p = d$, and elements ϖ of $\mathbb{C}_p$ and ϖ_0 of $\mathfrak{o}_p$ with $|\varpi|_p = p^{-\varrho_d(p)}$ and $|\varpi_0|_p = p^{(\chi_{K(P)}(p)-1)/2}$. For a series Θ that is integral on the null cone up to the degree $L = \mathrm{boxes}(\varkappa, n')$, a finite set Ξ of places and a family of *charges*, natural numbers $\mathrm{ch}(\tau, h, a)$ attached to a place, a height and an index, such that for every exponent e as above, every place τ, height h and index a

$$\tau \in \Xi \ \text{ or } \ (h = 1 \text{ and } [-p^e]_D \cdot \tau \in \Xi) \\ \Longrightarrow \quad \mathrm{ch}(\tau, 1, a) \neq 0 \ \text{ or } \ \mathrm{ch}([-p^e]_D \cdot \tau, h, a) \neq 0, \quad (3)$$

the entry (c', c) of the matrix of jets of Θ along the Schur matrix of $\varkappa$ has absolute value at most p to the power: the total charge of the boxes of the tableau c', minus

$$(\tfrac{1}{p-1} - \varrho_d(p))L - \tfrac{1-\chi_{K(P)}(p)}{2} \sum_{e,\tau} \mathrm{cc}_p(\varkappa, n'; \Xi; \tau, [-p^e]_D \cdot \tau; e) \\ - \#\,\mathrm{Pl}(D)\, n'^3 (\lfloor \log_p L \rfloor + 1).$$

For $\varkappa = \varkappa_D$ and $n' = n_{i+1}(D)$ the subtracted number is $\mathrm{je}_D(p, i, \Xi)$.

Input I9, continued. (iv) (*Hecke transfer.*) Let w be a finite place with $\overline{w} \neq w$; consider a theta lift from rank n to rank $n+k'$ whose two characters are unramified at w and whose datum is a vacuum at w and $\overline{w}$. Then the lift of a form φ fixed by the group at w is fixed by the group at w, and the Hecke polynomials $\mathrm{H}_w(X) = \sum_{n''} (-X)^{n''} \mathrm{q}_w^{n''(n''-1)/2} T_{w,n''}$ satisfy

$$\mathrm{H}_w(X)\,\mathrm{lift}(\varphi) = \prod_{l<k'} (1 - \mathrm{q}_w^{l}\,\alpha X) \cdot \mathrm{lift}\big(\mathrm{H}_w(\mathrm{q}_w^{k'}\,\alpha\alpha'^{-1} X)\varphi\big),$$

where α and α' are the values at a uniformiser of w of the Hecke characters of the two characters, normalised by $\mathrm{q}_w^{n/2}$ and $\mathrm{q}_w^{(n+k')/2}$.

LEAN unit 150: `Parameters.mem_raises_of_jet_bound`, `Parameters.norm_jetSeries_le`, `Parameters.norm_iota_periodFactor_eq_one`, `Parameters.thetaProducts_normal_form`, `norm_schurJet_le` · background library: `hecke_transfer_of_pairing`, a form of `hecke_transfer_of_equivariant`, which the catalogue records as the Eichler commutation relation through the theta correspondence (S. Rallis, *Langlands' functoriality and the Weil representation*, Amer. J. Math. 104 (1982); R. Howe, *θ-series and invariant theory*, Proc. Sympos. Pure Math. 33 (1979); T. Tamagawa, Ann. of Math. 77 (1963)), place not confirmed, relation: a special case (split places of unitary groups, unramified characters, the unramified vacuum)

Proposition 8.1 (Hecke transfer along the chain)**.** *In Settings 5.1 and 6.4, let $\mathfrak{l} \in \mathcal{L}$ with $\overline{\mathfrak{l}} \neq \mathfrak{l}$, and let i be a natural number. There are complex numbers c_n, $n \in \mathbb{N}$, with $c_n \in O_j$ for all n if $i < m$, such that for every x in the support of Ψ:*

(a) $\omega_{D,\vartheta,i}(1,a)\,\mathrm{td}_D(\mathsf{x},i) = \mathrm{td}_D(\mathsf{x},i)$ *for every* $a \in K_{X_{i+1}(D)}[\mathfrak{l}]$*; and*

(b) for every form $\varphi \in \mathrm{St}_0(D)$ *that is fixed by the group* $K_{X_0(D)}[\mathfrak{l}]$*, with* $C = \mathrm{C}_{D,\vartheta}(\mathrm{td}_D(\mathsf{x},\,\cdot\,))_{i+1}$*,*

$$\sum_{n \le \mathfrak{r}(D,p)} c_n\, T_{\mathfrak{l},n}(C\varphi) = p^{\Delta(D,p)} \cdot C(T_{\mathfrak{l},\mathfrak{r}(D,p)}\varphi).$$

Proof. By Lemma 6.5, p does not divide $g(D)N(D)$ and $p \in \mathfrak{l}$; by hypothesis p is prime to $\mathrm{nrm}(\vartheta)$. So $\mathfrak{l}$ and $\overline{\mathfrak{l}}$ are not chosen, do not divide $\pi_N(D)$ and do not lie in $Q(\vartheta)$ or its conjugate, and the characters of all the lifts are unramified at $\mathfrak{l}$. Since $\mathcal{L}$ is not empty, p is served; by Lemma 6.1(b) the order of $[-p]_D$ is even, say $2o'$, and the residue field of $\mathfrak{l}$ has $\mathrm{q} = p^{2o'}$ elements (the decomposition of an inert prime, unit 152). For x in the support, Lemma 5.2 shows that every datum $\mathrm{td}_D(\mathsf{x},i')$ is a vacuum at $\mathfrak{l}$ and $\overline{\mathfrak{l}}$; an element a as in (a) then acts on it through a value of an unramified character, which is 1. This is (a).

For (b), Input I9(iv) applies to each of the lifts of C, with $k' = t(D)+1$. Put $y_e = p^{o'(n_{i+1}(D) - n_e(D))}$. By induction on the number e of lifts, for the chain C_e of the first e lifts and all complex U,

$$\mathrm{H}_{\mathfrak{l}}(-U y_e)\, C_e\varphi = \prod_{e'<e} \prod_{l \le t(D)} (1 + U\,\mathrm{q}^{l}\,p^{o' n_i(D)}\,\alpha_{e'}) \cdot C_e(\mathrm{H}_{\mathfrak{l}}(-U y_0)\varphi),$$

where $\alpha_{e'}$ is the value at the uniformiser of the character of the lift e'. (Two facts enter the induction step. In Input I9(iv) for the lift e' the two characters take the same value $\alpha_{e'}$ at the uniformiser, so that $\alpha\alpha'^{-1} = \mathrm{q}^{-k'/2}$ and the variable is rescaled by $\mathrm{q}^{k'/2} = p^{o'(t(D)+1)}$, which turns $y_{e'+1}$ into $y_{e'}$. And the normalising factor of the lift e' times $y_{e'+1}$ is $\mathrm{q}^{n_{e'}(D)/2}\, y_{e'+1} = p^{o'(n_{e'}(D) + n_{i+1}(D) - n_{e'+1}(D))} = p^{o' n_i(D)}$, the

same for every e'.) These values are integral at j: the character of the lift e' has weights $-\mathrm{wt}_D(e',\tau)$, so $|j(\alpha_{e'})|_p$ is a product over τ of powers of p whose exponents are $\mathrm{wt}_D(e',\tau)$ at the places inducing $\mathfrak{l}$ and $-\mathrm{wt}_D(e',\tau)$ at those inducing $\bar{\mathfrak{l}}$, up to a common sign, and by Lemma 6.6(b), with $G(c,w) = -c\,[w = \mathfrak{l}]$, the exponents cancel. Taking $e = i+1$, the product is a polynomial $\Pi(U)$ with coefficients in O_j and constant term 1. Comparing the coefficients of $U^{\mathfrak{r}(D,p)}$ after multiplication by the inverse power series $\Pi(U)^{-1}$, which again has coefficients in O_j, expresses $C(T_{\mathfrak{l},\mathfrak{r}(D,p)}\varphi)$, times a power of p coming from y_0 and from $\mathrm{q}^{\mathfrak{r}(D,p)(\mathfrak{r}(D,p)-1)/2}$, as a combination of the $T_{\mathfrak{l},n}(C\varphi)$, $n \le \mathfrak{r}(D,p)$, with coefficients that are elements of O_j times powers of p; the identity needs $\mathfrak{r}(D,p) \le k(D)$, which holds by size. Multiplying by $p^{\Delta(D,p)}$ and dividing by the power of p on the left gives the numbers c_n. For $i < m$ they are integral: the exponents of p that are divided out are at most $2o'\,\mathfrak{r}(D,p)\,(n_m(D) + \mathfrak{r}(D,p))$, and $2o' \le f(D)$ because the order of an element of $\mathrm{Pl}(D)$ is at most $f(D)$; this is the definition of $\Delta(D,p)$. □

LEAN `Data.IsGoodDatum.exists_hecke_transfer` · unit 152: `Parameters.q_eq_inert` · unit 141: `Data.hasWeight_liftChar`

Proposition 8.2 (Hecke translates of the data raise gauges by a controlled amount)**.** *In Settings 5.1 and 6.4, let* x *be in the support of* Ψ*,* i *a natural number, and* $g = (g_{\mathfrak{l}})_{\mathfrak{l}\in\mathcal{L}}$ *a family with* $g_{\mathfrak{l}} \in \mathrm{GL}_{X_{i+1}(D)}(E(D)_{\mathfrak{l}})$ *such that every* $g_{\mathfrak{l}}$ *lies in the basic double coset of some index* $n \le \mathfrak{r}(D,p)$ *at* $\mathfrak{l}$*. Then*

$$\omega_{D,\vartheta,i}(1,\mathrm{at}(g))\,\mathrm{td}_D(\mathsf{x},i) \;\in\; \mathrm{Raise}_{\mathrm{k}_{D,\vartheta,i}}(1,1;p^{\sum_{\mathfrak{l}\in\mathcal{L}} A_D(\mathfrak{l})+\mathrm{ll}_{D,p}(\mathcal{L},i)}).$$

Proof. By Lemma 6.5, p is a uniformiser at each $\mathfrak{l} \in \mathcal{L}$, so $g_{\mathfrak{l}}$ is the product of an integral invertible matrix $y_{\mathfrak{l}}$, the diagonal matrix with entries p at the indices of a set $J_{\mathfrak{l}}$ and 1 elsewhere, and an integral invertible matrix $y'_{\mathfrak{l}}$; the set $J_{\mathfrak{l}}$ has at most $\mathfrak{r}(D,p)$ elements. Thus $\mathrm{at}(g) = y\,y''\,y'$ with $y, y' \in K_{X_{i+1}(D)}$ and a point y'' whose component at $\mathfrak{l} \in \mathcal{L}$ is the diagonal matrix with entries p on $J_{\mathfrak{l}}$ and 1 elsewhere, whose component at $\bar{\mathfrak{l}}$ is the inverse of that matrix (unitarity forces it), and which is 1 at the other places. The image of y'' at a place τ is taken under $x \mapsto j(\iota_\tau(\overline{x}))$ (Input I1). The prime of this embedding is the conjugate of the induced prime $\mathfrak{l}_j(\tau)$, so the image reads the component of y'' at $\overline{\mathfrak{l}_j(\tau)}$: it is the diagonal matrix with entries $u_\tau(a) = p^{\epsilon_\tau(a)}$,

$$\epsilon_\tau(a) = [\overline{\mathfrak{l}_j(\tau)} \in \mathcal{L},\, a \in J_{\overline{\mathfrak{l}_j(\tau)}}] - [\mathfrak{l}_j(\tau) \in \mathcal{L},\, a \in J_{\mathfrak{l}_j(\tau)}].$$

So $|u_\tau(a)|_p = p$ if $\mathfrak{l}_j(\tau) \in \mathcal{L}$ and $a \in J_{\mathfrak{l}_j(\tau)}$; $|u_\tau(a)|_p = p^{-1}$ if $\overline{\mathfrak{l}_j(\tau)} \in \mathcal{L}$ and $a \in J_{\overline{\mathfrak{l}_j(\tau)}}$; and $|u_\tau(a)|_p = 1$ otherwise (by (M1) the two conditions exclude each other). By Lemma 5.3 with $\delta = 1$ (recall $p \nmid g(D)$ and $p \ne N(D)$), the datum lies in $\mathrm{Std}_1(D,i)$, and Input I9(i) reduces the claim to a bound for the jets of all $\Phi \in \mathrm{Std}_1(D,i)$.

Let $\mathrm{ch}(\tau,h,a) = 1$ if ($\mathfrak{l}_j(\tau) \in \mathcal{L}$, $h = 1$ and $a \notin J_{\mathfrak{l}_j(\tau)}$) or ($\overline{\mathfrak{l}_j(\tau)} \in \mathcal{L}$ and $a \in J_{\overline{\mathfrak{l}_j(\tau)}}$), and 0 otherwise. Condition (3) holds for $\Xi = \Xi_j(\mathcal{L})$: if $\tau \in \Xi$ and $a \notin J_{\mathfrak{l}_j(\tau)}$, then $\mathrm{ch}(\tau,1,a) = 1$; if $\tau \in \Xi$ and $a \in J_{\mathfrak{l}_j(\tau)}$, then the place $[-p^e]_D \cdot \tau$ induces $\overline{\mathfrak{l}_j(\tau)}$ (Lemma 6.2(c); p is inert and e is odd), so its charge at a is 1; the case $h = 1$, $[-p^e]_D \cdot \tau \in \Xi$ is symmetric. By Input I9(ii) and (iii), applied with $c = \overline{\pi_g(D)}$, for which $|j(c\,s_1)|_p \le 1$ (indeed $|j(\overline{\pi_g(D)})|_p \le 1$ by Lemma 3.2(i), and $|j(s_1)|_p = 1$ by Input I9(ii), because p, which does not split in $K(P)$, does not divide $N_1(D)$), every

entry (c', c) of the jet matrix of $\Phi \in \mathrm{Std}_1(D, i)$ has absolute value at most p to the power (charges of c') $-\mathrm{je}_D(p, i, \Xi_j(\mathcal{L}))$.

It remains to count the exponent of p in the bound B of Input I9(i). The determinant factors contribute $\sum_\tau \mathrm{last}_D(i+1,\tau)\cdot(-\sum_a \epsilon_\tau(a))$, and $\mathrm{last}_D(i+1,\tau) = \lfloor n_i(D)/2\rfloor - \mathrm{wt}_D(i+1,\tau)$; by Lemma 6.6(b) with $G(c,w) = \#J_w\cdot(\lfloor n_i(D)/2\rfloor - c)$ this sum is 0. For a box of c' at τ with entry a, the factor $|u_\tau(a)|_p$ and the charge together contribute to the exponent: exactly 1 if $\mathfrak{l}_j(\tau)\in\mathcal{L}$ and the box lies in a column of height one (either $a \in J_{\mathfrak{l}_j(\tau)}$, $|u_\tau(a)|_p = p$ and the charge is 0, or $a \notin J_{\mathfrak{l}_j(\tau)}$, $|u_\tau(a)|_p = 1$ and the charge is 1); 1 if $\mathfrak{l}_j(\tau)\in\mathcal{L}$, the box lies in a higher column and $a \in J_{\mathfrak{l}_j(\tau)}$ ($|u_\tau(a)|_p = p$, charge 0); and 0 in the other cases (a charge 1 at a place inducing the conjugate of a marked prime comes with $|u_\tau(a)|_p = p^{-1}$, which cancels it). The entries of a column of a tableau are distinct, so a higher column at $\tau\in\Xi_j(\mathcal{L})$ contributes at most $\#J_{\mathfrak{l}_j(\tau)} \le \mathfrak{r}(D,p)$. Summing: the columns of height one give at most $\sum_{\tau\in\Xi_j(\mathcal{L})}\varkappa_D(\tau,1) = \sum_{\mathfrak{l}} A_D(\mathfrak{l})$, the others at most $\mathfrak{r}(D,p)\sum_{\tau\in\Xi_j(\mathcal{L})}\mathrm{lc}_D(i,\tau)$. With the jet exponent this is $\sum_{\mathfrak{l}} A_D(\mathfrak{l}) + \mathrm{ll}_{D,p}(\mathcal{L},i)$. □

LEAN `Data.IsGoodDatum.mem_raises`

Proposition 8.3 (moving mark operators into the data)**.** *In Settings 5.1 and 6.4, let i be a natural number and $\mathfrak{l}_1,\dots,\mathfrak{l}_n$ a list of distinct elements of $\mathcal{L}$. There is a $\mathbb{C}$-linear endomorphism t of $\mathcal{W}_i(D)$ such that for every x in the support of Ψ, with $\Phi = t\,\mathrm{td}_D(\mathsf{x},i)$:*

(a) $\omega_{D,\vartheta,i}(a,a')\Phi = t\,\omega_{D,\vartheta,i}(a,a')\mathrm{td}_D(\mathsf{x},i)$ *for all $a\in G_{X_i(D)}$ and $a'\in K_{X_{i+1}(D)}$;*

(b) for every column $\varphi\in\mathrm{Col}(D)$,

$$\mathrm{lift}_{D,\vartheta,i}(\Phi)\,\mathrm{C}_{D,\vartheta}(\mathrm{td}_D(\mathsf{x},\cdot))_i\,\varphi = \mathrm{C}_{D,\vartheta}(\mathrm{td}_D(\mathsf{x},\cdot))_{i+1}\,\Upsilon_{D,\mathcal{L}}(\mathfrak{l}_1)\cdots\Upsilon_{D,\mathcal{L}}(\mathfrak{l}_n)\varphi;$$

(c) if $i<m$: for every family $g=(g_{\mathfrak{l}})_{\mathfrak{l}\in\mathcal{L}}$ with $g_{\mathfrak{l}}=1$ for $\mathfrak{l}$ in the list and $g_{\mathfrak{l}}$ in the basic double coset of some index at most $\mathfrak{r}(D,p)$ for the other $\mathfrak{l}$,

$$\omega_{D,\vartheta,i}(1,\mathrm{at}(g))\Phi \;\in\; \mathrm{Raise}_{\mathrm{k}_{D,\vartheta,i}}\!\left(1,1;p^{\sum_{c=1}^{n}(\Delta(D,p)-A_D(\mathfrak{l}_c))+\sum_{\mathfrak{l}\in\mathcal{L}}A_D(\mathfrak{l})+\mathrm{ll}_{D,p}(\mathcal{L},i)}\right).$$

Proof. Induction on the list. For the empty list take $t=1$; (a), (b) are trivial and (c) is Proposition 8.2. Let the claim hold for $\mathfrak{l}_2,\dots,\mathfrak{l}_n$ with the operator t', and write $\mathfrak{l}=\mathfrak{l}_1$. Take the numbers c_n of Proposition 8.1 for $\mathfrak{l}$ and i, put $z = p^{A_D(\mathfrak{l})}/p^{\Delta(D,p)}$ and

$$t = \Big(\sum_{n\le\mathfrak{r}(D,p)} c_n\, z\, T_{\mathfrak{l},n}\Big)\circ t',$$

the Hecke operators acting on $\mathcal{W}_i(D)$ through $a'\mapsto\omega_{D,\vartheta,i}(1,a')$; by (a) for t' and Proposition 8.1(a), $t'\mathrm{td}_D(\mathsf{x},i)$ is fixed by $K_{X_{i+1}(D)}[\mathfrak{l}]$. (a) The Hecke operators at $\mathfrak{l}$ commute with $\omega_{D,\vartheta,i}(a,a')$ for $a'\in K_{X_{i+1}(D)}$ (Input I8(ii)). (b) By the equivariance of the lift (Input I4), Hecke operators on the datum become Hecke operators on the lifted form. With C_i, C_{i+1} the chains of the statement and $\varphi' = \Upsilon_{D,\mathcal{L}}(\mathfrak{l}_2)\cdots\Upsilon_{D,\mathcal{L}}(\mathfrak{l}_n)\varphi$, which is fixed by $K_{X_0(D)}[\mathfrak{l}]$ by Lemma 6.5(c),

$$\begin{aligned}\mathrm{lift}_{D,\vartheta,i}(\Phi)\,C_i\varphi &= z\sum_n c_n T_{\mathfrak{l},n}(C_{i+1}\varphi')\\ &= z\,p^{\Delta(D,p)}\,C_{i+1}(T_{\mathfrak{l},\mathfrak{r}(D,p)}\varphi') = C_{i+1}\,\Upsilon_{D,\mathcal{L}}(\mathfrak{l})\varphi',\end{aligned}$$

by the induction hypothesis, Proposition 8.1(b) and the definition of the mark operator. (c) Expand each $T_{\mathfrak{l},n}$ as a sum over coset representatives; each term is

$\omega_{D,\vartheta,i}(1, \mathrm{at}(g'))\, t' \mathrm{td}_D(\mathsf{x}, i)$ for a family g' that agrees with g except at $\mathfrak{l}$, where it is a representative of a basic double coset of index $n \le \mathfrak{r}(D,p)$. The induction hypothesis applies to g', and the scalar $c_n z$ has $|j(c_n z)|_p \le p^{\Delta(D,p) - A_D(\mathfrak{l})}$ because $c_n \in O_j$ for $i < m$; conclude by Input I4. □

LEAN `Data.IsGoodDatum.exists_move_markOperator`

Proposition 8.4 (the chain after the mark operators)**.** *In Settings 5.1 and 6.4, let Y be the product of the operators $\Upsilon_{D,\mathcal{L}}(\mathfrak{l})$ over all $\mathfrak{l} \in \mathcal{L}$, in a chosen order, and let $v \in \mathrm{Col}_j(D)$. Then the form $\mathrm{B}_D(\vartheta, \Psi)(Y^m v)$ has j-gauge at most $p^{\sum_{i<m}(\#\mathcal{L}\cdot\Delta(D,p) + \mathrm{ll}_{D,p}(\mathcal{L},i))}$ at 1.*

Proof. For each i take the operator t_i of Proposition 8.3 for the list of all elements of $\mathcal{L}$. By (b) and induction on i, the chain of i lifts with the data $t_{i'} \mathrm{td}_D(\mathsf{x}, i')$ applied to v equals the chain with the data $\mathrm{td}_D(\mathsf{x}, i')$ applied to $Y^i v$. By (a) the t_i commute with the levels, and by (c) for the family $g = 1$ the datum $t_i \mathrm{td}_D(\mathsf{x}, i)$ lies in $\mathrm{Raise}_{\mathrm{k}_{D,\vartheta,i}}(1, 1; p^{\#\mathcal{L}\Delta(D,p) + \mathrm{ll}_{D,p}(\mathcal{L},i)})$. So Proposition 5.5, with $d_i = 1$, bounds the gauge of the chain combination at $Y^m v$. The restriction restr_D does not increase gauges at 1 when $p \ne N(D)$ (unit 141), and $p \ne N(D)$ by Lemma 6.5(b). □

LEAN `Data.IsGoodDatum.gaugeLe_markOperator_pow` · unit 141: `Data.GaugeLe.restrictToY`

9 The image of the mark operators has small index

Input I10 (twisted Hecke operators; unit 144). Let $n \ge 1$ be a natural number, W a valuation ring and R a commutative ring. *Twisting data* T for (n, W, R) are a prime number p_T, a natural number o, ring homomorphisms $\phi_s : W \to R$ for $s \in \mathbb{Z}/n$ (the *letters*), a parity (even or odd) for each $s \in \mathbb{Z}/n$, and an element of R with square p_T, such that $5 \le p_T^o$, p_T divides $\phi_{s+1}(y) - \phi_s(y)^{p_T^o}$ for all s and y, the parities of s and $s+1$ differ, and the residue field of W has $(p_T^o)^n$ elements. For natural numbers q', o, n, k', a', r', l' and real numbers c_1, c_2, c_3, the *Hecke sizes* are the conditions: $0 < r'$, $2r' \le k'$, $c_1 \le 1 \le c_2$, $2l' = n$, $n\,a'k'(l'+2) + a' + k' < {q'}^{l'}$, $6(2(k'+k'^2)o\,n^2 + 2) + 2 \le c_2$, $2(n+2)k'/c_2 \le c_1$, $(c_2 + n(k'+1))\,k'\,{q'}^{3r'} + k' + 2 \le a'/c_2$ and $n\,a'\,(q'^{-2r'} + r'^2/(k'q')) \le c_3$. The theorem of unit 144 on the elementary divisors of a Frobenius-twisted Hecke operator says: under the Hecke sizes and if p_T is not a unit of R, an operator given on the image M of an R-linear map from functions on a finite set of classes by the twisted formula of that unit, with an injective source map, has type $(p_T, s, (c_3 + c_1 s + c_2) \cdot n_M)$ on M for every natural number s, where n_M is the number of classes times the number of Weyl indices.

LEAN unit 144: `TwistingData`, `HeckeSizes`, `TwistingData.hasType_twisted_hecke`

Proposition 9.1 (the type of a mark operator, from twisting data)**.** *In Setting 6.4, let $\mathfrak{l} \in \mathcal{L}$, let W be the valuation ring of $E(D)$ at $\mathfrak{l}$, $n \ge 1$ a natural number, and T twisting data for (n, W, O_j) with prime p_T, exponent o and letters ϕ_s, such that p_T is not a unit of O_j and $p_T = p$. Let t be a bijection from $\mathbb{Z}/n$ onto the set of the places $\tau \in \mathrm{tops}(D)$ with $\mathfrak{l}_j(\tau) = \mathfrak{l}$ or $\mathfrak{l}_j(\tau) = \bar{\mathfrak{l}}$, such that ϕ_s is the restriction of $\iota_{t(s)}$ to W if s is even and of $y \mapsto \iota_{t(s)}(\overline{y})$ if s is odd; assume $A_D(\mathfrak{l}) = a(D) \cdot \#\{s \in \mathbb{Z}/n : \ s \text{ even}\}$, and let l' be a natural number and c_1, c_2, c_3 real numbers such that the Hecke sizes*

hold for $(p^o, o, n, k(D), a(D), \mathfrak{r}(D,p), l'; c_1, c_2, c_3)$. *Then for every natural number* s *the mark operator* $\Upsilon_{D,\mathcal{L}}(\mathfrak{l})$ *has type* $(p, s, (c_3 + c_1 s + c_2)\cdot \dim_{\mathbb{C}} \mathrm{Col}(D))$ *on* $\mathrm{Col}_j(D)$.

Proof. This is a translation into the terms of Input I10. By Lemma 3.2(d) and Input I1, evaluation at representatives g_c of the classes of $K_0^{\mathrm{col}}(D)$, chosen integral above p, identifies $\mathrm{Col}(D)$ with the functions on the classes with values in $\mathbb{C}^{\mathrm{Wy}_{X_0(D)}(\varkappa_D)}$, and $\mathrm{Col}_j(D)$ with the functions with values in $O_j^{\mathrm{Wy}_{X_0(D)}(\varkappa_D)}$ (Input I1, last property of the gauge); this is the map of Input I10, and $\dim \mathrm{Col}(D)$ is the number n_M. The Hecke operator $T_{\mathfrak{l},\mathfrak{r}(D,p)}$ on a form fixed by $K_{X_0(D)}[\mathfrak{l}]$ is the sum, over the cosets β of the parahoric subgroup of $\mathrm{GL}_{k(D)}(W)$ of block sizes $(\mathfrak{r}(D,p), k(D)-\mathfrak{r}(D,p))$, of the translates by $\beta\cdot \mathrm{diag}(p1_{\mathfrak{r}(D,p)}, 1)$. Write $g_c\cdot \mathrm{at}(\beta\ \mathrm{diag}(p1_{\mathfrak{r}(D,p)}, 1)) = \gamma\, g_{c'}\, a$ with γ rational and a in the level; then the translate of the form at g_c is $\rho_{\varkappa_D, \mathrm{last}_D(0,\cdot)}(\gamma)$ applied to its value at $g_{c'}$. At the places $t(s)$ this matrix is, since the weight there is a single row of length $a(D)$, a symmetric power of the image of γ under the letter ϕ_s, or of its inverse transpose for odd s (unitarity); together with the factor $p^{A_D(\mathfrak{l})} = p^{a(D)\#\{s \text{ even}\}}$ this is the twisted formula. At the other places τ, either τ is not a top and the weight is trivial below the rank $k(D)$, or τ induces neither $\mathfrak{l}$ nor $\bar{\mathfrak{l}}$ and γ is integral and invertible there, so the matrix is integral and invertible. The source map $(c,\beta)\mapsto c'$ together with the coset is injective because the level is free. The formal proof checks these points in 35 steps, of which the last fifteen (the identification of the letter matrices and of the powers of p) were read and not replayed. □

LEAN `Columns.hasType_markOperator`

Proposition 9.2 (the type of a mark operator)**.** *In Setting 6.4, let* s *be a natural number and* $\mathfrak{l}\in\mathcal{L}$. *Then* $\Upsilon_{D,\mathcal{L}}(\mathfrak{l})$ *has type*

$$\Big(p,\ s,\ (128\, A_D(\mathfrak{l})\cdot \frac{\log b}{m\, p} + \frac{s}{x^{14}} + x^{40})\cdot \dim_{\mathbb{C}} \mathrm{Col}(D)\Big) \quad \text{on } \mathrm{Col}_j(D).$$

Proof. $\mathcal{L}$ is not empty, so p is served: p is inert, $o = o_0(D,p)$ is odd and $o \le \mathfrak{r}(D,p)$; moreover $p \ge 13$ (unit 152). Let τ_0 be a top inducing $\mathfrak{l}$ (M3) and $v = [-p]_D^o$, of order n; then $o\,n$ is the order of $[-p]_D$, which m divides (Lemma 6.1(b)), so n is even. Put $t(s) = v^s\tau_0$ for $s\in\mathbb{Z}/n$. By Lemmas 6.2(d) and 6.1(d), t is a bijection onto the tops that induce $\mathfrak{l}$ or $\bar{\mathfrak{l}}$, and by Lemma 6.2(c) consecutive places induce conjugate primes; call s even if $t(s)$ induces $\mathfrak{l}$. The letters ϕ_s of Proposition 9.1 map W into O_j, and $\phi_{s+1}\equiv \phi_s^{p^o}$ modulo p: both sides are ring homomorphisms from W to O_j/p that agree on a root of unity of order $N_1(D)$ and on $\sqrt{-d}$, by the Frobenius criterion of unit 152. The residue field of $\mathfrak{l}$ has $p^{o\,n}$ elements (unit 152, as $o n$ is even), a square root of p lies in O_j, and $A_D(\mathfrak{l}) = a(D)\cdot\#\{s \text{ even}\} = a(D)\,n/2$. These are twisting data. The Hecke sizes hold with $l' = n/2$, $c_1 = x^{-14}$, $c_2 = x^{40}$ and $c_3 = 128\, A_D(\mathfrak{l})\log b/(mp)$: the formal text derives the three conditions that are not immediate size comparisons from $p\ge 9$, from $o\,n \ge m \ge 2\log b$, from $p^{\,o\cdot 3\mathfrak{r}(D,p)} \le b^{1/2}$ (as $p < x_1$) and from $p^{\,2\mathfrak{r}(D,p)} \ge m^2$. Proposition 9.1 gives the claim. □

LEAN `Data.IsMarked.hasType_markOperator` · unit 152: `Parameters.frob_ext`, `Parameters.q_eq_inert`, `Parameters.thirteen_le_of_chiK_ne_one`

Corollary 9.3 (the type of the product)**.** *In Setting 6.4, let* Y *be the product of the operators* $\Upsilon_{D,\mathcal{L}}(\mathfrak{l})$ *over all* $\mathfrak{l}\in\mathcal{L}$, *in a chosen order. Then* Y^m *has type* $(p,\ \mathrm{jc}(D,p),\ 3\,\mathrm{loss}(D,p)\cdot \dim_{\mathbb{C}} \mathrm{Col}(D))$ *on* $\mathrm{Col}_j(D)$.

Proof. Types add (Input I2), so by Proposition 9.2 the operator Y^m has type $(p, \mathrm{jc}(D,p), c \cdot \dim \mathrm{Col}(D))$ with

$$c = m \sum_{\mathfrak{l} \in \mathcal{L}} \Big(128\, A_D(\mathfrak{l}) \frac{\log b}{mp} + \frac{\mathrm{jc}(D,p)}{x^{14}} + x^{40} \Big).$$

Now the sum of the numbers $A_D(\mathfrak{l})$ is at most $a(D)\,\#\mathrm{tops}(D) \le a(D) f(D)$, the set $\mathcal{L}$ has at most $f(D)$ elements, $\mathrm{jc}(D,p) \le 2mf(D)a(D)/p + 1$, and $f(D)(\log b)^2 a(D)/p \le \mathrm{loss}(D,p)$ (Lemma 6.1(c)); by size, $c \le 3\mathrm{loss}(D,p)$. □

LEAN `Data.IsMarked.hasType_prod_markOperator_pow`

10 The level prime

Setting 10.1 (the level prime as p)**.** Setting 2.1 holds, and Setting 2.2 holds with $p = N(D)$: $j\colon \mathbb{C} \to \mathbb{C}_{N(D)}$ is a ring homomorphism; we write N for $N(D)$ and $|\cdot|_N$ for the absolute value of $\mathbb{C}_N$. ϑ is a twist of D. The set $\mathcal{N}$ of the conjugates $\overline{\mathfrak{l}_j(\tau)}$, $\tau \in \mathrm{Pl}(D)$, of the induced primes is a *split level* for $N(D)$ in the sense of unit 141 (the properties of Lemma 3.1(c)), and this fact is an argument of the statements below.

Input I11 (names at the level prime; unit 141). In Setting 10.1: for a function F on matrices over $\prod_{w\in\mathcal{N}} E(D)_w$ and a datum v, the *name* $F\otimes v$ is the datum obtained from v by replacing its component at the places of $\mathcal{N}$ and their conjugates by F; $\mathbf{1}_{\mathrm{lc}}$ is the indicator function of the integral matrices whose last column is congruent to $(0,\dots,0,1)$ modulo N. For a natural number i, $d_i \in G_{X_i(D)}$ is the point with component $\mathrm{diag}(1,\dots,1,N)$ at the places of $\mathcal{N}\cap \mathrm{Pl}_\pi(D)$ and 1 at the other places of $\mathcal{N}$; it has the property that restr_D maps forms of j-gauge at most c at d_m to forms of j-gauge at most c at 1. F is the *Fourier operator* on $\mathrm{St}_0(D)$: a sum, over the vectors y modulo N with unit last entry, of an additive character of the last entry times the translate by an integral matrix with last row and last column y, followed by the translate by the inverse of $\mathrm{diag}(1,\dots,1,N)$, all through the embedding of the general linear group at $\mathcal{N}$ conjugated by d_0. A pair of general linear elements at $\mathcal{N}$ acts on a name through the function, up to the values $\det_{\chi,n}(g)=\prod_{w\in\mathcal{N}}\chi_w(\det g_w)\,|\det g_w|_w^{-n/2}$ of the characters χ of the lifts. (i) A chain of pairs of lifts whose steps are equivariant up to characters equals a scalar times the chain of the modified data applied to a power of the Fourier operator. The scalar is a product over the pairs; the factor of a pair is the product of two values $\det_{\chi,n}(\delta)$ at matrices δ that are $\mathrm{diag}(1,\dots,1,N)$ at some places of $\mathcal{N}$ and 1 at the others, of a natural number, and, for each of the lifts up to the second lift of the pair, of one quotient $\det_{\chi,n}(\delta)\cdot\det_{\chi,n'}(\delta')^{-1}$ of two such values for one character χ, with $n'\le n$ and matrices δ, δ' of this kind at the same places (their sizes may differ). Moreover, for such χ, δ, δ' and $n'\le n$: $|j(\det_{\chi,n}(\delta))|_N \le |j(\det_{\chi,n'}(\delta'))|_N$ (the absolute value does not increase with the exponent). (ii) There is a natural number $e'\le 4\,m\,f(D)\,b\cdot\dim_{\mathbb{C}}\mathrm{Col}(D)$ such that the module of the $\varphi\in\mathrm{Col}(D)$ for which $\mathrm{F}^{m/2}\varphi$ is a column of j-gauge at most 1 at d_0 has index dividing $N^{e'}$ in $\mathrm{Col}_j(D)$. (iii) The components at $\mathcal{N}$ of the elements of the column levels, conjugated by the points d_i, are integral with last column congruent to $(0,\dots,0,1)$ modulo N. (iv) For a modulus $\mathfrak{m}$ of $E(D)$, an integer g' of $E(D)$ with $g'\in\mathfrak{m}$ and an integral finite adele η with $\eta\in\mathfrak{m}\,\widehat{\mathcal{O}}_{E(D)}$, the name of a function modulo $\mathfrak{m}$ with integral values, applied to a datum u, is a finite combination $\sum_y c_y\mathrm{U}(y)\mathrm{P}u$ of translates of the projection $\mathrm{P}u$ of u at the places of $\mathfrak{m}$, with every entry of $\overline{\eta}\,y$ integral and every $\mathrm{N}(g')^{\mathrm{rank}}c_y$ integral at all primes; here $\mathrm{N}(g')$ is the absolute norm of g' and rank the number of entries of y (unit 150, the lemma on admissible combinations; the two elements play different roles: g' gives the denominator, η the support).

LEAN unit 141: `IsSplitLevel`, `Data.nonInducedPlaces`, `IsSplitLevel.name`, `IsSplitLevel.lastColDatum`, `Data.diagPointN`, `Data.GaugeLe.restrictToY_of_diag`, `IsSplitLevel.fourierOp`, `Data.nonInducedRep`, `detCharAt`, `IsSplitLevel.compUpTo_pairs_eq_smul`, `IsSplitLevel.norm_detCharAt_diagN_anti`, `Data.exists_indexDvd_fourierOp_pow`, `Data.atPlaces_conj_mem_mirabolic` · unit 150: `isAdmissibleComb_name`

Lemma 10.2 (character values at the level prime)**.** *In Setting 10.1, let $i\le 2m$, n and i' be natural numbers and S a set of finite places. Let χ be the character of the lift i twisted by ϑ^i, after base change, and δ_S the matrix over $\prod_{w\in\mathcal{N}}E(D)_w$, indexed by the elements of $X_{i'}(D)$ other than the last, with component $\mathrm{diag}(1,\dots,1,N)$ at the places of S and 1 elsewhere. Then $|j(\det_{\chi,n}(\delta_S))|_N \le N(D)^{2f(D)b}$.*

Proof. By Input I11(i) the absolute value does not increase when n is replaced by 0; so let $n = 0$, where $\det_{\chi,0}(\delta_S) = \prod_w \chi_w(\det \delta_{S,w})$. The character has weights $-\mathrm{wt}_D(i,\tau)$, so $|j(\chi(u))|_N$ is the product over τ of the absolute values of the components of u at the induced primes to the powers $\pm\mathrm{wt}_D(i,\tau)$; for the determinant of δ_S each component is N or 1. Hence each factor is 1 or $N^{\pm\mathrm{wt}_D(i,\tau)}$, which is at most $N^{\mathrm{wt}_D(i,\tau)} \le N^{2b}$ by the bound of unit 152 for $i \le 2m$, whatever the set S; and there are $f(D)$ factors. □

LEAN `Data.norm_detCharAt_diagN_le`

Lemma 10.3 (the indicator of the last column raises gauges little)**.** *In Setting 10.1, let* x *be a tuple of translations and* $i < m$ *a natural number. Then*

$$\mathbf{1}_{\mathrm{lc}} \otimes \mathrm{td}_D(\mathsf{x}, i) \;\in\; \mathrm{Raise}_{\mathrm{k}_{D,\vartheta,i}}(1, 1; N(D)^{\,f(D)b}).$$

Proof. Let $c_0 = (1+\sqrt{-d})/2$. The residue field of the prime induced at the unit place has N elements, so c_0 is congruent to an integer n modulo it; put $y_0 = c_0 - n \in \mathcal{O}_{K(P)}$. Then $j(y_0)$ is not a unit, and $j(\overline{y_0}\, d)$ is a unit: $\overline{y_0} = 1 - 2n - y_0$ and $(2c_0 - 1)^2 = -d$ is not zero modulo N. Apply Input I11(iv) to a modulus $\mathfrak{m}$ whose places are those of $\mathcal{N}$, to $g' = N$ and to $\eta = \overline{y_0}$. Both lie in every place of $\mathcal{N}$: for N this is Setting 10.1; and y_0 lies in the induced prime $\mathfrak{l}_j(\tau)$ for every τ, because $\iota_\tau(y_0) = y_0$ and $j(y_0)$ is not a unit, so $\overline{y_0}$ lies in $\overline{\mathfrak{l}_j(\tau)}$. Hence the name is a combination of translates $\mathrm{U}(y)\mathrm{Ptd}_D(\mathsf{x}, i)$ with $y_0 y$ integral, and with coefficients whose multiples by $\mathrm{N}(N)^{\,\mathrm{rank}}$ are integral. Each translate lies in $\mathrm{Std}_{y_0}(D, i)$: the projection of the datum is 0 or a translate $\mathrm{U}(y')\theta^{\circ}_{D,i}$ of the integral vacuum by a matrix that is integral at the places above N, hence a generator, and translations by the matrices y preserve the module (unit 150). So $\mathrm{N}(N)^{\,\mathrm{rank}}$ times the name lies in $\mathrm{Std}_{y_0}(D, i)$ and, by Input I8(i), raises gauges by at most $(|j(\overline{\pi_g(D)})|_N / |j(\overline{y_0} d)|_N)^L \le 1$. Now N is a rational integer and $E(D)$ has degree $2f(D)$, so $\mathrm{N}(N) = N^{2f(D)}$; and the rank of the space of homomorphisms is $n_i(D) n_{i+1}(D) \le n_m(D)^2$. Dividing by $\mathrm{N}(N)^{\,\mathrm{rank}}$ therefore costs at most $N^{2f(D) n_m(D)^2}$, which is at most $N^{f(D)b}$ because $2\, n_m(D)^2 \le b$ by size. (The norm of y_0 plays no part: its valuation at N is not bounded.) □

LEAN `Data.name_lastColDatum_mem_raises` · unit 150: `Parameters.finHeisOp_mem_standardData`

Proposition 10.4 (the paired chain at ramified columns)**.** *In Setting 10.1 there is a complex number* z *with* $|j(z)|_N \le N(D)^{\,2mf(D)b}$ *such that for every family* $v = (v_i)_{i \in \mathbb{N}}$ *with* $v_i \in \mathcal{W}_i(D)$ *and every* $\varphi \in \mathrm{Col}^{\mathrm{ram}}(D)$

$$\mathrm{C}^{(2)}_{D,\vartheta}(\nu \mapsto \mathrm{pt}_D(\mathrm{P}^{\mathrm{Pl}_\pi(D)} v)_\nu)_{m/2}\, \varphi = z \cdot \mathrm{C}_{D,\vartheta}((\omega_{D,\vartheta,i}(d_i, d_{i+1})(\mathbf{1}_{\mathrm{lc}} \otimes v_i))_i)_m\, \mathrm{F}^{m/2}\varphi.$$

Proof. The sum over y in Definition 2.8(b) is a sum of names: translating by $-y/N$ in the last column, or its conjugate in the last row, multiplies the component at $\mathcal{N}$ by a character of the last column, respectively row, and $\mathrm{diag}(1, \dots, 1, N)$ splits off the places of $\mathrm{Pl}_\pi(D)$. Each pair of lifts is then equivariant, up to the values $\det_{\chi,n}$, under the general linear groups at $\mathcal{N}$; the characters are unramified at the places above N. Input I11(i) gives the identity for ramified φ (ramified columns are ramified in the sense of unit 141, by Lemma 4.3(b)). The scalar z is a product over the $m/2$ pairs. By Input I11(i) the factor of a pair consists of two values $\det_{\chi,n}(\delta)$ of characters of the chain, each of absolute value at most $N^{2f(D)b}$ by Lemma 10.2; of a natural number, of absolute value at most 1; and of quotients $\det_{\chi,n}(\delta) \cdot \det_{\chi,n'}(\delta')^{-1}$ with $n' \le n$, each of absolute value at most 1 by the monotonicity in the exponent (an

upper bound for a value alone would say nothing about its inverse; the exponents are ranks $n_i(D)$, which increase with i). Hence $|j(z)|_N \le (N^{2f(D)b})^{2\cdot m/2} = N^{2mf(D)b}$. □

LEAN `Data.chain₂_pairTranslates_eq_smul_chain`

Proposition 10.5 (the chain at the level prime)**.** *In Setting 10.1, let Ψ be a finitely supported complex function on the tuples of translations of D such that Ψ is good for (D,ϑ), and let $\varphi \in \mathrm{Col}^{\mathrm{ram}}(D)$ be such that $\mathrm{F}^{m/2}\varphi$ is a column of j-gauge at most 1 at d_0. Then $\mathrm{B}_D(\vartheta,\Psi)\varphi$ has j-gauge at most $N(D)^{3mf(D)b}$ at 1.*

Proof. By (G3) and Proposition 10.4 with $v_i = \mathrm{td}_D(\mathsf{x},i)$, the chain combination at φ is z times the chain combination with the data $t_i\mathrm{td}_D(\mathsf{x},i)$, $t_i = \omega_{D,\vartheta,i}(d_i,d_{i+1})(\mathbf{1}_{\mathrm{lc}}\otimes\cdot\,)$, at $\mathrm{F}^{m/2}\varphi$. The operators t_i commute with the levels on these data: by Input I11(iii) the level elements, conjugated by the d_i, preserve the last column modulo N, hence the indicator. By Lemma 10.3 and the definition of the set of raising data, $t_i\mathrm{td}_D(\mathsf{x},i) \in \mathrm{Raise}_{\mathrm{k}_{D,\vartheta,i}}(d_i,d_{i+1};N^{f(D)b})$. Now N is prime to $\mathrm{nrm}(\vartheta)$ (Input I3), so Proposition 5.5 applies and gives gauge at most $N^{mf(D)b}$ at d_m; multiply by $|j(z)|_N \le N^{2mf(D)b}$ and restrict (Input I11). □

LEAN `Data.gaugeLe_Bmu_at_N`

11 Proof of Theorem B

Lemma 11.1 (four gauge bounds for the chain)**.** *In Setting 5.1, let $v \in \mathrm{Col}_j(D)$.*

(a) If $p \neq N(D)$ and $p \neq d$, then $\mathrm{B}_D(\vartheta,\Psi)v$ has j-gauge at most 1 at 1.

(b) If $\mathcal{L}$ is a full mark and Y is the product of the operators $\Upsilon_{D,\mathcal{L}}(\mathfrak{l})$ over all $\mathfrak{l}\in\mathcal{L}$, in a chosen order, then $\mathrm{B}_D(\vartheta,\Psi)(Y^m v)$ has j-gauge at most $p^{2\mathrm{loss}(D,p)-\mathrm{gain}(D,p)}$ at 1.

(c) If p is small and not split for D, then $\mathrm{B}_D(\vartheta,\Psi)v$ has j-gauge at most p^c at 1, where $c = m\cdot\mathrm{coll}(D,p,\emptyset) + \mathrm{loss}(D,p) - \mathrm{gain}(D,p)$.

(d) If p is small and not split for D, then $\mathrm{B}_D(\vartheta,\Psi)(p^{\mathrm{jc}(D,p)}\cdot v)$ has j-gauge at most $p^{\mathrm{loss}(D,p)-\mathrm{gain}(D,p)}$ at 1.

Proof. (a) By Lemma 5.3 with $\delta = \pi_g(D)$ every datum lies in $\mathrm{Std}_{\pi_g(D)}(D,i)$, so by Input I8(i) it lies in $\mathrm{Raise}_{\mathrm{k}_{D,\vartheta,i}}(1,1;c)$ with $c = (|j(\overline{\pi_g(D)})|_p/|j(\overline{\pi_g(D)}d)|_p)^L = |j(d)|_p^{-L} = 1$, as $p \neq d$. Proposition 5.5 with $t_i = 1$, $d_i = 1$, $r_i = 0$ and the restriction (unit 141, $p \neq N(D)$) give the claim.

(b) By Proposition 8.4 the exponent is at most

$$\sum_{i<m}\Big(\#\mathcal{L}\cdot\Delta(D,p) + \mathfrak{r}(D,p)\sum_{\tau\in\Xi_j(\mathcal{L})}\mathrm{lc}_D(i,\tau) - \mathrm{je}_D(p,i,\Xi_j(\mathcal{L}))\Big).$$

The place 1 is a top, so $\mathcal{L}$ is not empty and p is served, $\chi_{K(P)}(p) = -1$. By Lemma 7.4, Proposition 7.3 with $\Xi = \Xi_j(\mathcal{L})$ and Lemma 7.1(d), each summand is at most

$$\frac{\mathrm{loss}(D,p)}{m} - \frac{\mathrm{gain}(D,p)}{m} + f(D)(\log b)^2\frac{a(D)}{pm} \;\le\; \frac{2\mathrm{loss}(D,p)-\mathrm{gain}(D,p)}{m}\,;$$

the two terms $\#(\mathrm{tops}(D)\cap\Xi_j(\mathcal{L}))\,a(D)/x_1$ cancel, and the last step is Lemma 6.1(c).

(c) The empty set is marked, by (M4); for it $Y = 1$, $\Xi_j(\emptyset) = \emptyset$ and $\mathrm{ll}_{D,p}(\emptyset, i) = -\mathrm{je}_D(p, i, \emptyset)$. By Propositions 8.4 and 7.3, the exponent is at most

$$\sum_{i<m}\Big(-\frac{\mathrm{gain}(D,p)}{m} + \mathrm{coll}(D,p,\emptyset) + f(D)(\log b)^2\frac{a(D)}{pm}\Big),$$

and Lemma 6.1(c) applies again.

(d) $\mathrm{B}_D(\vartheta, \Psi)$ is linear, so the gauge of (c) is multiplied by $p^{-\mathrm{jc}(D,p)}$. By Lemma 7.1(b) with $o = 1$, $m\,\mathrm{coll}(D,p,\emptyset) \le 2mf(D)a(D)/p \le \mathrm{jc}(D,p)$. □

LEAN Data.IsGoodDatum.gaugeLe_one · Data.IsGoodDatum.gaugeLe_Bmu_of_isMarked · Data.IsGoodDatum.gaugeLe_collisionSum · Data.IsGoodDatum.gaugeLe_pow_jetCeil_smul

Proof of Theorem B. If a form has j-gauge at most p^c at 1 with $c \le -\beta$, then its multiple by $p^{-\beta}$ has j-gauge at most 1 at 1. Three cases, following the definition of $\mathrm{err}(D,p)$.

$p \in \mathrm{NS}(D)$. Then p is a prime below x_1 with $\chi_{K(P)}(p) \ne 1$, $\mathrm{err}(D,p) = 3\mathrm{loss}(D,p)$, $\mathrm{loss}(D,p) \ge 1$ by size and Lemma 6.1(c), and the hypothesis reads $\beta < \mathrm{gain}(D,p) - 3\mathrm{loss}(D,p) + 1$; so $2\mathrm{loss}(D,p) - \mathrm{gain}(D,p) \le -\beta$. If p is served, take a full mark $\mathcal{L}$ (Lemma 6.3), the operator Y of Lemma 11.1(b) and

$$S = Y^m\,\mathrm{Col}_j(D) + p^{\mathrm{jc}(D,p)}\,\mathrm{Col}_j(D).$$

By Corollary 9.3 there is a natural number $l \le 3\mathrm{loss}(D,p)\dim\mathrm{Col}(D)$ such that S has index dividing p^l in $\mathrm{Col}_j(D)$. An element of S is $Y^m u + p^{\mathrm{jc}(D,p)}v$ with $u, v \in \mathrm{Col}_j(D)$, and by Lemma 11.1(b) and (d) both parts are mapped to forms of gauge at most $p^{-\beta}$. If p is not served, take $l = 0$ and $S = \mathrm{Col}_j(D)$: by Lemma 11.1(c) and Lemma 7.1(c) the gauge is at most $p^{2\mathrm{loss}(D,p)-\mathrm{gain}(D,p)}$. In this case the columns need not be ramified.

$p \notin \mathrm{NS}(D)$ and $p = N(D)$. Then $\mathrm{err}(D,p) = 4mf(D)b$ and $\beta < 1 - 4mf(D)b$, so $3mf(D)b \le -\beta$, as $mf(D)b \ge 1$. Take for S the module of Input I11(ii) and $l = e'$. For a ramified column $\varphi \in S$, Proposition 10.5 gives gauge at most $N^{3mf(D)b}$. (Here $N(D)$ is prime to $\mathrm{nrm}(\vartheta)$ for every twist, and Setting 10.1 holds with the given j.)

The other primes. Then $\mathrm{err}(D,p) = 0$, $\beta \le 0$, and we take $l = 0$, $S = \mathrm{Col}_j(D)$. Since $p \ne N(D)$, and $p \ne d$ because d is a prime below x_1 with $\chi_{K(P)}(d) = 0$ and so belongs to $\mathrm{NS}(D)$, Lemma 11.1(a) applies. □

LEAN Data.exists_indexDvd_Bmu_divisible

12 Sizes: Theorems C and D

Lemma 12.1 (the prime sum, prime by prime)**.** *In Setting 2.1, let U be a natural number with $\lceil x_1\rceil \le U$, y a real number with $y \le a(D)$ and $\tau \in \mathrm{Pl}(D)$. Then*

$$\mathrm{Ask}_D(y,\tau) = \sum_{p<U,\ p\ prime} 2\,\mathrm{ask}_{D,p}(y,\tau)\cdot\log p.$$

Proof. The weight is $\lambda_{d,1}(\nu) = \Lambda(\nu)(1-\chi_{K(P)}(\nu))$, with Λ the von Mangoldt function. The term of index $\nu \ge \lceil x_1\rceil$ of $\mathrm{Ask}_D(y,\tau)$ vanishes, because $y/\nu \le a(D)/x_1$ and every nominal weight is at least $a(D)/x_1$; and $\lceil x_1\rceil \le a(D)$ by size. So the sum may be taken over all $\nu \ge 1$, and since Λ is supported on the prime powers it is $\sum_p\sum_{e\ge1}\log p\,(1-\chi_{K(P)}(p)^e)\,(y/p^e - w_D([-p^e]_D\cdot\tau))_+$. In each of the cases $p = d$, $\chi_{K(P)}(p) = 1$, $\chi_{K(P)}(p) = -1$ one has $1-\chi_{K(P)}(p)^e = 2\cdot\frac{1-\chi_{K(P)}(p)}{2}\cdot[e \text{ odd or } p = d]$ for $e \ge 1$. The primes $p \ge \lceil x_1\rceil$ contribute 0. □

LEAN `Data.primeSum_eq_sum_localAsk`

Lemma 12.2 (row lengths)**.** *In Setting 2.1, let $i < m$ and ν be natural numbers and $\tau \in \mathrm{Pl}(D)$. The row of index ν at τ of the diagram of $\varkappa_D$ for the rank $n_{i+1}(D)$ has length*

$$\sum_{\nu<h<n_{i+1}(D)} \varkappa_D(\tau,h) \;\le\; \big([\nu=0]\,\varkappa_D(\tau,1) + [0<\nu<k(D)]\,\mathrm{wt}_D(0,\tau)\big) + \big([\nu=0]\,(b_1(D)+b_2(D)) + r^{16}\big),$$

where $b_1(D)$ and $b_2(D)$ are the least multiples of $N_1(D)$ that are at least $a(D)/x$ and $a(D)/x_1$.

Proof. Bound each $\varkappa_D(\tau,h)$ with $h \neq 1$ by $[h=k(D)]\mathrm{wt}_D(1,\tau)+\mathrm{gap}(D)$ (unit 150); here $\mathrm{wt}_D(1,\tau) = \mathrm{wt}_D(0,\tau)+\mathrm{gap}(D)$, $\mathrm{wt}_D(0,\tau) \le b_1(D)+b_2(D)+r^8$, the height $k(D)$ occurs only for $\nu < k(D)$, and $(n_m(D)+2)\mathrm{gap}(D) \le r^{16}$ by size. □

LEAN `Data.diagramWeight_le_by_index`

Lemma 12.3 (two sizes)**.** *In Setting 2.1:*

(a) For every $\sigma \in \mathrm{Aut}(\mathbb{C})$, $|\sigma(\overline{\pi_g(D)})|^2 = g(D)$.

(b) For natural numbers n, n', e with $n \le n'+e$, $\mathrm{st}_D(n) \le \mathrm{st}_D(n') + (\log(n'+e+1) - \log(\Omega(P,\mu(D))/g(D)))\cdot e$.

Proof. (a) σ maps $K(P)$ to itself, by the identity or by conjugation, and $\pi_g(D)\overline{\pi_g(D)} = g(D)$. (b) First $n! \le (n'+e)! \le n'!\,(n'+e+1)^e$. Secondly the number $-\log(\Omega(P,\mu(D))/g(D))$ is non-negative by size, so its multiple by n is at most its multiple by $n'+e$. □

LEAN `Data.norm_map_star_piG_sq` · `Data.jetSize_le_add`

Lemma 12.4 (the number of classes)**.** *In Setting 2.1, let I be a finite set and K a subgroup of G_I.*

(a) If $G \ge 1$ is a natural number, $|I| \le 2\,n_m(D)$ and $K_I(G) \subseteq K$, then $\#\mathrm{Cl}(K) \le \exp(x^{29}(1+\log G))$.

(b) If $K_I(\overline{\pi_g(D)}\,\pi_N(D)) \subseteq K$ and $|I| \le n_m(D)$, then $\#\mathrm{Cl}(K) \le \exp(x^{30})$.

Proof. (a) The classes of K map onto those of K_I with fibres of at most $[K_I : K] \le [K_I : K_I(G)]$ elements; this index is at most a power of the norm of G, with exponent $|I|^2$ (background library), and the number of classes of K_I is at most $(1+2(6N'(f(D)|I|)^2)^3)^{4f(D)|I|^2}$ for $N' = N_1(D)\,d$, a number in the different (unit 149, from the short bases of Input I7). Take logarithms and compare by size. (b) Apply (a) with $G = g(D)\,N(D)$, which $\overline{\pi_g(D)}\pi_N(D)$ divides, and $x^{29}(1+\log(g(D)N(D))) \le x^{30}$ by size. □

LEAN `Data.card_stdClasses_le_exp_mul_log` · `Data.card_stdClasses_le_exp` · unit 149: `Space.card_classes_le` · background library: `relIndex_principalLevel_le`

Input I12 (sizes of one lift and of the restriction; units 143 and 141). In Setting 2.1, let $\sigma \in \mathrm{Aut}(\mathbb{C})$. (i) For every form $\varphi \in \mathrm{St}_m(D)$ and every $h \in G_{Y_m(D)}$, $\|\sigma \circ (\mathrm{restr}_D\,\varphi)(h)\|^2 \le \exp(x^{70} - \sum_\tau \mathrm{be}_D(\tau) \cdot \log N(D)) \cdot \|\sigma \circ \varphi(\mathrm{bd}_D(h))\|^2$. (ii) Let i be a natural number, $\Phi = \mathrm{U}(y)\theta^{\circ}_{D,i}$ a translate of the integral vacuum, g, h points and $L \ge 1$ a natural number. Then for every vector ξ, with $l(\tau,\nu)$ the length of the row ν at τ for the rank $n_{i+1}(D)$,

$$\sqrt{\|\sigma \circ \mathrm{jm}_{D,1,i}(\omega_{D,1,i}(g,h)\Phi)\xi\|^2} \le \exp\Big(c_i + \tfrac{1}{2}\sum_\tau \sum_{\nu<n_i(D)} \big(\tfrac{l(\tau,\nu)}{L} + \mathrm{st}_D(l(\tau,\nu))\big)\Big) \cdot \sqrt{\|\sigma \circ \xi\|^2},$$

where c_i is the constant of the parameters times $(N_1(D)\, n_i(D) n_{i+1}(D) + 1)^2$, plus $f(D)\, n_i(D) n_{i+1}(D) \log(2L+2)$.

LEAN unit 141: `Data.weylNormSq_restrictToY_le` · unit 143: `Parameters.norm_jet_le_of_isAdmissibleComb`, second part, for an admissible combination of one term

Proof of Theorem C. The claim without prime sums. By Lemma 12.1 with $U = \lceil x_1 \rceil$ and the definitions of $\mathrm{gain}(D,p)$ and $\mathrm{forced}(D,p)$, and because $\mathrm{gain}(D,p) = 0$ when $\chi_{K(P)}(p) = 1$,

$$2 \sum_{p \in \mathrm{NS}(D)} \mathrm{gain}(D,p) \log p = m\Big(\sum_{\tau \in \mathrm{tops}(D)} \mathrm{Ask}_D(a(D),\tau) + k(D) \sum_\tau \mathrm{Ask}_D(w_D(\tau),\tau)\Big).$$

These are exactly the prime sums of $h_{\mathrm{row}}(D)$, so the exponent in Theorem C is

$$B + m\Big(\#\mathrm{tops}(D) \cdot \mathrm{st}_D(a(D)) + k(D) \sum_\tau \mathrm{st}_D(\mathrm{wt}_D(0,\tau))\Big) - \#\mathrm{tops}(D) \cdot a(D) \log N(D) + \mathrm{budget}(D),$$

a statement about sizes only.

The restriction. By Input I12(i) and Lemma 3.3(a), the restriction costs the factor $\exp(x^{70} - \sum_\tau (\varkappa_D(\tau,1) - 2b) \log N(D)) = \exp(x^{70} + 2bf(D) \log N(D) - \#\mathrm{tops}(D) a(D) \log N(D))$.

The chain. Apply Lemma 5.4 to the data $\mathrm{td}_D(\mathsf{x}, i)$, jointly invariant by (G2), with $\mathrm{N}_i(g,\xi) = \sqrt{\|\sigma \circ \xi\|^2}$ and with r_i the factor of Input I12(ii) for $L = a(D) + 1$. The value of the chain combination at a point is a sum of at most $\prod_{i<m} \#\mathrm{Cl}(K_i^{\mathrm{col}}(D)) \le \exp(m\, x^{30})$ terms (Lemmas 12.4(b) and 3.2(a)) for each x, each of norm at most $|\sigma(\Psi(\mathsf{x}))| \cdot \prod_i r_i \cdot \mathrm{e}^{B/2}$; the norm $\sqrt{\|\sigma \circ \cdot\|^2}$ satisfies the triangle inequality, and $\sum_{\mathsf{x}} |\sigma(\Psi(\mathsf{x}))| \le \exp(x^{60})$ by (G1), since $Q(1) = \emptyset$.

The exponent. It remains to bound $\prod_i r_i^2$. In stage $i < m$ the rows are bounded by Lemma 12.2: the row $\nu = 0$ at a top has length at most $a(D) + e$ with $e = b_1(D) + b_2(D) + r^{16}$, the rows $0 < \nu < k(D)$ have length at most $\mathrm{wt}_D(0,\tau) + r^{16}$, and the other rows have length at most r^{16} (and $b_1(D) + b_2(D) + r^{16}$ for $\nu = 0$ off the tops). By Lemma 12.3(b) each $\mathrm{st}_D(l(\tau,\nu))$ exceeds $\mathrm{st}_D(a(D))$, $\mathrm{st}_D(\mathrm{wt}_D(0,\tau))$ or $\mathrm{st}_D(0) = 0$ by at most the excess of the row length times the factor $\log(2a(D)+1) - \log(\Omega(P,\mu(D))/g(D))$; this factor is at most m by size. Summing over $\nu < n_i(D)$, over τ and over the m stages gives $m(\#\mathrm{tops}(D) \cdot \mathrm{st}_D(a(D)) +$

$(k(D)-1)\sum_\tau \mathrm{st}_D(\mathrm{wt}_D(0,\tau)))$, since the rows $0<\nu<k(D)$ are $k(D)-1$ in number. This is at most the main term of the display above, because $\mathrm{st}_D(n)\ge 0$ for every n: $-\log(\Omega(P,\mu(D))/g(D))\ge 0$ by size. The remaining terms, namely c_i, $l/L\le 2$, the excesses, $m\,x^{30}$, x^{60}, x^{70} and $2bf(D)\log N(D)$, are bounded by $\mathrm{budget}(D)$; the formal text does this by a chain of size comparisons (through b and $\sqrt{b}$) that was read and not replayed. □

LEAN `Data.weylNormSq_Bmu_le`

Proof of Theorem D. Let $p\in U$. If $p\in \mathrm{NS}(D)$ then $p\le\lfloor x_1\rfloor$, $\log p\le\log x_1$ and $\mathrm{err}(D,p)\log p\le 3f(D)(\log b)^2a(D)\log x_1\cdot m^*/p$, where $m^*=m$ if p is short for D and 1 otherwise. If p is short, $p=mc\pm1$ for a natural number c with $1\le c\le x_1$, and $m/p\le 2/c$; if not, $m^*/p=1/p$, and put $c=p$. If $p=N(D)\notin\mathrm{NS}(D)$, the term is $4mf(D)b\log N(D)$. So every term is bounded by a term attached to a pair (c,n), $1\le c\le\lfloor x_1\rfloor$, $n\in\{0,1,2,3\}$, namely $3f(D)a(D)(\log b)^2\log x_1\cdot 2/c$, the prime p being the n-th entry of $(N(D),mc-1,c,mc+1)$; different primes of U have different pairs. Summing over all pairs, with $\sum_{c\le x_1}1/c\le 1+\log x_1$,

$$\sum_{p\in U}\mathrm{err}(D,p)\log p\;\le\;4\cdot 6\,f(D)a(D)(\log b)^2\log x_1\,(1+\log x_1)\;\le\;f(D)a(D)(\log b)^3,$$

the last step by the condition $(\log x_1)^6\le\log b$ of the data and by size; the term of $N(D)$ is bounded by the term of the pair $(1,0)$, by $m\log N(D)\le 30m^2\log b$, $m^2\le 4x$ and $x\,b\le a(D)$. □

LEAN `Data.sum_allowedLoss_mul_log_le`

13 A by-product: the determinant period

The first part of the file on the chain serves another unit (the value families of the unit on Selmer groups). It is independent of the data D.

Definition 13.1 (determinant kernel and determinant period)**.** Let E be a CM field with a CM type and Weil data, I a finite set, α a continuous character of the norm-one adeles of E that is trivial on the norm-one elements of E, c a function $\mathrm{Pl}(E)\to\mathbb{N}$, $\mathfrak{m}$ a modulus of E (a non-zero ideal divisible by no prime together with its conjugate), Ψ' a complex function on the matrices indexed by I over $\mathcal{O}_E/\mathfrak{m}$, and $(\psi_w)_w$ a family of characters of the residue fields of the finite places.

(a) $\mathrm{dk}_I^{\alpha,c}[\mathfrak{m};\Psi']$ is the function $h\mapsto\theta_{\alpha,\alpha}[\Psi',\prod_\tau\det_\tau^{c(\tau)}](1,h)$ on G_I: the theta kernel of the pair (E^I,E^I) with both characters equal to α, the finite datum given by Ψ' modulo $\mathfrak{m}$ and the polynomial $\prod_\tau\det_\tau^{c(\tau)}$ ($\det_\tau$ the determinant of the matrix of variables at τ), at the points $(1,h)$.

(b) $\mathrm{dp}_I^{\alpha,c}[\mathfrak{m};\psi]$ is the sum over the classes of K_I, weighted by the inverses of the orders of their stabilisers, of the function $\mathrm{dk}_I^{\alpha,c}[\mathfrak{m};\chi_{\mathfrak{m},\psi}\circ\det]$, where $\chi_{\mathfrak{m},\psi}(y)=\prod_{w\mid\mathfrak{m}}\psi_w(y \bmod w)$.

LEAN `detKernel` · `detPeriod` · background library: `modKernelI`, `letterMinor`, `stdClassSum`, `Modulus.charOfTamePart`

Lemma 13.2 (invariance and expansion)**.** *In the situation of Definition 13.1, assume that α has weight $-c$ at the infinite places and tame part $(\psi_w)_w$.*

(a) For every Ψ', every $\gamma\in\Gamma_I$ and every $g\in G_I$: $\mathrm{dk}_I^{\alpha,c}[\mathfrak{m};\Psi'](\gamma g)=\mathrm{dk}_I^{\alpha,c}[\mathfrak{m};\Psi'](g)$.

(b) *If $\psi_w = 1$ for every w that does not divide $\mathfrak{m}$, then $\mathrm{dp}_I^{\alpha,c}[\mathfrak{m};\psi]$ is the weighted sum over the classes of K_I of the function $g \mapsto \sum_y \mathrm{dk}_I^{\alpha,c}[\mathfrak{m};\mathbf{1}_1](g\tilde{y})$, where y runs over the invertible matrices over $\mathcal{O}_E/\mathfrak{m}$, $\tilde{y} \in K_I$ is a chosen lift of y, and $\mathbf{1}_1$ is the indicator function of the unit matrix.*

Proof. (a) The polynomial is a product of determinants of the stacked matrix of variables, and for such polynomials the kernel is invariant under the rational points on the left (unit 143), the characters having the weight $-c$. (b) Expand $\chi_{\mathfrak{m},\psi} \circ \det$, which vanishes on the singular matrices, as $\sum_y \chi_{\mathfrak{m},\psi}(\det y)\mathbf{1}_y$; right translation by $\tilde{y}$ moves $\mathbf{1}_1$ to $\mathbf{1}_y$ and multiplies by $\alpha(\det \tilde{y})$, which is $\chi_{\mathfrak{m},\psi}(\det y)$ by the hypothesis on the tame part (background library). □

LEAN `detKernel_embFin_mul` · `detPeriod_eq_stdClassSum_sum` · unit 143: `modKernel_embFin_mul` · background library: `modKernel_mul_Kf_right`

Proposition 13.3 (adding a place to the modulus)**.** *In the situation of Definition 13.1, assume that α has weight $-c$ at the infinite places and tame part $(\psi_w)_w$, and that $\psi_w = 1$ for every w that does not divide $\mathfrak{m}$. Let v be a finite place that does not divide $\mathfrak{m}$ and $\mathfrak{m}'$ a modulus whose places are those of $\mathfrak{m}$ together with v. Then*

$$\mathrm{dp}_I^{\alpha,c}[\mathfrak{m}';\psi] = \prod_{l<|I|} (1 - \mathrm{q}_v^l\, \alpha_v^{-1}) \cdot \mathrm{dp}_I^{\alpha,c}[\mathfrak{m};\psi],$$

where α_v is the value at a uniformiser of v of the Hecke character of α, normalised by $\mathrm{q}_v^{|I|/2}$.

Proof. α is unramified at v, and v is not conjugate to a place of $\mathfrak{m}'$. The local computation behind Input I9(iv), which the catalogue marks as substantial (the action of the Hecke coset sums on the unramified vacuum, by counting subspaces of the residue space), shows that the datum for $\mathfrak{m}'$, which differs from that for $\mathfrak{m}$ by the indicator of the invertible matrices at v, is obtained from it by the Hecke polynomial at $X = \alpha_v^{-1}$ acting through the Weil representation. Both class sums are class integrals of functions that are invariant on the left by Lemma 13.2(a) and on the right under K_I; the Hecke polynomial then becomes a sum over coset representatives, each contributing the same integral, and the count of the cosets is the q-binomial theorem $\sum_V (-X)^{\dim V} \mathrm{q}_v^{\binom{\dim V}{2}} = \prod_{l<|I|}(1 - \mathrm{q}_v^l X)$, the sum over the subspaces V of the residue space. □

LEAN `detPeriod_insert` · background library: `sum_heckePolyCoeff_smul_pairWeil`, `q_binomial_submodule` (the q-binomial theorem; the catalogue: the theorem itself, place not confirmed)

14 Where the results are used

By a search of the sources for the names of the declarations: Theorems A, B, C and D, with the definitions of $h_{\mathrm{row}}(D)$, $\mathrm{NS}(D)$ and $\mathrm{err}(D,p)$, are used in the file of the endgame that proves Theorem A of the handout (short forms). Proposition 4.5 is used by the unit on the filling, for the dimension of the ramified columns in Theorem B of the handout. Proposition 4.2 is used by the unit on the exchange, for its crude rows; Lemma 12.1 by the unit on the rows; the determinant period of §13 by the unit on Selmer groups (its value families). The definitions of the columns, of the levels, of the restriction and of good data (Definitions 2.3, 2.4, 2.7, 2.8) are used by the units on the seesaw and on the exchange; the seesaw unit proves that good

data exist and that the rows of the handout are of the form $\mathrm{B}_D(\vartheta,\Psi)\psi$. I read the statements of these consumers only where this note quotes them.

15 Notes on the formal text

1. *Arguments: the text reading and Lean.* Nothing was compiled, so the argument lists of the declarations are those that a program reads from the text, and in this folder they differ from Lean's in several places, because section variables are declared again and included or omitted by commands. The statements above follow the Lean text, as I read it; the points that only Lean can settle are these. *Entries of the text reading that are not arguments*: a place w dividing $\pi_N(D)$ in Definition 2.7(d) (the operator takes the data and a set of places; the note writes $\mathrm{ram}_D(\Sigma)$ through a macro of its own with these two arguments), in Lemma 3.2(e), (g) and in Lemma 5.2(a); a number called j with its primality in Definition 2.8(b), Lemma 5.4 and Proposition 5.5 (in Lemma 3.2(h) the same variable is the prime that the note calls ℓ); the exponent e in Definition 2.9(a) and Lemma 7.1(b); the set $\mathcal{L}$ in Lemma 6.3; the prime p and its place in Theorem C; ϑ and Ψ in Theorem B, which the statement binds; a second record of data in Lemma 5.2(a), where the formal statement names only one; the Weil data and other instances of variables that were declared again, in Definition 2.5, Lemma 4.1, Proposition 4.2 and Definition 2.7(a). *Arguments that the text reading does not show*: that Ψ is good and p is prime to $\mathrm{nrm}(\vartheta)$ in the four parts of Lemma 11.1 (a command includes these hypotheses again); that $\mathcal{L}$ is marked in Proposition 8.3 (used after the first local definition of the statement); the prime p and the homomorphism j in Lemma 3.2(i), Lemma 6.2(b), Lemma 6.5(a), (c) and Lemma 6.6. The note states all of them, through the Settings.

2. *Statements that the line reading cuts.* The formal statements of Propositions 8.1 and 8.3 contain local definitions, and the program takes their first sign of definition for the end of the statement; the two propositions were compared with the full Lean text. The same program counts the command that follows Lemma 10.3 and the one that follows Theorem C into these declarations; they make some definitions opaque and are no part of the proofs.

3. *Hypotheses that follow from others, or are used in one case only.* In Proposition 8.1 the hypothesis $\bar{\mathfrak{l}} \neq \mathfrak{l}$ follows from (M1). In Theorem B the columns have to be ramified only for $p = N(D)$. In the definition of $\mathrm{err}(D,p)$ the case $p = N(D)$ is tested after $p \in \mathrm{NS}(D)$; by size $N(D) \geq x_1$, so the order does not matter. Lemma 5.4 takes functions N_i and numbers r_i for all i and uses those with $i \leq m$. In Proposition 9.1 the equalities "W is the valuation ring at $\mathfrak{l}$" and $p_T = p$ are hypotheses of the formal statement.

4. *Forms of the statements.* Integrality of a column is the condition "j-gauge at most 1 at 1"; Theorem B(ii) is, in the formal text, membership of $p^{-\beta}\mathrm{B}_D(\vartheta,\Psi)\varphi$ in the module of such forms on $Y_m(D)$. Theorem B(i) uses the notion of index of unit 149 (Input I2), which is stated without quotient modules. In (G3) the numbers of pairs and of lifts are written with $\lfloor m/2 \rfloor - 1$. The jet exponent contains the term $e = 0$ for $p = d$. The definition of the restriction contains the proof that it maps forms to forms. Lemma 3.2(b) and Lemma 6.1(a) are

instances in the formal text: the first is found by Lean whenever a level is needed, the second feeds the tactic of size comparisons.

5. *What was not replayed.* The dossier marks as of lower confidence the claims made for dense automatic steps, whose intermediate goals are not visible without Lean. I read these lines; I could not replay them. They are: the extraction of the coefficients and the exponent count in Proposition 8.1; the identification of the letter matrices in Proposition 9.1; which of the Hecke sizes each line proves in Proposition 9.2; the rewriting of the paired chain in Proposition 10.4; the size comparisons at the end of the proofs of Theorems C and D; and the two computations in Proposition 13.3. For Proposition 7.3 the formal text proves the termwise inequality (2) by one automatic step after a case split; the proof given above is mine and was checked against the formal definitions, not against the automatic step. Three counts that an earlier version of this note left to the formal text are now written out and were compared with the source: the constant of Proposition 4.2, the exponents of the diagonal point in Proposition 8.2 (the image at τ reads the component at the conjugate of the induced prime), and the denominator and the scalar at the level prime (Lemma 10.3, Proposition 10.4).
6. *Size comparisons.* Many steps are comparisons of sizes of the data, proved in the formal text by a tactic from the scale conditions; the dossier does not record the exponents, and the note says "by size". They belong to the note on the admissible data.
7. *What the final theorems use.* The dossier found no member of the unit that is evidently unused; the count of the members reached from the final theorems by name alone is not meaningful here, and a kernel export will decide.
8. *References.* For the inputs from the background library the catalogue of the handout gives a reference only for the Hecke transfer (Input I9(iv), place not confirmed, a special case) and for the q-binomial theorem (the theorem, place not confirmed); for the others: *reference to be confirmed.* The headers of the files of this folder name as related work, without places: for the scaled Hecke operators, Rallis (1982), V. Lafforgue (2011) and Z. Liu (2025); for the divisibility of theta lifts, X. Zhang (2022), Berger (2009) and Prasanna (2006); for short columns, Gross (1999), Greenberg–Voight (2014) and the geometry of numbers of Minkowski and Mahler; for the elementary divisors of twisted Hecke operators, Arsovski (2019), Liu–Truong–Xiao–Zhao (2023) and Fakhruddin–Pilloni (2021). I did not check these.

How this note was written. From the dossier of the unit: the readings of its 105 declarations and its 46 structured proofs, all of which I read, with the reader's list of doubts. The program check of the dossier covered 331 leaf steps of 46 proofs. I read the seven Lean files of the folder completely, statements and proofs, and compared every statement of this note with its Lean text, argument by argument, with the program's list of arguments beside it. Outside the folder I read the Lean definitions of the objects of Inputs I2 to I6 (index and type, the level prime and the twists, the lift kernel, the data and the chain, induced primes and marked sets, the bookkeeping of unit 150 in its file), the definitions of gauges, levels, classes, kernels, lifts and Hecke operators at split places in the background library, and the statements, not

the proofs, of the results quoted in Inputs I7 to I12. The arrangement into sections, the remark that the prime sums cancel in Theorem C, the proof of (2) and the presentation of the proof of Theorem D are mine; elsewhere the proofs follow the structured proofs, with bookkeeping steps merged.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Definition 13.1	`detKernel`	ChainValuation.lean	24–25
Definition 13.1	`detPeriod`	ChainValuation.lean	29–29
Lemma 13.2	`detKernel_embFin_mul`	ChainValuation.lean	35–39
Lemma 13.2	`detPeriod_eq_stdClassSum_sum`	ChainValuation.lean	44–49
Proposition 13.3	`detPeriod_insert`	ChainValuation.lean	55–83
Definition 2.5	`IsShort`	ChainValuation.lean	91–92
Lemma 4.1	`IsShort.sub_translate`	ChainValuation.lean	94–99
Proposition 4.2	`span_isShort_eq_top`	ChainValuation.lean	105–147
Definition 2.7	`algFormCentralRep`	ChainValuation.lean	151–153
Lemma 3.2	`Data.not_isChosen_of_N_mem`	ChainValuation.lean	157–159
Lemma 3.2	`Data.mem_columnLevel`	ChainValuation.lean	161–164
Lemma 3.2	`Data.mem_chosenLevel_of_supportedOn`	ChainValuation.lean	166–168
Lemma 2.6	`Data.exists_centralGen`	ChainValuation.lean	173–181
Definition 2.7	`Data.centralGen`	ChainValuation.lean	185–185
Lemma 4.3	`Data.scalarFin_centralGen_mem`	ChainValuation.lean	187–189
Definition 2.7	`Data.centralRep`	ChainValuation.lean	193–193
Definition 2.7	`Data.ramifyingOp`	ChainValuation.lean	197–198
Definition 2.7	`Data.ramifiedColumns`	ChainValuation.lean	202–202
Lemma 4.3	`Data.exists_eq_sub_centralGen`	ChainValuation.lean	207–209
Lemma 3.3	`Data.sum_height_mul_kappa_le`	ChainValuation.lean	211–212
Lemma 4.4	`Data.finrank_ker_one_sub_centralGen_le`	ChainValuation.lean	214–232
Proposition 4.5	`Data.ramifyingOp_finrank_comap_and_short`	ChainValuation.lean	236–246
Theorem A	`Data.span_short_ramifiedColumns`	ChainValuation.lean	251–257
Definition 2.8	`Data.pairTranslates`	ChainValuation.lean	264–265
Definition 2.8	`Data.admissibleTranslations`	ChainValuation.lean	269–269
Definition 2.8	`Data.IsGoodDatum`	ChainValuation.lean	275–279
Lemma 3.2	`Data.notMem_piPlaces_of_ne`	ChainValuation.lean	281–283
Lemma 5.2	`Data.mem_integralAfter`	ChainValuation.lean	288–292
Lemma 5.2	`Data.projOn_translateDatum`	ChainValuation.lean	294–297
Lemma 5.4	`Data.exists_chain_eq_sum`	ChainValuation.lean	302–312
Lemma 5.3	`Data.IsGoodDatum.translateDatum_mem_standardData`	ChainValuation.lean	314–319
Proposition 5.5	`Data.IsGoodDatum.gaugeLe_chain`	ChainValuation.lean	324–335
Lemma 3.2	`Data.norm_star_piG_le_one`	ChainValuation.lean	337–338
Lemma 3.1	`Data.π_ne_zero`	ColumnDefs.lean	22–22
Definition 2.3	`Data.piPlaces`	ColumnDefs.lean	25–26
Lemma 3.1	`Data.mem_piPlaces`	ColumnDefs.lean	28–28
Lemma 3.1	`Data.isSplitLevel_piPlaces`	ColumnDefs.lean	30–32
Lemma 3.1	`Data.card_piPlaces`	ColumnDefs.lean	35–38
Definition 2.3	`Data.columnLevel`	ColumnDefs.lean	42–43
Lemma 3.2	`Data.principalLevel_le_columnLevel`	ColumnDefs.lean	45–47
Lemma 3.2	`Data.isLevel_columnLevel`	ColumnDefs.lean	49–50
Definition 2.4	`Data.lastIndexEquiv`	ColumnDefs.lean	54–54
Definition 2.4	`Data.branching`	ColumnDefs.lean	58–62
Lemma 3.2	`Data.card_X`	ColumnDefs.lean	64–64
Lemma 3.2	`Data.classAut_columnLevel_eq_bot`	ColumnDefs.lean	66–67

In this note	Lean declaration	File	Lines
Definition 2.3	`Data.Column`	ColumnDefs.lean	71–71
Definition 2.3	`Data.Stage`	ColumnDefs.lean	74–74
Definition 2.4	`Data.branchExp`	ColumnDefs.lean	78–79
Lemma 3.3	`Data.le_branchExp`	ColumnDefs.lean	83–84
Definition 2.4	`Data.borderU`	ColumnDefs.lean	88–90
Definition 2.4	`Data.restrictToY`	ColumnDefs.lean	93–101
Lemma 6.1	`Data.instUbCastRealRrLogLx₁`	JetExponentBound.lean	15–17
Lemma 6.1	`Data.IsServed.m_dvd_orderOf`	JetExponentBound.lean	21–28
Lemma 6.1	`Data.le_loss`	JetExponentBound.lean	30–30
Lemma 6.1	`Data.placeOfInt_mul_mem_tops_iff`	JetExponentBound.lean	34–36
Lemma 6.2	`Data.p_mem_inducedPlace`	JetExponentBound.lean	40–40
Lemma 6.2	`Data.inducedPlace_mul`	JetExponentBound.lean	42–43
Lemma 6.2	`Data.inducedPlace_neg_pow_mul_of_odd`	JetExponentBound.lean	51–53
Lemma 6.3	`Data.exists_isMarked`	JetExponentBound.lean	57–62
Lemma 6.2	`Data.inducedPlace_eq_or_star_iff`	JetExponentBound.lean	66–68
Lemma 6.5	`Data.IsMarked.card_le`	JetExponentBound.lean	70–71
Lemma 6.5	`Data.IsMarked.not_dvd_gProd_mul_N`	JetExponentBound.lean	73–75
Lemma 6.6	`Data.IsMarked.notMem_thick`	JetExponentBound.lean	79–84
Lemma 6.6	`Data.IsMarked.sum_inducedPlace_sub_star_eq_zero`	JetExponentBound.lean	88–92
Lemma 6.5	`Data.IsMarked.p_mem`	JetExponentBound.lean	94–94
Lemma 6.5	`Data.IsMarked.valuation_p`	JetExponentBound.lean	96–97
Lemma 6.5	`Data.IsMarked.star_ne`	JetExponentBound.lean	99–99
Lemma 6.5	`Data.IsMarked.maxCompactAt_le_columnLevel`	JetExponentBound.lean	101–104
Definition 2.9	`Data.IsMarked.markOperator`	JetExponentBound.lean	108–111
Definition 2.9	`Data.jetExponent`	JetExponentBound.lean	115–118
Lemma 7.1	`Data.IsCollision.ne_and_odd_and_dvd`	JetExponentBound.lean	121–122
Lemma 7.1	`Data.collisionSum_le`	JetExponentBound.lean	128–133
Lemma 7.1	`Data.mul_collisionSum_le_loss`	JetExponentBound.lean	138–144
Lemma 7.1	`Data.IsFullMark.collisionSum_evenPlaces_eq_zero`	JetExponentBound.lean	148–151
Lemma 7.2	`Data.lowColumns_lowBoxes_bounds`	JetExponentBound.lean	155–163
Proposition 7.3	`Data.le_jetExponent`	JetExponentBound.lean	168–204
Lemma 7.4	`Data.IsMarked.sum_lowColumns_le`	JetExponentBound.lean	208–221
Definition 2.9	`Data.liftLoss`	LocalStatement.lean	42–43
Proposition 8.1	`Data.IsGoodDatum.exists_hecke_transfer`	LocalStatement.lean	50–101
Proposition 8.2	`Data.IsGoodDatum.mem_raises`	LocalStatement.lean	106–143
Proposition 8.3	`Data.IsGoodDatum.exists_move_markOperator`	LocalStatement.lean	148–179
Proposition 8.4	`Data.IsGoodDatum.gaugeLe_markOperator_pow`	LocalStatement.lean	184–194
Theorem C	`Data.weylNormSq_Bmu_le`	LocalStatement.lean	201–228
Corollary 9.3	`Data.IsMarked.hasType_prod_markOperator_pow`	LocalStatement.lean	233–240
Lemma 11.1	`Data.IsGoodDatum.gaugeLe_one`	LocalStatement.lean	246–250
Lemma 11.1	`Data.IsGoodDatum.gaugeLe_Bmu_of_isMarked`	LocalStatement.lean	255–260
Lemma 11.1	`Data.IsGoodDatum.gaugeLe_collisionSum`	LocalStatement.lean	264–268
Lemma 11.1	`Data.IsGoodDatum.gaugeLe_pow_jetCeil_smul`	LocalStatement.lean	272–276
Theorem B	`Data.exists_indexDvd_Bmu_divisible`	LocalStatement.lean	285–307
Proposition 9.2	`Data.IsMarked.hasType_markOperator`	MarkedPrimeType.lean	15–67
Definition 2.10	`Data.heightPerRow`	SizeEstimates.lean	20–21
Definition 2.10	`Data.nonsplitPrimes`	SizeEstimates.lean	25–25
Lemma 12.1	`Data.primeSum_eq_sum_localAsk`	SizeEstimates.lean	29–39
Lemma 12.2	`Data.diagramWeight_le_by_index`	SizeEstimates.lean	43–48
Lemma 12.3	`Data.norm_map_star_pi6_sq`	SizeEstimates.lean	50–51
Lemma 12.3	`Data.jetSize_le_add`	SizeEstimates.lean	55–60
Lemma 12.4	`Data.card_stdClasses_le_exp_mul_log`	SizeEstimates.lean	64–73
Lemma 12.4	`Data.card_stdClasses_le_exp`	SizeEstimates.lean	75–79

In this note	Lean declaration	File	Lines
Definition 2.10	`Data.allowedLoss`	SizeEstimates.lean	83–84
Theorem D	`Data.sum_allowedLoss_mul_log_le`	SizeEstimates.lean	88–103
Lemma 10.2	`Data.norm_detCharAt_diagN_le`	TwistedHeckeType.lean	31–41
Lemma 10.3	`Data.name_lastColDatum_mem_raises`	TwistedHeckeType.lean	46–76
Proposition 10.4	`Data.chain₂_pairTranslates_eq_smul_chain`	TwistedHeckeType.lean	82–106
Proposition 10.5	`Data.gaugeLe_Bmu_at_N`	TwistedHeckeType.lean	113–135
Proposition 9.1	`Columns.hasType_markOperator`	TwistedHeckeType.lean	146–239

Four statements as they are written in the artifact.

```
lemma span_short_ramifiedColumns : span ℂ {φ : D.ramifiedColumns | IsShort D.budget.exp φ.1.1} =
  ⊤

lemma Data.exists_indexDvd_Bmu_divisible {β : ℤ} (hc : β < ite (p ∈ D.nonsplitPrimes) (D.gain p) 0 -
  D.allowedLoss p + 1) : ∃ l : ℕ, l ≤ D.allowedLoss p * finrank ℂ D.Column ∧ ∃ S,
  IndexDvd integralColumns S (p ^ l) ∧ ∀ θ Ψ, IsGoodDatum θ Ψ → p.Coprime θ.levelNorm → ∀ φ
  ∈ S, φ ∈ D.ramifiedColumns → (p : ℂ) ^ (-β) • Bμ θ Ψ φ ∈ integralFormsAt ⊤ 1

lemma Data.weylNormSq_Bmu_le (H : IsGoodDatum 1 Ψ) {σ : ℂ ≃+* ℂ} {φ B} (hB : ∀ g,
  weylNormSq (σ ∘ φ.1.1 g) ≤ exp B) h : weylNormSq (σ ∘ (D.Bμ .one Ψ φ).1 h) ≤
  exp (B + D.heightPerRow + 2 * ∑ p ∈ D.nonsplitPrimes, D.gain p * log p + D.budget)

lemma sum_allowedLoss_mul_log_le U : ∑ p ∈ U, D.allowedLoss p * .log p ≤ D.budget
```

In these texts `D.ramifiedColumns` is $\mathrm{Col}^{\mathrm{ram}}(D)$, `IsShort` the notion of Definition 2.5, `D.budget` is $\mathrm{budget}(D)$, `integralColumns` is $\mathrm{Col}_j(D)$, `IndexDvd` the index of Input I2, `Bμ θ Ψ` is $\mathrm{B}_D(\vartheta, \Psi)$, `IsGoodDatum θ Ψ` says that Ψ is good for (D, ϑ), `θ.levelNorm` is $\mathrm{nrm}(\vartheta)$, `integralFormsAt ⊤ 1` is the module of the forms of gauge at most 1 at 1, `weylNormSq` is the Gram norm, `D.heightPerRow` is $h_{\mathrm{row}}(D)$, `D.nonsplitPrimes` is $\mathrm{NS}(D)$ and `D.allowedLoss` is $\mathrm{err}(D, p)$. Elsewhere the letters of the formal text were changed where the write-up reserves them: the stage `j` is i and the homomorphism `ιₚ` is j; the element `θ` of the null cone is ϖ_0, as in the note on the CM curve; the exponent `s` of the data is $\mu(D)$; a place `σ` is τ; a marked set `∧` is $\mathcal{L}$ and its elements `l` are $\mathfrak{l}$; a prime `w` dividing `D.π` is $\mathfrak{p}$; a tuple of translations `x` is x; the set `ev` is Ξ; `κ` is $\varkappa$; the number of boxes `N` is L; `D.ϱ + 1` is $m/2$.

Notation

$P = (N_0, d)$, D parameters; admissible data for P (Setting 2.1)

x, x_1, b, m, r components of D: the three scales, the number of lifts, the rank of the rows

$\mu(D)$, $E(D)$, $N_1(D)$, $f(D)$ the exponent, the CM field, the order of its roots of unity, the number of its infinite places

$K(P)$, $\chi_{K(P)}$ $\mathbb{Q}(\sqrt{-d})$ and its quadratic character

$\mathrm{Pl}(D)$, τ, τ', τ_0, τ_e the group of infinite places of $E(D)$; places; $\tau_e = [-p^e]_D \cdot \tau$

ι_τ, $[\nu]_D$ the embedding at τ; the place named by the integer ν

$k(D)$, $t(D)$, $a(D)$, $n_i(D)$ $r + 1 - m$; $5r$; the order of the jets; the ranks $k(D) + i(t(D) + 1)$

$\mathrm{tops}(D)$, $\mathrm{thick}(D)$, $w_D(\tau)$, $\mathrm{wt}_D(i, \tau)$, $\mathrm{gap}(D)$ tops, thick places, nominal weight, weights, the gap $2r^8$

$\mathrm{budget}(D)$ $f(D)a(D)(\log b)^3$

E, $\mathrm{Pl}(E)$, $\mathcal{O}_E$, $\widehat{\mathcal{O}}_E$, $\mathbb{A}_{E,\mathrm{f}}$ a CM field; its infinite places, integers, integral adeles, finite adeles

w, $\overline{w}$, E_w, $\mathfrak{q}_w$, v a finite place, its conjugate, the completion, the size of the residue field; v also a finite place, or a column, or a family of data, where said

p, j, $\mathbb{C}_p$, $|\cdot|_p$, $\mathfrak{o}_p$, O_j a prime (a natural number where said); a ring homomorphism $\mathbb{C} \to \mathbb{C}_p$; the absolute value; the valuation rings (Setting 2.2)

ℓ, ι a prime number; a ring homomorphism $\mathbb{C} \to \mathbb{C}_\ell$

σ, $\mathrm{Aut}(\mathbb{C})$ an automorphism of $\mathbb{C}$; the group of all of them

I, I' finite index sets

f the number of infinite places of a CM field E

$\varkappa$, $\mathfrak{e}$, h, c, c' column counts; determinant exponents; a height; Weyl indices (tableaux); c also a real or complex constant, a class, a list index, a function $\mathrm{Pl}(E) \to \mathbb{N}$, where said

G_I, Γ_I, K_I, $K_I(a)$, $K_I[w]$ finite adelic unitary group, its rational points, the standard level, principal levels, the group at w (Input I1)

K, J, $\mathrm{Cl}(K)$, χ levels; classes; a character of a level, or of a lift

g, h, k, a, a', d, d', d_i, y, y', y'' points of the groups; y also a matrix over the adeles, a vector of residues, a real number, or an integer, where said

$\mathrm{Wy}_I(\varkappa)$, $\rho_{\varkappa,\mathfrak{e}}$, ξ, ξ' Weyl indices, the weight representation, vectors of the weight

$\mathrm{Alg}_I(\varkappa,\mathfrak{e})$, φ, φ', φ_0, ψ, $\mathrm{R}(g)$ algebraic forms; forms and columns; right translation

$\mathrm{gauge}_j^{\varkappa,\mathfrak{e}}(g;\xi)$, $\|\xi\|^2$, G the gauge at j; the Gram norm and its matrix

$\mathrm{boxes}(\varkappa,n')$, L, $l(\tau,\nu)$ numbers of boxes; row lengths

O, V, M, S, u, z, s, l Input I2: a ring, a module, submodules, a matrix, a scalar, an exponent, a natural number; S also a set of places, M a natural number, z a complex number, s a number of lifts or an element of $\mathbb{Z}/n$, V a subspace, where said

$X_i(D)$, $Y_m(D)$, i, i' index sets of the definite spaces; stages

$N(D)$, N, $\pi_N(D)$, $g(D)$, g_0, g_1, $\pi_g(D)$ the level prime (written N where said), an integer of norm N, the product of the two primes of the data, an integer of $K(P)$ of norm $g(D)$

N' a natural number in the different of a CM field; for $E(D)$ it is $N_1(D)\,d$

$K_I^{\mathrm{ch}}(D)$ the chosen level

ϑ, $Q(\vartheta)$, ψ_w, $\mathrm{nrm}(\vartheta)$ a twist, its set of places, its tame part, the product of the norms

$\mathcal{W}_i(D)$, $\mathrm{U}(y)$, $\theta^\circ_{D,i}$, P^Σ finite data, Heisenberg operators, the integral vacuum, a projection (Input I4)

$\mathrm{k}_{D,\vartheta,i}$, $\omega_{D,\vartheta,i}$, $\mathrm{jm}_{D,\vartheta,i}$, $\mathrm{lift}_{D,\vartheta,i}$ the lift kernel, its representation, its jet matrix, the lift

$\mathrm{Raise}_k(d,d';c)$ data that multiply gauges by at most c from d to d'

x, $\mathrm{td}_D(\mathsf{x},i)$, Φ, Φ', $\Phi^{(2)}$, Ψ, Ψ' tuples of translations, their data; data; a function on tuples; a function on matrices

$\mathrm{C}_{D,\vartheta}(\Phi)_s$, $\mathrm{C}^{(2)}_{D,\vartheta}(\Phi)_s$, C, C_i, $\mathrm{B}_D(\vartheta,\Psi)$ chains of lifts and of pairs; the chain followed by the restriction

$\mathrm{Col}(D)$, $\mathrm{Col}^{\mathrm{ram}}(D)$, $\mathrm{Col}_j(D)$, $\mathrm{St}_i(D)$ columns, ramified columns, j-integral columns, stages

$\mathrm{Pl}_\pi(D)$, $\mathfrak{p}$, Σ, $K_i^{\mathrm{col}}(D)$ the primes dividing $\pi_N(D)$, one of them, a set of them; the column levels

lst_D, bd_D, $\mathrm{be}_D(\tau)$, $\mathrm{Br}(D)$, restr_D Definition 2.4

φ is B-short, B, B' short forms; real bounds

$\mathfrak{c}_D(\mathfrak{p})$, cen_D, $\mathrm{ram}_D(\Sigma)$, ν Definition 2.7; ν an adele, an integer, or the index of a row or of a pair, where said

$\mathrm{Adm}_D(q)$, q, $\mathrm{pt}_D(v)_\nu$, Ψ is good for (D,ϑ) Definition 2.8

$\mathfrak{l}_j(\tau)$, $\mathcal{L}$, $\mathfrak{l}$, $\Xi_j(\mathcal{L})$, $A_D(\mathfrak{l})$, $\mathrm{lc}_D(i,\tau)$, $\mathrm{lb}_D(i,\tau)$ Input I5

$\mathrm{ask}_{D,p}(y,\tau)$, $\mathrm{forced}(D,p)$, $\mathrm{gain}(D,p)$, $\mathrm{loss}(D,p)$, $\mathfrak{r}(D,p)$, $\Delta(D,p)$, $\mathrm{jc}(D,p)$, $o_0(D,p)$, $\mathrm{coll}(D,p,\Xi)$, Ξ, e, $\mathrm{cc}_p(\varkappa,n';\Xi;\tau,\tau';e)$, $\varkappa_D(\tau,h)$, $\mathrm{last}_D(i,\tau)$, $\mathrm{kept}_D(h)$, $\varkappa_D^Y(\tau,h)$, $\varrho_d(p)$ Input I6

$\mathrm{je}_D(p,i,\Xi)$, $\mathrm{ll}_{D,p}(\mathcal{L},i)$, $\Upsilon_{D,\mathcal{L}}(\mathfrak{l})$, $T_{w,s}$, Y Definition 2.9; Hecke operators; the product of the mark operators

$h_{\mathrm{row}}(D)$, $\mathrm{NS}(D)$, $\mathrm{err}(D,p)$, $\mathrm{st}_D(n)$, $\mathrm{Ask}_D(y,\tau)$, $\mathsf{U}(D)$, $\lambda_{d,1}(\nu)$, $\Omega(P,\mu(D))$, Λ Definition 2.10; Λ the von Mangoldt function

β, U, η, δ, δ_S, ω an integer; a finite set of numbers, a natural number or a complex variable; $a(D)/(pm)$; an integer of $K(P)$; a diagonal matrix; in $\omega_{D,\vartheta,i}$

$\mathrm{Lat}(\delta;S)$, $\mathrm{Std}_\delta(D,i)$, $\mathrm{ThProd}_{P,\mu}(\delta,r;I)$, s_1, $\mathrm{per}(P,\mu)$, Θ, ϖ, ϖ_0, T objects of unit 150 (Inputs I8, I9); ϖ also a uniformiser (Input I8); T also twisting data

N_i, r_i, t_i, t, t', β_c, $\mathcal{C}_i$ sizes, factors, operators on the data, linear maps, properties (Inputs I8, Lemma 5.4); t also a bijection, r a real number

n, n', n'', o, o', k', a', r', l', q', c_0, c_1, c_2, c_3, c_n, e' natural numbers and constants defined where they occur

one_τ, low_τ, bx_τ; lv_e, $\mathrm{cl}_{\tau,e}$ the three counts in the coupling count (Input I6); the indicators of live pairs and of collisions (proof of Proposition 7.3)

$\mathrm{col}(z)$, $\mathrm{row}(z)$, $\mathrm{at}(g)$, $\mathrm{ev}(y)$, $\mathrm{sq}(y)$ matrices with one non-zero column or row; the point of a family of local matrices; evaluation vectors; the square matrices of the first columns of a point (Input I7)

ch, ϵ_τ, $J_{\mathfrak{l}}$, u_τ charges; exponents of a diagonal point; sets of indices; diagonal entries

p_T, ϕ_s, ϕ, W, R, X, Π, α, α', α_v, α_e, H twisting data and Hecke polynomials: a prime, the letters, ring homomorphisms, a valuation ring (also a subspace), a ring or a bound, a variable, a polynomial, values of characters; α also a character

$\mathcal{N}$, $\mathbf{1}_{\mathrm{lc}}$, F, F, $\mathfrak{m}$, $\mathfrak{m}'$, g', y_0 Setting 10.1 and Input I11: the split level, an indicator, a function, the Fourier operator, moduli, integers; in Input I11(iv) also an integral adele η, and $\mathrm{N}(g')$ the absolute norm

$b_1(D)$, $b_2(D)$, m^*, γ, A, A_ϕ, $Y(g)$, $c_{\tau,\nu}$ multiples of $N_1(D)$; a factor of the loss; a rational point; matrices of the proof of Proposition 4.2

$\mathrm{dk}_I^{\alpha,c}[\mathfrak{m};\Psi']$, $\mathrm{dp}_I^{\alpha,c}[\mathfrak{m};\psi]$, $\chi_{\mathfrak{m},\psi}$, $\tilde{y}$ Definition 13.1

e, π, θ the base of the natural logarithm; in $\pi_N(D)$, $\pi_g(D)$; in $\theta^{\circ}_{D,i}$

READBACK NOTE 40

Elementary divisors of a Frobenius-twisted Hecke operator

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. A group $\mathrm{GL}_k(W)$ over a local ring with finite residue field acts on forms in L sets of k variables, through L ring homomorphisms that are linked in a circle by a power of Frobenius modulo a prime p, alternately directly and by the inverse transpose. A Hecke operator of this action, scaled by powers of p so as to be integral, is shown to have small elementary divisors: modulo p^e its cokernel has length at most a small share of e times the rank, plus a constant times the rank. Four ideas carry the proof. A *root lemma*: a term of a form that is least, for valuation plus a weight, among all translates under a Levi subgroup has exponents that obey linear inequalities, by a count of zeros against degree. Two *dimension bounds* modulo p: forms whose exponents nearly obey these inequalities in all frames fill about one part in the degree of the operator, by a count of flags against the volume of a cone of cyclically interlacing vectors; and forms that the scaling makes divisible in all frames are rare. An *approximate projector* onto the first kind of forms, made from conjugation-invariant means on polynomials in generic matrices. And normal forms of matrices over a valuation ring, which turn the three into a statement about indices.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles with Lean v4.35.0 against a pinned version of the library Mathlib and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note treats one unit of the artifact, the folder on the twisted Hecke operator: 75 of its 78 declarations, which are all that carry a name (the others are three instances, described in §11). Proofs are given in full, *given* four inputs from other units: the calculus of index divisibility (unit 149), the Hermite and Smith normal forms over a valuation ring, a combinatorial Nullstellensatz with multiplicity, and the order of the general linear group of a finite field. Standard facts of algebra and of measure theory are used without proof: Nakayama's lemma, the freeness of a finitely generated torsion-free module over a valuation domain, the vanishing of the trace of a nilpotent endomorphism of a free module of finite rank over a domain, Legendre's formula for the power of a prime in a factorial, the change of variables for Lebesgue measure under a linear map, and the convergence of normalised counts of lattice points to the volume of a bounded convex set. Every statement of the unit and of the Inputs was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 86 declarations.

1 The statements

The notions are defined in §2; here is what is needed to read the five theorems. *Twisting data* T of length L over a local ring W and a ring $\mathcal{O}$ are a prime p, a power $P_T = p^o \geq 5$, ring homomorphisms $\iota_s \colon W \to \mathcal{O}$ for $s \in \mathbb{Z}/L$ (the *letters*) with $\iota_{s+1}(v) \equiv \iota_s(v)^{P_T}$ modulo p for v in W, a parity of the letters that alternates, and an element ϖ of $\mathcal{O}$ with $\varpi^2 = p$; the residue field of W has $q_T = P_T^L$ elements. An element γ of $\mathrm{GL}_k(W)$ acts on the variables $x_{i,s}$ $(0 \leq i < k,\ s \in \mathbb{Z}/L)$ of letter s through the matrix $\gamma_{T,s}$, which is $\iota_s(\gamma)$ for even s and $\iota_s((\gamma^{-1})^{\mathsf{T}})$ for odd s; $[\gamma]_T$ is this action on polynomials, and $\rho_{T,n}(\gamma)$ its matrix on the forms of degree n in each letter, in the basis of the monomials x^β, $\beta \in \mathrm{Mon}^L_{k,n}$. For a pair of indices (i_0, i_1) and a letter s_0, $\mathrm{ex}^{i_0,i_1}_{T,s_0}(\cdot)$ is a real weight on the variables, the *excess weight*; for a threshold $\mathfrak{t}$, $\mathrm{depth}^{k,n}_{T,\mathfrak{t}}(\beta)$ is the degree of x^β in the variables that are *deep* for $\mathfrak{t}$. $\mathrm{Ex}^{i_0,i_1}_{T,n}(I;h)$ is the module of the forms F such that every translate of F by a block diagonal element has, modulo the ideal I, only monomials of excess at most h at every letter; $\mathrm{Dp}^{k,n}_{T,\mathfrak{t}}(j)$ is the module of the forms all of whose translates have only monomials of depth at least j. $\mathrm{Sz}(P_T, o, L, k, n, \mathfrak{t}; \lambda, \kappa, y, z)$ is a list of ten conditions on sizes (Definition 2.9).

Theorem 1 (the root lemma). *In Setting 2.2, let s_0 be a letter, let $i_0 \neq i_1$ be indices, let τ be a labelling of the indices with $\tau(i_0) = \tau(i_1)$, and let ω be a real weight on the variables with*

$$0.4 \ \leq\ \omega(\mathrm{rt}_{T,s}(i_0, i_1)) - \omega(\mathrm{rt}_{T,s}(i_1, i_0)) \ \leq\ 1.6 \qquad \text{for every letter } s.$$

Let Φ be a polynomial in the variables x_v over $\mathcal{O}$, let α be an exponent and e a natural number. Suppose that $[\gamma^{-1}]_T\Phi$ lies in $\mathrm{Gs}_{\mathrm{ad}_{\varpi}(\cdot),\omega}(e + \langle\omega, \alpha\rangle)$ for every γ in $\mathrm{Lev}_\tau(W)$: the coefficient of $x^{\alpha'}$ in every such translate is divisible by ϖ to the power $\max(\lceil e + \langle\omega, \alpha\rangle - \langle\omega, \alpha'\rangle\rceil, 0)$. Suppose that ϖ^{e+1} does not divide the coefficient of x^α in Φ. Then

$$\langle \mathrm{ex}^{i_0,i_1}_{T,s_0}(\cdot), \alpha\rangle \ \leq\ 0.$$

LEAN `TwistingData.root_lemma`

Theorem 2 (the cyclic interlacing cone fills the share of one flag). *Let $k \geq 0$ and $L \geq 1$ be integers,* par *a map from $\mathbb{Z}/L$ to the signs $\{+, -\}$, and ϱ a real number with $1 < \varrho$. Let $\delta = (\delta_s)_{s \in \mathbb{Z}/L}$ be real numbers and $\mathcal{C}$ a subset of $\mathbb{R}^{k \times L}$ such that, for every ξ in $\mathbb{R}^{k \times L}$: $\mathrm{St}_{\mathrm{par},\varrho,1}(\xi)$ lies in $\mathcal{C}$ if and only if ξ lies in $\mathrm{Pc}_{\mathrm{par},\varrho}$. Then*

$$\prod_{l=1}^{k} \frac{\varrho^{Ll} - 1}{\varrho^L - 1} \ \cdot\ \mathrm{vol}(\mathcal{C} \cap \mathrm{Sl}_k(\delta)) \ =\ \mathrm{vol}(\mathrm{Sl}_k(\delta)).$$

LEAN `prod_mul_volume_inter_slab`

The identity holds for every real $\varrho > 1$. When ϱ^L is the number q of elements of a finite field, the product is the number of full flags in a vector space of dimension k over that field.

Theorem 3 (two dimension bounds in characteristic p). *In Setting 2.2, let the ring of T be a field $\mathbb{F}$ of characteristic p, the prime of T. Let n, λ be natural numbers and κ, z real numbers.*
(a) *Let i_0, i_1 be indices with $i_0 + 1 = i_1$, and h a real number such that $\mathrm{Sz}(P_T, o, L, k, n, i_1; \lambda, \kappa, h, z)$. Then*

$$\left[\mathrm{GL}_k(W) : \mathrm{Par}_{\tau_{k,i_1}(\cdot)}(W)\right] \cdot \dim_{\mathbb{F}} \mathrm{Ex}^{i_0,i_1}_{T,n}(0; h) \;\le\; (1+\kappa) \cdot \left(\#\mathrm{Mon}_{k,n}\right)^L.$$

(b) *Let $\mathfrak{t}$ be a natural number and y a real number such that $\mathrm{Sz}(P_T, o, L, k, n, \mathfrak{t}; \lambda, \kappa, y, z)$, and let j_0 be a natural number. Then*

$$\sum_{j<j_0} \dim_{\mathbb{F}} \mathrm{Dp}^{k,n}_{T,\mathfrak{t}}(j) \;\le\; (1+z) \cdot \left(\#\mathrm{Mon}_{k,n}\right)^L.$$

LEAN TwistingData.index_mul_finrank_excessCond_le · TwistingData.sum_finrank_deepForms_le

Here $(\#\mathrm{Mon}_{k,n})^L$ is the number of the monomials x^β, the rank of the module of forms. In (a) the index of the parahoric subgroup is the number of cosets over which a Hecke operator sums, its degree; so the forms obeying the excess condition fill about one part in the degree.

Theorem 4 (an approximate projector). *In Setting 2.2, let $\mathcal{O}$ be a valuation domain of characteristic zero in which the prime p of T is not a unit. Let c, n, λ be natural numbers, i_0, i_1 indices with $i_0 + 1 = i_1$, and h a real number. Suppose that $\lambda + \lambda = L$ and $L\,n\,k\,(\lambda+2) + n + k < P_T^\lambda$. Then there are a function w from $\mathrm{Mon}^L_{k,n}$ to $\mathcal{O}$ and a square matrix Z over $\mathcal{O}$ indexed by $\mathrm{Mon}^L_{k,n}$ such that:*

(i) *ZF lies in $\mathrm{Ex}^{i_0,i_1}_{T,n}(\varpi^c\mathcal{O}; h)$ for every F in $\mathcal{O}^{\mathrm{Mon}^L_{k,n}}$;*

(ii) *for every F' in $\mathrm{Ex}^{i_0,i_1}_{T,n}(\varpi^c\mathcal{O}; h-(4c+2))$ and every F in $\mathcal{O}^{\mathrm{Mon}^L_{k,n}}$, ϖ^c divides*

$$\sum_\beta F'(\beta)\, w(\beta)\, (ZF)(\beta) \;-\; p^{(k+k^2)\,oL\cdot L} \sum_\beta F'(\beta)\, F(\beta);$$

(iii) *$w(\beta)\,\rho_{T,n}(\gamma^{\mathsf{T}})_{\beta,\beta'} = w(\beta')\,\rho_{T,n}(\gamma)_{\beta',\beta}$ for every γ in $\mathrm{GL}_k(W)$ and all β, β' in $\mathrm{Mon}^L_{k,n}$.*

LEAN TwistingData.exists_projector

Theorem 5 (elementary divisors of the twisted Hecke operator). *In Setting 2.2, let $\mathcal{O}$ be a valuation domain of characteristic zero in which the prime p of T is not a unit. Let n, $\mathfrak{t}$, λ be natural numbers and κ, y, z real numbers with $\mathrm{Sz}(P_T, o, L, k, n, \mathfrak{t}; \lambda, \kappa, y, z)$. Let $\mathcal{X}$, $\mathcal{H}$, $\mathcal{I}$ be finite sets, write $\mathcal{Q} = \mathrm{GL}_k(W)/\mathrm{Par}_{\tau_{k,\mathfrak{t}}(\cdot)}(W)$ for the finite set of left cosets, and $\mathcal{E} = \mathcal{X} \times \mathcal{Q}$. Let V be an $\mathcal{O}$-module, $\Psi\colon \mathcal{O}^{\mathcal{X}\times\mathcal{I}} \to V$ an $\mathcal{O}$-linear map, and Y an $\mathcal{O}$-linear endomorphism of V. Let $\mathrm{src}\colon \mathcal{E} \to \mathcal{X}$ be a map, $(\gamma_{\mathfrak{e}})_{\mathfrak{e}\in\mathcal{E}}$ elements of $\mathrm{GL}_k(W)$, $(u_{\mathfrak{e}})_{\mathfrak{e}\in\mathcal{E}}$ elements of $\mathrm{GL}_{\mathcal{H}}(\mathcal{O})$ (invertible matrices over $\mathcal{O}$ indexed by $\mathcal{H}$), and $\jmath\colon \mathcal{I} \to \mathrm{Mon}^L_{k,n} \times \mathcal{H}$, $\mathfrak{i} \mapsto (\beta_{\mathfrak{i}}, \mathfrak{h}_{\mathfrak{i}})$, a bijection. Suppose that the map $\mathfrak{e} \mapsto (\mathrm{src}(\mathfrak{e}), (\gamma_{\mathfrak{e}}^{\mathsf{T}})^{-1}\mathrm{Par}_{\tau_{k,\mathfrak{t}}(\cdot)}(W))$ from $\mathcal{E}$ to $\mathcal{E}$ is injective, and that for every F in $\mathcal{O}^{\mathcal{X}\times\mathcal{I}}$*

$$Y(\Psi(F)) = \Psi(F^\flat),$$
$$F^\flat(\mathfrak{x}, \mathfrak{i}) = \sum_{\mathfrak{q}\in\mathcal{Q}} \sum_{\mathfrak{i}'\in\mathcal{I}} (u_{(\mathfrak{x},\mathfrak{q})})_{\mathfrak{h}_{\mathfrak{i}}, \mathfrak{h}_{\mathfrak{i}'}} \left(H_{(\mathfrak{x},\mathfrak{q})}\right)_{\beta_{\mathfrak{i}'}, \beta_{\mathfrak{i}}} F(\mathrm{src}(\mathfrak{x}, \mathfrak{q}), \mathfrak{i}'),$$

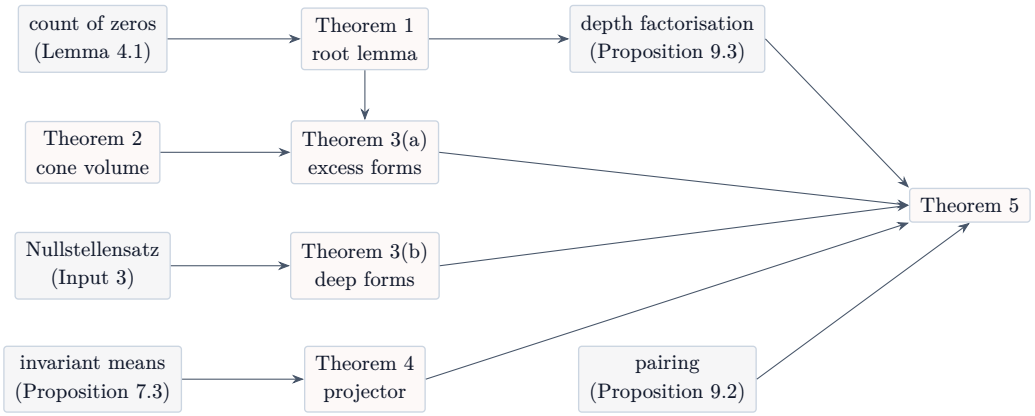
where $H_{(\mathfrak{x},\mathfrak{q})} = \rho_{T,n}(\gamma_{(\mathfrak{x},\mathfrak{q})}) \cdot \mathrm{diag}(p^{\mathrm{depth}^{k,n}_{T,\mathfrak{t}}(\beta)})_\beta \cdot \rho_{T,n}(\tilde{\mathfrak{q}}^{-1})$ and $\tilde{\mathfrak{q}}$ is the chosen representative of the coset $\mathfrak{q}$. Then, for every natural number e:

$$Y \text{ has type at most } (z + \kappa e + y) \cdot \#\mathcal{X} \cdot \#\mathcal{I} \text{ at } p^e \text{ on } \Psi(\mathcal{O}^{\mathcal{X}\times\mathcal{I}}).$$

LEAN `TwistingData.hasType_twisted_hecke`

The last line says (Input 1): Y maps the image of Ψ into itself, and the index of $Y(\operatorname{im}\Psi) + p^e \operatorname{im}\Psi$ in $\operatorname{im}\Psi$ divides p^l for a natural number $l \le (z + \kappa e + y)\,\#\mathcal{X}\,\#\mathcal{I}$. The rank of $\mathcal{O}^{\mathcal{X}\times\mathcal{I}}$ is $\#\mathcal{X} \cdot \#\mathcal{I}$; so of the $e \cdot \#\mathcal{X} \cdot \#\mathcal{I}$ units of length that p^e could cost, the operator loses the share κ, and a constant $z + y$ for each basis vector. The size conditions make κ small when y is large and n larger still. The operator $F \mapsto F^\flat$ is a sum over the cosets of the parahoric subgroup, as a Hecke operator is, of the scaling p^{depth} between two translations, twisted by the matrices $u_{\mathfrak{e}}$ on an auxiliary space and by the map src on the points.

The note on the unit of the columns (unit 145) quotes this theorem as its Input I10, with other letters: it writes n, k', a', q', r', l', c_1, c_2, c_3, ϕ_s and s for what is here L, k, n, P_T, $\mathfrak{t}$, λ, κ, y, z, ι_s and e, and it states the theorem for a valuation ring W.



How the five theorems depend on each other and on the main intermediate results.

2 The setting

Conventions. The natural numbers are $0, 1, 2, \dots$. Every ring is a commutative ring with 1. A *local ring* W has one maximal ideal $\mathfrak{m}_W$ and the residue field $\mathbb{F}_W = W/\mathfrak{m}_W$. A *valuation domain* is an integral domain in which, of any two elements, one divides the other; it is a local ring, and its maximal ideal is written $\mathfrak{m}$. The number of elements of a finite set $\mathcal{S}$ is $\#\mathcal{S}$; for the residue field of a local ring the formal text uses a count that is 0 for an infinite field (§11, note 5).

Indices, letters, variables. k and L are natural numbers, $L \geq 1$ unless said otherwise. The *indices* are $0, \dots, k-1$, written i, i', l. The *letters* are the elements of $\mathbb{Z}/L$, written s, s', s_0; $s+1$ is the next letter. A *variable* is a pair $v = (i, s)$ of an index and a letter; the set of variables is finite. A *labelling* is a map τ from the indices to the natural numbers. For a labelling τ and a local ring W, $\mathrm{Lev}_\tau(W)$ is the *Levi subgroup*, of the γ in $\mathrm{GL}_k(W)$ with $\gamma_{ii'} = 0$ whenever $\tau(i) \neq \tau(i')$, and $\mathrm{Par}_\tau(W)$ is the *parahoric subgroup*, of the γ with $\gamma_{ii'} \in \mathfrak{m}_W$ whenever $\tau(i) < \tau(i')$. For the constant labelling $\mathrm{Lev}_\tau(W) = \mathrm{GL}_k(W)$; for the labelling $i \mapsto i$, $\mathrm{Par}_\tau(W)$ consists of the matrices that are lower triangular modulo $\mathfrak{m}_W$ (the lower Iwahori subgroup).

Polynomials. For a set $\mathcal{V}$ of variables and a ring $\mathcal{O}$, $\mathcal{O}[x_\mathcal{V}]$ is the polynomial ring in the variables x_v, $v \in \mathcal{V}$. An *exponent* is a function α from $\mathcal{V}$ to the natural numbers with finite support; $x^\alpha = \prod_v x_v^{\alpha(v)}$, $|\alpha| = \sum_v \alpha(v)$ is its degree, and for a real function ω on $\mathcal{V}$, a *weight*, $\langle \omega, \alpha \rangle = \sum_v \omega(v)\alpha(v)$. The same notation is used for vectors α with real coordinates. For a square matrix G over $\mathcal{O}$ indexed by a finite set $\mathcal{V}$, the *linear substitution* of G is the endomorphism of the $\mathcal{O}$-algebra $\mathcal{O}[x_\mathcal{V}]$ with $x_{v'} \mapsto \sum_v G_{vv'} x_v$; it is multiplicative in G. $\mathrm{Mon}_{k,n}$ is the set of the exponents of degree n on the indices, and $\mathrm{Sym}^n(G)$, for G in $\mathrm{M}_k(\mathcal{O})$, is the matrix indexed by $\mathrm{Mon}_{k,n}$ whose (β, β') entry is the coefficient of x^β in the linear substitution of G applied to $x^{\beta'}$. A *family* is an element $\beta = (\beta_s)_s$ of $\mathrm{Mon}_{k,n}^L = (\mathrm{Mon}_{k,n})^{\mathbb{Z}/L}$; it is also read as the exponent $(i, s) \mapsto \beta_s(i)$ on the variables, of degree n in each letter. A *form* is a vector $F = (F(\beta))_\beta$ in $\mathcal{O}^{\mathrm{Mon}_{k,n}^L}$, also read as the polynomial $\sum_\beta F(\beta) x^\beta$. For matrices G_s indexed by $\mathrm{Mon}_{k,n}$, one for each letter, $\bigotimes_s G_s$ is the matrix indexed by $\mathrm{Mon}_{k,n}^L$ with entries $\prod_s (G_s)_{\beta_s, \beta'_s}$. Matrices act on column vectors; G^T is the transpose.

Volumes. $\mathbb{R}^{k \times L}$ is the space of the real functions on the variables, with Lebesgue measure vol; a volume is a real number, finite in all cases that occur.

Definition 2.1 (twisting data)**.** Let L be a natural number, W a local ring and $\mathcal{O}$ a ring. *Twisting data of length L over W and $\mathcal{O}$* are a tuple $T = (p, o, (\iota_s)_{s \in \mathbb{Z}/L}, \mathrm{par}_T, \varpi)$ of natural numbers p and o, ring homomorphisms $\iota_s \colon W \to \mathcal{O}$, a map par_T from $\mathbb{Z}/L$ to the signs $\{+, -\}$, the *parity* (the letter s is *even* if $\mathrm{par}_T(s) = +$ and *odd* otherwise), and an element ϖ of $\mathcal{O}$, such that: p is prime; $5 \leq p^o$; $\varpi^2 = p$ in $\mathcal{O}$; p divides $\iota_{s+1}(v) - \iota_s(v)^{p^o}$ in $\mathcal{O}$ for every letter s and every v in W; $\mathrm{par}_T(s+1) \neq \mathrm{par}_T(s)$ for every letter s; and $\#\mathbb{F}_W = (p^o)^L$. The set of all twisting data is $\mathrm{TD}_L(W, \mathcal{O})$. For T in it we put

$$P_T = p^o, \qquad q_T = P_T^L,$$

and for a ring homomorphism $\chi \colon \mathcal{O} \to \mathcal{O}'$ we write

$$\chi_* T = (p, o, (\chi \circ \iota_s)_s, \mathrm{par}_T, \chi(\varpi)),$$

an element of $\mathrm{TD}_L(W, \mathcal{O}')$.

LEAN TwistingData · TwistingData.P · TwistingData.q · TwistingData.map

For $L = 0$ there are no twisting data, as a field has at least two elements. For $L \geq 1$ the alternation forces L to be even. The components of T keep their names: p, o, ι_s, par_T and ϖ always belong to the twisting data called T in the statement at hand, and those of $\chi_* T$ are as just said.

Setting 2.2. $L \geq 1$ and $k \geq 0$ are integers, W is a local ring, $\mathcal{O}$ is a ring, and T is an element of $\mathrm{TD}_L(W, \mathcal{O})$, with components $p, o, (\iota_s)_s, \mathrm{par}_T, \varpi$. Indices, letters and variables are those of the conventions, for this k and L.

Letters. The letters of this note are local to it. The threshold of the depth is written $\mathfrak{t}$ and the parity par, because the write-up keeps μ and ς for the exponent of the data and for a root of unity. The letters λ, κ, τ, σ, m, q, θ and the polynomials Φ have no relation to the objects that the overview of the handout writes with them (an arithmetic function and a constant of Chapter I, the infinite places, the automorphisms of the complex numbers, the number of lifts, a rational number of Theorem A, the family of twists, a function of Chapter I), and P_T none to the parameters. In the application (§10) k and n become the rank of the columns and the number $a(D)$ of the data D.

Definition 2.3 (the action on forms)**.** In Setting 2.2, let γ be an element of $\mathrm{GL}_k(W)$, s a letter and n a natural number.

(a) $\gamma_{T,s}$ is the matrix in $\mathrm{M}_k(\mathcal{O})$ obtained by applying ι_s to the entries of γ if s is even, and of $(\gamma^{-1})^{\mathsf{T}}$ if s is odd. The map $\gamma \mapsto \gamma_{T,s}$ is multiplicative.

(b) $[\gamma]_T$ is the endomorphism of the $\mathcal{O}$-algebra of polynomials in the variables $x_{i,s}$ that acts on the variables of each letter s by the linear substitution of $\gamma_{T,s}$. The map $\gamma \mapsto [\gamma]_T$ is multiplicative: $\mathrm{GL}_k(W)$ acts on the left.

(c) $\rho_{T,n}(\gamma) = \bigotimes_s \mathrm{Sym}^n(\gamma_{T,s})$, a square matrix over $\mathcal{O}$ indexed by $\mathrm{Mon}^L_{k,n}$, multiplicative in γ.

(d) For natural numbers k, $\mathfrak{t}$ and an index i, $\tau_{k,\mathfrak{t}}(i)$ is 0 if $i < \mathfrak{t}$ and 1 otherwise. The labelling $\tau_{k,\mathfrak{t}}(\cdot)$ has the two blocks $\{i < \mathfrak{t}\}$ and $\{i \geq \mathfrak{t}\}$.

LEAN TwistingData.letterMatrix · TwistingData.polyAction · TwistingData.letterRep · twoBlock

Definition 2.4 (deep variables and depth)**.** In Setting 2.2, let $\mathfrak{t}$ and n be natural numbers. For a variable $v = (i, s)$ put $\mathrm{dp}_{T,\mathfrak{t}}(v) = 1$ if s is even and $i \geq \mathfrak{t}$, or s is odd and $i < \mathfrak{t}$, and $\mathrm{dp}_{T,\mathfrak{t}}(v) = 0$ otherwise; in the first case v is *deep* for $\mathfrak{t}$. For a family β in $\mathrm{Mon}^L_{k,n}$, its *depth* is

$$\mathrm{depth}^{k,n}_{T,\mathfrak{t}}(\beta) = \sum_{v=(i,s)} \beta_s(i)\, \mathrm{dp}_{T,\mathfrak{t}}(v) = \langle \mathrm{dp}_{T,\mathfrak{t}}(\cdot), \beta \rangle,$$

the degree of x^β in the deep variables. (A reading of the formal text by program gives the depth a sixth argument: it takes the name under which the degree is passed on for a further one.)

LEAN TwistingData.deepIndicator · TwistingData.depth

The scaling of the Hecke operator is the diagonal matrix $\mathrm{diag}(p^{\mathrm{depth}^{k,n}_{T,\mathfrak{t}}(\beta)})_\beta$. It is the action on forms of the diagonal matrix with entries 1 at the indices $i < \mathfrak{t}$ and p at

the others, read through the letters: directly at the even ones, and by the inverse transpose, up to the scalar p, at the odd ones.

Definition 2.5 (root variables and excess)**.** In Setting 2.2, let i_0, i_1 be indices and s, s_0 letters. The *root variable* is

$$\mathrm{rt}_{T,s}(i_0,i_1)=(i_1,s)\ \text{ if } s \text{ is even}, \qquad \mathrm{rt}_{T,s}(i_0,i_1)=(i_0,s)\ \text{ if } s \text{ is odd}.$$

For a letter s let $[s-s_0]$ be the representative of $s-s_0$ in $\{0,\dots,L-1\}$. The *excess weight* at s_0 is the real function on the variables $v=(i,s)$ given by

$$\mathrm{ex}^{i_0,i_1}_{T,s_0}(v)=\frac{1}{q_T}\Big([v=\mathrm{rt}_{T,s}(i_0,i_1)]\cdot P_T^{1+[s-1-s_0]}-[v=\mathrm{rt}_{T,s}(i_1,i_0)]\cdot P_T^{[s-s_0]}\Big),$$

where a bracket around a statement is 1 if it holds and 0 if not. The *excess* of an exponent α at s_0 is $\langle \mathrm{ex}^{i_0,i_1}_{T,s_0}(\cdot),\alpha\rangle$.

LEAN `TwistingData.rootVar` · `TwistingData.excessWeight`

Since $1+[s-1-s_0]$ is L for $s=s_0$ and $[s-s_0]$ otherwise, the excess of α at s_0 is, for $i_0\neq i_1$ and with $\alpha^+_s=\alpha(\mathrm{rt}_{T,s}(i_0,i_1))$ and $\alpha^-_s=\alpha(\mathrm{rt}_{T,s}(i_1,i_0))$,

$$\langle \mathrm{ex}^{i_0,i_1}_{T,s_0}(\cdot),\alpha\rangle=\alpha^+_{s_0}-\frac{\alpha^-_{s_0}}{q_T}+\sum_{s\neq s_0}\frac{P_T^{[s-s_0]}}{q_T}\,(\alpha^+_s-\alpha^-_s). \tag{1}$$

The two root variables of a letter are the two variables that the elementary matrix $1+vE_{i_0i_1}$, with v in W, couples at that letter; here $E_{ii'}$ is the matrix with the entry 1 at (i,i') and 0 elsewhere. In words: at an even letter it adds a multiple of $x_{i_0,s}$ to $x_{i_1,s}$, at an odd letter a multiple of $x_{i_1,s}$ to $x_{i_0,s}$.

Definition 2.6 (conditions on forms)**.** In Setting 2.2, let n be a natural number.

(a) For a labelling τ, an ideal I of $\mathcal{O}$ and a subset $\mathcal{B}$ of $\mathrm{Mon}^L_{k,n}$, $\mathrm{Tr}_{T,n}(\tau;I,\mathcal{B})$ is the $\mathcal{O}$-submodule of the forms F in $\mathcal{O}^{\mathrm{Mon}^L_{k,n}}$ such that $(\rho_{T,n}(\gamma)F)(\beta)\in I$ for every γ in $\mathrm{Lev}_\tau(W)$ and every β in $\mathcal{B}$.

(b) For indices i_0, i_1, an ideal I of $\mathcal{O}$ and a real number h,

$$\mathrm{Ex}^{i_0,i_1}_{T,n}(I;h)=\mathrm{Tr}_{T,n}(\tau_{k,i_1}(\cdot);I,\mathcal{B}),\qquad \text{where}$$
$$\mathcal{B}=\{\beta:\ h<\langle \mathrm{ex}^{i_0,i_1}_{T,s_0}(\cdot),\beta\rangle \text{ for some letter } s_0\}.$$

(c) For natural numbers k, $\mathfrak{t}$ and j, $\mathrm{Dp}^{k,n}_{T,\mathfrak{t}}(j)=\mathrm{Tr}_{T,n}(0;0,\{\beta:\mathrm{depth}^{k,n}_{T,\mathfrak{t}}(\beta)<j\})$, with the constant labelling 0 and the zero ideal.

LEAN `TwistingData.translatesIn` · `TwistingData.excessCond` · `TwistingData.deepForms`

So F lies in $\mathrm{Ex}^{i_0,i_1}_{T,n}(I;h)$ when, for every γ that is block diagonal with the blocks $\{i<i_1\}$ and $\{i\geq i_1\}$, the form $\rho_{T,n}(\gamma)F$ has modulo I only monomials whose excess is at most h at every letter; and F lies in $\mathrm{Dp}^{k,n}_{T,\mathfrak{t}}(j)$ when every translate of F under $\mathrm{GL}_k(W)$ has only monomials of depth at least j.

Definition 2.7 (filtrations)**.** Let $\mathcal{O}$ be a ring.

(a) Let $\mathcal{F}=(\mathcal{F}(\theta))_{\theta\in\mathbb{R}}$ be a family of additive subgroups of $\mathcal{O}$. We say that $\mathcal{F}$ is a real filtration if $1\in\mathcal{F}(0)$, if $\mathcal{F}(\theta)\cdot\mathcal{F}(\theta')\subseteq\mathcal{F}(\theta+\theta')$ for all θ,θ', and if $\mathcal{F}(\theta')\subseteq\mathcal{F}(\theta)$ whenever $\theta\leq\theta'$.

(b) For ϖ in $\mathcal{O}$ and a real number θ, $\mathrm{ad}_{\varpi}(\theta)$ is the additive group of the elements of $\mathcal{O}$ divisible by $\varpi^{\max(\lceil\theta\rceil,0)}$.

(c) For a family $\mathcal{F}$ of additive subgroups of $\mathcal{O}$, a set $\mathcal{V}$ of variables, a weight ω on $\mathcal{V}$ and a real number θ, $\mathrm{Gs}_{\mathcal{F},\omega}(\theta)$ is the additive group of the polynomials in $\mathcal{O}[x_{\mathcal{V}}]$ whose coefficient at each x^{α} lies in $\mathcal{F}(\theta-\langle\omega,\alpha\rangle)$.

LEAN `IsRealFiltration` · `adicFiltration` · `gaussFiltration`

In the sense of (a), $\mathrm{ad}_{\varpi}(\cdot)$ is a real filtration, because $\lceil\theta+\theta'\rceil\le\lceil\theta\rceil+\lceil\theta'\rceil$; and if $\mathcal{F}$ is a real filtration, then $\mathrm{Gs}_{\mathcal{F},\omega}(\cdot)$ is a real filtration, on the polynomial ring, because the coefficient of x^{α} in a product is a sum of products of coefficients at exponents with sum α. These are the two instances of the unit (§11, note 1). In $\mathrm{Gs}_{\mathrm{ad}_{\varpi}(\cdot),\omega}(\theta)$ a variable x_v sits at level $\omega(v)$: the level of a polynomial is the least value of *valuation plus weight* over its terms. For $\varpi=0$ the group $\mathrm{ad}_0(\theta)$ is $\mathcal{O}$ for $\theta\le 0$ and 0 for $\theta>0$.

Definition 2.8 (cones and slabs)**.** Let $k\ge 0$, $L\ge 1$, let par be a map from $\mathbb{Z}/L$ to the signs, ϱ a real number, and $\sigma=(\sigma_s)_{s\in\mathbb{Z}/L}$ permutations of the indices.

(a) $\mathrm{St}_{\mathrm{par},\varrho,\sigma}$ is the linear endomorphism of $\mathbb{R}^{k\times L}$ sending ξ to the vector with the coordinates $\mathrm{par}(s)\cdot\big(\varrho\,\xi(i,s+1)-\xi(\sigma_s(i),s)\big)$ at (i,s). For the identity permutations we write $\mathrm{St}_{\mathrm{par},\varrho,1}$.

(b) For reals $\delta=(\delta_s)_s$, $\mathrm{Sl}_k(\delta)$ is the set of the η in $\mathbb{R}^{k\times L}$ with all coordinates ≥ 0 and $\delta_s-1\le\sum_i\eta(i,s)\le\delta_s$ for every letter s.

(c) $\mathrm{Pc}_{\mathrm{par},\varrho}$ is the set of the ξ in $\mathbb{R}^{k\times L}$ such that for every letter s and all indices i, $i+1$: $\varrho\,\xi(i+1,s+1)\le\xi(i,s)$ if $\mathrm{par}(s)=+$, and $\xi(i+1,s)\le\varrho\,\xi(i,s+1)$ if $\mathrm{par}(s)=-$.

(d) In Setting 2.2 with $\mathcal{O}$ a field, for an index i_1 and a real number h, $\mathrm{Ec}_T(i_1;h)$ is the set of the η in $\mathbb{R}^{k\times L}$ such that for every letter s_0 and all indices i, $i+1$: $\langle\mathrm{ex}^{i,i+1}_{T,s_0}(\cdot),\eta\rangle\le h$ if $i+1=i_1$, and $\langle\mathrm{ex}^{i,i+1}_{T,s_0}(\cdot),\eta\rangle\le 0$ otherwise.

LEAN `permStep` · `slab` · `pieriCone` · `TwistingData.excessCone`

With $\eta=\mathrm{St}_{\mathrm{par},\varrho,1}(\xi)$, the condition of (c) at an even letter, together with $\eta(i,s)>0$, reads $\varrho\,\xi(i,s+1)>\xi(i,s)\ge\varrho\,\xi(i+1,s+1)$: the column $\xi(\cdot,s)$ interlaces the column $\varrho\,\xi(\cdot,s+1)$ from below. At an odd letter the roles are exchanged. Hence the name: a cone of cyclically interlacing columns.

Definition 2.9 (the size conditions)**.** For natural numbers P, o, L, k, n, $\mathfrak{t}$, λ and real numbers κ, y, z we write $\mathrm{Sz}(P,o,L,k,n,\mathfrak{t};\lambda,\kappa,y,z)$ for the conjunction of the ten conditions

$$0<\mathfrak{t},\quad 2\mathfrak{t}\le k,\quad \kappa\le 1,\quad 1\le y,\quad \lambda+\lambda=L,\quad L\,n\,k\,(\lambda+2)+n+k<P^{\lambda},$$
$$6\big(2(k+k^2)\,oL\cdot L+2\big)+2\le y,\qquad \frac{2(L+2)k}{y}\le\kappa,$$
$$\big(y+L(k+1)\big)\,k\,P^{3\mathfrak{t}}+k+2\le\frac{n}{y},\qquad L\,n\left(\frac{1}{P^{2\mathfrak{t}}}+\frac{\mathfrak{t}^2}{kP}\right)\le z.$$

LEAN `HeckeSizes` · in the formal text the ten arguments are called P, d, N, k, n, r, ℓ, κ, y, z

Definition 2.10 (Frobenius operators and conjugations)**.** In Setting 2.2, let $\mathfrak{t}$, λ, n, j be natural numbers, s a letter, γ an element of $\mathrm{GL}_k(W)$, and i_0, i_1, i indices.

(a) The *generic variables* are $x_{s,i,i'}$, for a letter and two indices; $X_s = (x_{s,i,i'})_{i,i'}$ is the generic matrix of the letter s. $X_T^{\mathrm{Fr}}(s,j)$ is the matrix over the polynomial ring in the generic variables over $\mathcal{O}$ whose (i,i') entry is $x_{s-j,i,i'}^{P_T^j}$: the generic matrix of the letter $s-j$ with its entries raised to the power P_T^j.

(b) Let $\mathcal{R}$ be a ring and $\psi = (\psi(s,j))$ matrices in $\mathrm{M}_k(\mathcal{R})$, for s a letter and j a natural number. For a letter s and a natural number m put $\Pi_s^m = \psi(s,m-1)\cdots\psi(s,1)\,\psi(s,0)$, and let B_s be the matrix in $\mathrm{M}_k(\mathcal{R})$ whose row i is row i of Π_s^m if s is even, and of the transpose of the adjugate of Π_s^m if s is odd, where $m = \lambda + \mathrm{dp}_{T,\mathrm{t}}((i,s))$. Then

$$\mathrm{Fr}_{T,\mathrm{t},\lambda,n}(\psi) = \bigotimes_s \mathrm{Sym}^n(B_s),$$

a square matrix over $\mathcal{R}$ indexed by $\mathrm{Mon}_{k,n}^L$.

(c) $\mathrm{cj}_{T,\gamma,s}$ is the multiplicative map $G \mapsto \iota_s(\gamma)\,G\,\iota_s(\gamma^{-1})$ of $\mathrm{M}_k(\mathcal{O})$.

(d) $\mathrm{iw}_{T,\lambda}^{i_0,i_1}(i) = ([i=i_1]-[i=i_0])/P_T^\lambda$, a real number.

(e) For a ring homomorphism $\chi\colon W \to \mathcal{O}$, $\mathrm{cs}_\chi(\gamma)$ is the endomorphism of the $\mathcal{O}$-algebra of polynomials in the entries $x_{i,i'}$ of one generic matrix X that substitutes the entries of $\chi(\gamma^{-1})\,X\,\chi(\gamma)$ for those of X.

LEAN `TwistingData.frobGenericMatrix` · `TwistingData.frobOperator` · `TwistingData.letterConj` · `TwistingData.indexWeight` · `ClassMean.conjSubst`

If the generic matrix of every letter s is specialised to $\iota_s(\gamma)$, for one γ in $\mathrm{GL}_k(W)$, then $X_T^{\mathrm{Fr}}(s,j)$ becomes congruent to $\iota_s(\gamma)$ modulo p, and the matrices B_s of (b), for $\psi = X_T^{\mathrm{Fr}}(\cdot,\cdot)$, become congruent to matrices whose rows are rows of $\iota_s(\gamma)^\lambda$ or of $\iota_s(\gamma)^{\lambda+1}$ (for odd letters, of their transposed adjugates). The operator on generic matrices keeps more: the exponents of the generic variables of each letter.

Inputs from other units. The results below are taken for granted in this note.

Input 1 (index divisibility; unit 149). *Let $\mathcal{O}$ be a commutative ring, V and V' modules over $\mathcal{O}$, let M_1, M_2, M_3 be submodules of V and M_1', M_2' submodules of V', and let π, π', π_0 be elements of $\mathcal{O}$. One writes $[M_1 : M_2] \mid \pi$, the index of M_2 in M_1 divides π, if for every natural number n' and every family of n' elements of M_1 there is a square matrix of size n' over $\mathcal{O}$ whose determinant divides π and whose rows, used as coefficients of combinations of the family, give n' elements of M_2. For an element p of $\mathcal{O}$, a natural number e, an $\mathcal{O}$-linear endomorphism Y of V and a real number θ one says that Y has type at most θ at p^e on M_1 if $Y(M_1) \subseteq M_1$ and $[M_1 : Y(M_1) + p^e M_1] \mid p^l$ for some natural number $l \leq \theta$. Then:*

(a) *if $\mathcal{O}$ is a valuation domain, $\mathcal{J}$ a finite set, G a square matrix over $\mathcal{O}$ indexed by $\mathcal{J}$ whose determinant divides π, and M_2 a submodule of $\mathcal{O}^{\mathcal{J}}$ that contains $G(\mathcal{O}^{\mathcal{J}})$, then $[\mathcal{O}^{\mathcal{J}} : M_2] \mid \pi$;*

(b) *if $[M_1 : M_2] \mid \pi$ and $[M_2 : M_3] \mid \pi'$, then $[M_1 : M_3] \mid \pi\pi'$;*

(c) *if $\Psi : V \to V'$ is an $\mathcal{O}$-linear map, $[M_1 : M_2] \mid \pi$, $M_1' \subseteq \Psi(M_1) + M_2'$ and $\Psi(M_2) \subseteq M_2'$, then $[M_1' : M_2'] \mid \pi$;*

(d) *if $Y : V \to V$ is an $\mathcal{O}$-linear map with $Y(M_1) \subseteq M_1$, if $[M_1 : Y(M_1) + \pi_0 M_1] \mid \pi$ and $[M_1 : M_2 + \pi_0 M_1] \mid \pi'$, then $[M_1 : Y(M_2) + \pi_0 M_1] \mid \pi\pi'$.*

Input 2 (normal forms over a valuation ring; units 120 and 112, background). *Let $\mathcal{A}$ be a commutative ring and $\mathcal{R}$ a subring such that $(\mathcal{A}, \mathcal{R})$ is a valuation pair: for any two elements of $\mathcal{A}$ there is an idempotent of $\mathcal{R}$ such that the $\mathcal{R}$-module spanned by the two elements lies in the $\mathcal{R}$-multiples of the first element minus the idempotent times the difference of the first and the second. If $\mathcal{A}$ is a valuation domain, then $(\mathcal{A}, \mathcal{A})$ is a valuation pair. Let $\mathcal{N}$ be a finite set.*

(a) (Hermite) *Let $\mathcal{K}$ be a finite set with an injective map to $\mathcal{N}$, by which $\mathcal{K}$ is regarded as a subset of $\mathcal{N}$, let X be a matrix over $\mathcal{A}$ with rows indexed by $\mathcal{N}$ and columns by $\mathcal{K}$, and let weights, natural numbers, be given on $\mathcal{K}$. Then there are an invertible matrix U over $\mathcal{A}$ indexed by $\mathcal{N}$, with the entries of U and of U^{-1} in $\mathcal{R}$, and a square matrix Δ over $\mathcal{A}$ indexed by $\mathcal{K}$, whose entry at (l, l') vanishes whenever the weight of l' is less than that of l, such that $X = U_{\cdot,\mathcal{K}}\, \Delta$ and $(U^{-1})_{\mathcal{K},\cdot}\, X = \Delta$; here $U_{\cdot,\mathcal{K}}$ consists of the columns of U in $\mathcal{K}$ and $(U^{-1})_{\mathcal{K},\cdot}$ of the rows of U^{-1} in $\mathcal{K}$.*

(b) (Smith) *Every square matrix X over $\mathcal{A}$ indexed by $\mathcal{N}$ is a product $U \operatorname{diag}(\varepsilon_l)_{l \in \mathcal{N}} U'$ with U, U' invertible, the entries of U, U^{-1}, U', U'^{-1} in $\mathcal{R}$, and ε_l in $\mathcal{A}$.*

(The reading of the Hermite form by program also lists a ring with involution, from an earlier part of its file; the statement does not involve it.)

LEAN `IndexDvd` · `HasType` · `IndexDvd.of_det_dvd` · `IndexDvd.trans` · `IndexDvd.map` · `IndexDvd.comp` · `IsValuationPair.hermite` · `IsValuationPair.smith_normal_form` · `isValuationPair_top`

Input 3 (a combinatorial Nullstellensatz with multiplicity; unit 57, background). *Let $\mathbb{F}$ be a field, $\mathcal{V}$ and $\mathcal{S}$ finite sets, and $\varkappa$ a map from $\mathcal{V}$ to the natural numbers with $0 < \varkappa(v)$ for every v; the* cost *of an exponent α' on $\mathcal{V}$ is $\sum_v \alpha'(v)/\varkappa(v)$. Let $(\upsilon_{\mathfrak{s}})_{\mathfrak{s}\in\mathcal{S}}$ be pairwise distinct points of $\mathbb{F}^{\mathcal{V}}$ with $\prod_v \varkappa(v) \leq \#\mathcal{S}$. Suppose that for every v in $\mathcal{V}$ there is a polynomial Φ_v in $\mathbb{F}[x_{\mathcal{V}}]$, all of whose monomials $x^{\alpha'}$ have cost less than 1, with $\Phi_v(\upsilon_{\mathfrak{s}}) = \upsilon_{\mathfrak{s}}(v)^{\varkappa(v)}$ for every $\mathfrak{s}$. Let j be a natural number and Φ a nonzero polynomial in $\mathbb{F}[x_{\mathcal{V}}]$ that lies, for every $\mathfrak{s}$, in the j-th power of the ideal of the polynomials vanishing at $\upsilon_{\mathfrak{s}}$. Then Φ has a monomial x^{α}, with nonzero coefficient, of cost at least j.* (The reading by program also lists a polynomial and an exponent of an earlier part of the file; they are the bound objects Φ_v and x^{α} of the statement.)

Input 4 (the order of the general linear group; unit 24, background). *For a finite field $\mathbb{F}$ and a finite set of k' elements, the group of the invertible matrices over $\mathbb{F}$ indexed by the set has $\prod_{l=0}^{k'-1}(\#\mathbb{F})^{l}((\#\mathbb{F})^{l+1}-1)$ elements; the formal text states this as an equation in an arbitrary field.*

LEAN `exists_le_cost_of_vanishing` · `card_GL_eq_prod`

Input 1 is the subject of the note on unit 149 of this write-up; there the definitions are stated with their arguments in the formal order and the four properties are proved. Input 2(a) is also an input of that note (its Input 4), where it is not proved either, and the fact that a valuation domain gives a valuation pair has a proof of two lines there; Input 2(b), the Smith form, comes from unit 112. The background library also supplies the vocabulary of the conventions (linear substitutions, symmetric powers, Kronecker products, Levi and parahoric subgroups; units 57 and 112).

References for the Inputs. The catalogue of quoted results of this write-up lists the Hermite form as an elementary lemma with no confirmed place in the literature; it has no entry yet for the Smith form, for Input 3 or for Input 4. What can be said of each:

- *Input 2.* This note uses both forms only for a valuation domain $\mathcal{O}$ and the pair $(\mathcal{O},\mathcal{O})$ (Propositions 9.2 and 9.3). There they are classical: among the entries of a nonzero matrix over a valuation domain one divides all the others, and operations on rows and columns move it to a corner and clear its row and its column. The rings over which every matrix has a triangular or a diagonal reduction are the subject of I. Kaplansky, *Elementary divisors and modules*, Trans. Amer. Math. Soc. 66 (1949), 464–491, and valuation rings are among them; the place in that paper was not confirmed. For the form with a valuation pair, which the artifact needs elsewhere: *reference to be confirmed.*
- *Input 3.* The artifact calls it a known result, a combinatorial Nullstellensatz with multiplicity (a nonzero polynomial that vanishes to high order on a large grid-like set of points has a term of large cost), and the header of its file names as sources N. Alon, *Combinatorial Nullstellensatz*, Combin. Probab. Comput. 8 (1999), 7–29, and S. Ball and O. Serra, *Punctured combinatorial Nullstellensätze*, Combinatorica 29 (2009), 511–522. Those papers concern the zeros of a polynomial on a product of finite subsets of the field, the second with multiplicity. Input 3 replaces the product by any set of points on which each power $x_v^{\varkappa(v)}$ agrees with a polynomial

of cost less than 1; this note did not compare the statements, and for the form stated here: *reference to be confirmed.*

- *Input 4* is classical, and its proof is one line: the column of index l of an invertible matrix, $l = 0, \dots, k' - 1$, is any vector outside the span of the l columns before it, so there are $\prod_{l=0}^{k'-1} ((\#\mathbb{F})^{k'} - (\#\mathbb{F})^{l})$ invertible matrices, which is the product of Input 4. A place in a textbook: *reference to be confirmed.*

3 First properties

Lemma 3.1 (elementary properties)**.** *In Setting 2.2:*

(a) *for every natural number j, every letter s and every v in W, p divides $\iota_{s+j}(v) - \iota_s(v)^{P_T^j}$;*
(b) *for γ in $\mathrm{GL}_k(W)$ and a variable $v = (i', s)$, $[\gamma]_T x_v = \sum_i (\gamma_{T,s})_{ii'} \, x_{i,s}$;*
(c) *for a natural number n, γ in $\mathrm{GL}_k(W)$ and a form F in $\mathcal{O}^{\mathrm{Mon}_{k,n}^L}$, $[\gamma]_T$ sends the polynomial $\sum_\beta F(\beta) x^\beta$ to the polynomial of the form $\rho_{T,n}(\gamma)F$;*
(d) *for a ring homomorphism $\chi \colon \mathcal{O} \to \mathcal{O}'$, a natural number n and γ in $\mathrm{GL}_k(W)$, the matrix $\rho_{\chi_* T, n}(\gamma)$ is obtained by applying χ to the entries of $\rho_{T,n}(\gamma)$;*
(e) *for natural numbers n, $\mathfrak{t}$ and an element π of $\mathcal{O}$, the diagonal matrix with the entries $\pi^{\mathrm{depth}_{T,\mathfrak{t}}^{k,n}(\beta)}$, $\beta \in \mathrm{Mon}_{k,n}^L$, equals $\bigotimes_s \mathrm{Sym}^n(\mathrm{diag}(\pi^{\mathrm{dp}_{T,\mathfrak{t}}((i,s))})_i)$.*

Moreover $1 < P_T$ and $1 < q_T$; the residue field of W is finite; and $\varpi \neq 0$ if $\mathcal{O}$ has characteristic zero.

Proof. (a) By induction on j: if $\iota_{s+j}(v) \equiv \iota_s(v)^{P_T^j}$ modulo p, then, modulo p,

$$\iota_{s+j+1}(v) \equiv \iota_{s+j}(v)^{P_T} \equiv \iota_s(v)^{P_T^{j+1}},$$

because congruent elements have congruent powers. (b) is the definition of the linear substitution. (c) By (b) and the definition of Sym^n, $[\gamma]_T$ sends the monomial $\prod_s x^{\beta_s}$, with x^{β_s} in the variables of the letter s, to

$$\prod_s \sum_{\beta'_s} \mathrm{Sym}^n(\gamma_{T,s})_{\beta'_s, \beta_s} x^{\beta'_s} = \sum_{\beta'} \rho_{T,n}(\gamma)_{\beta', \beta} \, x^{\beta'}.$$

(d) The entries of $\mathrm{Sym}^n(G)$ are polynomials with integer coefficients in the entries of G. (e) The linear substitution of a diagonal matrix multiplies each monomial by the product of the diagonal entries to the powers of the exponent. The last assertions: $P_T \geq 5$; $\#\mathbb{F}_W = q_T$ is a positive natural number; and $\varpi^2 = p \neq 0$. □

LEAN `TwistingData.frobenius_pow_congr` · `TwistingData.polyAction_X` · `TwistingData.polyAction_polyOfCoeffs` · `TwistingData.letterRep_map` · `TwistingData.diagonal_pow_depth` · `TwistingData.one_lt_P` · `TwistingData.one_lt_q` · `TwistingData.finite_residueField` · `TwistingData.w_ne_zero`

Lemma 3.2 (entries of parahoric elements)**.** *In Setting 2.2, let τ be a labelling, γ an element of $\mathrm{Par}_\tau(W)$, s a letter, and i, i' indices such that $\tau(i) < \tau(i')$ if s is even and $\tau(i') < \tau(i)$ if s is odd. Then p divides the entry $(\gamma_{T,s})_{ii'}$ in $\mathcal{O}$.*

Proof. Let v be the (i, i') entry of γ if s is even and of $(\gamma^{-1})^{\mathsf{T}}$ if s is odd, so that the entry in question is $\iota_s(v)$. In both cases v lies in $\mathfrak{m}_W$: for even s by the definition of the parahoric subgroup, for odd s because γ^{-1} lies in it and $v = (\gamma^{-1})_{i'i}$. Hence $1 - v^{q_T - 1}$

is a unit of W and its image under ι_s a unit of $\mathcal{O}$. By Lemma 3.1(a) with $j = L$, where $s + L = s$ and $P_T^L = q_T$, the prime p divides $\iota_s(v) - \iota_s(v)^{q_T} = \iota_s(v) \cdot \iota_s(1 - v^{q_T - 1})$, hence $\iota_s(v)$. □

LEAN TwistingData.p_dvd_letterMatrix_of_mem_parahoric

Lemma 3.3 (size of the excess weights)**.** *In Setting 2.2, for a letter s_0, indices i_0, i_1 and a variable v: $-\frac{1}{5} \le \mathrm{ex}_{T,s_0}^{i_0,i_1}(v) \le 1$.*

Proof. Let $v = (i, s)$. As $[s - s_0] \le L - 1$ and $P_T \ge 5$,

$$5\, P_T^{[s-s_0]} \le q_T;$$

and $1 + [s - 1 - s_0]$ is L if $s = s_0$ and $[s - s_0]$ otherwise. So the first term of the definition is 0, or 1, or lies in $(0, \frac{1}{5}]$, and the second is 0 or lies in $[-\frac{1}{5}, 0)$. □

LEAN TwistingData.excessWeight_mem_Icc · TwistingData.one_add_val_sub_one_sub · TwistingData.five_mul_P_pow_le_q

Lemma 3.4 (half of the letters are deep)**.** *In Setting 2.2, let $\mathfrak{t}$ be a natural number and i an index. Then $\sum_s \mathrm{dp}_{T,\mathfrak{t}}((i, s)) = L/2$. (The formal statement asks that $\mathcal{O}$ be a field and does not use it; it writes the quotient of natural numbers.)*

Proof. The parity alternates, so $\mathrm{dp}_{T,\mathfrak{t}}((i, s)) + \mathrm{dp}_{T,\mathfrak{t}}((i, s + 1)) = 1$ for every letter; summing over the letters gives twice the sum on the left and L on the right. □

LEAN TwistingData.sum_deepIndicator

4 The root lemma

Lemma 4.1 (a count of zeros against degree)**.** *Let W be a local ring, $\mathcal{O}$ a ring, $\chi\colon W \to \mathcal{O}$ a ring homomorphism, Φ a nonzero polynomial in one variable X over $\mathcal{O}$, and e a natural number. Suppose that $(X - \chi(v))^e$ divides Φ for every v in W; equivalently, X^e divides $\Phi(X + \chi(v))$. Then $e \cdot \#\mathbb{F}_W \le \deg \Phi$.*

Proof. If $\mathbb{F}_W$ is infinite the count on the left is 0. Otherwise choose one element of W in each residue class. The difference of two of them is a unit of W, so its image is a unit of $\mathcal{O}$, and the polynomials $(X - \chi(v))^e$ for the chosen v are pairwise coprime. Each divides Φ; so does their product, which is monic of degree $e \cdot \#\mathbb{F}_W$. A nonzero polynomial is not divisible by a monic polynomial of larger degree. □

LEAN stepanov_count

Lemma 4.2 (filtrations under substitution)**.** *Let $\mathcal{O}$ be a ring, $\mathcal{A}$ a commutative $\mathcal{O}$-algebra, $\mathcal{V}$ a set of variables, $\mathcal{F} = (\mathcal{F}(\theta))_\theta$ a family of additive subgroups of $\mathcal{O}$, ω a weight on $\mathcal{V}$ and θ a real number. Let χ and χ' be homomorphisms of $\mathcal{O}$-algebras from $\mathcal{O}[x_\mathcal{V}]$ to $\mathcal{A}$, and let $\mathcal{G} = (\mathcal{G}(\theta))_\theta$ be a family of additive subgroups of $\mathcal{A}$ such that $\mathcal{G}$ is a real filtration. Let Φ be a polynomial in $\mathrm{Gs}_{\mathcal{F},\omega}(\theta)$. Suppose that*

(i) *$\chi(x_v) \in \mathcal{G}(\omega(v))$ for every v in $\mathcal{V}$;*
(ii) *$\chi'(x_v) - \chi(x_v) \in \mathcal{G}(\omega(v) + \frac{1}{20})$ for every v in $\mathcal{V}$;*
(iii) *the image in $\mathcal{A}$ of every element of $\mathcal{F}(\theta')$ lies in $\mathcal{G}(\theta')$, for every real θ'.*

Then $\chi'(\Phi) \in \mathcal{G}(\theta)$ and $\chi'(\Phi) - \chi(\Phi) \in \mathcal{G}(\theta + \frac{1}{20})$.

Proof. It suffices to treat one term πx^α with π in $\mathcal{F}(\theta - \langle\omega, \alpha\rangle)$. By (i) and (ii) and since $\mathcal{G}$ is decreasing, $\chi'(x_v)$ and $\chi(x_v)$ lie in $\mathcal{G}(\omega(v))$; so $\chi'(x^\alpha)$ lies in $\mathcal{G}(\langle\omega, \alpha\rangle)$,

and with (iii) $\chi'(\pi x^\alpha)$ lies in $\mathcal{G}(\theta)$. The difference $\chi'(x^\alpha)-\chi(x^\alpha)$, expanded as a product of sums $\chi(x_v)+(\chi'(x_v)-\chi(x_v))$ less the product of the first summands, is a sum of products of $|\alpha|$ factors of which at least one is a difference; each such product lies in $\mathcal{G}(\langle\omega,\alpha\rangle+\frac{1}{20})$. □

LEAN `aeval_sub_mem_of_gaussFiltration` · used with it: `mem_adicFiltration_of_nonpos` (an element lies in $\mathrm{ad}_\varpi(\theta)$ if it is 0 or $\theta\le 0$), `monomial_mem_gaussFiltration` (a term πx^α with π in $\mathcal{F}(\theta-\langle\omega,\alpha\rangle)$ lies in $\mathrm{Gs}_{\mathcal{F},\omega}(\theta)$), `TwistingData.X_mem_gaussFiltration` ($x_v$ lies in $\mathrm{Gs}_{\mathrm{ad}_\varpi(\cdot),\omega}(\omega(v))$)

Proof of Theorem 1. Throughout, s runs over the letters, and we write

$$x_s^+=x_{\mathrm{rt}_{T,s}(i_0,i_1)},\qquad x_s^-=x_{\mathrm{rt}_{T,s}(i_1,i_0)},$$

and $\omega_s^\pm$, $\alpha_s'^\pm$ for the values of ω and of an exponent α' at these two variables. For every letter the two variables differ, as $i_0\neq i_1$. Put

$$\mathrm{w}^+(\alpha')=q_T\,\alpha_{s_0}'^+ +\sum_{s\neq s_0}P_T^{[s-s_0]}\alpha_s'^+,\qquad \mathrm{w}^-(\alpha')=q_T\,\alpha_{s_0}'^- +\sum_{s\neq s_0}q_T\,P_T^{[s-s_0]}\alpha_s'^-.$$

By (1), $q_T^2\cdot\langle \mathrm{ex}_{T,s_0}^{i_0,i_1}(\cdot),\alpha\rangle=q_T\,\mathrm{w}^+(\alpha)-\mathrm{w}^-(\alpha)$, so the claim is $q_T\,\mathrm{w}^+(\alpha)\le\mathrm{w}^-(\alpha)$. Put $\epsilon=(\omega_{s_0}^+-\omega_{s_0}^-)/q_T$; then $\epsilon>0$. The hypothesis on ω, with $P_T^{[s-s_0]}\ge 5$ and $5\,P_T^{[s-s_0]}\le q_T$ for $s\neq s_0$, gives for these letters

$$\epsilon\,q_T\,P_T^{[s-s_0]}\ \ge\ 2,\qquad \epsilon\,P_T^{[s-s_0]}\ \le\ 0.32,\qquad 0.4\le\omega_s^+-\omega_s^-\le 1.6. \tag{2}$$

Step 1: the elements of the Levi subgroup. For v in W let γ_v be the product of the permutation matrix that exchanges i_0 and i_1 with the elementary matrix $1\mp vE_{i_0i_1}$, the sign being $-$ if s_0 is even. It lies in $\mathrm{Lev}_\tau(W)$ because $\tau(i_0)=\tau(i_1)$. Let $\mathrm{sg}_s=-1$ if s has the parity of s_0 and $\mathrm{sg}_s=+1$ otherwise. A computation with Lemma 3.1(b), separately for even and odd letters, gives

$$[\gamma_v]_T x_s^+=x_s^-+\mathrm{sg}_s\,\iota_s(v)\,x_s^+,\qquad [\gamma_v]_T x_s^-=x_s^+, \tag{3}$$

and $[\gamma_v]_T$ fixes the other variables.

Step 2: two substitutions. Let $\mathcal{A}=\mathcal{O}[x_v][X]$, with one further variable X. Let χ_0 be the homomorphism of $\mathcal{O}$-algebras from $\mathcal{O}[x_v]$ to $\mathcal{A}$ with

$$\chi_0(x_{s_0}^+)=x_{s_0}^-,\qquad \chi_0(x_s^-)=x_s^-\,X^{\mathrm{w}^-(s)}\ \text{ for all } s,\qquad \chi_0(x_v)=x_v\ \text{ otherwise,}$$

where $\mathrm{w}^-(s)=q_T\,P_T^{[s-s_0]}$ is the coefficient of $\alpha_s'^-$ in $\mathrm{w}^-(\alpha')$; and let χ_1 agree with χ_0 except that, for $s\neq s_0$,

$$\chi_1(x_s^+)=x_s^+-\mathrm{sg}_s\,x_s^-,\qquad \chi_1(x_s^-)=x_s^-\,X^{\mathrm{w}^-(s)}-\mathrm{sg}_s\,x_s^+\,X^{P_T^{[s-s_0]}}.$$

For an exponent α' let α'^* be the exponent obtained by moving the value at $\mathrm{rt}_{T,s_0}(i_0,i_1)$ onto $\mathrm{rt}_{T,s_0}(i_1,i_0)$; then $\chi_0(x^{\alpha'})=x^{\alpha'^*}X^{\mathrm{w}^-(\alpha')}$.

On $\mathcal{A}$ consider two families of additive subgroups. An element lies in $\mathcal{G}_-(\theta)$, respectively in $\mathcal{G}_+(\theta)$, if its coefficient at each $x^{\alpha'}X^j$ is divisible by ϖ to the power

$$\begin{aligned}&\max(\lceil\theta-\langle\omega,\alpha'\rangle-\epsilon\,\mathrm{w}^-(\alpha')+\epsilon j\rceil,0) &&\text{for } \mathcal{G}_-(\theta),\\ &\max(\lceil\theta-\langle\omega,\alpha'\rangle+\epsilon\,\mathrm{w}^+(\alpha')-\epsilon j\rceil,0) &&\text{for } \mathcal{G}_+(\theta).\end{aligned}$$

Both are of the kind of Definition 2.7(c), with a weight on the variables x_v and X, so each has the three properties of Definition 2.7(a), and each contains the image of $\mathrm{ad}_\varpi(\theta)$ at level θ. Since $\epsilon q_T = \omega_{s_0}^+ - \omega_{s_0}^-$, moving the exponent does not change the two weights:

$$\langle \omega, \alpha'^* \rangle + \epsilon\, \mathrm{w}^-(\alpha'^*) = \langle \omega, \alpha' \rangle + \epsilon\, \mathrm{w}^-(\alpha'), \qquad \langle \omega, \alpha'^* \rangle - \epsilon\, \mathrm{w}^+(\alpha'^*) = \langle \omega, \alpha' \rangle - \epsilon\, \mathrm{w}^+(\alpha'). \tag{4}$$

Step 3: the degree. Put $\theta_0 = e + \langle \omega, \alpha \rangle$. We apply Lemma 4.2 to Φ, which lies in $\mathrm{Gs}_{\mathrm{ad}_{\varpi}(\cdot),\omega}(\theta_0)$ by the hypothesis for $\gamma = 1$, with $\chi = \chi_0$, $\chi' = \chi_1$ and $\mathcal{G} = \mathcal{G}_-$. A term $x_{v'} X^j$ lies in $\mathcal{G}_-(\theta)$ when $\theta \le \omega(v') + \epsilon(\mathrm{w}^-(v') - j)$, with $\mathrm{w}^-(v')$ the coefficient of v'. Condition (i) holds with equality in each case, for $x_{s_0}^+$ by the choice of ϵ. Condition (ii) concerns the letters $s \ne s_0$: for x_s^+ it reads $\omega_s^+ + \frac{1}{20} \le \omega_s^- + \epsilon\, q_T P_T^{[s-s_0]}$, and for x_s^- it reads $\omega_s^- + \frac{1}{20} \le \omega_s^+ - \epsilon\, P_T^{[s-s_0]}$; both follow from (2). The lemma gives

$$\chi_1(\Phi) \in \mathcal{G}_-(\theta_0), \qquad \chi_1(\Phi) - \chi_0(\Phi) \in \mathcal{G}_-(\theta_0 + \tfrac{1}{20}). \tag{5}$$

Step 4: the zeros. Fix v in W, write $\pi = \iota_{s_0}(v)$, and let t_v be the automorphism $X \mapsto X + \pi$ of $\mathcal{A}$. By hypothesis the polynomial $\Phi_v = [\gamma_v^{-1}]_T \Phi$ lies in $\mathrm{Gs}_{\mathrm{ad}_{\varpi}(\cdot),\omega}(\theta_0)$, and $\mathrm{t}_v(\chi_1(\Phi)) = \chi_v(\Phi_v)$ for the homomorphism $\chi_v = \mathrm{t}_v \circ \chi_1 \circ [\gamma_v]_T$. We apply Lemma 4.2 with $\chi = \chi' = \chi_v$ and $\mathcal{G} = \mathcal{G}_+$, and have to check that $\chi_v(x_v)$ lies in $\mathcal{G}_+(\omega(v))$ for every variable. A term $\pi' x_{v'} X^j$ lies in $\mathcal{G}_+(\theta)$ when ϖ to the power $\max(\lceil \theta - \omega(v') + \epsilon(\mathrm{w}^+(v') - j) \rceil, 0)$ divides π', with $\mathrm{w}^+(v')$ the coefficient of v' in w^+. By Lemma 3.1(a), $\iota_s(v) \equiv \pi^{P_T^{[s-s_0]}}$ and $\pi^{q_T} \equiv \pi$ modulo p, and $(X+\pi)^{P_T^j} \equiv X^{P_T^j} + \pi^{P_T^j}$ modulo p because P_T is a power of p. From (3) and the definition of χ_1 one finds, with $P' = P_T^{[s-s_0]}$:

$$\begin{aligned}
\chi_v(x_{s_0}^+) &= x_{s_0}^-\big((X+\pi)^{q_T} - \pi\big) = x_{s_0}^-\big(X^{q_T} + p\,(\dots)\big), \qquad \chi_v(x_{s_0}^-) = x_{s_0}^-, \\
\chi_v(x_s^+) &= x_s^-\big((X+\pi)^{q_T P'} - \iota_s(v)\big) - \mathrm{sg}_s\, x_s^+\big((X+\pi)^{P'} - \iota_s(v)\big) \\
&= x_s^-\big(X^{q_T P'} + p\,(\dots)\big) - \mathrm{sg}_s\, x_s^+\big(X^{P'} + p\,(\dots)\big) \qquad (s \ne s_0), \\
\chi_v(x_s^-) &= x_s^+ - \mathrm{sg}_s\, x_s^- \qquad (s \ne s_0),
\end{aligned}$$

where the dots are polynomials in X of degree less than that of the leading power, and χ_v fixes the other variables. In the third line $\mathrm{sg}_s^2 = 1$ was used. Now each term is checked against the criterion. The leading terms: $x_{s_0}^- X^{q_T}$ is at level $\omega_{s_0}^- + \epsilon q_T = \omega_{s_0}^+$; $x_s^+ X^{P'}$ is at level ω_s^+; $x_s^- X^{q_T P'}$ is at level $\omega_s^- + \epsilon q_T P' \ge \omega_s^+$ by (2); x_s^+ is at level $\omega_s^+ - \epsilon P' \ge \omega_s^-$ by (2), and x_s^- and $x_{s_0}^-$ are at their own levels. The terms with the factor $p = \varpi^2$ need the power of ϖ to be at most 2: for $x_{s_0}^- X^j$ it is at most $\lceil \omega_{s_0}^+ - \omega_{s_0}^- \rceil \le 2$; for $x_s^- X^j$ at most $\lceil \omega_s^+ - \omega_s^- \rceil \le 2$; for $x_s^+ X^j$ with $j < P'$ at most $\lceil \epsilon P' \rceil \le 1$. So the lemma applies:

$$\mathrm{t}_v(\chi_1(\Phi)) \in \mathcal{G}_+(\theta_0) \qquad \text{for every } v \text{ in } W. \tag{6}$$

Step 5: the count. Let Φ^*, a polynomial in X over $\mathcal{O}$, be the coefficient of x^{α^*} in $\chi_1(\Phi)$. By (5) and (4) the coefficient of X^j in Φ^* is divisible by ϖ to the power $\lceil e + \epsilon(j - \mathrm{w}^-(\alpha)) \rceil$, which is at least $e+1$ for $j > \mathrm{w}^-(\alpha)$. By the second part of (5), the coefficient of $X^{\mathrm{w}^-(\alpha)}$ in Φ^* differs by a multiple of ϖ^{e+1} from the coefficient of $x^{\alpha^*} X^{\mathrm{w}^-(\alpha)}$ in $\chi_0(\Phi)$. A term $x^{\alpha'}$ of Φ contributes to the latter only if $\alpha'^* = \alpha^*$ and $\mathrm{w}^-(\alpha') = \mathrm{w}^-(\alpha)$; then α' and α agree outside the two root variables of s_0, have the

same sum there, and the same value at $\mathrm{rt}_{T,s_0}(i_1,i_0)$, so $\alpha'=\alpha$. Thus that coefficient is the coefficient of x^α in Φ, which ϖ^{e+1} does not divide. Finally, taking the coefficient of x^{α^*} commutes with t_υ, and by (6) and (4) the coefficient of X^j in $\Phi^*(X+\iota_{s_0}(\upsilon))$ is divisible by ϖ to the power $\lceil e+\epsilon(\mathrm{w}^+(\alpha)-j)\rceil \geq e+1$ for $j<\mathrm{w}^+(\alpha)$.

So the image of Φ^* in $(\mathcal{O}/\varpi^{e+1}\mathcal{O})[X]$ is not zero, has degree at most $\mathrm{w}^-(\alpha)$, and is divisible by $(X-\iota_{s_0}(\upsilon))^{\mathrm{w}^+(\alpha)}$ for every υ in W. Lemma 4.1, for the homomorphism from W to $\mathcal{O}/\varpi^{e+1}\mathcal{O}$ induced by ι_{s_0}, gives $\mathrm{w}^+(\alpha)\cdot\#\mathbb{F}_W\leq\mathrm{w}^-(\alpha)$, and $\#\mathbb{F}_W=q_T$. □

LEAN `TwistingData.root_lemma` · `stepanov_count` · `aeval_sub_mem_of_gaussFiltration`

5 The volume of the cyclic interlacing cone

Lemma 5.1 (the determinant of an interlacing pattern)**.** *Let $k\geq 0$, and let $\xi_0\geq\ldots\geq\xi_{k-1}$ and $\eta_0\geq\ldots\geq\eta_{k-1}$ be weakly decreasing sequences of real numbers. The determinant of the $k\times k$ matrix whose (i,i') entry is 1 if $\eta_{i'}<\xi_i$ and 0 otherwise equals 1 if $\eta_i<\xi_i$ for all i and $\xi_{i+1}\leq\eta_i$ for all $i<k-1$, and equals 0 otherwise.*

Proof. By induction on k; for $k=0$ both sides are 1. Let $k\geq 1$. If $\eta_0<\xi_i$ for some $i\geq 1$, then the rows 0 and i consist of ones, since $\eta_{i'}\leq\eta_0<\xi_i\leq\xi_0$; the determinant is 0, and the condition fails because $\xi_1\geq\xi_i>\eta_0$. Otherwise $\xi_i\leq\eta_0$ for all $i\geq 1$, column 0 has at most its entry in row 0 different from zero, and the determinant is that entry, 1 or 0 according as $\eta_0<\xi_0$ or not, times the determinant of the same kind for the sequences without their first members. The induction hypothesis evaluates the latter, and $\xi_1\leq\eta_0$ holds. □

LEAN `det_interlacing_indicator`

Proposition 5.2 (the volume as a signed sum)**.** *Let k and m be natural numbers and $L=m+1$, let ϱ be a real number with $1<\varrho$,* par *a map from $\mathbb{Z}/L$ to the signs, $\delta=(\delta_s)_s$ real numbers, and $\mathcal{C}$ a subset of $\mathbb{R}^{k\times L}$ such that, for every ξ: $\mathrm{St}_{\mathrm{par},\varrho,1}(\xi)\in\mathcal{C}$ if and only if $\xi\in\mathrm{Pc}_{\mathrm{par},\varrho}$. For a permutation σ' of the indices let σ'^* be the linear map $f\mapsto f\circ\sigma'$ of $\mathbb{R}^k$ and $\mathrm{dt}(\sigma')=|\det(1-\varrho^{-L}\sigma'^*)|$. Then*

$$\mathrm{vol}(\mathcal{C}\cap\mathrm{Sl}_k(\delta))=\frac{\mathrm{dt}(\mathrm{id})}{k!}\left(\sum_{\sigma'}\frac{\mathrm{sgn}(\sigma')}{\mathrm{dt}(\sigma')}\right)\mathrm{vol}(\mathrm{Sl}_k(\delta)),$$

the sum being over all permutations of the indices. (The formal statement is for the letters $0,\ldots,m$ with addition modulo $m+1$.)

Proof. For a tuple $\sigma=(\sigma_s)_s$ of permutations write $S_\sigma=\mathrm{St}_{\mathrm{par},\varrho,\sigma}$, and S_1 for the identity permutations. The set $\mathrm{Sl}_k(\delta)$ is bounded, and so is its intersection with any set; all volumes below are finite.

The determinant of S_σ. Up to the signs $\mathrm{par}(s)$, S_σ is $\varrho\cdot\mathrm{Sh}\circ(1-\mathrm{N})$, where $(\mathrm{Sh}\,\xi)(i,s)=\xi(i,s+1)$ and $(\mathrm{N}\xi)(\cdot,s)=N_s\,\xi(\cdot,s-1)$ with $N_s=\varrho^{-1}\sigma_{s-1}^*$. For such a cyclic operator,

$$\det(1-\mathrm{N})=\det(1-N_{L-1}\cdots N_1N_0).$$

Indeed, order the letters as $0,\ldots,L-1$ and let El be the operator with

$$(\mathrm{El}\eta')(\cdot,s)=\sum_{s'\leq s}N_sN_{s-1}\cdots N_{s'+1}\,\eta'(\cdot,s');$$

then

$$(\mathrm{El}(1-\mathrm{N})\xi)(\cdot,s)=\xi(\cdot,s)-N_s\cdots N_0\,\xi(\cdot,L-1).$$

The operator El has determinant 1, and $\mathrm{El}(1-\mathrm{N})$ is triangular with the diagonal blocks $1,\dots,1$ and $1-N_{L-1}\cdots N_0$. Hence

$$|\det S_\sigma| = \varrho^{kL}\cdot \mathrm{dt}(\sigma_0\sigma_1\cdots\sigma_{L-1}), \tag{7}$$

because $N_{L-1}\cdots N_0 = \varrho^{-L}(\sigma_{L-1}\sigma_0\cdots\sigma_{L-2})^*$ and dt takes the same value on conjugate permutations. As $\varrho>1$, the right side is not zero: $\mathrm{dt}(\sigma')$ is the product of the numbers $1-\varrho^{-Ll}$ over the cycles of σ', of lengths l.

Two families of sets. Let $\mathcal{W}$ be the set of the ξ such that the column sums $\sum_i (S_1\xi)(i,s)$ lie in $[\delta_s-1,\delta_s]$ for every letter. The column sums of $S_\sigma\xi$ are those of $S_1\xi$, the permutations only reordering the summands. For a tuple σ let $\mathcal{W}_\sigma$ be the set of the ξ in $\mathcal{W}$ with all coordinates of $S_\sigma\xi$ positive. It is the preimage under S_σ of the slab without its walls, which have measure zero, so by (7) $\mathrm{vol}(\mathcal{W}_\sigma) = \varrho^{-kL}\,\mathrm{dt}(\sigma_0\cdots\sigma_{L-1})^{-1}\,\mathrm{vol}(\mathrm{Sl}_k(\delta))$. Let Υ be the set of the ξ in $\mathrm{Pc}_{\mathrm{par},\varrho}$ with all coordinates of $S_1\xi$ positive. By the hypothesis on $\mathcal{C}$, $\mathcal{W}\cap\Upsilon$ is the preimage under S_1 of $\mathcal{C}\cap\mathrm{Sl}_k(\delta)$ without the walls, so $\mathrm{vol}(\mathcal{W}\cap\Upsilon) = \varrho^{-kL}\,\mathrm{dt}(\mathrm{id})^{-1}\,\mathrm{vol}(\mathcal{C}\cap\mathrm{Sl}_k(\delta))$. For a tuple $\tau' = (\tau'_s)_s$ of permutations let $\xi\circ\tau'$ be the vector with coordinates $\xi(\tau'_s(i),s)$; the map $\xi\mapsto\xi\circ\tau'$ preserves volume and maps $\mathcal{W}$ to itself.

A pointwise identity. We claim that for every ξ

$$\#\{\tau' :\ \xi\circ\tau'\in\Upsilon\} \;=\; \sum_\sigma \mathrm{sgn}(\sigma_0\cdots\sigma_{L-1})\cdot\big[\text{all coordinates of } S_\sigma\xi \text{ are positive}\big]. \tag{8}$$

The right side is the product over the letters s of

$$\sum_{\sigma_s}\mathrm{sgn}(\sigma_s)\prod_i\big[\mathrm{par}(s)\big(\varrho\,\xi(i,s+1)-\xi(\sigma_s(i),s)\big)>0\big],$$

that is, of the determinants of the matrices A_s whose entry at (i,i') is

$$\big[\mathrm{par}(s)\big(\varrho\,\xi(i,s+1)-\xi(i',s)\big)>0\big].$$

Reorder the rows of A_s so that $\xi(\cdot,s+1)$ becomes weakly decreasing, and its columns so that $\xi(\cdot,s)$ does. This multiplies $\det A_s$ by the signs of two permutations, and in the product over the circle of letters each sign occurs twice. After the reordering Lemma 5.1 applies, to the transposed matrix when $\mathrm{par}(s)=-$: the determinant is 1 if the decreasing rearrangements $\xi^\downarrow$ of the columns satisfy, for all i,

$$\begin{aligned}\varrho\,\xi^\downarrow(i,s+1) > \xi^\downarrow(i,s) \ge \varrho\,\xi^\downarrow(i+1,s+1) \qquad &\text{when } \mathrm{par}(s)=+,\\ \xi^\downarrow(i,s) > \varrho\,\xi^\downarrow(i,s+1) \ge \xi^\downarrow(i+1,s) \qquad &\text{when } \mathrm{par}(s)=-,\end{aligned}$$

and it is 0 otherwise. These are the conditions that define Υ, at the letter s. So the right side of (8) is 1 if $\xi^\downarrow\in\Upsilon$ and 0 otherwise. On the left, the columns of a vector in Υ are strictly decreasing, by the same inequalities; so $\xi\circ\tau'\in\Upsilon$ forces $\xi\circ\tau' = \xi^\downarrow$ and determines τ', the entries of each column being distinct. Hence the left side is also 1 if $\xi^\downarrow\in\Upsilon$ and 0 otherwise.

Integration. Integrate (8) over $\mathcal{W}$. The left side gives $(k!)^L\,\mathrm{vol}(\mathcal{W}\cap\Upsilon)$, the right side $\sum_\sigma \mathrm{sgn}(\sigma_0\cdots\sigma_{L-1})\,\mathrm{vol}(\mathcal{W}_\sigma)$. Each permutation σ' is the product $\sigma_0\cdots\sigma_{L-1}$ for $(k!)^{L-1}$ tuples. Inserting the two volumes and cancelling $\varrho^{-kL}(k!)^{L-1}$ gives the formula. □

LEAN `volume_inter_slab_eq_signed_sum` · `isBounded_inter_slab` (the intersection of any set with a slab is bounded)

Proof of Theorem 2. By Proposition 5.2, $\mathrm{vol}(\mathcal{C} \cap \mathrm{Sl}_k(\delta))$ is $\mathrm{vol}(\mathrm{Sl}_k(\delta))$ times a number that depends only on k and on $q = \varrho^L$. We evaluate this number in the case of one letter, $L = 1$, with the sign $+$, the real number q in the place of ϱ, $\delta = 1$ and $\mathcal{C} = \mathrm{Pc}_{+,q}$, the set of the ξ in $\mathbb{R}^k$ with $q\,\xi(i+1) \le \xi(i)$ for all i; the hypothesis holds because $S_1\xi = (q-1)\xi$ and $\mathcal{C}$ is a cone. Put $\mathrm{f}_l = (q^{l+1}-1)/(q-1)$ and let G be the linear endomorphism of $\mathbb{R}^k$ with $(G\xi)(l) = \mathrm{f}_l\,(\xi(l) - q\,\xi(l+1))$, where $\xi(k) = 0$. It is triangular with the diagonal entries f_l, so $\det G = \prod_{l=0}^{k-1} \mathrm{f}_l > 0$. It preserves the sum of the coordinates, because $\mathrm{f}_l - q\,\mathrm{f}_{l-1} = 1$. And $G\xi$ has all coordinates ≥ 0 if and only if $\xi(l) \ge q\,\xi(l+1)$ for all l, which implies $\xi \ge 0$ by descending induction. Hence the preimage of $\mathrm{Sl}_k(1)$ under G is $\mathcal{C} \cap \mathrm{Sl}_k(1)$, and $\mathrm{vol}(\mathcal{C} \cap \mathrm{Sl}_k(1)) = \mathrm{vol}(\mathrm{Sl}_k(1))/\det G$. The volume of $\mathrm{Sl}_k(1)$ is finite and positive, as the set contains the cube $[0, 1/k]^k$; only here is a volume cancelled. So the number is $1/\prod_{l=1}^{k} \frac{q^l-1}{q-1}$, and the theorem follows, also when the slab $\mathrm{Sl}_k(\delta)$ has volume zero. □

LEAN `prod_mul_volume_inter_slab`

The proof evaluates the signed sum of Proposition 5.2 without computing it: the identity $\frac{\mathrm{dt(id)}}{k!} \sum_{\sigma'} \mathrm{sgn}(\sigma')/\mathrm{dt}(\sigma') = \prod_{l=1}^{k} \frac{q-1}{q^l-1}$ is obtained from the geometry of one letter.

6 Two dimension bounds modulo p

Lemma 6.1 (the number of flags)**.** *Let $\mathbb{F}$ be a finite field and $k \ge 0$. The index in $\mathrm{GL}_k(\mathbb{F})$ of the subgroup of the lower triangular matrices is at most $\prod_{l=1}^{k} ((\#\mathbb{F})^l - 1)/(\#\mathbb{F} - 1)$.*

Proof. The subgroup has $(\#\mathbb{F}-1)^k (\#\mathbb{F})^{k(k-1)/2}$ elements, the lower triangular matrices with units on the diagonal, and Input 4 gives the order of the group; the quotient is the product. □

LEAN `index_borel_le` · `card_GL_eq_prod`

Lemma 6.2 (the excess of a step)**.** *In Setting 2.2 with $\mathcal{O}$ a field, let i_0, i_1 be indices, s_0 a letter and ξ a vector in $\mathbb{R}^{k\times L}$. Put $\eta = \mathrm{St}_{\mathrm{par}_T, P_T, 1}(\xi)$. Then*

$$\langle \mathrm{ex}^{i_0,i_1}_{T,s_0}(\cdot), \eta \rangle = \Big(1 - \frac{1}{q_T}\Big) \cdot \Big(\eta(\mathrm{rt}_{T,s_0}(i_0,i_1)) - (\xi(i_0,s_0) - \xi(i_1,s_0))\Big).$$

Proof. Let $\mathrm{df}_s = \xi(i_0,s) - \xi(i_1,s)$, and write η^+_s, η^-_s for the values of η at $\mathrm{rt}_{T,s}(i_0,i_1)$ and $\mathrm{rt}_{T,s}(i_1,i_0)$. For every letter, even or odd, the definition of the step gives $\eta^+_s - \eta^-_s = \mathrm{df}_s - P_T\,\mathrm{df}_{s+1}$. By the definition of the excess weight, in which the root variable of s_0 has the coefficient $P_T^L/q_T = 1$, the excess of η at s_0 is

$$\Big(1 - \frac{1}{q_T}\Big)\eta^+_{s_0} + \frac{1}{q_T} \sum_s P_T^{[s-s_0]}(\eta^+_s - \eta^-_s) = \Big(1 - \frac{1}{q_T}\Big)\eta^+_{s_0} + \frac{1}{q_T} \sum_s P_T^{[s-s_0]}(\mathrm{df}_s - P_T\,\mathrm{df}_{s+1}),$$

and the last sum, taken around the circle from s_0, telescopes to

$$\mathrm{df}_{s_0} - P_T^L\,\mathrm{df}_{s_0} = (1 - q_T)\,\mathrm{df}_{s_0}.$$ □

LEAN `TwistingData.excessWeight_dotProduct_permStep`

By this lemma, $\mathrm{St}_{\mathrm{par}_T,P_T,1}(\xi)$ lies in $\mathrm{Ec}_T(i_1;0)$ if and only if ξ lies in $\mathrm{Pc}_{\mathrm{par}_T,P_T}$: the set $\mathcal{C} = \mathrm{Ec}_T(i_1;0)$ satisfies the hypothesis of Theorem 2 for $\mathrm{par} = \mathrm{par}_T$ and $\varrho = P_T$, whatever i_1 is.

Lemma 6.3 (a point deep inside the cone)**.** *In Setting 2.2 with $\mathcal{O}$ a field, let i_0, i_1 be indices with $i_0 + 1 = i_1$. There is a vector σ° in $\mathrm{Ec}_T(i_1;-1)$ with $0 \le \sigma^\circ(v) \le P_T^{3i_1}$ for every variable v.*

Proof. Let

$$\mathrm{g}_i = 2\,P_T^{2(i_0-i)} \quad \text{for } i \le i_0 \qquad \text{and} \qquad \mathrm{g}_i = 0 \quad \text{for } i > i_0,$$

so that $\mathrm{g}_i = P_T^2 \mathrm{g}_{i+1} + 2\,[i+1 = i_1]$. Let $\xi(i,s) = \mathrm{g}_i$ for even s and $\xi(i,s) = P_T \mathrm{g}_i$ for odd s, and

$$\sigma^\circ = \mathrm{St}_{\mathrm{par}_T,P_T,1}(\xi).$$

Then $\sigma^\circ(i,s) = (P_T^2 - 1)\mathrm{g}_i$ for even s and 0 for odd s, which lies between 0 and $P_T^{3i_1}$. By Lemma 6.2, the excess of σ° at s_0 for the indices i, $i+1$ is $(1 - 1/q_T)$ times

$$\begin{aligned} (P_T^2 - 1)\mathrm{g}_{i+1} - (\mathrm{g}_i - \mathrm{g}_{i+1}) = -2\,[i+1 = i_1] \qquad & \text{if } s_0 \text{ is even,} \\ -P_T(\mathrm{g}_i - \mathrm{g}_{i+1}) \le -2\,[i+1 = i_1] \qquad & \text{if } s_0 \text{ is odd;} \end{aligned}$$

and $2(1 - 1/q_T) \ge 1$. □

LEAN `TwistingData.exists_mem_excessCone_mem_Icc`

Lemma 6.4 (the volume of a slice of a simplex)**.** *Let $n' \ge 0$ be a natural number and $\delta \le \delta'$ real numbers. The set of the η in $\mathbb{R}^{n'+1}$ with all coordinates ≥ 0 and $\delta - 1 \le \sum_i \eta_i \le \delta$ has volume at most $|{\delta'}^{n'}/n'!|$.*

Proof. If $\delta' < 0$ the set is empty. Otherwise argue by induction on n'. For $n' = 0$ the set lies in an interval of length 1. For the step, the slice at a value η_0 of the first coordinate is empty unless $0 \le \eta_0 \le \delta \le \delta'$, and is then a set of the same kind in one dimension less, for $\delta - \eta_0 \le \delta' - \eta_0$; and $\int_0^{\delta'} (\delta' - \eta_0)^{n'-1}/(n'-1)!\, d\eta_0 = {\delta'}^{n'}/n'!$. □

LEAN `volume_sum_mem_Icc_le`

Proposition 6.5 (forms obeying the excess condition, against a volume)**.** *In Setting 2.2, let the ring of T be a field $\mathbb{F}$ of characteristic p, let i_0, i_1 be indices with $i_0 + 1 = i_1$, n a natural number and h a real number. Write Lev° for the Levi subgroup $\mathrm{Lev}_{\tau_{k,i_1}(\cdot)}(W)$, and Par° for the parahoric subgroup of the labelling $i \mapsto i$, and let $\delta_s = n + k + 1$ for every letter s. Then*

$$\dim_{\mathbb{F}} \mathrm{Ex}_{T,n}^{i_0,i_1}(0;h) \;\le\; [\mathrm{Lev}^\circ : \mathrm{Lev}^\circ \cap \mathrm{Par}^\circ] \cdot \mathrm{vol}\Big(\mathrm{Ec}_T(i_1; h + L(k+1)) \cap \mathrm{Sl}_k(\delta)\Big).$$

Proof. Write $\Xi = \mathrm{Ex}_{T,n}^{i_0,i_1}(0;h)$ and $\mathcal{C}$ for the set on the right; $\mathcal{C}$ is bounded and convex. The number of the points of $\mathcal{C}$ with coordinates in $\frac{1}{Q}\mathbb{Z}$, divided by Q^{kL}, tends to $\mathrm{vol}(\mathcal{C})$ as the natural number Q grows, a bounded convex set having a boundary of measure zero. So it suffices to show, for $Q = q_T^l$ with $l \ge 1$, that

$$Q^{kL} \cdot \dim_{\mathbb{F}} \Xi \;\le\; [\mathrm{Lev}^\circ : \mathrm{Lev}^\circ \cap \mathrm{Par}^\circ] \cdot \#(\mathcal{C} \cap \tfrac{1}{Q}\mathbb{Z}^{k\times L}). \tag{9}$$

The ring of T being a field of characteristic p, we have $p = 0$ and $\varpi = 0$ in $\mathbb{F}$, and $\iota_{s+j}(v) = \iota_s(v)^{P_T^j}$ by Lemma 3.1(a); in particular $\iota_s(v)^Q = \iota_s(v)$. Hence $[\gamma]_T$

commutes with the substitution $x_v \mapsto x_v^Q$, and for a form F the polynomial $F(x^Q)$ is sent by $[\gamma]_T$ to $(\rho_{T,n}(\gamma)F)(x^Q)$ (Lemma 3.1(c)).

Amplification. Let $\mathcal{U}$ be the set of the functions ν from the variables to $\{0, \dots, Q-1\}$. For ν in $\mathcal{U}$ let $\hat{\nu}$ be the exponent with

$$\hat{\nu}(i,s) = \nu(i,s) + Q\,[i = i_0]\left(k - \left\lfloor \sum_{i'} \nu(i',s)/Q \right\rfloor\right);$$

it is congruent to ν modulo Q, and its degree in each letter lies in $[kQ, kQ+Q)$. Send a tuple $(F_\nu)_{\nu\in\mathcal{U}}$ of elements of Ξ to the polynomial $\Phi = \sum_\nu x^{\hat{\nu}} F_\nu(x^Q)$. The exponents of the ν-th summand are congruent to ν modulo Q, so the coefficient of $x^{Q\beta+\hat{\nu}}$ in Φ is $F_\nu(\beta)$, and the map is injective. Let Γ be a set of representatives of the right cosets $(\mathrm{Lev}^\circ \cap \mathrm{Par}^\circ)\gamma$ in Lev°. We show that the linear map sending $(F_\nu)_\nu$ to the family of the coefficients of $x^{Q\eta}$ in $[\gamma]_T\Phi$, for γ in Γ and η in $\mathcal{C} \cap \frac{1}{Q}\mathbb{Z}^{k\times L}$, is injective; comparing dimensions, this is (9).

A least term. Let ω be the weight with $\omega(i,s) = i$ for even s and $\omega(i,s) = -i$ for odd s. Suppose $\Phi \neq 0$. Among the pairs of a γ in Γ and an exponent α with nonzero coefficient in $[\gamma]_T\Phi$, choose one, (γ_0, α), with $\langle\omega, \alpha\rangle$ least. We show that α/Q lies in $\mathcal{C}$; then the coefficient of x^α in $[\gamma_0]_T\Phi$ is one of the coordinates of the image and is not zero.

By the commutation above,

$$[\gamma_0]_T\Phi = \sum_\nu ([\gamma_0]_T x^{\hat{\nu}}) \cdot (\rho_{T,n}(\gamma_0)F_\nu)(x^Q),$$

so $\alpha = Q\beta + \alpha''$ for some ν, a family β with $(\rho_{T,n}(\gamma_0)F_\nu)(\beta) \neq 0$, and an exponent α'' of $[\gamma_0]_T x^{\hat{\nu}}$. The action preserves the degree in each letter, so the degree of α'' in each letter lies in $[kQ, kQ+Q]$. Hence α/Q has coordinates ≥ 0 and its sums over the indices lie in $[n+k, n+k+1]$: it lies in the slab.

The excess at (i_0, i_1). As F_ν lies in Ξ and γ_0 in Lev°, the family β has excess at most h at every letter. The excess of α'' is at most its degree, at most $L(kQ+Q)$, by Lemma 3.3. So the excess of α/Q is at most $h + L(k+1)$.

The excess at the other pairs. Let i, $i+1$ be indices with $i+1 \neq i_1$, and s_0 a letter. We apply Theorem 1 to the indices i, $i+1$, the labelling $\tau_{k,i_1}(\cdot)$, which takes the same value at both, the weight ω, for which the differences of the hypothesis are equal to 1, the polynomial $[\gamma_0]_T\Phi$, the exponent α and $e = 0$. Its coefficient at x^α is not zero, that is, not divisible by $\varpi = 0$. For γ in Lev° write $\gamma^{-1}\gamma_0 = \gamma'\gamma_1$ with γ' in $\mathrm{Lev}^\circ \cap \mathrm{Par}^\circ$ and γ_1 in Γ. All exponents of $[\gamma_1]_T\Phi$ have weight at least $\langle\omega, \alpha\rangle$, by the choice of the pair. By Lemma 3.2, for the labelling $i \mapsto i$, and since $p = 0$ in $\mathbb{F}$, the element γ' sends a variable $x_{i',s}$ to a combination of the variables $x_{l,s}$ with $l \geq i'$ if s is even and with $l \leq i'$ if s is odd: of variables of weight at least $\omega(i',s)$. So all exponents of $[\gamma^{-1}]_T[\gamma_0]_T\Phi$ have weight at least $\langle\omega, \alpha\rangle$, which is the hypothesis of Theorem 1 for $\varpi = 0$. The theorem gives that the excess of α at s_0 for the indices i, $i+1$ is at most 0. □

LEAN `TwistingData.finrank_excessCond_le_volume` · `TwistingData.root_lemma` · `TwistingData.p_dvd_letterMatrix_of_mem_parahoric` · `TwistingData.excessWeight_mem_Icc`

Proof of Theorem 3(a). There is an index, so $k \geq 1$. Of the size conditions we use $\kappa \leq 1$, $1 \leq h$,

$$\frac{2(L+2)k}{h} \leq \kappa \qquad \text{and} \qquad (h + L(k+1))\,k\,P_T^{3i_1} + k + 2 \leq \frac{n}{h}. \tag{10}$$

Let σ° be the vector of Lemma 6.3 and $\sigma^\bullet = (h + L(k+1))\,\sigma^\circ$. The conditions that define the cones are linear, so the translation by $\sigma^\bullet$ maps $\mathrm{Ec}_T(i_1; h + L(k+1)) \cap \mathrm{Sl}_k(\delta)$, with $\delta_s = n + k + 1$, into $\mathrm{Ec}_T(i_1; 0) \cap \mathrm{Sl}_k(\delta')$ with $\delta'_s = n + k + 1 + \sum_i \sigma^\bullet(i,s)$. Let Lev° and Par° be as in Proposition 6.5. Both lie in the parahoric subgroup $\mathrm{Par}_{\tau_{k,i_1}(\cdot)}(W)$, so its index times $[\mathrm{Lev}^\circ : \mathrm{Lev}^\circ \cap \mathrm{Par}^\circ]$ is at most $[\mathrm{GL}_k(W) : \mathrm{Par}^\circ]$. Now Par° is the preimage of the group of the lower triangular matrices under the reduction map from $\mathrm{GL}_k(W)$ to $\mathrm{GL}_k(\mathbb{F}_W)$. So its index is the index of the lower triangular matrices of the image in the image, which is at most their index in $\mathrm{GL}_k(\mathbb{F}_W)$. (The two are equal, the reduction map being surjective: a matrix over the local ring W is invertible when its reduction is. The formal proof uses the inequality only.) By Lemma 6.1 and $\#\mathbb{F}_W = q_T$, that index is at most the product

$$\prod_{l=1}^{k} \frac{q_T^l - 1}{q_T - 1}.$$

By Theorem 2, with the remark after Lemma 6.2, this product times the volume of $\mathrm{Ec}_T(i_1; 0) \cap \mathrm{Sl}_k(\delta')$ is $\mathrm{vol}(\mathrm{Sl}_k(\delta'))$. With Proposition 6.5 we get

$$\left[\mathrm{GL}_k(W) : \mathrm{Par}_{\tau_{k,i_1}(\cdot)}(W)\right] \cdot \dim_{\mathbb{F}} \mathrm{Ex}_{T,n}^{i_0,i_1}(0; h) \;\le\; \mathrm{vol}(\mathrm{Sl}_k(\delta')).$$

The slab is contained in the product over the letters of the sets of Lemma 6.4 in $\mathbb{R}^k$. By the second condition of (10) and $\sigma^\circ \le P_T^{3i_1}$, $\delta'_s \le n + n/h \le n\exp(1/h)$. So the volume is at most $\left((n\exp(1/h))^{k-1}/(k-1)!\right)^L$, and

$$\frac{n^{k-1}}{(k-1)!} \;\le\; \binom{n+k-1}{k-1} = \#\mathrm{Mon}_{k,n}.$$

Finally $\exp((k-1)L/h) \le 1 + \kappa$: the exponent is at most $\kappa/2 \le \frac{1}{2}$ by the first condition of (10), and $\exp(\theta) \le 1 + 2\theta$ for $0 \le \theta \le \frac{1}{2}$. □

LEAN `TwistingData.index_mul_finrank_excessCond_le`

Proposition 6.6 (a nonzero deep form has a costly monomial)**.** *In Setting 2.2, let the ring of T be a field $\mathbb{F}$ of characteristic p, let n, $\mathfrak{t}$, j be natural numbers with $2\mathfrak{t} \le k$ and $2\mathfrak{t} < L$, and let F be a nonzero form in $\mathrm{Dp}_{T,\mathfrak{t}}^{k,n}(j)$. Then there is a family β with $F(\beta) \ne 0$ and*

$$j \;\le\; \sum_{v=(i,s)} \beta_s(i)\,\frac{\mathrm{dp}_{T,\mathfrak{t}}(v)}{P_T^{e_i}}, \qquad e_i = 2\mathfrak{t} \;\; \textit{if } i \ge 2\mathfrak{t}, \quad e_i = 1 \;\; \textit{if } i < 2\mathfrak{t}.$$

Proof. Let $\mathbb{F}'$ be the field of rational functions over $\mathbb{F}$ in the variables that are not deep, and regard the polynomial of F as a polynomial Φ over $\mathbb{F}'$ in the deep variables; it is not zero. We apply Input 3 over $\mathbb{F}'$, with $\mathcal{V}$ the set of the deep variables and $\varkappa(i,s) = P_T^{e_i}$. Then $\prod_v \varkappa(v) = P_T^{(L/2)\sum_i e_i} = q_T^{\mathfrak{t}(k-2\mathfrak{t}+1)}$ by Lemma 3.4. A monomial of Φ of cost at least j comes from a family β with $F(\beta) \ne 0$ whose deep part it is, and the cost is the sum of the statement. It remains to give the points and the polynomials.

The points. Let $\mathcal{K}$ be the set of the pairs (l, l') of indices with $l < \mathfrak{t}$ and either $l' = l + \mathfrak{t}$ or $l' \ge 2\mathfrak{t}$; it has $\mathfrak{t}(k - 2\mathfrak{t} + 1)$ elements. Let $\mathcal{S}$ be the set of the functions $\mathfrak{s}$ from $\mathcal{K}$ to $\mathbb{F}_W$, so that $\#\mathcal{S} = q_T^{\mathfrak{t}(k-2\mathfrak{t}+1)}$. For $\mathfrak{s}$ in $\mathcal{S}$ choose a lift $\tilde{\mathfrak{s}}_{l,l'}$ in W of each value and let $\gamma_{\mathfrak{s}} = 1 + \sum_{(l,l')\in\mathcal{K}} \tilde{\mathfrak{s}}_{l,l'} E_{ll'}$, an element of $\mathrm{GL}_k(W)$ with inverse $1 - \sum \tilde{\mathfrak{s}}_{l,l'} E_{ll'}$,

the rows and the columns that occur being disjoint. By Lemma 3.1(b), $[\gamma_{\mathfrak{s}}^{-1}]_T$ fixes the variables that are not deep and sends a deep variable x_v to $x_v - \upsilon_{\mathfrak{s}}(v)$, where

$$\upsilon_{\mathfrak{s}}(l', s) = \sum_{l} \iota_s(\tilde{\mathfrak{s}}_{l,l'})\, x_{l,s} \quad (s \text{ even}), \qquad \upsilon_{\mathfrak{s}}(l, s) = -\sum_{l'} \iota_s(\tilde{\mathfrak{s}}_{l,l'})\, x_{l',s} \quad (s \text{ odd}),$$

the sums being over the pairs in $\mathcal{K}$ with the given member; these are elements of $\mathbb{F}'$, the variables on the right not being deep. As F lies in $\mathrm{Dp}^{k,n}_{T,\mathfrak{t}}(j)$, every monomial of $\rho_{T,n}(\gamma_{\mathfrak{s}})F$ has at least j deep variables, counted with multiplicity; applying $[\gamma_{\mathfrak{s}}^{-1}]_T$ and Lemma 3.1(c), Φ is a combination of products of at least j factors $x_v - \upsilon_{\mathfrak{s}}(v)$. So Φ lies in the j-th power of the ideal of the point $\upsilon_{\mathfrak{s}}$.

The polynomials. Fix a letter s'. For a pair (l, l') in $\mathcal{K}$ choose a natural number $j_{l,l'}$: it is 0 if $l' = l + \mathfrak{t}$; if $l' \geq 2\mathfrak{t}$ it is the one of $2l$, $2l+1$ for which the letter $s' - j_{l,l'}$ is even. Let $v_{l,l'}$ be the deep variable $(l', s' - j_{l,l'})$ if that letter is even, and (l, s') if $j_{l,l'} = 0$ and s' is odd. In each case $j_{l,l'} < e$, where $P_T^e = \varkappa(v_{l,l'})$: for $l' \geq 2\mathfrak{t}$ we have $j_{l,l'} \leq 2\mathfrak{t} - 1$ and $e = 2\mathfrak{t}$. By Lemma 3.1(a) and $p = 0$,

$$\upsilon_{\mathfrak{s}}(v_{l,l'})^{P_T^{j_{l,l'}}} = \sum_{(l_1,l_1')\in\mathcal{K}} C_{(l,l'),(l_1,l_1')} \cdot \iota_{s'}(\tilde{\mathfrak{s}}_{l_1,l_1'}),$$

with a matrix C over $\mathbb{F}'$ that does not depend on $\mathfrak{s}$: its entries are, up to sign, the $P_T^{j_{l,l'}}$-th powers of the variables that occur in the formula for $\upsilon_{\mathfrak{s}}(v_{l,l'})$. We claim that $\det C \neq 0$. If s' is even, C is block diagonal for the second members: the block of the pairs $(l, l+\mathfrak{t})$ is diagonal with the entries $x_{l,s'}$, and the block of each $l' \geq 2\mathfrak{t}$ is the matrix $(x_{l_1,s'-2l}^{P_T^{2l}})_{l,l_1}$, whose entries are powers of $\mathfrak{t}^2$ distinct variables, the letters $s' - 2l$ being distinct because $2\mathfrak{t} < L$. If s' is odd, order the pairs $(l, l+\mathfrak{t})$ first: C is block triangular, with a diagonal block of entries $-x_{l+\mathfrak{t},s'}$ and blocks $(x_{l_1,s'-2l-1}^{P_T^{2l+1}})_{l,l_1}$. A matrix whose entries are powers of distinct variables has a nonzero determinant. So each $\iota_{s'}(\tilde{\mathfrak{s}}_{l_1,l_1'})$ is a combination over $\mathbb{F}'$, with coefficients independent of $\mathfrak{s}$, of the values at $\upsilon_{\mathfrak{s}}$ of the monomials $x_{v_{l,l'}}^{P_T^{j_{l,l'}}}$, which cost $P_T^{j_{l,l'}}/\varkappa(v_{l,l'}) < 1$.

Now let v be a deep variable of the letter s, and $P_T^e = \varkappa(v)$. By Lemma 3.1(a), $\upsilon_{\mathfrak{s}}(v)^{\varkappa(v)}$ is a combination, with coefficients in $\mathbb{F}'$ independent of $\mathfrak{s}$, of the $\iota_{s+e}(\tilde{\mathfrak{s}}_{l_1,l_1'})$; by the last paragraph for $s' = s + e$ it is the value at $\upsilon_{\mathfrak{s}}$ of a polynomial Φ_v whose monomials cost less than 1. The same paragraph, for any letter s', shows that the point $\upsilon_{\mathfrak{s}}$ determines the $\iota_{s'}(\tilde{\mathfrak{s}}_{l_1,l_1'})$, hence $\mathfrak{s}$: two elements of W with the same image under $\iota_{s'}$ differ by a non-unit. So the points are pairwise distinct, and Input 3 applies. □

LEAN `TwistingData.exists_le_weight_of_mem_deepForms` · `exists_le_cost_of_vanishing` · `TwistingData.sum_deepIndicator`

Proof of Theorem 3(b). From the size conditions: $0 < \mathfrak{t}$, $2\mathfrak{t} \leq k$, $1 \leq y$, $n < P_T^\lambda$ and $L = 2\lambda$, and $P_T^{3\mathfrak{t}} \leq n$ from the ninth condition; so $3\mathfrak{t} < \lambda$ and $2\mathfrak{t} < L$. For a family β let $\mathrm{cost}(\beta)$ be the sum of Proposition 6.6. By that proposition, a form in $\mathrm{Dp}^{k,n}_{T,\mathfrak{t}}(j)$ that vanishes at all β with $j \leq \mathrm{cost}(\beta)$ is zero; so $\dim_{\mathbb{F}} \mathrm{Dp}^{k,n}_{T,\mathfrak{t}}(j)$ is at most the number of these β. Summing over $j < j_0$, each β is counted at most $\mathrm{cost}(\beta) + 1$ times:

$$\sum_{j<j_0} \dim_{\mathbb{F}} \mathrm{Dp}^{k,n}_{T,\mathfrak{t}}(j) \;\leq\; \sum_{\beta} (1 + \mathrm{cost}(\beta)).$$

The cyclic shifts of the indices permute the families. Averaging over them replaces $\beta_s(i)$ by n/k, so, by Lemma 3.4,

$$\sum_{\beta} \operatorname{cost}(\beta) = \left(\#\mathrm{Mon}_{k,n}\right)^{L} \cdot \frac{n}{k} \cdot \frac{L}{2} \cdot \left(\frac{k-2\mathfrak{t}}{P_T^{2\mathfrak{t}}} + \frac{2\mathfrak{t}}{P_T} \right) \;\le\; \left(\#\mathrm{Mon}_{k,n}\right)^{L} \cdot L\, n \left(\frac{1}{P_T^{2\mathfrak{t}}} + \frac{\mathfrak{t}^2}{k\, P_T} \right),$$

which is at most $z\,(\#\mathrm{Mon}_{k,n})^{L}$ by the tenth condition. □

LEAN `TwistingData.sum_finrank_deepForms_le`

7 Conjugation-invariant means

In this section the variables are the entries $x_{i,i'}$ of one generic matrix, and for a finite set $\mathcal{V}$ of variables $\mathrm{Mon}_{\mathcal{V}}(n)$ is the set of the exponents of degree n on $\mathcal{V}$.

Lemma 7.1 (density of the points where a form is a unit)**.** *Let W be a local ring, $\mathcal{O}$ a ring, $\chi\colon W \to \mathcal{O}$ a ring homomorphism, $\mathcal{V}$ a finite set, and n, n' natural numbers. Let Φ_0 be a homogeneous polynomial of degree n' in $W[x_{\mathcal{V}}]$, with $n + n' \le \#\mathbb{F}_W$, and let v_0 be a point of $W^{\mathcal{V}}$ at which Φ_0 takes a unit value. Let M_1 be an $\mathcal{O}$-submodule of $\mathcal{O}^{\mathrm{Mon}_{\mathcal{V}}(n)}$ that contains the vector $(\chi(v^{\alpha}))_{\alpha}$ for every point v of $W^{\mathcal{V}}$ with $\Phi_0(v)$ a unit. Then $M_1 = \mathcal{O}^{\mathrm{Mon}_{\mathcal{V}}(n)}$.*

Proof. Let M_0 be the W-submodule of $W^{\mathrm{Mon}_{\mathcal{V}}(n)}$ of the vectors whose image under χ lies in M_1. It suffices that M_0 is everything, for then M_1 contains the unit vectors. By Nakayama's lemma, the finitely generated W-module $W^{\mathrm{Mon}_{\mathcal{V}}(n)}/M_0$ is zero if every W-linear map $\mathcal{L}$ from it to $\mathbb{F}_W$ is zero. Given such a map, let Φ be the polynomial over $\mathbb{F}_W$ with the coefficient $\mathcal{L}$ of the α-th unit vector at x^{α}; it is homogeneous of degree n. If v is a point at which Φ_0 is a unit, then $(v^{\alpha})_{\alpha}$ lies in M_0, so Φ vanishes at the residue of v. Hence the product of Φ with the reduction of Φ_0 vanishes at every point of $\mathbb{F}_W^{\mathcal{V}}$. It is homogeneous of degree $n + n' \le \#\mathbb{F}_W$, and such a polynomial is zero, by the zero test that follows; the reduction of Φ_0 is not zero, because of v_0; so $\Phi = 0$ and $\mathcal{L} = 0$.

The zero test. Let $\mathbb{F}$ be a field and Φ' a polynomial in $\mathbb{F}[x_{\mathcal{V}}]$ that vanishes at every point of $\mathbb{F}^{\mathcal{V}}$. First, if $\mathbb{F}$ is finite and the degree of Φ' in each variable is less than $\#\mathbb{F}$, then $\Phi' = 0$. This is proved by induction on $\#\mathcal{V}$: write Φ' as a polynomial in one of the variables, with coefficients in the others; at each point of the other variables it has more zeros than its degree, so the coefficients vanish at every point, and they are zero by the induction hypothesis. Secondly, if $\mathbb{F}$ is finite and Φ' is homogeneous of a degree $l \le \#\mathbb{F}$, then $\Phi' = 0$. For $l < \#\mathbb{F}$ this is the first statement. Let $l = \#\mathbb{F}$. The monomials of degree l in which some variable has an exponent that is not less than $\#\mathbb{F}$ are the powers $x_v^{\#\mathbb{F}}$. Replace each of them by x_v. The values on $\mathbb{F}^{\mathcal{V}}$ do not change, as $v^{\#\mathbb{F}} = v$ for v in $\mathbb{F}$. The monomials stay distinct and keep their coefficients, as $\#\mathbb{F} \ge 2$: the new ones have degree 1, the others degree $\#\mathbb{F}$. The new polynomial has degree less than $\#\mathbb{F}$ in each variable, so it is zero by the first statement, and then every coefficient of Φ' is zero. In the proof above $\mathbb{F}$ is the residue field $\mathbb{F}_W$. If it is infinite, the count $\#\mathbb{F}_W$ is 0 by the convention, so $n + n' = 0$ and the product is a constant; it vanishes at a point, so it is zero. The case $l = \#\mathbb{F}$ belongs to the statement of the lemma, as in the formal text; there the polynomial $x_v^{\#\mathbb{F}} - x_v$ shows that homogeneity cannot be dropped. □

LEAN `ClassMean.eq_top_of_eval_monomial`

Lemma 7.2 (an integral invariant diagonal form)**.** *Let $\mathcal{O}$ be a valuation domain of characteristic zero, $\mathcal{V}$ a finite set, p a prime number that is not a unit in $\mathcal{O}$, and e a natural number. There is a function w from the exponents on $\mathcal{V}$ to $\mathcal{O}$ such that*

(i) *for every exponent α with $|\alpha| < p^e$: $p^{\#\mathcal{V}\cdot e} = \binom{|\alpha|}{\alpha} w(\alpha)$, where the multinomial coefficient $\binom{|\alpha|}{\alpha}$ is $|\alpha|!/\prod_v \alpha(v)!$;*

(ii) *for every square matrix G over $\mathcal{O}$ indexed by $\mathcal{V}$ and all exponents α, α': the coefficient of x^α in the linear substitution of G applied to $x^{\alpha'}$, times $w(\alpha)$, equals $w(\alpha')$ times the coefficient of $x^{\alpha'}$ in the linear substitution of G^{T} applied to x^α.*

Proof. A multinomial coefficient of degree l is a product of $\#\mathcal{V}$ binomial coefficients $\binom{l'}{j}$ with $j \le l' \le l$. Let $l' < p^e$. By Legendre's formula for the power of p in a factorial, the exponent of p in $\binom{l'}{j}$ is

$$\sum_{r\ge 1}\Big(\Big\lfloor\frac{l'}{p^r}\Big\rfloor - \Big\lfloor\frac{j}{p^r}\Big\rfloor - \Big\lfloor\frac{l'-j}{p^r}\Big\rfloor\Big).$$

Each term is 0 or 1, and it is 0 when $p^r > l'$, hence for $r \ge e$; so the exponent is at most e. The part of $\binom{l'}{j}$ prime to p is a unit of $\mathcal{O}$. Indeed, an integer prime to p has a multiple that is congruent to 1 modulo p; as p is not a unit, it lies in the maximal ideal $\mathfrak{m}$ of the valuation domain $\mathcal{O}$; so that multiple, and with it the integer, does not lie in $\mathfrak{m}$, and the integer is a unit. Hence $\binom{l'}{j}$ divides p^e in $\mathcal{O}$, and $\binom{|\alpha|}{\alpha}$ divides $p^{\#\mathcal{V}\cdot e}$ when $|\alpha| < p^e$; let $w(\alpha)$ be the quotient, and $w(\alpha) = 0$ for the other α. For (ii), expand the product $\prod_{v'}(\sum_v G_{vv'}x_v)^{\alpha'(v')}$: the coefficient of x^α is the sum, over the matrices $(m_{vv'})$ of natural numbers with row sums α and column sums α', of $\prod_{v'}\alpha'(v')! \cdot \prod_{v,v'} G_{vv'}^{m_{vv'}}/m_{vv'}!$. Multiplied by $\prod_v \alpha(v)!$ this is symmetric under transposition. So both coefficients of (ii) vanish unless $|\alpha| = |\alpha'|$, and then they differ by the factor $\prod \alpha'(v)!/\prod\alpha(v)! = \binom{|\alpha|}{\alpha}/\binom{|\alpha'|}{\alpha'}$, which is $w(\alpha')/w(\alpha)$ if the degree is less than p^e; otherwise both values of w are zero. □

LEAN `ClassMean.exists_invariant_diagonal_form`

Proposition 7.3 (an invariant mean with prescribed values)**.** *Let W be a local ring, $\mathcal{O}$ a valuation domain of characteristic zero, $\chi\colon W \to \mathcal{O}$ a ring homomorphism, k and n_0 natural numbers, and τ a labelling of the indices. Let p be a prime number that is not a unit in $\mathcal{O}$ and e a natural number with $n_0 < p^e$ and $n_0 + k < \#\mathbb{F}_W$. Let ω be a real function on the indices and Ω a set of real numbers. For an exponent α on the generic variables put $\mathrm{rw}(\alpha) = \sum_{i,i'} \alpha(i,i')\,\omega(i)$ and $\mathrm{cw}(\alpha) = \sum_{i,i'} \alpha(i,i')\,\omega(i')$. Then there is an $\mathcal{O}$-linear form $\mathcal{M}$ on the polynomials in the generic variables over $\mathcal{O}$ such that:*

(a) *$\mathcal{M}(\mathrm{cs}_\chi(\gamma)\Phi) = \mathcal{M}(\Phi)$ for every γ in $\mathrm{Lev}_\tau(W)$ and every polynomial Φ;*

(b) *$\mathcal{M}(x^\alpha) = 0$ for every α with $\mathrm{rw}(\alpha) \ne \mathrm{cw}(\alpha)$;*

(c) *for every α with $\mathrm{rw}(\alpha) \in \Omega$: $\mathcal{M}(x^\alpha)$ is $p^{k^2 e}$ times the value of x^α at the identity matrix if $|\alpha| \le n_0$, and $\mathcal{M}(x^\alpha) = 0$ if $|\alpha| > n_0$;*

(d) *$\mathcal{M}$ is an $\mathcal{O}$-linear combination of forms $\mathcal{L} \circ \mathrm{cs}_\chi(\gamma)$, with γ in $\mathrm{Lev}_\tau(W)$ and $\mathcal{L}$ an $\mathcal{O}$-linear form that vanishes at every x^α with $\mathrm{rw}(\alpha) \notin \Omega$.*

Proof. We construct, for each $j \le n_0$, a form $\mathcal{M}_j$ that vanishes on the monomials of degree other than j and has the four properties for the monomials of degree j; their sum is $\mathcal{M}$. Fix j.

The algebra. Let V_j be the free $\mathcal{O}$-module with the basis $e_{\mathbf{l}}$, for the sequences $\mathbf{l} = (l_1, \dots, l_j)$ of indices, and put $|\mathbf{l}|_\omega = \sum_r \omega(l_r)$. Let X^τ be the matrix with the entry $x_{i',i}$ at (i,i') if $\tau(i) = \tau(i')$ and 0 otherwise. An entry of its j-th Kronecker power is $\prod_r X^\tau_{l_r,l'_r}$, which is zero or the monomial x^α with $\alpha = \sum_r 1_{(l'_r,l_r)}$; then $\mathrm{rw}(\alpha) = |\mathbf{l}'|_\omega$ and $\mathrm{cw}(\alpha) = |\mathbf{l}|_\omega$. For α of degree j let S_α be the endomorphism of V_j whose matrix has the entry 1 where the Kronecker power has the entry x^α, and 0 elsewhere; for a vector $c = (c_\alpha)$ put $S(c) = \sum_\alpha c_\alpha S_\alpha$. Then for every linear form $\mathcal{L}$, $S((\mathcal{L}(x^\alpha))_\alpha)$ is $\mathcal{L}$ applied to the entries of the Kronecker power. In particular, for γ in $\mathrm{Lev}_\tau(W)$, the point with $x_{i',i} = \gamma_{ii'}$ gives a vector c_γ with $S(c_\gamma) = \chi(\gamma)^{\otimes j}$, the j-th Kronecker power of the matrix $\chi(\gamma)$.

Density. We claim: a submodule of the module of the vectors c that contains every c with $S(c) = \chi(\gamma)^{\otimes j}$ for some γ in $\mathrm{Lev}_\tau(W)$ is everything. This is Lemma 7.1 for the generic variables, the degree j and the polynomial $\Phi_0 = \det X^\tau$, which is homogeneous of degree k and equal to 1 at the point of the identity matrix; here $j + k \le n_0 + k < \#\mathbb{F}_W$. Indeed, at a point υ where $\det X^\tau$ is a unit, the value of X^τ is an element γ of the Levi subgroup, and $S((\chi(\upsilon^\alpha))_\alpha) = \chi(\gamma)^{\otimes j}$.

The module. Let V_j^Ω be the submodule of V_j spanned by the vectors $S(c)e_{\mathbf{l}}$, for all c and all $\mathbf{l}$ with $|\mathbf{l}|_\omega \in \Omega$. It is mapped into itself by every $S(c)$: by the density claim, used twice, it suffices to see that $\chi(\gamma)^{\otimes j}\chi(\gamma')^{\otimes j}e_{\mathbf{l}} = \chi(\gamma\gamma')^{\otimes j}e_{\mathbf{l}}$ lies in V_j^Ω. Being finitely generated and without torsion over the valuation domain $\mathcal{O}$, V_j^Ω is free.

The form. Let w be the function of Lemma 7.2 for the k^2 generic variables, the prime p and the exponent e. For a polynomial Φ with coefficients Φ_α let $\mathrm{K}(\Phi) = (w(\alpha)\Phi_\alpha)_{|\alpha|=j}$, and define

$$\mathcal{M}_j(\Phi) = \mathrm{tr}(S(\mathrm{K}(\Phi)) \,\big|\, V_j^\Omega), \qquad \text{so that} \quad \mathcal{M}_j(x^\alpha) = w(\alpha)\,\mathrm{tr}(S_\alpha | V_j^\Omega) \ \text{ for } |\alpha| = j.$$

Equivariance. For γ in $\mathrm{Lev}_\tau(W)$,

$$S(\mathrm{K}(\mathrm{cs}_\chi(\gamma)\Phi)) = \chi(\gamma)^{\otimes j}\, S(\mathrm{K}(\Phi))\, \chi(\gamma^{-1})^{\otimes j}. \tag{11}$$

Indeed $\mathrm{cs}_\chi(\gamma)$ is the linear substitution of a matrix G indexed by the generic variables. By Lemma 7.2(ii) the α-th coordinate of $\mathrm{K}(\mathrm{cs}_\chi(\gamma)\Phi)$ is $\sum_{\alpha'} w(\alpha')\Phi_{\alpha'}$ times the coefficient of $x^{\alpha'}$ in the substitution of G^{T} applied to x^α. So the left side of (11) is $\sum_{\alpha'} w(\alpha')\Phi_{\alpha'}$ times the matrix of the coefficients of $x^{\alpha'}$ in the Kronecker power of the image of X^τ under the substitution of G^{T}. That substitution replaces the matrix $(x_{i,i'})$ by $\chi(\gamma^{-1})^{\mathsf{T}}(x_{i,i'})\chi(\gamma)^{\mathsf{T}}$, and therefore X^τ, which is its transpose cut down to the pattern of the Levi subgroup, by $\chi(\gamma)X^\tau\chi(\gamma)^{-1}$, the matrices $\chi(\gamma)$ and $\chi(\gamma)^{-1}$ having that pattern. Taking coefficients gives the right side.

The four properties. (a) By (11) and because $\chi(\gamma)^{\otimes j}$ and its inverse map V_j^Ω to itself, the two traces agree. (b) S_α sends $e_{\mathbf{l}'}$ to zero unless $|\mathbf{l}'|_\omega = \mathrm{rw}(\alpha)$, and then to a combination of the $e_{\mathbf{l}}$ with $|\mathbf{l}|_\omega = \mathrm{cw}(\alpha)$. If the two weights differ, $S_\alpha^2 = 0$, and a nilpotent endomorphism of a free module over a domain has trace zero. (c) If $\mathrm{rw}(\alpha) \in \Omega$, then S_α maps V_j into V_j^Ω, so its trace on V_j^Ω is its trace on V_j, the number of the sequences $\mathbf{l}$ with $\sum_r 1_{(l_r,l_r)} = \alpha$. That number is $\binom{|\alpha|}{\alpha}$ if α is supported on the variables $x_{i,i}$ and 0 otherwise; and the value of x^α at the identity matrix is 1 or 0 in the same cases. With Lemma 7.2(i), $j \le n_0 < p^e$, this is (c). (d) Choose a basis of V_j^Ω; the trace is the sum, over the basis, of forms $\Phi \mapsto \mathcal{L}'(S(\mathrm{K}(\Phi))\,\mathbf{v})$ with $\mathbf{v}$ in V_j^Ω and $\mathcal{L}'$ a linear form

on V_j^{Ω}. By the definition of V_j^{Ω} and the density claim we may take $\mathbf{v} = \chi(\gamma)^{\otimes j} e_{\mathbf{l}}$ with $|\mathbf{l}|_\omega \in \Omega$. By (11) for γ^{-1}, $S(\mathrm{K}(\Phi))\chi(\gamma)^{\otimes j} e_{\mathbf{l}} = \chi(\gamma)^{\otimes j} S(\mathrm{K}(\mathrm{cs}_\chi(\gamma^{-1})\Phi))e_{\mathbf{l}}$, so the form is $\mathcal{L} \circ \mathrm{cs}_\chi(\gamma^{-1})$ with $\mathcal{L}(\Phi) = \mathcal{L}'(\chi(\gamma)^{\otimes j} S(\mathrm{K}(\Phi))e_{\mathbf{l}})$, and $\mathcal{L}(x^\alpha) = 0$ unless $\mathrm{rw}(\alpha) = |\mathbf{l}|_\omega$, which lies in Ω. □

LEAN `ClassMean.exists_mean` · `ClassMean.eq_top_of_eval_monomial` · `ClassMean.exists_invariant_diagonal_form`

The image of S is the Schur algebra of the Levi subgroup in degree j, an order spanned by the Kronecker powers of the group elements; V_j^{Ω} is the part of the tensor space generated by the weight vectors with weight in Ω, and $\mathcal{M}_j$ is its character, read as a function on the dual of the Schur algebra. The function w is the price of integrality: the identification of polynomials of degree j with that dual has the multinomial coefficients as denominators.

8 The Frobenius operator and the projector

Lemma 8.1 (the Frobenius operator under a ring homomorphism)**.** *In Setting 2.2, let* $\mathfrak{t}$, λ, n *be natural numbers,* $\mathcal{R}$ *a ring,* $\psi = (\psi(s,j))$ *matrices in* $\mathrm{M}_k(\mathcal{R})$*,* $\chi\colon \mathcal{R} \to \mathcal{O}$ *a ring homomorphism, and* γ *an element of* $\mathrm{Lev}_{\tau_{k,\mathfrak{t}}(\cdot)}(W)$*. Then* χ*, applied to the entries of* $\mathrm{Fr}_{T,\mathfrak{t},\lambda,n}(\psi)$*, gives*

$$\rho_{T,n}(\gamma^{-1}) \cdot \mathrm{Fr}_{T,\mathfrak{t},\lambda,n}(\psi') \cdot \rho_{T,n}(\gamma), \qquad \psi'(s,j) = \mathrm{cj}_{T,\gamma,s}(\chi(\psi(s,j))).$$

Proof. Let Π_s^m and $\Pi_s'^m$ be the products of Definition 2.10(b) for ψ and ψ'; then

$$\chi(\Pi_s^m) = \iota_s(\gamma^{-1})\, \Pi_s'^m\, \iota_s(\gamma).$$

For an even letter this says that $\chi(\Pi_s^m)$ is $\Pi_s'^m$ conjugated by the letter matrices of γ. For an odd letter, taking transposed adjugates gives

$$\chi(\mathrm{adj}(\Pi_s^m))^{\mathsf{T}} = \iota_s(\gamma)^{\mathsf{T}}\, \mathrm{adj}(\Pi_s'^m)^{\mathsf{T}}\, \iota_s(\gamma^{-1})^{\mathsf{T}},$$

the determinants of $\iota_s(\gamma)$ and $\iota_s(\gamma^{-1})$ cancelling; and $\iota_s(\gamma)^{\mathsf{T}}$ is the letter matrix of γ^{-1} at an odd letter. The number $m = \lambda + \mathrm{dp}_{T,\mathfrak{t}}((i,s))$ depends only on the block of i, and the letter matrices of γ are block diagonal; so choosing the rows according to m commutes with these conjugations, and

$$\chi(B_s) = \gamma_{T,s}^{-1}\, B_s'\, \gamma_{T,s}.$$

Symmetric powers and Kronecker products are multiplicative. □

LEAN `TwistingData.frobOperator_map`

Lemma 8.2 (the Frobenius operator in a filtration)**.** *In Setting 2.2, let* $\mathfrak{t}$, λ, n *be natural numbers,* $\mathcal{R}$ *a ring,* $\psi = (\psi(s,j))$ *matrices in* $\mathrm{M}_k(\mathcal{R})$*, and* β, β' *families in* $\mathrm{Mon}_{k,n}^L$*. Let* $\mathcal{G}$ *be a family of additive subgroups of* $\mathcal{R}$ *such that* $\mathcal{G}$ *is a real filtration. Let* $\eta_{s,j}(i)$ *be real numbers, for a letter, a natural number and an index, let* ω *be a weight on the variables and* ϵ *a real number, such that*

(i) $|\eta_{s,0}(i)| \le \epsilon$ *for all* s*,* i*, and* $\sum_i \eta_{s,j}(i) = 0$ *for all* s*,* j*;*
(ii) $\omega(i,s) = \eta_{s,m}(i)$ *if* s *is even and* $\omega(i,s) = -\eta_{s,m}(i)$ *if* s *is odd, where* $m = \lambda + \mathrm{dp}_{T,\mathfrak{t}}((i,s))$*;*

(iii) $\psi(s,j)_{ii'} \in \mathcal{G}(\eta_{s,j+1}(i) - \eta_{s,j}(i') - \epsilon)$ *for every letter* s*, every* $j \le \lambda$ *and all indices* i*,* i'*;*

(iv) $0 \le \epsilon$.

Then the (β, β') *entry of* $\mathrm{Fr}_{T,\mathfrak{t},\lambda,n}(\psi)$ *lies in* $\mathcal{G}(\langle\omega,\beta\rangle - L\,n\,k\,(\lambda+2)\,\epsilon)$.

Proof. First, $(\Pi_s^m)_{ii'}$ lies in $\mathcal{G}(\eta_{s,m}(i) - (m+1)\epsilon)$ for $m \le \lambda+1$: for $m=0$ the entries are 0 and 1 and $\eta_{s,0}(i) - \epsilon \le 0$; and $(\psi(s,m)\Pi_s^m)_{ii'} = \sum_l \psi(s,m)_{il}(\Pi_s^m)_{li'}$, where the levels add to $\eta_{s,m+1}(i) - (m+2)\epsilon$. Next, every entry $(B_s)_{ii'}$ lies in $\mathcal{G}(\omega(i,s) - k(\lambda+2)\epsilon)$. For even s this is the first step. For odd s the entry is, up to sign, a minor of Π_s^m without the row i, a sum of products of one entry from each other row; the levels add to $\sum_{l\neq i}\eta_{s,m}(l) - (k-1)(m+1)\epsilon = -\eta_{s,m}(i) - (k-1)(m+1)\epsilon$, by (i). Finally an entry of $\mathrm{Sym}^n(B_s)$ in the row β_s is a sum of products of n entries of B_s, of which $\beta_s(i)$ lie in the row i; and the entry of the Kronecker product is the product over the L letters. □

LEAN `TwistingData.frobOperator_mem`

Lemma 8.3 (conjugation and Frobenius modulo p)**.** *In Setting 2.2, let* γ *be an element of* $\mathrm{GL}_k(W)$*,* s *a letter,* j *a natural number,* G *a matrix in* $\mathrm{M}_k(\mathcal{O})$ *and* i*,* i' *indices. Write* $G^{(j)}$ *for the matrix of the* P_T^j*-th powers of the entries of* G*. Then* p *divides*

$$\Big(\mathrm{cj}_{T,\gamma,s}\big((\mathrm{cj}_{T,\gamma^{-1},s-j}(G))^{(j)}\big)\Big)_{ii'} - (G_{ii'})^{P_T^j}.$$

Proof. Modulo p the map $\pi \mapsto \pi^{P_T^j}$ of $\mathcal{O}$ is a ring homomorphism, so it commutes with products of matrices; and it sends $\iota_{s-j}(\upsilon)$ to $\iota_s(\upsilon)$, by Lemma 3.1(a). Hence $(\iota_{s-j}(\gamma^{-1})\,G\,\iota_{s-j}(\gamma))^{(j)} \equiv \iota_s(\gamma^{-1})\,G^{(j)}\,\iota_s(\gamma)$ modulo p. □

LEAN `TwistingData.p_dvd_cj_frobenius_sub`

Proposition 8.4 (exponents of the Frobenius operator track the excess)**.** *In Setting 2.2, let* i_0*,* i_1 *be indices with* $i_0+1 = i_1$*, let* s_0 *be a letter, and let* λ*,* n *be natural numbers with* $L\,n\,k\,(\lambda+2) \le P_T^\lambda$ *and* $\lambda \le L$*. Let* γ *be an element of* $\mathrm{GL}_k(W)$*, let* T' *be the twisting data* $\chi_* T$ *for the inclusion* χ *of* $\mathcal{O}$ *in the polynomial ring in the generic variables* $x_{s,i,i'}$*, and suppose that* ψ *is given by*

$$\psi(s,j) = \mathrm{cj}_{T',\gamma,s}\Big((\mathrm{cj}_{T',\gamma^{-1},s-j}(X_{s-j}))^{(j)}\Big)$$

for every letter s *and every natural number* j*, in the notation of Lemma 8.3. Let* c *be a natural number,* β, β' *families, and* ν *an exponent on the generic variables, with parts* ν_s *on the variables of the letter* s*. With* rw *and* cw *as in Proposition 7.3 for the function* $\omega_0 = \mathrm{iw}_{T,\lambda}^{i_0,i_1}(\cdot)$ *on the indices, suppose that* $\mathrm{rw}(\nu_s) = \mathrm{cw}(\nu_s)$ *for every letter* $s \neq s_0$*, and that*

$$2c+1 < \big|\langle \mathrm{ex}_{T,s_0+\lambda}^{i_0,i_1}(\cdot), \beta\rangle - \mathrm{rw}(\nu_{s_0})\big|.$$

Then ϖ^c *divides the coefficient of* x^ν *in the* (β,β') *entry of* $\mathrm{Fr}_{T,i_1,\lambda,n}(\psi)$.

Proof. Let $\mathrm{sg} = \pm 1$ be the sign of the difference in the hypothesis, and put

$$\mathrm{a}_m(i) = \omega_0(i)\cdot\tfrac{\mathrm{sg}}{2}\cdot P_T^m/q_T; \qquad \text{so} \quad |\mathrm{a}_m(i)| \le \tfrac{1}{2}P_T^{-\lambda} \ \text{ for } m \le L, \qquad \sum_i \mathrm{a}_m(i) = 0.$$

Give the generic variable $x_{s,i,i'}$ the weight $\mathrm{a}_L(i)$ if $s = s_0$, and $\mathrm{a}_{[s-s_0]}(i) - \mathrm{a}_{[s-s_0]}(i')$ otherwise, and let $\mathcal{G}$ be the corresponding filtration $\mathrm{Gs}_{\mathrm{ad}_{\varpi^{(\cdot)}},\cdot}(\cdot)$ of the polynomial

ring. The weight of x^ν is then $\frac{\mathrm{sg}}{2}\mathrm{rw}(\nu_{s_0})$, the other letters contributing multiples of $\mathrm{rw}(\nu_s)-\mathrm{cw}(\nu_s)=0$. We apply Lemma 8.2, for $\mathfrak{t}=i_1$, with

$$\eta_{s,j}(i)=\mathrm{a}_{j+[s-j-s_0]}(i),\qquad \omega=\tfrac{\mathrm{sg}}{2}\,\mathrm{ex}^{i_0,i_1}_{T,s_0+\lambda}(\cdot),\qquad \epsilon=\tfrac{1}{2}P_T^{-\lambda}.$$

Condition (i) holds by the two properties of a_m. Condition (ii) is the definition of the excess weight at the letter $s_0+\lambda$, read at the two root variables of each letter, using $i_0<i_1$: these are the variables (i_1,s) and (i_0,s), of which the deep one for the threshold i_1 is the root variable $\mathrm{rt}_{T,s}(i_0,i_1)$. For condition (iii) let $j\le\lambda$. By Lemma 8.3 for T', the entry $\psi(s,j)_{ii'}$ is $x^{P_T^j}_{s-j,i,i'}$ plus p times a polynomial; and it is homogeneous of degree P_T^j in the variables of the letter $s-j$. The weights of these variables are at most $P_T^{-\lambda}$ in absolute value, so every monomial of the entry has weight within $2P_T^{j-\lambda}\le 2$ of the weight of the leading one; as $p=\varpi^2$, the entry lies in $\mathcal{G}$ at the level of its leading monomial. That level is P_T^j times the weight of $x_{s-j,i,i'}$, which is $\eta_{s,j+1}(i)-\eta_{s,j}(i')$ if $s-j\ne s_0$ and $\eta_{s,j+1}(i)-\eta_{s,j}(i')+\mathrm{a}_j(i')$ if $s-j=s_0$; and $|\mathrm{a}_j(i')|\le\epsilon$. The lemma gives the level $\frac{\mathrm{sg}}{2}\langle \mathrm{ex}^{i_0,i_1}_{T,s_0+\lambda}(\cdot),\beta\rangle - L\,n\,k\,(\lambda+2)\epsilon$ for the entry, where the last term is at most $\frac12$ by hypothesis. So the coefficient of x^ν is divisible by ϖ to a power that is at least $\frac12\big|\langle \mathrm{ex}^{i_0,i_1}_{T,s_0+\lambda}(\cdot),\beta\rangle-\mathrm{rw}(\nu_{s_0})\big|-\frac12>c$. □

LEAN `TwistingData.pow_dvd_coeff_conj_frobOperator` · `TwistingData.frobOperator_mem` · `TwistingData.p_dvd_cj_frobenius_sub`

Lemma 8.5 (degrees in the Frobenius operator)**.** *In Setting 2.2, let* $\mathfrak{t}$, λ, n *be natural numbers,* β,β' *families, and* ν *an exponent on the generic variables. If the coefficient of* x^ν *in the* (β,β') *entry of* $\mathrm{Fr}_{T,\mathfrak{t},\lambda,n}(X^{\mathrm{Fr}}_T(\cdot,\cdot))$ *is not zero, then for every letter* s' *the part of* ν *on the variables of* s' *has degree at most* $L\,n\,k\,(\lambda+2)\,P_T^\lambda$.

Proof. The entries of $X^{\mathrm{Fr}}_T(s,j)$ with $j\le\lambda$ have degree at most P_T^λ; those of the products Π_s^m, $m\le\lambda+1$, and of their adjugates have degree at most $k(\lambda+1)P_T^\lambda$; an entry of the n-th symmetric power has n times that degree at most, and an entry of the Kronecker product over L letters L times as much. □

LEAN `TwistingData.degree_le_of_coeff_frobOperator` · the formal proof is Lemma 8.2 for a filtration that records the degree in the variables of one letter

Lemma 8.6 (products of linear forms)**.** *Let* $\mathcal{S}$ *be a finite set,* $\mathcal{V}$ *a set and* $\mathcal{O}$ *a ring. There is a map, linear in each argument, that sends a family* $(\mathcal{L}_\mathfrak{s})_{\mathfrak{s}\in\mathcal{S}}$ *of* $\mathcal{O}$*-linear forms on* $\mathcal{O}[x_\mathcal{V}]$ *to an* $\mathcal{O}$*-linear form* $\bigotimes_\mathfrak{s}\mathcal{L}_\mathfrak{s}$ *on the polynomial ring in the variables* $x_{\mathfrak{s},v}$*, such that* $(\bigotimes_\mathfrak{s}\mathcal{L}_\mathfrak{s})(\prod_\mathfrak{s}\Phi_\mathfrak{s})=\prod_\mathfrak{s}\mathcal{L}_\mathfrak{s}(\Phi_\mathfrak{s})$ *whenever each* $\Phi_\mathfrak{s}$ *is a polynomial in the variables* $x_{\mathfrak{s},v}$*,* $v\in\mathcal{V}$*, read as a polynomial in* $\mathcal{O}[x_\mathcal{V}]$.

Proof. Define the form on a monomial x^ν, with parts $\nu_\mathfrak{s}$, as $\prod_\mathfrak{s}\mathcal{L}_\mathfrak{s}(x^{\nu_\mathfrak{s}})$, and extend linearly. □

LEAN `exists_tensor_dual`

Proof of Theorem 4. Write $q=q_T=p^{oL}$. The hypotheses give

$$n_0+k<q\quad\text{for}\quad n_0=L\,n\,k\,(\lambda+2)\,P_T^\lambda,\qquad n<q,\qquad L\,n\,k\,(\lambda+2)\le P_T^\lambda,\quad(12)$$

because $(L\,n\,k\,(\lambda+2)+n+k+1)\,P_T^\lambda\le P_T^{2\lambda}=q$.

The diagonal form. Let w_0 be the function of Lemma 7.2 for the k indices as variables, the prime p and the exponent oL, and put $w(\beta)=\prod_s w_0(\beta_s)$. As $n<q$, each $w_0(\beta_s)$ divides $p^{k\,oL}$; choose $w'(\beta)$ with $w(\beta)\,w'(\beta)=p^{k\,oL\cdot L}$. Property (iii) of the theorem is

Lemma 7.2(ii) for each letter, since $\rho_{T,n}(\gamma)_{\beta',\beta} = \prod_s \mathrm{Sym}^n(\gamma_{T,s})_{\beta'_s,\beta_s}$ and the letter matrices of γ^{T} are the transposes of those of γ.

The operator. Let Fr be the matrix $\mathrm{Fr}_{T,i_1,\lambda,n}(X_T^{\mathrm{Fr}}(\cdot,\cdot))$, with entries in the polynomial ring in the generic variables. For a family $\mathcal{L} = (\mathcal{L}_s)_s$ of $\mathcal{O}$-linear forms on the polynomials in one generic matrix let $Z(\mathcal{L})$ be the matrix over $\mathcal{O}$ obtained by applying the form $\bigotimes_s \mathcal{L}_s$ of Lemma 8.6 to the entries of Fr; it is linear in each $\mathcal{L}_s$.

The means. For every letter s let $\mathcal{M}_s^+$ and $\mathcal{M}_s^-$ be the forms of Proposition 7.3 for the homomorphism ι_s, the labelling $\tau_{k,i_1}(\cdot)$, the degree n_0, the prime p with the exponent oL, the function $\omega_0 = \mathrm{iw}_{T,\lambda}^{i_0,i_1}(\cdot)$, and the set Ω of all real numbers, respectively of the real numbers greater than $h-(2c+1)$; the hypotheses hold by (12). Put $\mathcal{L}_s^\circ = \mathcal{M}_s^+ - \mathcal{M}_s^-$ and

$$Z = Z(\mathcal{L}^\circ)\cdot \mathrm{diag}(w'(\beta))_\beta.$$

By (a), (b), (c) of the proposition, $\mathcal{L}_s^\circ$ is invariant under $\mathrm{cs}_{\iota_s}(\gamma)$ for γ in the Levi subgroup, and $\mathcal{L}_s^\circ(x^\alpha) = 0$ unless $\mathrm{rw}(\alpha) = \mathrm{cw}(\alpha) \le h-(2c+1)$.

A criterion. Let s_0 be a letter, γ an element of $\mathrm{Lev}_{\tau_{k,i_1}(\cdot)}(W)$, and $\mathcal{L}$, $\mathcal{L}'$ families of forms with $\mathcal{L}'_s = \mathcal{L}_s \circ \mathrm{cs}_{\iota_s}(\gamma)$ for every letter. Let F_1, F_2 be forms. Suppose that for every family β at which the row vector $F_1^{\mathsf{T}}\rho_{T,n}(\gamma^{-1})$ has a coordinate not divisible by ϖ^c, and for every exponent ν on the generic variables, either $\prod_s \mathcal{L}_s(x^{\nu_s}) = 0$, or the two hypotheses of Proposition 8.4 hold for s_0, β and ν. Then ϖ^c divides $F_1^{\mathsf{T}} Z(\mathcal{L}')F_2$. Indeed, $\bigotimes_s \mathcal{L}'_s$ is $\bigotimes_s \mathcal{L}_s$ after the substitution of $\iota_s(\gamma^{-1})X_s\iota_s(\gamma)$ for each X_s. By Lemma 8.1, for the twisting data T' and the substitution as the ring homomorphism, the substituted matrix Fr is $\rho_{T,n}(\gamma^{-1})\cdot \mathrm{Fr}_{T,i_1,\lambda,n}(\psi)\cdot\rho_{T,n}(\gamma)$ with ψ as in Proposition 8.4; the constant matrices pass through the linear form. So

$$F_1^{\mathsf{T}} Z(\mathcal{L}')F_2 = \sum_{\beta,\beta'} (F_1^{\mathsf{T}}\rho_{T,n}(\gamma^{-1}))(\beta)\cdot\Big(\sum_\nu \mathrm{cf}_\nu \prod_s \mathcal{L}_s(x^{\nu_s})\Big)\cdot(\rho_{T,n}(\gamma)F_2)(\beta'),$$

with cf_ν the coefficient of x^ν in the (β,β') entry of $\mathrm{Fr}_{T,i_1,\lambda,n}(\psi)$, and every term is divisible by ϖ^c, by the hypothesis or by Proposition 8.4.

Property (i). Let F be a form, γ in the Levi subgroup, and β a family whose excess at some letter s_1 exceeds h. The coordinate $(\rho_{T,n}(\gamma)ZF)(\beta)$ is $F_1^{\mathsf{T}}Z(\mathcal{L}^\circ)F_2$ for F_1^{T} the β-th row of $\rho_{T,n}(\gamma)$ and $F_2 = w'F$. Apply the criterion with $\mathcal{L} = \mathcal{L}' = \mathcal{L}^\circ$ and $s_0 = s_1 - \lambda$. The row vector $F_1^{\mathsf{T}}\rho_{T,n}(\gamma^{-1})$ is the β-th unit vector. If $\prod_s \mathcal{L}_s^\circ(x^{\nu_s}) \neq 0$, then every ν_s has equal row and column weights, at most $h-(2c+1)$, and the excess of β at $s_0+\lambda$ exceeds $\mathrm{rw}(\nu_{s_0})$ by more than $2c+1$.

Property (ii). Let F' lie in $\mathrm{Ex}_{T,n}^{i_0,i_1}(\varpi^c\mathcal{O}; h-(4c+2))$, let F be a form, and put $\mathrm{bil}(\mathcal{L}) = (wF')^{\mathsf{T}}Z(\mathcal{L})(w'F)$, with the products wF' and $w'F$ taken coordinate by coordinate. The quantity of (ii) is $\mathrm{bil}(\mathcal{L}^\circ)$ less $p^{(k+k^2)oL\cdot L}\sum_\beta F'(\beta)F(\beta)$. Expanding in each letter, $\mathrm{bil}(\mathcal{L}^\circ)$ is the sum, over the subsets $\mathcal{S}$ of the letters, of $\mathrm{bil}(\mathcal{L}^{\mathcal{S}})$ with $\mathcal{L}_s^{\mathcal{S}} = -\mathcal{M}_s^-$ for s in $\mathcal{S}$ and $\mathcal{L}_s^{\mathcal{S}} = \mathcal{M}_s^+$ otherwise.

For the empty subset: by Lemma 8.5 and (c), $\mathcal{M}_s^+$ takes at every monomial that occurs in Fr the value $p^{k^2 oL}$ times its value at the identity; and Fr with all generic matrices equal to the identity is the identity matrix. So $Z(\mathcal{M}^+) = p^{k^2 oL\cdot L}\cdot 1$, and $\mathrm{bil}(\mathcal{M}^+) = p^{k^2 oL\cdot L}\sum_\beta w(\beta)w'(\beta)F'(\beta)F(\beta)$, which is the term subtracted.

For a nonempty subset choose s_0 in it. By (d), $\mathcal{M}^-_{s_0}$ is a combination of forms $\mathcal{L}_0 \circ \mathrm{cs}_{\iota_{s_0}}(\gamma)$ with γ in the Levi subgroup and $\mathcal{L}_0$ vanishing at the x^α with $\mathrm{rw}(\alpha) \le h-(2c+1)$. So it suffices that ϖ^c divides $\mathrm{bil}(\mathcal{L}')$ when $\mathcal{L}'_{s_0} = \mathcal{L}_0 \circ \mathrm{cs}_{\iota_{s_0}}(\gamma)$ and $\mathcal{L}'_s = \mathcal{L}^{\mathcal{S}}_s$ for $s \neq s_0$. Apply the criterion with $\mathcal{L}_{s_0} = \mathcal{L}_0$, $\mathcal{L}_s = \mathcal{L}'_s$ otherwise, which is allowed by the invariance (a), with $F_1 = wF'$ and $F_2 = w'F$. By property (iii), $((wF')^{\mathsf{T}}\rho_{T,n}(\gamma^{-1}))(\beta) = w(\beta)\cdot(\rho_{T,n}((\gamma^{-1})^{\mathsf{T}})F')(\beta)$, and $(\gamma^{-1})^{\mathsf{T}}$ lies in the Levi subgroup; so, by the choice of F', this coordinate is divisible by ϖ^c unless the excess of β is at most $h-(4c+2)$ at every letter. For such β, and ν with $\prod_s \mathcal{L}_s(x^{\nu_s}) \neq 0$: the parts ν_s, $s \neq s_0$, have equal row and column weights by (b), and $\mathrm{rw}(\nu_{s_0}) > h-(2c+1)$, which exceeds the excess of β at $s_0+\lambda$ by more than $2c+1$. □

LEAN `TwistingData.exists_projector` · `ClassMean.exists_mean` · `TwistingData.pow_dvd_coeff_conj_frobOperator` · `TwistingData.frobOperator_map` · `TwistingData.degree_le_of_coeff_frobOperator` · `exists_tensor_dual` · `ClassMean.map_C_mul_mul_C` (a linear form applied to the entries commutes with multiplication by constant matrices)

9 Elementary divisors

In this section $\mathcal{O}$ is a valuation domain, with maximal ideal $\mathfrak{m}$, and $\bar{T} = \mathrm{res}_* T$ is the twisting data over the residue field $\mathcal{O}/\mathfrak{m}$ obtained from T by the residue map $\mathrm{res}\colon \mathcal{O} \to \mathcal{O}/\mathfrak{m}$. For a finite set $\mathcal{X}$ a vector in $\mathcal{O}^{\mathcal{X}\times \mathrm{Mon}^L_{k,n}}$ has the *blocks* $F(\mathfrak{x},\cdot)$, forms in $\mathcal{O}^{\mathrm{Mon}^L_{k,n}}$.

Lemma 9.1 (independent columns)**.** *In Setting 2.2, let $\mathcal{O}$ be a valuation domain, n a natural number, $\mathcal{X}$ a finite set and $\mathcal{N}$ a set. Let A be a matrix over $\mathcal{O}$ with rows indexed by $\mathcal{X}\times\mathrm{Mon}^L_{k,n}$ and columns by $\mathcal{N}$, and A' a matrix with rows indexed by $\mathcal{N}$ and columns by $\mathcal{X}\times\mathrm{Mon}^L_{k,n}$. Let $\mathcal{S}$ be a finite subset of $\mathcal{N}$, τ a labelling and $\mathcal{B}$ a subset of $\mathrm{Mon}^L_{k,n}$. Suppose that $(A'A)_{l,l'}$ is 1 if $l=l'$ and 0 otherwise, for every l' in $\mathcal{S}$ and every l in $\mathcal{N}$, and that for every l' in $\mathcal{S}$ every block of the column l' of A lies in $\mathrm{Tr}_{T,n}(\tau;\mathfrak{m},\mathcal{B})$. Then*

$$\#\mathcal{S} \;\le\; \#\mathcal{X}\cdot\dim_{\mathcal{O}/\mathfrak{m}}\mathrm{Tr}_{\bar{T},n}(\tau;0,\mathcal{B}).$$

Proof. Reduce the columns $l' \in \mathcal{S}$ of A modulo $\mathfrak{m}$. Their blocks lie in $\mathrm{Tr}_{\bar{T},n}(\tau;0,\mathcal{B})$, by Lemma 3.1(d); and they are linearly independent over $\mathcal{O}/\mathfrak{m}$, because the reduction of A' recovers the coefficients of a combination of them. □

LEAN `TwistingData.card_le_finrank_translatesIn`

Proposition 9.2 (two matrices that pair almost perfectly)**.** *In Setting 2.2, let $\mathcal{O}$ be a valuation domain, n a natural number, and $\mathcal{X}$, $\mathcal{N}$ finite sets. Let A and B be matrices over $\mathcal{O}$ with rows indexed by $\mathcal{X}\times\mathrm{Mon}^L_{k,n}$ and columns by $\mathcal{N}$, and let A' and C be matrices with rows indexed by $\mathcal{N}$ and columns by $\mathcal{X}\times\mathrm{Mon}^L_{k,n}$. Let π_1, π_2, π_0 be elements of $\mathcal{O}$, τ a labelling and $\mathcal{B}$ a subset of $\mathrm{Mon}^L_{k,n}$. Suppose that*

(i) $A'A = 1$;
(ii) π_0 *is not a unit;*
(iii) $\pi_1\pi_0$ *divides every entry of* $CB - \pi_1\cdot 1$;
(iv) *whenever* $AF_1 - BF_2 = \pi_1\pi_0 F$ *for vectors* F_1, F_2 *in* $\mathcal{O}^{\mathcal{N}}$ *and* F *in* $\mathcal{O}^{\mathcal{X}\times\mathrm{Mon}^L_{k,n}}$, *every block of* F *lies in* $\mathrm{Tr}_{T,n}(\tau;\mathfrak{m},\mathcal{B})$.

Then there is a natural number l with

$$\#\mathcal{N}+l \;\le\; \#\mathcal{X}\cdot\dim_{\mathcal{O}/\mathfrak{m}}\mathrm{Tr}_{\bar{T},n}(\tau;0,\mathcal{B}) \qquad \text{and} \qquad [\mathcal{O}^{\mathcal{N}}:CA(\mathcal{O}^{\mathcal{N}})+\pi_2\,\mathcal{O}^{\mathcal{N}}]\;\big|\;\pi_1^{\#\mathcal{N}}\pi_2^{l}.$$

Proof. Write $\mathcal{Y}=\mathcal{X}\times\mathrm{Mon}_{k,n}^{L}$. By Input 2(b), for the valuation pair $(\mathcal{O},\mathcal{O})$, there are invertible matrices U, U' over $\mathcal{O}$ and elements ε_y of $\mathcal{O}$, $y\in\mathcal{Y}$, with

$$(AA'B-B)\,A'=U\,\mathrm{diag}(\varepsilon_y)\,U'. \tag{13}$$

Multiplying by A' on the left and using (i) gives $A'U\,\mathrm{diag}(\varepsilon_y)=0$; so the column y of $A'U$ is zero whenever $\varepsilon_y\neq 0$. Let $\mathcal{Y}_0$ be the set of the y with $\varepsilon_y\neq 0$ and ε_y dividing $\pi_1\pi_0$, and $l=\#\mathcal{Y}_0$.

The inequality. Apply Lemma 9.1 to the matrix $(A\ U)$, with columns indexed by the disjoint union of $\mathcal{N}$ and $\mathcal{Y}$, to the matrix with the rows of A' above those of $U^{-1}-U^{-1}AA'$, and to the columns in $\mathcal{N}$ and in $\mathcal{Y}_0$. The product of the two matrices has the blocks 1, $A'U$, 0 and $1-U^{-1}AA'U$; its columns in $\mathcal{N}$ and in $\mathcal{Y}_0$ are unit vectors, by the remark on $A'U$. The blocks of a column $Ae_{l'}$ lie in $\mathrm{Tr}_{T,n}(\tau;\mathfrak{m},\mathcal{B})$ by (iv) with $F_1=\pi_1\pi_0e_{l'}$ and $F_2=0$. For y in $\mathcal{Y}_0$ write $\pi_1\pi_0=\varepsilon_y\pi'$; by (13), with $F_2=\pi'A'U'^{-1}e_y$, we have $A(A'BF_2)-BF_2=\pi_1\pi_0\,Ue_y$, and (iv) applies to the column Ue_y.

The index. For every y one of $\pi_1\pi_0$, ε_y divides the other; choose ε'_y with $\pi_1\pi_0\varepsilon'_y=\varepsilon_y$, or with $\varepsilon_y\varepsilon'_y=\pi_1\pi_0$ when the former is impossible. Put $C_1=CU\,\mathrm{diag}(\pi_1\pi_0\varepsilon'_y-\varepsilon_y)$; its column y is zero for y outside $\mathcal{Y}_0$. Write $CB-\pi_1=\pi_1\pi_0C_0$ by (iii). Using (13) and (i),

$$C_1U'A+CAA'B=\pi_1\pi_0\,CU\,\mathrm{diag}(\varepsilon'_y)U'A+CB=\pi_1(1+\pi_0\,(C_0+CU\,\mathrm{diag}(\varepsilon'_y)U'A)).$$

The determinant of the last bracket is congruent to 1 modulo π_0, hence a unit by (ii); so the determinant of the left side divides $\pi_1^{\#\mathcal{N}}$, and its image lies in $C_1(\mathcal{O}^{\mathcal{Y}})+CA(\mathcal{O}^{\mathcal{N}})$. By Input 1(a), $[\mathcal{O}^{\mathcal{N}}:C_1(\mathcal{O}^{\mathcal{Y}})+CA(\mathcal{O}^{\mathcal{N}})+\pi_2\mathcal{O}^{\mathcal{N}}]\;\big|\;\pi_1^{\#\mathcal{N}}$. The diagonal matrix on $\mathcal{O}^{\mathcal{Y}}$ with the entries π_2 at $\mathcal{Y}_0$ and 1 elsewhere has determinant π_2^{l}, and C_1 maps its image into $\pi_2\mathcal{O}^{\mathcal{N}}$. By Input 1(a) and (c), for the map C_1, $[C_1(\mathcal{O}^{\mathcal{Y}})+CA(\mathcal{O}^{\mathcal{N}})+\pi_2\mathcal{O}^{\mathcal{N}}:CA(\mathcal{O}^{\mathcal{N}})+\pi_2\mathcal{O}^{\mathcal{N}}]\;\big|\;\pi_2^{l}$, and Input 1(b) gives the claim. □

LEAN `TwistingData.of_pairing` · `TwistingData.card_le_finrank_translatesIn` · `IsValuationPair.smith_normal_form` · `IndexDvd.of_det_dvd` · `IndexDvd.map` · `IndexDvd.trans`

Proposition 9.3 (the depth factorisation)**.** *In Setting 2.2, let $\mathcal{O}$ be a valuation domain of characteristic zero, c and n natural numbers, i_0, i_1 indices with $i_0<i_1$, and j_0 a natural number with $\mathrm{depth}_{T,i_1}^{k,n}(\beta)\le j_0$ for every family β. Let $\mathcal{X}$ and $\mathcal{E}$ be finite sets, $\mathrm{pt}\colon\mathcal{E}\to\mathcal{X}$ a map and $(\gamma_{\mathfrak{e}})_{\mathfrak{e}\in\mathcal{E}}$ elements of $\mathrm{GL}_k(W)$ such that the map $\mathfrak{e}\mapsto(\mathrm{pt}(\mathfrak{e}),\gamma_{\mathfrak{e}}^{-1}\mathrm{Par}_{\tau_{k,i_1}(\cdot)}(W))$ from $\mathcal{E}$ to $\mathcal{X}\times\mathrm{GL}_k(W)/\mathrm{Par}_{\tau_{k,i_1}(\cdot)}(W)$ is surjective. Let J be the matrix with rows indexed by $\mathcal{E}\times\mathrm{Mon}_{k,n}^{L}$ and columns by $\mathcal{X}\times\mathrm{Mon}_{k,n}^{L}$ whose entry at $((\mathfrak{e},\beta),(\mathfrak{x},\beta'))$ is $\varpi^{\mathrm{depth}_{T,i_1}^{k,n}(\beta)}\cdot\rho_{T,n}(\gamma_{\mathfrak{e}})_{\beta,\beta'}$ if $\mathrm{pt}(\mathfrak{e})=\mathfrak{x}$, and 0 otherwise. Then there are a matrix A of the shape of J, a square matrix Δ indexed by $\mathcal{X}\times\mathrm{Mon}_{k,n}^{L}$ and a matrix A' of the shape of the transpose of J, all over $\mathcal{O}$, such that:*

(i) $A\Delta=J$;
(ii) $\det\Delta$ *divides* ϖ *to the power* $\#\mathcal{X}\cdot\sum_{j<j_0}\dim_{\mathcal{O}/\mathfrak{m}}\mathrm{Dp}_{\bar{T},i_1}^{k,n}(j)$;

(iii) $A'A = 1$;

(iv) *for every* F *in* $\mathcal{O}^{\mathcal{X}\times \mathrm{Mon}^L_{k,n}}$, *every* $\mathfrak{e}$ *in* $\mathcal{E}$ *and every real number* h *with* $2c \le h$, *the block* $(AF)(\mathfrak{e},\cdot)$ *lies in* $\mathrm{Ex}^{i_0,i_1}_{T,n}(\varpi^c\mathcal{O}; h)$.

Proof. Write $\mathcal{Y} = \mathcal{X}\times \mathrm{Mon}^L_{k,n}$, $\mathrm{Par} = \mathrm{Par}_{\tau_{k,i_1}(\cdot)}(W)$ and $\mathrm{d}(\beta) = \mathrm{depth}^{k,n}_{T,i_1}(\beta)$. So

$$(JF)(\mathfrak{e},\beta) = \varpi^{\mathrm{d}(\beta)}\,(\rho_{T,n}(\gamma_{\mathfrak{e}})F(\mathrm{pt}(\mathfrak{e}),\cdot))(\beta).$$

Step 1: divisibility passes to all translates. Let π in $\mathcal{O}$ divide every coordinate of JF. Then π divides $\varpi^{\mathrm{d}(\beta)}(\rho_{T,n}(\gamma)F(\mathfrak{x},\cdot))(\beta)$ for every $\mathfrak{x}$, every γ in $\mathrm{GL}_k(W)$ and every β. Indeed, choose $\mathfrak{e}$ with $\mathrm{pt}(\mathfrak{e}) = \mathfrak{x}$ and $\gamma' = \gamma\gamma_{\mathfrak{e}}^{-1}$ in Par. For a letter s and indices i, i' such that (i,s) is not deep and (i',s) is deep, for $\mathfrak{t} = i_1$, the entry $(\gamma'_{T,s})_{ii'}$ is divisible by $p = \varpi^2$ (Lemma 3.2). So there are matrices G_s over $\mathcal{O}$ with $\mathrm{diag}(\varpi^{\mathrm{dp}_{T,i_1}((i,s))})_i \cdot \gamma'_{T,s} = G_s \cdot \mathrm{diag}(\varpi^{\mathrm{dp}_{T,i_1}((i,s))})_i$, and by Lemma 3.1(e), $\mathrm{diag}(\varpi^{\mathrm{d}(\beta)})\,\rho_{T,n}(\gamma') = (\bigotimes_s \mathrm{Sym}^n(G_s))\,\mathrm{diag}(\varpi^{\mathrm{d}(\beta)})$. Hence the vector $(\varpi^{\mathrm{d}(\beta)}(\rho_{T,n}(\gamma)F(\mathfrak{x},\cdot))(\beta))_\beta$ is the image of the block $(JF)(\mathfrak{e},\cdot)$ under a matrix over $\mathcal{O}$.

Step 2: the factorisation. The map pt is surjective; a section of it makes $\mathcal{Y}$ a subset of $\mathcal{E}\times\mathrm{Mon}^L_{k,n}$. Input 2(a), for $(\mathcal{O},\mathcal{O})$ and the weights 0, gives an invertible matrix U and a square matrix Δ with $J = U_{\cdot,y}\Delta$. Put $A = U_{\cdot,y}$ and $A' = (U^{-1})_{y,\cdot}$; then (i) and (iii) hold. By Input 2(b), $\Delta = U_1\,\mathrm{diag}(\varepsilon_y)U_2$ with U_1, U_2 invertible over $\mathcal{O}$. The column y of $JU_2^{-1} = AU_1\,\mathrm{diag}(\varepsilon_y)$ is divisible by ε_y; so by Step 1, for all $\mathfrak{x}$, γ, β,

$$\varepsilon_y \ \text{ divides } \ \varpi^{\mathrm{d}(\beta)}\,(\rho_{T,n}(\gamma)\,(U_2^{-1}e_y)(\mathfrak{x},\cdot))(\beta). \tag{14}$$

In particular ε_y divides ϖ^{j_0}: this element is the (y,y) entry of $U_2\cdot\varpi^{j_0}\cdot U_2^{-1}$, a sum of terms each divisible by ε_y, by (14) for $\gamma = 1$ and $\mathrm{d}(\beta)\le j_0$. Choose ε'_y with $\varepsilon_y\varepsilon'_y = \varpi^{j_0}$.

Step 3: the determinant. Up to a unit, $\det\Delta = \prod_y \varepsilon_y$. For each y let m_y be the number of the $j < j_0$ such that ε_y does not divide ϖ^j. These j form an initial segment, and ε_y divides ϖ^{j_0}; so ε_y divides ϖ^{m_y}, and $\sum_y m_y$ is the sum over $j<j_0$ of the number of the y with $\varepsilon_y \nmid \varpi^j$. For a fixed j, apply Lemma 9.1 to the matrices U_2^{-1} and U_2, to these y, the constant labelling 0 and the set $\mathcal{B}$ of the β with $\mathrm{d}(\beta) < j$: if some coordinate in (14) with $\mathrm{d}(\beta)<j$ had a unit as second factor, ε_y would divide ϖ^j. So the number of these y is at most $\#\mathcal{X}\cdot\dim \mathrm{Dp}^{k,n}_{T,i_1}(j)$. This is (ii).

Step 4: the excess condition. Let F, $\mathfrak{e}$ and $h\ge 2c$ be as in (iv), let γ lie in $\mathrm{Lev}_{\tau_{k,i_1}(\cdot)}(W)$ and let β be a family whose excess at a letter s_0 exceeds h. Put $F^\circ = U_2^{-1}\,\mathrm{diag}(\varepsilon'_y)U_1^{-1}F$; then $JF^\circ = \varpi^{j_0}AF$. Let $F_1 = F^\circ(\mathrm{pt}(\mathfrak{e}),\cdot)$. The letter matrices of γ are block diagonal, so $\rho_{T,n}(\gamma)$ commutes with $\mathrm{diag}(\varpi^{\mathrm{d}(\beta)})$, and

$$\varpi^{j_0}\cdot(\rho_{T,n}(\gamma)(AF)(\mathfrak{e},\cdot))(\beta) = \varpi^{\mathrm{d}(\beta)}\cdot(\rho_{T,n}(\gamma\gamma_{\mathfrak{e}})F_1)(\beta).$$

As $\varpi\neq 0$ (Lemma 3.1), we have to show that ϖ^{j_0+c} divides the right side.

Let $\omega = \mathrm{dp}_{T,i_1}(\cdot) - \frac12\mathrm{ex}^{i_0,i_1}_{T,s_0}(\cdot)$, a weight on the variables. For a family β' let $\mathrm{ord}(\beta')$ be the largest natural number, at most $\max(\lceil j_0 - \langle\omega,\beta'\rangle\rceil, 0)$, such that $\varpi^{\mathrm{ord}(\beta')}$ divides $(\rho_{T,n}(\gamma')F_1)(\beta')$ for every γ' in $\mathrm{GL}_k(W)$. Choose a family β_0 for which $\theta_0 = \mathrm{ord}(\beta_0) + \langle\omega,\beta_0\rangle$ is least. Then, for every γ' and every β',

$$\varpi \ \text{ to the power } \ \max(\lceil\theta_0 - \langle\omega,\beta'\rangle\rceil, 0) \ \text{ divides } \ (\rho_{T,n}(\gamma')F_1)(\beta'), \tag{15}$$

because the natural number $\mathrm{ord}(\beta')$ is at least $\theta_0 - \langle\omega, \beta'\rangle$. We claim that $\theta_0 \geq j_0$. If $\mathrm{ord}(\beta_0)$ is the bound in its definition, this is clear. Otherwise there is γ_0 such that $\varpi^{\mathrm{ord}(\beta_0)+1}$ does not divide $(\rho_{T,n}(\gamma_0)F_1)(\beta_0)$. On the one hand, Step 1 for F° and $\pi = \varpi^{j_0}$ shows that ϖ^{j_0} divides $\varpi^{\mathrm{d}(\beta_0)}(\rho_{T,n}(\gamma_0)F_1)(\beta_0)$; so $j_0 \leq \mathrm{ord}(\beta_0) + \mathrm{d}(\beta_0)$. On the other hand, Theorem 1 applies to the indices $i_0 \neq i_1$, the letter s_0, the constant labelling 0, the weight ω, the polynomial of the form $\rho_{T,n}(\gamma_0)F_1$, the exponent β_0 and the order $e = \mathrm{ord}(\beta_0)$: its hypothesis on the translates is (15), by Lemma 3.1(c), and the hypothesis on ω holds because of the two root variables of a letter exactly $\mathrm{rt}_{T,s}(i_0, i_1)$ is deep for $\mathfrak{t} = i_1$, so that the difference in question is 1 less half a difference of two excess weights, which lies in $[0.4, 1.6]$ by Lemma 3.3. The theorem gives $\langle \mathrm{ex}^{i_0,i_1}_{T,s_0}(\cdot), \beta_0\rangle \leq 0$, that is, $\langle\omega, \beta_0\rangle \geq \mathrm{d}(\beta_0)$; and so $\theta_0 \geq \mathrm{ord}(\beta_0) + \mathrm{d}(\beta_0) \geq j_0$.

Finally, by (15) for $\gamma' = \gamma\gamma_{\mathfrak{e}}$ and β, the coordinate $(\rho_{T,n}(\gamma\gamma_{\mathfrak{e}})F_1)(\beta)$ is divisible by ϖ to a power that is at least $\theta_0 - \langle\omega, \beta\rangle \geq j_0 - \mathrm{d}(\beta) + \frac{1}{2}\langle \mathrm{ex}^{i_0,i_1}_{T,s_0}(\cdot), \beta\rangle > j_0 - \mathrm{d}(\beta) + c$, as the excess of β exceeds $h \geq 2c$. □

LEAN `TwistingData.exists_depth_factorization` · `TwistingData.root_lemma` · `TwistingData.p_dvd_letterMatrix_of_mem_parahoric` · `TwistingData.excessWeight_mem_Icc` · `IsValuationPair.hermite` · `IsValuationPair.smith_normal_form` · `TwistingData.card_le_finrank_translatesIn`

Proof of Theorem 5. Notation. Put $\kappa_0 = (k + k^2)\,oL \cdot L$ and $c = 2\kappa_0 + 2$; the seventh size condition reads $6c + 2 \leq y$. Let i_1 be the index $\mathfrak{t}$ and $i_0 = \mathfrak{t} - 1$; they exist because $0 < \mathfrak{t}$ and $2\mathfrak{t} \leq k$. Let j_0 be a bound for the depths $\mathrm{depth}^{k,n}_{T,\mathfrak{t}}(\beta)$. The residue field $\mathcal{O}/\mathfrak{m}$ has characteristic p, as p is not a unit, and $\bar{T}$ satisfies the same size conditions as T; also ϖ is not a unit, and $\varpi \neq 0$. Put $\mathcal{X}' = \mathcal{H} \times \mathcal{X}$, $\mathcal{E}' = \mathcal{H} \times \mathcal{E}$ and $\mathcal{Y} = \mathcal{X}' \times \mathrm{Mon}^L_{k,n}$; then $\#\mathcal{Y} = \#\mathcal{X} \cdot \#\mathcal{I}$, by the bijection $\jmath$. Write $\mathrm{Par} = \mathrm{Par}_{\tau_{k,\mathfrak{t}}(\cdot)}(W)$.

Two factorisations. The map of the hypothesis is an injective map of the finite set $\mathcal{E}$ to itself, hence bijective. So Proposition 9.3 applies, with c, (i_0, i_1), j_0, to the sets $\mathcal{X}'$ and $\mathcal{E}'$ with

$$\mathrm{pt}_1(\mathfrak{h}, \mathfrak{e}) = (\mathfrak{h}, \mathrm{src}(\mathfrak{e})) \ \text{ and the elements } \gamma^{\mathsf{T}}_{\mathfrak{e}}; \qquad \mathrm{pt}_2(\mathfrak{h}, (\mathfrak{x}, \mathfrak{q})) = (\mathfrak{h}, \mathfrak{x}) \ \text{ and the } \tilde{\mathfrak{q}}^{-1}.$$

Let $J_1 = A_1\Delta_1$ and $J_2 = A_2\Delta_2$ be the factorisations, with left inverses A'_1, A'_2.

The projector. Let w and Z be as in Theorem 4 for c, (i_0, i_1), λ and $h = y$; its hypotheses are the fifth and sixth size conditions. On vectors indexed by $\mathcal{E}' \times \mathrm{Mon}^L_{k,n}$ let $\mathbf{u}$ act by $u_{\mathfrak{e}}$ on the coordinate in $\mathcal{H}$, for each $\mathfrak{e}$, let $\mathbf{Z}$ act by Z and $\mathbf{w}$ by $\mathrm{diag}(w(\beta))$ on the coordinate in $\mathrm{Mon}^L_{k,n}$.

The pairing. We apply Proposition 9.2 to the sets $\mathcal{E}'$ (in the place of $\mathcal{X}$) and $\mathcal{Y}$ (in the place of $\mathcal{N}$), the matrices

$$A = A_1, \qquad B = \mathbf{u}^{-1}\mathbf{Z}\,A_2'^{\mathsf{T}}, \qquad A' = A'_1, \qquad C = A_2^{\mathsf{T}}\,\mathbf{u}\,\mathbf{w},$$

the elements $\pi_1 = \varpi^{2\kappa_0} = p^{\kappa_0}$, $\pi_2 = p^e$, $\pi_0 = \varpi$, the labelling $\tau_{k,\mathfrak{t}}(\cdot)$ and the set $\mathcal{B}$ of the families whose excess for (i_0, i_1) exceeds y at some letter; so the module of the conclusion is built from $\mathrm{Ex}^{i_0,i_1}_{\bar{T},n}(0; y)$. Hypotheses (i) and (ii) hold. For (iii): $CB = A_2^{\mathsf{T}}\,\mathbf{w}\mathbf{Z}\,A_2'^{\mathsf{T}}$, so its (y_1, y_2) entry is the sum over $\mathfrak{e}'$ in $\mathcal{E}'$ of $\sum_\beta F'(\beta)w(\beta)(ZF)(\beta)$ for the blocks $F' = (A_2 e_{y_1})(\mathfrak{e}', \cdot)$ and $F = (A_2'^{\mathsf{T}} e_{y_2})(\mathfrak{e}', \cdot)$. By Proposition 9.3(iv), and because $2c \leq y - (4c + 2)$, the block F' lies in

$$\mathrm{Ex}^{i_0,i_1}_{T,n}(\varpi^c\mathcal{O}; y - (4c + 2)).$$

By Theorem 4(ii) the entry is congruent modulo ϖ^{c} to p^{κ_0} times the (y_1, y_2) entry of $A_2^{\mathsf{T}} A_2'^{\mathsf{T}} = 1$; and $\pi_1 \pi_0 = \varpi^{c-1}$. For (iv): if $A_1 F_1 - B F_2 = \varpi^{c-1} F$, then each block of $A_1 F_1$ lies in $\mathrm{Ex}^{i_0,i_1}_{T,n}(\varpi^{c}\mathcal{O}; y)$ by Proposition 9.3(iv), and each block of $B F_2$ is Z applied to a form, so it lies there by Theorem 4(i). Hence, for γ in the Levi subgroup and β in $\mathcal{B}$, ϖ^{c} divides ϖ^{c-1} times the coordinate of the translate of a block of F, and that coordinate lies in $\mathfrak{m}$. The proposition gives a natural number l with

$$\#\mathcal{Y} + l \le \#\mathcal{E}' \cdot \dim_{\mathcal{O}/\mathfrak{m}} \mathrm{Ex}^{i_0,i_1}_{\bar{T},n}(0; y), \qquad [\mathcal{O}^{\mathcal{Y}} : C A_1(\mathcal{O}^{\mathcal{Y}}) + p^{e}\mathcal{O}^{\mathcal{Y}}] \;\big|\; p^{\kappa_0 \#\mathcal{Y} + e l}. \tag{16}$$

The chain. Let $e_0 = \#\mathcal{X}' \cdot \sum_{j<j_0} \dim_{\mathcal{O}/\mathfrak{m}} \mathrm{Dp}^{k,n}_{\bar{T},\mathfrak{t}}(j)$; the determinants of Δ_1 and of Δ_2^{T} divide ϖ^{e_0}. By Input 1(a), $[\mathcal{O}^{\mathcal{Y}} : \Delta_1(\mathcal{O}^{\mathcal{Y}}) + p^{e}\mathcal{O}^{\mathcal{Y}}] \mid \varpi^{e_0}$, and the same for Δ_2^{T}. Input 1(d), used for the endomorphism $C A_1$ with (16) and then for Δ_2^{T}, gives

$$[\mathcal{O}^{\mathcal{Y}} : \Delta_2^{\mathsf{T}} C A_1 \Delta_1(\mathcal{O}^{\mathcal{Y}}) + p^{e}\mathcal{O}^{\mathcal{Y}}] \mid p^{l'}, \qquad l' = e_0 + \kappa_0 \,\#\mathcal{Y} + e\, l,$$

because $\varpi^{2e_0} = p^{e_0}$; and $\Delta_2^{\mathsf{T}} C A_1 \Delta_1 = J_2^{\mathsf{T}}\, \mathbf{u}\, \mathbf{w}\, J_1$.

The exponent. By Theorem 3(b) for $\bar{T}$, $e_0 \le (1+z)\,\#\mathcal{X} \cdot \#\mathcal{J}$. By (16) and Theorem 3(a) for $\bar{T}$, with $h = y$ and the index $\#\mathcal{Q}$ of the parahoric subgroup,

$$l \;\le\; \#\mathcal{H} \cdot \#\mathcal{X} \cdot \left(\#\mathcal{Q} \cdot \dim \mathrm{Ex}^{i_0,i_1}_{\bar{T},n}(0; y) - (\#\mathrm{Mon}_{k,n})^{L}\right) \;\le\; \kappa \cdot \#\mathcal{X} \cdot \#\mathcal{J}.$$

As $1 + \kappa_0 = c/2 \le y$, we get $l' \le (z + \kappa e + y) \cdot \#\mathcal{X} \cdot \#\mathcal{J}$.

The operator. Let r be the isomorphism from $\mathcal{O}^{\mathcal{Y}}$ to $\mathcal{O}^{\mathcal{X}\times\mathcal{J}}$ with

$$(\mathrm{r}G)(\mathfrak{x}, \mathfrak{i}) = G((\mathfrak{h}_{\mathfrak{i}}, \mathfrak{x}), \beta_{\mathfrak{i}}).$$

We claim that, for every G in $\mathcal{O}^{\mathcal{Y}}$,

$$\mathrm{r}\big(J_2^{\mathsf{T}}\, \mathbf{u}\, \mathbf{w}\, J_1\, G\big) = \big(\mathrm{r}(\mathbf{w} G)\big)^{\flat}, \tag{17}$$

with $F \mapsto F^{\flat}$ the operator of the statement. By the definitions of J_1 and J_2, the coordinate of $J_2^{\mathsf{T}}\mathbf{u}\mathbf{w}J_1 G$ at $((\mathfrak{h}, \mathfrak{x}), \beta)$ is

$$\sum_{\mathfrak{q}} \sum_{\mathfrak{h}', \beta', \beta''} \varpi^{2\mathrm{d}(\beta')}\, \rho_{T,n}(\tilde{\mathfrak{q}}^{-1})_{\beta',\beta}\, (u_{(\mathfrak{x},\mathfrak{q})})_{\mathfrak{h},\mathfrak{h}'}\, w(\beta') \cdot \rho_{T,n}(\gamma^{\mathsf{T}}_{(\mathfrak{x},\mathfrak{q})})_{\beta',\beta''}\, G((\mathfrak{h}', \mathrm{src}(\mathfrak{x}, \mathfrak{q})), \beta''),$$

with $\mathrm{d}(\beta')$ the depth. By Theorem 4(iii), $w(\beta')\, \rho_{T,n}(\gamma^{\mathsf{T}})_{\beta',\beta''} = w(\beta'')\, \rho_{T,n}(\gamma)_{\beta'',\beta'}$; with $\varpi^2 = p$ the sum over β' becomes the (β'', β) entry of the matrix $H_{(\mathfrak{x},\mathfrak{q})}$ of the statement, and (17) follows.

Conclusion. By hypothesis $Y(\mathrm{im}\,\Psi) \subseteq \mathrm{im}\,\Psi$, and by (17) the map $\Psi \circ \mathrm{r}$ sends the image of $J_2^{\mathsf{T}}\mathbf{u}\mathbf{w}J_1$ into $Y(\mathrm{im}\,\Psi)$. Input 1(c), for the map $\Psi \circ \mathrm{r}$, carries the index statement of the chain to $[\mathrm{im}\,\Psi : Y(\mathrm{im}\,\Psi) + p^{e}\,\mathrm{im}\,\Psi] \mid p^{l'}$. □

LEAN TwistingData.hasType_twisted_hecke · TwistingData.exists_depth_factorization · TwistingData.exists_projector · TwistingData.of_pairing · TwistingData.index_mul_finrank_excessCond_le · TwistingData.sum_finrank_deepForms_le · IndexDvd.of_det_dvd · IndexDvd.comp · IndexDvd.map

10 Where the results are used

Theorem 5 is used once, in the unit of the columns (unit 145), for the operator attached to a marked prime of the field of the data. There W is the valuation ring of that field at the prime; the letters are the infinite places of a certain class (the tops) that induce the prime or its complex conjugate, with the embeddings of these places, or their composites with complex conjugation, as the maps ι_s; $\mathcal{O}$ is the ring of the complex numbers that are integral for an embedding into a p-adic field; k is the rank of the columns, n the number that the handout calls $a(D)$, and the threshold $\mathfrak{t}$ the least natural number such that p to that power is at least the number of lifts of the data; and the module is that of the integral columns. The size conditions are verified in the same unit. The result enters Theorem A of the write-up through the index statement (b) of that theorem: it is the price, in index, of regaining at some primes the divisibility that the theta lifts lose there. Theorems 1 to 4 and the other results of the unit are used only inside it.

11 Notes on the formal text

1. *The three declarations without a name.* The unit has three instances: that the family $\mathrm{ad}_{\varpi}(\cdot)$ has the three properties of Definition 2.7(a), that $\mathrm{Gs}_{\mathcal{F},\omega}(\cdot)$ has them when $\mathcal{F}$ has, and that a parahoric subgroup has finite index when the residue field is finite. The first two are stated after Definition 2.7; the third is used silently wherever an index of a parahoric subgroup is counted.

2. *Letters.* The formal text writes N for the number of letters (here L, the letter N being kept in this write-up for the level), r for the threshold (here $\mathfrak{t}$), ℓ for $L/2$ (here λ), a, a' for the pair of indices (here i_0, i_1), i for the base letter (here s_0), w for ϖ, o_0 for o, R, O, K for the ring of the data, g, h, u for group elements (here γ), c, x for weights (here ω), P, v, D, Ω for ϱ, par, δ, $\mathcal{C}$ in the statements on cones, ν, β for the points ξ, η there, Q, M, R, β and a, z, w for the matrices A, B, A', C and the elements π_1, π_2, π_0 of Proposition 9.2, Q, d, R, s, g, x for A, Δ, A', pt, γ, j_0 of Proposition 9.3, b, θ for w, h of Theorem 4, κ, P, D, f, x for $\mathcal{M}$, Ω, n_0, e, ω of Proposition 7.3, and f, M, J, m, c for Ψ, V, $\jmath$, e, $\mathfrak{x}$ of Theorem 5.

3. *Arguments that a reading by program adds.* In seventeen declarations of the unit a reading of the text by program lists as an argument a name that the declaration binds itself, because a variable of the same name is declared earlier in the file: the group element in the letter matrices, in the action, in the matrix on forms and in the conditions on forms; the base letter in the excess condition; the letter in the Frobenius operator and in three lemmas on it; the matrices and the exponent in Proposition 9.3; a family in Theorem 4; the point of $\mathcal{X}$ in the formula of Theorem 5; the element ϖ in Theorem 1, where the statement mentions only the component of T; the name under which the degree is passed on in the depth; and the index of the cone of Definition 2.8(d), listed twice. This note states every declaration with the arguments that its uses in the artifact show: the depth has five, the conditions on forms five and six, the Frobenius operator five. Lean was not asked.

4. *An argument that the reading by program misses.* In Proposition 5.2 the formal statement names the determinant by a local definition before the equation, and the reading by program stops there; so it does not see that the statement depends

on k. Lean takes k as the first, implicit, argument: the proof of Theorem 2 applies the proposition with eight arguments, of which k is the first.

5. *Counting an infinite set.* The number of elements of the residue field in Lemma 4.1, Lemma 7.1 and Proposition 7.3 is, in the formal text, a natural number that is 0 for an infinite field. For twisting data the field is finite, by the last condition of Definition 2.1.

6. *Hypotheses that are not used.* Setting 2.2 asks for $L \geq 1$. Some elementary lemmas of the unit do not ask for it (parts of Lemma 3.1, Lemma 8.3); nothing is lost, there being no twisting data for $L = 0$. Lemma 3.4, Lemma 6.2, Lemma 6.3 and Definition 2.8(d) are stated in the formal text for data over a field, which their proofs do not use. Of the ten size conditions, Theorem 3(a) uses the third, fourth, eighth and ninth, Theorem 3(b) the first, second, fourth, fifth, sixth, ninth and tenth, Theorem 4 the fifth and sixth; Theorem 5 uses all.

7. *Arguments with default values.* In Lemma 4.2 the second homomorphism is, by default, the first, and hypotheses (ii) and (iii) are then discharged by Lean; hypothesis (iv) of Lemma 8.2 and the weights of the Hermite form have defaults as well.

8. *Definitions written out.* The formal text defines the cone of Definition 2.8(c) through the step of (a); the two inequalities written there are that definition unfolded. Volumes are taken as real numbers. In Theorem 5 the formal statement takes an enumeration of the cosets as an argument and the chosen representative of a coset from the library; the module on which Ψ is defined is written as a module of functions of two arguments.

9. *The last line of Theorem 5.* It is the index divisibility of Input 1: for every finite family in the image of Ψ a square matrix with determinant dividing p^l moves it into $Y(\operatorname{im}\Psi) + p^e \operatorname{im}\Psi$. Nothing is assumed about torsion in V, and the residue field of $\mathcal{O}$, over which the dimensions of Theorem 3 are taken in the proof, is not the residue field of W.

10. *The constant of Theorem 5.* The proof fixes $c = 2(k + k^2)oL \cdot L + 2$, the number in the seventh size condition; an unrelated variable of the same name is in scope in the file.

11. *Assemblies.* Theorem 5 assembles Theorems 3 and 4 and Propositions 9.2 and 9.3; Theorem 3(a) assembles Theorem 2, Proposition 6.5 and Lemmas 6.1 to 6.4. Their own content is the bookkeeping written out in their proofs.

12. *Sources named by the artifact.* The headers of three files of the unit call parts of the argument known and name related work. This note has not compared any of it; the titles and years are those of the headers. For the root lemma: counts of zeros against degree, after Stepanov, and E. Bombieri, *Counting points on curves over finite fields*, Sém. Bourbaki 430 (1973); Ghate and Jana, *Modular representations of* $\mathrm{GL}_2(\mathbb{F}_q)$ *using calculus* (2025); Imai and Koskivirta, *Automorphic vector bundles on the stack of G-zips* (2021). For the dimension bounds: volumes of interlacing regions, Frobenius amplification and counting by leading terms along flags; Forrester and Zhang, *Corank-1 projections and the randomised Horn problem* (2021), after Dixon (1905) and Anderson (1991); Bombieri, as above; Okounkov, *Brunn–Minkowski inequality for multiplicities* (1996). For the

projector: the Schur algebra as the dual of the polynomials in a generic matrix, and truncation by weights; Benson and Doty, *Schur–Weyl duality over finite fields* (2009); J. A. Green, *Polynomial representations of* GL_n (1980); Donkin, *On Schur algebras and related algebras I* (1986). The file of Input 3 names the two papers cited after the Inputs.

How this note was written. From the dossier of the unit: its 78 statement readings and its structured proofs; the program check of the dossier covered 320 leaf steps of 32 proofs. A second reader compared all readings with the Lean text and sampled every proof; the corrections of that reading are followed here: in Proposition 9.3 the root lemma is applied with the order $\mathrm{ord}(\beta_0)$ and the inequality $j_0 \le \mathrm{ord}(\beta_0) + \mathrm{d}(\beta_0)$ is not strict, the numbers m_y are counts and not valuations, the first factorisation in the proof of Theorem 5 uses the transposed elements, the divisibility in the proof of Theorem 4 rests on the condition on coefficients and not on invariance alone, and Theorem 2 is an identity for every real $\varrho > 1$. I read the eight files of the unit completely, and in the background library the definitions of the conventions and the statements of the Inputs; I did not read the proofs of the Inputs. The places where the dossier's reader was unsure were read in the source: the seventeen declarations of note 3, the statement of Proposition 5.2 with the place where it is applied, and the long proofs of Theorems 1, 4 and 5 and of Propositions 6.5, 7.3, 8.4, 9.2 and 9.3, whose automation was not replayed. Each inequality that the formal text leaves to automation was checked by hand, among them the constants 0.4, 1.6, $\frac{1}{20}$ and 5 of Theorem 1. The proofs follow the formal ones in substance. They differ in form in these places: Lemma 4.2 is proved term by term instead of by induction on the polynomial; the determinant in Proposition 5.2 is computed by an elimination instead of a factorisation and Sylvester's identity; Lemma 7.2(ii) is proved by expanding the product instead of by an adjointness for a pairing of polynomials; in Proposition 6.6 the determinant is seen not to vanish from the shape of the matrix instead of by a specialisation; Lemma 8.5 is proved directly; in Proposition 7.3 the form is a sum over the degrees instead of the result of an induction; and Theorem 3(b) averages over cyclic shifts instead of reflections. The diagram, the remarks after the definitions and the paragraph after Proposition 7.3 are mine. Lean was not run for this note.

An independent referee read the note against the Lean text. In the revision of 26 November 2026, after that report: the threshold and the parity received the letters $\mathfrak{t}$ and par (they were μ and ς); three steps were written out, namely the zero test in the proof of Lemma 7.1, the index of the parahoric subgroup in the proof of Theorem 3(a), and the units in the proof of Lemma 7.2; the references for the Inputs were sorted; and the notation list was completed. For the zero test and for the power of p in a binomial coefficient the formal proofs call lemmas of the library Mathlib (a homogeneous polynomial of degree at most the number of elements of an integral domain that vanishes at every point is zero; the exponent of p in $\binom{l'}{j}$ is at most the logarithm of l' to the base p). The proofs written here are mine and were not compared with the library's.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Definition 2.10	`ClassMean.conjSubst`	ClassMeans.lean	23–24
Theorem 4	`ClassMean.map_C_mul_mul_C`	ClassMeans.lean	26–28
Lemma 7.1	`ClassMean.eq_top_of_eval_monomial`	ClassMeans.lean	36–50
Lemma 7.2	`ClassMean.exists_invariant_diagonal_form`	ClassMeans.lean	57–71
Proposition 7.3	`ClassMean.exists_mean`	ClassMeans.lean	80–136
Lemma 6.1	`index_borel_le`	Counting.lean	29–38
Definition 2.8	`TwistingData.excessCone`	Counting.lean	43–44
Proposition 6.5	`TwistingData.finrank_excessCond_le_volume`	Counting.lean	49–115
Lemma 6.2	`TwistingData.excessWeight_dotProduct_permStep`	Counting.lean	120–127
Lemma 6.3	`TwistingData.exists_mem_excessCone_mem_Icc`	Counting.lean	131–145
Lemma 6.4	`volume_sum_mem_Icc_le`	Counting.lean	149–159
Definition 2.9	`HeckeSizes`	Counting.lean	163–166
Theorem 3	`TwistingData.index_mul_finrank_excessCond_le`	Counting.lean	173–203
Theorem 3	`TwistingData.sum_finrank_deepForms_le`	Counting.lean	210–233
Lemma 3.4	`TwistingData.sum_deepIndicator`	DeepFormCost.lean	13–16
Proposition 6.6	`TwistingData.exists_le_weight_of_mem_deepForms`	DeepFormCost.lean	21–92
Lemma 9.1	`TwistingData.card_le_finrank_translatesIn`	ElementaryDivisors.lean	34–40
Proposition 9.2	`TwistingData.of_pairing`	ElementaryDivisors.lean	45–69
Proposition 9.3	`TwistingData.exists_depth_factorization`	ElementaryDivisors.lean	77–137
Lemma 8.6	`exists_tensor_dual`	ElementaryDivisors.lean	142–146
Theorem 4	`TwistingData.exists_projector`	ElementaryDivisors.lean	153–208
Theorem 5	`TwistingData.hasType_twisted_hecke`	ElementaryDivisors.lean	217–261
Definition 2.10	`TwistingData.frobGenericMatrix`	FrobeniusFactors.lean	20–20
Definition 2.10	`TwistingData.frobOperator`	FrobeniusFactors.lean	25–26
Definition 2.10	`TwistingData.letterConj`	FrobeniusFactors.lean	30–31
Definition 2.10	`TwistingData.indexWeight`	FrobeniusFactors.lean	35–35
Lemma 8.1	`TwistingData.frobOperator_map`	FrobeniusFactors.lean	44–56
Lemma 8.2	`TwistingData.frobOperator_mem`	FrobeniusFactors.lean	62–83
Lemma 4.2	`TwistingData.X_mem_gaussFiltration`	FrobeniusFactors.lean	85–86
Lemma 8.3	`TwistingData.p_dvd_cj_frobenius_sub`	FrobeniusFactors.lean	89–96
Proposition 8.4	`TwistingData.pow_dvd_coeff_conj_frobOperator`	FrobeniusFactors.lean	102–131
Lemma 8.5	`TwistingData.degree_le_of_coeff_frobOperator`	FrobeniusFactors.lean	133–138
Definition 2.8	`permStep`	InterlacingCone.lean	14–15
Definition 2.8	`slab`	InterlacingCone.lean	18–18
Definition 2.8	`pieriCone`	InterlacingCone.lean	22–23
Lemma 5.1	`det_interlacing_indicator`	InterlacingCone.lean	27–36
Proposition 5.2	`isBounded_inter_slab`	InterlacingCone.lean	40–42
Proposition 5.2	`volume_inter_slab_eq_signed_sum`	InterlacingCone.lean	47–106
Theorem 2	`prod_mul_volume_inter_slab`	InterlacingCone.lean	111–138
Lemma 4.1	`stepanov_count`	RootLemma.lean	27–34
Definition 2.7	`IsRealFiltration`	RootLemma.lean	38–38
Definition 2.7	`adicFiltration`	RootLemma.lean	44–44
Definition 2.7	`gaussFiltration`	RootLemma.lean	53–54
Lemma 4.2	`mem_adicFiltration_of_nonpos`	RootLemma.lean	58–59
Lemma 4.2	`monomial_mem_gaussFiltration`	RootLemma.lean	61–62
Lemma 4.2	`aeval_sub_mem_of_gaussFiltration`	RootLemma.lean	70–81
Theorem 1	`TwistingData.root_lemma`	RootLemma.lean	88–154
Definition 2.3	`twoBlock`	TwistingData.lean	21–21
Definition 2.1	`TwistingData`	TwistingData.lean	25–34
Definition 2.1	`TwistingData.P`	TwistingData.lean	42–42
Definition 2.1	`TwistingData.q`	TwistingData.lean	45–45
Definition 2.1	`TwistingData.map`	TwistingData.lean	49–51
Definition 2.3	`TwistingData.letterMatrix`	TwistingData.lean	55–56

In this note	Lean declaration	File	Lines
Definition 2.3	`TwistingData.polyAction`	TwistingData.lean	60–61
Definition 2.3	`TwistingData.letterRep`	TwistingData.lean	65–66
Definition 2.4	`TwistingData.deepIndicator`	TwistingData.lean	70–70
Definition 2.4	`TwistingData.depth`	TwistingData.lean	74–74
Definition 2.5	`TwistingData.rootVar`	TwistingData.lean	77–77
Definition 2.5	`TwistingData.excessWeight`	TwistingData.lean	81–82
Definition 2.6	`TwistingData.translatesIn`	TwistingData.lean	86–87
Definition 2.6	`TwistingData.excessCond`	TwistingData.lean	91–92
Definition 2.6	`TwistingData.deepForms`	TwistingData.lean	96–96
Lemma 3.1	`TwistingData.one_lt_P`	TwistingData.lean	103–103
Lemma 3.1	`TwistingData.one_lt_q`	TwistingData.lean	105–105
Lemma 3.1	`TwistingData.frobenius_pow_congr`	TwistingData.lean	108–109
Lemma 3.1	`TwistingData.w_ne_zero`	TwistingData.lean	114–114
Lemma 3.1	`TwistingData.finite_residueField`	TwistingData.lean	117–117
Lemma 3.2	`TwistingData.p_dvd_letterMatrix_of_mem_parahoric`	TwistingData.lean	119–124
Lemma 3.1	`TwistingData.polyAction_X`	TwistingData.lean	128–129
Lemma 3.1	`TwistingData.diagonal_pow_depth`	TwistingData.lean	131–133
Lemma 3.1	`TwistingData.polyAction_polyOfCoeffs`	TwistingData.lean	137–140
Lemma 3.1	`TwistingData.letterRep_map`	TwistingData.lean	142–142
Lemma 3.3	`TwistingData.one_add_val_sub_one_sub`	TwistingData.lean	144–145
Lemma 3.3	`TwistingData.five_mul_P_pow_le_q`	TwistingData.lean	147–148
Lemma 3.3	`TwistingData.excessWeight_mem_Icc`	TwistingData.lean	150–152
Input 1	`IndexDvd`	ModuleIndex.lean	20–20
Input 1	`HasType`	ModuleIndex.lean	55–56
Input 1	`IndexDvd.of_det_dvd`	ModuleIndex.lean	68–73
Input 1	`IndexDvd.trans`	ModuleIndex.lean	31–33
Input 1	`IndexDvd.map`	ModuleIndex.lean	37–41
Input 1	`IndexDvd.comp`	ModuleIndex.lean	50–51
Input 2	`IsValuationPair.hermite`	FrameAction.lean	58–69
Input 2	`IsValuationPair.smith_normal_form`	ColumnLattices.lean	23–44
Input 2	`isValuationPair_top`	FrameAction.lean	50–51
Input 3	`exists_le_cost_of_vanishing`	ZerosCostDegree.lean	271–313
Input 4	`card_GL_eq_prod`	CuspidalAtScalars.lean	17–21

The five theorems and the twisting data as they are written in the artifact.

```
structure TwistingData where
  (p o₀ : ℕ) ι : ZMod N → W →+* R
  isEven : ZMod N → _
  w : R
  prime : p.Prime
  five_le_P : 5 ≤ p ^ o₀
  w_sq : w ^ 2 = p
  frobenius_congr s x : ↑p ∣ ι (s + 1) x - ι s x ^ p ^ o₀
  isEven_succ s : isEven (s + 1) = !isEven s
  card_residueField : .card (ResidueField W) = (p ^ o₀) ^ N

lemma TwistingData.root_lemma i (ha : a ≠ a') τ c {f σ} (l : ℕ) (h : ∀ g ∈ levi W τ,
  T.polyAction g⁻¹ f ∈ gaussFiltration (adicFiltration T.w)
  c (l + weight c σ)) (h0 : ¬T.w ^ (l + 1) ∣ f.coeff σ) (hτ : τ a = τ a') (hc : ∀ s, c (T.rootVar a
  a' s) - c (T.rootVar a' a s) ∈ Set.Icc 0.4 1.6) : weight (T.excessWeight i a a') σ ≤ 0

lemma prod_mul_volume_inter_slab {N} [NeZero N] {v P} (hP : 1 < P) (D : ZMod N → ℝ) {Ω} (hΩ : ∀ v,
  permStep v P 1 v ∈ Ω ↔ v ∈ pieriCone v P) : (∏ l : Fin k, ((P ^ N) ^ (l.1 + 1) - 1) /
  (P ^ N - 1)) * volume.real (Ω ∩ slab k D) = volume.real (slab k D)

lemma TwistingData.index_mul_finrank_excessCond_le [CharP K T.p] (ha : a.1 + 1 = a') {n ℓ h κ z}
  (big : HeckeSizes T.P T.o₀ N k n a' ℓ κ h z) : (parahoric W (twoBlock k a')).index * finrank K
```

```
  (T.excessCond n a a' ⊥ h) ≤ (1 + κ) * Fintype.card (Monomials (Fin k) n) ^ N

lemma TwistingData.sum_finrank_deepForms_le [CharP K T.p] {n r ℓ κ h z} (big :
  HeckeSizes T.P T.o₀ N k n r ℓ κ h z) M : (∑ d ∈ range M, finrank K (T.deepForms k n r d) : ℝ) ≤
  (1 + z) * Fintype.card (Monomials (Fin k) n) ^ N

lemma exists_projector (hp : ¬IsUnit (T.p : O)) (ha : a + 1 = r.1) {ℓ} θ (e : ℓ + ℓ = N)
    (h : N * (n * (k * (ℓ + 2))) + n + k < T.P ^ ℓ) : ∃ b Z, (∀ F, Z *ᵥ F ∈ T.excessCond n a r
    (span {T.w ^ c}) θ) ∧ (∀ y ∈ T.excessCond n a r (span {T.w ^ c}) (θ - (4 * c + 2)), ∀ F,
    T.w ^ c ∣ y ·ᵥ (b * Z *ᵥ F) - T.p ^ ((k + k ^ 2) * (T.o₀ * N) * N) * y ·ᵥ F) ∧
    ∀ γ α β, b β * T.letterRep n (transposeGL γ) β α = b α * T.letterRep n γ α β

lemma hasType_twisted_hecke (hp : ¬IsUnit (T.p : O)) {r ℓ κ y z} (big : HeckeSizes T.P T.o₀ N k n r
  ℓ κ y z) {I M : Type} [Fintype H] [Fintype (_ ⧸ parahoric W (twoBlock k r))] [Fintype I]
  [AddCommGroup M] [Module O M] (f : (X → I → O) →ₗ[O] M) {Y : End O M} (src :
  X × _ ⧸ parahoric W (twoBlock k r) → X) (g : _ → GL (Fin k) W) (u : _ → GL H O) (J : I ≃ _ × H)
  (hs : Injective fun e ↦ ((src e, ↑(transposeGL (g e))⁻¹) : X × _ ⧸ parahoric W (twoBlock k r)))
  (hY : ∀ F, Y (f F) = f fun c i ↦ ∑ b, ∑ j, u (c, b) (J i).2 (J j).2 * (T.letterRep n (g (c, b)) *
  diagonal ((T.p : O) ^ T.depth k n r ·) * T.letterRep n b.out⁻¹ :) (J j).1 (J i).1 * F (src (c, b))
  j) m : HasType ↑T.p f.range m Y ((z + κ * m + y) * (Nat.card X * Nat.card I))
```

In these texts `ZMod N` is $\mathbb{Z}/L$, `Fin k` the set of the indices, `levi W τ` and `parahoric W τ` the two subgroups of the conventions, `⊥` the zero ideal, `weight c α` the sum $\langle\omega, \alpha\rangle$, and `volume.real` the volume as a real number; note 2 of §11 gives the letters.

Notation

L, s, s', s_0, s_1, $[s - s_0]$ the number of letters; letters, elements of $\mathbb{Z}/L$; the representative in $\{0, \dots, L-1\}$

k, i, i', l, l', l_1, i_0, i_1 the number of indices; indices $0, \dots, k-1$; a pair of indices (l, l' are also natural numbers)

v, x_v, $x_{i,s}$ a variable (i, s); the variables of the polynomial ring

W, $\mathfrak{m}_W$, $\mathbb{F}_W$ a local ring, its maximal ideal, its residue field

$\mathcal{O}$, $\mathcal{O}'$, $\mathfrak{m}$, $\mathbb{F}$, $\mathbb{F}'$ rings; the maximal ideal of a valuation domain; fields

T, p, o, ι_s, par_T, ϖ twisting data and their components (Definition 2.1)

$\mathrm{TD}_L(W, \mathcal{O})$, P_T, q_T, $\chi_* T$, $\bar{T}$ the set of twisting data; p^o; P_T^L; the image under a ring homomorphism; the reduction

γ, γ', γ_0, γ_1, γ_v, γ_ϵ, γ^{T} elements of $\mathrm{GL}_k(W)$; the transpose

$\gamma_{T,s}$, $[\gamma]_T$, $\rho_{T,n}(\gamma)$ the letter matrix, the action on polynomials, the matrix on forms (Definition 2.3)

τ, $\tau_{k,\mathfrak{t}}(\cdot)$, $\mathrm{Lev}_\tau(W)$, $\mathrm{Par}_\tau(W)$, Lev°, Par° labellings; Levi and parahoric subgroups

n, n', n_0, $\mathfrak{t}$, λ, j, j_0, e, e_0, e_i, c, m, m_y, r natural numbers: degrees, the threshold of the depth, $L/2$, levels and steps, exponents, counts, an index of summation

α, α', α'', α^*, $\alpha_s^\pm$, ν, $\hat{\nu}$, ν_s exponents; values at root variables; exponents on generic variables

β, β', β'', β_0, $\mathrm{Mon}_{k,n}$, $\mathrm{Mon}_{k,n}^L$ families; exponents of degree n; the set of families

F, F', F_1, F_2, F_ν, F°, $F^\flat$ forms and vectors of forms; the Hecke operator applied to F

Φ, Φ', Φ_0, Φ_v, Φ_υ, Φ^*, X polynomials; a further variable (also a matrix, in Input 2)

ω, ω_0, $\omega_s^\pm$, $\langle\omega, \alpha\rangle$ weights; the weighted sum

$\mathrm{dp}_{T,\mathfrak{t}}(v)$, $\mathrm{depth}_{T,\mathfrak{t}}^{k,n}(\beta)$ the indicator of deep variables; the depth

$\mathrm{rt}_{T,s}(i_0, i_1)$, $\mathrm{ex}_{T,s_0}^{i_0,i_1}(v)$ the root variable; the excess weight

$\mathrm{Tr}_{T,n}(\tau; I, \mathcal{B})$, $\mathrm{Ex}_{T,n}^{i_0,i_1}(I; h)$, $\mathrm{Dp}_{T,\mathfrak{t}}^{k,n}(j)$ the modules of forms of Definition 2.6; I an ideal, $\mathcal{B}$ a set of families, h a real number

$\mathcal{F}$, $\mathcal{G}$, $\mathcal{G}_\pm$, θ, θ', θ_0, $\mathrm{ad}_\varpi(\theta)$, $\mathrm{Gs}_{\mathcal{F},\omega}(\theta)$ families of subgroups; real levels; the filtrations of Definition 2.7

χ, χ', χ_0, χ_1, χ_υ, $\mathcal{A}$ ring homomorphisms; an algebra

υ, υ_0, $\upsilon_\mathfrak{s}$, π, π', π_0, π_1, π_2 elements of W and points; elements of $\mathcal{O}$

ϵ, ε_y, ε'_y a real number; diagonal entries of a Smith form

par, ϱ, σ, σ', τ', δ, δ', $\mathcal{C}$ a map from the letters to the signs (for twisting data, the parity par_T), a real number > 1, permutations, levels of a slab, a cone

$\mathrm{St}_{\mathrm{par},\varrho,\sigma}$, $\mathrm{Sl}_k(\delta)$, $\mathrm{Pc}_{\mathrm{par},\varrho}$, $\mathrm{Ec}_T(i_1; h)$ the step, the slab and the cones of Definition 2.8

ξ, η, η', $\xi^\downarrow$, σ°, $\sigma^\bullet$, vol real vectors; a rearrangement; two special vectors; Lebesgue measure

q, Q, P' ϱ^L or q_T; a power of q_T; a power of P_T

κ, y, z, κ_0, $\mathrm{Sz}(P,o,L,k,n,\mathfrak{t};\lambda,\kappa,y,z)$ real numbers; the size conditions (Definition 2.9); P stands for a natural number there

X_s, $x_{s,i,i'}$, $X_T^{\mathrm{Fr}}(s,j)$, $\mathrm{Fr}_{T,\mathfrak{t},\lambda,n}(\psi)$, ψ, Π_s^m, B_s generic matrices and the Frobenius operator (Definition 2.10)

$\mathrm{cj}_{T,\gamma,s}$, $\mathrm{iw}_{T,\lambda}^{i_0,i_1}(i)$, $\mathrm{cs}_\chi(\gamma)$, $G^{(j)}$ conjugation, index weights, the substitution by conjugation; entrywise powers

$E_{ii'}$ the matrix with the entry 1 at (i,i') and 0 elsewhere

G, G_s, S, S_α, N_s, A_s, U, U', U_1, U_2, Δ, J, J_1, J_2, H matrices

A, B, A', C, C_0, C_1, A_1, A_2, Z, w, w_0, w' the matrices of §9; the projector and the diagonal form

$\mathcal{M}$, $\mathcal{M}_j$, $\mathcal{M}_s^\pm$, $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{L}_0$, $\mathcal{L}^\circ$, Ω linear forms on polynomials; a set of real numbers

V, V', V_j, V_j^Ω, M_0, M_1, M_2, M_3, M_1', M_2', Ψ, Y modules and submodules; a linear map; an endomorphism

$[M_1:M_2] \mid \pi$ the index of M_2 in M_1 divides π (Input 1)

$\mathcal{X}$, $\mathcal{X}'$, $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{H}$, $\mathcal{I}$, $\mathcal{Q}$, $\mathcal{N}$, $\mathcal{K}$, $\mathcal{S}$, $\mathcal{V}$, $\mathcal{R}$, $\mathcal{Y}$, $\mathcal{Y}_0$, $\mathcal{U}$, $\mathcal{W}$, $\mathcal{J}$, Γ, Υ, Ξ finite sets, sets of indices and of points, a ring ($\mathcal{R}$), a module (Ξ)

$\mathfrak{x}$, $\mathfrak{q}$, $\tilde{\mathfrak{q}}$, $\mathfrak{e}$, $\mathfrak{e}'$, $\mathfrak{h}$, $\mathfrak{i}$, $\mathfrak{s}$, y, y_1, y_2 elements of $\mathcal{X}$, $\mathcal{Q}$, $\mathcal{E}$, $\mathcal{H}$, $\mathcal{I}$, $\mathcal{S}$, $\mathcal{Y}$ (the letter y is also a real number)

$u_{\mathfrak{e}}$, $\jmath$, $\beta_{\mathfrak{i}}$, $\mathfrak{h}_{\mathfrak{i}}$, $\mathbf{u}$, $\mathbf{Z}$, $\mathbf{w}$, $\mathbf{l}$, $\mathbf{v}$, e_y the data of Theorem 5; operators in its proof; a sequence of indices; a vector; unit vectors

$\mathrm{w}^\pm(\alpha')$, $\mathrm{w}^\pm(\upsilon)$, sg_s, sg, t_υ in the proof of Theorem 1: two weighted degrees of an exponent and their coefficients, a sign (also in Proposition 8.4), the translation $X \mapsto X + \iota_{s_0}(\upsilon)$

$\mathrm{dt}(\sigma')$, σ'^*, S_σ, Sh, N, El in Proposition 5.2: $|\det(1-\varrho^{-L}\sigma'^*)|$, a permutation acting on vectors, the step, the shift, the cyclic operator, the elimination

f_l, df_s, g_i the numbers $(q^{l+1}-1)/(q-1)$; differences of coordinates (Lemma 6.2); the numbers of Lemma 6.3

$\mathrm{cost}(\beta)$, $\varkappa$ the sum of Proposition 6.6; the map of Input 3 that gives the costs

$\mathrm{rw}(\alpha)$, $\mathrm{cw}(\alpha)$, $X^\intercal$, $e_{\mathbf{l}}$, $|\mathbf{l}|_\omega$, $\mathrm{K}(\Phi)$ in Proposition 7.3: the row and column weights of an exponent on the generic variables, the generic matrix of a Levi subgroup, a basis vector and its weight, the vector $(w(\alpha)\Phi_\alpha)_\alpha$

$\mathrm{a}_m(i)$, $\eta_{s,j}(i)$, Fr, $Z(\mathcal{L})$, cf_ν, $\mathrm{bil}(\mathcal{L})$ in §8: real numbers, the generic Frobenius operator, its image under a product of forms, a coefficient, the bilinear expression of property (ii)

res, src, pt, pt_1, pt_2, r the residue map; maps to $\mathcal{X}$ (Theorem 5, Proposition 9.3); an isomorphism in the proof of Theorem 5

$\mathrm{d}(\beta)$, $\mathrm{ord}(\beta')$ in §9: the depth $\mathrm{depth}_{T,i_1}^{k,n}(\beta)$; the order of divisibility defined in the proof of Proposition 9.3

μ, ς; k', a', q', r', c_1, c_2, c_3, ϕ_s letters of the write-up that this note does not use (after Setting 2.2); letters of the note on unit 145 (after Theorem 5)

N, r, ℓ, a, b, d, f, g, x, D, K, O, R, M letters of the formal text, used only in §11 and after Definition 2.9; $a(D)$ is a number of the handout (§10)

READBACK NOTE 15

A crude bound for a class number, and divisibility of indices of submodules

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The unit holds two pieces of bookkeeping that do not depend on each other. The first is an explicit, crude bound for the number of classes in the genus of the standard lattice of a definite hermitian space over a CM field: each class is recorded by two matrices of algebraic integers of bounded size, and such matrices are counted. The second is a way to say, over an arbitrary commutative ring, that the index of one submodule in another divides a given element: every finite family of the first is moved into the second by a square matrix whose determinant divides that element. The unit proves the calculus of this notion and a criterion by determinants over a valuation domain, which rests on the Hermite normal form. Both pieces serve Theorem A of the write-up (short forms), and the first also Theorem C: the class number enters the sizes of the columns and, through two bounds of the unit of the columns, the units of the rows and of the exchange; the index language is the one in which the losses of divisibility at single primes are stated.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles with Lean v4.35.0 against a pinned version of the library Mathlib and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note treats one unit of the artifact: all 11 of its 11 declarations. Their proofs are given in full, *given* four results of other units, which are stated as Inputs 1 to 4 and not proved here: two lemmas of reduction theory, a count of bounded integral vectors, and the Hermite normal form (its second part, Input 4(b), is stated with its proof of two lines). Every statement of the unit and of the Inputs was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 20 declarations.

1 The statements

The notions are defined in §2. In short: for a CM field E and a finite set $\mathcal{J}$, $\mathrm{Cl}(\mathrm{U}_{\mathcal{J}}(\widehat{\mathcal{O}}_E))$ is the set of classes in the genus of the standard lattice of the hermitian space $E^{\mathcal{J}}$ with the form $\sum_i \bar{v}_i v_i$; $f(E)$ is the number of infinite places of E; and $c_{\mathrm{sb}}(E,\mathcal{J},N') = 6N'(f(E)\cdot\#\mathcal{J})^2$. For submodules A, B of a module and a ring element z, $[A:B] \mid z$ is read *the index of B in A divides z*; it is one symbol, defined in Definition 2.1.

Theorem 1 (a crude bound for the class number). *Let E be a CM field with a CM type, let $N' \geq 1$ be an integer whose image in the ring of integers of E lies in the different of E, and let $\mathcal{J}$ be a finite set. Then*

$$\#\,\mathrm{Cl}(\mathrm{U}_{\mathcal{J}}(\widehat{\mathcal{O}}_E)) \;\leq\; \left(1 + 2\,c_{\mathrm{sb}}(E, \mathcal{J}, N')^3\right)^{4\,f(E)\,(\#\mathcal{J})^2}.$$

Theorem 2 (the determinant criterion). *Let $\mathcal{O}$ be a valuation domain, $\mathcal{J}$ a finite set, z an element of $\mathcal{O}$, and G a square matrix over $\mathcal{O}$ with rows and columns indexed by $\mathcal{J}$; in words, a matrix G in $\mathrm{M}_{\mathcal{J}}(\mathcal{O})$. Let B be a submodule of $\mathcal{O}^{\mathcal{J}}$. If $\det G$ divides z and $G(\mathcal{O}^{\mathcal{J}}) \subseteq B$, then $[\mathcal{O}^{\mathcal{J}} : B] \mid z$.*

LEAN `Space.card_classes_le` · `IndexDvd.of_det_dvd`

In Theorem 1 the number of elements is taken as the formal text takes it, a natural number that would be 0 for an infinite set; the proof shows that the set is finite (§7, note 1). Between the two theorems stands the calculus of §4: inclusion, pull-back, transitivity, push-forward, sums and composition (Lemmas 4.1 to 4.6), and the additivity of the *type* of an endomorphism under composition (Proposition 4.7).

2 The setting

Conventions. The natural numbers are $0, 1, 2, \ldots$. Every ring is a commutative ring with 1. In §4 and §5, $\mathcal{O}$ is a commutative ring, V and V' are $\mathcal{O}$-modules, and a submodule is an $\mathcal{O}$-submodule of a module. For submodules A, B of one module, $A + B$ is their sum; for an element w of $\mathcal{O}$, wA is the submodule of the multiples wv, $v \in A$; for a linear map G, $G(A)$ is the image and $G^{-1}(B)$ the preimage of a submodule. A *valuation domain* is an integral domain in which, of any two elements, one divides the other. A finite set indexes coordinates: for a finite set $\mathcal{J}$, $\mathcal{O}^{\mathcal{J}}$ is the module of families $(v_i)_{i \in \mathcal{J}}$, $\mathrm{M}_{\mathcal{J}}(\mathcal{O})$ is the ring of matrices with rows and columns indexed by $\mathcal{J}$, acting on $\mathcal{O}^{\mathcal{J}}$, and $\mathrm{GL}_{\mathcal{J}}(\mathcal{O})$ is its group of units. The number of elements of a finite set $\mathcal{S}$ is $\#\mathcal{S}$.

In §3, E is a CM field: a totally complex quadratic extension of a totally real number field. Complex conjugation of E is written $v \mapsto \bar{v}$. A *CM type* of E is a family $(\iota_\tau)_\tau$ of field embeddings $\iota_\tau \colon E \to \mathbb{C}$, one for each infinite place τ of E, such that ι_τ induces τ. The ring of integers of E is $\mathcal{O}_E$, its different (over $\mathbb{Z}$) is the ideal whose inverse is the dual of $\mathcal{O}_E$ for the trace form. The ring of finite adeles of E is $\mathbb{A}_{E,f}$, and $\widehat{\mathcal{O}}_E$ is the closure of $\mathcal{O}_E$ in it, the product of the rings of integers of the completions of E at the finite places; E is a subring of $\mathbb{A}_{E,f}$, and $E \cap \widehat{\mathcal{O}}_E = \mathcal{O}_E$. Complex conjugation extends to $\mathbb{A}_{E,f}$ and preserves $\widehat{\mathcal{O}}_E$. For a matrix h over E or over $\mathbb{A}_{E,f}$, h^* is its conjugate transpose.

Letters. Three letters differ from those of the formal text, because the write-up keeps M, K and Φ for a modulus, for the imaginary quadratic field and for a function of Chapter I: the integer in the different of E is N', as in the note on the unit of the columns (unit 145), a level is K, and the embeddings of the CM type are ι_τ, for the infinite places τ, as in the overview.

Definition 2.1 (index divisibility)**.** Let A and B be submodules of the $\mathcal{O}$-module V, and let z be an element of $\mathcal{O}$. We write

$$[A : B] \mid z$$

and say that *the index of B in A divides z*, if for every natural number n and all elements $y_1, \dots, y_n$ of A there is an $n \times n$ matrix $u = (u_{ii'})$ over $\mathcal{O}$ such that $\det u$ divides z and

$$\sum_{i'=1}^{n} u_{ii'}\, y_{i'} \in B \qquad \text{for } i = 1, \dots, n.$$

LEAN `IndexDvd`

The definition does not ask that B lie in A, nor that either module be free or finitely generated. Since the combinations $\sum_{i'} u_{ii'} y_{i'}$ lie in A, the statement $[A : B] \mid z$ depends on B only through $A \cap B$; it holds for $z = 0$ (take $u = 0$), and for every z when $A \subseteq B$ (Lemma 4.1). The bracket alone has no meaning. Remark 5.1 compares the notion with the length of a quotient in the one case where both are defined. The overview of the handout uses the same pattern for the index of a family of rows: there, too, a square matrix u moves a family into the integral forms, and what is bounded is the divisibility of its determinant: at each prime the absolute value of $\det u$ is bounded from below, as here $\det u$ divides z.

Definition 2.2 (type of an endomorphism)**.** Let p be an element of $\mathcal{O}$, A a submodule of the $\mathcal{O}$-module V, ν a natural number, Y an $\mathcal{O}$-linear endomorphism of V, and c a real number. We say that

$$Y \text{ has type at most } c \text{ at } p^{\nu} \text{ on } A$$

if $Y(A) \subseteq A$ and there is a natural number $i \le c$ with $[A : Y(A) + p^{\nu} A] \mid p^{i}$.

LEAN `HasType`

When $\mathcal{O}$ is a discrete valuation ring with uniformiser p and A is free of finite rank, this says that Y preserves A and that the cokernel of Y on $A/p^{\nu}A$ has length at most c (Remark 5.1): the elementary divisors of Y on A, cut off at p^{ν}, have exponents of sum at most c.

Definition 2.3 (the standard definite space and its classes)**.** Let E be a CM field and $\mathcal{J}$ a finite set.

(a) $\mathrm{U}_{\mathcal{J}}(\mathbb{A}_{E,f})$ is the group of the matrices h in $\mathrm{GL}_{\mathcal{J}}(\mathbb{A}_{E,f})$ with $h^{*}h = 1$: the finite adelic points of the unitary group of the hermitian form $\sum_{i \in \mathcal{J}} \bar{v}_i v_i$ on $E^{\mathcal{J}}$. $\mathrm{U}_{\mathcal{J}}(E)$ is the group of the matrices γ in $\mathrm{GL}_{\mathcal{J}}(E)$ with $\gamma^{*}\gamma = 1$; it is a subgroup of $\mathrm{U}_{\mathcal{J}}(\mathbb{A}_{E,f})$. The subgroup

$$\mathrm{U}_{\mathcal{J}}(\widehat{\mathcal{O}}_E)$$

of $\mathrm{U}_{\mathcal{J}}(\mathbb{A}_{E,f})$ consists of the elements h such that h and h^{-1} have all entries in $\widehat{\mathcal{O}}_E$; it is the stabiliser of the standard lattice $\widehat{\mathcal{O}}_E^{\mathcal{J}}$.

(b) For a subgroup K of $\mathrm{U}_{\mathcal{J}}(\mathbb{A}_{E,f})$, the set of *classes at level* K is the set of double cosets

$$\mathrm{Cl}(\mathsf{K}) = \mathrm{U}_{\mathcal{J}}(E) \backslash \mathrm{U}_{\mathcal{J}}(\mathbb{A}_{E,f}) / \mathsf{K}.$$

(c) $f(E)$ is the number of infinite places of E; this is defined for every number field.

(d) For a natural number N', $c_{\mathrm{sb}}(E, \mathcal{J}, N') = 6\, N' \, (f(E) \cdot \#\mathcal{J})^{2}$, a natural number.

LEAN `Kf` · `StdClasses` · `numInfPlaces` · `Space.shortBasisConst` · these four definitions belong to the background library (units 122, 102, 10 and 99)

The classes at level $\mathrm{U}_{\mathcal{J}}(\widehat{\mathcal{O}}_E)$ correspond to the isometry classes of the hermitian lattices in $E^{\mathcal{J}}$ that are everywhere locally isometric to $\mathcal{O}_E^{\mathcal{J}}$: the class of h is that of the lattice $E^{\mathcal{J}} \cap h\,\widehat{\mathcal{O}}_E^{\mathcal{J}}$. The number of classes is the class number of the genus.

Inputs from other units. The four results below are taken for granted in this note; only Input 4(b) is proved, in its place.

Input 1 (short bases; unit 99, background). *Let E be a CM field with a CM type, $\mathcal{J}$ a finite set, and $N' \geq 1$ an integer that, as an element of $\mathcal{O}_E$, lies in the different of E. For every h in $\mathrm{U}_{\mathcal{J}}(\mathbb{A}_{E,f})$ there is a matrix L in $\mathrm{GL}_{\mathcal{J}}(E)$ such that $h^{-1}L$ has all entries in $\widehat{\mathcal{O}}_E$ and $|\varphi(L_{ii'})| \leq c_{\mathrm{sb}}(E, \mathcal{J}, N')$ for every ring homomorphism $\varphi\colon E \to \mathbb{C}$ and all i, i' in $\mathcal{J}$.*

Input 2 (rational approximation of an adele; unit 99, background). *Let E be a CM field with a CM type $(\iota_\tau)_\tau$, and let δ be a nonzero element of the different of E. For every family $(z_\tau)_\tau$ of complex numbers indexed by the infinite places of E and every finite adele η in $\mathbb{A}_{E,f}$ there is α in E with $\eta - \alpha \in \widehat{\mathcal{O}}_E$ and $|z_\tau - \iota_\tau(\alpha)| \leq (2\,f(E))^2\,|\iota_\tau(\delta)|$ for every infinite place τ.*

Input 3 (bounded integral vectors are few; unit 148). *Let E' be a number field with an embedding $\iota_\tau\colon E' \to \mathbb{C}$ inducing τ for each infinite place τ of E'. Let $\mathcal{S}$ be a set, $\mathcal{S}'$ a subset of it, $\mathcal{J}$ a finite set, ω a real number with $0 \leq \omega$, and F a map from $\mathcal{S}$ to $E'^{\mathcal{J}}$. Suppose that for every c in $\mathcal{S}'$ and every i in $\mathcal{J}$ the coordinate $F(c)_i$ is an algebraic integer with $|\iota_\tau(F(c)_i)| \leq \omega$ for every τ, and that F is injective on $\mathcal{S}'$. Then $\mathcal{S}'$ is finite, with at most $(1+2\omega)^{2\,\#\mathcal{J}\,f(E')}$ elements.* (The reading of this lemma by program lists, besides these arguments, variables of its file that the statement does not mention, among them a matrix; by Lean's rules for such variables they are not arguments of the lemma.)

Input 4 (Hermite normal form; unit 120, background). (a) *Let $\mathcal{A}$ be a commutative ring and $\mathcal{R}$ a subring such that $(\mathcal{A}, \mathcal{R})$ is a valuation pair: for any two elements y, y' of $\mathcal{A}$ there is an idempotent ϵ of $\mathcal{R}$ with $\mathcal{R}y + \mathcal{R}y' \subseteq \mathcal{R}\,(y - \epsilon(y - y'))$. Let $\mathcal{K}$ and $\mathcal{N}$ be finite sets, $\jmath\colon \mathcal{K} \to \mathcal{N}$ an injective map, X a matrix over $\mathcal{A}$ with rows indexed by $\mathcal{N}$ and columns by $\mathcal{K}$, and ω a map from $\mathcal{K}$ to the natural numbers (weights). Then there are an invertible matrix T over $\mathcal{A}$, indexed by $\mathcal{N}$, such that T and T^{-1} have their entries in $\mathcal{R}$, and a square matrix Δ over $\mathcal{A}$, indexed by $\mathcal{K}$, with $\Delta_{ll'} = 0$ whenever $\omega(l') < \omega(l)$, such that*

$$X = T_{\cdot,\jmath}\,\Delta \qquad \text{and} \qquad (T^{-1})_{\jmath,\cdot}\,X = \Delta,$$

where $T_{\cdot,\jmath}$ consists of the columns $\jmath(l)$ of T and $(T^{-1})_{\jmath,\cdot}$ of the rows $\jmath(l)$ of T^{-1}, $l \in \mathcal{K}$. (b) *If $\mathcal{A}$ is a valuation domain, then $(\mathcal{A}, \mathcal{A})$ is a valuation pair.* (Proof of (b): take $\epsilon = 0$ if y divides y', so that $\mathcal{A}y + \mathcal{A}y' = \mathcal{A}y$; otherwise y' divides y, and $\epsilon = 1$ gives $y - (y - y') = y'$ and $\mathcal{A}y + \mathcal{A}y' = \mathcal{A}y'$. The reading of (a) by program also lists a ring with involution, from an earlier part of its file; the statement does not involve it.)

LEAN `Space.exists_short_basis_stdSpace` · `exists_rat_near_of_mem_differentIdeal` · `Filling.encard_integral_le` · `IsValuationPair.hermite` · `isValuationPair_top`

Inputs 1 and 2 are lemmas of the reduction theory that the artifact develops for the standard space: Input 1 comes from a transference argument (a basis of short vectors, from the covering radius of a lattice against the first minimum of its dual), Input 2 from a fundamental domain for $\mathcal{O}_E$ scaled by the different. Input 3 is a packing count: distinct algebraic integers are at distance at least 1 at some infinite place. Input 4(a), for a discrete valuation ring and $\mathcal{K} = \mathcal{N}$, is the Hermite form obtained by row operations: $\Delta = T^{-1}X$ is X multiplied on the left by an invertible matrix with entries in $\mathcal{R}$, and Δ is block triangular for the weights ω. The valuation pair is the form in which the artifact proves it once for a valuation domain and for $\widehat{\mathcal{O}}_E$ inside $\mathbb{A}_{E,f}$.

References. The catalogue of quoted results of this write-up lists Inputs 1, 2 and 4(a) as elementary lemmas, with no confirmed place in the literature. For Input 1 it names the theorem behind it, the general short-basis lemma of the same file of the artifact, as a known result, the transference theorem, with the references W. Banaszczyk, *New bounds in some transference theorems in the geometry of numbers*, Math. Ann. 296 (1993), Theorems (2.1) and (2.2), and J. C. Lagarias, H. W. Lenstra Jr. and C. P. Schnorr, *Korkin–Zolotarev bases and successive minima of a lattice and its reciprocal lattice*, Combinatorica 10 (1990), Theorems 2.4–2.5. The relation recorded there is that of a weaker form: those theorems, with a computation of the dual lattice for the trace form, give a smaller radius than the artifact's lemma; Input 1 is the case of the standard space of that lemma. For Input 2: *reference to be confirmed.* Input 3 is a result of the argument (unit 148), not of the literature. Input 4(a) is used in this note only for a valuation domain and the pair $(\mathcal{A}, \mathcal{A})$; there the reduction of a matrix to triangular form by an invertible matrix is classical. A source for rings with such reductions is I. Kaplansky, *Elementary divisors and modules*, Trans. Amer. Math. Soc. 66 (1949), 464–491; the place in it was not confirmed, and for the form with a valuation pair no place is known to us: *reference to be confirmed.*

3 The class number

Proof of Theorem 1. Write Cl for the set $\mathrm{Cl}(\mathrm{U}_{\mathcal{J}}(\widehat{\mathcal{O}}_E))$, and choose for every class $\mathfrak{c}$ in it a representative $h_{\mathfrak{c}}$ in $\mathrm{U}_{\mathcal{J}}(\mathbb{A}_{E,f})$.

Short bases. By Input 1 there is, for every class $\mathfrak{c}$, a matrix $L_{\mathfrak{c}}$ in $\mathrm{GL}_{\mathcal{J}}(E)$ such that

$$Q_{\mathfrak{c}} = h_{\mathfrak{c}}^{-1} L_{\mathfrak{c}} \quad \text{has its entries in } \widehat{\mathcal{O}}_E$$

and

$$|\varphi((L_{\mathfrak{c}})_{ii'})| \le c_{\mathrm{sb}}(E, \mathcal{J}, N') \quad \text{for every ring homomorphism } \varphi\colon E \to \mathbb{C} \text{ and all } i, i'.$$

Let $H_{\mathfrak{c}} = L_{\mathfrak{c}}^{*} L_{\mathfrak{c}}$, the Gram matrix of the columns of $L_{\mathfrak{c}}$; it lies in $\mathrm{GL}_{\mathcal{J}}(E)$.

An approximate inverse. Apply Input 2 with $\delta = N'$ and $z_\tau = 0$: for every finite adele η there is $\varrho(\eta)$ in E with

$$\eta - \varrho(\eta) \in \widehat{\mathcal{O}}_E, \qquad |\iota_\tau(\varrho(\eta))| \le (2\, f(E))^2 N' \quad \text{for every infinite place } \tau.$$

Let $Z_{\mathfrak{c}}$ be the matrix over E obtained by applying ϱ to each entry of $Q_{\mathfrak{c}}^{-1}$. Then $Q_{\mathfrak{c}}^{-1} - Z_{\mathfrak{c}}$ has its entries in $\widehat{\mathcal{O}}_E$, and so, for all classes $\mathfrak{c}$ and $\mathfrak{c}'$,

$$Q_{\mathfrak{c}'} Q_{\mathfrak{c}}^{-1} - Q_{\mathfrak{c}'} Z_{\mathfrak{c}} = Q_{\mathfrak{c}'}(Q_{\mathfrak{c}}^{-1} - Z_{\mathfrak{c}}) \quad \text{has its entries in } \widehat{\mathcal{O}}_E. \tag{1}$$

The map. Send a class $\mathfrak{c}$ to the pair of matrices $(H_{\mathfrak{c}},\, H_{\mathfrak{c}}Z_{\mathfrak{c}})$, a vector with $2(\#\mathcal{J})^2$ coordinates in E. We show that its coordinates are algebraic integers, that they are bounded by $c_{\mathrm{sb}}(E,\mathcal{J},N')^3$ at every infinite place, and that the map is injective. Input 3, for the field E with its CM type, the set $\mathcal{S}=\mathcal{S}'=\mathrm{Cl}$, an index set of $2(\#\mathcal{J})^2$ elements and $\omega=c_{\mathrm{sb}}(E,\mathcal{J},N')^3$, then shows that Cl is finite with at most $(1+2\,c_{\mathrm{sb}}(E,\mathcal{J},N')^3)^{2\cdot 2(\#\mathcal{J})^2\cdot f(E)}$ elements, which is the bound of the theorem.

Integrality. Since $h_{\mathfrak{c}}^{-1}$ is unitary, $H_{\mathfrak{c}}=(h_{\mathfrak{c}}^{-1}L_{\mathfrak{c}})^*(h_{\mathfrak{c}}^{-1}L_{\mathfrak{c}})=Q_{\mathfrak{c}}^*Q_{\mathfrak{c}}$, and the entries of $Q_{\mathfrak{c}}^*$ lie in $\widehat{\mathcal{O}}_E$ because conjugation preserves $\widehat{\mathcal{O}}_E$. So $H_{\mathfrak{c}}$ has its entries in $E\cap\widehat{\mathcal{O}}_E=\mathcal{O}_E$. In the same way $H_{\mathfrak{c}}Z_{\mathfrak{c}}=Q_{\mathfrak{c}}^*\,(Q_{\mathfrak{c}}Z_{\mathfrak{c}})$, and $Q_{\mathfrak{c}}Z_{\mathfrak{c}}=1-Q_{\mathfrak{c}}(Q_{\mathfrak{c}}^{-1}-Z_{\mathfrak{c}})$ has its entries in $\widehat{\mathcal{O}}_E$ by (1) with $\mathfrak{c}'=\mathfrak{c}$.

Size. Let τ be an infinite place. The composite of ι_τ with complex conjugation of E is again a ring homomorphism from E to $\mathbb{C}$, so every entry of $L_{\mathfrak{c}}$ and of $L_{\mathfrak{c}}^*$ has absolute value at most $c_{\mathrm{sb}}(E,\mathcal{J},N')$ under ι_τ. Hence

$$|\iota_\tau((H_{\mathfrak{c}})_{ii'})|\le \#\mathcal{J}\cdot c_{\mathrm{sb}}(E,\mathcal{J},N')^2,$$
$$|\iota_\tau((H_{\mathfrak{c}}Z_{\mathfrak{c}})_{ii'})|\le (\#\mathcal{J})^2\,c_{\mathrm{sb}}(E,\mathcal{J},N')^2\,(2\,f(E))^2N'.$$

Here $\#\mathcal{J}\ge 1$, as there is an entry, and $N'\ge 1$, $f(E)\ge 1$. So

$$\#\mathcal{J}\le c_{\mathrm{sb}}(E,\mathcal{J},N')\qquad\text{and}\qquad 4N'\,f(E)^2(\#\mathcal{J})^2\le 6N'(f(E)\cdot\#\mathcal{J})^2=c_{\mathrm{sb}}(E,\mathcal{J},N'),$$

and both bounds are at most the cube of $c_{\mathrm{sb}}(E,\mathcal{J},N')$.

Injectivity. Let $\mathfrak{c}$ and $\mathfrak{c}'$ be classes with $H_{\mathfrak{c}}=H_{\mathfrak{c}'}$ and $H_{\mathfrak{c}}Z_{\mathfrak{c}}=H_{\mathfrak{c}'}Z_{\mathfrak{c}'}$. As $H_{\mathfrak{c}}$ is invertible, $Z_{\mathfrak{c}}=Z_{\mathfrak{c}'}$. Put $\gamma=L_{\mathfrak{c}}L_{\mathfrak{c}'}^{-1}$, a matrix in $\mathrm{GL}_{\mathcal{J}}(E)$. Then

$$\gamma^*\gamma=(L_{\mathfrak{c}'}^{-1})^*\,H_{\mathfrak{c}}\,L_{\mathfrak{c}'}^{-1}=(L_{\mathfrak{c}'}^{-1})^*\,H_{\mathfrak{c}'}\,L_{\mathfrak{c}'}^{-1}=1,$$

so γ lies in $\mathrm{U}_{\mathcal{J}}(E)$. Put $\kappa=(\gamma h_{\mathfrak{c}'})^{-1}h_{\mathfrak{c}}$, an element of $\mathrm{U}_{\mathcal{J}}(\mathbb{A}_{E,f})$. Then

$$\kappa=h_{\mathfrak{c}'}^{-1}L_{\mathfrak{c}'}L_{\mathfrak{c}}^{-1}h_{\mathfrak{c}}=Q_{\mathfrak{c}'}Q_{\mathfrak{c}}^{-1},$$

and by (1), used for the pairs $(\mathfrak{c},\mathfrak{c}')$ and $(\mathfrak{c}',\mathfrak{c}')$, and by $Z_{\mathfrak{c}}=Z_{\mathfrak{c}'}$,

$$\kappa=1+(Q_{\mathfrak{c}'}Q_{\mathfrak{c}}^{-1}-Q_{\mathfrak{c}'}Z_{\mathfrak{c}})-(Q_{\mathfrak{c}'}Q_{\mathfrak{c}'}^{-1}-Q_{\mathfrak{c}'}Z_{\mathfrak{c}'})$$

has its entries in $\widehat{\mathcal{O}}_E$. The inverse of the unitary matrix κ is κ^*, which has its entries in $\widehat{\mathcal{O}}_E$ as well, so κ lies in $\mathrm{U}_{\mathcal{J}}(\widehat{\mathcal{O}}_E)$. From $h_{\mathfrak{c}}=\gamma\,h_{\mathfrak{c}'}\,\kappa$ the two representatives lie in one double coset, and $\mathfrak{c}=\mathfrak{c}'$. □

LEAN `Space.card_classes_le` · the formal proof has these five parts in this order

The bound is crude. (The mass formula of Smith, Minkowski and Siegel gives the sum of the inverses of the orders of the automorphism groups over the classes, hence a lower bound for their number; we make no comparison.) What the argument of the write-up needs is a bound that is explicit and at most exponential in a polynomial of $f(E)$, $\#\mathcal{J}$ and N'.

4 The calculus of index divisibility

In this section $\mathcal{O}$ is a commutative ring and V, V' are $\mathcal{O}$-modules, as in the conventions.

Lemma 4.1 (inclusion)**.** *Let A and B be submodules of V and z an element of $\mathcal{O}$. If $A \subseteq B$, then $[A : B] \mid z$.*

Proof. For $y_1, \dots, y_n$ in A the identity matrix serves: its determinant 1 divides z, and each y_i lies in B. □

LEAN `IndexDvd.of_le`

Lemma 4.2 (pull-back)**.** *Let A, B be submodules of V, let A', B' be submodules of V', let z be an element of $\mathcal{O}$, and let $G : V' \to V$ be an $\mathcal{O}$-linear map. Suppose that $[A : B] \mid z$, that $G(A') \subseteq A$, and that every v in A' with $G(v) \in B$ lies in B'. Then $[A' : B'] \mid z$.*

Proof. Let $y_1, \dots, y_n$ lie in A'. The elements $G(y_i)$ lie in A, so there is a matrix u with $\det u$ dividing z and $\sum_{i'} u_{ii'} G(y_{i'}) \in B$ for all i. The element $v_i = \sum_{i'} u_{ii'} y_{i'}$ lies in A' and $G(v_i) \in B$; hence $v_i \in B'$. □

LEAN `IndexDvd.comap`

Lemma 4.3 (transitivity)**.** *Let A, B, C be submodules of V and let z and w be elements of $\mathcal{O}$. If $[A : B] \mid z$ and $[B : C] \mid w$, then $[A : C] \mid zw$.*

Proof. Let $y_1, \dots, y_n$ lie in A. There is u with $\det u$ dividing z such that the elements $y'_i = \sum_{i'} u_{ii'} y_{i'}$ lie in B; and for these there is u' with $\det u'$ dividing w such that $\sum_i u'_{li} y'_i \in C$ for all l. The matrix $u'u$ has determinant $\det u' \cdot \det u$, which divides zw, and $\sum_{i'} (u'u)_{li'} y_{i'} = \sum_i u'_{li} y'_i$ lies in C. □

LEAN `IndexDvd.trans`

Lemma 4.4 (push-forward)**.** *Let A, B be submodules of V, let A', B' be submodules of V', let z be an element of $\mathcal{O}$, and let $G : V \to V'$ be an $\mathcal{O}$-linear map. Suppose that $[A : B] \mid z$, that $A' \subseteq G(A) + B'$, and that $G(B) \subseteq B'$. Then $[A' : B'] \mid z$.*

Proof. Let $y_1, \dots, y_n$ lie in A', and write $y_i = G(v_i) + b_i$ with v_i in A and b_i in B'. There is u with $\det u$ dividing z and $\sum_{i'} u_{ii'} v_{i'} \in B$ for all i. Then

$$\sum_{i'} u_{ii'} y_{i'} = G\Big(\sum_{i'} u_{ii'} v_{i'}\Big) + \sum_{i'} u_{ii'} b_{i'} \;\in\; G(B) + B' \subseteq B'. \qquad \square$$

LEAN `IndexDvd.map`

Lemma 4.5 (finite sums)**.** *Let $\mathcal{J}$ be a set, let $G_i : V \to V'$, for i in $\mathcal{J}$, be $\mathcal{O}$-linear maps, let A be a submodule of V, B' a submodule of V' and z an element of $\mathcal{O}$. Suppose that $[A : G_i^{-1}(B')] \mid z$ for every i in $\mathcal{J}$. Then for every finite subset $\mathcal{S}$ of $\mathcal{J}$,*

$$\Big[\textstyle\sum_{i \in \mathcal{S}} G_i(A) + B' : B'\Big] \mid z^{\#\mathcal{S}}.$$

Proof. By induction on $\mathcal{S}$. For the empty set the left module is B', and Lemma 4.1 applies. Let $\mathcal{S} = \mathcal{S}_0 \cup \{i\}$ with i not in $\mathcal{S}_0$, and put $A'_0 = \sum_{l \in \mathcal{S}_0} G_l(A) + B'$ and $A' = G_i(A) + A'_0$. Lemma 4.4, for the map G_i and the submodules A, $G_i^{-1}(B')$ of V and A', A'_0 of V', gives $[A' : A'_0] \mid z$: indeed $A' \subseteq G_i(A) + A'_0$ and $G_i(G_i^{-1}(B')) \subseteq B' \subseteq A'_0$. By the induction hypothesis $[A'_0 : B'] \mid z^{\#\mathcal{S}_0}$, and Lemma 4.3 gives the claim with $z \cdot z^{\#\mathcal{S}_0} = z^{\#\mathcal{S}}$. □

LEAN `IndexDvd.sup` · the formal statement writes the left module as the span of the union of the images $G_i(A)$, $i \in \mathcal{S}$, and of B'

Lemma 4.6 (composition)**.** *Let A and B be submodules of V, let z, z' and w be elements of $\mathcal{O}$, and let $G\colon V \to V$ be an $\mathcal{O}$-linear map with $G(A) \subseteq A$. If $[A : G(A) + wA] \mid z$ and $[A : B + wA] \mid z'$, then $[A : G(B) + wA] \mid zz'$.*

Proof. Apply Lemma 4.4 to the second hypothesis and the map G, with the submodules $A' = G(A) + wA$ and $B' = G(B) + wA$. The inclusion $A' \subseteq G(A) + B'$ is clear, and

$$G(B + wA) = G(B) + w\,G(A) \subseteq G(B) + wA$$

because $G(A) \subseteq A$. This gives $[G(A) + wA : G(B) + wA] \mid z'$. Lemma 4.3, with the first hypothesis, gives the claim. □

LEAN `IndexDvd.comp`

Proposition 4.7 (types add under composition)**.** *Let $\mathcal{J}$ be a set, and for every i in $\mathcal{J}$ let Y_i be an $\mathcal{O}$-linear endomorphism of V and c_i a real number. Let p be an element of $\mathcal{O}$ (the formal text calls it z), A a submodule of V and ν a natural number. Let $(i_1, \dots, i_e)$ be a finite list of elements of $\mathcal{J}$, with $e \geq 0$; repetitions are allowed. Suppose that Y_{i_l} has type at most c_{i_l} at p^ν on A for $l = 1, \dots, e$. Then $Y_{i_1} \circ \dots \circ Y_{i_e}$ has type at most $c_{i_1} + \cdots + c_{i_e}$ at p^ν on A; for $e = 0$ the composite is the identity and the sum is 0.*

Proof. By induction on e. For $e = 0$: the identity preserves A, and $[A : A + p^\nu A] \mid p^0$ by Lemma 4.1, with the natural number $0 \leq 0$. For the step put $Y = Y_{i_1}$ and $Y' = Y_{i_2} \circ \dots \circ Y_{i_e}$. By the hypothesis on Y there is a natural number $i \leq c_{i_1}$ with $[A : Y(A) + p^\nu A] \mid p^i$, and $Y(A) \subseteq A$; by the induction hypothesis $Y'(A) \subseteq A$ and there is a natural number $i' \leq c_{i_2} + \cdots + c_{i_e}$ with $[A : Y'(A) + p^\nu A] \mid p^{i'}$. Then $Y(Y'(A)) \subseteq Y(A) \subseteq A$, and Lemma 4.6, with $G = Y$, $B = Y'(A)$ and $w = p^\nu$, gives $[A : Y(Y'(A)) + p^\nu A] \mid p^{i+i'}$, where $i + i' \leq c_{i_1} + \cdots + c_{i_e}$. □

LEAN `HasType.prod`

5 The determinant criterion

Proof of Theorem 2. Let $y_1, \dots, y_n$ be elements of $\mathcal{O}^{\mathcal{J}}$; write $(y_i)_l$ for the coordinates, $l \in \mathcal{J}$, and let $\mathbf{y} = ((y_i)_l)_{i,l}$ be the matrix with rows indexed by $\{1, \dots, n\}$ and columns by $\mathcal{J}$. Let $\mathcal{N}$ be the disjoint union of $\{1, \dots, n\}$ and $\mathcal{J}$, and let X be the square matrix over $\mathcal{O}$, indexed by $\mathcal{N}$, with the blocks

$$X = \begin{pmatrix} 1 & -\mathbf{y} \\ 0 & G^{\mathsf{T}} \end{pmatrix},$$

where 1 is the identity matrix of size n and G^{T} is the transpose of G. By Input 4(b) the pair $(\mathcal{O}, \mathcal{O})$ is a valuation pair. Input 4(a), for $\mathcal{K} = \mathcal{N}$, the identity map $\jmath$, the matrix X and the weights $\omega = 1$ on $\{1, \dots, n\}$ and $\omega = 0$ on $\mathcal{J}$, gives an invertible matrix T over $\mathcal{O}$ whose inverse has entries in $\mathcal{O}$, and a matrix Δ with

$$X = T\Delta, \qquad T^{-1}X = \Delta, \qquad \Delta_{il} = 0 \ \text{ for } i \in \{1, \dots, n\},\ l \in \mathcal{J}.$$

Write $\Delta = \left(\begin{smallmatrix} \Delta_{11} & 0 \\ \Delta_{21} & \Delta_{22} \end{smallmatrix}\right)$ and $T^{-1} = \left(\begin{smallmatrix} T'_{11} & T'_{12} \\ T'_{21} & T'_{22} \end{smallmatrix}\right)$ in blocks. We claim that $u = \Delta_{11}$ is a matrix as Definition 2.1 asks for.

The determinant. As X is block upper triangular, $\det X = \det G^{\mathsf{T}} = \det G$; as Δ is block lower triangular, $\det X = \det T \cdot \det \Delta_{11} \cdot \det \Delta_{22}$. So $\det u$ divides $\det G$, which divides z.

The combinations. The first block row of $T^{-1}X = \Delta$ reads $T'_{11} = \Delta_{11}$ in the first block column and $-T'_{11}\mathbf{y} + T'_{12}\,G^{\mathsf{T}} = 0$ in the second, that is, $u\mathbf{y} = T'_{12}\,G^{\mathsf{T}}$. Hence, for $i = 1, \dots, n$ and $l \in \mathcal{J}$,

$$\sum_{i'=1}^{n} u_{ii'}\,(y_{i'})_l = \sum_{l' \in \mathcal{J}} (T'_{12})_{il'}\,G_{ll'},$$

that is, $\sum_{i'} u_{ii'} y_{i'} = G(t_i)$ for the vector $t_i = ((T'_{12})_{il'})_{l'}$ of $\mathcal{O}^{\mathcal{J}}$. It lies in $G(\mathcal{O}^{\mathcal{J}}) \subseteq B$. □

LEAN `IndexDvd.of_det_dvd` · `IsValuationPair.hermite` · `isValuationPair_top`

Remark 5.1 (comparison with the length of a quotient; not in the formal text)**.** Let $\mathcal{O}$ be a discrete valuation ring with uniformiser p, let A be a free $\mathcal{O}$-module of finite rank n and B a submodule of A with A/B of finite length $\operatorname{len}(A/B)$. Then, for z in $\mathcal{O}$: $[A : B] \mid z$ holds if and only if $p^{\operatorname{len}(A/B)}$ divides z. Indeed, B is free of rank n, so in a basis of A it is the image of a matrix G with $\det G$ a unit times $p^{\operatorname{len}(A/B)}$, and Theorem 2 gives one direction. Conversely, apply Definition 2.1 to a basis $y_1, \dots, y_n$ of A: if $z \neq 0$, the matrix u defines an injective endomorphism of A with image in B and cokernel of length the valuation of $\det u$, so $\operatorname{len}(A/B)$ is at most the valuation of z; and every power of p divides 0.

6 Where the results are used

Theorem 1 is used in one place, in the unit of the columns (unit 145), in a bound for the number of classes of standard definite spaces at the levels that contain a principal level: such a level has finite index in $\mathrm{U}_{\mathcal{J}}(\widehat{\mathcal{O}}_E)$, and the number of classes is at most that index times the bound of Theorem 1. That bound, and its corollary for the levels of the data, are used again in the unit of the columns and in the units of the rows and of the exchange (units 143 and 147), which belong to Theorem C of the write-up.

The index language is used in the units that prove Theorem A of the write-up. The determinant criterion is applied there with diagonal matrices (units 150 and 141); pull-back and push-forward transport index statements along the maps between spaces of forms (units 141 and 145); the lemma on finite sums is the step, in the unit of the endgame (unit 151), from one translate of a Poincaré series to finitely many, which is where the number of translates enters the index bound of Theorem A(b); and Proposition 4.7 adds the types of the operators attached to the marked primes (unit 145). Unit 144, on the elementary divisors of a Frobenius-twisted Hecke operator, uses Theorem 2, transitivity, push-forward and composition, and states its result with Definition 2.2.

7 Notes on the formal text

1. *The number of classes.* The formal statement of Theorem 1 bounds the cardinality of the set of classes as a natural number, which the library defines to be 0 for an infinite set. The formal proof bounds the cardinality as an extended natural number, through Input 3, and so proves that the set is finite; the statement does not say it.

2. *Letters.* The formal text calls the integer of Theorem 1 N and the index set n; here they are N' and $\mathcal{J}$, the letter N being kept in this write-up for the level. It writes K for a level and Φ for the places and their embeddings (here K, τ and ι_τ). The submodules that the formal text calls M, N, P, M' and (its own) N' are A, B, C, A', B' here; its modules V and U are V and V'; its scalar a is w; its linear maps f are G. In Proposition 4.7 the formal text writes z, s, T, t, L for what is here p, ν, (Y_i), (c_i) and the list.

3. *The arguments of the index relation.* The file declares M, N, z as implicit variables and then makes them explicit for this one definition. A reading of the text by program therefore lists each of the three twice. Lean takes each once, as an explicit argument: every use in the artifact gives exactly three arguments. This was not confirmed by asking Lean.

4. *Inferred types.* In the definition of the type the formal text does not write the types of p, of the submodule and of the exponent; Lean infers them. The uses (Proposition 4.7, where p is an element of the ring) show that p is a ring element and the exponent a natural number.

5. *Families of length zero.* Definition 2.1 quantifies over all n, including $n = 0$, where there is nothing to check. The families are indexed by $\{0, \dots, n-1\}$ in the formal text.

6. *Lemma 4.5.* Its hypothesis is asked for every i in $\mathcal{J}$, not only for those in $\mathcal{S}$, and $\mathcal{J}$ is an arbitrary set. The formal proof is one line of induction that calls push-forward and transitivity, as here.

7. *The injectivity step of Theorem 1.* The formal proof gives the rational unitary matrix and the integral adelic matrix as terms and leaves it to Lean to see that they relate the two representatives. The arrangement written above, $h_{\mathfrak{c}} = \gamma h_{\mathfrak{c}'} \kappa$ with $\gamma = L_{\mathfrak{c}} L_{\mathfrak{c}'}^{-1}$ and $\kappa = Q_{\mathfrak{c}'} Q_{\mathfrak{c}}^{-1}$, is read off these terms: the unitarity is proved for the product $L_{\mathfrak{c}} L_{\mathfrak{c}'}^{-1}$, and the integrality for the combination of the two instances of (1) displayed in the proof, which is $Q_{\mathfrak{c}'} Q_{\mathfrak{c}}^{-1}$. The library's lemma on double cosets, which fixes the order, was not opened: the library is not on the machine where this note was written.

8. *The size estimates.* The formal proof closes the two inequalities of the paragraph *Size* by automation. The intermediate bounds written there are ours; they were checked by hand.

9. *What an assembly assembles.* Lemma 4.6 is push-forward followed by transitivity, and Lemma 4.5 is an induction on these two; their content lies in Lemmas 4.3 and 4.4.

10. *The place of Theorem 1.* The header of its file counts it with the reduction theory of the background (Mahler and Godement compactness, finite covolume, adelic

Minkowski theorems), a known subject; it stands in this unit because its proof needs the count of Input 3, which belongs to the argument.

How this note was written. From the dossier of the unit: its eleven statement readings, its five structured proofs and the reader's list of doubts. The program check of the dossier covered 22 leaf steps of 5 proofs. A second reader compared the eleven readings with the Lean text and sampled the proofs; the remarks of that reading are taken up in the paragraph after Definition 2.1 and in the hypotheses of Theorem 2. I read both files of the unit completely, and in the background library the statements and definitions that the unit rests on: the unitary groups, the level subgroup, the rational points and the classes, the constant and the two lemmas of Inputs 1 and 2, the count of Input 3 with the variables of its file, the valuation pairs and the Hermite form. I did not read the proofs of the Inputs. The three lemmas without a structured proof (pull-back, inclusion, composition) and Lemma 4.5 were read in the source. The proofs above follow the formal ones; the proof of Theorem 2 writes out in blocks what the formal proof obtains by simplification, and Remark 5.1 is mine. Lean was not run for this note. An independent referee read the note against the Lean text; in the revision of 26 November 2026, after that report, three letters were changed (see the conventions), Input 4(b) received its proof, the comment on the Hermite form was corrected (row operations), the references for the Inputs were sorted, and the account of where Theorem 1 is used was completed.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Theorem 1	`Space.card_classes_le`	ClassCountBound.lean	14–37
Definition 2.1	`IndexDvd`	ModuleIndex.lean	20–20
Lemma 4.2	`IndexDvd.comap`	ModuleIndex.lean	24–25
Lemma 4.1	`IndexDvd.of_le`	ModuleIndex.lean	28–28
Lemma 4.3	`IndexDvd.trans`	ModuleIndex.lean	31–33
Lemma 4.4	`IndexDvd.map`	ModuleIndex.lean	37–41
Lemma 4.5	`IndexDvd.sup`	ModuleIndex.lean	46–48
Lemma 4.6	`IndexDvd.comp`	ModuleIndex.lean	50–51
Definition 2.2	`HasType`	ModuleIndex.lean	55–56
Proposition 4.7	`HasType.prod`	ModuleIndex.lean	60–64
Theorem 2	`IndexDvd.of_det_dvd`	ModuleIndex.lean	68–73
Definition 2.3	`numInfPlaces`	ZpowSum.lean	32–32
Definition 2.3	`Space.shortBasisConst`	MassBasic.lean	73–73
Definition 2.3	`Kf`	SpaceDefs.lean	109–109
Definition 2.3	`StdClasses`	ClassReps.lean	93–93
Input 1	`Space.exists_short_basis_stdSpace`	MassBasic.lean	101–107
Input 2	`exists_rat_near_of_mem_differentIdeal`	MassBasic.lean	116–124
Input 3	`Filling.encard_integral_le`	BoxPartition.lean	287–292
Input 4	`IsValuationPair.hermite`	FrameAction.lean	58–69
Input 4	`isValuationPair_top`	FrameAction.lean	50–51

The two theorems and the two definitions as they are written in the artifact.

```
lemma Space.card_classes_le {E} [Field E] [CMType E] {N}
    [NeZero N] (h : ↑N ∈ differentIdeal ℤ (𝒪 E)) {n} [Fintype n] : .card (StdClasses
    (Kf E n)) ≤ (1 + 2 * shortBasisConst E n N ^ 3) ^ (4 * numInfPlaces E * card n ^ 2)
```

```
lemma IndexDvd.of_det_dvd [IsDomain O] [ValuationRing O] [Fintype ι] {A : Matrix ι ι O} {N}
    (h : A.det ∣ z) (e : .map A.mulVecLin ⊤ ≤ N) : IndexDvd ⊤ N z

def IndexDvd := ∀ n (Φ : Fin n → V), (∀ k, Φ k ∈ M) → ∃ m, det m ∣ z ∧ ∀ i, ∑ k, m i k • Φ k ∈ N

def HasType p M n (y : End O V) (c : ℝ) :=
  M.map y ≤ M ∧ ∃ l : ℕ, l ≤ c ∧ IndexDvd M (M.map y ⊔ p ^ n • M) (p ^ l)
```

In these texts `⊤` is the whole module, `⊔` the sum of submodules, `M.map y` the image and `p ^ n • M` the multiples; `Fin n` is the set $\{0, \dots, n-1\}$.

Notation

$\mathcal{O}$ a commutative ring; in Theorem 2 a valuation domain; $\mathcal{O}$ with a subscript is a ring of integers

V, V' $\mathcal{O}$-modules

A, B, C, A', B', A'_0 submodules

z, z', w; v elements of $\mathcal{O}$; an element of a module or of E

$[A : B] \mid z$ the index of B in A divides z (Definition 2.1)

n; $y_1, \dots, y_n$; y'_i, v_i, b_i, t_i the length of a family; a family of elements of a module; further elements

u, u', $u_{ii'}$ square matrices over $\mathcal{O}$ and their entries

i, i', l, l' indices; also natural numbers bounding exponents

G, G_i, G^{T} a linear map, also a square matrix; a family of linear maps; the transpose

p, ν, Y, c a ring element, a natural number, an endomorphism, a real number (Definition 2.2)

Y_i, c_i, Y', e, $\mathrm{len}(A/B)$ endomorphisms, real numbers; a composite; the length of a list; the length of a module of finite length

$\mathcal{J}$, $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{S}_0$, $\#\mathcal{S}$ finite sets or sets of indices; the number of elements

$\mathcal{O}^{\mathcal{J}}$, $\mathrm{M}_{\mathcal{J}}(\mathcal{O})$, $\mathrm{GL}_{\mathcal{J}}(\mathcal{O})$ families, matrices and invertible matrices indexed by $\mathcal{J}$

E, $\mathcal{O}_E$, $\mathbb{A}_{E,f}$, $\widehat{\mathcal{O}}_E$ a CM field, its integers, its finite adeles, the integral adeles

τ, ι_τ, φ an infinite place; the embeddings of a CM type (in Input 3, chosen embeddings of a number field); a ring homomorphism to $\mathbb{C}$

N' an integer ≥ 1 in the different of E

$\mathrm{U}_{\mathcal{J}}(\mathbb{A}_{E,f})$, $\mathrm{U}_{\mathcal{J}}(E)$, $\mathrm{U}_{\mathcal{J}}(\widehat{\mathcal{O}_E})$ the unitary group of the standard form: adelic points, rational points, the stabiliser of the standard lattice

$\mathrm{Cl}(\mathsf{K})$, K the classes at level K; a subgroup of $\mathrm{U}_{\mathcal{J}}(\mathbb{A}_{E,f})$

$f(E)$ the number of infinite places of E

$c_{\mathrm{sb}}(E, \mathcal{J}, N')$ the constant $6N'(f(E) \cdot \#\mathcal{J})^2$

$\mathfrak{c}$, $\mathfrak{c}'$, $h_{\mathfrak{c}}$, h, h^* classes; a representative; an adelic matrix; the conjugate transpose

$L_{\mathfrak{c}}$, L, $Q_{\mathfrak{c}}$, $H_{\mathfrak{c}}$, $Z_{\mathfrak{c}}$ a short basis; $h_{\mathfrak{c}}^{-1} L_{\mathfrak{c}}$; the Gram matrix; the approximate inverse

ϱ, η, α, δ, z_τ the approximation map; an adele; an element of E; an element of the different; complex numbers

γ, κ a rational unitary matrix; an element of the level subgroup

E', F, ω in Input 3: a number field, a map, a bound; in Input 4, ω are weights

$\mathcal{A}$, $\mathcal{R}$, $\mathcal{K}$, $\mathcal{N}$, $\jmath$, ϵ in Input 4: a ring, a subring, finite sets, an injection, an idempotent

X, T, Δ, T'_{11}, Δ_{11}, $\mathbf{y}$ the matrix to be reduced, the transformation, the triangular form, their blocks; the matrix of the coordinates of a family

y, y', $\bar{v}$ in Input 4 two ring elements; the complex conjugate

N, M, K, Φ, P, U, a, f, s, t, L letters of the formal text, used only in the conventions and in §7, note 2

Chapter VI

Theorem B: enough forms

- **Translates of Poincaré series fill the rows: a Petersson-type second moment along a Bessel subgroup** (unit 148)
  read: note built, structured proofs checked. 497 declarations, 3\,016 proof lines.
- **The analysis at the real places: tower measures, gauges, open-cell charts and Plancherel bounds** (unit 132)
  read: note built, structured proofs checked. 239 declarations, 1\,476 proof lines.
- **Discrete series coefficients near 1, Laplace's method, and explicitly large Bessel functions** (unit 133)
  read: note built, structured proofs checked. 37 declarations, 501 proof lines.

READBACK NOTE 35

Translates of Poincaré series fill the rows: the second moment over the level group and the filling theorem

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The rows of the proof are cusp forms on an indefinite unitary group; the columns are algebraic forms on a definite one, and a column gives a row by forming its Poincaré series along a Bessel subgroup. This note proves that the translates of these Poincaré series by the finite group of the chosen primes are *spread*: every subspace F of the rows is avoided, by some translate, on a share of the columns of nearly $\frac{1}{2}(1 - \dim F / \dim \text{rows})$. The proof is a second moment of Petersson type. The mean over the group of the squared Hilbert–Schmidt norm of the Gram operators of two translates is unfolded into sums of amplitudes over rational elements; its main term is bounded with constant 1 by a positive kernel of local Bessel integrals at finitely many primes and a sieve; a variance inequality turns the moment into the spreading statement. Theorem B of the handout follows from it by a greedy choice of translates.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles and that Lean's kernel accepted the proofs; an independent replay is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note (Note F of its unit) treats 18 of the 497 declarations of the unit *Translates of Poincaré series fill the rows*: the modules on fixed lines, on the finite system, on the moments, on the dimension of the rows and the filling theorem itself. Proved here from the inputs: Theorems A to D and the lemmas of §§4–9; two limiting arguments (in Proposition 4.5 and in Step 2 of the proof of Theorem D) are described and not written out. Taken for granted and stated as Inputs 1 to 10: the set-up of the unit with its amplitude sums and its crude estimate, the unfolding at finitely many places (Note G), the local theory at a finite place (Note T, with the facts on standard levels and on the tower method that it rests on), the sieve, a variance inequality of the unit on heights, the Petersson unfolding of Poincaré series, and classical facts from the artifact's background library (Siegel's formula in Weil's form, uniqueness of Haar measure, Harish-Chandra's formal degree). Every statement of this note was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 18 declarations.

1 The statements

The objects are defined or recalled in §2. In brief: D is admissible data for the parameters (Setting 2.5); $\mathrm{Col}(D)$ is the space of columns and $\mathrm{Row}(D)$ the space of rows; $\mathsf{G}(D)$ is the finite group of the chosen primes; $\mathcal{P}_D^\circ(\cdot)$ is the Poincaré series map from columns to forms of principal level (Definition 2.6); $\Pi_D(s)$ translates such a form by $s \in \mathsf{G}(D)$ and projects it to the rows; and $\mathrm{Ram}_D(\mathrm{Div}_N(D))$ is the operator on columns whose image is the space of ramified columns.

Theorem A (the filling theorem). *In Setting 2.5, let $c \in \mathbb{C}^\times$. For $s \in \mathsf{G}(D)$ let $B_s \colon \mathrm{Col}(D) \to \mathrm{Row}(D)$ be the linear map*

$$B_s(\psi) = \Pi_D(s)\Big(\mathcal{P}_D^\circ(c \cdot \mathrm{Ram}_D(\mathrm{Div}_N(D))\,\psi)\Big).$$

Then $\mathrm{Spread}((B_s)_{s\in\mathsf{G}(D)}; b(D)^{-2})$: for every complex subspace F of $\mathrm{Row}(D)$, with $\phi = \dim F / \dim \mathrm{Row}(D)$ (and $\phi = 0$ if $\dim \mathrm{Row}(D) = 0$), there is $s \in \mathsf{G}(D)$ such that

$$\dim B_s^{-1}(F) \;\le\; \Big(\frac{1+\phi}{2} + b(D)^{-2}\Big) \cdot \dim \mathrm{Col}(D).$$

LEAN `Data.ramifiedColumns_fill_rows`

Corollary 1.1 (rows exist)**.** *In Setting 2.5:* $\dim_{\mathbb{C}} \mathrm{Row}(D) \neq 0$.

LEAN `Data.finrank_rows_ne_zero`

Relation to the overview of the handout. The map $\mathcal{P}_D^\circ(\cdot)$ of this note is defined on all columns and involves no translation, no projection and no constant. The map that the overview writes $\mathcal{P}_D(\gamma)$, for $\gamma \in \mathsf{G}(D)$, from the ramified columns to the rows, is $\psi \mapsto \Pi_D(\gamma)(\mathcal{P}_D^\circ(c_0 \cdot \psi))$, where $c_0 \in \mathbb{C}^\times$ is the constant of the seesaw identity (unit on the seesaw). The ramified columns are the image of $\mathrm{Ram}_D(\mathrm{Div}_N(D))$; so the map B_s of Theorem A for $c = c_0$ is the overview's map $\mathcal{P}_D(s)$ composed with $\mathrm{Ram}_D(\mathrm{Div}_N(D))$, and Theorem A for $c = c_0$ gives Theorem B of the overview (§10).

Theorem A is the case, for the filling data $S(D)$ of the admissible data and the representation of $\mathsf{G}(D)$ on the forms of principal level, of the following statement about any unitary representation whose matrix coefficients on a family of vectors are amplitude sums.

Theorem B (filling, abstract form). *In Setting 2.1, suppose that E carries Weil data and that S is admissible. Let V be a finite-dimensional complex inner product space, ρ a representation of $\mathsf{G}(S)$ on V, let $Q\colon \ell^2(\mathrm{Cl}(S)\times \mathrm{Bc}(S)) \to V$ be a linear map, $g=(g_0,g_1)$ with $g_i \in \mathbb{C}[\mathsf{G}_i(S)]$, and c a real number. Suppose that*

(i) *ρ is unitary: $\langle \rho(s)x, \rho(s)y\rangle = \langle x, y\rangle$ for all s, x, y;*

(ii) *for all $s \in \mathsf{G}(S)$ and all pairs $x=(x_1,x_2)$, $y=(y_1,y_2)$ in $\mathrm{Cl}(S)\times \mathrm{Bc}(S)$,*

$$\big\langle \rho(s)\, Q(e_y),\, Q(e_x)\big\rangle = c\cdot A_S^{\mathrm{full}}(G_S(E); x_1, s, y_1)_{(x_2,y_2)},$$

where e_x is the vector of the standard basis at x;

(iii) *each g_i is a central idempotent, and $\mathrm{Re}\,(1-g_i)(1) \le b(S)^{-8}/2$ for $i=0,1$;*

(iv) *$0<c$; and $\dim \rho(g_0\otimes g_1)V \le \#\mathsf{G}(S)\cdot \deg(S)\cdot \mathrm{mass}(E, r(\mathfrak{s}))$.*

For $s\in\mathsf{G}(S)$ let $R_s\colon \ell^2(\mathrm{Cl}(S)\times\mathrm{Bc}(S)) \to \rho(g_0\otimes g_1)V$ be $R_s = \rho(g_0\otimes g_1)\circ\rho(s)\circ Q$. Then $\mathrm{Spread}((R_s)_{s\in\mathsf{G}(S)}; \frac{19}{2}\,b(S)^{-4})$: for every subspace F of $\rho(g_0\otimes g_1)V$, with $\phi = \dim F/\dim \rho(g_0\otimes g_1)V$, there is s with

$$\dim R_s^{-1}(F) \;\le\; \Big(\frac{1+\phi}{2} + \frac{19}{2}\,b(S)^{-4}\Big)\cdot \mathrm{cd}(S).$$

LEAN `Filling.Fits.exists_translate_finrank_comap_le`

Its proof rests on a second moment in which the constant of the main term is 1 up to $O(b(S)^{-4})$.

Theorem C (second moment with constant one). *In Setting 2.1, suppose that E carries Weil data and that S is admissible. Let V be a complex inner product space, $\Theta\colon \mathsf{G}(S)\to\mathbb{R}$ a function with $\Theta(s)\ge 0$ for all s, and $u\colon \mathsf{G}(S)\to V$ a family of vectors with $\langle u(a), u(b)\rangle = \Theta(a^{-1}b)$ for all a, b. Let $\Sigma \subseteq \mathrm{Ch}_0(S)$ be a set of chosen places above $q_0(S)$ and $\mathbf{d}$ frame-vector data for S (Definition 2.2). Suppose that*

$$\sum_{s\in\Omega}\Theta(s) \;\le\; \mathrm{En}_S(A_S^{\mathrm{full}}(G_S(E);\cdot,\cdot,\cdot);\Omega) \qquad \text{for every subset } \Omega\subseteq\mathsf{G}(S),$$

and that $\Theta(s)\neq 0$ implies $s\in \mathrm{Fix}_{S,w}(\mathbf{d}_w)$ for every $w\in\Sigma$. Then

$$\sum_{s\in\mathsf{G}(S)}\Theta(s) \;\le\; (1+9\,b(S)^{-4})\cdot \mathsf{d}(S)\cdot b(S)^{-6k(\mathfrak{s})\cdot\#\Sigma}.$$

LEAN `Filling.sum_le_of_positive_type`

Theorem D (the main-term estimate). *In Setting 2.1, suppose that E carries Weil data and that S is admissible. For every set $\Sigma\subseteq\mathrm{Ch}_0(S)$ of chosen places above $q_0(S)$ and all frame-vector data $\mathbf{d}$ for S,*

$$\begin{aligned}&\mathrm{En}_S(A_S^{\mathrm{full}}(G_S(E);\cdot,\cdot,\cdot);\mathrm{Fix}_S(\Sigma,\mathbf{d}))\\ &\quad\le\; (1-b(S)^{-4})(1+9\,b(S)^{-4})\cdot \mathsf{d}(S)\cdot b(S)^{-6k(\mathfrak{s})\cdot\#\Sigma}.\end{aligned}$$

LEAN `Filling.energy_fixingSet_le`

Here $\mathsf{d}(S) = \mathrm{cd}(S)^2/(\deg(S)\,\mathrm{mass}(E, r(\mathfrak{s})))$ is the expected value: the square of the number of columns divided by the expected number of rows per element of the group. For $\Sigma = \emptyset$ Theorem C says that the sum over the whole group of the squared Hilbert–Schmidt norms of the amplitude sums exceeds this by a factor $1 + 9\,b(S)^{-4}$ at most; each chosen place at which the summand is forced to fix a line gains a further factor $b(S)^{-6k(\mathfrak{s})}$. The chain of the proof is

$$\text{Theorem D} \implies \text{Theorem C} \implies \text{Theorem B} \implies \text{Theorem A},$$

and Theorem D itself combines the crude estimate of the unit (Input 3), the unfolding of the main terms into local correlations (Input 4), a one-sided form of Siegel's formula at finitely many places (Proposition 4.5), the domination by the top kernel (Proposition 4.6) and the sieve (Input 6) applied to a finite sieve system (Proposition 4.7).

2 The setting

Conventions. The natural numbers are $0, 1, 2, \ldots$. E is a CM field with its CM type: a number field that is a CM field, with complex conjugation $x \mapsto \bar{x}$, together with the choice of one embedding at each infinite place τ; $x_\tau \in \mathbb{C}$ is the image of $x \in E$ under it, and $f(E)$ is the number of infinite places. A *finite place* w of E is a non-zero prime ideal; $\mathbb{F}_w$ is its residue field, E_w the completion, $\bar{w}$ the conjugate place. $\mathbb{A}_{E,f}$ is the ring of finite adeles and $\hat{O}_E$ its subring of integral adeles. The field E is *stably unramified* if it contains a primitive root of unity of order 5040, and E *carries Weil data* if it is stably unramified and a splitting character in the sense of Chapter III of the handout is given. A *shape* is a triple $\mathfrak{s} = (\varrho, \kappa, k_0)$ of a natural number, a finite set and an element of it; we write $\varrho(\mathfrak{s})$, $k(\mathfrak{s}) = \#\kappa$, $n(\mathfrak{s}) = \varrho(\mathfrak{s}) + 1$ and $r(\mathfrak{s}) = 2\varrho(\mathfrak{s}) + k(\mathfrak{s}) + 1$. For a commutative ring with involution, X^* is the conjugate transpose of a matrix X. A *semilocal field* A is a commutative topological ring with a Haar measure vol in the sense of the background library (here always a finite product of completions of E); with an involution, an *integral place structure* on it provides a ring of integers O_A of volume 1, with unit group $O_A^\times$, a unitary additive character ψ_A and a modulus mod_A; the place structure is never called P. Index sets written n, j, ι are finite sets or arbitrary sets as stated. For a finite group G, $\mathbb{C}[G]$ is the complex group algebra and $x(s)$ the coefficient of $x \in \mathbb{C}[G]$ at $s \in G$; x is a *central idempotent* if $x^2 = x$ and $s \mapsto x(s)$ is a class function. A complex *inner product space* V has the inner product $\langle \cdot, \cdot \rangle$, conjugate-linear in the first variable, and is *finite-dimensional* where this is said. Integrals of functions with values in $[0, \infty]$ are Lebesgue integrals. P denotes the parameters of the proof (Chapter III), and the letters T, T_0 of the formal text for sets of places are written T, T_0; F, a level in the formal text, is written $\mathcal{L}$, and Z is an intermediate group of it. A *side* is one of the two words full, cut. Four letters have in this note a meaning other than in the handout's overview, and each use says which: c, c' are classes of the filling data, and c is also the scalar of Theorem A and the constant of Theorem B; s, t are elements of the finite group of the chosen places (the complex variable does not occur); μ is a measure (the exponent of the data does not occur); and $\zeta_{\hat{O}_E}$ is the ideal zeta function of the integral adeles (Riemann's zeta function does not occur).

Setting 2.1 (filling data)**.** E is a CM field with its CM type, $\mathfrak{s}$ is a shape, and S is filling data over E of shape $\mathfrak{s}$, in the sense of the set-up of this unit. Its components and the facts about them that this note uses are:

(i) a real number $b(S) \geq 1000$; a prime number $N(S) \geq 5$ and a finite idele $\bar{\pi}(S) \in \hat{O}_E$ with $\bar{\pi}(S)\,\overline{\bar{\pi}(S)} = N(S)$ (the bar in $\bar{\pi}(S)$ is part of the name of this idele, as in the formal text, and not a conjugation; its conjugate is $\overline{\bar{\pi}(S)}$); two primes $q_0(S)$, $q_1(S)$ different from $N(S)$ with $b(S)^{10} \leq q_i(S) \leq b(S)^{11}$, and $g(S) = q_0(S)q_1(S)$, with $b(S)^{19} \leq g(S) \leq b(S)^{23}$ (two further conditions of the formal record; they follow from the bounds for the primes, which give $b(S)^{20} \leq g(S) \leq b(S)^{22}$);

(ii) for $i = 0, 1$ a set $\mathrm{Ch}_i(S)$ of $f(E)$ finite places above $q_i(S)$, the *chosen places*, with residue fields of $q_i(S)$ elements, such that the conjugate of a chosen place is never chosen, and an integral finite adele r is divisible by $g(S)$ exactly if r and $\bar{r}$ vanish modulo every chosen place;

(iii) a generator $\varpi(S)$ of the different of E with $\overline{\varpi(S)} = -\varpi(S)$; an open compact subgroup $K^{\mathrm{col}}(S)$ of the finite adelic unitary group of the standard hermitian form on κ (the *column level*), with trivial class automorphism groups; for every infinite place τ a Bessel system $\mathfrak{B}_\tau(S)$ on the real group, with a finite index set, a full and a cut Bessel function, a representation and a positive number, its degree;

(iv) the inequalities $10^5 \leq \varrho(\mathfrak{s})$, $2^{-\varrho(\mathfrak{s})} \leq b(S)^{-6}$, $n(\mathfrak{s})\,[E:\mathbb{Q}] \leq 2^{2\varrho(\mathfrak{s})}$, $1000\,(2\varrho(\mathfrak{s}) + 2) \leq k(\mathfrak{s}) \leq f(E) \leq b(S)/1000$, $k(\mathfrak{s}) \leq b(S)$, and the chain $10^{10} n(\mathfrak{s})^4 \leq n(\mathfrak{s})^6$, $3\,n(\mathfrak{s})^6 + 12 \leq n(\mathfrak{s})^7 \leq k(\mathfrak{s})$.

Forms and groups. These notions belong to the background library and to the set-up of this unit; they are recalled with their arguments. The index set $I(\mathfrak{s})$ consists of $e_0, \dots, e_{\varrho(\mathfrak{s})-1}$, of the elements of κ, of z, and of $e_0^*, \dots, e_{\varrho(\mathfrak{s})-1}^*$; $M(\mathfrak{s}) = \kappa \sqcup \{z\}$. The *frame* is $\mathfrak{f}_a = e_a$ for $a < \varrho(\mathfrak{s})$ and $\mathfrak{f}_{\varrho(\mathfrak{s})} = z$. For a commutative ring R with involution, an $\mathfrak{s}$*-form* over R is a hermitian matrix T indexed by $M(\mathfrak{s})$; its *split Gram matrix* $J(T)$, indexed by $I(\mathfrak{s})$, pairs e_a with e_a^* to 1 and is T on the middle; $G(T)$ is the unitary group of $J(T)$ over R and $h_T(v, v') = v^* J(T) v'$. For $g \in G(T)$, $Z_T(g) = (h_T(\mathfrak{f}_a, g\,\mathfrak{f}_{a'}))_{a,a' \leq \varrho(\mathfrak{s})}$ is its frame Gram matrix. In $G(T)$: $\mathcal{H}(T)$ is the subgroup of the elements that fix all e_a, all e_a^* and z; $\mathcal{H}_\varrho(T)$ the subgroup of those that fix all e_a and e_a^*; $\mathcal{U}(T)$ is the Bessel unipotent subgroup of the library and $\mathcal{S}(T) = \mathcal{U}(T)\,\mathcal{H}(T)$ the *Bessel subgroup*: an element s lies in $\mathcal{S}(T)$ exactly if, for every a, $s\,\mathfrak{f}_a - \mathfrak{f}_a$ lies in the span of the $\mathfrak{f}_{a'}$ with $a' < a$. For $u \in \mathcal{U}(T)$, $\mathrm{sd}(u)$ is the sum of the entries of u just above the diagonal of its frame block. For a family $p = (p_0, p_1, \dots)$ of units of R, $w_\varrho(p) \in G(T)$ is the Weyl element of length $\varrho(\mathfrak{s})$: it sends e_a to $p_a e_a^*$ and e_a^* to $\bar{p}_a^{-1} e_a$ for $a < \varrho(\mathfrak{s})$, and fixes the middle; for $p\colon \{0, \dots, \varrho(\mathfrak{s}) - 1\} \to R^\times$ and a unit t, $(p \,|\, t)$ is the sequence $p_0, \dots, p_{\varrho(\mathfrak{s})-1}, t, 1, 1, \dots$.

In Setting 2.1: $T_S^{\mathrm{rat}}(R)$ is the $\mathfrak{s}$-form over R that is the identity on κ and $-N(S)$ at z, and $G_S(R) = G(T_S^{\mathrm{rat}}(R))$; $T_S^{\mathrm{lat}}(R)$ is the Gram matrix of the same hermitian form in the *lattice basis*, in which the basis vector x_{k_0} of k_0 is replaced by $\overline{\bar{\pi}(S)}^{-1} x_{k_0} + N(S)^{-1} z$ (for a unit π of another ring in the place of $\bar{\pi}(S)$: by $\bar{\pi}^{-1} x_{k_0} + (\pi\bar{\pi})^{-1} z$); $\Lambda(S)$ is the matrix of this change of basis over $\mathbb{A}_{E,f}$, and $G_S^{\mathrm{lat}} = G(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$. The homomorphism $\mathrm{r}_S^f\colon G_S(E) \to G_S^{\mathrm{lat}}$ is the diagonal em-

bedding followed by the change of basis, $\mathrm{r}_S^\infty \colon G_S(E) \to \prod_\tau G_1(\mathfrak{s})$ is the family of the images at the infinite places, rescaled to the real form $G_1(\mathfrak{s})$ of the library, and inc_S embeds the finite adelic points of the unitary group of the standard form on κ into G_S^{lat} as $\mathcal{H}$. For $\gamma \in G_S(E)$, $Z_S(\gamma) = Z_{T_S^{\mathrm{rat}}(E)}(\gamma)$ and $\mathsf{g}_S(\gamma)$ is the *term discriminant*, formed from the last two leading minors of $Z_S(\gamma)$.

$K(S) \subseteq G_S^{\mathrm{lat}}$ is the subgroup of the elements that are integral with integral inverse; the finite group of the chosen places is

$$\mathsf{G}(S) = \mathsf{G}_0(S) \times \mathsf{G}_1(S), \qquad \mathsf{G}_i(S) = \prod_{w \in \mathrm{Ch}_i(S)} \mathrm{GL}_{I(\mathfrak{s})}(\mathbb{F}_w),$$

with $\iota_i \colon \mathsf{G}_i(S) \to \mathsf{G}(S)$ the inclusion of the i-th factor; $\mathrm{red}_S \colon K(S) \to \mathsf{G}(S)$ reduces the entries modulo the chosen places, $K_g(S)$ is its kernel, and for a subgroup B of G_S^{lat}, $\overline{B}^{S} = \mathrm{red}_S(B \cap K(S))$. $\hat{g}(S)$ is the integer $g(S)$ as a finite idele, and $\psi_S^{\mathcal{U}}(u) = \psi_f(-\mathrm{sd}(u)/(g(S)\varpi(S)))$ for $u \in G_S^{\mathrm{lat}}$, with ψ_f the standard additive character of $\mathbb{A}_{E,f}$. $\mathrm{Cl}(S)$ is the finite set of double cosets of the finite adelic unitary group of κ by its rational points and $K^{\mathrm{col}}(S)$; each class c has a chosen representative, and $h_S(c)$ is its image under inc_S. For classes c, c' and $\gamma \in G_S(E)$,

$$\mathrm{pos}_S(c, c'; \gamma) = h_S(c)^{-1}\, \mathrm{r}_S^f(\gamma)\, h_S(c') \in G_S^{\mathrm{lat}},$$

and for $s \in \mathsf{G}(S)$ the set of *terms over* (c, s, c') is $\mathrm{Tm}_S(c, s, c') = \{\gamma \in G_S(E) : \mathrm{pos}_S(c, c'; \gamma) \in K(S),\ \mathrm{red}_S(\mathrm{pos}_S(c, c'; \gamma)) = s\}$. A set $\Xi \subseteq G_S(E)$ is *stable* if $u^{-1}\gamma u' \in \Xi \iff \gamma \in \Xi$ for all rational $u, u' \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$. $\mathcal{U}_{S,c}$ is the group of the rational $u \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$ with $\mathrm{pos}_S(c, c; u) \in K_g(S)$.

Amplitudes, energies, constants. $\mathrm{Bc}(S)$ is the set of the families $(i_\tau)_\tau$ of indices of the Bessel systems. For a side σ, a point y of $\prod_\tau G_1(\mathfrak{s})$ and $i \in \mathrm{Bc}(S)$, $W_S^\sigma(y)_i$ is the product over τ of the i_τ-th coordinates at y_τ of the full or the cut Bessel function of $\mathfrak{B}_\tau(S)$. With the cutoffs $\beta_{S,c}$ of the set-up, the *amplitude* of $\gamma \in G_S(E)$ over c, c' is the vector of $\ell^2(\mathrm{Bc}(S) \times \mathrm{Bc}(S))$

$$a_S^\sigma(c, c'; \gamma)_{(i,i')} = \int \beta_{S,c}(y)\, \beta_{S,c'}(\mathrm{r}_S^\infty(\gamma)^{-1} y)\, W_S^\sigma(y)_i\, \overline{W_S^\sigma(\mathrm{r}_S^\infty(\gamma)^{-1} y)_{i'}}\, dy,$$

and for $\Xi \subseteq G_S(E)$ the *amplitude sum* is

$$A_S^\sigma(\Xi; c, s, c') = \sum_{\gamma \in \Xi \cap \mathrm{Tm}_S(c,s,c')} a_S^\sigma(c, c'; \gamma).$$

For a family $A(c, s, c')$ of vectors of a normed space and $\Omega \subseteq \mathsf{G}(S)$, its *energy* is

$$\mathrm{En}_S(A; \Omega) = \nu(S)^{-2} \sum_{c, c' \in \mathrm{Cl}(S)} \sum_{s \in \Omega} \|A(c, s, c')\|^2.$$

The constants of the set-up are: the covolume $\nu(S) > 0$; $\mathrm{cd}(S) = \#\mathrm{Cl}(S) \cdot \#\mathrm{Bc}(S)$; $\deg(S)$, the product of the degrees of the Bessel systems; the mass $\mathrm{mass}(E, r(\mathfrak{s}))$ of the genus of the standard hermitian lattice of rank $r(\mathfrak{s})$ over E; and

$$\mathsf{d}(S) = \frac{\mathrm{cd}(S)^2}{\deg(S)\ \mathrm{mass}(E, r(\mathfrak{s}))}, \qquad \mathsf{d}_1(S) = \#\overline{\mathcal{S}}^{S} \cdot \mathrm{cd}(S), \qquad \mathrm{tl}(S) = b(S)^{k(\mathfrak{s}) f(E)/4},$$

where $\overline{\mathcal{S}}^{S}$ is formed with $\mathcal{S} = \mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$; further the codimension factor and the order

$$\mathrm{cf}(S) = (b(S)/g(S))^{k(\mathfrak{s})f(E)} \cdot 2\,\mathrm{tl}(S)^2, \qquad \mathrm{uo}(S) = g(S)^{f(E)\varrho(\mathfrak{s})(2k(\mathfrak{s})+2\varrho(\mathfrak{s})+1)},$$

Siegel's constant

$$c_*(S) = \Big(\frac{\pi^{k(\mathfrak{s})}}{(k(\mathfrak{s})-1)!\,N(S)^{k(\mathfrak{s})}}\Big)^{f(E)} |d_E|^{1/4}\ \mathrm{vol}(\hat{O}_E)^{k(\mathfrak{s})}$$

(with d_E the discriminant of E and $\mathrm{vol}(\hat{O}_E)$ the volume of the integral adeles for the self-dual measure), the packet constant $a_*(S) = \#\overline{\mathcal{S}}^{S} \cdot \mathrm{uo}(S) \cdot \#\mathrm{Cl}(S) \cdot c_*(S)$, the frame scales $\mathrm{fs}_S(\tau, a) = \kappa_\tau(S)^{2(\varrho(\mathfrak{s})-a)}/N(S) > 0$ for $a \le \varrho(\mathfrak{s})$, where $\kappa_\tau(S)$ is the real part of $g(S)\,\varpi(S)_\tau/(2\pi i)$ (in particular $\mathrm{fs}_S(\tau, \varrho(\mathfrak{s})) = N(S)^{-1}$), with their product $\mathrm{fs}(S)$, and the constant $\mathrm{oc}(\mathfrak{s}) > 0$ of the open cell.

S is *admissible* if the standing hypotheses of the set-up hold. This note uses these of them: $K^{\mathrm{col}}(S) = \mathrm{inc}_S^{-1}(K_g(S))$; red_S is surjective; and

$$\mathrm{tl}(S)^{44}\,\mathsf{d}_1(S) \le \mathsf{d}(S), \qquad \mathrm{tl}(S)^{18}\,\mathrm{cf}(S)\,\mathsf{d}(S) \le \mathsf{d}_1(S),$$
$$a_*(S) \ \le\ (1 + b(S)^{-9})\,\mathsf{d}(S)\ \deg(S)\ \frac{\mathrm{oc}(\mathfrak{s})^{f(E)}}{\mathrm{vol}(\hat{O}_E)^{n(\mathfrak{s})}}\Big(\frac{\nu(S)\,\mathrm{fs}(S)}{\#\mathrm{Bc}(S)}\Big)^2.$$

Definition 2.2 (fixing sets; recalled from the set-up)**.** In Setting 2.1. *Frame-vector data* $\mathbf{d}$ for S assign to every $w \in \mathrm{Ch}_0(S)$ a pair $\mathbf{d}_w = (\epsilon_w, v_w)$ of a Boolean and a non-zero vector $v_w \in \mathbb{F}_w^{\{0,\dots,\varrho(\mathfrak{s})\}}$; the same word is used for a single pair at a place of $\mathrm{Ch}_i(S)$. For $w \in \mathrm{Ch}_i(S)$ and such a pair (ϵ, v), $\mathrm{Fix}_{S,w}((\epsilon, v))$ is the subgroup of the $s \in \mathsf{G}(S)$ for which $s_w^{(\epsilon)}$ fixes the vector $\sum_a v_a \mathfrak{f}_a$ of $\mathbb{F}_w^{I(\mathfrak{s})}$, where $s_w^{(\epsilon)} = s_w$ if ϵ is false and $s_w^{(\epsilon)} = J^{-1}(s_w^{-1})^t J$ if ϵ is true, J being the split Gram matrix of $T_S^{\mathrm{lat}}(\mathbb{F}_w)$. For a natural number X, s *fixes norm at least* X if there is a finite set of triples (i, w, ϵ), $w \in \mathrm{Ch}_i(S)$, with $X \le \prod q_i(S)$ over the set, such that for each of them $s \in \mathrm{Fix}_{S,w}((\epsilon, v))$ for some $v \ne 0$. For $\Sigma \subseteq \mathrm{Ch}_0(S)$ and frame-vector data $\mathbf{d}$, $\mathrm{Fix}_S(\Sigma, \mathbf{d})$ is the set of the $s \in \mathsf{G}(S)$ such that $s \in \mathrm{Fix}_{S,w}(\mathbf{d}_w)$ for every $w \in \Sigma$, and such that $\Sigma = \mathrm{Ch}_0(S)$ or, for some $w \in \mathrm{Ch}_0(S)$, all leading minors of the frame Gram matrix $Z_{T_S^{\mathrm{lat}}(\mathbb{F}_w)}(s_w)$ are non-zero.

Levels at finitely many places. For a finite set T of finite places, $E_{\mathsf{T}} = \prod_{w \in \mathsf{T}} E_w$; $x_{\mathsf{T}} \in E_{\mathsf{T}}$ is the family of the components at T of a finite adele x, or of an element of E; x is *integral off* T if its components at the places outside T are integral, and an element of G_S^{lat} is integral off T if its entries and those of its inverse are. The homomorphism $\mathrm{loc}_{S,\mathsf{T}} \colon G_S^{\mathrm{lat}} \to G(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$ takes components at T. When E_{T} carries an integral place structure, O_{T} is its ring of integers and, for a unit u of E_{T}, $K_{S,\mathsf{T}}(u)$ is the group of the $y \in G(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$ that are integral with integral inverse and congruent to 1 modulo u; so $K_{S,\mathsf{T}}(1)$ is the integral unitary group.

A *level* $\mathcal{L}$ of $T_S^{\mathrm{lat}}(E_{\mathsf{T}})$, in the sense of the unit on the real place, consists of a hyperbolic basis of the middle form with an orthonormal part, a unit $g_{\mathcal{L}}$ and measures on the floors of the tower $\mathcal{H}_0 \supseteq \cdots \supseteq \mathcal{H}_\varrho \supseteq \mathcal{H}$; of these this note uses the measure $\mathsf{M}(\mathcal{L})$ on $G(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$, which for a standard level is a Haar measure (Input 10(a)), the measure $\theta_n(\mathcal{L})$ on $\mathcal{H}$, the measure $\mu_{\varrho,2}^{\mathcal{L}}$ on $\mathcal{H}_\varrho$ and the measure *quadric measure* $\delta^{(\varrho)}(\mathcal{L})$ (written so in Note T and in the set-up) on the vectors e of $E_{\mathsf{T}}^{I(\mathfrak{s})}$ in the orbit

of z (the quadric $h(e,e) = -N(S)$). Following Note G, $\mathcal{L} \in \mathrm{FL}(S, \mathsf{T})$ ($\mathcal{L}$ *is a filling level for* S *at* T) means: $\mathcal{L}$ is a standard level with $3\varrho(\mathfrak{s}) + 6$ further hyperbolic pairs and an orthonormal part of size $k(\mathfrak{s}) - 6\varrho(\mathfrak{s}) - 13$; the character of the place structure is $\psi_{S,\mathsf{T}}(x) = \psi_f(-\tilde{x}/\varpi(S))$, $\tilde{x}$ the extension of x by zero; and $g_{\mathcal{L}} = \hat{g}(S)_{\mathsf{T}}$. Then the congruence group of $\mathcal{L}$ is $K_{S,\mathsf{T}}(g_{\mathcal{L}})$.

Setting 2.3 (a filling level)**.** In Setting 2.1, with E, $\mathfrak{s}$ and S as there: T is a finite set of finite places of E that is stable under complex conjugation; E_{T} carries an integral place structure; and $\mathcal{L}$ is a level of $T^{\mathrm{lat}}_S(E_{\mathsf{T}})$ with $\mathcal{L} \in \mathrm{FL}(S, \mathsf{T})$, a filling level.

In Setting 2.3 the following objects of Note T are used; y, x, l are elements of $G(T^{\mathrm{lat}}_S(E_{\mathsf{T}}))$.

(a) The *finite Bessel function*: for a unit g, $\chi_g(y) = \psi_{\mathsf{T}}(g^{-1}\,\mathrm{sd}(u))$ if $y \in u\,K_{S,\mathsf{T}}(g)$ with $u \in \mathcal{U}$, and $\chi_g(y) = 0$ if there is no such u.

(b) $\mathrm{Int}(\mathcal{L})$ is the set of the subgroups Z with $K_{S,\mathsf{T}}(g_{\mathcal{L}}) \le Z \le K_{S,\mathsf{T}}(1)$, the *intermediate groups*; $\mathrm{Cos}(Z)$ is the set of the cosets of $K_{S,\mathsf{T}}(g_{\mathcal{L}})$ in Z, each with a representative a_c; $\mathsf{v}(\mathcal{L})$ is the product of the floor moduli of $\mathcal{L}$ below ϱ, and

$$\mathrm{mr}(Z) = \mathsf{M}(\mathcal{L})(Z) \cdot \mathsf{M}(\mathcal{L})(K_{S,\mathsf{T}}(g_{\mathcal{L}})) \cdot \mathsf{v}(\mathcal{L})^{-2},$$
$$\mathcal{J}_0(\mathcal{L}; Z) = \int \Big| \int_Z \chi_{g_{\mathcal{L}}}(yxy^{-1})\, d\mathsf{M}(\mathcal{L})(y) \Big|^2 d\mathsf{M}(\mathcal{L})(x).$$

(c) The *finite Bessel vector* $W_Z(y) \in \ell^2(\mathrm{Cos}(Z))$ has the coordinate $\chi_{g_{\mathcal{L}}}(y\,a_c^{-1})$ at the coset c.

(d) For measures and vector-valued functions, $\mathrm{J}(\mu, W; \gamma)$ is the tensor with coordinates $\int W(y)_i\, \overline{W(\gamma^{-1}y)_{i'}}\, d\mu(y)$. For a sequence p of units of E_{T}, the *correlation* is $\mathcal{J}(Z; p, l) = \mathsf{h}_{\varrho}(\mathcal{L}; p, 1) \cdot \mathrm{J}(\mathsf{M}(\mathcal{L}), W_Z(\cdot); w_{\varrho}(p)\, l^{-1})$, where $\mathsf{h}_{\varrho}(\mathcal{L}; p, 1)$ is the head density of the unit on the real place; for $L \subseteq E_{\mathsf{T}}$ and sequences p, q the *top kernel* is

$$\mathcal{K}(L; Z; p, q) = \int_{\{l \in \mathcal{H}_{\varrho}:\ h(z, l^{-1}z) - p_{\varrho} \in L\}} \big\langle \mathcal{J}(Z; p, l), \mathcal{J}(Z; q, l) \big\rangle\, d\mu^{\mathcal{L}}_{\varrho,2}(l);$$

and for a complex matrix R and a family ω of sequences of units, both indexed by a finite set, and a vector e, the *energy* $\mathcal{E}(Z; R, \omega, e)$ is the infimum over the $l \in \mathcal{H}_{\varrho}$ with $l^{-1}z = e$ of $\int \sum_x (\mathrm{Re}\ v^* R v)^+\, d\theta_n(\mathcal{L})(h)$, where $v_p = \mathcal{J}(Z; \omega_p, hl)_x$ and x runs over $\mathrm{Cos}(Z) \times \mathrm{Cos}(Z)$.

(e) For $j \le \varrho(\mathfrak{s})$, sequences p, q, a vector-valued function W and a function ω, $\mathcal{I}_j(\mathcal{L}; p, q; W, \omega) \in \mathbb{C}$ is the floor form of the unit on the real place, and for a natural number K, $\Lambda_K(j; p) = \prod_{l<j} [p_0 \cdots p_l\,]^{K/2}$ if all the partial products are integral, and 0 otherwise, where $[\xi]$ is the generalised index of ξ, equal to the index of ξO_{T} in O_{T} for integral ξ; $\mathrm{den}(\xi)$ is the index of $O_{\mathsf{T}} \cap \xi^{-1} O_{\mathsf{T}}$ in O_{T}.

The *coset amplitude* of Note G, for a side σ, classes c, c', $s \in \mathsf{G}(S)$ and $\gamma \in G_S(E)$, is

$$a^{\sigma}_{S,\mathsf{T}}(c, s, c'; \gamma) = \sum_{(u,u')} a^{\sigma}_S(c, c'; u\gamma u'^{-1}),$$

the sum over the pairs of rational elements of $\mathcal{U}(T^{\mathrm{rat}}_S(E))$ with $\mathrm{pos}_S(c, c; u)$ integral off T and $u\gamma u'^{-1} \in \mathrm{Tm}_S(c, s, c')$. The *frequency gain* of the set-up is $\beta_{S,\mathsf{T}}(\xi) = [\xi]^{(k(\mathfrak{s}) - n(\mathfrak{s})^7)/2}\, \mathrm{den}(\xi)^{-6}\, b(S)^{n(\mathfrak{s})^9 f(E)}$ for $\xi \in E_{\mathsf{T}}$.

Definition 2.4 (sieve systems; recalled from the set-up)**.** Let C be a commutative ring with a measure, U a topological space, $u\colon C \to U$ and $\iota_U\colon U \to C$ maps, n a natural number, $J(i,p,q) \in \mathbb{C}$ a function of a natural number and two sequences in U, $\psi\colon C \to \mathbb{C}$, $\mathcal{E}_0$ a real number, $\Lambda(i,p)$ and $\beta(\xi)$ real functions. These data form a *sieve system* of depth n if: $\psi(0) = 1$; for $i \le n$, all p, q and $\xi \in C$, the function $f(c) = \psi(\xi c)\, J(i+1, p[i \mapsto u(c)], q[i \mapsto u(c)])$ is integrable, $\int f = J(i,p,q)$ if $\xi = 0$, and $|\int f| \le \beta(\xi)\Lambda(i,p)\Lambda(i,q)$ if ξ is a unit; for every finite family of sequences with a common n-th term the matrix of the values of $J(n+1,\cdot,\cdot)$ is positive semidefinite; $J(n+1,\cdot,\cdot)$ is continuous; $|J(0,p,q)| \le \mathcal{E}_0$; and for $i < n$, if $\xi = \iota_U(p_i) - \iota_U(q_i)$ is a unit, then $|J(i+1,p,q)| \le \beta(\xi)\Lambda(i+1,p)\Lambda(i+1,q)$.

Setting 2.5 (the admissible data)**.** $P = (N_0, d)$ are the parameters of the proof and D is admissible data for P (Chapter III of the handout; Setting 1 of its overview). Attached to D are, as in the note *Good forms and rows*: the CM field $E(D)$, the ranks $m(D)$, $r(D)$, $k(D)$, the third scale $b(D)$, the level $N(D)$, a prime, with an integer $\pi_N(D)$ of $E(D)$ such that $\pi_N(D)\overline{\pi_N(D)} = N(D)$; the finite group $\mathsf{G}(D)$ of the chosen primes; the space $Y_0(D)$; the space $\mathrm{Prin}(D, Y_0(D))$ of the forms of principal level with the representation $\mathrm{rep}_{D,Y_0(D)}$ of $\mathsf{G}(D)$; the good idempotent $e_0(D) \in \mathbb{C}[\mathsf{G}(D)]$; and the rows $\mathrm{Row}(D)$, the image of $e_0(D)$ on $\mathrm{Prin}(D, Y_0(D))$. From the same note and from the units on the rows and on the exchange, for §9: the hermitian space $W_{m(D)}(D)$ of rank $r(D)$ that carries the good forms; the finite set $I(D)$ of the row indices, of $r(D)$ elements, with its numbering $\mathrm{num}_D\colon I(D) \to \{0, \dots, r(D)-1\}$; for every infinite place τ the row parameter $\varkappa_D(\tau,\cdot)$, a real function on $\{0,\dots,r(D)-1\}$; the gap $h(D)$ of the doubled row parameters; and, in the finite adelic unitary group of $W_{m(D)}(D)$, the maximal level K_0 and inside it the principal level $K_g(D)$ at the chosen primes. From the units on the columns and on the seesaw: the space $\mathrm{Col}(D)$ of columns; the set $\mathrm{Div}_N(D)$ of the primes of $E(D)$ that divide $\pi_N(D)$; for $w \in \mathrm{Div}_N(D)$ a central element $\epsilon_D(w)$, a norm-one integral finite idele that is 1 away from w and $\bar{w}$; for $\mathcal{W} \subseteq \mathrm{Div}_N(D)$ the operator

$$\mathrm{Ram}_D(\mathcal{W}) = \prod_{w \in \mathcal{W}} (1 - \text{translation by } \epsilon_D(w)) \quad \text{on } \mathrm{Col}(D);$$

for $s \in \mathsf{G}(D)$ the map $\Pi_D(s)\colon \mathrm{Prin}(D, Y_0(D)) \to \mathrm{Row}(D)$,

$$\Pi_D(s)(y) = \#\mathsf{G}(D) \cdot e_0(D)(\mathrm{rep}_{D,Y_0(D)}(s)\, y);$$

the filling data $S(D)$ of D (Note G), which are filling data over $E(D)$ whose group $\mathsf{G}(S(D))$ is identified with $\mathsf{G}(D)$; and the linear isomorphism $\mathrm{cc}_D\colon \mathrm{Col}(D) \to \ell^2(\mathrm{Cl}(S(D)) \times \mathrm{Bc}(S(D)))$ of column coordinates.

For a family $(B_s)_{s\in\mathcal{G}}$ of linear maps from a finite-dimensional complex space X to a complex space Y and a real number η, the unit on heights writes $\mathrm{Spread}((B_s)_{s\in\mathcal{G}}; \eta)$ (*the family is spread with defect* η) if for every subspace F of Y, with $\phi = \dim F / \dim Y$ (read as 0 when $\dim Y = 0$), there is s with $\dim B_s^{-1}(F) \le ((1+\phi)/2 + \eta)\dim X$.

Definition 2.6 (the Poincaré series map)**.** In Setting 2.5, for a column $\psi \in \mathrm{Col}(D)$,

$$\mathcal{P}^\circ_D(\psi) = \mathcal{P}^{\mathrm{fit}}_{S(D)}(\mathrm{cc}_D(\psi)) \;\in\; \mathrm{Prin}(D, Y_0(D)),$$

where $\mathcal{P}^{\mathrm{fit}}_{S(D)}$ is the Poincaré series map of the set-up for the filling data $S(D)$, which fit the space $Y_0(D)$: it sends the basis vector at $(c,i) \in \mathrm{Cl}(S(D)) \times \mathrm{Bc}(S(D))$ to the

form $\sum_\gamma \beta_{c,i}(\gamma\,\cdot)$, the sum over the rational points of the unitary group of $Y_0(D)$, of the cut Bessel function $\beta_{c,i}$ attached to the class c and the coordinate i. The map $\psi \mapsto \mathcal{P}_D^\circ(\psi)$ is $\mathbb{C}$-linear.

LEAN `Data.poincare`

3 What is taken for granted

The following statements are used and not proved here. Each was compared with its Lean statement. Inputs 1 to 3 and 6 belong to the set-up of this unit (chunks a and b of its dossier, the subject of other notes of the unit), Input 4 to Note G of this unit (*From amplitude sums to local correlations*), Input 5 to Note T of this unit (*The top floor at a finite place*); Inputs 7 to 9 come from other units and from the background library; Input 10 collects what the local theory needs of standard levels (set-up of this unit) and of the tower method (unit on the real place), facts that Note T recalls and does not prove.

Input 1 (terms and the Bessel subgroup; set-up of this unit). In Setting 2.1, with $\mathcal{U} = \mathcal{U}(T_S^{\mathrm{rat}}(E))$, $\mathcal{S} = \mathcal{S}(T_S^{\mathrm{rat}}(E))$:

(a) *(Size facts.)* $2 \le g(S)$, $2 \le \mathrm{tl}(S)$, $0 \le \mathsf{d}_1(S)$, $0 \le \mathsf{d}(S)$, $0 < \nu(S)$, $0 < \#\overline{\mathcal{S}}^S$; and $\mathrm{tl}(S)^4 = b(S)^{k(\mathfrak{s})f(E)}$.

(b) The set of the rational points of the Bessel subgroup is stable, and so is its complement.

(c) Suppose S is admissible. If $\gamma \in \mathcal{S}$ and $\mathrm{pos}_S(c,d;\gamma) \in K_g(S)$, then $c = d$ and $\gamma \in \mathcal{U}_{S,c}$. If $t \in \overline{\mathcal{S}}^S$ and c is a class, there are a class d and $\gamma \in \mathcal{S}$ with $\gamma \in \mathrm{Tm}_S(c,t,d)$.

(d) *(Freeness.)* Suppose E is stably unramified. Let n be a finite set with $\#n \le k(\mathfrak{s})$, $i \in \{0,1\}$, γ a rational element of the unitary group of the standard form on n and y a finite adelic element of it such that $y^{-1}\gamma y$ is integral and reduces to 1 at every place of $\mathrm{Ch}_i(S)$. Then $\gamma = 1$.

(e) For $\gamma \in G_S(E)$ and classes c, c': the frame vectors satisfy $\mathrm{pos}_S(c,c';\gamma)\,\mathfrak{f}_a = h_S(c)^{-1} \cdot (\gamma \mathfrak{f}_a)$; the frame Gram matrix of $\mathrm{pos}_S(c,c';\gamma)$ for $T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})$ is the image of $Z_S(\gamma)$; and if $\gamma \in \mathcal{U}$, then $\mathrm{pos}_S(c,c;\gamma) \in \mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$.

(f) For an integral finite adele r: $\hat{g}(S)^{-1}r$ is integral exactly if r and $\bar{r}$ vanish modulo every chosen place. For y in the finite adelic unitary group of κ with $\mathrm{inc}_S(y) \in K(S)$: y is integral with integral inverse, and for a chosen place w the reduction of $\mathrm{inc}_S(y)$ at w is 1 exactly if that of y is.

(g) *($\mathcal{U}$ times $\mathcal{H}$.)* Let $u \in \mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ and let h be in the image of inc_S. Every entry of h is the corresponding entry of uh or of the identity matrix. If $uh \in K(S)$, then $h \in K(S)$ and $u \in K(S)$; and if moreover the reduction of uh at a chosen place is 1, so is that of h. For a rational element δ of the unitary group of the standard form on κ, read as an element of $\mathcal{H}(T_S^{\mathrm{rat}}(E))$: $\mathrm{r}_S^f(\delta) = \mathrm{inc}_S(\delta)$, so that $\mathrm{pos}_S(c,c;\delta) = \mathrm{inc}_S(y_c^{-1}\delta\, y_c)$ for a class c with representative y_c.

Input 2 (amplitude sums; set-up of this unit). In Setting 2.1, suppose that E is stably unramified and S admissible; let σ be a side and $\Xi, \Xi' \subseteq G_S(E)$.

(a) *(Splitting.)* $A_S^\sigma(\Xi; c, s, c') = A_S^\sigma(\Xi \setminus \Xi'; c, s, c') + A_S^\sigma(\Xi \cap \Xi'; c, s, c')$ for all c, s, c'; the families of amplitudes are summable.

(b) *(The identity term.)* For classes c, c' and $i, j \in \mathrm{Bc}(S)$: $A_S^{\mathrm{full}}(\mathcal{S}(T_S^{\mathrm{rat}}(E)); c, 1, c')_{(i,j)}$ equals $\nu(S)$ if $c = c'$ and $i = j$, and 0 otherwise.

(c) *(Equivariance under $\mathcal{U}$.)* If Ξ is stable and $u, u' \in K(S)$ lie in $\mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$, then $A_S^\sigma(\Xi; c, \mathrm{red}_S(u)\, s\, \mathrm{red}_S(u')^{-1}, c') = \psi_S^{\mathcal{U}}(u)^{-1}\psi_S^{\mathcal{U}}(u') \cdot A_S^\sigma(\Xi; c, s, c')$.

(d) *(Moves.)* Let Ξ be stable, let $\gamma_0 \in \mathcal{S}(T_S^{\mathrm{rat}}(E)) \cap \mathrm{Tm}_S(c, t, c_1)$ and $\gamma_0' \in \mathcal{S}(T_S^{\mathrm{rat}}(E)) \cap \mathrm{Tm}_S(c', t', c_1')$ be such that $\gamma \in \Xi \iff \gamma_0\gamma\gamma_0'^{-1} \in \Xi$ for all γ, and let L be a continuous linear map of $\ell^2(\mathrm{Bc}(S) \times \mathrm{Bc}(S))$ that takes the tensor $W(y) \otimes \overline{W(\mathrm{r}_S^\infty(\gamma)^{-1}y)}$ of the Bessel functions of the side σ to the same tensor formed for $\gamma_0\gamma\gamma_0'^{-1}$ at the point $\mathrm{r}_S^\infty(\gamma_0)y$, for all γ and y. Then $A_S^\sigma(\Xi; c, t\, s\, t'^{-1}, c') = L\big(A_S^\sigma(\Xi; c_1, s, c_1')\big)$ for all s.

(e) *(The Bessel functions under the Bessel subgroup.)* For rational $u \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$ and $\eta \in \mathcal{H}(T_S^{\mathrm{rat}}(E))$ and every point y,

$$W_S^{\mathrm{full}}(\mathrm{r}_S^\infty(u\eta)\, y)_\cdot = \psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u))^{-1} \cdot A_\eta\, W_S^{\mathrm{full}}(y)_\cdot,$$

where $|\psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u))| = 1$ and A_η is a unitary operator of $\ell^2(\mathrm{Bc}(S))$: the tensor product over τ of the values at the image of η of the unitary representations of the Bessel systems $\mathfrak{B}_\tau(S)$.

Input 3 (the crude estimate; set-up of this unit). In Setting 2.1, suppose that E is stably unramified and S admissible. Let $C \geq 0$ and $\omega_0 \geq 0$ be real numbers, $\Omega \subseteq \mathsf{G}(S)$, $\Xi \subseteq G_S(E)$ a stable set, and $X \neq 0$ a natural number such that every $s \in \Omega$ fixes norm at least X. Suppose that $\mathsf{d}_1(S) \leq C$; that $\mathrm{tl}(S)^{20}\mathsf{d}_1(S) \leq C$ if some $\gamma \in \Xi$ that is a term over some (c, s, c') with $s \in \Omega$ is in normal form of index 0; let $\mathcal{R}$ be a set of links such that $\mathrm{tl}(S)^{12}X^{-k(s)}\mathsf{d}(S) \leq C$ if some link is not in $\mathcal{R}$; and suppose that the link energy of the cut side satisfies

$$\sum_{c,c'} \sum_{s \in \Omega} \sum_{\Lambda \in \mathcal{R}} \big\| A_S^{\mathrm{cut}}(\{\gamma \in \Xi :\ \mathsf{g}_S(\gamma) \neq 0,\ \mathrm{link}(\gamma) = \Lambda\}; c, s, c') \big\|^2 \ \leq\ \omega_0\, \nu(S)^2.$$

Then $\mathrm{En}_S(A_S^{\mathrm{full}}(\Xi; \cdot, \cdot, \cdot); \Omega) \leq (1 + b(S)^{-4})^2\, \omega_0 + b(S)^{-12}\, C$. Here $\mathrm{link}(\gamma)$ is the link of γ, a subspace attached to γ in the set-up; and γ is *in normal form of index* 0 if there is $v \in E$ with $v\bar{v} = 1$ such that for every frame index a there are coefficient vectors x, y, supported on the indices $\leq a$, with $x_a = v$, $y_a = 1$ and $\sum_{a'} x_{a'}\mathsf{f}_{a'} = \gamma \sum_{a'} y_{a'}\mathsf{f}_{a'}$.

Input 4 (levels at finitely many places and the unfolding; Note G). In Setting 2.1:

(a) If E carries Weil data, then for every finite set T of finite places that is stable under complex conjugation, E_{T} carries an integral place structure and a level $\mathcal{L}$ with $\mathcal{L} \in \mathrm{FL}(S, \mathsf{T})$. For every finite set T_0 of units of E there are arbitrarily large such T off which all $t^{\pm 1}$ ($t \in \mathsf{T}_0$), $\hat{g}(S)^{-1}$, $N(S)^{-1}$, $\Lambda(S)$ and $\Lambda(S)^{-1}$ are integral; and every element of G_S^{lat} is integral off all sufficiently large T.

(b) In Setting 2.3: every $y \in G(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$ is $\mathrm{loc}_{S,\mathsf{T}}(x)$ for some x integral off T, with $x \in \mathcal{U}$ if $y \in \mathcal{U}$ and $x \in \mathcal{H}$ if $y \in \mathcal{H}$; an x lies in $K(S)$ exactly if it is integral off T and $\mathrm{loc}_{S,\mathsf{T}}(x) \in K_{S,\mathsf{T}}(1)$. If moreover $\hat{g}(S)^{-1}$ is integral off T: x lies in $K_g(S)$ exactly if it is integral off T and $\mathrm{loc}_{S,\mathsf{T}}(x) \in K_{S,\mathsf{T}}(g_{\mathcal{L}})$; for $k, l \in K(S)$, $\mathrm{red}_S(k) = \mathrm{red}_S(l) \iff \mathrm{loc}_{S,\mathsf{T}}(k)^{-1}\,\mathrm{loc}_{S,\mathsf{T}}(l) \in K_{S,\mathsf{T}}(g_{\mathcal{L}})$; for x integral off T, $\psi_{\mathsf{T}}(g_{\mathcal{L}}^{-1}\,\mathrm{sd}(\mathrm{loc}_{S,\mathsf{T}}(x))) = \psi_S^{\mathcal{U}}(x)$; $\mathrm{uo}(S) \cdot \mathsf{v}(\mathcal{L}) = 1$; $\mathrm{mod}_{\mathsf{T}}(g_{\mathcal{L}}) = g(S)^{-2f(E)}$; and for a unit x of E with x, x^{-1} integral off T, $\mathrm{mod}_{\mathsf{T}}(x_{\mathsf{T}})^{1/2} = \prod_\tau |x_\tau|^{-1}$.

(c) In Setting 2.3 with $\hat{g}(S)^{-1}$ integral off T: if B is a subgroup of G_S^{lat} and μ an s-finite measure on $G(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$ such that μ-almost every point is $\mathrm{loc}_{S,\mathsf{T}}(x)$ for some $x \in B$ that is integral off T, and such that μ is invariant under right translation by $\mathrm{loc}_{S,\mathsf{T}}(x)$ for *every* $x \in B$, then $\#\overline{B}^{S} \cdot \mu(K_{S,\mathsf{T}}(g_{\mathcal{L}})) = \mu(K_{S,\mathsf{T}}(1))$. If E carries Weil data, $\#\overline{\mathcal{S}}^{S} = \#\overline{\mathcal{H}}^{S} \cdot \mathrm{uo}(S)$.

(d) *(The floor group.)* For $\gamma \in \mathcal{H}_{\varrho}(T_S^{\mathrm{rat}}(E))$ and classes c, c', put $l = \mathrm{pos}_S(c, c'; \gamma)$, $t = Z_S(\gamma)_{\varrho\varrho}$ and let $\omega \in E^{\kappa}$ be the κ-part of γz. Then $l \in \mathcal{H}_{\varrho}$, $l\,z = \mathrm{tv}(\bar{\pi}(S), t, y_c^{-1}\omega)$ with y_c the representative of c, $N(S)\,\omega^*\omega = t\bar{t} - N(S)^2$, and $\mathrm{pos}_S(c, c'; w_{\varrho}(p)\gamma) = w_{\varrho}(p)\,l$ for every family p of units of E; at an infinite place τ, $\mathrm{r}_S^{\infty}(w_{\varrho}(p)\gamma)_\tau = w_{\varrho}(p^{(\tau)}) \cdot \mathrm{r}_S^{\infty}(\gamma)_\tau$ with $p_a^{(\tau)} = u(\mathrm{fs}_S(\tau, a) \cdot (p_a)_\tau)$. Here $\mathrm{tv}(\pi, t, x)$, for a unit π, an element t and a vector x indexed by κ, is the vector $x - (\pi\bar{\pi})^{-1} t\,z$ written in the lattice basis formed with π. Taking components commutes with it: for T stable under conjugation, a finite adele t and $x \in \mathbb{A}_{E,f}^{\kappa}$, $\mathrm{tv}(\bar{\pi}(S), t, x)_{\mathsf{T}} = \mathrm{tv}(\bar{\pi}(S)_{\mathsf{T}}, t_{\mathsf{T}}, x_{\mathsf{T}})$.

Input 4, continued (Note G). In Setting 2.1:

(e) *(Blocks as limits of coset amplitudes.)* Suppose E carries Weil data and S is admissible; let σ be a side, c, c' classes, $s \in \mathsf{G}(S)$ and Λ a link. If $A^{\sigma}_{S}(\{\gamma : \mathfrak{g}_S(\gamma) \neq 0,\ \mathrm{link}(\gamma) = \Lambda\}; c, s, c') \neq 0$ and all leading minors of $Z_S(\gamma)$ are non-zero for every $\gamma \in \mathrm{Tm}_S(c, s, c')$, then there are a unit t of E, an element $\gamma_1 \in \mathcal{H}_{\varrho}(T^{\mathrm{rat}}_S(E))$ and $k \in K(S)$ with $Z_S(\gamma_1)_{\varrho\varrho} = t$, $N(S)^2 < |t_\tau|^2$ for all τ, $\mathrm{red}_S(k) = s$, $\Lambda = \mathrm{link}(w_{\varrho}(1)\gamma_1)$, and a family (L_p) of vectors, indexed by the maps p from $\{0, \dots, \varrho(\mathfrak{s}) - 1\}$ to $E^{\times}$, which is summable with sum the amplitude sum above, such that for every p, $a^{\sigma}_{S,\mathsf{T}}(c, s, c'; w_{\varrho}((p \,|\, t))\gamma_1) \to L_p$ as T grows.

(f) *(Unfolding.)* In Setting 2.3 with $\hat{g}(S)^{-1}$ integral off T, suppose E carries Weil data and S is admissible; let σ be a side, $\gamma_1 \in \mathcal{H}_{\varrho}(T^{\mathrm{rat}}_S(E))$, p a family of units of E with $\mathrm{pos}_S(c, c'; w_{\varrho}(p)\gamma_1)$ integral off T, $Z \in \mathrm{Int}(\mathcal{L})$, $k \in K(S)$ and cosets $a, b \in \mathrm{Cos}(Z)$ with $\mathrm{loc}_{S,\mathsf{T}}(k)^{-1} a_a a_b^{-1} \in K_{S,\mathsf{T}}(g_{\mathcal{L}})$. Then, with $\gamma = w_{\varrho}(p)\gamma_1$ and $W^{\sigma} = W^{\sigma}_S(\cdot)$,,

$$\begin{aligned}\mathcal{J}(Z; p_{\mathsf{T}}, \mathrm{loc}_{S,\mathsf{T}}(\mathrm{pos}_S(c, c'; \gamma_1))^{-1})_{(a,b)} \cdot \mathrm{J}(dy, W^{\sigma}; \mathrm{r}^{\infty}_S(\gamma)) \\ = \mathsf{h}_{\varrho}(\mathcal{L}; p_{\mathsf{T}}, 1) \cdot \mathsf{M}(\mathcal{L})(K_{S,\mathsf{T}}(g_{\mathcal{L}})) \cdot a^{\sigma}_{S,\mathsf{T}}(c, \mathrm{red}_S(k), c'; \gamma).\end{aligned}$$

Input 5 (the local theory at T; Note T). In Setting 2.3, with $K_g = K_{S,\mathsf{T}}(g_{\mathcal{L}})$, $K_1 = K_{S,\mathsf{T}}(1)$ and $Z \in \mathrm{Int}(\mathcal{L})$:

(a) If $\chi_g(y) \neq 0$ then $u^{-1} y \in K_{S,\mathsf{T}}(g)$ for some $u \in \mathcal{U}$, and then $\chi_g(y) = \psi_{\mathsf{T}}(g^{-1}\,\mathrm{sd}(u))$ for every such u; $\chi_g(yk) = \chi_g(y)$ for $k \in K_{S,\mathsf{T}}(g)$; and $y\, a_c^{-1} \in K_g$ holds for exactly one coset $c \in \mathrm{Cos}(Z)$ when $y \in Z$.

(b) *(The bottom value.)* For every natural number t with $t + 10^{10} n(\mathfrak{s})^4 \leq k(\mathfrak{s})$,

$$\mathcal{I}_0(\mathcal{L}; Z) \ \leq\ \mathsf{M}(\mathcal{L})(Z)\, \mathsf{M}(\mathcal{L})(K_g) \Big(\mathsf{M}(\mathcal{L})(Z) + \mathrm{vol}(O_{\mathsf{T}} \setminus O_{\mathsf{T}}^{\times})^{t+1}\, 2^{-n(\mathfrak{s})} \Big);$$

and $\int_{K_g} \big| \int_Z \chi_{g_{\mathcal{L}}}(yxy^{-1}) d\mathsf{M}(\mathcal{L})(y) \big|^2 d\mathsf{M}(\mathcal{L})(x) = \mathsf{M}(\mathcal{L})(Z)^2 \mathsf{M}(\mathcal{L})(K_g)$.

(c) If $B \geq 2$ and $\mathrm{vol}(O_{\mathsf{T}} \setminus O_{\mathsf{T}}^{\times}) \leq B^{-1}$, then $2^{-n(\mathfrak{s})} \leq \mathsf{M}(\mathcal{L})(K_1)$. Always $0 < \mathrm{mr}(Z) < \infty$.

(d) *(Frequency bound.)* Let $j \leq \varrho(\mathfrak{s})$, K and t natural numbers with $10^{10} n(\mathfrak{s})^4 \leq K$ and $t + 10^{10} n(\mathfrak{s})^4 \leq k(\mathfrak{s})$, ξ a unit of E_{T}, p, q sequences of units, and ω a function such that either $\omega(x) = \psi_{\mathsf{T}}(\xi \cdot Z(x)_{jj})$, or $j = i + 1$, $\omega = 1$ and $\xi = g_{\mathcal{L}}^{-1}(p_i^{-1} - q_i^{-1})$ for some i. Then

$$\begin{aligned}&\big| \mathcal{J}_j(\mathcal{L}; p, q; W_Z(\cdot), \omega) \big| \\ &\qquad \leq\ \mathrm{mr}(Z) \cdot [\xi]^{t/2}\, \mathrm{den}(\xi)^{-6} \Big(\frac{2}{\mathrm{mod}_{\mathsf{T}}(g_{\mathcal{L}})} \Big)^{K} \Lambda_K(j; p)\, \Lambda_K(j; q).\end{aligned}$$

Input 5, continued (Note T). In Setting 2.3, with K_g, K_1 and $Z \in \mathrm{Int}(\mathcal{L})$ as before:

(e) *(The top floor.)* For every $\xi \in E_{\mathsf{T}}$ and sequences p, q, the function $l \mapsto \psi_{\mathsf{T}}(\xi\, h(z, l^{-1}z))\langle \mathcal{J}(Z;p,l), \mathcal{J}(Z;q,l)\rangle$ is integrable for $\mu^{\mathcal{L}}_{\varrho,2}$, with integral $\mathcal{I}_{\varrho}(\mathcal{L};p,q;W_Z(\cdot),\omega_\xi)$, $\omega_\xi(x) = \psi_{\mathsf{T}}(\xi\, h(z,xz))$. For an open additive subgroup L, $(p,q) \mapsto \mathcal{K}(L;Z;p,q)$ is continuous.

(f) *(Energy and the top kernel.)* Let L be open, R a positive semidefinite matrix and ω a family of sequences of units, indexed by the same finite set, with $(\omega_p)_{\varrho} = t$ for all p. Then $\mathcal{Q} = \sum_{p,q} R_{pq}\, \mathcal{K}(L;Z;\omega_p,\omega_q)$ is real and ≥ 0, and

$$\int \mathbf{1}\big[(Je)_z - t \in L\big]\ \mathcal{E}(Z;R,\omega,e)\ d\delta^{(\varrho)}(\mathcal{L})(e)\ \leq\ \mathcal{Q},$$

J the split Gram matrix of $T^{\mathrm{lat}}_S(E_{\mathsf{T}})$.

(g) *(Packets.)* Let e be a vector and $(l_\theta, n_\theta)_{\theta\in\Theta_0}$ a non-empty finite family with $l_\theta \in \mathcal{H}_\varrho$, $l_\theta^{-1}z = e$ and $n_\theta \in \mathrm{Cos}(Z) \times \mathrm{Cos}(Z)$, such that $l_\theta l_{\theta'}^{-1} \in K_g$ and $n_\theta = n_{\theta'}$ imply $\theta = \theta'$; let R be a complex matrix and ω a family of sequences of units, indexed by the same finite set. Then for all x in a neighbourhood of e, all $k \in \mathcal{H} \cap K_g$ and all y with $ky = x$,

$$\theta_n(\mathcal{L})(K_g) \cdot \sum_\theta \big(\mathrm{Re}\ v_\theta^* R v_\theta\big)^+\ \leq\ \mathcal{E}(Z;R,\omega,y), \qquad v_\theta(p) = \mathcal{J}(Z;\omega_p,l_\theta)_{n_\theta}.$$

(Note T assumes here that the units of the ring form an open set and that inversion is continuous at 1; both hold in E_{T}, a finite product of local fields.)

(h) *(What the tower method needs of the finite Bessel vector.)* The function $W = W_Z(\cdot)$ is a cut Bessel function for $\mathcal{L}$ in the sense of the unit on the real place: it is measurable and bounded, $W(uy) = \psi_{\mathsf{T}}(g_{\mathcal{L}}^{-1}\,\mathrm{sd}(u))\, W(y)$ for $u \in \mathcal{U}$, and $W(y) \neq 0$ only if $uy \in K_1$ for some $u \in \mathcal{U}$ (Note T, Lemma 4.5). For $j \leq \varrho(\mathfrak{s})$ and sequences p, q of units, the majorant of the floor form $\mathcal{J}_j(\mathcal{L};p,q;W,\cdot)$, a function on $G(T^{\mathrm{lat}}_S(E_{\mathsf{T}}))$ with values in $[0,\infty]$ that the unit on the real place defines, has a finite integral for $\mathsf{M}(\mathcal{L})$ (Note T, Corollary 5.5 for the number 0, which is admissible because $10^8 n(\mathfrak{s})^4 \leq k(\mathfrak{s})$). And for every x,

$$\Big|\int_Z \chi_{g_{\mathcal{L}}}(yxy^{-1})\, d\mathsf{M}(\mathcal{L})(y)\Big| = \mathsf{v}(\mathcal{L}) \cdot \big|\mathsf{A}(\mathcal{L};W,x)\big|$$

(Note T, Lemma 4.8), with the average $\mathsf{A}(\mathcal{L};W,x)$ of Input 10(d).

Input 6 (the sieve, and slabs near a quadric; set-up of this unit). (a) In Setting 2.1, suppose E carries Weil data and S is admissible. Let T be a finite set of finite places, stable under complex conjugation, with $\hat{g}(S)^{-1}$ integral off T; let T_0 be a finite set of units of E with t, t^{-1} integral off T for $t \in \mathsf{T}_0$, and $U \subseteq \mathsf{T}_0$. Let J' and $\mathcal{E}_0$ be such that the maps $c \mapsto u(c)$, $x \mapsto g(S)_{\mathsf{T}}^{-1} x^{-1}$, the depth $\varrho(\mathfrak{s})$, J', the character $\psi_{S,\mathsf{T}}$, the number $\mathcal{E}_0$, the weights $\Lambda_{n(\mathfrak{s})^6}(\cdot;\cdot)$ and the frequency gain $\beta_{S,\mathsf{T}}(\cdot)$ form a sieve system on E_{T}. Then

$$\begin{aligned} &\operatorname{Re} \sum_{t \in U} \sum_{p,q} \Big(\prod_{\tau} \mathrm{bl}_\tau(x_\tau(p,t), x_\tau(q,t)) \Big) J'\big(n(\mathfrak{s}), (p \,|\, t)_{\mathsf{T}}, (q \,|\, t)_{\mathsf{T}}\big) \\ &\leq \prod_{a < n(\mathfrak{s})} \frac{\mathrm{vol}(\hat{O}_E)}{\prod_\tau \mathrm{fs}_S(\tau, a)^2} \cdot \Big(\prod_\tau \frac{1 + b(S)^{-10}}{\deg \mathfrak{B}_\tau(S)} \cdot \mathcal{E}_0 + \frac{\deg(S)^{-1}}{b(S)^{\,6k(\mathfrak{s})f(E)+10}} \Big), \end{aligned}$$

where p, q run over the maps from $\{0, \dots, \varrho(\mathfrak{s}) - 1\}$ to T_0, bl_τ is the band limit of the Bessel system $\mathfrak{B}_\tau(S)$ at level $n(\mathfrak{s})$, and $x_\tau(p,t)$ is the sequence $a \mapsto u\big(\mathrm{fs}_S(\tau, a) \cdot ((p \,|\, t)_a)_\tau\big)$. The band limit of a Bessel system with index set I_τ and cut Bessel function W_τ^{cut} satisfies: for $\gamma \in \mathcal{H}_\varrho$ in the real group $G_1(\mathfrak{s})$ and sequences p, q of non-zero complex numbers with $p_\varrho = q_\varrho = Z(\gamma)_{\varrho\varrho}$, $\mathrm{bl}_\tau(p,q) = \langle \mathrm{nc}_\tau(\gamma, p), \mathrm{nc}_\tau(\gamma, q) \rangle$, where

$$\begin{aligned} \mathrm{nc}_\tau(\gamma, p) = \frac{\sqrt{\mathrm{oc}(\mathfrak{s})}\,(|p_\varrho|^2 - 1)^{(k(\mathfrak{s})-1)/2}}{\#I_\tau} \prod_{a < \varrho(\mathfrak{s})} |p_a|^{k(\mathfrak{s})+1+2(\varrho(\mathfrak{s})-a-1)} \\ \cdot \mathrm{J}(dy, W_\tau^{\mathrm{cut}}; w_\varrho(p)\gamma). \end{aligned}$$

Input 6, continued (slabs near a quadric; set-up of this unit). (b) Let A be a semilocal field with involution and an integral place structure, π a unit with $\pi\bar{\pi} = N$, $\mathcal{B}$ a hyperbolic basis of the lattice form of the shape $\mathfrak{s}$ for N, with its quadric measure $\delta^{(\varrho)}(\mathcal{B})$, $\Phi \geq 0$ a continuous function with compact support on $A^{I(\mathfrak{s})}$ and $c > \int \Phi \, d\delta^{(\varrho)}(\mathcal{B})$. Then for all sufficiently small measurable sets $M \subseteq A$,

$$\begin{aligned} \int \int_{\{y:\ y^* y - ((\pi\bar{\pi})^{-1}\tau\bar{\tau} - N) \in M\}} \Phi\big(\mathrm{tv}(\pi, \tau, y)\big)\, dy\, d\tau \\ \leq\ \mathrm{mod}_A(\pi) \cdot \nu_A(\delta_A M) \cdot c, \end{aligned}$$

with ν_A the measure on the skew elements and δ_A the skew unit of the place structure. Further, for a unit π with $\pi\bar{\pi} = N$ and $e = \mathrm{tv}(\pi, t, x)$: $(Je)_z = t$ for the split Gram matrix J of the lattice form.

Input 7 (positive type and variance; unit on heights). Let G be a finite group, H a complex inner product space.

(a) If $u, v\colon G \to H$ satisfy $\langle u(a), v(b)\rangle = f(a^{-1}b)$, then $\langle \sum_a u(a), \sum_a v(a)\rangle = \#G \cdot \sum_s f(s)$.

(b) Let k be a finite field, n, m finite sets, $\pi\colon G \to \mathrm{GL}_n(k)$ a homomorphism, A_0, A_1 matrices over k with rows indexed by n and columns by m, $\Theta\colon G \to \mathbb{R}$ with $\Theta \geq 0$ and $\langle u(a), u(b)\rangle = \Theta(a^{-1}b)$ for a family $u\colon G \to H$, and $Y \geq 0$ such that for every $v \neq 0$ in k^m the sum of Θ over the s with $\pi(s)A_0 v = A_0 v$, and the sum over the s with $(\pi(s)^{-1})^t A_1 v = A_1 v$, are at most Y. Then

$$\#k \cdot \sum_{s\colon \det(A_1^t \pi(s) A_0)=0} \Theta(s) \;\leq\; 2\sum_s \Theta(s) + \#k^{\#m}\sqrt{\#k^{\#n}} \cdot Y.$$

(c) For an orthonormal basis $\mathbf{b}$ of H (finite-dimensional) and a finite family $v = (v_x)$ in H, let $\mathrm{dy}_{\mathbf{b}}(v) = \sum_x \bar{v}_x \otimes v_x$, the conjugates taken in the coordinates of $\mathbf{b}$; then $\langle \mathrm{dy}_{\mathbf{b}}(v), \mathrm{dy}_{\mathbf{b}}(v')\rangle = \sum_{x,y} |\langle v'_y, v_x\rangle|^2$. Let $\mathsf{P}\colon \mathbb{C}^n \to H$ and $\mathsf{R}\colon H \to H$ be linear, U a subspace of $\mathbb{C}^n$ and A one of H, with $\mathsf{P}v - \mathsf{RP}v \in A$ for $v \in U$, and $\sum_x \|(\mathsf{P}^*\mathsf{P} - 1)e_x\| \leq \varepsilon$. Then

$$\dim U - \varepsilon - \sum_x \|\mathsf{RP}e_x\|^2 \;\leq\; \mathrm{Re}\langle \mathrm{dy}_{\mathbf{b}}((\mathsf{P}e_x)_x), \mathrm{dy}_{\mathbf{b}}(\mathbf{a})\rangle$$

for an orthonormal basis $\mathbf{a}$ of A that the library chooses.

Input 8 (background library).

(a) *(Central idempotents.)* If x_i are central idempotents of the group algebras of finite groups G_i, so are $1 - x_i$ and the tensor $\otimes_i x_i \in \mathbb{C}[\prod_i G_i]$. For a central idempotent x and a unitary representation, the operator of x is self-adjoint. If x is a central idempotent of $\mathbb{C}[G]$, then $\#G \cdot x(\cdot)$ is the character of a representation of G; in particular $x(s^{-1}) = \overline{x(s)}$ and $\sum_s |x(s)|^2 = (x \cdot x)(1) = x(1)$. For a homomorphism $\varphi\colon G \to G'$ of finite groups, a central idempotent x of $\mathbb{C}[G]$ and a family f of vectors indexed by G': $\sum_s \| \sum_t x(t) f(\varphi(t)^{-1} s) \|^2 \le \sum_s \|f(s)\|^2$. Elementary; reference to be confirmed.

(b) *(Siegel's formula for one vector.)* Let E carry Weil data, κ be a finite set with $\#\kappa \ge 3$, K an open compact subgroup of the finite adelic unitary group of the standard form on κ, Φ_0 a locally constant function with compact support on $\mathbb{A}_{E,f}^{\kappa}$, invariant under K, with values ≥ 0, and $\alpha \in E$ with $\bar{\alpha} = \alpha$ and $\operatorname{Re} \alpha_\tau > 0$ for all τ. Then, with the sum over the classes h of K, weighted by the inverses of the orders of their automorphism groups,

$$\sum_h \sum_{x \in E^\kappa,\ x^* x = \alpha} \Phi_0(h^{-1} x) = \Big(\sum_h 1 \Big) \cdot |d_E|^{1/4} \prod_\tau \frac{\pi^{\#\kappa}}{(\#\kappa - 1)!} (\operatorname{Re} \alpha_\tau)^{\#\kappa - 1} \cdot \int \int \psi_f(\tfrac{1}{2} z (x^* x - \alpha)) \Phi_0(x) \, dx \, dz,$$

z over the self-adjoint finite adeles. This is a special case of Weil's mean-value identity on a non-zero index fibre (A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*, §52, Theorem 5, equation (41)), with the constants of one vector worked out in the formal text; the comparison is the second reader's.

Input 8, continued (background library).

(c) *(Local densities off* T*.)* Let E carry Weil data and κ be a finite set. Let $M \geq 1$ be an integer, $m \in \mathbb{A}_{E,f}$ with $Mm = 1$, and $\mathfrak{c}_M$ the *codifferent scaled by* M: the group of the self-adjoint finite adeles z with Mz in the inverse different of E, a compact open subgroup of the self-adjoint finite adeles. Let $\beta \in \mathbb{A}_{E,f}$ with $\bar{\beta} = \beta$, let T be stable under conjugation with an integral place structure on E_{T}, and let a_0 be a natural number with $\zeta_{\hat{O}_E}(a_0) \leq 2$ and $2(a_0+1) \leq \#\kappa$, where $\zeta_{\hat{O}_E}$ is the ideal zeta function of $\hat{O}_E$.
(i) For every finite measure μ on $\mathbb{A}_{E,f}^{\kappa}$,

$$\Big| \int_{\mathfrak{c}_M} \int \psi_f(\tfrac{1}{2} z(x^*x - \beta))\, d\mu(x)\, dz \Big| = \mathrm{vol}(\mathfrak{c}_M) \cdot \mu\{x : \ m(x^*x - \beta) \in \hat{O}_E\}.$$

(ii) There is a finite number $v_0 \geq 0$ with

$$\mathrm{vol}(\mathfrak{c}_M) \cdot \nu_{\mathsf{T}}\{\Gamma : \ m_{\mathsf{T}} \delta_{\mathsf{T}}^{-1} \Gamma \in O_{\mathsf{T}}\} \cdot v_0 \ \leq \ \zeta_{\hat{O}_E}(a_0) \cdot \mathrm{vol}(\hat{O}_E)^{\#\kappa}$$

such that for all $a \in E_{\mathsf{T}}$, with extension by zero $\tilde{a}$, and all measurable $f \geq 0$ on E_{T}^{κ},

$$\int_{\{x:\ m(x^*x - \beta - \tilde{a}) \in \hat{O}_E\}} \mathbf{1}[x \text{ integral off } \mathsf{T}]\, f(x_{\mathsf{T}})\, dx$$
$$= v_0 \int_{\{y:\ m_{\mathsf{T}}(y^*y - \beta_{\mathsf{T}} - a) \in O_{\mathsf{T}}\}} f(y)\, dy.$$

Here ν_{T} is the measure on the skew elements and δ_{T} the skew unit of the place structure.
(iii) $\zeta_{\hat{O}_E}(u+v+4)^{n'} \leq 1 + 2^{-u}$ for natural numbers u, v, n' with $n'[E:\mathbb{Q}] \leq 2^v$; and for a unit u of a semilocal field A with an integral place structure, $\mathrm{vol}\{x : ux \in O_A\} = \mathrm{mod}_A(u)^{-1}$. Reference to be confirmed.

Input 8, continued (background library).

(d) *(Haar measure.)* On a locally compact second countable group with a right-invariant Haar measure μ, every s-finite right-invariant measure is $d \cdot \mu$ for some $d \in [0, \infty]$. The regular case is classical (compare F. van Doorn, *Formalized Haar measure*, §6, Theorem 12); the relation of the s-finite form with an infinite multiplier to the literature is recorded as unresolved by the second reader.

(e) *(Finite groups.)* The stabiliser of a non-zero vector of k^n in $\mathrm{GL}_n(k)$, k a finite field, has index $\#k^{\#n} - 1$. Elementary.

(f) *(Forms, classes, mass.)* For a level K with trivial class automorphism groups, the space of forms of level K of an algebraic type has dimension the number of classes times the dimension of the weight representation; the number of classes of $K' \leq K$ is the index times that of K when counted with weights; the weighted number of classes of the maximal level of rank r is $\mathrm{mass}(E, r)$; right translation is unitary for the Petersson product; a level has positive finite volume. Reference to be confirmed.

(g) *(Formal degree.)* For a Harish-Chandra parameter with a large gap, the formal degree of the discrete series is given by Weyl's dimension polynomial (Harish-Chandra, *Discrete series for semisimple Lie groups II*, Acta Math. 116 (1966), §41, Theorem 16; the exact comparison of measures and coefficients with the formal statement is recorded as open by the second reader).

(h) *(Place structures.)* In a semilocal field A with a place structure almost every element, for the Haar measure, is a unit, and $\mathrm{mod}_A(\bar{x}) = \mathrm{mod}_A(x)$. Both are among the conditions that the library puts on a place structure; they hold for a finite product of local fields.

Input 9 (the data). In Setting 2.5:

(a) *(Note G.)* $S(D)$ is admissible, $E(D)$ carries Weil data, $b(S(D)) = b(D)$, $N(S(D)) = N(D)$, $r(\mathfrak{s}) = r(D)$ for the shape of $S(D)$, and $\deg(S(D))$ is the product over the infinite places of the degrees of Bessel systems that fit the type of the rows.

(b) *(Petersson unfolding; set-up of this unit and unit on the seesaw.)* For $s \in \mathsf{G}(D)$ and $a = (c, i)$, $a' = (c', i')$ in $\mathrm{Cl}(S(D)) \times \mathrm{Bc}(S(D))$,

$$\begin{aligned}&\langle \mathrm{rep}_{D, Y_0(D)}(s)\, \mathcal{P}^{\mathrm{fit}}_{S(D)}(e_{a'}),\ \mathcal{P}^{\mathrm{fit}}_{S(D)}(e_a)\rangle \\ &\qquad = \mathrm{vol}(K_g) \cdot A^{\mathrm{full}}_{S(D)}(G_{S(D)}(E(D)); c, s, c')_{(i,i')},\end{aligned}$$

with $\mathrm{vol}(K_g) > 0$ the volume of the principal level at the chosen primes.

(c) *(Good idempotent; units on labels and good forms.)* $e_0(D) = g_0 \otimes g_1$, where g_i is the sum of the label idempotents over the good tuples at the chosen primes above $g_i(D)$; each g_i is a central idempotent, and $\mathrm{Re}\,(1 - g_i)(1) \le b(D)^{-8}/2$. (The first is a restricted form of Green's classification, J. A. Green, *The characters of the finite general linear groups*, Trans. Amer. Math. Soc. 80 (1955), §8, Theorems 13–14, in the second reader's comparison; the second is the bound for the bad share of the note *Primitive labels*, with the sizes of the data.)

(d) *(Ramified columns; unit on the columns.)* For $\mathcal{W} \subseteq \mathrm{Div}_N(D)$ and a subspace X of $\mathrm{Col}(D)$: $\dim \mathrm{Ram}_D(\mathcal{W})^{-1}(X) \le \dim X + \#\mathcal{W}\,(N(D) - 1)^{-1} \dim \mathrm{Col}(D)$. Moreover $\#\mathrm{Div}_N(D) = f(E(D))$.

(e) *(Sizes; Chapter III.)* $b(D)^9 \le N(D) - 1$; $b(D)^{-2} < 1/2$; and $\frac{19}{2}\, b(D)^{-4} + f(E(D))\,(N(D) - 1)^{-1} \le b(D)^{-2}$, a comparison of the logarithmic sizes of the data.

(f) *(Rows; note Good forms and rows, units on the rows and on the exchange.)* $\dim \mathrm{Row}(D)$ equals the dimension of the good forms on the space $W_{m(D)}(D)$, which is standard, hence definite; the row parameter $\varkappa_D(\tau, \cdot)$ is strictly decreasing, num_D is a bijection from the row index set $I(D)$ to $\{0, \dots, r(D) - 1\}$, and the doubled row parameters have the gap $h(D)$; a parameter with a gap is a Harish-Chandra parameter, through which the forms of the type of the data on a standard space are algebraic forms; the class automorphism groups of the principal level $K_g(D)$ at the chosen primes are trivial; and its index in the maximal level K_0 is $\#\mathsf{G}(D)$.

Input 10 (standard levels and the tower method). In Setting 2.3; a filling level is a standard level in the sense of the note *Poincaré series of cut Bessel functions* of this unit. Write $K_1 = K_{S,\mathsf{T}}(1)$.

(a) *(Standard levels; note Poincaré series of cut Bessel functions, set-up of this unit.)* $\mathsf{M}(\mathcal{L})$ is a Haar measure of $G(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$, invariant under left and right translations (its corollary on the group measure); $\mathsf{M}(\mathcal{L})(K_1) \le 1 + 2^{-n(\mathfrak{s})}$ (its proposition on the mass of the integral group, from above); $\theta_n(\mathcal{L})(K_1) = 1$ (condition (LL2) of a lattice level); and $\mathcal{L}$ has a gauge in the sense of the unit on the real place (the gauge of depth 1 of a standard level).

(b) *(Levels; unit on the real place, note The tower method at a place.)* The measure $\theta_n(\mathcal{L})$ is carried by $\mathcal{H}$, and it is invariant under right translation by every element of $\mathcal{H}$. Both are conditions in the definition of a level.

(c) *(Descent; unit on the real place, note The local theorem for floor forms, Theorem B.)* Let W be a cut Bessel function for $\mathcal{L}$, $j < \varrho(\mathfrak{s})$, $\xi \in E_{\mathsf{T}}$ and p, q sequences of units such that the majorant of the floor form $\mathcal{J}_j(\mathcal{L}; p, q; W, \cdot)$ has a finite integral for $\mathsf{M}(\mathcal{L})$. Then the function $c \mapsto \psi_{\mathsf{T}}(\xi c)\, \mathcal{J}_{j+1}(\mathcal{L}; p[j \mapsto u(c)], q[j \mapsto u(c)]; W, 1)$ on E_{T} is integrable, with integral $\mathcal{J}_j(\mathcal{L}; p, q; W, \omega_\xi)$, $\omega_\xi(x) = \psi_{\mathsf{T}}(\xi\, Z(x)_{jj})$; here $u(c) = c$ if c is a unit and $u(c) = 1$ otherwise, and $p[j \mapsto u]$ is the sequence p with its entry j replaced by u.

(d) *(The floor 0; unit on the real place, the same note.)* Let W be a cut Bessel function for $\mathcal{L}$ and p, q sequences of units. For a point x put

$$\mathsf{A}(\mathcal{L}; W, x) = \int \big\langle W(\sigma), W(\sigma x) \big\rangle \, d\mathsf{M}_{\ge 0}(\mathcal{L})(\sigma),$$

where $\mathsf{M}_{\ge 0}(\mathcal{L})$ is the convolution of all the floor measures of $\mathcal{L}$, from the measure $\theta_n(\mathcal{L})$ down to that of the floor 0. The unit on the real place bounds every floor form by its head density times the integral for $\mathsf{M}(\mathcal{L})$ of its modulus (lemma on form, modulus and majorant). At the floor 0 the head density $\mathsf{h}_0(\mathcal{L}; p, q)$ is an empty product, equal to 1, and the modulus of $\mathcal{J}_0(\mathcal{L}; p, q; W, 1)$ at x is $|\mathsf{A}(\mathcal{L}; W, x)|^2$, by its definition, in which no floor lies below the floor 0, and by the lemma on the integrand at weight 1. Hence

$$\big|\mathcal{J}_0(\mathcal{L}; p, q; W, 1)\big| \;\le\; \int \big|\mathsf{A}(\mathcal{L}; W, x)\big|^2 \, d\mathsf{M}(\mathcal{L})(x).$$

This last inequality is not a formal statement of its own: the formal proof of Proposition 4.7 assembles it from the three facts just named.

4 At finitely many places: fixed lines, Siegel's formula, the finite system

Throughout this section $K_1 = K_{S,\mathsf{T}}(1)$ and $K_g = K_{S,\mathsf{T}}(g_{\mathcal{L}})$ in Setting 2.3.

Intermediate groups that fix lines.

Lemma 4.1 (non-units are rare)**.** *In Setting 2.1, let T be a finite set of finite places of E, stable under complex conjugation, and let E_{T} carry an integral place structure. Suppose that every place of T is a chosen place of S, in $\mathrm{Ch}_0(S) \cup \mathrm{Ch}_1(S)$, or the conjugate of one. Then $2 \le b(S)^9$ and*

$$\mathrm{vol}(O_{\mathsf{T}} \setminus O_{\mathsf{T}}^{\times}) \;\le\; b(S)^{-9}.$$

Proof. The first assertion holds because $b(S) \ge 1000$. Almost every element of E_{T} is a unit of E_{T} (Input 8(h)). Such an element of O_{T} that is not a unit of O_{T} has a component, at some $w \in \mathsf{T}$, of absolute value less than 1; so it lies in $\{x : \varpi_w^{-1} x \in O_{\mathsf{T}}\}$, where ϖ_w is the unit of E_{T} that is a uniformiser at w and 1 elsewhere. This set has volume $\mathrm{mod}_{\mathsf{T}}(\varpi_w^{-1})^{-1} = \#\mathbb{F}_w^{-1}$ (Input 8(c)), and $\#\mathbb{F}_w = q_i(S) \ge b(S)^{10}$ for the index i of the chosen place that is w or $\bar{w}$. As $\#\mathsf{T} \le 2\,(\#\mathrm{Ch}_0(S) + \#\mathrm{Ch}_1(S)) = 4f(E)$ and $4f(E) \le b(S)$,

$$\mathrm{vol}(O_{\mathsf{T}} \setminus O_{\mathsf{T}}^{\times}) \le 4f(E)\, b(S)^{-10} \le b(S)^{-9}. \qquad \square$$

LEAN `Filling.volume_nonunits_le`

Lemma 4.2 (one prime)**.** *In Setting 2.3, suppose that S is admissible. Let $\Sigma \subseteq \mathrm{Ch}_0(S)$, let o be either the word* none *or a place $w_o \in \Sigma$, let $\mathbf{d}$ be frame-vector data for S, and T_0 a finite set of finite places of E such that $\hat{g}(S)^{-1}$ is integral off T_0. Suppose that, if o is a place w_o, then T consists of the places $j \in \mathsf{T}_0$ with $j = w_o$ or $\bar{j} = w_o$. Then there is $Z \in \mathrm{Int}(\mathcal{L})$ such that*

(i) $\mathrm{loc}_{S,\mathsf{T}}(k) \in Z$ *for every $k \in K(S)$ with $\mathrm{red}_S(k) \in \mathrm{Fix}_S(\Sigma, \mathbf{d})$, and*

(ii) $\mathcal{J}_0(\mathcal{L}; Z) \cdot \mathsf{M}(\mathcal{L})(K_1) \cdot c_o \le \mathcal{J}_0(\mathcal{L}; K_1) \cdot \mathsf{M}(\mathcal{L})(Z)$, *where $c_o = 1$ if o is* none *and $c_o = b(S)^{6k(\mathfrak{s})}$ if o is a place.*

Proof. Let $\mathsf{p} \colon K(S) \to G(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$ be the restriction of $\mathrm{loc}_{S,\mathsf{T}}$. Its image is K_1 (Input 4(b)). If o is *none*, take $Z = K_1$: then (i) holds for every $k \in K(S)$ and (ii) is an equality.

Let o be a place $w = w_o \in \Sigma$, and write $b = b(S)$, $k = k(\mathfrak{s})$, $n = n(\mathfrak{s})$, $\varrho = \varrho(\mathfrak{s})$. By the hypothesis every place of T is w or $\bar{w}$, so Lemma 4.1 gives $2 \le b^9$ and $\mathrm{vol}(O_{\mathsf{T}} \setminus O_{\mathsf{T}}^{\times}) \le b^{-9}$. Let

$$H = \mathrm{red}_S^{-1}(\mathrm{Fix}_{S,w}(\mathbf{d}_w)) \subseteq K(S), \qquad Z = \mathsf{p}(H).$$

Z is an intermediate group. We show $\mathsf{p}^{-1}(K_g) \subseteq H$. Let $x \in K(S)$ with $\mathsf{p}(x) \in K_g$, so that the entries of $g_{\mathcal{L}}^{-1}(\mathsf{p}(x) - 1)$ are integral, where $g_{\mathcal{L}} = \hat{g}(S)_{\mathsf{T}}$. The place w lies in T: otherwise $w \notin \mathsf{T}_0$, and $\hat{g}(S)^{-1}$ would be integral at w, which is false because w lies above $q_0(S)$ (formally, by Input 1(f) for the idempotent of w). Hence the entries of $\hat{g}(S)^{-1}(x - 1)$ are integral at w, so by Input 1(f) the entries of $x - 1$ vanish modulo w: the component at w of $\mathrm{red}_S(x)$ is 1, which fixes every vector, and $x \in H$. Therefore $K_g = \mathsf{p}(\mathsf{p}^{-1}(K_g)) \subseteq Z \subseteq K_1$, using $K_g \subseteq K_1 = \mathsf{p}(K(S))$.

(i). If $\mathrm{red}_S(x) \in \mathrm{Fix}_S(\Sigma, \mathbf{d})$ then, as $w \in \Sigma$, $\mathrm{red}_S(x) \in \mathrm{Fix}_{S,w}(\mathbf{d}_w)$, so $x \in H$ and $\mathsf{p}(x) \in Z$.

The index of Z. Since $\ker \mathsf{p} \subseteq \mathsf{p}^{-1}(K_g) \subseteq H$, we have $\mathsf{p}^{-1}(Z) = H$ and $[K_1 : Z] = [K(S) : H]$. As red_S is surjective (admissibility), this is the index of $\mathrm{Fix}_{S,w}(\mathbf{d}_w)$ in $\mathsf{G}(S)$; and since $s \mapsto s_w^{(\epsilon_w)}$ maps $\mathsf{G}(S)$ onto $\mathrm{GL}_{I(\mathfrak{s})}(\mathbb{F}_w)$, it is the index of the stabiliser of the non-zero vector $\sum_a (v_w)_a \mathsf{f}_a$ in that group, which is $\#\mathbb{F}_w^{2\varrho+k+1} - 1 = q_0(S)^{2\varrho+k+1} - 1$ (Input 8(e)). Put $t = k - 10^{10} n^4$. Then $t + 1 < 2\varrho + k + 1$, hence $q_0(S)^{t+1} \le q_0(S)^{2\varrho+k+1} - 1$; with $b^9 \le b^{10} \le q_0(S)$ and $\mathsf{M}(\mathcal{L})(K_1) = [K_1 : Z]\, \mathsf{M}(\mathcal{L})(Z)$ (the measure is a Haar measure, Input 10(a), and Z is open in the compact group K_1),

$$\mathsf{M}(\mathcal{L})(Z) \cdot b^{9(t+1)} \;\le\; \mathsf{M}(\mathcal{L})(K_1). \tag{1}$$

(ii). From $10^{10} n^4 \le n^6$ and $3n^6 + 12 \le n^7 \le k$ we get $3 \cdot 10^{10} n^4 \le k$, that is $6k \le 9t$, so $b^{6k} \le b^{9t}$. The bottom value (Input 5(b), with this t), the bound for the non-units and $2^{-n} \le \mathsf{M}(\mathcal{L})(K_1)$ (Input 5(c) with $B = b^9$) give

$$\mathcal{J}_0(\mathcal{L}; Z) \;\le\; \mathsf{M}(\mathcal{L})(Z)\, \mathsf{M}(\mathcal{L})(K_g)\, \big(\mathsf{M}(\mathcal{L})(Z) + b^{-9(t+1)} \mathsf{M}(\mathcal{L})(K_1)\big).$$

By (1) and $2 \le b^9$, $2\,(\mathsf{M}(\mathcal{L})(Z) + b^{-9(t+1)}\mathsf{M}(\mathcal{L})(K_1))\, b^{9t} \le \mathsf{M}(\mathcal{L})(Z)\, b^{9(t+1)} + \mathsf{M}(\mathcal{L})(K_1) \le 2\, \mathsf{M}(\mathcal{L})(K_1)$. Hence

$$\mathcal{J}_0(\mathcal{L}; Z) \cdot \mathsf{M}(\mathcal{L})(K_1) \cdot b^{6k} \;\le\; \mathsf{M}(\mathcal{L})(Z)\, \mathsf{M}(\mathcal{L})(K_g)\, \mathsf{M}(\mathcal{L})(K_1)^2.$$

On the other side, restricting the outer integral of $\mathcal{J}_0(\mathcal{L}; K_1)$ to K_g and using the second statement of Input 5(b) for the intermediate group K_1, we get $\mathcal{J}_0(\mathcal{L}; K_1) \ge \mathsf{M}(\mathcal{L})(K_1)^2\, \mathsf{M}(\mathcal{L})(K_g)$. Multiplying by $\mathsf{M}(\mathcal{L})(Z)$ gives (ii). □

LEAN `Filling.IsFillingLevel.exists_intermediateGroup_one_prime`

Proposition 4.3 (lines at several primes)**.** *In Setting 2.3, suppose that S is admissible, that E carries Weil data and that $\hat{g}(S)^{-1}$ is integral off T. Let $\Sigma \subseteq \mathrm{Ch}_0(S)$ and let $\mathbf{d}$ be frame-vector data for S. Then there is $Z \in \mathrm{Int}(\mathcal{L})$ such that*

(i) $\mathrm{loc}_{S,\mathsf{T}}(k) \in Z$ *for every* $k \in K(S)$ *with* $\mathrm{red}_S(k) \in \mathrm{Fix}_S(\Sigma, \mathbf{d})$, *and*
(ii) $\mathcal{J}_0(\mathcal{L}; Z) \cdot \mathsf{M}(\mathcal{L})(K_1) \cdot b(S)^{6k(\mathfrak{s}) \cdot \#\Sigma} \le \mathcal{J}_0(\mathcal{L}; K_1) \cdot \mathsf{M}(\mathcal{L})(Z)$.

Proof. A place of E equals, or is conjugate to, at most one place of Σ, because the conjugate of a chosen place is never chosen. For o in $\Sigma \cup \{\text{none}\}$ let T_o be the set of the $w \in \mathsf{T}$ that are equal or conjugate to o (for $o = $ none: to no place of Σ). The sets T_o partition T and are stable under conjugation. By Input 4(a) each E_{T_o} carries an integral place structure and a filling level $\mathcal{L}_o \in \mathrm{FL}(S, \mathsf{T}_o)$, and Lemma 4.2, applied to T_o, $\mathcal{L}_o$, o and $\mathsf{T}_0 = \mathsf{T}$, gives intermediate groups Z_o of $\mathcal{L}_o$ with (i) and with

$$\mathcal{J}_0(\mathcal{L}_o; Z_o)\, \mathsf{M}(\mathcal{L}_o)(K_1^o)\, c_o \;\le\; \mathcal{J}_0(\mathcal{L}_o; K_1^o)\, \mathsf{M}(\mathcal{L}_o)(Z_o), \tag{2}$$

where $K_1^o = K_{S,\mathsf{T}_o}(1)$. The projections $E_{\mathsf{T}} \to E_{\mathsf{T}_o}$ induce a homomorphism

$$\varphi \colon G(T_S^{\mathrm{lat}}(E_{\mathsf{T}})) \longrightarrow \prod_o G(T_S^{\mathrm{lat}}(E_{\mathsf{T}_o})),$$

which is bijective, continuous and a measurable embedding, since a matrix over E_{T} is a family of matrices over the E_{T_o} and the unitarity condition holds componentwise. Under φ:

(a) for every unit u of E_{T}, $K_{S,\mathsf{T}}(u)$ is the preimage of the product of the groups $K_{S,\mathsf{T}_o}(u_o)$, integrality and congruences being conditions at each place; an element lies in $\mathcal{U}$ exactly if all its components do;

(b) $\psi_{S,\mathsf{T}}(x) = \prod_o \psi_{S,\mathsf{T}_o}(x_o)$, the extension by zero of x being the sum of those of its parts; hence, with Input 5(a), $\chi_g(y) = \prod_o \chi_{g_o}(\varphi(y)_o)$: if one side is non-zero, y lies in $u\,K_{S,\mathsf{T}}(g)$ with $u \in \mathcal{U}$, and then both sides are the value of the character at $g^{-1}\,\mathrm{sd}(u)$, which is additive over the parts;

(c) the measures $\mathsf{M}(\mathcal{L})$ and $\mathsf{M}(\mathcal{L}_o)$ are Haar measures (Input 10(a)); the image of $\mathsf{M}(\mathcal{L})$ under φ is right-invariant, so by the uniqueness of Haar measure (Input 8(d)) it is $\lambda \cdot \prod_o \mathsf{M}(\mathcal{L}_o)$ for some $\lambda \in [0,\infty]$. As both measures give the compact open group K_1, which corresponds to $\prod_o K_1^o$ by (a), a positive finite mass, $0 < \lambda < \infty$.

For a family (Z_o') of subgroups put $Q(Z') = \varphi^{-1}(\prod_o Z_o')$. By (b), (c) and Fubini's theorem,

$$\mathsf{M}(\mathcal{L})(Q(Z')) = \lambda \prod_o \mathsf{M}(\mathcal{L}_o)(Z_o'), \qquad \mathcal{J}_0(\mathcal{L}; Q(Z')) = \lambda^3 \prod_o \mathcal{J}_0(\mathcal{L}_o; Z_o'),$$

the inner integral contributing λ inside the square and the outer one λ once more. Take $Z = Q((Z_o)_o)$. By (a), $K_g = Q((K_g^o)_o) \subseteq Z \subseteq Q((K_1^o)_o) = K_1$, so $Z \in \mathrm{Int}(\mathcal{L})$; (i) holds because it holds on every part; and the product of the inequalities (2) over o, multiplied by λ^4, is (ii), since $\prod_o c_o = b(S)^{6k(\mathfrak{s})\#\Sigma}$. □

LEAN `Filling.IsFillingLevel.exists_intermediateGroup_lines`

A one-sided form of Siegel's formula at T.

Lemma 4.4 (step functions)**.** *Let A be a semilocal field with involution and an integral place structure, n a finite set, $\mathcal{G}$ a subgroup of $\mathrm{GL}_n(A)$ and $\mathcal{G}_0$ a subgroup of $\mathcal{G}$ all of whose elements are integral with integral inverse. Let ι be an index set, $p\colon \iota \to A^n$ a family of points, $f\colon A^n \to [0,\infty]$ a function, $s \subseteq \iota$ a finite set and $\mathrm{wt}\colon \iota \to \mathbb{R}_{\geq 0}$. Suppose that for every $z \in s$ there is a neighbourhood of $p(z)$ such that $\mathrm{wt}(z) \leq f(y)$ for every x in it, every $k \in \mathcal{G}_0$ and every y with $ky = x$. Then there is a function $\Phi\colon A^n \to \mathbb{R}_{\geq 0}$ with compact support such that $\Phi(x+v) = \Phi(x)$ for all x and all v in a neighbourhood of 0, $\Phi(kx) = \Phi(x)$ for all $k \in \mathcal{G}_0$, $\Phi \leq f$, and $\mathrm{wt}(z) \leq \Phi(p(z))$ for every $z \in s$.*

Proof. Almost every element of A is a unit (Input 8(h)), so the units are dense and 0 is a limit of units. As O_A^n is compact and s finite, there is a unit u so small that for every $z \in s$ all points $p(z) + ur$, $r \in O_A^n$, lie in the neighbourhood of the hypothesis. Put

$$\Phi(x) = \max\bigl\{\mathrm{wt}(z) :\ z \in s,\ \exists k \in \mathcal{G}_0,\ u^{-1}(kx - p(z)) \in O_A^n\bigr\},$$

the maximum of the empty set being 0. If z occurs for x, then $kx = p(z) + ur$, so $\mathrm{wt}(z) \leq f(x)$: thus $\Phi \leq f$. The elements of $\mathcal{G}_0$ preserve O_A^n and commute with the scalar u; therefore Φ is invariant under translation by uO_A^n (an open neighbourhood of 0), invariant under $\mathcal{G}_0$ (replace k by kk'), and supported in the union over $z \in s$ of the sets $\mathcal{G}_0^{-1}(p(z) + uO_A^n)$, which is relatively compact because $\mathcal{G}_0$ lies in a compact group of integral matrices. Finally $k = 1$ shows $\Phi(p(z)) \geq \mathrm{wt}(z)$. □

LEAN `Filling.exists_step_function`

Proposition 4.5 (Siegel's formula at T, one-sided)**.** *In Setting 2.3, suppose that E carries Weil data, that S is admissible, that $\hat{g}(S)^{-1}$ is integral off T, and that $\Lambda(S)$, $\Lambda(S)^{-1}$ and $N(S)^{-1}$ are integral off T. Let $f\colon E_{\mathsf{T}}^{I(\mathfrak{s})} \to [0,\infty]$ be a function, $\mathcal{Y}$ a finite set of pairs (c,y) of a class $c \in \mathrm{Cl}(S)$ and a vector $y \in E^{\kappa}$, $t \in E$ with $N(S)^2 < |t_\tau|^2$ for every τ, and $\mathrm{wt}\colon \mathrm{Cl}(S) \times E^{\kappa} \to \mathbb{R}_{\geq 0}$. For a pair (c,y) let $x(c,y) = y_c^{-1} y \in \mathbb{A}_{E,f}^{\kappa}$, y_c the representative of c. Suppose that for every $(c,y) \in \mathcal{Y}$: $N(S)\, y^* y = t\bar{t} - N(S)^2$; every coordinate of $x(c,y)$ is integral off T; and there is a neighbourhood of $\mathrm{tv}(\bar{\pi}(S), t, x(c,y))_{\mathsf{T}}$ such that $\mathrm{wt}(c,y) \leq f(e')$ for every e in it, every $k \in \mathcal{H} \cap K_g$ and every e' with $ke' = e$. Then there is a neighbourhood of 0 in E_{T} such that for every closed and open set L contained in it,*

$$\begin{aligned}\Big(\sum_{(c,y)\in\mathcal{Y}} \mathrm{wt}(c,y)\Big)\cdot \mathrm{vol}(L) \\ \leq\ (1+b(S)^{-6})\,\#\mathrm{Cl}(S)\cdot c_*(S)\cdot\Big(\prod_{\tau}(|t_\tau|^2 - N(S)^2)\Big)^{k(\mathfrak{s})-1} \\ \cdot\ \int \mathbf{1}\big[(Je)_z - t_{\mathsf{T}} \in L\big]\, f(e)\, d\delta^{(\varrho)}(\mathcal{L})(e),\end{aligned}$$

where J is the split Gram matrix of $T_S^{\mathrm{lat}}(E_{\mathsf{T}})$.

Proof. Write $N = N(S)$, $k = k(\mathfrak{s})$, $n = n(\mathfrak{s})$, $\pi_{\mathsf{T}} = \bar{\pi}(S)_{\mathsf{T}}$ (a unit of E_{T}; the letter π alone is the number) and $w_0 = \sum_y \mathrm{wt}$. If $w_0 = 0$ there is nothing to prove. Let $\beta = (t\bar{t} - N^2)/N \in E$; then $\bar{\beta} = \beta$ and $\beta_\tau = (|t_\tau|^2 - N^2)/N > 0$.

The test function. Lemma 4.4, for $A = E_{\mathsf{T}}$, the points $\mathrm{tv}(\bar{\pi}(S), t, x(c,y))_{\mathsf{T}}$, the weights, f and the group $\mathcal{H} \cap K_g$, gives Ψ with compact support, invariant under small translations and under $\mathcal{H} \cap K_g$, with $\Psi \leq f$ and $\mathrm{wt}(c,y) \leq \Psi(\mathrm{tv}(\bar{\pi}(S), t, x(c,y))_{\mathsf{T}})$ on $\mathcal{Y}$. Define Φ_0 on $\mathbb{A}_{E,f}^{\kappa}$ by $\Phi_0(x) = \Psi(\mathrm{tv}(\pi_{\mathsf{T}}, t_{\mathsf{T}}, x_{\mathsf{T}}))$ if x is integral off T, and $\Phi_0(x) = 0$ otherwise. It is locally constant with compact support. It is invariant under $K^{\mathrm{col}}(S)$: an element κ_0 of this group is integral, and the tail vector of $\kappa_0 x$ at T is $\mathrm{loc}_{S,\mathsf{T}}(\mathrm{inc}_S(\kappa_0))$ applied to that of x (Input 4(d), and the definition of inc_S through the lattice basis), where $\mathrm{loc}_{S,\mathsf{T}}(\mathrm{inc}_S(\kappa_0)) \in \mathcal{H} \cap K_g$ because $K^{\mathrm{col}}(S) = \mathrm{inc}_S^{-1}(K_g(S))$ (admissibility and Input 4(b)).

Siegel's formula. Input 8(b), for $K = K^{\mathrm{col}}(S)$, Φ_0 and $\alpha = \beta$ (note $k \geq 3$), in which the weighted sum over the classes is the sum over $\mathrm{Cl}(S)$ because the class automorphism groups are trivial, reads

$$\sum_{c\in\mathrm{Cl}(S)}\ \sum_{x\in E^{\kappa},\ x^*x=\beta} \Phi_0(y_c^{-1}x) = C\cdot\mathfrak{F}, \qquad C = \#\mathrm{Cl}(S)\,|d_E|^{1/4}\prod_{\tau}\frac{\pi^k}{(k-1)!}\beta_\tau^{k-1},$$

with $\mathfrak{F} = \iint \psi_f(\tfrac{1}{2}z(x^*x-\beta))\Phi_0(x)\,dx\,dz$. The pairs of $\mathcal{Y}$ contribute distinct terms on the left, each at least $\mathrm{wt}(c,y)$: indeed $y^*y = \beta$, $x(c,y)$ is integral off T, and $\Phi_0(y_c^{-1}y) = \Psi(\mathrm{tv}(\pi_{\mathsf{T}}, t_{\mathsf{T}}, x(c,y)_{\mathsf{T}})) = \Psi(\mathrm{tv}(\bar{\pi}(S), t, x(c,y))_{\mathsf{T}})$ by the last assertion of Input 4(d). Hence $w_0 \leq C\,|\mathfrak{F}|$, and with $r_0 = (1+b(S)^{-6})/(1+2^{-n}) > 1$ (recall $2^{-n} < 2^{-\varrho(\mathfrak{s})} \leq b(S)^{-6}$) we get the strict inequality $w_0 < r_0\, C\,|\mathfrak{F}|$.

Localising. For $\tau' \in E_{\mathsf{T}}$ put $\mathsf{b}(\tau') = (\pi_{\mathsf{T}}\bar{\pi}_{\mathsf{T}})^{-1}\tau'\bar{\tau}' - N - \beta_{\mathsf{T}}$; as $\pi_{\mathsf{T}}\bar{\pi}_{\mathsf{T}} = N$ in E_{T} and $t\bar{t} = N(\beta+N)$, $\mathsf{b}(t_{\mathsf{T}}) = 0$. For a natural number $M \geq 1$ let $\mathfrak{F}(M,\tau')$ be the integral $\mathfrak{F}$ with z restricted to the codifferent scaled by M and β replaced by $\beta + \widetilde{\mathsf{b}(\tau')}$. By dominated convergence $\mathfrak{F}(M,\tau') \to \mathfrak{F}$ as the scaled codifferents exhaust the self-

adjoint adeles and $\tau' \to t_{\mathsf T}$; so there are a neighbourhood B_0 of $t_{\mathsf T}$ and an integer $M_0 \ge 1$ such that for every multiple M of M_0

$$w_0 < r_0\, C\, |\mathfrak F(M,\tau')| \qquad (\tau' \in B_0). \tag{3}$$

Fix such an M; it will be enlarged once, in the next step. Let m be the finite adele with $Mm = 1$ and $\mathfrak c_M$ the codifferent scaled by M. The measure $\Phi_0(x)\,dx$ is finite, and $\beta + \widetilde{\mathsf b(\tau')}$ is self-adjoint; so by Input 8(c)(i)

$$|\mathfrak F(M,\tau')| = \mathrm{vol}(\mathfrak c_M) \cdot \int \mathbf 1\big[m(x^*x - \beta - \widetilde{\mathsf b(\tau')}) \in \hat O_E\big]\, \Phi_0(x)\,dx.$$

Now, by the definitions of Φ_0 and of $\mathsf b$,

$$\Phi_0(x) = \mathbf 1[x \text{ integral off } \mathsf T] \cdot \Psi(\mathrm{tv}(\pi_{\mathsf T}, t_{\mathsf T}, x_{\mathsf T})),$$
$$y^*y - \beta_{\mathsf T} - \mathsf b(\tau') = y^*y - (\pi_{\mathsf T}\bar\pi_{\mathsf T})^{-1}\tau'\bar\tau' + N.$$

So Input 8(c)(ii), for $a = \mathsf b(\tau')$, for the function f with $f(y) = \Psi(\mathrm{tv}(\pi_{\mathsf T}, t_{\mathsf T}, y))$ and for $a_0 = 3\varrho(\mathfrak s) + 5$, gives a finite constant v_0, which does not depend on τ', with

$$|\mathfrak F(M,\tau')| = \mathrm{vol}(\mathfrak c_M) \cdot v_0 \cdot \int_{\mathcal D(\tau')} \Psi(\mathrm{tv}(\pi_{\mathsf T}, t_{\mathsf T}, y))\,dy,$$

where $\mathcal D(\tau')$ is the set of the $y \in E_{\mathsf T}^{\kappa}$ with $M^{-1}(y^*y - (\pi_{\mathsf T}\bar\pi_{\mathsf T})^{-1}\tau'\bar\tau' + N) \in O_{\mathsf T}$, and with

$$\mathrm{vol}(\mathfrak c_M) \cdot \nu_{\mathsf T}(\delta_{\mathsf T} M O_{\mathsf T}) \cdot v_0 \;\le\; \zeta_{\hat O_E}(3\varrho(\mathfrak s) + 5)\ \mathrm{vol}(\hat O_E)^k \;\le\; (1 + 2^{-n})\ \mathrm{vol}(\hat O_E)^k.$$

The hypotheses of Input 8(c)(ii) hold. First $2(a_0 + 1) = 6\varrho(\mathfrak s) + 12 \le k$, because $1000(2\varrho(\mathfrak s) + 2) \le k$. Secondly, Input 8(c)(iii) for $(u, v, n') = (n(\mathfrak s), 2\varrho(\mathfrak s), 1)$, where $u + v + 4 = 3\varrho(\mathfrak s) + 5$ and $[E : \mathbb Q] \le n(\mathfrak s)[E : \mathbb Q] \le 2^{2\varrho(\mathfrak s)}$, gives $\zeta_{\hat O_E}(3\varrho(\mathfrak s) + 5) \le 1 + 2^{-n} \le 2$; this is also the last inequality of the display.

Averaging over L. Let L be closed and open, so small that $t_{\mathsf T} + L \subseteq B_0$ and that $\Psi(e + \mathrm{tv}(\pi_{\mathsf T}, l, 0)) = \Psi(e)$ for $l \in L$; then $\Psi(\mathrm{tv}(\pi_{\mathsf T}, t_{\mathsf T}, y)) = \Psi(\mathrm{tv}(\pi_{\mathsf T}, t_{\mathsf T} + l, y))$, the tail vector being additive. Let $a < w_0$. Integrating (3) over $\tau' \in t_{\mathsf T} + L$ and inserting the evaluation,

$$a \cdot \mathrm{vol}(L) \;\le\; r_0\, C\, \mathrm{vol}(\mathfrak c_M)\, v_0 \int\!\!\int_{y \in \mathcal D(\tau')} \mathbf 1_L(\tau' - t_{\mathsf T})\, \Psi(\mathrm{tv}(\pi_{\mathsf T}, \tau', y))\,dy\,d\tau'.$$

The function $\Phi(e) = \mathbf 1_L((Je)_z - t_{\mathsf T})\Psi(e)$ is continuous with compact support, and $(J\,\mathrm{tv}(\pi_{\mathsf T}, \tau', y))_z = \tau'$; so by Input 6(b), if the scaled set $MO_{\mathsf T}$ is small enough (replace M by a multiple), the double integral is at most $\mathrm{mod}_{\mathsf T}(\pi_{\mathsf T})\,\nu_{\mathsf T}(\delta_{\mathsf T} M O_{\mathsf T})\,c'$ for every $c' > \int \Phi\, d\delta^{(\varrho)}(\mathcal L)$. Letting $a \to w_0$ and $c' \to \int \Phi\, d\delta^{(\varrho)}(\mathcal L)$, and using $\Psi \le f$,

$$w_0 \cdot \mathrm{vol}(L) \;\le\; r_0\, C\, (1 + 2^{-n})\ \mathrm{vol}(\hat O_E)^k\, \mathrm{mod}_{\mathsf T}(\pi_{\mathsf T}) \int \mathbf 1[(Je)_z - t_{\mathsf T} \in L] f(e)\, d\delta^{(\varrho)}(\mathcal L)(e).$$

The constant. The modulus is multiplicative and takes the same value at $\bar\pi_{\mathsf T}$ and at $\pi_{\mathsf T}$ (Input 8(h)); so

$$\mathrm{mod}_{\mathsf T}(\pi_{\mathsf T})^2 = \mathrm{mod}_{\mathsf T}(\pi_{\mathsf T}\bar\pi_{\mathsf T}) = \mathrm{mod}_{\mathsf T}(N_{\mathsf T}) = N^{-2f(E)}$$

(Input 4(b); N and N^{-1} are integral off $\mathsf T$), and $\mathrm{mod}_{\mathsf T}(\pi_{\mathsf T}) = N^{-f(E)}$. Further $\prod_\tau \beta_\tau^{k-1} = (\prod_\tau (|t_\tau|^2 - N^2))^{k-1} N^{-(k-1)f(E)}$. Hence the constant is $(1 + b(S)^{-6})\,\#\mathrm{Cl}(S)\,c_*(S)\,(\prod_\tau (|t_\tau|^2 - N^2))^{k-1}$. □

LEAN `Filling.IsFillingLevel.siegel_sum_le_lintegral_quadric`

The top kernel dominates the phase sums.

Proposition 4.6 (blocks and the top kernel)**.** *In Setting 2.3, suppose that E carries Weil data, that S is admissible, that $\hat{g}(S)^{-1}$ is integral off T, and that $\Lambda(S)$, $\Lambda(S)^{-1}$ and $N(S)^{-1}$ are integral off T. Let σ be a side, $Z \in \mathrm{Int}(\mathcal{L})$, T_0 a finite set of units of E, $\mathcal{J}$ a finite index set, and for $j \in \mathcal{J}$ let c_j, c_j' be classes, $\gamma_j \in G_S(E)$ and $k_j \in K(S)$; let t be a unit of E. Suppose that*

(a) *the tuples $(c_j, c_j', \mathrm{red}_S(k_j), \gamma_j)$ are pairwise distinct;*
(b) *$\gamma_j \in \mathcal{H}_\varrho(T_S^{\mathrm{rat}}(E))$ and $Z_S(\gamma_j)_{\varrho\varrho} = t$ for all j, and $N(S)^2 < |t_\tau|^2$ for all τ;*
(c) *$\mathrm{loc}_{S,\mathsf{T}}(k_j) \in Z$ and $\mathrm{pos}_S(c_j, c_j'; \gamma_j)$ is integral off T, for all j.*

Let $j_0 \in \mathcal{J}$, let R be a complex matrix and ρ a real function, both indexed by the maps p from $\{0, \dots, \varrho(\mathfrak{s}) - 1\}$ to T_0, and r a real number. Put $\omega_p = (p \,|\, t)_{\mathsf{T}}$,

$$y_j = r^2 \Big\| \sum_p a^\sigma_{S,\mathsf{T}}(c_j, \mathrm{red}_S(k_j), c_j'; w_\varrho((p \,|\, t))\gamma_j) \Big\|^2,$$
$$G = \Big(\prod_\tau (|t_\tau|^2 - N(S)^2) \Big)^{k(\mathfrak{s})-1},$$

and suppose further that

(d) *for every j, R is the Gram matrix of the vectors $\rho(p) \cdot \mathrm{J}(dy, W_S^\sigma(\cdot).; \gamma_{j,p})$, where $\gamma_{j,p} = \mathrm{r}_S^\infty(w_\varrho((p \,|\, t))\gamma_j)$;*
(e) *$\rho(p) \cdot \mathsf{h}_\varrho(\mathcal{L}; \omega_p, 1) = r\sqrt{G}$ for all p;*
(f) *$\mathrm{pos}_S(c_j, c_j'; w_\varrho((p \,|\, t))\gamma_j)$ is integral off T for all j and p.*

Then there is a neighbourhood of 0 in E_{T} such that for every open and compact set L contained in it with $0 \in L$,

$$\sum_{j \in \mathcal{J}} y_j \;\le\; (1 + b(S)^{-6}) \cdot a_*(S) \cdot \mathrm{Re} \sum_{p,q} R_{pq} \, \frac{\mathcal{K}(L; Z; \omega_p, \omega_q)}{\mathrm{vol}(L) \cdot \mathrm{mr}(Z)}.$$

Proof. Write $l_j = \mathrm{pos}_S(c_j, c_j'; \gamma_j)$ and let $\omega^{(j)} \in E^\kappa$ be the κ-part of $\gamma_j z$. By Input 4(d), $l_j \in \mathcal{H}_\varrho$, $l_j z = \mathrm{tv}(\bar{\pi}(S), t, y_{c_j}^{-1}\omega^{(j)})$ and $N(S)\,\omega^{(j)*}\omega^{(j)} = t\bar{t} - N(S)^2$.

Step 1: the constant. Let $D_0 = \#\overline{\mathcal{S}}^S \cdot \mathrm{uo}(S)$, so that $a_*(S) = D_0 \cdot \#\mathrm{Cl}(S) \cdot c_*(S)$. We claim

$$\mathrm{mr}(Z) \;\le\; \theta_n(\mathcal{L})(K_g) \cdot \mathsf{M}(\mathcal{L})(K_g)^2 \cdot \#\,\mathrm{Cos}(Z) \cdot D_0. \tag{4}$$

Indeed Z is the union of $\#\,\mathrm{Cos}(Z)$ translates of K_g, so its measure $\mathsf{M}(\mathcal{L})(Z)$ is at most $\#\,\mathrm{Cos}(Z)$ times $\mathsf{M}(\mathcal{L})(K_g)$; and $\mathsf{v}(\mathcal{L})^{-2} = \mathrm{uo}(S)^2$ by Input 4(b). We apply Input 4(c) to the subgroup $B = \mathcal{H}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ of G_S^{lat}, written $\mathcal{H}$, and to the measure $\mu = \theta_n(\mathcal{L})$. Its two hypotheses hold: $\theta_n(\mathcal{L})$ is carried by $\mathcal{H}$ at T (Input 10(b)), and every element of $\mathcal{H}$ at T is the image of an element of $\mathcal{H}$ that is integral off T (Input 4(b)); and $\theta_n(\mathcal{L})$ is invariant under right translation by every element of $\mathcal{H}$ at T (Input 10(b)), in particular by the image of every element of B, which is the formal hypothesis. Hence $\#\overline{\mathcal{H}}^S \cdot \theta_n(\mathcal{L})(K_g) = \theta_n(\mathcal{L})(K_1) = 1$, the last by Input 10(a); with $\#\overline{\mathcal{S}}^S = \#\overline{\mathcal{H}}^S\,\mathrm{uo}(S)$ this gives $\theta_n(\mathcal{L})(K_g)\,D_0 = \mathrm{uo}(S)^2$, whence (4).

Step 2: packets. Group the indices by $\mathsf{g}(j) = (c_j, \omega^{(j)})$. Fix a value z_0 of g and let Θ_0 be the set of the pairs $\theta = (j, b)$ with $\mathsf{g}(j) = z_0$ and $b \in \mathrm{Cos}(Z)$. For such θ put

$l_\theta = \mathrm{loc}_{S,\mathsf{T}}(l_j)^{-1} \in \mathcal{H}_\varrho$; then $l_\theta^{-1} z = e_0$, the tail vector of z_0 at T, the same for all θ. As $\mathrm{loc}_{S,\mathsf{T}}(k_j) \in Z$, there is exactly one coset $A(j,b)$ with $\mathrm{loc}_{S,\mathsf{T}}(k_j)^{-1} a_b a_{A(j,b)}^{-1} \in K_g$ (Input 5(a)); put $n_\theta = (b, A(j,b))$.

The pairs $(K_g l_\theta, n_\theta)$ are distinct. Let $\theta = (j,b)$, $\theta' = (j',b')$ with $l_\theta l_{\theta'}^{-1} \in K_g$ and $n_\theta = n_{\theta'}$. Then $b = b'$ and $A(j,b) = A(j',b)$, so $\mathrm{loc}_{S,\mathsf{T}}(k_j)^{-1} \mathrm{loc}_{S,\mathsf{T}}(k_{j'}) \in K_g$ and $\mathrm{red}_S(k_j) = \mathrm{red}_S(k_{j'})$ (Input 4(b)); also $c_j = c_{j'}$. The vectors $\gamma_j z$ and $\gamma_{j'} z$ lie in the span of κ and z, have the same κ-part and, as $h(z, \gamma_j z) = t = h(z, \gamma_{j'} z)$, the same coefficient at z; so the element $\gamma_j^{-1}\gamma_{j'} \in \mathcal{H}_\varrho$ fixes z, and it lies in $\mathcal{H}$, hence in the Bessel subgroup. Its position $\mathrm{pos}_S(c'_j, c'_{j'}; \gamma_j^{-1}\gamma_{j'}) = l_j^{-1} l_{j'}$ is integral off T and lies in K_g at T, hence in $K_g(S)$ (Input 4(b)); so $c'_j = c'_{j'}$ and $\gamma_j^{-1}\gamma_{j'}$ lies in $\mathcal{U}_{S,c'_j} \subseteq \mathcal{U}$ (Input 1(c)). But $\mathcal{H}_\varrho \cap \mathcal{U} = 1$, so $\gamma_j = \gamma_{j'}$, and $j = j'$ by (a).

By Input 5(g), with $v_\theta(p) = \mathcal{J}(Z; \omega_p, l_\theta)_{n_\theta}$: for all e near e_0, all $k \in \mathcal{H} \cap K_g$ and all e' with $ke' = e$,

$$\theta_n(\mathcal{L})(K_g) \sum_{\theta \in \Theta_0} \left(\mathrm{Re}\; v_\theta^* R v_\theta\right)^+ \;\le\; \mathcal{E}(Z; R, \omega, e').$$

By (d), $v_\theta^* R v_\theta = \left\| \sum_p \rho(p)\, v_\theta(p)\, \mathrm{J}(dy, W_S^\sigma(\cdot).; \mathrm{r}_S^\infty(w_\varrho((p\,|\,t))\gamma_j)) \right\|^2$; by the unfolding (Input 4(f), with (f) and the choice of $A(j,b)$) and by (e), the vector inside the norm equals $r\sqrt{G} \cdot \mathsf{M}(\mathcal{L})(K_g) \cdot \sum_p a_{S,\mathsf{T}}^\sigma(c_j, \mathrm{red}_S(k_j), c'_j; w_\varrho((p\,|\,t))\gamma_j)$. Hence $v_\theta^* R v_\theta = G\, \mathsf{M}(\mathcal{L})(K_g)^2\, y_j$ for every b, and with (4)

$$\begin{aligned} \mathcal{E}(Z; R, \omega, e') \;&\ge\; \theta_n(\mathcal{L})(K_g)\, \#\,\mathrm{Cos}(Z)\, G\, \mathsf{M}(\mathcal{L})(K_g)^2 \sum_{\mathrm{g}(j) = z_0} y_j \\ &\ge\; \frac{G \cdot \mathrm{mr}(Z)}{D_0} \sum_{\mathrm{g}(j) = z_0} y_j =: \mathrm{wt}(z_0). \end{aligned} \tag{5}$$

Step 3: Siegel's formula. Apply Proposition 4.5 to the set $\mathcal{Y}$ of the values of g, the weights wt and $f = \mathcal{E}(Z; R, \omega, \cdot)$. The quadric condition was noted; $y_{c_j}^{-1}\omega^{(j)}$ is integral off T because l_j is and $\Lambda(S)^{\pm 1}$ are; the orbit condition is (5). For L small, open and compact with $0 \in L$ (so closed, with $0 < \mathrm{vol}(L) < \infty$),

$$\begin{aligned} \frac{G\ \mathrm{mr}(Z)}{D_0} \Big(\sum_j y_j \Big)\, \mathrm{vol}(L) \;&\le\; (1 + b(S)^{-6}) \#\mathrm{Cl}(S)\, c_*(S)\, G \\ &\quad \cdot \int \mathbf{1}\big[(Je)_z - t_\mathsf{T} \in L\big] \mathcal{E}(Z; R, \omega, e)\, d\delta^{(\varrho)}(\mathcal{L})(e). \end{aligned}$$

The matrix R is positive semidefinite, being a Gram matrix (here j_0 is used), and $(\omega_p)_\varrho = t_\mathsf{T}$; so by Input 5(f) the integral is at most $\mathrm{Re} \sum_{p,q} R_{pq} \mathcal{K}(L; Z; \omega_p, \omega_q)$. Dividing by $G\ \mathrm{mr}(Z)\ \mathrm{vol}(L)/D_0$, which is positive and finite (Input 5(c)), gives the claim. □

LEAN `Filling.IsFillingLevel.sum_blocks_le_topKernel`

The finite sieve system.

Proposition 4.7 (the system at T)**.** *In Setting 2.3, suppose that E carries Weil data, that S is admissible and that $\hat{g}(S)^{-1}$ is integral off T. Let $\Sigma \subseteq \mathrm{Ch}_0(S)$ and let $\mathbf{d}$ be frame-vector data for S. Then there is $Z \in \mathrm{Int}(\mathcal{L})$ such that*

(i) *$\mathrm{loc}_{S,\mathsf{T}}(k) \in Z$ for every $k \in K(S)$ with $\mathrm{red}_S(k) \in \mathrm{Fix}_S(\Sigma, \mathbf{d})$, and*

(ii) *for every open additive subgroup L of $E_{\mathsf T}$ that is compact, the kernels*

$$J_L(j,p,q) = \mathcal{I}_j(\mathcal{L};p,q;W_Z(\cdot),1)\cdot \mathrm{mr}(Z)^{-1} \qquad (j \le \varrho(\mathfrak{s})),$$
$$J_L(j,p,q) = \mathcal{K}(L;Z;p,q)\cdot \mathrm{vol}(L)^{-1}\ \mathrm{mr}(Z)^{-1} \quad (j > \varrho(\mathfrak{s}))$$

form, with the maps $c \mapsto u(c)$ and $x \mapsto g(S)_{\mathsf T}^{-1}x^{-1}$, the character $\psi_{S,\mathsf T}$, the bottom bound $(1+b(S)^{-6})\,b(S)^{-6k(\mathfrak{s})\cdot\#\Sigma}$, the weights $\Lambda_{n(\mathfrak{s})^6}(\cdot;\cdot)$ and the frequency gain $\beta_{S,\mathsf T}(\cdot)$, a sieve system of depth $\varrho(\mathfrak{s})$ on $E_{\mathsf T}$ (Definition 2.4).

Proof. Let Z be the group of Proposition 4.3; (i) is its first property. Write $b = b(S)$, $k = k(\mathfrak{s})$, $n = n(\mathfrak{s})$, $\varrho = \varrho(\mathfrak{s})$, $W = W_Z(\cdot)$, $\psi = \psi_{\mathsf T} = \psi_{S,\mathsf T}$.

Frequency bounds. Let ξ be a unit and ω as in Input 5(d). With $K = n^6$ and $t = k - n^7$ (admissible choices, as $10^{10}n^4 \le n^6 \le n^7 \le k$), Input 5(d) gives $|\mathcal{I}_j(\mathcal{L};p,q;W,\omega)|/\,\mathrm{mr}(Z) \;\le\; [\xi]^{(k-n^7)/2}\,\mathrm{den}(\xi)^{-6}(2/\mathrm{mod}_{\mathsf T}(g_{\mathcal L}))^{n^6}\Lambda_{n^6}(j;p)\Lambda_{n^6}(j;q)$. By Input 4(b), $2/\mathrm{mod}_{\mathsf T}(g_{\mathcal L}) = 2g(S)^{2f(E)} \le b^{1+46f(E)} \le b^{47f(E)}$, and $47 \le n^3$; so

$$\frac{|\mathcal{I}_j(\mathcal{L};p,q;W,\omega)|}{\mathrm{mr}(Z)} \;\le\; \beta_{S,\mathsf T}(\xi)\cdot\Lambda_{n^6}(j;p)\,\Lambda_{n^6}(j;q). \tag{6}$$

The axioms. $\psi(0)=1$. *Steps below the top* ($i<\varrho$): W is a cut Bessel function for $\mathcal{L}$ and the majorant of the floor form at (i,p,q) has a finite integral (Input 5(h)), and $\mathcal{L}$ has a gauge (Input 10(a)); so by the descent (Input 10(c)) the function $c \mapsto \psi(\xi c)J_L(i+1,p[i\mapsto u(c)],q[i\mapsto u(c)])$ is integrable with integral $\mathcal{I}_i(\mathcal{L};p,q;W,\omega_\xi)/\,\mathrm{mr}(Z)$; for $\xi=0$ this is $J_L(i,p,q)$, and for a unit ξ it is bounded by (6). *The step at the top* ($i=\varrho$): the correlations $\mathcal{J}(Z;p,l)$ do not depend on the entry p_ϱ, and almost every c is a unit (Input 8(h)); so by Fubini's theorem and the invariance of the Haar measure

$$\int \psi(\xi c)\,\mathcal{K}(L;Z;p[\varrho\mapsto c],q[\varrho\mapsto c])\,dc$$
$$= \Big(\int_L \psi(-\xi x)\,dx\Big)\int \psi(\xi\,h(z,l^{-1}z))\big\langle \mathcal{J}(Z;p,l),\mathcal{J}(Z;q,l)\big\rangle\,d\mu^{\mathcal L}_{\varrho,2}(l),$$

and the last integral is $\mathcal{I}_\varrho(\mathcal{L};p,q;W,\omega_\xi)$ (Input 5(e)). After division by $\mathrm{vol}(L)\,\mathrm{mr}(Z)$: for $\xi=0$ the first factor is $\mathrm{vol}(L)$ and we get $J_L(\varrho,p,q)$; for a unit ξ the first factor has modulus at most $\mathrm{vol}(L)$ and (6) applies. *Positivity*: for sequences $a_1,\dots,a_m$ with a common ϱ-th term the matrix $(\mathcal{K}(L;Z;a_i,a_{i'}))$ is hermitian, and for a vector x its quadratic form is the number $\mathcal{Q}$ of Input 5(f) for the positive semidefinite matrix $R=(\bar{x}_i x_{i'})$, hence ≥ 0. *Continuity* is Input 5(e). *Unequal entries* ($i<\varrho$, $\xi = g(S)_{\mathsf T}^{-1}(p_i^{-1}-q_i^{-1})$ a unit): this is (6) for $j=i+1$, $\omega=1$, the second alternative of Input 5(d), as $g_{\mathcal L} = \hat g(S)_{\mathsf T}$.

The bottom bound. By the last assertion of Input 5(h) the inner integral of $\mathcal{I}_0(\mathcal{L};Z)$ at x has the modulus $\mathsf{v}(\mathcal{L})\cdot|\mathsf{A}(\mathcal{L};W,x)|$; with Input 10(d) this gives

$$\big|\mathcal{I}_0(\mathcal{L};p,q;W,1)\big|\cdot\mathsf{v}(\mathcal{L})^2 \;\le\; \mathsf{v}(\mathcal{L})^2\int\big|\mathsf{A}(\mathcal{L};W,x)\big|^2\,d\mathsf{M}(\mathcal{L})(x) = \mathcal{I}_0(\mathcal{L};Z).$$

By Proposition 4.3(ii), $\mathcal{I}_0(\mathcal{L};Z)\,\mathsf{M}(\mathcal{L})(K_1)\,b^{6k\#\Sigma} \le \mathcal{I}_0(\mathcal{L};K_1)\,\mathsf{M}(\mathcal{L})(Z)$; and by the bottom value for the intermediate group K_1 with $t=0$, by $\mathrm{vol}(O_{\mathsf T}\setminus O_{\mathsf T}^\times)\le 1$ and by $\mathsf{M}(\mathcal{L})(K_1)\le 1+2^{-n}$ (Inputs 5(b) and 10(a)),

$$\mathcal{I}_0(\mathcal{L};K_1) \le \mathsf{M}(\mathcal{L})(K_1)\,\mathsf{M}(\mathcal{L})(K_g)\,(1+2^{-n}+2^{-n}) \le \mathsf{M}(\mathcal{L})(K_1)\,\mathsf{M}(\mathcal{L})(K_g)\,(1+b^{-6}),$$

since $2^{-n}+2^{-n}=2^{-\varrho}\le b^{-6}$. Together,

$$|\mathcal{J}_0(\mathcal{L};p,q;W,1)|\,\mathsf{v}(\mathcal{L})^2\,b^{6k\#\Sigma}\le \mathsf{M}(\mathcal{L})(Z)\,\mathsf{M}(\mathcal{L})(K_g)\,(1+b^{-6}),$$

that is $|J_L(0,p,q)|\le(1+b^{-6})\,b^{-6k\#\Sigma}$. □

LEAN `Filling.IsFillingLevel.exists_finite_sieveSystem`

5 The main-term estimate: proof of Theorem D

Write $b=b(S)$, $k=k(\mathfrak{s})$, $n=n(\mathfrak{s})$, $\varrho=\varrho(\mathfrak{s})$, $\nu=\nu(S)$ and $y_\Sigma=b^{-6k\#\Sigma}$; recall $\#\Sigma\le\#\mathrm{Ch}_0(S)=f(E)$. The amplitude sum of the statement is $A=A_S^{\mathrm{full}}(G_S(E);\cdot,\cdot,\cdot)$.

Proof of Theorem D. Step 1: the crude estimate. We apply Input 3 with $\Xi=G_S(E)$ (which is stable), $\Omega=\mathrm{Fix}_S(\Sigma,\mathbf{d})$, $X=q_0(S)^{\#\Sigma}$, $C=y_\Sigma\mathsf{d}(S)$, $\omega_0=(1+b^{-4})\,y_\Sigma\mathsf{d}(S)$, and with $\mathcal{R}$ the set of all links if $\Sigma\neq\mathrm{Ch}_0(S)$ and the empty set if $\Sigma=\mathrm{Ch}_0(S)$. Its hypotheses hold:

- every $s\in\Omega$ fixes norm at least X, by the triples $(0,w,\epsilon_w)$, $w\in\Sigma$;
- $\mathrm{tl}(S)^{20}\mathsf{d}_1(S)\le C$: by admissibility $\mathrm{tl}(S)^{44}\mathsf{d}_1(S)\le\mathsf{d}(S)$, and $\mathrm{tl}(S)^{-24}=b^{-6kf(E)}\le y_\Sigma$; hence also $\mathsf{d}_1(S)\le C$, as $\mathrm{tl}(S)\ge 2$;
- if some link is not in $\mathcal{R}$, then $\Sigma=\mathrm{Ch}_0(S)$, so $X=q_0(S)^{f(E)}\ge b^{10f(E)}$ and $\mathrm{tl}(S)^{12}X^{-k}\mathsf{d}(S)\le b^{3kf(E)-10kf(E)}\mathsf{d}(S)\le y_\Sigma\mathsf{d}(S)=C$.

If moreover the link energy is at most $\omega_0\nu^2$, Input 3 gives

$$\mathrm{En}_S(A;\Omega)\le\big((1+b^{-4})^3+b^{-12}\big)\,y_\Sigma\mathsf{d}(S)\le(1-b^{-4})(1+9b^{-4})\,y_\Sigma\mathsf{d}(S),$$

the last step because, with $\varepsilon=b^{-4}\le 10^{-12}$, $(1-\varepsilon)(1+9\varepsilon)-(1+\varepsilon)^3-\varepsilon^3=5\varepsilon-12\varepsilon^2-2\varepsilon^3\ge 0$. So it remains to show

$$\sum_{c,c'}\sum_{s\in\Omega}\sum_{\Lambda\in\mathcal{R}}\big\|\mathrm{Bl}(c,s,c',\Lambda)\big\|^2\;\le\;(1+b^{-4})\,y_\Sigma\,\mathsf{d}(S)\,\nu^2, \tag{7}$$

where $\mathrm{Bl}(c,s,c',\Lambda)$ is the *block* $A_S^{\mathrm{cut}}(\{\gamma:\mathfrak{g}_S(\gamma)\neq 0,\ \mathrm{link}(\gamma)=\Lambda\};c,s,c')$. If $\Sigma=\mathrm{Ch}_0(S)$ the sum is empty. Assume $\Sigma\neq\mathrm{Ch}_0(S)$; it suffices to bound the sum over a finite set $\mathcal{F}$ of tuples $j=(c_j,s_j,c_j',\Lambda_j)$ with non-zero blocks.

Step 2: blocks as limits. Let $j\in\mathcal{F}$. As $s_j\in\Omega$ and $\Sigma\neq\mathrm{Ch}_0(S)$, there is $w\in\mathrm{Ch}_0(S)$ at which all leading minors of the frame Gram matrix of $(s_j)_w$ are non-zero. For a term γ over (c_j,s_j,c_j') the frame Gram matrix of $\mathrm{pos}_S(c_j,c_j';\gamma)$ is the image of $Z_S(\gamma)$ and reduces to that of $(s_j)_w$ at w (Input 1(e)); so all leading minors of $Z_S(\gamma)$ are non-zero. Input 4(e) gives a unit t_j, an element $\gamma_j\in\mathcal{H}_\varrho(T_S^{\mathrm{rat}}(E))$ with $Z_S(\gamma_j)_{\varrho\varrho}=t_j$ and $N(S)^2<|(t_j)_\tau|^2$, an element $k_j\in K(S)$ with $\mathrm{red}_S(k_j)=s_j$, such that $\Lambda_j=\mathrm{link}(w_\varrho(1)\gamma_j)$, and vectors $L_{j,p}$ with

$$\mathrm{Bl}(j)=\sum_p L_{j,p},\qquad L_{j,p}=\lim_{\mathsf{T}}\ a_{S,\mathsf{T}}^{\mathrm{cut}}(c_j,s_j,c_j';w_\varrho((p\,|\,t_j))\gamma_j).$$

Hence it suffices to prove, for every finite set T_0 of units of E containing all t_j, with p running over the maps from $\{0,\dots,\varrho-1\}$ to T_0, and for arbitrarily large T,

$$\sum_{j\in\mathcal{F}}\Big\|\sum_p a_{S,\mathsf{T}}^{\mathrm{cut}}(c_j,s_j,c_j';w_\varrho((p\,|\,t_j))\gamma_j)\Big\|^2\;\le\;(1+b^{-4})\,y_\Sigma\,\mathsf{d}(S)\,\nu^2\,: \tag{8}$$

the partial sums over such p exhaust the sum over all p, and a limit of quantities that are bounded for arbitrarily large indices is bounded.

Step 3: the places. By Input 4(a) there are arbitrarily large T, stable under conjugation, off which $t^{\pm1}$ ($t \in \mathsf{T}_0$), $\hat{g}(S)^{-1}$, $N(S)^{-1}$ and $\Lambda(S)^{\pm1}$ are integral, and also the finitely many positions $\mathrm{pos}_S(c_j, c_j'; \gamma_j)$ and $\mathrm{pos}_S(c_j, c_j'; w_\varrho((p\,|\,t_j))\gamma_j)$. Fix such T, an integral place structure and a filling level $\mathcal{L} \in \mathrm{FL}(S, \mathsf{T})$ (Input 4(a)), and let Z be the intermediate group of Proposition 4.7 for Σ and $\mathbf{d}$. Then $\mathrm{loc}_{S,\mathsf{T}}(k_j) \in Z$ for all j, because $\mathrm{red}_S(k_j) = s_j \in \mathrm{Fix}_S(\Sigma, \mathbf{d})$.

Step 4: the top kernel. Fix a value t of $j \mapsto t_j$ and let $\mathcal{J}_t$ be its fibre. For a map p let $x_\tau(p,t)$ be the sequence $a \mapsto u(\mathrm{fs}_S(\tau,a)\cdot((p\,|\,t)_a)_\tau)$ of Input 6(a), and put

$$R^{(t)}_{pq} = \prod_\tau \mathrm{bl}_\tau(x_\tau(p,t), x_\tau(q,t)),$$

$$r = \frac{1}{\#\mathrm{Bc}(S)} \prod_\tau \Big(\sqrt{\mathrm{oc}(\mathfrak{s})} \prod_{a \le \varrho} \mathrm{fs}_S(\tau,a)^{\,k-1+2(\varrho-a)} \Big).$$

We check the hypotheses of Proposition 4.6 for $\mathcal{J}_t$, the side cut, the elements k_j, the matrix $R^{(t)}$ and r. (a): if two indices of $\mathcal{J}_t$ have the same tuple $(c, c', \mathrm{red}_S(k), \gamma)$, they have the same s and the same link $\mathrm{link}(w_\varrho(1)\gamma)$, so they are equal. (b), (c), (f) hold by Steps 2 and 3. (d): the rescaled image at τ of $w_\varrho((p\,|\,t))\gamma_j$ is $w_\varrho(x_\tau(p,t)) \cdot \mathrm{r}^\infty_S(\gamma_j)_\tau$ (Input 4(d)), the element $\mathrm{r}^\infty_S(\gamma_j)_\tau$ lies in $\mathcal{H}_\varrho$, and its frame Gram matrix has the entry $\mathrm{fs}_S(\tau,\varrho)\,t_\tau = t_\tau/N(S) = x_\tau(p,t)_\varrho$ at (ϱ,ϱ); so by the band limits (Input 6(a)) and because the correlation of a product of functions on a product of groups is the product of the correlations, $R^{(t)}$ is the Gram matrix of the vectors $\rho_t(p) \cdot \mathrm{J}(dy, W_S^{\mathrm{cut}}(\cdot).; \mathrm{r}^\infty_S(w_\varrho((p\,|\,t))\gamma_j))$, for every $j \in \mathcal{J}_t$, where

$$\rho_t(p) = \prod_\tau \frac{\sqrt{\mathrm{oc}(\mathfrak{s})}\,(|t_\tau|^2/N(S)^2 - 1)^{(k-1)/2}}{\#I_\tau} \prod_{a<\varrho} \big(\mathrm{fs}_S(\tau,a)\,|(p_a)_\tau|\big)^{k+1+2(\varrho-a-1)}.$$

(e): the head density of the unit on the real place is

$$\mathsf{h}_\varrho(\mathcal{L}; \omega_p, 1) = \prod_{a<\varrho} \mathrm{mod}_{\mathsf{T}}((p_a)_{\mathsf{T}})^{(k+1+2(\varrho-a-1))/2},$$

and by Input 4(b), for the unit p_a of E, which is integral off T with its inverse, $\mathrm{mod}_{\mathsf{T}}((p_a)_{\mathsf{T}})^{1/2} = \prod_\tau |(p_a)_\tau|^{-1}$. As $k+1+2(\varrho-a-1) = k-1+2(\varrho-a)$,

$$\mathsf{h}_\varrho(\mathcal{L}; \omega_p, 1) = \prod_{a<\varrho} \prod_\tau |(p_a)_\tau|^{-(k-1+2(\varrho-a))},$$

which cancels the powers of the $|(p_a)_\tau|$ in $\rho_t(p)$. What remains of $\rho_t(p)$ is the product over τ of $\sqrt{\mathrm{oc}(\mathfrak{s})}\,(\#I_\tau)^{-1} \prod_{a<\varrho} \mathrm{fs}_S(\tau,a)^{\,k-1+2(\varrho-a)}$ and of

$$(|t_\tau|^2/N(S)^2 - 1)^{(k-1)/2} = (|t_\tau|^2 - N(S)^2)^{(k-1)/2} \cdot \mathrm{fs}_S(\tau,\varrho)^{\,k-1},$$

by $\mathrm{fs}_S(\tau,\varrho) = N(S)^{-1}$; the last factor is the term $a = \varrho$ of the product in r. With $\prod_\tau \#I_\tau = \#\mathrm{Bc}(S)$ this gives $\rho_t(p) \cdot \mathsf{h}_\varrho(\mathcal{L}; \omega_p, 1) = r\sqrt{G}$, where $G = (\prod_\tau (|t_\tau|^2 - N(S)^2))^{k-1}$. Proposition 4.6 gives, for every sufficiently small open compact $L \ni 0$,

$$\begin{aligned} r^2 \sum_{j \in \mathcal{J}_t} \Big\| \sum_p a^{\mathrm{cut}}_{S,\mathsf{T}}(c_j, s_j, c_j'; w_\varrho((p\,|\,t))\gamma_j) \Big\|^2 \\ \le\ (1 + b^{-6})\, a_*(S) \cdot \mathrm{Re} \sum_{p,q} R^{(t)}_{pq}\, \frac{\mathcal{K}(L; Z; \omega_p, \omega_q)}{\mathrm{vol}(L)\ \mathrm{mr}(Z)}. \end{aligned} \tag{9}$$

As E_{T} has a basis of neighbourhoods of 0 consisting of compact open additive subgroups, one such subgroup L serves all the finitely many t.

Step 5: the sieve. By Proposition 4.7(ii) the kernels J_L form a sieve system with bottom bound $\mathcal{E}_0 = (1+b^{-6})\,y_\Sigma$, and $J_L(n,\omega_p,\omega_q)$ is the quotient on the right of (9). Summing (9) over the values t and applying the sieve (Input 6(a), with U the set of these values),

$$\begin{aligned} r^2\cdot(\text{left side of (8)}) \;\le\; (1+b^{-6})\,a_*(S)\cdot\prod_{a\le\varrho}\frac{\operatorname{vol}(\hat O_E)}{\prod_\tau \operatorname{fs}_S(\tau,a)^2} \\ \cdot\Big(\prod_\tau\frac{1+b^{-10}}{\deg\mathfrak{B}_\tau(S)}\,(1+b^{-6})\,y_\Sigma+\frac{\deg(S)^{-1}}{b^{6kf(E)+10}}\Big). \end{aligned}$$

Step 6: the numbers. By admissibility,

$$a_*(S)\le(1+b^{-9})\,\mathsf{d}(S)\,\deg(S)\,\operatorname{oc}(\mathfrak{s})^{f(E)}\operatorname{vol}(\hat O_E)^{-n}(\nu\operatorname{fs}(S)/\#\mathrm{Bc}(S))^2,$$

and $\operatorname{fs}(S)=\prod_\tau\prod_{a\le\varrho}\operatorname{fs}_S(\tau,a)^{\,k+2(\varrho-a)}$, so that

$$\operatorname{oc}(\mathfrak{s})^{f(E)}\cdot\operatorname{fs}(S)^2\cdot\#\mathrm{Bc}(S)^{-2}\cdot\prod_{a\le\varrho}\prod_\tau\operatorname{fs}_S(\tau,a)^{-2}=r^2.$$

Further $(1+b^{-10})^{f(E)}\le\exp(f(E)\,b^{-10})\le 1+b^{-9}$ because $f(E)\le b/1000$, and $b^{-6kf(E)-10}\le b^{-10}y_\Sigma$. Hence

$$r^2\cdot(\text{left side of (8)}) \;\le\; r^2\,\nu^2\,\mathsf{d}(S)\,y_\Sigma\cdot(1+b^{-6})(1+b^{-9})((1+b^{-9})(1+b^{-6})+b^{-10}),$$

and the last factor is at most $1+b^{-4}$ for $b\ge1000$. As $r>0$, this is (8). □

6 The second moment with constant one: proof of Theorem C

Proof of Theorem C. By downward induction on Σ in the finite set of the subsets of $\mathrm{Ch}_0(S)$: we assume the statement for all $\Sigma'\supsetneq\Sigma$ and all V, Θ, u, $\mathbf{d}$. Write $b=b(S)$, $k=k(\mathfrak{s})$, $\varrho=\varrho(\mathfrak{s})$, $q_0=q_0(S)$, $\Theta_{\mathrm{tot}}=\sum_s\Theta(s)$ and $y_\Sigma=b^{-6k\#\Sigma}$, and suppose, for a contradiction, that

$$\Theta_{\mathrm{tot}}>(1+9b^{-4})\,\mathsf{d}(S)\,y_\Sigma. \tag{10}$$

Claim: $\sum_{s\notin\mathrm{Fix}_S(\Sigma,\mathbf{d})}\Theta(s)\le b^{-4}\,\Theta_{\mathrm{tot}}$. Granting this, the hypothesis on the energies and Theorem D give

$$\begin{aligned}(1-b^{-4})\,\Theta_{\mathrm{tot}}\le\sum_{s\in\mathrm{Fix}_S(\Sigma,\mathbf{d})}\Theta(s)&\le\mathrm{En}_S(A_S^{\mathrm{full}}(G_S(E);\cdot,\cdot,\cdot);\mathrm{Fix}_S(\Sigma,\mathbf{d}))\\&\le(1-b^{-4})(1+9b^{-4})\,\mathsf{d}(S)\,y_\Sigma,\end{aligned}$$

which contradicts (10).

Proof of the claim. If $\Sigma=\mathrm{Ch}_0(S)$, every s with $\Theta(s)\neq0$ lies in $\mathrm{Fix}_S(\Sigma,\mathbf{d})$ by the hypothesis on the support, and the sum is 0. Otherwise fix $w\in\mathrm{Ch}_0(S)\setminus\Sigma$. Let $s\notin\mathrm{Fix}_S(\Sigma,\mathbf{d})$ with $\Theta(s)\neq0$. The first condition of Definition 2.2 holds for s, so the second fails: some leading minor of the frame Gram matrix of s_w, of an order $l\in\{0,\dots,\varrho+1\}$, vanishes. It therefore suffices to show, for each l,

$$\sum_{s:\ \text{the minor of order } l \text{ of } s_w \text{ vanishes}}\Theta(s)\;\le\;3\,b^{-10}\,\Theta_{\mathrm{tot}}, \tag{11}$$

since $3(\varrho+2)\,b^{-10} \le b^{-4}$.

Let F_0 be the matrix over $\mathbb{F}_w$ whose l columns are the frame vectors $\mathfrak{f}_0, \dots, \mathfrak{f}_{l-1}$, and $F_1 = JF_0$ with J the split Gram matrix of $T_S^{\mathrm{lat}}(\mathbb{F}_w)$; the minor of order l of s_w is $\det(F_1^t s_w F_0)$. We apply Input 7(b) to the group $\mathsf{G}(S)$, the homomorphism $s \mapsto s_w$, the field $\mathbb{F}_w$ of q_0 elements, the index sets $I(\mathfrak{s})$ (of $2\varrho+k+1$ elements) and $\{0,\dots,l-1\}$, the matrices F_0, F_1, and $Y = b^{-6k}\,\Theta_{\mathrm{tot}}$. Its hypothesis concerns, for a non-zero $v \in \mathbb{F}_w^l$, the sets of the s with $s_w F_0 v = F_0 v$, respectively $(s_w^{-1})^t F_1 v = F_1 v$; these are the groups $\mathrm{Fix}_{S,w}((\epsilon, v'))$ for ϵ false, respectively true, where v' is v extended by zeros. Fix such (ϵ, v') and put $\Theta' = \Theta \cdot \mathbf{1}_{\mathrm{Fix}_{S,w}((\epsilon,v'))}$. Then $\Theta' \ge 0$; $\Theta'(a^{-1}b) = \langle u'(a), u'(b)\rangle$ for the vectors $u'(a) = \delta_{a_w^{(\epsilon)} F_0 v} \otimes u(a)$ in the tensor product of the space of functions on $\mathbb{F}_w^{I(\mathfrak{s})}$ with V, where δ_x is the indicator of the point x; the sums of Θ' are dominated by those of Θ, hence by the energies; and $\Theta'(s) \ne 0$ implies $s \in \mathrm{Fix}_{S,w'}(\mathbf{d}'_{w'})$ for all $w' \in \Sigma \cup \{w\}$, where $\mathbf{d}'$ is $\mathbf{d}$ with the pair at w replaced by (ϵ, v'). The induction hypothesis for $\Sigma \cup \{w\}$ and (10) give

$$\sum_s \Theta'(s) \le (1 + 9b^{-4})\,\mathsf{d}(S)\,y_\Sigma\,b^{-6k} < b^{-6k}\,\Theta_{\mathrm{tot}} = Y.$$

So Input 7(b) applies:

$$q_0 \cdot \sum_{s:\ \det(F_1^t s_w F_0)=0} \Theta(s) \;\le\; 2\,\Theta_{\mathrm{tot}} + q_0^l \sqrt{q_0^{2\varrho+k+1}} \cdot b^{-6k}\,\Theta_{\mathrm{tot}}.$$

Since $q_0 \le b^{11}$ and $l \le \varrho+1$, the square of $q_0^l\sqrt{q_0^{2\varrho+k+1}}$ is at most $b^{11(4\varrho+k+3)} \le b^{12k}$, because $44\varrho + 33 \le k$; so the right side is at most $3\,\Theta_{\mathrm{tot}}$, and $q_0 \ge b^{10}$ gives (11). □

7 Terms in and off the Bessel subgroup

In this section $\mathcal{S} = \mathcal{S}(T_S^{\mathrm{rat}}(E))$, $\mathcal{U} = \mathcal{U}(T_S^{\mathrm{rat}}(E))$, and $\mathcal{S}^c$ is the complement of $\mathcal{S}$ in $G_S(E)$.

Lemma 7.1 (the trace of a small central idempotent on the Bessel terms)**.** *In Setting 2.1, suppose that E carries Weil data and that S is admissible. Let $i \in \{0,1\}$ and let $x \in \mathbb{C}[\mathsf{G}_i(S)]$ be a central idempotent with* $\mathrm{Re}\ x(1) \le b(S)^{-8}/2$*. Then*

$$\mathrm{Re} \sum_{s \in \mathsf{G}_i(S)} x(s) \sum_{(c,a) \in \mathrm{Cl}(S) \times \mathrm{Bc}(S)} \frac{A_S^{\mathrm{full}}(\mathcal{S}; c, \iota_i(s)^{-1}, c)_{(a,a)}}{\nu(S)} \;\le\; b(S)^{-4} \cdot \mathrm{cd}(S).$$

Proof. Let T be the set of all chosen places of S and their conjugates. It is stable under conjugation, and $\hat{g}(S)^{-1}$ is integral off T by Input 1(f), applied to the idempotent adele of the complement of T. Take a place structure and a filling level $\mathcal{L} \in \mathrm{FL}(S, \mathsf{T})$ (Input 4(a)); write K_1, K_g as in §4, $\chi = \chi_{g_{\mathcal{L}}}(\cdot)$, and for $s \in \mathsf{G}_i(S)$ choose $\sigma_s \in K(S)$ with $\mathrm{red}_S(\sigma_s) = \iota_i(s)$ (admissibility). Denote by $\mathcal{L}_0$ the sum on the left, before taking real parts.

Pointwise identity. For every $y \in K_1$,

$$\mathcal{L}_0 = \mathrm{cd}(S) \cdot \sum_s x(s)\,\chi(y\,\mathrm{loc}_{S,\mathsf{T}}(\sigma_s)\,y^{-1}). \tag{12}$$

Write $y = \mathrm{loc}_{S,\mathsf{T}}(\kappa_1)$ with $\kappa_1 \in K(S)$ (Input 4(b)). As x is a class function, we may replace s by its conjugate under the i-th component of $\mathrm{red}_S(\kappa_1)$; so it suffices to show, for $l = \kappa_1\sigma_s\kappa_1^{-1} \in K(S)$, that

$$\sum_{(c,a)} A_S^{\mathrm{full}}(\mathcal{S}; c, \mathrm{red}_S(l)^{-1}, c)_{(a,a)} = \mathrm{cd}(S)\,\nu(S)\,\chi(\mathrm{loc}_{S,\mathsf{T}}(l)).$$

Case 1: $\mathrm{loc}_{S,\mathsf{T}}(l) \in u' K_g$ with $u' \in \mathcal{U}$. Lift u' to an element κ' of $\mathcal{U}$, integral off T (Input 4(b)); it lies in $K(S)$, since $\mathrm{loc}_{S,\mathsf{T}}(\kappa') \in \mathrm{loc}_{S,\mathsf{T}}(l) K_g \subseteq K_1$, and $\mathrm{red}_S(\kappa') = \mathrm{red}_S(l)$. Then $\chi(\mathrm{loc}_{S,\mathsf{T}}(l)) = \psi_S^{\mathcal{U}}(\kappa')$ (Inputs 5(a) and 4(b)), and by the equivariance and the identity term (Input 2(c),(b)) $A_S^{\mathrm{full}}(\mathcal{S}; c, \mathrm{red}_S(\kappa')^{-1}, c) = \psi_S^{\mathcal{U}}(\kappa') \cdot A_S^{\mathrm{full}}(\mathcal{S}; c, 1, c)$, whose diagonal entries are $\psi_S^{\mathcal{U}}(\kappa')\,\nu(S)$. Summing over the $\mathrm{cd}(S)$ pairs (c, a) gives the identity.

Case 2: no such u' exists. Then $\chi(\mathrm{loc}_{S,\mathsf{T}}(l)) = 0$, and we show that no $\gamma \in \mathcal{S}$ is a term over $(c, \mathrm{red}_S(l)^{-1}, c)$, so that the amplitude sums are empty. Suppose γ is one, with position $x_0 = \mathrm{pos}_S(c, c; \gamma) \in K(S)$. Write $\gamma = u\,\delta$ with $u \in \mathcal{U}$ and δ in $\mathcal{H}$, a rational element of the unitary group of κ, and let $y' = y_c^{-1} \delta y_c$. Then $x_0 = \mathrm{pos}_S(c, c; u) \cdot \mathrm{pos}_S(c, c; \delta) = \mathrm{pos}_S(c, c; u) \cdot \mathrm{inc}_S(y')$ (Input 1(g)), with $\mathrm{pos}_S(c, c; u) \in \mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ (Input 1(e)). As $x_0 \in K(S)$, Input 1(g) gives $\mathrm{inc}_S(y') \in K(S)$. The element $\mathrm{red}_S(l)$ is trivial at the chosen places above the other prime $q_{i+1}(S)$ (indices modulo 2), because l is conjugate in $K(S)$ to σ_s, which reduces to $\iota_i(s)$; so the reduction $\mathrm{red}_S(l)^{-1}$ of x_0 is 1 at these places, and by Input 1(g) so is that of $\mathrm{inc}_S(y')$. By Input 1(f), y' is integral and reduces to 1 at these places, and by freeness (Input 1(d), with $n = \kappa$ and the index $i+1$) $\delta = 1$. Thus $x_0 \in \mathcal{U}$ and $\mathrm{red}_S(x_0^{-1}) = \mathrm{red}_S(l)$, whence $\mathrm{loc}_{S,\mathsf{T}}(l) \in \mathrm{loc}_{S,\mathsf{T}}(x_0^{-1}) K_g$: the excluded case.

Integration. Integrating (12) over K_1 for $\mathsf{M}(\mathcal{L})$, $\mathsf{M}(\mathcal{L})(K_1)\,\mathcal{L}_0 = \mathrm{cd}(S) \sum_s x(s)\, I_s$ with $I_s = \int_{K_1} \chi(y\, \mathrm{loc}_{S,\mathsf{T}}(\sigma_s)\, y^{-1})\, d\mathsf{M}(\mathcal{L})(y)$. By the Cauchy–Schwarz inequality $|\sum_s x(s) I_s|^2 \le \sum_s |x(s)|^2 \cdot \sum_s |I_s|^2$. For a central idempotent $x(s^{-1}) = \overline{x(s)}$ (Input 8(a)), so $\sum_s |x(s)|^2 = (x \cdot x)(1) = x(1) \le b(S)^{-8}/2$; here $x(1)$ is real and non-negative, being this sum. The cosets $\mathrm{loc}_{S,\mathsf{T}}(\sigma_s) K_g$ are pairwise disjoint (Input 4(b)), and on such a coset the inner integral of $\mathcal{J}_0(\mathcal{L}; K_1)$ is constant, equal to I_s, because K_1 normalises K_g and χ is right-invariant under K_g. Hence, by the bottom value with $t = 0$ (Input 5(b)),

$$\begin{aligned} \mathsf{M}(\mathcal{L})(K_g) \sum_s |I_s|^2 \;&\le\; \mathcal{J}_0(\mathcal{L}; K_1) \;\le\; \mathsf{M}(\mathcal{L})(K_1)\, \mathsf{M}(\mathcal{L})(K_g)\, \big(\mathsf{M}(\mathcal{L})(K_1) + 2^{-n(\mathfrak{s})}\big) \\ &\le\; 2\, \mathsf{M}(\mathcal{L})(K_1)^2\, \mathsf{M}(\mathcal{L})(K_g), \end{aligned}$$

where $2^{-n(\mathfrak{s})} \le \mathsf{M}(\mathcal{L})(K_1)$ by Input 5(c) and Lemma 4.1. Therefore

$$\mathsf{M}(\mathcal{L})(K_1)^2\, |\mathcal{L}_0|^2 \;\le\; \mathrm{cd}(S)^2 \cdot \tfrac{1}{2} b(S)^{-8} \cdot 2\, \mathsf{M}(\mathcal{L})(K_1)^2,$$

that is $|\mathcal{L}_0| \le b(S)^{-4} \mathrm{cd}(S)$. □

LEAN `Filling.re_trace_besselS_le`

Lemma 7.2 (second moment off the Bessel subgroup)**.** *In Setting 2.1, suppose that E is stably unramified and that S is admissible, and let $i \in \{0, 1\}$. Let Ω_i be the set of the $s \in \mathsf{G}(S)$ whose i-th component is that of an element of $\overline{\mathcal{S}}^S$. Then*

$$\mathrm{En}_S(A_S^{\mathrm{full}}(\mathcal{S}^c; \cdot, \cdot, \cdot); \Omega_i) \;\le\; b(S)^{-12} \cdot \mathsf{d}_1(S).$$

Proof. We apply Input 3 with $\Xi = \mathcal{S}^c$ (stable, Input 1(b)), $\Omega = \Omega_i$, $C = \mathsf{d}_1(S)$, $\omega_0 = 0$, $\mathcal{R} = \emptyset$ and $X = q_i(S)^{2f(E)}$; its conclusion is the claim. Write $b = b(S)$, $k = k(\mathfrak{s})$.

Every $s \in \Omega_i$ fixes norm at least X. Let $s_i = t_i$ with $t = \mathrm{red}_S(\kappa_1)$, $\kappa_1 \in \mathcal{S} \cap K(S)$. An element of the Bessel subgroup fixes the first frame vector $\mathfrak{f}_0$; so for every

$w \in \mathrm{Ch}_i(S)$ the reduction of κ_1 at w fixes $\mathfrak{f}_0$, and so does the reduction of its entrywise conjugate, which is the inverse adjoint of the former. Hence $s \in \mathrm{Fix}_{S,w}((\epsilon, e_0))$ for both Booleans ϵ, e_0 the first coordinate vector; these are $2f(E)$ triples.

No term is in normal form of index 0. Suppose $\gamma \in \mathcal{S}^c$ is a term over (c, s, c') with $s \in \Omega_i$ and is in normal form of index 0 with the scalar υ. For the frame index 0 this says $\gamma \mathfrak{f}_0 = \upsilon \mathfrak{f}_0$, so by Input 1(e) the $(\mathfrak{f}_0, \mathfrak{f}_0)$-entry of $\mathrm{pos}_S(c, c'; \gamma)$ is υ: it is integral, and it reduces at every $w \in \mathrm{Ch}_i(S)$ to the corresponding entry of $s_w = t_w$, which is 1. By freeness for the unitary group of rank one (Input 1(d)), $\upsilon = 1$. Then, by induction on the frame index a, $\gamma \mathfrak{f}_a - \mathfrak{f}_a$ lies in the span of the earlier frame vectors; so $\gamma \in \mathcal{S}$, a contradiction. Thus the condition on normal forms is void.

The remaining conditions. $\mathsf{d}_1(S) \le C$ holds with equality, and the link energy over the empty set of links is 0. As no link lies in $\mathcal{R}$, we need $\mathrm{tl}(S)^{12} X^{-k} \mathsf{d}(S) \le \mathsf{d}_1(S)$. By admissibility $\mathrm{tl}(S)^{18} \mathrm{cf}(S) \mathsf{d}(S) \le \mathsf{d}_1(S)$, where

$$\mathrm{tl}(S)^{18} \mathrm{cf}(S) = 2\,\mathrm{tl}(S)^{20} (b/g(S))^{kf(E)} \;\ge\; 2\,\mathrm{tl}(S)^{12}\, b^{3kf(E) - 23kf(E)},$$

using $\mathrm{tl}(S)^8 = b^{2kf(E)}$ and $g(S) \le b^{23}$; and $X^{-k} = q_i(S)^{-2kf(E)} \le b^{-20kf(E)}$. □

LEAN `Filling.energy_compl_besselS_le`

Lemma 7.3 (rows of the terms off the Bessel subgroup)**.** *In Setting 2.1, suppose that E is stably unramified and that S is admissible. Let $i \in \{0, 1\}$ and let $x \in \mathbb{C}[\mathsf{G}_i(S)]$ be a central idempotent. For $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathrm{Cl}(S) \times \mathrm{Bc}(S)$ put*

$$\mathsf{m}_{ab} = \sum_{t \in \mathsf{G}_i(S)} x(t)\, \frac{A_S^{\mathrm{full}}(\mathcal{S}^c; a_1, \iota_i(t)^{-1}, b_1)_{(a_2, b_2)}}{\nu(S)}.$$

Then the sum of the ℓ^2-norms of the rows of this matrix satisfies

$$\sum_{a \in \mathrm{Cl}(S) \times \mathrm{Bc}(S)} \big\| (\mathsf{m}_{ab})_{b \in \mathrm{Cl}(S) \times \mathrm{Bc}(S)} \big\|_{\ell^2} \;\le\; b(S)^{-4} \cdot \mathrm{cd}(S).$$

Proof. For $\sigma \in \mathsf{G}(S)$ put $Y(\sigma) = \nu(S)^{-2} \sum_{c, c'} \big\| \sum_t x(t)\, A_S^{\mathrm{full}}(\mathcal{S}^c; c, \iota_i(t)^{-1}\sigma, c') \big\|^2$. The sum of the squares of the row norms in the statement is $Y(1)$.

$Y(\sigma) = Y(1)$ for $\sigma \in \overline{\mathcal{S}}^S$. For every class c' choose a class $p(c')$ and $\gamma_{c'} \in \mathcal{S} \cap \mathrm{Tm}_S(c', \sigma, p(c'))$ (Input 1(c)). The map p is injective, hence bijective: if $p(c') = p(d')$, then $\gamma_{c'} \gamma_{d'}^{-1} \in \mathcal{S}$ has its position over (c', d') in $K_g(S)$, so $c' = d'$ (Input 1(c)). Write $\gamma_{c'} = u\,\eta$ with $u \in \mathcal{U}$, $\eta \in \mathcal{H}$. By Input 2(e) the product Bessel function transforms under $\mathrm{r}_S^\infty(u)$ by a scalar of modulus 1 and under $\mathrm{r}_S^\infty(\eta)$ by a unitary operator; so the tensor $W(y) \otimes \overline{W(\gamma_{c'}\gamma^{-1} y)}$ is the image of $W(y) \otimes \overline{W(\gamma^{-1} y)}$ under a unitary operator $L_{c'}$ of $\ell^2(\mathrm{Bc}(S) \times \mathrm{Bc}(S))$, the identity in the first factor. By the moves (Input 2(d), with the trivial move on the left and $\gamma_{c'}$ on the right; $\mathcal{S}^c$ is preserved), $A_S^{\mathrm{full}}(\mathcal{S}^c; c, s\,\sigma^{-1}, c') = L_{c'}(A_S^{\mathrm{full}}(\mathcal{S}^c; c, s, p(c')))$ for all s. Taking $s = \iota_i(t)^{-1}\sigma$, summing against $x(t)$ and using that $L_{c'}$ is unitary and p bijective gives $Y(1) = Y(\sigma)$.

The sum over $\overline{\mathcal{S}}^S$. Fix c, c' and let $f(s) = A_S^{\mathrm{full}}(\mathcal{S}^c; c, s, c')$ for $s \in \Omega_{i+1}$ and $f(s) = 0$ otherwise, with Ω_{i+1} as in Lemma 7.2 (indices modulo 2). For $\sigma \in \overline{\mathcal{S}}^S$ and $t \in \mathsf{G}_i(S)$ the element $\iota_i(t)^{-1}\sigma$ lies in Ω_{i+1}. So by Input 8(a)

$$\sum_{\sigma \in \overline{\mathcal{S}}^S} \Big\| \sum_t x(t) f(\iota_i(t)^{-1}\sigma) \Big\|^2 \;\le\; \sum_{s \in \Omega_{i+1}} \|f(s)\|^2,$$

and summing over c, c', with Lemma 7.2,

$$\#\overline{\mathcal{S}}^{S}\cdot Y(1)=\sum_{\sigma\in\overline{\mathcal{S}}^{S}}Y(\sigma)\ \le\ \mathrm{En}_S(A_S^{\mathrm{full}}(\mathcal{S}^c;\cdot,\cdot,\cdot);\Omega_{i+1})\ \le\ b(S)^{-12}\,\#\overline{\mathcal{S}}^{S}\cdot\mathrm{cd}(S).$$

Hence $Y(1)\le b(S)^{-12}\mathrm{cd}(S)$, and by the Cauchy–Schwarz inequality over the $\mathrm{cd}(S)$ rows the sum of the row norms is at most $\sqrt{\mathrm{cd}(S)\cdot Y(1)}\le b(S)^{-6}\mathrm{cd}(S)\le b(S)^{-4}\mathrm{cd}(S)$. □

LEAN `Filling.sum_norm_amplitudeSum_le`

8 The variance argument: proof of Theorem B

Proof of Theorem B. Write $b=b(S)$, $\nu=\nu(S)$, $e=g_0\otimes g_1\in\mathbb{C}[\mathsf{G}(S)]$, $V_e=\rho(e)V$ and $A(s)_{xy}=A_S^{\mathrm{full}}(G_S(E);x_1,s,y_1)_{(x_2,y_2)}$ for $x,y\in\mathrm{Cl}(S)\times\mathrm{Bc}(S)$. By Input 8(a), e and the $1-g_i$ are central idempotents, and for a central idempotent z the operator $\rho(z)$ is a self-adjoint idempotent that commutes with all $\rho(s)$.

Normalisation. Let $t_0>0$ with $t_0^2=(c\,\nu)^{-1}$ and $\mathsf{P}=t_0Q$. By (i) and (ii),

$$\big\langle\rho(b')\mathsf{P}e_y,\ \rho(a')\mathsf{P}e_x\big\rangle=\frac{A(a'^{-1}b')_{xy}}{\nu}\qquad(a',b'\in\mathsf{G}(S)).\tag{13}$$

In particular the Gram matrix $(\langle\mathsf{P}e_y,\mathsf{P}e_x\rangle)_{x,y}$ minus the identity has the entries $A(1)_{xy}/\nu-\delta_{xy}$, which by the splitting and the identity term (Input 2(a),(b)) are $A_S^{\mathrm{full}}(\mathcal{S}^c;x_1,1,y_1)_{(x_2,y_2)}/\nu$. This matrix is hermitian and is, up to complex conjugation, that of $\mathsf{P}^*\mathsf{P}-1$; so by Lemma 7.3 for $i=0$ and the central idempotent 1, $\sum_x\|(\mathsf{P}^*\mathsf{P}-1)e_x\|\le b^{-4}\mathrm{cd}(S)$.

The variance inequality. Fix F and ϕ, and for $s\in\mathsf{G}(S)$ put $U_s=R_s^{-1}(F)$ and $\mathsf{P}_s=\rho(s)\mathsf{P}$. For $v\in U_s$, $\mathsf{P}_sv-\rho(1-e)\mathsf{P}_sv=\rho(e)\rho(s)\mathsf{P}v=t_0R_s(v)\in F$, and $\mathsf{P}_s^*\mathsf{P}_s=\mathsf{P}^*\mathsf{P}$. Input 7(c), for P_s, $\mathsf{R}=\rho(1-e)$, U_s and F, gives

$$\dim U_s-b^{-4}\mathrm{cd}(S)-\sum_x\|\rho(1-e)\mathsf{P}_se_x\|^2\ \le\ \mathrm{Re}\big\langle\mathrm{dy}_{\mathbf{b}}(\mathsf{P}_s),\mathrm{dy}_{\mathbf{b}}(\mathbf{a})\big\rangle,\tag{14}$$

where $\mathrm{dy}_{\mathbf{b}}(\mathsf{P}_s)$ is formed with the vectors P_se_x and $\mathbf{a}$ is an orthonormal basis of F.

The mean over s. By Input 7(c) and (13),

$$\big\langle\mathrm{dy}_{\mathbf{b}}(\mathsf{P}_{a'}),\mathrm{dy}_{\mathbf{b}}(\mathsf{P}_{b'})\big\rangle=\sum_{x,y}\frac{|A(a'^{-1}b')_{xy}|^2}{\nu^2}=\Theta(a'^{-1}b'),$$

where $\Theta(s)=\mathrm{En}_S(A_S^{\mathrm{full}}(G_S(E);\cdot,\cdot,\cdot);\{s\})$. So Θ is ≥0 and of positive type, its sums over subsets are energies, and Theorem C for $\Sigma=\emptyset$ gives $\sum_s\Theta(s)\le(1+9b^{-4})\mathsf{d}(S)$. By Input 7(a), $\|\sum_s\mathrm{dy}_{\mathbf{b}}(\mathsf{P}_s)\|^2=\#\mathsf{G}(S)\sum_s\Theta(s)$, and by (iv)

$$\|\,\mathrm{dy}_{\mathbf{b}}(\mathbf{a})\|^2=\dim F=\phi\dim V_e\ \le\ \phi\cdot\#\mathsf{G}(S)\ \deg(S)\ \mathrm{mass}(E,r(\mathfrak{s})).$$

With $\mathsf{d}(S)=\mathrm{cd}(S)^2/(\deg(S)\,\mathrm{mass}(E,r(\mathfrak{s})))$ and $4\alpha\beta\le(\alpha+\beta)^2$,

$$\begin{aligned}\sum_s\mathrm{Re}\big\langle\mathrm{dy}_{\mathbf{b}}(\mathsf{P}_s),\mathrm{dy}_{\mathbf{b}}(\mathbf{a})\big\rangle&\le\#\mathsf{G}(S)\cdot\mathrm{cd}(S)\sqrt{(1+9b^{-4})\,\phi}\\&\le\ \#\mathsf{G}(S)\cdot\frac{\mathrm{cd}(S)}{2}(1+9b^{-4}+\phi).\end{aligned}$$

Hence there is s for which the right side of (14) is at most $\frac{1}{2}\mathrm{cd}(S)(1+9b^{-4}+\phi)$.

The bad part. For every s, $\|\rho(1-e)\mathsf{P}_s e_x\|^2 = \langle \mathsf{P}e_x, \rho(1-e)\mathsf{P}e_x\rangle$. In $\mathbb{C}[\mathsf{G}(S)]$,

$$1-e = \iota_0(1-g_0)+\iota_1(1-g_1)-(1-g_0)\otimes(1-g_1),$$

where $\iota_i(z)$ is z at the factor i tensored with 1; and $\langle \mathsf{P}e_x, \rho((1-g_0)\otimes(1-g_1))\mathsf{P}e_x\rangle \geq 0$, the operator of the central idempotent $(1-g_0)\otimes(1-g_1)$ being a self-adjoint idempotent. So by (13)

$$\sum_x \|\rho(1-e)\mathsf{P}_s e_x\|^2 \;\leq\; \sum_i \mathrm{Re}\sum_{t\in\mathsf{G}_i(S)} (1-g_i)(t)\sum_x \frac{A(\iota_i(t)^{-1})_{xx}}{\nu}.$$

Split A into its parts over $\mathcal{S}$ and over $\mathcal{S}^c$ (Input 2(a)). The first contributes at most $b^{-4}\mathrm{cd}(S)$ by Lemma 7.1 and (iii); the second at most the sum of the row norms of Lemma 7.3, again $b^{-4}\mathrm{cd}(S)$. So the bad part is at most $4\,b^{-4}\mathrm{cd}(S)$.

Conclusion. For the s found above, $\dim U_s \leq \frac{1}{2}\mathrm{cd}(S)(1+9b^{-4}+\phi)+b^{-4}\mathrm{cd}(S)+4b^{-4}\mathrm{cd}(S) = (\frac{1+\phi}{2}+\frac{19}{2}b^{-4})\mathrm{cd}(S)$. □

9 The dimension of the rows and the filling theorem

Lemma 9.1 (positions of the row parameters)**.** *In Setting 2.5, let τ be an infinite place of $E(D)$ and ι an element of the row index set $I(D)$. With $l_D(\tau) = (2\,\varkappa_D(\tau, \mathrm{num}_D(\iota')))_{\iota'\in I(D)}$ the doubled row parameters, the number of the $\iota' \in I(D)$ with $l_D(\tau)_\iota < l_D(\tau)_{\iota'}$ is $\mathrm{num}_D(\iota)$.*

Proof. The row parameter $\varkappa_D(\tau,\cdot)$ is strictly decreasing (Input 9(f)), so $l_D(\tau)_\iota < l_D(\tau)_{\iota'}$ holds exactly if $\mathrm{num}_D(\iota') < \mathrm{num}_D(\iota)$; and num_D is a bijection from $I(D)$ to $\{0,\dots,r(D)-1\}$. □

LEAN `Data.decrPosition_paramW`

Proposition 9.2 (the rows are at most the expected number)**.** *In Setting 2.5:*

$$\dim_{\mathbb{C}} \mathrm{Row}(D) \;\leq\; \#\mathsf{G}(D)\cdot\deg(S(D))\cdot\mathrm{mass}(E(D), r(D)).$$

Proof. By Input 9(f), $\dim \mathrm{Row}(D)$ is the dimension of the good forms on the definite space $W_{m(D)}(D)$, which is at most that of all forms of principal level on it. The doubled row parameters have the gap $h(D)$, so they are a Harish-Chandra parameter in the sense of the unit on the rows, and through it the forms of principal level on $W_{m(D)}(D)$ are algebraic forms with values in the weight representation. As the class automorphism groups of the level are trivial, their dimension is the number of classes of the principal level $K_g(D)$ times the dimension of the weight representation (Input 8(f)). The number of classes is $[K_0 : K_g(D)]$ times the weighted number of classes of the maximal level K_0, that is $\#\mathsf{G}(D)\cdot\mathrm{mass}(E(D), r(D))$ (Inputs 8(f) and 9(f)). The weight representation is a tensor product over the infinite places, and the dimension of each factor is the formal degree of the discrete series of the parameter (Input 8(g)), which is the degree of the Bessel system of $S(D)$ at that place (Input 9(a)). So the dimension of the weight representation is $\deg(S(D))$. □

LEAN `Data.finrank_rows_le_mass`

Proof of Theorem A. Let $S = S(D)$, of shape $\mathfrak{s}$; it is admissible, $E(D)$ carries Weil data, $b(S) = b(D)$ and $r(\mathfrak{s}) = r(D)$ (Input 9(a)). We apply Theorem B to the space $V = \mathrm{Prin}(D, Y_0(D))$ with the Petersson product (it is finite-dimensional), to the representation $\rho = \mathrm{rep}_{D,Y_0(D)}$ of $\mathsf{G}(D) = \mathsf{G}(S)$, to $Q = \mathcal{P}_S^{\mathrm{fit}}$, to the idempotents g_0, g_1 of Input 9(c) and to the number $\mathrm{vol}(K_g)$ in the place of the constant c of Theorem B (the letter c stays the scalar of Theorem A). Its hypotheses hold: (i) right translation is unitary (Input 8(f)); (ii) is the Petersson unfolding (Input 9(b)); (iii) is Input 9(c); and (iv) holds because $\mathrm{vol}(K_g) > 0$ and, as $\rho(g_0 \otimes g_1)V = e_0(D)\,\mathrm{Prin}(D, Y_0(D)) = \mathrm{Row}(D)$, by Proposition 9.2. Theorem B gives: for every subspace F of $\mathrm{Row}(D)$, with $\phi = \dim F / \dim \mathrm{Row}(D)$, there is $s \in \mathsf{G}(D)$ with

$$\dim R_s^{-1}(F) \;\le\; \Big(\frac{1+\phi}{2} + \frac{19}{2}\,b(D)^{-4}\Big)\mathrm{cd}(S), \qquad R_s = e_0(D) \circ \rho(s) \circ \mathcal{P}_S^{\mathrm{fit}}.$$

The column coordinates cc_D are a linear isomorphism from $\mathrm{Col}(D)$ onto the space $\ell^2(\mathrm{Cl}(S) \times \mathrm{Bc}(S))$; so $\dim \mathrm{Col}(D) = \mathrm{cd}(S)$, and by Definition 2.6 the map

$$R'_s = R_s \circ \mathrm{cc}_D = e_0(D) \circ \rho(s) \circ \mathcal{P}_D^{\circ}(\cdot)$$

satisfies the same bound with $\dim \mathrm{Col}(D)$ on the right. Now $B_s = \#\mathsf{G}(D) \cdot c \cdot R'_s \circ \mathrm{Ram}_D(\mathrm{Div}_N(D))$, and the non-zero scalars do not change preimages of subspaces: $B_s^{-1}(F) = \mathrm{Ram}_D(\mathrm{Div}_N(D))^{-1}(R_s'^{-1}(F))$. By Input 9(d), with $\#\mathrm{Div}_N(D) = f(E(D))$,

$$\begin{aligned}\dim B_s^{-1}(F) \;&\le\; \dim R_s'^{-1}(F) + \frac{f(E(D))}{N(D)-1}\dim \mathrm{Col}(D)\\ &\le\; \Big(\frac{1+\phi}{2} + \frac{19}{2}\,b(D)^{-4} + \frac{f(E(D))}{N(D)-1}\Big)\dim \mathrm{Col}(D),\end{aligned}$$

and the sizes of the data (Input 9(e), with $N(D) - 1 \ge b(D)^9$) bound the bracket by $\frac{1+\phi}{2} + b(D)^{-2}$. □

Proof of Corollary 1.1. Suppose $\dim \mathrm{Row}(D) = 0$. Theorem A for $c = 1$ and $F = \mathrm{Row}(D)$, for which $\phi = 0$ by the convention on the quotient, gives s with

$$\dim \mathrm{Col}(D) = \dim B_s^{-1}(\mathrm{Row}(D)) \le (\tfrac{1}{2} + b(D)^{-2})\dim \mathrm{Col}(D).$$

But $\mathrm{Col}(D) \neq 0$, being isomorphic to $\ell^2(\mathrm{Cl}(S(D)) \times \mathrm{Bc}(S(D)))$ with non-empty index sets, and $b(D)^{-2} < \frac{1}{2}$ (Input 9(e)): a contradiction. □

10 Where the results are used

Theorem A and Corollary 1.1 are used in the unit on the endgame: with the lemma on few translates of the unit on heights (a family that is spread with defect η has a subfamily of at most $5 \dim Y \log(1+\eta^{-1})/\dim X$ members whose images span a subspace of dimension at least $(1-4\eta)\dim Y$), applied with $\eta = b(D)^{-2}$ and with the constant of the seesaw identity for c, they give Theorem B of the handout's overview (enough forms). Corollary 1.1 is used twice more: in the main module of the proof, for the positivity of the dimension of the rows, and in the unit on the rows, in the module on the comparison of forms. Definition 2.6 is used in the unit on the seesaw (the seesaw theorem) and in the endgame, in the map $\mathcal{P}_D(\gamma)$ of the overview (§1). Proposition 9.2 is used only in the proof of Theorem A. Lemma 9.1 is not used in

the chain of this chapter: its only use is in the unit on the Selmer groups, in the module that shows that rows exist; it stands in §9 because the formal text has it in the module on the dimension of the rows. The module on link pairings, the fourth module of this chunk of the dossier, feeds the crude estimate (Input 3) and is the subject of Note P.

11 Notes on the formal text

1. *Theorem A.* The formal statement is the predicate "spread with defect $(b(D)^2)^{-1}$" of the unit on heights, applied to the family $s \mapsto \Pi_D(s) \circ \mathcal{P}^\circ_D(\cdot) \circ (c \cdot \mathrm{Ram}_D(\mathrm{Div}_N(D)))$. The domain is the space of all columns, not that of the ramified columns; the argument of the ramifying operator is the set of all primes dividing $\pi_N(D)$. The quotient $\dim F / \dim \mathrm{Row}(D)$ is Lean's division, which is 0 when the denominator is 0; the corollary uses exactly this case. The second reader confirmed the statement and the order of its quantifiers independently.
2. *Arguments that the census reads differently from Lean; to be settled by* `#check` *on a build.* The census lists for the Poincaré map a second explicit argument, which is the domain in the type of the linear map; the notation $\mathcal{P}^\circ_D(\psi)$ shows it as the column to which the map is applied. Proposition 4.6 takes the side as an implicit argument, which the census omits; it is stated here as the side σ, and the proof of Theorem D uses it for the cut side. Its explicit argument j_0, an element of the index set, serves only to know that the matrix R is a Gram matrix. In Proposition 4.7 the place structure is an implicit argument and not an instance.
3. *Statements cut by the census.* The statements of Lemma 4.2, Proposition 4.3, Proposition 4.5 and Proposition 4.6 contain local definitions, at which the census ends the statement; they were read to their true end in the source, with the second reader's table of corrected spans.
4. *Small sets.* "There is a neighbourhood of 0 such that for every set L contained in it" renders the formal "for all L eventually in the filter of small sets at 0".
5. *Sides.* In Theorems B, C and D and in the lemmas of §7 the amplitude sums are those of the full Bessel functions: the formal text names no side there and Lean takes its default instance. The link energy in the hypothesis of the crude estimate (Input 3) is that of the cut side, by a local instance of the file that proves it.
6. *Exponents computed here.* The following were computed from the Lean text and not taken from the dossier. (a) In Lemma 4.2: $6k \le 9(k - 10^{10}n^4)$ and $k - 10^{10}n^4 + 1 < 2\varrho + k + 1$. (b) In the proof of Theorem C the matrix of the frame vectors has l columns, for an order $l \le \varrho + 1$, the bound needed is $q_0^{2l+2\varrho+k+1} \le b^{12k}$, which follows from $44\varrho + 33 \le k$, and each order contributes at most $3b^{-10}$ of the total; the dossier's steps 13 to 15 of this proof state other exponents ($l + 1$ columns, a bound $1/b^{10}$ for the second term, and an inequality with $6k$), which the formal text does not need. (c) In Proposition 4.5 the ideal zeta function is taken at $3\varrho + 5$, with the conditions $[E : \mathbb{Q}] \le 2^{2\varrho}$ and $6\varrho + 12 \le k$; the dossier's steps 1.25.3 and 1.25.4 name $\varrho + 1$ and $2(\varrho + 2) \le k$. (d) In the proof of Theorem D the factors $(1 + b^{-4})^3 + b^{-12} \le (1 - b^{-4})(1 + 9b^{-4})$ and the final factor $\le 1 + b^{-4}$ were recomputed, and so was the identity $\rho_t(p)\mathsf{h}_\varrho(\mathcal{L}; \omega_p, 1) = r\sqrt{G}$ of Step 4, which the formal text verifies by expanding both sides; the computation is in the proof.

7. *The defect of Theorem B.* $\frac{19}{2} = \frac{9}{2} + 1 + 4$: $\frac{9}{2}$ from the constant $1 + 9b^{-4}$ of the second moment, 1 from the defect of orthonormality of the Poincaré series, 4 from the two bad parts.
8. *Inputs and their owners.* Input 4(c) is stated with the formal hypothesis, invariance of the measure under the images of all elements of the subgroup; the proof in Note G uses it only for integral elements, and the one application here, to $\mathcal{H}$ and $\theta_n(\mathcal{L})$, satisfies the formal hypothesis. Input 5 holds only what Note T proves; the facts about standard levels and about the tower method that the proofs of §4 use as well are Input 10, with the notes that own them. The inequality at the end of Input 10(d) is not a formal statement: it is a step of the formal proof of Proposition 4.7.
9. *The threshold of a fixing condition.* In "s fixes norm at least X" the number X is a natural number in the formal text, compared with a product of primes.
10. *Hypotheses.* Lemma 4.1 does not assume admissibility (the formal text omits it there). Lemmas 7.2 and 7.3 assume that E is stably unramified, the other results of §§5–8 that it carries Weil data. Lemma 7.1 takes the filling data as an explicit argument; the others take them implicitly.

How this note was written. From the structured proofs of the fourth chunk of the unit's dossier, which a second reader had re-read completely. I read in the Lean text, line by line, the five modules treated here (the filling theorem, the dimension of the rows, the moments, the fixed lines, the finite system) and the set-up module with the record of the filling data and its standing hypotheses, and compared every statement of this note with them argument by argument. Of the inputs, I read in the Lean text the statements of the crude estimate, of the fixing sets and energies, of sieve systems and the sieve, of the band limits, of the declarations of Notes G and T that are cited (with the corrected spans), of the variance lemmas of the unit on heights, of the central idempotents, and of the lemmas on the ramifying operator, the number of primes dividing $\pi_N(D)$ and the monotonicity of the row parameters; the other inputs are stated from the dossier's readings of the quoted statements. In the revision after the referee's report I read in the Lean text also the statements behind Inputs 1(g), 2(e), 4(c), 8(a), 8(c), 8(h) and 10, and the lattice basis. The proofs follow the formal ones. Shorter here: the support of the step function in Lemma 4.4; the limiting arguments of Proposition 4.5 and of Step 2 of the proof of Theorem D, which are described and not written out. The identity of hypothesis (e) in Step 4 of that proof is computed in the text; the computation is the referee's and was checked again. Nothing was compiled for this note. The program check of the unit's dossier covered 1371 leaf steps of 179 proofs.

Appendix A. Concordance with the Lean artifact

The formal names of the objects and the letters of this note: `S` is S, `𝔰` is $\mathfrak{s}$, `T` (a set of places) is T, `T` (a set of rational elements) is Ξ, `F` (a level) is $\mathcal{L}$, `Sg` and `A` (sets of chosen places) are Σ, `d` and `D` (frame-vector data) are $\mathbf{d}$, `v` (an infinite place) is τ, `Φ v x` is x_τ, `GRes` is $\mathsf{G}(S)$, `Data.G` is $\mathsf{G}(D)$, `besselS`, `besselU`, `besselH`, `GT ϱ` are $\mathcal{S}$, $\mathcal{U}$, $\mathcal{H}$, $\mathcal{H}_\varrho$, `unipCovolume` is $\nu(S)$, `mainEnergy` and `besselEnergy` are $\mathsf{d}(S)$ and $\mathsf{d}_1(S)$, `tailScale` is $\mathrm{tl}(S)$, `columnDim` is $\mathrm{cd}(S)$. Four statements as the artifact has them:

```
lemma Data.ramifiedColumns_fill_rows :
```

```
  IsSpread (D.goodProjTranslate · ∘ₗ D.poincare ∘ₗ (c.1 • D.ramifyingOp .univ)) (D.b ^ 2)⁻¹

lemma Fits.exists_translate_finrank_comap_le [WeilData E] {g c} [Module.Finite ℂ V] {ρ :
  Representation ℂ _ V} (hρ : ∀ s x y, ⟪ρ s x, ρ s y⟫_ℂ = ⟪x, y⟫_ℂ) {Q :
  EuclideanSpace ℂ (_ × _) →ₗ[ℂ] V} (hQ : ∀ s x y, ⟪ρ s (Q (.single y 1)), Q (.single x 1)⟫_ℂ =
  ofReal c * S.amplitudeSum .univ x.1 s y.1 (x.2, y.2)) (hg : ∀ i, IsCentralProj (g i)) (hb : ∀ i,
  ((1 - g i).coeff 1).re ≤ (S.b ^ 8)⁻¹ / 2) (hc : 0 < c) (hD : finrank ℂ (ρ.asAlgebraHom
  (piTensorElem g)).range ≤ Nat.card S.GRes * S.besselDegree * mass E 𝔰.r) : IsSpread
  ((ρ.asAlgebraHom (piTensorElem g)).rangeRestrict ∘ₗ ρ · ∘ₗ Q) (19 / 2 * (S.b ^ 4)⁻¹)

lemma sum_le_of_positive_type [WeilData E] {θ : _ → ℝ} (hθ : ∀ s, 0 ≤ θ s) (u : _ → V) (hu :
  ∀ a b, ⟪u a, u b⟫_ℂ = θ (a⁻¹ * b)) A (D : ∀ _, _) (hB : ∀ Ω, ∑ s with s ∈ Ω, θ s ≤ energy
  (S.amplitudeSum .univ) Ω) (hA : ∀ s, θ s ≠ 0 → ∀ w ∈ A, s ∈ S.vectorFixer (i := 0) w (D w)) :
  ∑ s, θ s ≤ (1 + 9 / S.b ^ 4) * S.mainEnergy * (S.b ^ (6 * 𝔰.k))⁻¹ ^ #A

lemma energy_fixingSet_le [WeilData E] {𝔰} {S : Filling E 𝔰} [S.IsAdmissible] Sg D :
  energy (S.amplitudeSum .univ) (fixingSet Sg D) ≤ (1 - (S.b ^ 4)⁻¹) *
  ((1 + 9 / S.b ^ 4) * S.mainEnergy * (S.b ^ (6 * 𝔰.k))⁻¹ ^ #Sg)
```

In this note	Lean declaration	File	Lines
Definition 2.6	`Data.poincare`	FillingTheorem.lean	13–14
Theorem A	`Data.ramifiedColumns_fill_rows`	FillingTheorem.lean	22–40
Corollary 1.1	`Data.finrank_rows_ne_zero`	FillingTheorem.lean	45–47
Lemma 4.4	`Filling.exists_step_function`	FiniteSystem.lean	16–33
Proposition 4.5	`Filling.IsFillingLevel.siegel_sum_le_lintegral_quadric`	FiniteSystem.lean	47–148
Proposition 4.6	`Filling.IsFillingLevel.sum_blocks_le_topKernel`	FiniteSystem.lean	156–223
Proposition 4.7	`Filling.IsFillingLevel.exists_finite_sieveSystem`	FiniteSystem.lean	235–293
Lemma 4.1	`Filling.volume_nonunits_le`	FixedLines.lean	19–34
Lemma 4.2	`Filling.IsFillingLevel.exists_intermediateGroup_one_prime`	FixedLines.lean	40–81
Proposition 4.3	`Filling.IsFillingLevel.exists_intermediateGroup_lines`	FixedLines.lean	86–143
Lemma 7.1	`Filling.re_trace_besselS_le`	Moments.lean	40–100
Theorem D	`Filling.energy_fixingSet_le`	Moments.lean	105–182
Lemma 7.2	`Filling.energy_compl_besselS_le`	Moments.lean	189–218
Theorem C	`Filling.sum_le_of_positive_type`	Moments.lean	226–257
Lemma 7.3	`Filling.sum_norm_amplitudeSum_le`	Moments.lean	262–296
Theorem B	`Filling.Fits.exists_translate_finrank_comap_le`	Moments.lean	301–345
Lemma 9.1	`Data.decrPosition_paramW`	RowDimension.lean	10–11
Proposition 9.2	`Data.finrank_rows_le_mass`	RowDimension.lean	15–25

Notation

P, D the parameters (N_0, d) and admissible data for them; $E(D)$, $m(D)$, $r(D)$, $k(D)$, $b(D)$, $N(D)$, $\pi_N(D)$, $h(D)$, $I(D)$, num_D, $\varkappa_D$, $K_g(D)$, K_0, $W_{m(D)}(D)$ as in Setting 2.5; $l_D(\tau)$ in Lemma 9.1

E, τ, x_τ a CM field with its CM type; an infinite place; the image of x at τ

$f(E)$, d_E, $\mathbb{Q}$ the number of infinite places; the discriminant; the rational numbers

w, $\bar{w}$, $\mathbb{F}_w$, E_w a finite place, its conjugate, residue field, completion

$\mathbb{A}_{E,f}$, $\hat{O}_E$, ψ_f finite adeles, integral adeles, the standard additive character

$\mathfrak{s} = (\varrho, \kappa, k_0)$ a shape; $\varrho(\mathfrak{s})$, $k(\mathfrak{s})$, $n(\mathfrak{s})$, $r(\mathfrak{s})$, $I(\mathfrak{s})$, $M(\mathfrak{s})$; in proofs k, n, ϱ abbreviate them where this is said

e_a, e_a^*, z, $\mathfrak{f}_a$ the hyperbolic basis vectors, the last basis vector, the frame

S filling data: $b(S)$, $N(S)$, $q_i(S)$, $g(S)$, $\bar{\pi}(S)$ (an idele; the bar belongs to the name), $\varpi(S)$, $\mathrm{Ch}_i(S)$, $K^{\mathrm{col}}(S)$, $\mathfrak{B}_\tau(S)$; $S(D)$ those of D; $\kappa_\tau(S)$ the real part of $g(S)\varpi(S)_\tau/(2\pi i)$

T, $J(T)$, $G(T)$, h_T, $Z_T(g)$ an $\mathfrak{s}$-form, its split Gram matrix, unitary group, hermitian form, frame Gram matrix; J a split Gram matrix; R a commutative ring in §2

$\mathcal{U}$, $\mathcal{H}$, $\mathcal{S}$, $\mathcal{H}_\varrho$ Bessel unipotent group, $\mathrm{U}(X_0)$, Bessel subgroup, the subgroup fixing all e_a, e_a^*;

$\mathrm{sd}(u)$ the sum of the superdiagonal; $\mathcal{S}^c$ the complement

$w_\varrho(p)$, $(p\,|\,t)$ Weyl element of length $\varrho(\mathfrak{s})$; the sequence $p_0, \dots, t, 1, \dots$

$T_S^{\mathrm{rat}}(R)$, $T_S^{\mathrm{lat}}(R)$, $G_S(R)$, G_S^{lat} the middle form in the standard and the lattice basis; the unitary groups; $\Lambda(S)$ the change of basis; x_{k_0} the basis vector of k_0

r_S^f, r_S^∞, inc_S the maps to the finite adelic group, to the real groups $G_1(\mathfrak{s})$, and of $\mathrm{U}(\kappa)$ into G_S^{lat}

$K(S)$, red_S, $K_g(S)$, $\overline{B}^S$ integral subgroup, reduction, principal level, reduced image of a subgroup B

$\mathsf{G}(S)$, $\mathsf{G}_i(S)$, ι_i; $\mathsf{G}(D)$ the finite groups of the chosen places and the inclusions of the factors

$\mathrm{Cl}(S)$, c, c', y_c, $h_S(c)$ classes, their representatives and images

$\mathrm{pos}_S(c, c'; \gamma)$, $\mathrm{Tm}_S(c, s, c')$ position of a rational element; terms over (c, s, c')

$Z_S(\gamma)$, $\mathfrak{g}_S(\gamma)$, $\mathrm{link}(\gamma)$, Λ frame Gram matrix, term discriminant, link; a link

$\mathrm{Bc}(S)$, $W_S^\sigma(y)_i$, σ Bessel coordinates, product Bessel function, a side (full or cut)

$a_S^\sigma(c, c'; \gamma)$, $A_S^\sigma(\Xi; c, s, c')$ amplitude, amplitude sum over a set Ξ; $\beta_{S,c}$ the cutoffs; $\mathcal{U}_{S,c}$; Bl a block

$\mathrm{En}_S(A; \Omega)$ energy of a family A over $\Omega \subseteq \mathsf{G}(S)$; Ω_i in Lemma 7.2

$\nu(S)$, $\mathrm{cd}(S)$, $\deg(S)$, $\mathrm{mass}(E, r)$ covolume, number of column coordinates, product of degrees, mass

$\mathsf{d}(S)$, $\mathsf{d}_1(S)$, $\mathrm{tl}(S)$, $\mathrm{cf}(S)$ main energy, Bessel energy, tail scale, codimension factor

$c_*(S)$, $a_*(S)$, $\mathrm{uo}(S)$, $\mathrm{fs}_S(\tau, a)$, $\mathrm{fs}(S)$, $\mathrm{oc}(\mathfrak{s})$ Siegel's constant, packet constant, unipotent order, frame scales and their product, the constant of the open cell

$\hat{g}(S)$, $\psi_S^{\mathcal{U}}(u)$ the idele of $g(S)$; the unipotent character

$\mathbf{d}$, $\mathrm{Fix}_{S,w}((\epsilon, v))$, $\mathrm{Fix}_S(\Sigma, \mathbf{d})$ frame-vector data, fixers, fixing sets; Σ a set of chosen places above $q_0(S)$; X a norm; y_Σ

T, T_0, E_T, O_T, ψ_T, mod_T a finite set of finite places, a finite set of places or of units, the product of completions with its integers, character and modulus; A, O_A, ψ_A, ν_A, δ_A for a general semilocal field

$\mathrm{loc}_{S,\mathsf{T}}$, $K_{S,\mathsf{T}}(u)$, K_1, K_g components at T; congruence groups

$\mathcal{L}$, $g_\mathcal{L}$, $\mathrm{FL}(S, \mathsf{T})$ a level, its unit, filling levels; $\mathsf{M}(\mathcal{L})$, $\theta_n(\mathcal{L})$, $\mu_{\varrho,2}^{\mathcal{L}}$, the quadric measure $\delta^{(\varrho)}(\mathcal{L})$, $\mathsf{M}_{\geq 0}(\mathcal{L})$ its measures; $\mathsf{A}(\mathcal{L}; W, x)$ the average of Input 10(d)

$\chi_g(y)$, $\mathrm{Int}(\mathcal{L})$, Z, $\mathrm{Cos}(Z)$, a_c finite Bessel function, intermediate groups, cosets and representatives

$\mathsf{v}(\mathcal{L})$, $\mathrm{mr}(Z)$, $\mathcal{J}_0(\mathcal{L}; Z)$, $W_Z(y)$ unipotent mass, mass ratio, bottom form, finite Bessel vector

$\mathrm{J}(\mu, W; \gamma)$, $\mathcal{J}(Z; p, l)$, $\mathcal{K}(L; Z; p, q)$, $\mathcal{E}(Z; R, \omega, e)$ correlation tensors, top kernel, local energy

$\mathsf{h}_j(\mathcal{L}; p, q)$, $\mathcal{J}_j(\mathcal{L}; p, q; W, \omega)$, $\Lambda_K(j; p)$ head density, floor form, head weights; $[\xi]$, $\mathrm{den}(\xi)$ generalised and denominator index

$a_{S,\mathsf{T}}^\sigma(c, s, c'; \gamma)$, $\psi_{S,\mathsf{T}}$, $\beta_{S,\mathsf{T}}(\xi)$ coset amplitude, character at T, frequency gain

J_L, J', $\mathcal{E}_0$, $u(c)$, ι_U, β, C, U kernels and data of a sieve system

$\mathrm{tv}(\pi, t, x)$, bl_τ, nc_τ, I_τ tail vector, band limit, normalised correlation, index set of a Bessel system

$\mathrm{Col}(D)$, $\mathrm{Row}(D)$, $\mathrm{Prin}(D, \mathcal{V})$, $\mathrm{rep}_{D,\mathcal{V}}$, $Y_0(D)$, $e_0(D)$ columns, rows, forms of principal level, their representation, the space of the rows, the good idempotent

$\mathcal{P}_D^\circ(\psi)$, $\Pi_D(s)$, $\mathrm{Ram}_D(\mathcal{W})$, $\mathrm{Div}_N(D)$, $\epsilon_D(w)$, cc_D Poincaré series map, translate and project, ramifying operator, primes dividing $\pi_N(D)$, central elements, column coordinates; $\mathcal{P}^{\mathrm{fit}}$ the Poincaré map of the set-up; $\mathcal{P}_D(\gamma)$ the map of the overview, c_0 the constant of the seesaw identity

$\mathrm{Spread}((B_s)_s; \eta)$, F, ϕ, η spread with defect η; a subspace; its share

V, ρ, Q, g_i, e, V_e, R_s, B_s, U_s the data of Theorem B and the maps of Theorems A, B; $\ell^2(\cdot)$ the Hilbert space of functions on a finite set, e_x its standard basis

Θ, u, Θ_{tot}, Y a function of positive type, its vectors, its total, a bound

P, R, $\mathrm{dy}_\mathbf{b}$, $\mathbf{a}$, $\mathbf{b}$, t_0, ε operators, the dyad sum, orthonormal bases, a normalisation, a small number

G, H; G in §§4, 5 a finite group and an inner product space in Input 7; the number $(\prod_\tau(|t_\tau|^2 - N(S)^2))^{k(\mathfrak{s})-1}$

$\mathcal{Y}$, wt, f, L, M, Ψ, Φ, Φ_0, $\mathfrak{F}$ pairs, weights, a function, a small set, an integer, step and test functions, the Fourier integral of Proposition 4.5; β, α, v_0, w_0, r_0, a, c', b, B_0, M_0, $\mathcal{D}(\tau')$, $\pi_\mathsf{T} = \bar{\pi}(S)_\mathsf{T}$ there; $\mathfrak{c}_M$, m, a_0, n' in Input 8(c)

$\mathcal{J}$, $\mathcal{F}$, j, j_0, γ_j, k_j, l_j, t, R, ρ, r, ω_p, y_j, $\gamma_{j,p}$ the data of Proposition 4.6 and of the proof of Theorem D; m_{ab} the matrix of Lemma 7.3; D_0, g, Θ_0, θ, v_θ, n_θ, z_0, e_0 in its proof

o, c_o, H, $\mathfrak{p}$, φ, λ, $Q(Z')$ in §4: none or a place, a constant, a subgroup, maps, the constant of the Haar measure, a product subgroup

x, i, q, p, m, l, s, b, k, n, h, d, N, K, X, I, O, W, A elements, indices, sequences, the usual meanings as stated where they occur; $\mathbb{C}$, $\mathbb{R}$; υ a scalar of norm one; $\zeta_{\hat{O}_E}$ the ideal zeta function; π the number; A_η the operator of Input 2(e); δ, ϵ, κ_0, κ_1, κ', σ_s, ϖ_w, ξ, ν, μ, ψ, χ, ω, ω_0, Γ, Ω, Λ as introduced

References

The places are those given in the second reader's comparisons of the formal statements with the literature; none was typed from memory.

- A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*: §52, Theorem 5, equation (41), pp. 76–77. (Input 8(b): a special case.)
- J. A. Green, *The characters of the finite general linear groups*, Trans. Amer. Math. Soc. 80 (1955): §8, Theorems 13–14, pp. 441–444. (Input 9(c): a restricted, weaker form.)
- F. van Doorn, *Formalized Haar measure*: §6, Lemma 11 and Theorem 12. (Input 8(d): the relation is recorded as unresolved.)
- Harish-Chandra, *Discrete series for semisimple Lie groups II*, Acta Math. 116 (1966): §41, Theorem 16, pp. 96–100. (Input 8(g): the exact comparison is recorded as open.)

READBACK NOTE 37

The top floor at a finite place: finite Bessel functions, shell bounds and the top kernel

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. At finitely many finite places the second moment of the Poincaré series is governed by one function on a unitary group over a product A of local fields: the *finite Bessel function*, which is a character of the Bessel unipotent subgroup on the cosets of a principal congruence subgroup and zero elsewhere. This note feeds the vector of its translates, the *finite Bessel vector* of a group Z between the congruence subgroup and the integral group, into the tower method of the unit on the real place. It proves three things. The floor forms of this vector decay in the frequency: a non-integral frequency ξ costs a power of the index of ξ. The form at the bottom of the tower is the square of the measure of Z up to an error that is a high power of the volume of the non-units. And at the top floor the form unfolds to an integral of inner products of *correlations*, which gives a kernel that is continuous, of positive type, and bounds from above an energy that counts packets of rational terms. The estimates rest on one *shell bound*: the positions are sorted by the denominator ideal of $(x-1)/g$, and a sum over ideals is paid for by an ideal zeta function.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles and that Lean's kernel accepted the proofs; an independent replay is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note (Note T of its unit) treats 45 of the 497 declarations of the unit *Translates of Poincaré series fill the rows*: the module on the top floor at a finite place. Proved here from the inputs: Theorems A to D and the statements of §§4–6. Taken for granted and stated as Inputs 1 to 5 in §3: the Bessel subgroup and the congruence subgroups from the background library, the tower of floor measures and its estimates from the unit on the real place, the standard levels from another note of this unit, and three facts on ideals, valuations and Eichler elements. Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 45 declarations.

1 The statements

The objects are defined in §2. In brief: A is a finite product of local fields with an involution, T a hermitian form of a shape $\mathfrak{s}$, $G(T)$ the unitary group of its split Gram matrix, $\mathcal{L}$ a standard level with unit $g_{\mathcal{L}}$ and congruence subgroup $\mathrm{K}^\circ(\mathcal{L})$, and Z a group between $\mathrm{K}^\circ(\mathcal{L})$ and the group $K_T(1)$ of the integral isometries. The finite Bessel vector $W_Z(y)$ has one coordinate for each coset of $\mathrm{K}^\circ(\mathcal{L})$ in Z; $\mathcal{I}_j(\mathcal{L};p,q;W,\omega)$ is the floor form of the unit on the real place, $\mathrm{mr}(Z)$ the mass ratio, $[\xi]$ the generalised index and $\mathrm{den}(\xi)$ the denominator index of a unit ξ.

Theorem A (the frequency bound). *In Setting 2.4, let $j \le \varrho(\mathfrak{s})$, let K and t be natural numbers with $10^{10}n(\mathfrak{s})^4 \le K$ and $t + 10^{10}n(\mathfrak{s})^4 \le k(\mathfrak{s})$, let ξ be a unit of A, let p, q be sequences of units of A, and let $\omega\colon G(T) \to \mathbb{C}$ be a function such that either $\omega(x) = \psi_A(\xi \cdot Z_T(x)_{jj})$ for all x, or there is i with $j = i+1$, $\omega = 1$ and $\xi = g_{\mathcal{L}}^{-1}(p_i^{-1} - q_i^{-1})$. Then*

$$\left|\mathcal{I}_j(\mathcal{L};p,q;W_Z(\cdot),\omega)\right| \;\le\; \mathrm{mr}(Z)\cdot[\xi]^{t/2}\,\mathrm{den}(\xi)^{-6}\left(\frac{2}{\mathrm{mod}_A(g_{\mathcal{L}})}\right)^{K}\Lambda_K(j;p)\,\Lambda_K(j;q).$$

LEAN `IntegralPlace.IsStdLevel.enorm_floorForm_le`

Theorem B (the bottom value). *In Setting 2.4, let t be a natural number with $t + 10^{10}n(\mathfrak{s})^4 \le k(\mathfrak{s})$. Then*

$$\mathcal{I}_0(\mathcal{L};Z) \;\le\; \mathsf{M}(\mathcal{L})(Z)\cdot\mathsf{M}(\mathcal{L})(\mathrm{K}^\circ(\mathcal{L}))\cdot\Big(\mathsf{M}(\mathcal{L})(Z) + \mathrm{vol}(O_A\setminus O_A^\times)^{t+1}\cdot 2^{-n(\mathfrak{s})}\Big).$$

LEAN `IntegralPlace.IsStdLevel.floorZeroForm_le_main_term`

Theorem C (the top floor: unfolding and continuity). *In Setting 2.4:*

(i) *Let p, q be sequences of units of A and $\xi \in A$; put $\varphi_\xi(x) = \psi_A(\xi \cdot Z_T(x)_{\varrho\varrho})$ for $x \in G(T)$. Then the function $F_\xi(l) = \varphi_\xi(l^{-1})\cdot\langle \mathcal{J}(Z;p,l), \mathcal{J}(Z;q,l)\rangle$ is integrable for the measure $\mu^{\mathcal{L}}_{\varrho,2}$, and*

$$\mathcal{I}_\varrho(\mathcal{L};p,q;W_Z(\cdot),\varphi_\xi) \;=\; \int F_\xi(l)\,d\mu^{\mathcal{L}}_{\varrho,2}(l).$$

(ii) *Let L be an open additive subgroup of A. Then $(p,q) \mapsto \mathcal{K}(L;Z;p,q)$ is a continuous function on the pairs of sequences of units of A.*

LEAN `IntegralPlace.IsStdLevel.floorForm_top_eq_integral` · `IntegralPlace.IsStdLevel.continuous_topKernel`

Theorem D (the top floor: energy and packets). *In Setting 2.4, let $\mathcal{C}$ be a finite set, R a complex matrix with rows and columns indexed by $\mathcal{C}$, and $\omega = (\omega_i)_{i\in\mathcal{C}}$ a family of sequences of units of A.*

(i) *Let $L \subseteq A$ be an open subset, suppose that R is positive semidefinite, and let t be a unit of A with $(\omega_i)_\varrho = t$ for all indices $i \in \mathcal{C}$. Then $\mathcal{Q} = \sum_{i,i'\in\mathcal{C}} R_{ii'} \cdot \mathcal{K}(L; Z; \omega_i, \omega_{i'})$ is real and $\mathcal{Q} \geq 0$, and*

$$\int \mathbf{1}\big[(J(T)\,e)_z - t \in L\big] \cdot \mathcal{E}(Z; R, \omega, e)\; d\delta^{(\varrho)}(\mathcal{L})(e) \;\leq\; \mathcal{Q}.$$

(ii) *Suppose that the units of A form an open set and that inversion is continuous at 1. Let e be a vector, Θ_0 a non-empty finite set, and for $\theta \in \Theta_0$ let $l_\theta \in \mathcal{H}_\varrho(T)$ with $l_\theta^{-1} z = e$ and $n_\theta \in \mathrm{Cos}(Z) \times \mathrm{Cos}(Z)$, such that $l_\theta l_{\theta'}^{-1} \in \mathrm{K}^\circ(\mathcal{L})$ and $n_\theta = n_{\theta'}$ together imply $\theta = \theta'$. For $\theta \in \Theta_0$ let v_θ be the vector with the coordinate $\mathcal{J}(Z; \omega_i, l_\theta)_{n_\theta}$ for every index $i \in \mathcal{C}$. Then there is a neighbourhood of e such that for every x in it, every $k \in \mathcal{H}(T) \cap \mathrm{K}^\circ(\mathcal{L})$ and every vector y with $k\,y = x$,*

$$\theta_n(\mathcal{L})(\mathrm{K}^\circ(\mathcal{L})) \cdot \sum_{\theta\in\Theta_0} \big(\mathrm{Re}\; v_\theta^* R v_\theta\big)^+ \;\leq\; \mathcal{E}(Z; R, \omega, y).$$

LEAN `IntegralPlace.IsStdLevel.lintegral_energy_le_topKernel` · `IntegralPlace.IsStdLevel.eventually_sum_packets_le_energy`

Theorem A is the decay of the finite sieve in the frequency, Theorem B its bottom value, and Theorems C and D are what the top floor of the sieve needs: the last form is an integral of a positive kernel, and that kernel dominates the sum over the packets of rational terms that the main note of the unit (Note F, *Translates of Poincaré series fill the rows*) compares with Siegel's formula.

2 The setting

Conventions. The natural numbers are $0, 1, 2, \ldots$, and a difference of natural numbers is read as 0 when it would be negative. A is a *semilocal field*: a locally compact topological ring with a Haar measure vol and a compact open subring O_A of volume 1, in this proof always a finite product of non-archimedean local fields. It carries an involution $x \mapsto \bar{x}$ and an *integral place structure*: O_A is stable under the involution; $O_A^\times$ is the group of the units of O_A; ψ_A is a unitary additive character of A which is *self-dual*, that is, $\psi_A(ra) = 1$ for all $r \in O_A$ exactly if $a \in O_A$; for a unit u of A, $\mathrm{mod}_A(u)$ is the modulus, $\mathrm{vol}(uX) = \mathrm{mod}_A(u)\,\mathrm{vol}(X)$, and $[u] = \mathrm{mod}_A(u)^{-1}$ is the *generalised index*, which for $u \in O_A$ is the index of uO_A in O_A, so that then $[u] \geq 1$; for a unit ξ, $\mathrm{den}(\xi)$ is the index of $O_A \cap \xi^{-1} O_A$ in O_A. The place structure has further data (a trace-one element, a measure on the skew elements) that enter only through the inputs; its Plancherel constant is 1. A *shape* is a triple $\mathfrak{s} = (\varrho, \kappa, k_0)$ of a natural number, a finite set and an element of it; $\varrho(\mathfrak{s}) = \varrho$, $k(\mathfrak{s}) = \#\kappa$, $n(\mathfrak{s}) = \varrho(\mathfrak{s}) + 1$. T is an *$\mathfrak{s}$-form* over A whose split Gram matrix is a *hermitian unit* (hermitian and invertible); ι is a finite set, m a natural number, and $\mathcal{L}$ denotes a level of T in $\mathrm{Lev}(\iota, T, m+1)$, in the sense recalled below. A *sequence*

of units is a map $p\colon \mathbb{N} \to A^\times$, $a \mapsto p_a$; the sequences of units carry the product topology, and 1 is the constant sequence. For a finite set X, $\ell^2(X)$ is $\mathbb{C}^X$ with the inner product $\langle v, w\rangle = \sum_x \bar{v}_x w_x$, conjugate-linear in the first variable; X^* is the conjugate transpose of a matrix. Integrals of functions with values in $[0,\infty]$ are Lebesgue integrals; for measures μ, ν on a group, $\mu * \nu$ is the image of $\mu \times \nu$ under the multiplication. Elements written x, y, u, k are elements of $G(T)$, and g is a unit of A, where nothing else is said.

Forms and groups. These notions belong to the background library. The index set $I(\mathfrak{s})$ consists of $e_0, \dots, e_{\varrho-1}$, of the elements of κ, of z, and of $e_0^*, \dots, e_{\varrho-1}^*$. The *frame* is $\mathfrak{f}_a = e_a$ for $a < \varrho$ and $\mathfrak{f}_\varrho = z$. An $\mathfrak{s}$-form T is a hermitian matrix indexed by $\kappa \sqcup \{z\}$; its *split Gram matrix* $J(T)$, indexed by $I(\mathfrak{s})$, pairs e_a with e_a^* to 1 and is T on the middle; $G(T)$ is the unitary group of $J(T)$ over A, with the form $h(v,v') = v^* J(T) v'$, so that the coordinate of a vector w at e_a^* is $h(e_a, w)$. For $x \in G(T)$, $Z_T(x) = (h(\mathfrak{f}_a, x\,\mathfrak{f}_{a'}))_{a,a' \le \varrho}$ is its *frame Gram matrix*; its corner entry is $Z_T(x)_{\varrho\varrho} = h(z, xz)$. The *level* of an index is a for e_a, ϱ for the middle indices, and $2\varrho - a$ for e_a^*. In $G(T)$: $\mathcal{U}(T)$ is the *Bessel unipotent subgroup*, the elements that are block upper triangular for the levels with identity diagonal blocks; $\mathcal{H}(T)$ is the subgroup of the elements that fix all e_a, all e_a^* and z; $\mathcal{H}_l(T)$, for $l \le \varrho$, the subgroup of those that fix e_a and e_a^* for $a < l$, and $\mathcal{H}_{\varrho+1}(T) = \mathcal{H}(T)$; $\mathcal{Q}_l(T)$ is the group of the elements that are unitriangular on the first l floors; and $\mathcal{S}(T) = \mathcal{U}(T)\mathcal{H}(T)$ is the *Bessel subgroup*. For $u \in G(T)$, $\mathrm{sd}(u) = \sum_{a<\varrho} h(e_a^*, u\,\mathfrak{f}_{a+1})$ is the sum of the entries of u just above the diagonal of its frame block. For a sequence p of units and $l \le \varrho$, $w_l(p) \in G(T)$ is the *Weyl element* of length l: it sends e_a to $p_a e_a^*$ and e_a^* to $\bar{p}_a^{-1} e_a$ for $a < l$ and fixes the other basis vectors. For a unit g of A, $K_T(g)$ is the group of the $k \in G(T)$ that have entries in O_A, as has k^{-1}, and for which $g^{-1}(k-1)$ has entries in O_A; so $K_T(1)$ is the group of the *integral isometries*.

Levels and the tower. These notions belong to the unit on the real place (the notes *The tower method at a place*); they are recalled with their arguments. A *level* $\mathcal{L} \in \mathrm{Lev}(\iota, T, m+1)$ consists of a hyperbolic basis of the middle form with $m+1$ further hyperbolic pairs and an orthonormal part indexed by ι; a unit $g_\mathcal{L}$ of A with $\overline{g_\mathcal{L}} = g_\mathcal{L}$; s-finite measures $\theta_\varrho(\mathcal{L})$, carried by $\mathcal{H}_\varrho(T)$, and $\theta_n(\mathcal{L}) \neq 0$, carried by $\mathcal{H}(T)$ and invariant under right translation by its elements; an open set of vectors, the *ball*; a compact set $\mathrm{Box}(\mathcal{L}) \subseteq G(T)$, the *box*; and a measurable set $\mathrm{Posn}(\mathcal{L}) \subseteq G(T)$, the *positions*. Its *congruence subgroup* is $\mathrm{K}^\circ(\mathcal{L}) = K_T(g_\mathcal{L})$. The library provides measures η_l on $G(T)$, the *Heisenberg measures* of the floors; $\eta_{<j}$ is the convolution of the first j of them, and $\eta_{<\varrho+2}$ is carried by $\mathcal{U}(T)$. The *floor measures* are $\theta_l(\mathcal{L})$ for $l < \varrho$ (the law of the inverse Weyl element of floor l, convolved with η_l), $\theta_\varrho(\mathcal{L})$, and $\theta_{\varrho+1}(\mathcal{L}) = \theta_n(\mathcal{L})$. For natural numbers l, i the *stack measure* is

$$\mu_{l,i}^{\mathcal{L}} = (\eta_{l+i-1} * \theta_{l+i-1}(\mathcal{L})) * \cdots * (\eta_l * \theta_l(\mathcal{L})),$$

with $\mathsf{M}(\mathcal{L}) = \mu_{0,\varrho+2}^{\mathcal{L}}$, the *group measure*, and $\mathsf{M}_{\ge 0}(\mathcal{L}) = \theta_{\varrho+1}(\mathcal{L}) * \cdots * \theta_0(\mathcal{L})$. The *quadric measure* $\delta^{(\varrho)}(\mathcal{L})$ is the image of $\theta_\varrho(\mathcal{L})$ under $\sigma \mapsto \sigma^{-1} z$, a measure on the vectors $e \in A^{I(\mathfrak{s})}$ with $h(e,e) = h(z,z)$. For a unit g and $l < \varrho$ the *floor modulus* is

$\mathsf{d}_{l}(\mathcal{L};g)=\operatorname{mod}_{A}(g)^{k(\mathfrak{s})+1+2(\varrho-l-1)+1/2}$, and for sequences p, q of units and $j\le\varrho$ the *head density* is

$$\mathsf{h}_{j}(\mathcal{L};p,q)=\prod_{a<j}\operatorname{mod}_{A}(p_{a}q_{a})^{(k(\mathfrak{s})+1+2(\varrho-a-1))/2}.$$

The *volume exponent* is $\mathsf{s}(\mathcal{L})=\#\iota-(3n(\mathfrak{s})+m+2)$. For $a<\varrho$ and $x,y\in G(T)$ the set of *floor values* $\operatorname{Val}_{a}(\mathcal{L};x,y)\subseteq A$ consists of the coordinates $h(e_{a},w)$ of the vectors w whose coordinates at e_{b}^{*}, $b<a$, vanish and for which $x^{-1}w$ and $g_{\mathcal{L}}^{-1}(x^{-1}w-(xy)^{-1}w)$ lie in the ball. A point x is *near at* j if $\xi x\in\operatorname{Box}(\mathcal{L})$ for some $\xi\in\mathcal{Q}_{j}(T)$.

For a function $k\colon G(T)\times G(T)\to\mathbb{C}$ and $j\le\varrho+1$ put

$$\begin{aligned}\mathsf{Av}_{j}(\mathcal{L};k)(g,y)&=\int k(\sigma g,\sigma y)\,d(\theta_{\varrho+1}(\mathcal{L})*\cdots*\theta_{j}(\mathcal{L}))(\sigma),\\ \mathsf{Pv}_{j}(\mathcal{L};k)(g,y)&=\int\bigl|\mathsf{Av}_{j+1}(\mathcal{L};k)(\sigma g,\sigma y)\bigr|\,d\theta_{j}(\mathcal{L})(\sigma),\end{aligned}$$

here with $g,y\in G(T)$. For a finite set X, functions $W,W'\colon G(T)\to\ell^{2}(X)$ and $d\in G(T)$, write $k^{d}_{W,W'}(x,y)=\langle W(x),W'(dy)\rangle$. The *pairing norm* is $\mathrm{PN}(\mathcal{L};W,W')=\iint|\langle W(x),W'(y)\rangle|^{2}\,d\mathsf{M}_{\ge0}(\mathcal{L})(y)\,d\mathsf{M}_{\ge0}(\mathcal{L})(x)$. For $j\le\varrho$, sequences p, q of units, $x,g\in G(T)$ and $\omega\colon G(T)\to\mathbb{C}$, with $d=w_{j}(q)^{-1}w_{j}(p)$, put

$$\begin{aligned}\Phi_{j}(x,g)=\int\!\!\int&\omega(\sigma\tau^{-1})\cdot\mathsf{Av}_{j+1}(\mathcal{L};k^{1}_{W,W})(\sigma g,\sigma gx)\\&\cdot\overline{\mathsf{Av}_{j+1}(\mathcal{L};k^{d}_{W,W})(\tau w_{j}(p)^{-1}g,\tau w_{j}(p)^{-1}gx)}\,d\theta_{j}(\mathcal{L})(\tau)\,d\theta_{j}(\mathcal{L})(\sigma).\end{aligned}$$

Then the *floor form*, its *modulus* at x and its *majorant* at x are

$$\begin{aligned}\mathcal{I}_{j}(\mathcal{L};p,q;W,\omega)&=\mathsf{h}_{j}(\mathcal{L};p,q)\cdot\int\!\!\int\Phi_{j}(x,g)\,d\mathsf{M}(\mathcal{L})(x)\,d\mu^{\mathcal{L}}_{0,j}(g),\\ \mathcal{I}^{\mathrm{mod}}_{j}(\mathcal{L};p,q;W,x,\omega)&=\int|\Phi_{j}(x,g)|\,d\mu^{\mathcal{L}}_{0,j}(g),\\ \mathcal{I}^{\mathrm{maj}}_{j}(\mathcal{L};p,q;W,W',x)&=\int\mathsf{Pv}_{j}(\mathcal{L};k^{1}_{W,W})(g,gx)\\&\qquad\cdot\mathsf{Pv}_{j}(\mathcal{L};k^{d}_{W,W'})(w_{j}(p)^{-1}g,w_{j}(p)^{-1}gx)\,d\mu^{\mathcal{L}}_{0,j}(g).\end{aligned}$$

A function $W\colon G(T)\to\ell^{2}(X)$ is a *cut Bessel function for* $\mathcal{L}$ if it is measurable and bounded, $W(ux)=\psi_{A}(g_{\mathcal{L}}^{-1}\operatorname{sd}(u))W(x)$ for $u\in\mathcal{U}(T)$, and $W(x)\neq0$ implies $ux\in\operatorname{Box}(\mathcal{L})$ for some $u\in\mathcal{U}(T)$. For a measure μ on $G(T)$, such a W and $\gamma\in G(T)$, the *Bessel correlation* $\mathrm{J}(\mu,W;\gamma)\in\ell^{2}(X\times X)$ has the coordinates $\int W(y)_{a}\,\overline{W(\gamma^{-1}y)_{b}}\,d\mu(y)$.

A *gauge* $\Gamma\in\mathrm{Gau}(\mathcal{L})$ provides, for every natural number i, a compact set of the elements of A *of size at most* i, written $\mathrm{sz}_{\Gamma}(x)\le i$, increasing with i, and a constant $\Lambda(\Gamma)\in[2,\infty)$; a vector or a matrix has size at most i if its entries have. A sequence p of units is a *head of length* j *for* Γ, $p\in\mathrm{Hd}_{j}(\Gamma)$, if $(p_{0}\cdots p_{l-1})^{-1}$ has size at most 1 for all $l\le j$. The *self-norm product* $\mathrm{SNP}_{j}(\Gamma;p,q;W,W',x)\in[0,\infty]$ of a gauge is the quantity of that name of the unit on the real place; it enters here only through the inputs.

Standard levels. From the note *Poincaré series of cut Bessel functions* of this unit: $\mathcal{L} \in \mathrm{StdLev}$ (*$\mathcal{L}$ is a standard level*) means that $\mathcal{L}$ is the level of the lattice $O_A^{I(\mathfrak{s})}$ with a bound for an ideal zeta function; what this note uses of it is Input 3. For a unit α of A with $\alpha \in O_A$ and $\bar{\alpha} = \alpha$, $\Gamma_{\mathcal{L}}(\alpha) \in \mathrm{Gau}(\mathcal{L})$ is the *gauge of depth* α of a standard level: x has size at most i exactly if $\alpha^i x \in O_A$, and $\Lambda(\Gamma_{\mathcal{L}}(\alpha)) = 2/\operatorname{mod}_A(\alpha)$. For a natural number K, $j \le \varrho$ and a sequence p of units, the *head weight* is $\Lambda_K(j;p) = \prod_{l<j}[p_0 \cdots p_l]^{K/2}$ if all the partial products $p_0 \cdots p_l$, $l < j$, lie in O_A, and $\Lambda_K(j;p) = 0$ otherwise; $\zeta_A(s) = \sum_I [O_A : I]^{-s}$ is the ideal zeta function of O_A, the sum over the ideals of finite index.

Setting 2.1 (a level)**.** A is a semilocal field with an involution and an integral place structure, $\mathfrak{s}$ is a shape, T is an $\mathfrak{s}$-form over A whose split Gram matrix $J(T)$ is a hermitian unit, ι is a finite set, m a natural number, and $\mathcal{L} \in \mathrm{Lev}(\iota, T, m+1)$ is a level.

Definition 2.2 (the finite Bessel function)**.** Let A be a semilocal field with an involution and an integral place structure, $\mathfrak{s}$ a shape and T an $\mathfrak{s}$-form over A whose split Gram matrix is a hermitian unit. For a unit g of A and $z \in G(T)$,

$$\chi_g(z) = \begin{cases} \psi_A(g^{-1}\,\mathrm{sd}(u)) & \text{if } u^{-1}z \in K_T(g) \text{ for some } u \in \mathcal{U}(T), \\ 0 & \text{if there is no such } u, \end{cases}$$

where in the first case u is a chosen such element; by Lemma 4.1(c) the value does not depend on the choice.

LEAN `IntegralPlace.phase`

Definition 2.3 (intermediate groups and the finite Bessel vector)**.** In Setting 2.1:

(a) $\mathrm{Int}(\mathcal{L})$ is the set of the subgroups Z of $G(T)$ with $\mathrm{K}^\circ(\mathcal{L}) \le Z \le K_T(1)$, the *intermediate groups* of $\mathcal{L}$.

(b) The *unipotent mass* is $\mathsf{v}(\mathcal{L}) = \prod_{l<\varrho} \mathsf{d}_l(\mathcal{L}; g_{\mathcal{L}}) \in [0, \infty]$.

(c) For a subgroup Z of $G(T)$,

$$\mathcal{I}_0(\mathcal{L}; Z) = \int \Big| \int_Z \chi_{g_{\mathcal{L}}}(yxy^{-1})\, d\mathsf{M}(\mathcal{L})(y) \Big|^2 d\mathsf{M}(\mathcal{L})(x) \ \in [0, \infty].$$

(d) For $Z \in \mathrm{Int}(\mathcal{L})$, $\mathrm{Cos}(Z)$ is the set of the cosets $\mathrm{K}^\circ(\mathcal{L})a$, $a \in Z$; it is finite, because $\mathrm{K}^\circ(\mathcal{L})$ is open and Z lies in the compact group $K_T(1)$. Each coset c has a chosen representative $a_c \in Z$.

(e) For $Z \in \mathrm{Int}(\mathcal{L})$ the *mass ratio* is $\mathrm{mr}(Z) = \mathsf{M}(\mathcal{L})(Z) \cdot \mathsf{M}(\mathcal{L})(\mathrm{K}^\circ(\mathcal{L})) \cdot \mathsf{v}(\mathcal{L})^{-2} \in [0, \infty]$.

(f) For $Z \in \mathrm{Int}(\mathcal{L})$ and $y \in G(T)$ the *finite Bessel vector* $W_Z(y) \in \ell^2(\mathrm{Cos}(Z))$ has the coordinate $\chi_{g_{\mathcal{L}}}(y\, a_c^{-1})$ at the coset c. We write $\mathsf{k}_Z(x, y) = \langle W_Z(x), W_Z(y) \rangle$ for its kernel.

LEAN `Level.IntermediateGroup` · `Level.unipotentMass` · `Level.floorZeroForm` · `Level.IntermediateGroup.Cosets` · `instance : Fintype Z.Cosets` (the finiteness, an instance without a name) · `Level.IntermediateGroup.massRatio` · `Level.IntermediateGroup.besselVector`

Setting 2.4 (a standard level with an intermediate group)**.** In Setting 2.1, with A, $\mathfrak{s}$, T, ι, m and $\mathcal{L}$ as there: $\mathcal{L}$ is a standard level, $\mathcal{L} \in \mathrm{StdLev}$, and $Z \in \mathrm{Int}(\mathcal{L})$ is an intermediate group of $\mathcal{L}$.

Definition 2.5 (correlations, the top kernel, the energy)**.** In Setting 2.4, write $\varrho = \varrho(\mathfrak{s})$.

(a) For a sequence p of units of A and $l \in G(T)$, the *correlation* is

$$\mathcal{J}(Z;p,l) = \mathsf{h}_{\varrho}(\mathcal{L};p,1)\cdot \mathsf{J}(\mathsf{M}(\mathcal{L}), W_Z(\cdot); w_{\varrho}(p)\,l^{-1}) \;\in\; \ell^2(\mathrm{Cos}(Z)\times\mathrm{Cos}(Z)).$$

(b) For a set $L \subseteq A$ and sequences p, q of units of A, the *top kernel* is

$$\mathcal{K}(L;Z;p,q) = \int_{\{l:\ h(z,l^{-1}z)-p_{\varrho}\in L\}} \langle \mathcal{J}(Z;p,l), \mathcal{J}(Z;q,l)\rangle \, d\mu^{\mathcal{L}}_{\varrho,2}(l) \;\in \mathbb{C}.$$

(c) For a finite set $\mathcal{C}$, a complex matrix R indexed by $\mathcal{C}$, a family $\omega = (\omega_i)_{i\in\mathcal{C}}$ of sequences of units of A and a vector $e \in A^{I(\mathfrak{s})}$, the *energy* is

$$\mathcal{E}(Z;R,\omega,e) = \inf_{l} \int \sum_{x\in \mathrm{Cos}(Z)\times\mathrm{Cos}(Z)} (\mathrm{Re}\ v^* R v)^{+}\, d\theta_n(\mathcal{L})(h) \;\in [0,\infty],$$

where the infimum is over the $l \in \mathcal{H}_{\varrho}(T)$ with $l^{-1}z = e$ (it is ∞ if there is none), v is the vector with the coordinates $v_i = \mathcal{J}(Z;\omega_i,hl)_x$, the index i running over $\mathcal{C}$, and $r^+ = \max(r,0)$. (The index, called p in the formal text, is bound inside the definition; see §8, item 1.)

LEAN `Level.correlation` · `Level.topKernel` · `Level.energy`

3 What is taken for granted

The following statements are used and not proved here. Inputs 1 and 5 belong to the background library, Inputs 2 and 4 to the unit on the real place (the notes *The tower method at a place*), Input 3 to the note *Poincaré series of cut Bessel functions* of this unit. They are stated from the dossier's readings of their quoted Lean statements; the statements of Inputs 2 to 4 I also read in the Lean text. No reference to the literature is claimed for any of them: they are statements about the objects of this proof.

Input 1 (the Bessel subgroup and the congruence subgroups; background library). Let A, $\mathfrak{s}$, T be as in the conventions and g a unit of A.

(a) $\mathcal{U}(T) \subseteq \mathcal{S}(T)$, and $\mathcal{S}(T)$ normalises $\mathcal{U}(T)$. For $s, s' \in \mathcal{S}(T)$: $\mathrm{sd}(ss') = \mathrm{sd}(s) + \mathrm{sd}(s')$; and $\mathrm{sd}(h) = 0$ for $h \in \mathcal{H}(T)$.

(b) $K_T(g)$ is an open compact subgroup of $G(T)$, contained in $K_T(1)$ and normalised by it; for $k \in K_T(g)$, $g^{-1}\,\mathrm{sd}(k) \in O_A$.

(c) If $J(T)$ and its inverse have entries in O_A and $g \in O_A$, then every $x \in G(T)$ for which $g^{-1}(x-1)$ has entries in O_A lies in $K_T(g)$.

(d) For a sequence p of units and $l \le \varrho$, $w_l(p)$ commutes with the elements of $\mathcal{H}_l(T)$. $\mathcal{H}(T) \subseteq \mathcal{H}_{\varrho}(T)$; if $\sigma, \tau \in \mathcal{H}_{\varrho}(T)$ and $\sigma^{-1}z = \tau^{-1}z$, then $\sigma\tau^{-1} \in \mathcal{H}(T)$; and $Z_T(hxh')_{\varrho\varrho} = Z_T(x)_{\varrho\varrho}$ for $h, h' \in \mathcal{H}(T)$ and $x \in G(T)$.

(e) For $u \in \mathcal{U}(T)$, $a < \varrho$ and a vector w whose coordinates at e^*_b, $b < a$, vanish, the coordinates of uw and of w at e^*_a agree.

Input 2 (the tower; unit on the real place). In Setting 2.1, write $\varrho = \varrho(\mathfrak{s})$.

(a) *(Measures.)* $\mathsf{M}(\mathcal{L}) = \eta_{<\varrho+2} * \mathsf{M}_{\geq 0}(\mathcal{L})$; $\mathsf{M}(\mathcal{L}) = \mu^{\mathcal{L}}_{\varrho,2} * \mu^{\mathcal{L}}_{0,\varrho}$; $\mu^{\mathcal{L}}_{\varrho,2} = \theta_n(\mathcal{L}) * \theta_\varrho(\mathcal{L})$, and this measure is invariant under right translation by the elements of $\mathcal{H}_\varrho(T)$; $\mu^{\mathcal{L}}_{l,i}$ is carried by $\mathcal{H}_l(T)$, and $\mu^{\mathcal{L}}_{0,0}$ is the unit mass at 1.

(b) *(Pairing kernels.)* Let W be a cut Bessel function for $\mathcal{L}$, $k = k^1_{W,W}$, and suppose that $\mathcal{L}$ has a gauge. Then $|\mathsf{Av}_j(\mathcal{L}; k)(g, y)| \leq \mathsf{Pv}_j(\mathcal{L}; k)(g, y)$ for $j \leq \varrho$ and $g, y \in G(T)$; there is $c < \infty$ with $\int |k(\sigma g, \sigma y)|\, d\mu^{\mathcal{L}}_{\varrho,2}(\sigma) \leq c$ for all g, y; and for every function k', if $\mathsf{Pv}_j(\mathcal{L}; k')(g, y) \neq 0$ then $k'(\sigma g, \sigma y) \neq 0$ for some $\sigma \in \mathcal{H}_j(T)$.

(c) *(Forms.)* For $j \leq \varrho$, sequences p, q of units, W and ω: $|\mathcal{I}_j(\mathcal{L}; p, q; W, \omega)| \leq \mathsf{h}_j(\mathcal{L}; p, q) \cdot \int \mathcal{I}^{\mathrm{mod}}_j(\mathcal{L}; p, q; W, x, \omega)\, d\mathsf{M}(\mathcal{L})(x)$; and if W is a cut Bessel function for $\mathcal{L}$ and $|\omega| \leq 1$, then $\mathcal{I}^{\mathrm{mod}}_j(\mathcal{L}; p, q; W, x, \omega) \leq \mathcal{I}^{\mathrm{maj}}_j(\mathcal{L}; p, q; W, W, x)$ for all x.

(d) *(Near points.)* If W is a cut Bessel function for $\mathcal{L}$ and $W(x) \neq 0$, then x is near at every j; if x is near at j and $\sigma \in \mathcal{H}_j(T)$, then σx is near at j. If Γ is a gauge of $\mathcal{L}$, $j \leq \varrho$, p is a sequence of units, and x and $w_j(p)^{-1}x$ are near at j, then $\mathrm{sz}_\Gamma(p_0 \cdots p_{l-1}) \leq 9n(\mathfrak{s})$ for all $l \leq j$.

(e) *(Correlations.)* Let Γ be a gauge of $\mathcal{L}$, W a continuous cut Bessel function for $\mathcal{L}$, p, q heads of length ϱ for Γ, $x \in \mathcal{H}_\varrho(T)$ and μ a Haar measure of $G(T)$. Then

$$\begin{aligned}&\langle \mathrm{J}(\mu, W; w_\varrho(p)x), \mathrm{J}(\mu, W; w_\varrho(q)x)\rangle \\ &\quad = \int\int \langle W(y_1), W(y_2)\rangle\, \overline{\langle W((w_\varrho(p)x)^{-1}y_1), W((w_\varrho(q)x)^{-1}y_2)\rangle}\, d\mu\, d\mu,\end{aligned}$$

the integral over (y_1, y_2). If $W(y) \neq 0$ and $W((w_\varrho(p)x)^{-1}y) \neq 0$, then $\mathrm{sz}_\Gamma(y) \leq 60n(\mathfrak{s})^2$; and for every i the set of the $y \in G(T)$ with $\mathrm{sz}_\Gamma(y) \leq i$ is compact.

Input 3 (standard levels; note *Poincaré series of cut Bessel functions* of this unit). In Setting 2.1, suppose $\mathcal{L} \in \mathrm{StdLev}$, and write $\varrho = \varrho(\mathfrak{s})$, $n = n(\mathfrak{s})$.

(a) $J(T)$ and its inverse have entries in O_A; $g_{\mathcal{L}} \in O_A$; $\mathrm{Box}(\mathcal{L}) = K_T(1)$; the ball is $O_A^{I(\mathfrak{s})}$; $\mathrm{Posn}(\mathcal{L}) = \{y^{-1}uy : \ y \in G(T),\ u \in \mathcal{U}(T)\} \cdot \mathrm{K}^\circ(\mathcal{L})$; and there is a vector u_0 with $h(u_0, w) = w_z$ for all vectors w and with z-coordinate 0.

(b) $\zeta_A(m)^n \le 1 + 2^{-n}$, hence $\zeta_A(m) \le 2$; $m < 9n$; $13n \le \#\iota$; and $k(\mathfrak{s}) = \#\iota + 2m + 3$.

(c) $\mathsf{M}(\mathcal{L})$ is a Haar measure of $G(T)$, invariant under left and right translations; and $\mathrm{vol}(O_A^\times)^{i+1} \le \mu^{\mathcal{L}}_{l,i+2}(K_T(1))$ whenever $l + i = \varrho$.

(d) *(Unfolding.)* For a unit g of A with $g \in O_A$ and $\bar{g} = g$, an s-finite measure θ on $G(T)$ and $c \in G(T)$:

$$\begin{aligned}&(\eta_{<\varrho+2} * \theta)(\{y : \ yc \in K_T(g)\}) \\ &\qquad = \prod_{l<\varrho} \mathsf{d}_l(\mathcal{L}; g) \cdot \theta(\{y : \ uyc \in K_T(g) \text{ for some } u \in \mathcal{U}(T)\}).\end{aligned}$$

(e) *(Gauges.)* For a unit $\alpha \in O_A$ with $\bar{\alpha} = \alpha$, $\Gamma_{\mathcal{L}}(\alpha)$ is a gauge of $\mathcal{L}$, as described in §2; in every gauge Γ of $\mathcal{L}$ the elements of the box and of the ball have size at most 1, the sizes add under multiplication, and $\mathrm{mod}_A(u) \le \Lambda(\Gamma)^i$ for a unit u with $\mathrm{sz}_\Gamma(u) \le i$.

Input 4 (the estimates of the tower method; unit on the real place). In Setting 2.1, suppose $\mathcal{L} \in \mathrm{StdLev}$; let W be a cut Bessel function for $\mathcal{L}$, $k = k^{1}_{W,W}$, $j \le \varrho(\mathfrak{s})$, p, q sequences of units, Γ a gauge of $\mathcal{L}$, and write $n = n(\mathfrak{s})$, $\varrho = \varrho(\mathfrak{s})$.

(a) *(Positions.)* Let q be a head of length j for Γ, Γ' a second gauge, S a set of positions x with $\mathrm{sz}_{\Gamma'}(g_{\mathcal{L}}^{-1}(x-1)) \le 1$, and $f\colon G(T) \to [0,\infty]$, $\epsilon \in [0,\infty]$ with $f(x)^2 \le \epsilon^2 \cdot \mathrm{SNP}_j(\Gamma; p, q; W, W, x)$ on S. Then

$$\mathsf{h}_j(\mathcal{L}; p, q) \int_S f \, d\mathsf{M}(\mathcal{L}) \;\le\; \epsilon \cdot \mathsf{v}(\mathcal{L}) \cdot \mathrm{PN}(\mathcal{L}; W, W) \cdot \Lambda(\Gamma)^{N_1} \cdot \Lambda(\Gamma')^{N_2},$$

where $N_1 = 1500000n^3 + 1$ and $N_2 = 50000n^2$.

(b) *(The majorant.)* Let p be a head of length j for Γ, x a position and $\phi_0 \in [0,\infty]$. Suppose that, if $\phi_0 < 1$, for every y with $\langle W(y), W(yx)\rangle \neq 0$ such that $w_j(p)^{-1}y$ and $w_j(q)^{-1}yx$ are near at j there is a with $j \le a < \varrho$ and $\mathrm{vol}(\mathrm{Val}_a(\mathcal{L}; y, x)) \le \phi_0$. Then $\mathcal{J}^{\mathrm{maj}}_j(\mathcal{L}; p, q; W, W, x)^2 \le \phi_0^{\mathfrak{s}(\mathcal{L})} \cdot V_0^2 \cdot \mathrm{SNP}_j(\Gamma; p, q; W, W, x)$ with $V_0 = 2^{99 \cdot 13\, n}$, the floor-mass constant of the gauge of depth 1.

(c) *(Frequencies.)* Let p be a head of length j for Γ and x a position. If η, ξ are units with $\psi_A(\xi c) = \psi_A(\eta c)$ for all c with $\mathrm{sz}_\Gamma(c) \le 200n^2$, and $\omega(y) = \psi_A(\xi \cdot Z_T(y)_{jj})$, then $\mathcal{J}^{\mathrm{mod}}_j(\mathcal{L}; p, q; W, x, \omega)^2 \le [\eta]^{\#\iota} \cdot \mathrm{SNP}_j(\Gamma; p, q; W, W, x)$. If $j = i+1$ and $\xi = g_{\mathcal{L}}^{-1}(p_i^{-1} - q_i^{-1})$ is a unit, then $\mathcal{J}^{\mathrm{mod}}_j(\mathcal{L}; p, q; W, x, 1)^2 \le [\xi]^{\#\iota} \cdot \mathrm{SNP}_j(\Gamma; p, q; W, W, x)$.

(d) *(Integrality from floor values.)* Let p, q be heads of length j for Γ and Γ'' a second gauge such that, if $j \neq 0$, $\mathrm{sz}_{\Gamma''}(g_{\mathcal{L}}^{-1}c) \le 1$ for every c with $\mathrm{sz}_\Gamma(c) \le 10n$. Let $x, y \in G(T)$ be such that: there are $u, v \in \mathcal{U}(T)$ with ux, $vxy \in \mathrm{Box}(\mathcal{L})$ and $\mathrm{sz}_{\Gamma''}(g_{\mathcal{L}}^{-1}(vxy - ux)) \le 2$; $w_j(p)^{-1}x$ and $w_j(q)^{-1}xy$ are near at j; and for every a with $j \le a < \varrho$ there is a unit $c \in \mathrm{Val}_a(\mathcal{L}; x, y)$ with $\mathrm{sz}_{\Gamma''}(c^{-1}) \le 1$. Then $\mathrm{sz}_{\Gamma''}(g_{\mathcal{L}}^{-1}(y-1)) \le 99n$.

(e) *(The floor 0.)* $\mathsf{h}_0(\mathcal{L}; p, q) = 1$, $\mathcal{J}^{\mathrm{maj}}_0(\mathcal{L}; 1, 1; W, W, x) = \mathsf{Pv}_0(\mathcal{L}; k)(1, x)^2$, and there is a number $\Sigma_0(x) \in [0,\infty]$ with $\mathrm{SNP}_0(\Gamma; 1, 1; W, W, x) = \Lambda(\Gamma)^{3 \cdot 10^6 n^3} \cdot \Sigma_0(x)^2$ such that, for a position x and $\varepsilon \in [0,\infty]$: if for every $\sigma \in G(T)$ with $k(\sigma, \sigma x) \neq 0$ there is $a < \varrho$ with $\mathrm{vol}(\mathrm{Val}_a(\mathcal{L}; \sigma, x)) \le \varepsilon$, then $\mathsf{Pv}_0(\mathcal{L}; k)(1, x)^2 \le \varepsilon^{\mathfrak{s}(\mathcal{L})} \cdot \Lambda(\Gamma)^{10^4 n\varrho} \cdot \Sigma_0(x)$.

Input 5 (ideals, valuations, Eichler elements; background library). Let A be as in the conventions.

(a) *(Summation over ideals.)* Let s be a natural number with $\zeta_A(s) \le 2$, ρ a measure on a measurable space, S a subset of it, f a function with values in $[0,\infty]$, $V \in [0,\infty]$, and $I \mapsto S(I)$ a family of subsets indexed by the ideals of O_A. If for almost every $x \in S$ there is an ideal I of finite index with $x \in S(I)$ and $[O_A : I]^s \cdot f(x) \cdot \rho(S(I)) \le V$, then $\int_S f\, d\rho \le \zeta_A(s) \cdot V$.

(b) *(Content.)* For a finite family (d_i) of elements of A one of which is 1, there is a unit c of A with $c \in O_A$ such that $\{\varepsilon \in A : \ \varepsilon d_i \in O_A \text{ for all } i\} = cO_A$; this is an ideal of O_A of index $[c]$, and $\bar{c} \in O_A$.

(c) *(Greatest common divisors.)* For units ξ, β of A there are a unit η and $x, c \in O_A$ with $\eta = \xi + c\beta$, $\xi = x\eta$ and $\beta \in \eta O_A$.

(d) *(Primitive matrices.)* If a matrix X over A, with columns indexed by a finite set with a weight function, has a left inverse, there is an invertible matrix b, block upper triangular for the weights, such that Xb has entries in O_A and a left inverse with entries in O_A.

(e) *(Volumes.)* For a unit u, $\mathrm{vol}(\{x : \ ux \in O_A\}) = \mathrm{mod}_A(u)^{-1}$; for a unit $u \in O_A$ the ideal uO_A has index $[u]$; and $\mathrm{vol}(O_A \cap \xi^{-1} O_A) = \mathrm{den}(\xi)^{-1}$. Almost every element of A is a unit.

(f) *(Eichler elements.)* In Setting 2.1, let u_0 be a vector with $h(u_0, w) = w_z$ for all w and with z-coordinate 0. For every vector v there is $\mathrm{E}(v) \in G(T)$, given by a formula that is polynomial in the coordinates of v and in the ring inverse of v_z, with $\mathrm{E}(z) = 1$, such that $\mathrm{E}(v) \in \mathcal{H}_\varrho(T)$ if the coordinates of v at all e_a and e_a^* vanish, and $\mathrm{E}(v)^{-1} z = v$ if $h(v,v) = h(z,z)$ and v_z is a unit.

4 The finite Bessel function and its vector

Lemma 4.1 (the finite Bessel function)**.** *Let A be a semilocal field with an involution and an integral place structure, $\mathfrak{s}$ a shape, T an $\mathfrak{s}$-form over A, g a unit of A, and $x, y, u, k \in G(T)$.*

(a) *If $\chi_g(y) \neq 0$, there is $u' \in \mathcal{U}(T)$ with $u'^{-1} y \in K_T(g)$.*

(b) *$|\chi_g(y)| \le 1$.*

(c) *If $u \in \mathcal{U}(T)$ and $u^{-1} y \in K_T(g)$, then $\chi_g(y) = \psi_A(g^{-1} \,\mathrm{sd}(u))$.*

(d) *If $x \in \mathcal{S}(T)$, $u \in \mathcal{U}(T)$ and $u^{-1} x \in K_T(g)$, then $\chi_g(xy) = \psi_A(g^{-1} \,\mathrm{sd}(u)) \cdot \chi_g(y)$.*

(e) *If $k \in K_T(g)$, then $\chi_g(yk) = \chi_g(y)$.*

(f) *If $u \in \mathcal{U}(T)$, then $\chi_g(uy) = \psi_A(g^{-1} \,\mathrm{sd}(u)) \cdot \chi_g(y)$.*

(g) *The function $z \mapsto \chi_g(z)$ on $G(T)$ is locally constant, hence continuous.*

(h) *$K_T(g) \le K_T(1)$.*

Proof. (a) and (b) are read off Definition 2.2, as ψ_A is unitary. (h) is part of the definition of $K_T(g)$.

(c) Let u_0 be the chosen element, so that $u_0^{-1} y \in K_T(g)$. Then $u_0^{-1} u = (u_0^{-1} y)(u^{-1} y)^{-1}$ lies in $K_T(g)$, so $g^{-1} \,\mathrm{sd}(u_0^{-1} u) \in O_A$ by Input 3(b), and $\psi_A(g^{-1} \,\mathrm{sd}(u_0^{-1} u)) = 1$ because

ψ_A is self-dual. As $\mathrm{sd}(u) = \mathrm{sd}(u_0) + \mathrm{sd}(u_0^{-1}u)$ by Input 3(a), the two values of the character agree.

(d) Suppose first that $u_0^{-1}y \in K_T(g)$ for some $u_0 \in \mathcal{U}(T)$. Then $v = xu_0x^{-1}u$ lies in $\mathcal{U}(T)$, because x normalises it, and $v^{-1}xy = (u^{-1}x)(u_0^{-1}y) \in K_T(g)$. By (c), $\chi_g(xy) = \psi_A(g^{-1}\,\mathrm{sd}(v))$, and $\mathrm{sd}(v) = \mathrm{sd}(x)+\mathrm{sd}(u_0)+\mathrm{sd}(x^{-1})+\mathrm{sd}(u) = \mathrm{sd}(u_0)+\mathrm{sd}(u)$ by additivity. With $\chi_g(y) = \psi_A(g^{-1}\,\mathrm{sd}(u_0))$ this is the claim. If there is no such u_0, then $\chi_g(y) = 0$, and $\chi_g(xy) = 0$ as well: from $v^{-1}xy \in K_T(g)$ with $v \in \mathcal{U}(T)$ one gets $u_0 = x^{-1}vu^{-1}x \in \mathcal{U}(T)$ with $u_0^{-1}y = (u^{-1}x)^{-1}(v^{-1}xy) \in K_T(g)$.

(e) The condition $u^{-1}yk \in K_T(g)$ is the condition $u^{-1}y \in K_T(g)$; so both sides vanish together, and otherwise they are equal by (c). (f) is (d) for $x = u$. (g) By (e) the function is constant on the cosets of the open subgroup $K_T(g)$ (Input 3(b)). □

LEAN `IntegralPlace.exists_besselU_of_phase_ne_zero` · `IntegralPlace.norm_phase_le_one` · `IntegralPlace.phase_eq` · `IntegralPlace.phase_besselS_mul` · `IntegralPlace.phase_mul_of_mem_unitaryLevel` · `IntegralPlace.phase_besselU_mul` · `IntegralPlace.continuous_phase` · `IntegralPlace.unitaryLevel_le_unitaryLevel_one`

Lemma 4.2 (intermediate groups)**.** *In Setting 2.1:*

(a) $\mathsf{v}(\mathcal{L}) \neq 0$ *and* $\mathsf{v}(\mathcal{L}) \neq \infty$.
(b) *Every* $Z \in \mathrm{Int}(\mathcal{L})$ *is an open subgroup of* $G(T)$.
(c) *For* $Z \in \mathrm{Int}(\mathcal{L})$ *and* $y \in Z$ *there is a coset* $d \in \mathrm{Cos}(Z)$ *such that, for every* $c \in \mathrm{Cos}(Z)$, $y\,a_c^{-1} \in \mathrm{K}^\circ(\mathcal{L})$ *holds exactly if* $c = d$.
(d) *For every unit* u *of* A, $\mathrm{mod}_A(u)^{-1} = \left(\sqrt{[u]}\right)^2$.

Suppose moreover that $\mathcal{L}$ *is a standard level,* $\mathcal{L} \in \mathrm{StdLev}$.

(e) *If* S *is a subgroup of* $G(T)$ *contained in* $K_T(1)$, *then* $\mathsf{M}(\mathcal{L})(S) \neq \infty$.
(f) *For every* $Z \in \mathrm{Int}(\mathcal{L})$: $\mathrm{mr}(Z) \neq 0$ *and* $\mathrm{mr}(Z) \neq \infty$.

Proof. (a) Each floor modulus is a product of powers of $\mathrm{mod}_A(g_{\mathcal{L}})$, the modulus of a unit, which is neither 0 nor ∞. (b) Z contains the open subgroup $\mathrm{K}^\circ(\mathcal{L})$ (Input 3(b)). (c) Take for d the coset of y: $y\,a_c^{-1} \in \mathrm{K}^\circ(\mathcal{L})$ says that y and a_c lie in the same coset. (d) is the definition of the generalised index. (e) $\mathsf{M}(\mathcal{L})$ is a Haar measure (Input 3(c)) and $K_T(1)$ is compact. (f) Z and $\mathrm{K}^\circ(\mathcal{L})$ are open, hence of positive measure, and of finite measure by (e); with (a) this gives the claim. □

LEAN `Level.unipotentMass_ne_zero_and_ne_top` · `Level.IntermediateGroup.isOpen` · `Level.IntermediateGroup.exists_coset` · `IntegralPlace.inv_md_eq_sqrt_genIndex_sq` · `IntegralPlace.IsStdLevel.groupMeasure_ne_top` · `IntegralPlace.IsStdLevel.massRatio_ne_zero_and_ne_top`

Lemma 4.3 (non-orthogonal Bessel vectors)**.** *In Setting 2.1, let* $Z \in \mathrm{Int}(\mathcal{L})$ *and* $x, y \in G(T)$ *with* $\langle W_Z(x), W_Z(y)\rangle \neq 0$. *Then there are* $u, v \in \mathcal{U}(T)$ *with* $ux \in K_T(1)$ *and* $(ux)^{-1}(vy) \in \mathrm{K}^\circ(\mathcal{L})$.

Proof. For some coset c both $\chi_{g_{\mathcal{L}}}(x\,a_c^{-1})$ and $\chi_{g_{\mathcal{L}}}(y\,a_c^{-1})$ are not zero. By Lemma 4.1(a) there are $u, v \in \mathcal{U}(T)$ such that $k_1 = ux\,a_c^{-1}$ and $k_2 = vy\,a_c^{-1}$ lie in $\mathrm{K}^\circ(\mathcal{L})$. Then $ux = k_1a_c \in K_T(1)$, as $a_c \in Z \leq K_T(1)$, and $(ux)^{-1}(vy) = a_c^{-1}(k_1^{-1}k_2)a_c \in \mathrm{K}^\circ(\mathcal{L})$, because $K_T(1)$ normalises $\mathrm{K}^\circ(\mathcal{L})$ (Input 3(b)). □

LEAN `Level.exists_of_inner_besselVector_ne_zero`

Lemma 4.4 (the form on the congruence subgroup)**.** *In Setting 2.4, with x, y the variables of integration and $g = g_{\mathcal{L}}$,*

$$\int_{\mathrm{K}^\circ(\mathcal{L})} \Big| \int_Z \chi_g(yxy^{-1})\, d\mathsf{M}(\mathcal{L})(y) \Big|^2 d\mathsf{M}(\mathcal{L})(x) = \mathsf{M}(\mathcal{L})(Z)^2 \cdot \mathsf{M}(\mathcal{L})(\mathrm{K}^\circ(\mathcal{L})).$$

Proof. For $x \in \mathrm{K}^\circ(\mathcal{L})$ and $y \in Z \le K_T(1)$ the conjugate yxy^{-1} lies in $\mathrm{K}^\circ(\mathcal{L})$, so $\chi_g(yxy^{-1}) = \psi_A(g^{-1}\,\mathrm{sd}(1)) = 1$ by Lemma 4.1(c) with $u = 1$. The inner integral is $\mathsf{M}(\mathcal{L})(Z)$, which is finite by Lemma 4.2(e). □

LEAN `IntegralPlace.IsStdLevel.floorZeroForm_on_congGroup`

Lemma 4.5 (the finite Bessel vector is cut)**.** *In Setting 2.4, the function $y \mapsto W_Z(y)$ is a continuous cut Bessel function for $\mathcal{L}$.*

Proof. Its coordinates are continuous by Lemma 4.1(g) and of modulus at most 1, so its norm is at most $\sqrt{\#\,\mathrm{Cos}(Z)}$. The rule $W_Z(uy) = \psi_A(g_{\mathcal{L}}^{-1}\,\mathrm{sd}(u))W_Z(y)$ for $u \in \mathcal{U}(T)$ is Lemma 4.1(f) in each coordinate. If $W_Z(x) \neq 0$, then $\langle W_Z(x), W_Z(x)\rangle \neq 0$, and Lemma 4.3 gives $u \in \mathcal{U}(T)$ with $ux \in K_T(1) = \mathrm{Box}(\mathcal{L})$ (Input 3(a)). □

LEAN `IntegralPlace.IsStdLevel.isCut_besselVector`

Proposition 4.6 (unfolding along the unipotent subgroup)**.** *In Setting 2.4, for all $y, x \in G(T)$, with $g = g_{\mathcal{L}}$ and u the variable of integration,*

$$\mathsf{v}(\mathcal{L}) \cdot \langle W_Z(y), W_Z(yx)\rangle = \int \mathbf{1}_Z(uy) \cdot \chi_g(uyx(uy)^{-1})\, d\eta_{<\varrho+2}(u).$$

Proof. The measure $\eta_{<\varrho+2}$ is carried by $\mathcal{U}(T)$; let $u \in \mathcal{U}(T)$. We claim that the integrand at u equals

$$\sum_{c \in \mathrm{Cos}(Z)} \mathbf{1}[uy\,a_c^{-1} \in \mathrm{K}^\circ(\mathcal{L})] \cdot \overline{W_Z(y)_c} \cdot W_Z(yx)_c.$$

If $uy \notin Z$, both vanish, since $uy\,a_c^{-1} \in \mathrm{K}^\circ(\mathcal{L})$ would give $uy \in \mathrm{K}^\circ(\mathcal{L})a_c \subseteq Z$. If $uy \in Z$, by Lemma 4.2(c) exactly one coset d has $k = uy\,a_d^{-1} \in \mathrm{K}^\circ(\mathcal{L})$, and the sum is $\overline{\chi_g(y\,a_d^{-1})} \cdot \chi_g(yx\,a_d^{-1})$. Now $y\,a_d^{-1} = u^{-1}k$, so $\chi_g(y\,a_d^{-1}) = \psi_A(g^{-1}\,\mathrm{sd}(u^{-1})) = \overline{\psi_A(g^{-1}\,\mathrm{sd}(u))}$ by Lemma 4.1(c) and Input 3(a); and $uyx(uy)^{-1} = u \cdot (yx\,a_d^{-1}) \cdot k^{-1}$, so $\chi_g(uyx(uy)^{-1}) = \psi_A(g^{-1}\,\mathrm{sd}(u)) \cdot \chi_g(yx\,a_d^{-1})$ by Lemma 4.1(e), (f). This proves the claim. Integrating it, the coset c contributes $\eta_{<\varrho+2}(\{u : uy\,a_c^{-1} \in \mathrm{K}^\circ(\mathcal{L})\}) \cdot \overline{W_Z(y)_c}\, W_Z(yx)_c$. By the unfolding of Input 3(d) for the unit mass at 1, this measure is $\mathsf{v}(\mathcal{L})$ if $u'y\,a_c^{-1} \in \mathrm{K}^\circ(\mathcal{L})$ for some $u' \in \mathcal{U}(T)$, which is the case when $W_Z(y)_c \neq 0$ (Lemma 4.1(a)); and when $W_Z(y)_c = 0$ the term vanishes. The sum over c is $\mathsf{v}(\mathcal{L})\langle W_Z(y), W_Z(yx)\rangle$. □

LEAN `IntegralPlace.IsStdLevel.unipotentMass_mul_inner_besselVector`

Proposition 4.7 (the pairing norm)**.** *In Setting 2.4,* $\mathrm{PN}(\mathcal{L}; W_Z(\cdot), W_Z(\cdot)) \cdot \mathsf{v}(\mathcal{L}) \le \mathrm{mr}(Z)$.

Proof. Write $W = W_Z(\cdot)$, $\mathsf{v} = \mathsf{v}(\mathcal{L})$, and $\nu(x) = \eta_{<\varrho+2}(\{u : ux \in Z\})$ for $x \in G(T)$. By Proposition 4.6 (for the pair x, $x^{-1}y$) and Lemma 4.1(b),

$$\mathsf{v} \cdot |\langle W(x), W(y)\rangle| \le \nu(x) \qquad (x, y \in G(T)). \tag{1}$$

Moreover $\nu(x) \le 1$: as $Z \le K_T(1)$, Input 3(d) for the unit 1 and the unit mass at 1 bounds $\nu(x)$ by $\prod_{l<\varrho} \mathsf{d}_l(\mathcal{L};1) = 1$. Since $\mathsf{M}(\mathcal{L}) = \eta_{<\varrho+2} * \mathsf{M}_{\ge 0}(\mathcal{L})$ (Input 3(a)), $\mathsf{M}(\mathcal{L})(Z) = \int \nu(x)\, d\mathsf{M}_{\ge 0}(\mathcal{L})(x)$. So it suffices to show, for every x,

$$\mathsf{v}^3 \int |\langle W(x), W(y)\rangle|^2 \, d\mathsf{M}_{\ge 0}(\mathcal{L})(y) \;\le\; \nu(x) \cdot \mathsf{M}(\mathcal{L})(\mathrm{K}^\circ(\mathcal{L})), \tag{2}$$

and to integrate over x. If $W(x) = 0$ this is clear. Otherwise, as W is cut (Lemma 4.5), there is $u_0 \in \mathcal{U}(T)$ with $b = u_0 x \in K_T(1)$. By the right invariance of $\mathsf{M}(\mathcal{L})$ and Input 3(d) for $\theta = \mathsf{M}_{\ge 0}(\mathcal{L})$ and $c = b^{-1}$,

$$\mathsf{M}(\mathcal{L})(\mathrm{K}^\circ(\mathcal{L})) = \mathsf{M}(\mathcal{L})(\{y : \; yb^{-1} \in \mathrm{K}^\circ(\mathcal{L})\}) = \mathsf{v} \cdot \mathsf{M}_{\ge 0}(\mathcal{L})(Y_x),$$
$$Y_x = \{y : \; uyb^{-1} \in \mathrm{K}^\circ(\mathcal{L}) \text{ for some } u \in \mathcal{U}(T)\}.$$

For $y \in Y_x$, (1) and $\nu(x) \le 1$ give $\mathsf{v}^2 |\langle W(x), W(y)\rangle|^2 \le \nu(x)^2 \le \nu(x)$. For $y \notin Y_x$ the inner product vanishes: otherwise Lemma 4.3 gives $u', v' \in \mathcal{U}(T)$ with $u'x \in K_T(1)$ and $v'y \in u'x\,\mathrm{K}^\circ(\mathcal{L})$; then $v'yb^{-1} \in u'u_0^{-1} \cdot b\,\mathrm{K}^\circ(\mathcal{L})\,b^{-1} = u'u_0^{-1}\mathrm{K}^\circ(\mathcal{L})$, since b normalises $\mathrm{K}^\circ(\mathcal{L})$, and $u = u_0 u'^{-1} v'$ shows $y \in Y_x$. Hence the left side of (2) is at most $\mathsf{v} \cdot \nu(x) \cdot \mathsf{M}_{\ge 0}(\mathcal{L})(Y_x) = \nu(x)\mathsf{M}(\mathcal{L})(\mathrm{K}^\circ(\mathcal{L}))$. Integrating (2) over x gives $\mathsf{v}^3 \cdot \mathrm{PN}(\mathcal{L}; W, W) \le \mathsf{M}(\mathcal{L})(Z)\, \mathsf{M}(\mathcal{L})(\mathrm{K}^\circ(\mathcal{L}))$, which is the claim after division by v^2 (Lemma 4.2(a)). □

LEAN `IntegralPlace.IsStdLevel.pairingNorm_mul_le_massRatio`

Lemma 4.8 (the inner integral as an average)**.** *In Setting 2.4, for every $x \in G(T)$, with y the variable of integration and $g = g_{\mathcal{L}}$,*

$$\Big| \int_Z \chi_g(yxy^{-1})\, d\mathsf{M}(\mathcal{L})(y) \Big| = \big|\mathsf{Av}_0(\mathcal{L}; \mathsf{k}_Z)(1, x)\big| \cdot \mathsf{v}(\mathcal{L}).$$

Proof. The integrand is bounded on Z, which has finite measure. By $\mathsf{M}(\mathcal{L}) = \eta_{<\varrho+2} * \mathsf{M}_{\ge 0}(\mathcal{L})$ and Fubini's theorem the integral is

$$\int \int \mathbf{1}_Z(u\sigma)\, \chi_g(u\sigma x (u\sigma)^{-1})\, d\eta_{<\varrho+2}(u)\, d\mathsf{M}_{\ge 0}(\mathcal{L})(\sigma),$$

and by Proposition 4.6 the inner integral is $\mathsf{v}(\mathcal{L}) \cdot \mathsf{k}_Z(\sigma, \sigma x)$. The integral of $\mathsf{k}_Z(\sigma, \sigma x)$ over σ is $\mathsf{Av}_0(\mathcal{L}; \mathsf{k}_Z)(1, x)$ by definition. □

LEAN `IntegralPlace.IsStdLevel.enorm_integral_phase_eq_upperPairing`

Lemma 4.9 (a lower bound for the mass of the integral group)**.** *In Setting 2.1, suppose $\mathcal{L} \in \mathrm{StdLev}$, and let $B \in [0, \infty]$ with $2 \le B$ and $\mathrm{vol}(O_A \setminus O_A^\times) \le B^{-1}$. Then $2^{-n(\mathfrak{s})} \le \mathsf{M}(\mathcal{L})(K_T(1))$.*

Proof. By Input 3(c) for $l = 0$, $i = \varrho$: $\mathrm{vol}(O_A^\times)^{\varrho+1} \le \mathsf{M}(\mathcal{L})(K_T(1))$; and $\mathrm{vol}(O_A^\times) = 1 - \mathrm{vol}(O_A \setminus O_A^\times) \ge 1 - B^{-1} \ge \frac{1}{2}$. □

LEAN `IntegralPlace.IsStdLevel.inv_two_pow_le_groupMeasure`

5 The shell bound, the frequency bound and the bottom value

In this section $n = n(\mathfrak{s})$ and $\varrho = \varrho(\mathfrak{s})$; in the proofs $g = g_{\mathcal{L}}$ and $W = W_Z(\cdot)$. For a standard level and the gauge $\Gamma_{\mathcal{L}}(\alpha)$, "size at most i" means $\alpha^i x \in O_A$; for $\alpha = 1$ it means integrality.

Lemma 5.1 (floor values)**.** *In Setting 2.4, let $a < \varrho(\mathfrak{s})$ and $x, z, y \in G(T)$ with $\langle W_Z(x), W_Z(z)\rangle \neq 0$. Then there is a unit c of A with $c \in O_A$, $c \in \mathrm{Val}_a(\mathcal{L}; x, y)$ and $\mathrm{vol}(\mathrm{Val}_a(\mathcal{L}; x, y)) \leq \mathrm{mod}_A(c)$.*

Proof. Let X be the matrix with the rows of x^{-1} above the rows of $g^{-1}(x^{-1} - (xy)^{-1})$; it has a left inverse. As the ball is $O_A^{I(\mathfrak{s})}$ (Input 3(a)), $\mathrm{Val}_a(\mathcal{L}; x, y)$ is the set of the coordinates $w_{e_a^*}$ of the vectors w with Xw integral and $w_{e_b^*} = 0$ for $b < a$. By Input 3(d), with the levels of the indices as weights, there is an invertible matrix b, block upper triangular for the levels, such that Xb is integral with an integral left inverse; so Xw is integral exactly if $b^{-1}w$ is. Put $i = e_a^*$. Its level $2\varrho - a$ is shared by no other index, and the indices of higher level are the e_b^* with $b < a$. As b and b^{-1} are block upper triangular, the column w^0 of b at i vanishes at these indices, and $(b^{-1}w)_i = (b^{-1})_{ii}\, w_i$ for every w that vanishes at them. For $w = w^0$ this gives $(b^{-1})_{ii}\, b_{ii} = 1$; let $c = b_{ii}$, a unit. Since Xw^0 is a column of Xb, $c = w_i^0 \in \mathrm{Val}_a(\mathcal{L}; x, y)$. Every element of $\mathrm{Val}_a(\mathcal{L}; x, y)$ is a w_i with $b^{-1}w$ integral and w vanishing at the e_b^*, $b < a$; then $c^{-1}w_i = (b^{-1}w)_i \in O_A$. So $\mathrm{Val}_a(\mathcal{L}; x, y) \subseteq cO_A$, a set of volume $\mathrm{mod}_A(c)$. Finally, by Lemma 4.3 there is $u \in \mathcal{U}(T)$ with $ux \in K_T(1)$; $x^{-1}w^0$ is integral, hence so is $uw^0 = (ux)(x^{-1}w^0)$, and its coordinate at e_a^* is $w_i^0 = c$ by Input 3(e). So $c \in O_A$. □

LEAN `IntegralPlace.IsStdLevel.exists_unit_mem_floorValues`

Lemma 5.2 (non-orthogonal vectors are close)**.** *In Setting 2.4, let Γ be a gauge of $\mathcal{L}$ and $x, y \in G(T)$ with $\langle W_Z(x), W_Z(y)\rangle \neq 0$. Then there are $u, v \in \mathcal{U}(T)$ with $ux \in \mathrm{Box}(\mathcal{L})$, $vy \in \mathrm{Box}(\mathcal{L})$ and $\mathrm{sz}_\Gamma(g_{\mathcal{L}}^{-1}(vy - ux)) \leq 2$.*

Proof. Take u, v from Lemma 4.3. Then $ux \in K_T(1) = \mathrm{Box}(\mathcal{L})$, and $k = (ux)^{-1}(vy)$ lies in $\mathrm{K}^\circ(\mathcal{L})$, so that $vy = (ux)k \in K_T(1)$. Further $g^{-1}(vy - ux) = (ux) \cdot g^{-1}(k - 1)$ is a product of integral matrices; its columns lie in the ball, whose elements have size at most 1 in every gauge (Input 3(e)). □

LEAN `IntegralPlace.IsStdLevel.exists_close_of_inner_ne_zero`

Proposition 5.3 (a gauge adapted to two heads)**.** *In Setting 2.4, let $j \leq \varrho(\mathfrak{s})$, let K be a natural number and p, q sequences of units of A, and suppose that $\mathcal{I}_j^{\mathrm{maj}}(\mathcal{L}; p, q; W_Z(\cdot), W_Z(\cdot), x) \neq 0$ for some $x \in G(T)$. Then there is a unit α of A with $\alpha \in O_A$ and $\bar{\alpha} = \alpha$ such that, for the gauge $\Gamma = \Gamma_{\mathcal{L}}(\alpha)$:*

(i) *$p \in \mathrm{Hd}_j(\Gamma)$ and $q \in \mathrm{Hd}_j(\Gamma)$, and if $j \neq 0$ then $\mathrm{sz}_\Gamma(g_{\mathcal{L}}^{-1}) \leq 1$;*
(ii) *$\mathrm{sz}_\Gamma(p_i^{-1} - q_i^{-1}) \leq 1$ for all $i < j$;*
(iii) *$\Lambda(\Gamma)^E \leq (2/\,\mathrm{mod}_A(g_{\mathcal{L}}))^K \cdot \Lambda_K(j; p) \cdot \Lambda_K(j; q)$ for every natural number E with $4E \leq K$.*

Proof. Write $P_l = p_0 \cdots p_{l-1}$ and $Q_l = q_0 \cdots q_{l-1}$. Since the majorant at x is not zero, there is a point y in the support of $\mu_{0,j}^{\mathcal{L}}$ at which both pivot moduli of its integrand are not zero, hence (Input 3(b)) $\sigma, \tau \in \mathcal{H}_j(T)$ with $\langle W(\sigma y), W(\sigma y x)\rangle \neq 0$ and $\langle W(\tau w_j(p)^{-1} y), W(w_j(q)^{-1} w_j(p) \tau w_j(p)^{-1} y x)\rangle \neq 0$. As τ commutes with the Weyl elements (Input 3(d)), W does not vanish at σy, $\tau w_j(p)^{-1} y$, $\sigma y x$ and $\tau w_j(q)^{-1} y x$.

By Input 3(d), with Lemma 4.5, the points y and $w_j(p)^{-1}y$ are near at j, and so are yx and $w_j(q)^{-1}yx$; hence, for the gauge of depth 1, all P_l and Q_l with $l \le j$ lie in O_A. Put $\beta = \prod_{l\le j} P_l$ and $\gamma = \prod_{l\le j} Q_l$. Then $\beta \in O_A$, $\beta/P_l \in O_A$ for $l \le j$, $\beta/p_i = P_i \cdot (\beta/P_{i+1}) \in O_A$ for $i < j$, and $\Lambda_K(j;p) = [\beta]^{K/2}$ by the multiplicativity of the index; likewise for γ and q. Let

$$\alpha = g \cdot \overline{\beta\gamma} \cdot \beta\gamma,$$

a unit in O_A with $\bar\alpha = \alpha$ (Input 3(a)). Then αP_l^{-1}, αQ_l^{-1} $(l \le j)$, αg^{-1} and $\alpha(p_i^{-1} - q_i^{-1})$ $(i < j)$ lie in O_A, which is (i) and (ii). For (iii), $\Lambda(\Gamma) = 2[\alpha] = 2[g]\,[\beta]^2[\gamma]^2$, and the three indices are at least 1; so for $4E \le K$, $\Lambda(\Gamma)^E \le 2^K[g]^K([\beta][\gamma])^{K/2}$. □

LEAN `IntegralPlace.IsStdLevel.exists_gauge_depth_of_ne_zero`

Proposition 5.4 (the shell bound)**.** *In Setting 2.4, let $j \le \varrho(\mathfrak{s})$, let t be a natural number with $t + 10^8 n(\mathfrak{s})^4 \le k(\mathfrak{s})$, let α be a unit of A with $\alpha \in O_A$ and $\bar\alpha = \alpha$, and let Γ be the gauge $\Gamma = \Gamma_{\mathcal{L}}(\alpha)$. Let p, q be sequences of units of A with $p \in \mathrm{Hd}_j(\Gamma)$, $q \in \mathrm{Hd}_j(\Gamma)$ and, if $j \ne 0$, $\mathrm{sz}_\Gamma(g_{\mathcal{L}}^{-1}) \le 1$. Let $f\colon G(T) \to [0,\infty]$ and $e \in [0,\infty]$ be such that $f(x) \le \mathcal{I}_j^{\mathrm{maj}}(\mathcal{L};p,q;W_Z(\cdot),W_Z(\cdot),x)$ for all x and such that, if $e < 1$, then $f(x)^2 \le \left(e^{\mathfrak{s}(\mathcal{L})}\right)^2 \cdot \mathrm{SNP}_j(\Gamma;p,q;W_Z(\cdot),W_Z(\cdot),x)$ for all $x \in \mathrm{Posn}(\mathcal{L})$. Then*

$$\mathsf{h}_j(\mathcal{L};p,q) \cdot \int f\,d\mathsf{M}(\mathcal{L}) \;\le\; \min(1,e)^t \cdot \mathrm{mr}(Z) \cdot \Lambda(\Gamma)^{10^9 n(\mathfrak{s})^4}.$$

Proof. Write $\Lambda = \Lambda(\Gamma)$, $s = \mathfrak{s}(\mathcal{L})$. By Input 3(b) and the hypothesis on t,

$$s = k(\mathfrak{s}) - 3m - 3n - 5 \;\ge\; 8 \cdot 10^7 n^4 + t. \tag{3}$$

Step 1: f lives on the positions. If $f(x) \ne 0$, the majorant at x is not zero; as in the proof of Proposition 5.3 there is y' with $\langle W(y'), W(y'x)\rangle \ne 0$, and Lemma 4.3 gives $u, v \in \mathcal{U}(T)$ with $(uy')^{-1}(vy'x) \in \mathrm{K}^\circ(\mathcal{L})$, that is, $x \in y'^{-1}(v^{-1}u)y' \cdot \mathrm{K}^\circ(\mathcal{L}) \subseteq \mathrm{Posn}(\mathcal{L})$ (Input 3(a)).

Step 2: shells. For a position x let $I(x)$ be the set of the $\varepsilon \in A$ that multiply 1 and all entries of $g^{-1}(x-1)$ into O_A. By Input 3(b), $I(x) = cO_A$ for a unit $c = c(x) \in O_A$, an ideal of index $[c]$. For an ideal I let $S(I)$ be the set of the positions with $I(x) = I$. By Input 3(a) for the measure $f\,d\mathsf{M}(\mathcal{L})$, the constant function $\mathsf{h}_j(\mathcal{L};p,q)$ and the exponent m (recall $\zeta_A(m) \le 2$), the claim follows, with $2 \cdot 2^{10^5 n^2} \le \Lambda^{10^8 n^4}$, once we show for every position x, with $c = c(x)$ and $S = S(cO_A)$:

$$[c]^m \cdot \mathsf{h}_j(\mathcal{L};p,q) \cdot \int_S f\,d\mathsf{M}(\mathcal{L}) \;\le\; \min(1,e)^t \cdot \mathrm{mr}(Z) \cdot \Lambda^{9\cdot 10^8 n^4} \cdot 2^{10^5 n^2}. \tag{4}$$

Step 3: one shell. Let $\Gamma' = \Gamma_{\mathcal{L}}(\bar c c)$, so $\Lambda(\Gamma') = 2[c]^2$, and let $\phi \in [0,\infty]$ be the number with

$$\phi^{400n^2} = \mathrm{mod}_A(c) \cdot \Lambda^{(10n+1)\cdot 99n}.$$

For $x' \in S$ the matrix $\bar c c \cdot g^{-1}(x'-1)$ is integral, so S consists of positions x' with $\mathrm{sz}_{\Gamma'}(g^{-1}(x'-1)) \le 1$. Put

$$\epsilon = \min(\phi, 1, e)^s \cdot \Lambda^{9999n}.$$

We show in Step 4 that $f(x')^2 \le \epsilon^2 \cdot \mathrm{SNP}_j(\Gamma; p, q; W, W, x')$ for all $x' \in S$. Then Input 3(a), with Lemma 4.5 and Proposition 4.7, gives

$$\mathsf{h}_j(\mathcal{L}; p, q) \cdot \int_S f \, d\mathsf{M}(\mathcal{L}) \;\le\; \epsilon \cdot \mathrm{mr}(Z) \cdot \Lambda^{1500000n^3+1} \cdot \big(2[c]^2\big)^{50000n^2}.$$

By (3),

$$\begin{aligned} \min(\phi, 1, e)^s &\le \phi^{400n^2 \cdot 200000n^2} \cdot \min(1, e)^t \\ &= \mathrm{mod}_A(c)^{200000n^2} \cdot \Lambda^{(10n+1)99n \cdot 200000n^2} \cdot \min(1, e)^t. \end{aligned}$$

The powers of $[c] = \mathrm{mod}_A(c)^{-1}$ are $m + 100000n^2 - 200000n^2 \le 0$, as $m < 9n$; they are at most 1. The exponent of Λ is

$$9999n + 1500000n^3 + 1 + (10n+1) \cdot 99n \cdot 200000n^2 \;\le\; 9 \cdot 10^8 n^4.$$

This is (4).

Step 4: the bound on a shell. Let $x' \in S$. If $e < \min(\phi, 1)$, then $e < 1$ and the hypothesis gives $f(x')^2 \le (e^s)^2\ \mathrm{SNP}_j(\Gamma; p, q; W, W, x') \le \epsilon^2\ \mathrm{SNP}_j(\Gamma; p, q; W, W, x')$. Otherwise $\min(\phi, 1, e) = \min(\phi, 1)$, and by $f \le \mathcal{J}_j^{\mathrm{maj}}(\mathcal{L}; p, q; W, W, \cdot)$ and Input 3(b) for $\phi_0 = \min(\phi, 1)^2$, with $V_0^2 \le \Lambda^{2 \cdot 9999n}$, it suffices to verify the hypothesis of that input. So let $\phi < 1$, and let y be such that $\langle W(y), W(yx')\rangle \ne 0$ and $w_j(p)^{-1}y$, $w_j(q)^{-1}yx'$ are near at j; suppose, for a contradiction, that $\mathrm{vol}(\mathrm{Val}_a(\mathcal{L}; y, x')) > \phi^2$ for all a with $j \le a < \varrho$. By Lemma 5.1 there are units $d_a \in O_A$ with $d_a \in \mathrm{Val}_a(\mathcal{L}; y, x')$ and $\mathrm{vol}(\mathrm{Val}_a(\mathcal{L}; y, x')) \le \mathrm{mod}_A(d_a)$. Let $C = \prod_{j \le a < \varrho} d_a$ and let Γ'' be the gauge of depth $\alpha^{10n+1}\bar{C}C$. Then $C/d_a \in O_A$, so d_a^{-1} has size at most 1 for Γ''; if $j \ne 0$ and c' has size at most $10n$ for Γ, then $\alpha^{10n+1}g^{-1}c' \in O_A$, so $g^{-1}c'$ has size at most 1 for Γ''; and Lemma 5.2 provides the two unipotent elements. Input 3(d), for the pair (y, x'), shows that $\varepsilon = (\alpha^{10n+1}\bar{C}C)^{99n}$ multiplies $g^{-1}(x'-1)$ into O_A; it also lies in O_A. So $\varepsilon \in I(x') = cO_A$, whence $\mathrm{mod}_A(\varepsilon) \le \mathrm{mod}_A(c)$, that is,

$$\mathrm{mod}_A(C)^{198n} \;\le\; \mathrm{mod}_A(c) \cdot \mathrm{mod}_A(\alpha)^{-(10n+1)99n} \;\le\; \mathrm{mod}_A(c) \cdot \Lambda^{(10n+1)99n} = \phi^{400n^2}.$$

On the other hand $\mathrm{mod}_A(C) = \prod_a \mathrm{mod}_A(d_a) > \phi^{2(\varrho - j)}$, or $\mathrm{mod}_A(C) = 1$ if $j = \varrho$; as $\phi < 1$ and $396n(\varrho - j) \le 400n^2$, this gives $\mathrm{mod}_A(C)^{198n} > \phi^{400n^2}$, a contradiction. □

LEAN `IntegralPlace.IsStdLevel.lintegral_shells_le`

Corollary 5.5 (the majorant is integrable)**.** *In Setting 2.4, let $j \le \varrho(\mathfrak{s})$, let r be a natural number with $r + 10^8 n(\mathfrak{s})^4 \le k(\mathfrak{s})$, and let p, q be sequences of units of A. Then $\int \mathcal{J}_j^{\mathrm{maj}}(\mathcal{L}; p, q; W_Z(\cdot), W_Z(\cdot), x)\, d\mathsf{M}(\mathcal{L})(x) \ne \infty$.*

Proof. If the integral is infinite, the majorant is not identically zero; Proposition 5.3, with $K = 0$, gives a gauge $\Gamma = \Gamma_{\mathcal{L}}(\alpha)$ that satisfies the hypotheses of Proposition 5.4, which we apply to f the majorant, $e = 1$ and $t = r$. The right side is finite (Lemma 4.2(f)) and $\mathsf{h}_j(\mathcal{L}; p, q) \ne 0$; a contradiction. □

LEAN `IntegralPlace.IsStdLevel.lintegral_formMajorant_ne_top`

Proof of Theorem A. The function ω has modulus at most 1, so by Input 3(c)

$$\begin{aligned} |\mathcal{J}_j(\mathcal{L}; p, q; W, \omega)| &\le \mathsf{h}_j(\mathcal{L}; p, q) \cdot \int f \, d\mathsf{M}(\mathcal{L}), \\ f(x) = \mathcal{J}_j^{\mathrm{mod}}(\mathcal{L}; p, q; W, x, \omega) &\le \mathcal{J}_j^{\mathrm{maj}}(\mathcal{L}; p, q; W, W, x). \end{aligned} \tag{5}$$

If the majorant vanishes identically, the form is zero. Otherwise let $\Gamma = \Gamma_{\mathcal{L}}(\alpha)$ be the gauge of Proposition 5.3, $\Lambda = \Lambda(\Gamma)$, and $\beta = \alpha^{200n^2}$.

A unit η. We claim that there are units η, w of A with

$$w \in O_A, \quad \xi w \in O_A, \quad \xi\eta^{-1} \in O_A, \quad \min(1,[\eta]) \le \mathrm{mod}_A(w)\cdot \mathrm{mod}_A(\beta)^{-1}, \tag{6}$$

such that $f(x)^2 \le [\eta]^{\#\iota}\cdot \mathrm{SNP}_j(\Gamma;p,q;W,W,x)$ for all positions x. In the second case of the theorem ($j=i+1$, $\omega=1$), take $\eta=\xi$ and $w=\beta$: by (i) and (ii) of Proposition 5.3, ξ has size at most 2 for Γ, so $\xi\beta\in O_A$, and the bound is the second part of Input 3(c). In the first case, Input 3(c) gives a unit η and $x_0, c \in O_A$ with $\eta = \xi + c\beta$, $\xi = x_0\eta$ and $\beta \in \eta O_A$; put $w=\eta^{-1}\beta$. Then $w\in O_A$, $\xi w = x_0\beta \in O_A$, $\xi\eta^{-1}=x_0$, and $\mathrm{mod}_A(w)\,\mathrm{mod}_A(\beta)^{-1}=[\eta]$. For every c' of size at most $200n^2$ for Γ, $(\eta-\xi)c' = c\,\beta c' \in O_A$, so $\psi_A(\eta c') = \psi_A(\xi c')$, and the bound is the first part of Input 3(c).

The shell bound. We apply Proposition 5.4 to f, the number $e=[\eta]^{1/2}$ and $t+12$ in the place of t; note $t+12+10^8n^4 \le k(\mathfrak{s})$. Its second hypothesis holds: if $e<1$, then $[\eta]^{\#\iota} \le [\eta]^{\mathsf{s}(\mathcal{L})} = (e^{\mathsf{s}(\mathcal{L})})^2$, because $\mathsf{s}(\mathcal{L}) \le \#\iota$. With (5) we get

$$|\mathcal{J}_j(\mathcal{L};p,q;W,\omega)| \;\le\; \min(1,e)^t\cdot\min(1,[\eta])^6\cdot \mathrm{mr}(Z)\cdot\Lambda^{10^9n^4}.$$

Now $[\eta]\le[\xi\eta^{-1}][\eta]=[\xi]$, as $\xi\eta^{-1}\in O_A$; so $\min(1,e)^t \le [\xi]^{t/2}$. By (6), $\min(1,[\eta])^6 \le \mathrm{mod}_A(w)^6\,\mathrm{mod}_A(\beta)^{-6}$; here $\mathrm{mod}_A(\beta)^{-6}\le\Lambda^{1200n^2}$ (Input 3(e), as β^{-1} has size at most $200n^2$), and $\mathrm{mod}_A(w)=\mathrm{vol}(wO_A)\le \mathrm{vol}(O_A\cap\xi^{-1}O_A) = \mathrm{den}(\xi)^{-1}$ by Input 3(e), since wO_A is contained in $O_A\cap\xi^{-1}O_A$. Finally $\Lambda^{10^9n^4+1200n^2}\le\Lambda^{2\cdot10^9n^4}\le (2/\mathrm{mod}_A(g))^K\Lambda_K(j;p)\Lambda_K(j;q)$ by Proposition 5.3(iii), as $8\cdot10^9n^4\le K$. This is the bound of Theorem A. □

Proof of Theorem B. Let $\Gamma=\Gamma_{\mathcal{L}}(1)$, so $\Lambda(\Gamma)=2$, and $\epsilon_0=\min(\mathrm{vol}(O_A\setminus O_A^\times),\frac12)$. Split the outer integral of $\mathcal{J}_0(\mathcal{L};Z)$ over $\mathrm{K}^\circ(\mathcal{L})$ and its complement. The first part is $\mathsf{M}(\mathcal{L})(Z)^2\mathsf{M}(\mathcal{L})(\mathrm{K}^\circ(\mathcal{L}))$ by Lemma 4.4. In the second, by Lemma 4.8 and Input 3(b) (with Lemma 4.5), the integrand is at most $\mathsf{v}(\mathcal{L})^2\cdot f(x)$, where

$$f(x) = \mathbf{1}[x\notin \mathrm{K}^\circ(\mathcal{L})]\cdot \mathsf{Pv}_0(\mathcal{L};\mathsf{k}_Z)(1,x)^2 = \mathbf{1}[x\notin \mathrm{K}^\circ(\mathcal{L})]\cdot \mathcal{J}_0^{\mathrm{maj}}(\mathcal{L};1,1;W,W,x)$$

(Input 3(e)). We apply Proposition 5.4 with $j=0$, $p=q=1$, $e=\epsilon_0$ and $t+1+n+10^9n^4$ in the place of t; the sequence 1 is a head of length 0, and $t+1+n+10^9n^4+10^8n^4\le k(\mathfrak{s})$. Granting its second hypothesis, and since $\mathsf{h}_0(\mathcal{L};1,1)=1$,

$$\begin{aligned}\mathsf{v}(\mathcal{L})^2\int f\,d\mathsf{M}(\mathcal{L}) \;&\le\; \mathsf{v}(\mathcal{L})^2\,\mathrm{mr}(Z)\cdot\epsilon_0^{t+1}\cdot\epsilon_0^n\cdot(2\epsilon_0)^{10^9n^4}\\ &\le\; \mathsf{M}(\mathcal{L})(Z)\,\mathsf{M}(\mathcal{L})(\mathrm{K}^\circ(\mathcal{L}))\cdot\mathrm{vol}(O_A\setminus O_A^\times)^{t+1}\cdot 2^{-n},\end{aligned}$$

which with the first part is the claim.

It remains to show $f(x)^2\le(\epsilon_0^{\mathsf{s}(\mathcal{L})})^2\cdot\mathrm{SNP}_0(\Gamma;1,1;W,W,x)$ for a position $x\notin\mathrm{K}^\circ(\mathcal{L})$. By Input 3(e), and since $2\cdot10^4n\varrho\le3\cdot10^6n^3$, this follows once we show: for every σ with $\mathsf{k}_Z(\sigma,\sigma x)\neq0$ there is $a<\varrho$ with $\mathrm{vol}(\mathrm{Val}_a(\mathcal{L};\sigma,x))\le\epsilon_0$. By Lemma 5.1 there are units $u_a\in O_A$ with $u_a\in\mathrm{Val}_a(\mathcal{L};\sigma,x)$ and $\mathrm{vol}(\mathrm{Val}_a(\mathcal{L};\sigma,x))\le\mathrm{mod}_A(u_a)$. If some u_a is not a unit of O_A, then u_aO_A is a proper ideal contained in $O_A\setminus O_A^\times$, so $\mathrm{mod}_A(u_a)=\mathrm{vol}(u_aO_A)\le\epsilon_0$, and a is the floor we want. Otherwise all u_a^{-1}

are integral. Then Input 3(d), for $\Gamma'' = \Gamma$, $j = 0$ and the pair (σ, x), applies: the unipotent elements come from Lemma 5.2, and every point is near at 0, because $\mathcal{Q}_0(T) = G(T)$. It shows that $g^{-1}(x-1)$ is integral, so $x \in \mathrm{K}^\circ(\mathcal{L})$ by Inputs 3(c) and 3(a), against the choice of x. □

6 The top floor

Again $n = n(\mathfrak{s})$, $\varrho = \varrho(\mathfrak{s})$, $g = g_{\mathcal{L}}$ and $W = W_Z(\cdot)$.

Lemma 6.1 (one gauge for two sequences)**.** *In Setting 2.1, suppose $\mathcal{L} \in \mathrm{StdLev}$, and let p, q be sequences of units of A. Then there is a gauge $\Gamma \in \mathrm{Gau}(\mathcal{L})$ such that all sequences r of units in a neighbourhood of p, and all sequences r in a neighbourhood of q, are heads of length $\varrho(\mathfrak{s})$ for Γ.*

Proof. For $l \le \varrho$ the map $(c, r) \mapsto \bar{c}c \cdot (r_0 \cdots r_{l-1})^{-1}$ from pairs of an element of A and a sequence of units to A is continuous and sends $(0, p)$ and $(0, q)$ to $0 \in O_A$, and O_A is open. So there are a neighbourhood N of 0 and neighbourhoods of p and of q on whose products with N all these maps have values in O_A. Almost every element of A is a unit (Input 3(e)), so N contains a unit c, and $\Gamma = \Gamma_{\mathcal{L}}(\bar{c}c)$ has the property; note $\bar{c}c \in O_A$, the case $l = 0$. □

LEAN `IntegralPlace.IsStdLevel.exists_gauge_isHead`

Proposition 6.2 (the correlations are locally constant)**.** *In Setting 2.4, let p be a sequence of units of A. Then there is a neighbourhood of $(p, 1)$ in the product of the space of sequences of units with $G(T)$ such that for every (p', l') in it with $l' \in \mathcal{H}_\varrho(T)$ and every $l \in \mathcal{H}_\varrho(T)$: $\mathcal{J}(Z; p', ll') = \mathcal{J}(Z; p, l)$.*

Proof. Let Γ be the gauge of Lemma 6.1 for p and 1, and C the set of the $y \in G(T)$ with $\mathrm{sz}_\Gamma(y) \le 60n^2$, a compact set (Input 3(e)). The neighbourhood consists of the (p', l') such that p' is a head of length ϱ for Γ, $p_a^{-1} p'_a \in O_A^\times$ for $a < \varrho$, and

$$k_y = y^{-1}\, w_\varrho(p)\, l'\, w_\varrho(p')^{-1}\, y \in \mathrm{K}^\circ(\mathcal{L}) \qquad \text{for all } y \in C. \tag{7}$$

These are neighbourhood conditions: $O_A^\times$ and $\mathrm{K}^\circ(\mathcal{L})$ are open, the expression in (7) is continuous in (p', l', y) and equal to 1 at $(p, 1, y)$, and C is compact. Let (p', l') be such a pair with $l' \in \mathcal{H}_\varrho(T)$, and $l \in \mathcal{H}_\varrho(T)$. First, $\mathrm{mod}_A(p'_a) = \mathrm{mod}_A(p_a)$ for $a < \varrho$, so $\mathsf{h}_\varrho(\mathcal{L}; p', 1) = \mathsf{h}_\varrho(\mathcal{L}; p, 1)$. Secondly, we show for every $y \in G(T)$

$$W(y) = 0 \qquad \text{or} \qquad W(ll'\, w_\varrho(p')^{-1} y) = W(l\, w_\varrho(p)^{-1} y),$$

which gives the equality of the two Bessel correlations. Suppose $W(y) \neq 0$ and one of the two values on the right is not zero. By Input 3(e), for the head p' and $(ll')^{-1}$, or for the head p and l^{-1}, we get $y \in C$. Then $ll' w_\varrho(p')^{-1} y = l\, w_\varrho(p)^{-1} y \cdot k_y$ with $k_y \in \mathrm{K}^\circ(\mathcal{L})$, and for every coset c the element $a_c k_y a_c^{-1}$ lies in $\mathrm{K}^\circ(\mathcal{L})$ too, so the coordinates at c of the two vectors agree by Lemma 4.1(e). □

LEAN `IntegralPlace.IsStdLevel.eventually_correlation_eq`

Proof of Theorem C. (i) Let Γ be the gauge of Lemma 6.1 for p and q, so that p and q are heads of length ϱ. For $l \in \mathcal{H}_\varrho(T)$, Input 3(e) with $x = l^{-1}$ and $\mu = \mathsf{M}(\mathcal{L})$ (Lemma 4.5, Input 3(c)) gives

$$\begin{aligned}&\langle \mathcal{J}(Z;p,l), \mathcal{J}(Z;q,l)\rangle \\ &\quad = \mathsf{h}_\varrho(\mathcal{L};p,q)\int\int \langle W(x), W(y)\rangle \\ &\qquad\qquad \cdot \langle W(l\,w_\varrho(q)^{-1}y), W(l\,w_\varrho(p)^{-1}x)\rangle \, d\mathsf{M}(\mathcal{L})(x)\, d\mathsf{M}(\mathcal{L})(y),\end{aligned}$$

because $\mathsf{h}_\varrho(\mathcal{L};p,1)\mathsf{h}_\varrho(\mathcal{L};q,1) = \mathsf{h}_\varrho(\mathcal{L};p,q)$. Let $\mathcal{A}(l,x,y)$ be $\varphi_\xi(l^{-1})$ times the integrand on the right. It is integrable for $\mu^{\mathcal{L}}_{\varrho,2}\times \mathsf{M}(\mathcal{L})\times\mathsf{M}(\mathcal{L})$: it vanishes unless x and y lie in the compact set of Input 3(e), and for fixed x, y the integral of its modulus over l is bounded, by the bound for pairing kernels of Input 3(b), since $|\langle W(lw_1), W(lw_2)\rangle| = |k^1_{W,W}(lw_1, lw_2)|$. So F_ξ is integrable, and $\int F_\xi\, d\mu^{\mathcal{L}}_{\varrho,2} = \mathsf{h}_\varrho(\mathcal{L};p,q)\int \mathcal{A}$. We transform $\int\mathcal{A}$. Replace y by xx' (left invariance of $\mathsf{M}(\mathcal{L})$); write $x = sy'$ with $s = h\sigma$, where (h,σ,y') has the law $\theta_n(\mathcal{L})\times\theta_\varrho(\mathcal{L})\times\mu^{\mathcal{L}}_{0,\varrho}$ (Input 3(a)); and replace l by $l's^{-1}$, $l' = k\tau$ with (k,τ) of law $\theta_n(\mathcal{L})\times\theta_\varrho(\mathcal{L})$, which is allowed since $s\in\mathcal{H}_\varrho(T)$ almost surely and $\mu^{\mathcal{L}}_{\varrho,2}$ is invariant under right translation by it. As the Weyl elements commute with $\mathcal{H}_\varrho(T)$ (Input 3(d)),

$$l\,w_\varrho(p)^{-1}x = k\tau w_\varrho(p)^{-1}y' =: z_0, \qquad l\,w_\varrho(q)^{-1}xx' = w_\varrho(q)^{-1}w_\varrho(p)\,z_0\,x',$$
$$\varphi_\xi(l^{-1}) = \varphi_\xi(h\sigma\tau^{-1}k^{-1}) = \varphi_\xi(\sigma\tau^{-1}),$$

the last by Input 3(d). Hence, with $d = w_\varrho(q)^{-1}w_\varrho(p)$,

$$\int\mathcal{A} = \int \varphi_\xi(\sigma\tau^{-1})\cdot\langle W(h\sigma y'), W(h\sigma y'x')\rangle\cdot\overline{\langle W(z_0), W(d\,z_0x')\rangle},$$

the integral over (k,τ,h,σ,x',y'). Integrating first over h and over k turns the two inner products into $\mathsf{Av}_{\varrho+1}(\mathcal{L};k^1_{W,W})(\sigma y',\sigma y'x')$ and the conjugate of $\mathsf{Av}_{\varrho+1}(\mathcal{L};k^d_{W,W})(\tau w_\varrho(p)^{-1}y', \tau w_\varrho(p)^{-1}y'x')$; integrating then over σ and τ gives $\Phi_\varrho(x',y')$ for $\omega = \varphi_\xi$, and the integral over (x',y') for $\mathsf{M}(\mathcal{L})\times\mu^{\mathcal{L}}_{0,\varrho}$, times the head density, is the floor form.

(ii) Let (p_0,q_0) be a pair of sequences. By Proposition 6.2 with $l' = 1$, for (p,q) near (p_0,q_0) the correlations $\mathcal{J}(Z;p,l) = \mathcal{J}(Z;p_0,l)$ and $\mathcal{J}(Z;q,l) = \mathcal{J}(Z;q_0,l)$ agree for all $l\in\mathcal{H}_\varrho(T)$, which carries the measure; and $p_\varrho - (p_0)_\varrho \in L$ for p near p_0, because L is open, so the domain of integration is the same, L being a subgroup. The kernel is locally constant. □

Proof of Theorem D. (i) For $l\in G(T)$ and $x\in\mathrm{Cos}(Z)\times\mathrm{Cos}(Z)$ let $v(l,x)$ be the vector with the coordinates $\mathcal{J}(Z;\omega_i,l)_x$, $i\in\mathcal{C}$. Then

$$\sum_{i,i'} R_{ii'}\langle \mathcal{J}(Z;\omega_i,l), \mathcal{J}(Z;\omega_{i'},l)\rangle = \sum_x v(l,x)^* R\, v(l,x) \;\geq 0,$$

as R is positive semidefinite. Each summand of $\mathcal{Q}$ is an integral of an integrable function (Theorem C(i) for $\xi = 0$), over the same set $D = \{l:\ h(z,l^{-1}z) - t\in L\}$ because $(\omega_i)_\varrho = t$. So $\mathcal{Q} = \int_D \sum_x v(l,x)^* R v(l,x)\, d\mu^{\mathcal{L}}_{\varrho,2}(l)$ is real and ≥ 0, and equals

the integral of $G(l)=\sum_x(\mathrm{Re}\ v(l,x)^{*}Rv(l,x))^{+}$ over D. By $\mu^{\mathcal{L}}_{\varrho,2}=\theta_n(\mathcal{L})*\theta_{\varrho}(\mathcal{L})$, and since $(h\sigma)^{-1}z=\sigma^{-1}z$ for $h\in\mathcal{H}(T)$,

$$\begin{aligned}\mathcal{Q}&=\int \mathbf{1}\big[h(z,\sigma^{-1}z)-t\in L\big]\int G(h\sigma)\,d\theta_n(\mathcal{L})(h)\,d\theta_{\varrho}(\mathcal{L})(\sigma)\\&\geq\int \mathbf{1}\big[h(z,\sigma^{-1}z)-t\in L\big]\cdot\mathcal{E}(Z;R,\omega,\sigma^{-1}z)\,d\theta_{\varrho}(\mathcal{L})(\sigma),\end{aligned}$$

because $\sigma\in\mathcal{H}_{\varrho}(T)$ almost surely and the energy at $\sigma^{-1}z$ is an infimum over a set that contains $l=\sigma$. The right side is at least the integral of the theorem: $\delta^{(\varrho)}(\mathcal{L})$ is the image of $\theta_{\varrho}(\mathcal{L})$ under $\sigma\mapsto\sigma^{-1}z$, and $h(z,e)=(J(T)e)_z$; the integral of a function for an image measure is at most the integral of its composite, with equality for measurable functions, so no measurability of the energy is needed.

(ii) Fix $o\in\Theta_0$; then $l_o e=z$. Let u_0 be the vector of Input 3(a) and E the Eichler elements of Input 3(f). For a vector x put $m(x)=l_o^{-1}\,\mathrm{E}(l_o x)\,l_o$. The z-coordinate of $l_o x$ is 1 at $x=e$; as the units are open and inversion is continuous at 1, for x near e this coordinate is a unit and $m(x)$ is close to $m(e)=1$. By Proposition 6.2, applied to each of the finitely many ω_i, there is a neighbourhood N of e such that for $x\in N$: the z-coordinate of $l_o x$ is a unit, and

$$\mathcal{J}(Z;\omega_i,l\,m(x))=\mathcal{J}(Z;\omega_i,l)\qquad\text{for all } i\in\mathcal{C}\text{ and } l\in\mathcal{H}_{\varrho}(T),\ \text{if } m(x)\in\mathcal{H}_{\varrho}(T).\tag{8}$$

Let $x\in N$, $k\in\mathcal{H}(T)\cap\mathrm{K}^{\circ}(\mathcal{L})$ and y with $ky=x$, and let $l\in\mathcal{H}_{\varrho}(T)$ with $l^{-1}z=y$; we must show $\theta_n(\mathcal{L})(\mathrm{K}^{\circ}(\mathcal{L}))\sum_{\theta}c_{\theta}\leq\int\sum_{x'}G_{x'}(hl)\,d\theta_n(\mathcal{L})(h)$, where $G_{x'}(l')=(\mathrm{Re}\ v^{*}Rv)^{+}$ for the vector $v_i=\mathcal{J}(Z;\omega_i,l')_{x'}$ and $c_{\theta}=G_{n_{\theta}}(l_{\theta})$.

The vector $l_o x=l_o k\,l^{-1}z$ has vanishing coordinates at all e_a, e_a^{*} and satisfies $h(l_o x,l_o x)=h(z,z)$; so $m=m(x)\in\mathcal{H}_{\varrho}(T)$ and $\mathrm{E}(l_o x)^{-1}z=l_o x$, that is, $m\,x=e$. For $\theta\in\Theta_0$ the elements l and $l_{\theta}mk$ of $\mathcal{H}_{\varrho}(T)$ satisfy $(l_{\theta}mk)^{-1}z=k^{-1}m^{-1}e=y=l^{-1}z$, so $b_{\theta}=l\,(l_{\theta}mk)^{-1}\in\mathcal{H}(T)$ by Input 3(d). Thus

$$l=b_{\theta}\,l_{\theta}\,m\,k\qquad(\theta\in\Theta_0).\tag{9}$$

Two invariances of the correlations: for $k'\in\mathcal{H}(T)\cap\mathrm{K}^{\circ}(\mathcal{L})$, $l'\in G(T)$ and a sequence p, $\mathcal{J}(Z;p,k'l')=\mathcal{J}(Z;p,l')=\mathcal{J}(Z;p,l'k')$. Indeed $W(k'y')=W(y')$ for all y', by Lemma 4.1(d) with $u=1$ and $\mathrm{sd}(1)=0$; this gives the first equality in the definition, and the second after the substitution $y'\mapsto k'^{-1}y'$, since k' commutes with $w_{\varrho}(p)$ and $\mathsf{M}(\mathcal{L})$ is left invariant.

Now let $h\in\mathcal{H}(T)$, which carries $\theta_n(\mathcal{L})$, and let θ be such that $hb_{\theta}\in\mathrm{K}^{\circ}(\mathcal{L})$. By (9), the two invariances and (8), $\mathcal{J}(Z;\omega_i,hl)=\mathcal{J}(Z;\omega_i,(hb_{\theta})\,l_{\theta}\,m\,k)=\mathcal{J}(Z;\omega_i,l_{\theta})$ for all i, so $G_{n_{\theta}}(hl)=c_{\theta}$. If also $hb_{\theta'}\in\mathrm{K}^{\circ}(\mathcal{L})$, then $l_{\theta}l_{\theta'}^{-1}=b_{\theta}^{-1}b_{\theta'}\in\mathrm{K}^{\circ}(\mathcal{L})$ by (9); so $\theta\mapsto n_{\theta}$ is injective on the set of these θ, by the hypothesis. Hence

$$\sum_{\theta}\mathbf{1}\big[hb_{\theta}\in\mathrm{K}^{\circ}(\mathcal{L})\big]\cdot c_{\theta}\ \leq\ \sum_{x'}G_{x'}(hl).$$

Integrating over h and using the invariance of $\theta_n(\mathcal{L})$ under right translation by $b_{\theta}\in\mathcal{H}(T)$, the left side becomes $\theta_n(\mathcal{L})(\mathrm{K}^{\circ}(\mathcal{L}))\sum_{\theta}c_{\theta}$. $\square$

7 Where the results are used

All results are used inside the unit. Note F (*Translates of Poincaré series fill the rows*) quotes them as its Input 5, for the form T in the lattice basis of the filling data over the product of the completions at a finite set of places: Lemma 4.1(a), (c), (e) and Lemma 4.2(c) (its part (a)); Theorem B and Lemma 4.4 (b); Lemma 4.9 and Lemma 4.2(f) (c); Theorem A (d); Theorem C (e); Theorem D(i) (f); Theorem D(ii) (g); Lemma 4.5, Corollary 5.5 and Lemma 4.8 (h). The facts that this note recalls and does not prove, and that Note F uses as well (the Haar property and the upper bound for the mass of the integral group, the conditions on the measure on $\mathcal{H}(T)$, the descent and the bound of a floor form by its modulus), are Input 10 of Note F, with the notes that own them. Note G (*From amplitude sums to local correlations*) uses Definition 2.2, Lemma 4.1, the finite Bessel vector and the correlations in its unfolding theorem, Lemma 4.2(a) and Lemma 4.2(e), and the notion of an intermediate group.

8 Notes on the formal text

1. *Arguments that the census reads differently from Lean; to be settled by* `#check` *on a build.* The census reads the arguments from the text, and counts a variable of the file as an argument when its name occurs in a declaration. For the energy (Definition 2.5(c)) it lists five explicit arguments, among them a sequence p; in the definition p is the index of the family, bound inside, and both uses of the energy in the Lean text (the two statements of Theorem D) apply it to four arguments: Z, R, ω and the vector. The note writes these four. The same happens with the letter r in Lemma 6.1, with p in both parts of Theorem D, and with u in Definition 2.2: each is bound in the statement and has no counterpart as an argument. In the other direction, the census does not list Z, p, q for Theorem C(i), nor Z for Theorem D(i), although the statements name these variables of the file; the note states them. Implicit elements x, y, k, g that the census lists for many statements are bound variables, or parts of other names, and have no counterpart either.
2. *Generality.* The form $\mathcal{J}_0(\mathcal{L}; Z)$ is defined for every subgroup Z and every level; the theorems take an intermediate group of a standard level. Lemma 4.1 is stated in the formal text without the hypothesis that the split Gram matrix is a hermitian unit.
3. *The bound for the Bessel vector.* In Lemma 4.5 the bound of the norm is $\sqrt{\#\,\mathrm{Cos}(Z)}$, from coordinates of modulus at most 1; the dossier's first reading said that the norm is at most 1, and the second reader corrected it. Nothing depends on the value of the bound.
4. *Arguments that a conclusion does not name.* In Corollary 5.5 the number r occurs only in the hypothesis $r + 10^8 n(\mathfrak{s})^4 \le k(\mathfrak{s})$, which thus says $10^8 n(\mathfrak{s})^4 \le k(\mathfrak{s})$. In Theorem B the number t is free, so the bound holds with the largest t allowed. In Proposition 5.3 the hypothesis is the existence of one point where the majorant does not vanish; it is used only to see that the partial products of p and q are integral.
5. *Hypotheses on the ring.* Theorem D(ii) assumes that the units of A form an open set and that inversion is continuous at 1. Both hold for a finite product of local

fields; they are hypotheses here because the formal notion of a semilocal field is axiomatic. In Theorem D(ii) the matrix R is arbitrary.

6. *Forms of the formal statements.* Proposition 5.4 is stated in the formal text for a gauge Γ together with an equation that identifies it with a gauge of some depth; $[\xi]^{t/2}$ in Theorem A is written as a power of $\sqrt{[\xi]}$, and the denominator index enters through its inverse; the inequality $j \le \varrho(\mathfrak{s})$ is needed to name the diagonal entry $Z_T(x)_{jj}$. Differences of natural numbers (the volume exponent, the exponents of the floor moduli) are truncated at 0 in the formal text. In Theorem D(i) the formal statement says $0 \le \mathcal{Q}$ for the order of the complex numbers, and bounds the integral by the real part of $\mathcal{Q}$.
7. *Constants.* The exponents 10^8, 10^9, 10^{10} are those of the formal text; the proofs above show which smaller numbers are used: Proposition 5.4 needs $\mathsf{s}(\mathcal{L}) \ge 8 \cdot 10^7 n(\mathfrak{s})^4 + t$, and Theorem A applies it with $t + 12$, which gives the sixth power of the denominator index.
8. *What was not re-derived.* The statements of Inputs 2 and 4 are the results of the tower method; this note uses them as stated. The formal proof of Theorem C(i) verifies the change of variables as an equality of the laws of two functions on product spaces; the text above gives the substitutions. The measurability of the functions that are integrated is checked in the formal text and not discussed here.

How this note was written. From the structured proofs of the module on the top floor in the third chunk of the unit's dossier, all of whose statements and claims a second reader had re-read; her remarks were taken into account. I read the module in the Lean text line by line and compared every statement of this note with it argument by argument; I read in the Lean text the definitions of the unit on the real place and of the note on standard levels that are recalled in §2, and the statements of Inputs 2 to 4. Inputs 1 and 5 are stated from the dossier's readings and from the way the formal proofs use them; of these I read in the Lean text the statements on primitive matrices, on Eichler elements, on the summation over ideals and the definition of a valuation pair. Arguments that are mine and replace computations of the formal text: the proof of Lemma 5.1 in terms of the lattice of the vectors w with Xw integral, the bookkeeping of the exponents in Proposition 5.4, and the description of the change of variables in Theorem C(i). Nothing was compiled for this note. The program check of the unit's dossier covered 1371 leaf steps of 179 proofs.

Appendix A. Concordance with the Lean artifact

The formal names of the objects and the letters of this note: `phase` is $\chi_g(z)$; `unitaryLevel`, `congGroup` are $K_T(g)$, $\mathrm{K}^\circ(\mathcal{L})$; `besselU`, `besselH`, `besselS`, `GT`, `unitriGroup` are $\mathcal{U}(T)$, $\mathcal{H}(T)$, $\mathcal{S}(T)$, $\mathcal{H}_l(T)$, $\mathcal{Q}_l(T)$; `simpleRootSum`, `frameGram`, `weylElem` are $\mathrm{sd}(u)$, $Z_T(x)$, $w_l(p)$; `md`, `genIndex`, `denomIndex`, `idealZeta` are mod_A, $[u]$, $\mathrm{den}(\xi)$, $\zeta_A(s)$; `groupMeasure`, `upperMeasure`, `floorMeasure`, `measureH`, `stackMeasure`, `floorQuadric` are $\mathsf{M}(\mathcal{L})$, $\mathsf{M}_{\ge 0}(\mathcal{L})$, $\theta_l(\mathcal{L})$, $\theta_n(\mathcal{L})$, $\mu^{\mathcal{L}}_{l,i}$, $\delta^{(\varrho)}(\mathcal{L})$; `ascConv` `heisenbergMeasure` is $\eta_{<j}$; `floorModulus`, `headDensity`, `volumeExponent`, `floorValues` are $\mathsf{d}_l(\mathcal{L};g)$, $\mathsf{h}_j(\mathcal{L};p,q)$, $\mathsf{s}(\mathcal{L})$, $\mathrm{Val}_a(\mathcal{L};x,y)$; `upperPairing`, `pivotModulus`, `pairingNorm` are $\mathsf{Av}_j(\mathcal{L};k)(g,y)$, $\mathsf{Pv}_j(\mathcal{L};k)(g,y)$, $\mathrm{PN}(\mathcal{L};W,W')$; `floorForm`, `formModulus`, `formMajorant`, `selfNormProduct` are $\mathcal{J}_j(\mathcal{L};p,q;W,\omega)$ and its modulus,

majorant and self-norm product; `besselCorrelation`, `finHeadWeight` are $\mathrm{J}(\mu, W; \gamma)$, $\Lambda_K(j; p)$; `IsStdLevel.gauge`, `Gauge.Λ`, `IsHead` are $\Gamma_{\mathcal{L}}(\alpha)$, $\Lambda(\Gamma)$, $p \in \mathrm{Hd}_j(\Gamma)$; the level `F` is $\mathcal{L}$, the index set `j` of Theorem D is $\mathcal{C}$, and `Level.energy` is $\mathcal{E}(Z; R, \omega, e)$. Three statements as the artifact has them:

```
lemma enorm_floorForm_le {ω} (hK : 10 ^ 10 * ϰ.n ^ 4 ≤ K)
    (H : (ω = fun x ↦ P.ψ (ξ * (T.frameGram x.1.1).diag ⟨j, by lia⟩)) ∨ ∃ i, j = i + 1 ∧ ω = 1 ∧ ξ =
    F.g⁻¹ * ((p i)⁻¹.1 - (q i)⁻¹.1)) : ‖F.floorForm j p q Z.besselVector ω‖ₑ ≤
  Z.massRatio * (.ofReal √(_root_.genIndex ξ.1) ^ t * (denomIndex R ξ.1 : ℝ≥0∞)⁻¹ ^ 6 *
  (2 / P.md F.g) ^ K * .ofReal (finHeadWeight K j p * finHeadWeight K j q))

lemma floorZeroForm_le_main_term : F.floorZeroForm Z ≤ F.groupMeasure Z * F.groupMeasure
  F.congGroup * (F.groupMeasure Z + volume (P.R \ P.intUnits) ^ (t + 1) * 2⁻¹ ^ ϰ.n)

lemma eventually_sum_packets_le_energy (hi : IsOpen {x : A | IsUnit x}) (hc :
  ContinuousAt Ring.inverse (1 : A)) {i} [Fintype i] {l : i → _} (o : i) (hl : ∀ θ, l θ ∈ GT ϰ.ϱ ∧
  orbitPoint ϰ.top (l θ) = e) {n : i → _} (hn : ∀ θ θ', l θ * (l θ')⁻¹ ∈ F.congGroup → n θ = n θ' →
  θ = θ') : let v θ p := correlation Z (ω p) (l θ) (n θ)
  ∀ᶠ x in nhds e, ∀ k ∈ besselH ⊓ F.congGroup, ∀ y, k *ᵥ y = x →
            F.measureH F.congGroup * ∑ θ, .ofReal (star (v θ) ⬝ᵥ R *ᵥ v θ).re ≤ energy Z R ω y
```

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Lemma 4.2	`Level.unipotentMass_ne_zero_and_ne_top`	TopFloor.lean	99–100
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Definition 2.5	`Level.correlation`	TopFloor.lean	425–426
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Definition 2.5	`Level.energy`	TopFloor.lean	436–437
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Theorem C	`IntegralPlace.IsStdLevel.floorForm_top_eq_integral`	TopFloor.lean	513–556
Theorem D	`IntegralPlace.IsStdLevel.lintegral_energy_le_topKernel`	TopFloor.lean	561–582
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Notation

A, $x \mapsto \bar{x}$, O_A, $O_A^\times$, ψ_A, mod_A, vol a semilocal field with involution, its integers, their units, the character, the modulus, Haar measure

$[u]$, $\mathrm{den}(\xi)$, $\zeta_A(s)$, I, $I(x)$, $S(I)$ generalised index, denominator index, ideal zeta function; an ideal, the content ideal of a position, its shell

$\mathfrak{s} = (\varrho, \kappa, k_0)$, $\varrho(\mathfrak{s})$, $k(\mathfrak{s})$, $n(\mathfrak{s})$, n a shape and its numbers; in proofs n, ϱ abbreviate $n(\mathfrak{s})$, $\varrho(\mathfrak{s})$

$I(\mathfrak{s})$, e_a, e_a^*, z, $\mathfrak{f}_a$, h index set, basis vectors, the frame, the hermitian form

T, $J(T)$, $G(T)$, $Z_T(x)$ an $\mathfrak{s}$-form, its split Gram matrix, unitary group, frame Gram matrix

$\mathcal{U}(T)$, $\mathcal{H}(T)$, $\mathcal{S}(T)$, $\mathcal{H}_l(T)$, $\mathcal{Q}_l(T)$ Bessel unipotent subgroup, the subgroup fixing the frame and its dual, Bessel subgroup, floor groups, unitriangular groups

$\mathrm{sd}(u)$, $w_l(p)$, $K_T(g)$ sum of the entries above the diagonal, Weyl element, congruence subgroup modulo g

ι, m, $\mathcal{L}$, $\mathrm{Lev}(\iota, T, m+1)$, $g_{\mathcal{L}}$, $\mathrm{K}^\circ(\mathcal{L})$ index set of the orthonormal part, number of further pairs, a level, the levels, its unit, its congruence subgroup

$\mathrm{Box}(\mathcal{L})$, $\mathrm{Posn}(\mathcal{L})$, $\mathcal{L} \in \mathrm{StdLev}$ box, positions, standard levels

η_l, $\eta_{<j}$, $\theta_l(\mathcal{L})$, $\theta_n(\mathcal{L})$, $\mu_{l,i}^{\mathcal{L}}$, $\mathsf{M}(\mathcal{L})$, $\mathsf{M}_{\geq 0}(\mathcal{L})$, $\delta^{(\varrho)}(\mathcal{L})$ Heisenberg, floor, stack and group measures, the quadric measure; μ, ν, ρ, θ measures

$\mathsf{d}_l(\mathcal{L}; g)$, $\mathsf{h}_j(\mathcal{L}; p, q)$, $\mathsf{s}(\mathcal{L})$, s, $\mathrm{Val}_a(\mathcal{L}; x, y)$ floor modulus, head density, volume exponent, floor values

$\mathsf{Av}_j(\mathcal{L}; k)(g, y)$, $\mathsf{Pv}_j(\mathcal{L}; k)(g, y)$, $k^d_{W,W'}$, $\mathrm{PN}(\mathcal{L}; W, W')$ averaged kernel, pivot modulus, the kernel of two functions, pairing norm

$\mathcal{I}_j(\mathcal{L}; p, q; W, \omega)$, Φ_j, $\mathcal{I}_j^{\mathrm{mod}}(\mathcal{L}; p, q; W, x, \omega)$, $\mathcal{I}_j^{\mathrm{maj}}(\mathcal{L}; p, q; W, W', x)$ floor form, its integrand, modulus, majorant

Γ, Γ', Γ'', $\mathrm{Gau}(\mathcal{L})$, $\mathrm{sz}_\Gamma(x) \leq i$, $\Lambda(\Gamma)$, Λ, $p \in \mathrm{Hd}_j(\Gamma)$, $\Gamma_{\mathcal{L}}(\alpha)$ gauges, size, the constant, heads, gauge of depth α

$\mathrm{SNP}_j(\Gamma; p, q; W, W', x)$, Σ_0, V_0 self-norm product, its factor at floor 0, the floor-mass constant

$\Lambda_K(j; p)$, $\mathrm{J}(\mu, W; \gamma)$, $\ell^2(X)$ head weight, Bessel correlation, the space $\mathbb{C}^X$ with its inner product

$\chi_g(z)$, $\mathrm{Int}(\mathcal{L})$, Z, $\mathrm{Cos}(Z)$, a_c, $\mathsf{v}(\mathcal{L})$, v, $\mathrm{mr}(Z)$ finite Bessel function, intermediate groups, cosets, representatives, unipotent mass, mass ratio

$\mathcal{J}_0(\mathcal{L}; Z)$, $W_Z(y)$, W, k_Z the form at the bottom, finite Bessel vector, its kernel

$\mathcal{J}(Z; p, l)$, $\mathcal{K}(L; Z; p, q)$, $\mathcal{E}(Z; R, \omega, e)$, G, $G_{x'}$ correlation, top kernel, energy, the integrands of the energy

p, q, r, P_l, Q_l, ω_i, $\mathcal{C}$, R, $\mathcal{Q}$ sequences of units, partial products, a family, its index set, a matrix, the kernel sum

ξ, η, α, β, γ, ϕ, ϕ_0, ϵ, ϵ_0, ε units and real numbers of the proofs (frequency, its modification, depths, products, thresholds)

ω, φ_ξ, F_ξ, $\mathcal{A}$, f, e, E, K, t, B, V, C, D, N, L functions and numbers of the statements; C also a product of floor values or a compact set, D a domain, N a neighbourhood, L a subset of A

x, y, u, v, k, l, b, σ, τ, w, d, c, X, Y_x, i, j, a elements of $G(T)$, vectors, matrices, units, indices, as said at each place

Θ_0, θ, l_θ, n_θ, v_θ, c_θ, b_θ, o, $m(x)$, E, u_0 packets and the objects of the proof of Theorem D

READBACK NOTE 38

From amplitude sums to local correlations: filling levels at finitely many places, and the filling data of the admissible data

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The second moment of the Poincaré series is a sum of *amplitudes* of rational elements of a unitary group, sorted by classes and by an element of a finite level group. This note is the dictionary that turns such sums into integrals at a finite set T of finite places. It shows that the product of the completions at T carries a standard level whose congruence subgroup reads the principal level of the filling data, that integrality off T and the components at T describe the level groups and the unipotent character, and it proves the *unfolding*: the sum of the amplitudes over a double coset of the unipotent group, cut off at T, is the product of an archimedean Bessel correlation and one coordinate of the local correlation of Note T. A second theorem writes a block of amplitude sums with a given link as a sum of limits of such coset amplitudes as T grows. The last section attaches the filling data to the admissible data of the proof, proves that Bessel systems of the required type exist at the infinite places, and verifies the standing hypotheses of the filling theorem.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles and that Lean's kernel accepted the proofs; an independent replay is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note (Note G of its unit) treats 34 of the 497 declarations of the unit *Translates of Poincaré series fill the rows*: the modules on the passage from global data to the places of T, on the filling data of the admissible data and on their admissibility. Proved here from the inputs: Theorems A to C and the statements of §§4–6. Theorem D and Proposition 7.3 are assemblies; the note says which inputs they assemble and does not reproduce their arithmetic. Taken for granted and stated as Inputs 1 to 6 in §3: the set-up of the unit, the local theory of Note T, the standard levels, facts on adeles from the background library, and the results on the data from Chapter III and from the units on the columns, the rows and the discrete series. Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 34 declarations.

1 The statements

The objects are recalled in §2. In brief: S is filling data over a CM field E, $G_S(E)$ the rational unitary group, G_S^{lat} its finite adelic points in the lattice basis, with the level groups $K_g(S) \le K(S)$ and the reduction red_S to the finite group $\mathsf{G}(S)$; $\mathrm{pos}_S(c, c'; \gamma)$ is the position of a rational element for two classes, $\mathrm{loc}_{S,\mathsf{T}}$ takes the components at T, and $a^{\sigma}_{S,\mathsf{T}}(c, s, c'; \gamma)$ is the coset amplitude of the side σ (Definition 2.5).

Theorem A (unfolding). *In Setting 2.4, suppose that $\hat{g}(S)^{-1}$ is integral off T, that E carries Weil data and that S is admissible. Let σ be a side, $\gamma_1 \in \mathcal{H}_\varrho(T_S^{\mathrm{rat}}(E))$, p a sequence of units of E, and c, c' classes such that $\mathrm{pos}_S(c, c'; w_\varrho(p)\gamma_1) \in \mathrm{Int}^{\mathsf{T}}(S)$. Let $Z \in \mathrm{Int}(\mathcal{L})$, $k \in K(S)$, and let $a, b \in \mathrm{Cos}(Z)$ be cosets with $\mathrm{loc}_{S,\mathsf{T}}(k)^{-1} a_a a_b^{-1} \in \mathrm{K}^\circ(\mathcal{L})$. Put $\gamma = w_\varrho(p)\gamma_1$, let p_{T} be the sequence of the components of p at T, and $W^\sigma = W_S^\sigma(\cdot)$.. Then*

$$\begin{aligned} &\mathcal{J}(Z; p_{\mathsf{T}}, \mathrm{loc}_{S,\mathsf{T}}(\mathrm{pos}_S(c, c'; \gamma_1))^{-1})_{(a,b)} \cdot \mathrm{J}(dy, W^\sigma; \mathrm{r}_S^\infty(\gamma)) \\ &\quad = \mathsf{h}_\varrho(\mathcal{L}; p_{\mathsf{T}}, 1) \cdot \mathsf{M}(\mathcal{L})(\mathrm{K}^\circ(\mathcal{L})) \cdot a^{\sigma}_{S,\mathsf{T}}(c, \mathrm{red}_S(k), c'; \gamma). \end{aligned}$$

LEAN `Filling.IsFillingLevel.correlation_smul_eq_cosetAmplitude`

Theorem B (blocks as limits of coset amplitudes). *In Setting 2.1, suppose that E carries Weil data and that S is admissible. Let σ be a side, c, c' classes, $s \in \mathsf{G}(S)$ and Λ a link, such that $A^\sigma_S(\mathrm{Tm}^{\mathrm{lk}}_S(\Lambda); c, s, c') \neq 0$ and such that for every $\gamma \in \mathrm{Tm}_S(c, s, c')$ all leading minors of $Z_S(\gamma)$ are non-zero. Then there are a unit t of E, an element $\gamma_1 \in G_S(E)$ and $k \in K(S)$ such that*

(i) *$\gamma_1 \in \mathcal{H}_\varrho(T_S^{\mathrm{rat}}(E))$, $Z_S(\gamma_1)_{\varrho\varrho} = t$, $N(S)^2 < |t_\tau|^2$ for every infinite place τ, $\mathrm{red}_S(k) = s$ and $\Lambda = \mathrm{link}(w_\varrho(1)\gamma_1)$; and*

(ii) *there is a family (L_p) of vectors of $\ell^2(\mathrm{Bc}(S) \times \mathrm{Bc}(S))$, indexed by the maps $p\colon \{0, \dots, \varrho(\mathfrak{s}) - 1\} \to E^\times$, which is summable with sum $A^\sigma_S(\mathrm{Tm}^{\mathrm{lk}}_S(\Lambda); c, s, c')$, such that for every p*

$$a^{\sigma}_{S,\mathsf{T}}(c, s, c'; w_\varrho((p \mid t))\gamma_1) \longrightarrow L_p$$

as T grows in the directed set of the finite sets of finite places of E.

LEAN `Filling.exists_hasSum_cosetAmplitude`

Theorem C (filling levels exist). *In Setting 2.1:*

(i) *Suppose that E carries Weil data, and let T be a finite set of finite places of E that is stable under complex conjugation. Then there are an integral place structure on E_{T} and a level $\mathcal{L}$ of $T_S^{\mathrm{lat}}(E_{\mathsf{T}})$ with $\mathcal{L} \in \mathrm{FL}(S, \mathsf{T})$.*

(ii) *For every finite set T_0 of units of E there are arbitrarily large finite sets T of finite places, stable under complex conjugation, such that t and t^{-1} are integral off T for all $t \in \mathsf{T}_0$, $\hat{g}(S)^{-1}$ is integral off T, $\Lambda(S)$ and its inverse have entries integral off T, and $N(S)^{-1}$ is integral off T.*

(iii) *Every finite adele a is integral off all sufficiently large finite sets T of finite places; and for every $x \in G_S^{\mathrm{lat}}$, $x \in \mathrm{Int}^{\mathsf{T}}(S)$ for all sufficiently large T.*

LEAN `Filling.exists_isFillingLevel` · `Filling.exists_places_integralOff` · `Filling.eventually_mem_integralOff` · `Filling.eventually_isIntegralOff`

Theorem D (the filling data of the admissible data). *In Setting 2.6, let $S = S(D)$ be the filling data of D (Definition 7.1), and let K_f be the group of the integral points of the finite adelic unitary group of the standard form on the index set of the columns. Then:*

(i) $K_0^{\mathrm{col}}(D) = \mathrm{inc}_{S(D)}^{-1}(K_g(S(D)))$;

(ii) *the index of* $\mathrm{inc}_{S(D)}^{-1}(K(S(D))) \cap K_f$ *in* K_f *is* $(N(D)^{k(D)} - 1)^{f(D)}$;

(iii) $\#\mathrm{Cl}(S(D)) = \#\overline{\mathcal{H}}^{S(D)} \cdot (N(D)^{k(D)} - 1)^{f(D)} \cdot \mathrm{mass}(E(D), k(D))$, *where* $\mathcal{H} = \mathcal{H}(T_{S(D)}^{\mathrm{lat}}(\mathbb{A}_{E,f}))$; *and*

(iv) $S(D)$ *is admissible.*

LEAN `Data.columnLevel_zero_eq_comap` · `Data.relIndex_latStabilizer` · `Data.card_classes_filling` · `instance : D.filling.IsAdmissible` (an instance without a name)

Theorems A and B are Inputs 4(f) and 4(e) of the main note of the unit (Note F, *Translates of Poincaré series fill the rows*); Theorem C and the lemmas of §4 are its Inputs 4(a) to (c), the lemmas of §5 its Input 4(d), and Theorem D with Definition 7.1 and Proposition 7.3 its Input 9(a).

2 The setting

Conventions. The natural numbers are $0, 1, 2, \ldots$. E is a CM field with its CM type: a number field that is a CM field, with complex conjugation $x \mapsto \bar{x}$, together with the choice of one embedding at each infinite place τ; $x_\tau \in \mathbb{C}$ is the image of $x \in E$ under it, and $f(E)$ is the number of infinite places. A *finite place* w of E is a non-zero prime ideal; $\mathbb{F}_w$ is its residue field, E_w the completion, O_w its integers, $\bar{w}$ the conjugate place. $\mathbb{A}_{E,f}$ is the ring of finite adeles, $\hat{O}_E$ its subring of integral adeles, and ψ_f the standard additive character of $\mathbb{A}_{E,f}$. A finite set T of finite places is *stable under complex conjugation* if it contains $\bar{w}$ with every w; for a finite set T of finite places, $E_{\mathsf{T}} = \prod_{w \in \mathsf{T}} E_w$, $x_{\mathsf{T}} \in E_{\mathsf{T}}$ is the family of the components at T of a finite adele or of an element of E, and $\tilde{y}$ is the extension by zero of $y \in E_{\mathsf{T}}$ to a finite adele. A finite adele is *integral off* T if its components at the places outside T are integral, and an element of E if its image is. The field E is *stably unramified* if it contains a primitive root of unity of order 5040, and E *carries Weil data* if

it is stably unramified and a splitting character in the sense of Chapter III of the handout is given. A *shape* is a triple $\mathfrak{s} = (\varrho, \kappa, k_0)$ of a natural number, a finite set and an element of it; $\varrho(\mathfrak{s}) = \varrho$, $k(\mathfrak{s}) = \#\kappa$, $n(\mathfrak{s}) = \varrho(\mathfrak{s}) + 1$, $r(\mathfrak{s}) = 2\varrho(\mathfrak{s}) + k(\mathfrak{s}) + 1$. A *semilocal field*, an *integral place structure* on it, with its ring of integers, character and modulus, a *level* $\mathcal{L}$, its unit $g_{\mathcal{L}}$ and congruence subgroup $\mathrm{K}^\circ(\mathcal{L})$, the group measure $\mathsf{M}(\mathcal{L})$ and a *standard level* are as in Note T of this unit (*The top floor at a finite place*); on E_{T} the integers are $O_{\mathsf{T}} = \prod_{w \in \mathsf{T}} O_w$, the character is written ψ_{T} and the modulus $\mathrm{mod}_{\mathsf{T}}$. A *side* is one of the two words full, cut; in this note σ always denotes a side. P denotes the parameters of the proof (Chapter III). X^* is the conjugate transpose of a matrix, $\ell^2(X)$ the space $\mathbb{C}^X$ with its inner product, conjugate-linear in the first variable. The letters T, F of the formal text for a set of places and a level are written T and $\mathcal{L}$.

Forms and groups. As in Notes F and T: $I(\mathfrak{s})$ is the index set with $e_0, \dots, e_{\varrho-1}$, the elements of κ, z, and $e_0^*, \dots, e_{\varrho-1}^*$; the frame is $\mathfrak{f}_a = e_a$ $(a < \varrho)$, $\mathfrak{f}_\varrho = z$. For a commutative ring R with involution and an $\mathfrak{s}$-form T over R, $J(T)$ is the split Gram matrix, $G(T)$ its unitary group, $h(v, v') = v^* J(T) v'$, $Z_T(x)$ the frame Gram matrix with corner entry $h(z, xz)$; $\mathcal{U}(T)$, $\mathcal{H}(T)$, $\mathcal{S}(T) = \mathcal{U}(T)\mathcal{H}(T)$ are the Bessel unipotent subgroup, the subgroup fixing all e_a, e_a^* and z, and the Bessel subgroup; $\mathcal{H}_\varrho(T)$ is the subgroup fixing all e_a and e_a^*; $\mathrm{sd}(u)$ is the sum of the entries of u just above the diagonal of its frame block; and for a sequence p of units, $w_\varrho(p)$ is the Weyl element of length ϱ, which sends e_a to $p_a e_a^*$ and e_a^* to $\bar{p}_a^{-1} e_a$ for $a < \varrho$. For $p \colon \{0, \dots, \varrho - 1\} \to R^\times$ and a unit t, $(p \,|\, t)$ is the sequence $p_0, \dots, p_{\varrho-1}, t, 1, 1, \dots$. For a unit π, an element t and $x \in R^\kappa$, the *tail vector* $\mathrm{tv}(\pi, t, x)$ is the vector $x - (\pi\bar{\pi})^{-1} t\, z$, written in the lattice basis below.

Setting 2.1 (filling data)**.** E is a CM field with its CM type, $\mathfrak{s}$ is a shape, and S is filling data over E of shape $\mathfrak{s}$, in the sense of the set-up of this unit (the note *The filling data*). Its components are: a real number $b(S) \geq 1000$; a prime number $N(S) \geq 5$ and a finite idele $\bar{\pi}(S) \in \hat{O}_E$ with $\bar{\pi}(S)\,\overline{\bar{\pi}(S)} = N(S)$; two primes $q_0(S)$, $q_1(S)$ different from $N(S)$, with $g(S) = q_0(S) q_1(S)$; for $i = 0, 1$ a set $\mathrm{Ch}_i(S)$ of $f(E)$ finite places above $q_i(S)$, the *chosen places*, with residue fields of $q_i(S)$ elements and such that the conjugate of a chosen place is never chosen; a generator $\varpi(S)$ of the different of E with $\overline{\varpi(S)} = -\varpi(S)$; an open compact subgroup $K^{\mathrm{col}}(S)$ of the finite adelic unitary group of the standard form on κ, with trivial class automorphism groups; and for every infinite place τ a Bessel system $\mathfrak{B}_\tau(S)$. Among its inequalities: $10^5 \leq \varrho(\mathfrak{s})$, $n(\mathfrak{s})\,[E : \mathbb{Q}] \leq 2^{2\varrho(\mathfrak{s})}$ and $1000\,(2\varrho(\mathfrak{s}) + 2) \leq k(\mathfrak{s})$.

The groups of the set-up. In Setting 2.1, for a commutative ring R with involution over the integers of E: $T_S^{\mathrm{rat}}(R)$ is the $\mathfrak{s}$-form that is the identity on κ and $-N(S)$ at z, and $G_S(R) = G(T_S^{\mathrm{rat}}(R))$; $T_S^{\mathrm{lat}}(R)$ is the Gram matrix of the same form in the *lattice basis*, in which the basis vector x_{k_0} of k_0 is replaced by $\overline{\bar{\pi}(S)}^{-1} x_{k_0} + N(S)^{-1} z$ (the bar in $\bar{\pi}(S)$ is part of the name of the idele; the coefficient is the inverse of its conjugate); $\Lambda(S)$ is the matrix of this change of basis over $\mathbb{A}_{E,f}$, and $G_S^{\mathrm{lat}} = G(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$. The homomorphism $\mathrm{r}_S^f \colon G_S(E) \to G_S^{\mathrm{lat}}$ is the diagonal embedding followed by the change of basis; $\mathrm{r}_S^\infty \colon G_S(E) \to \prod_\tau G_1(\mathfrak{s})$ is the family of the images at the infinite places, rescaled to the real form $G_1(\mathfrak{s})$ of the library by positive numbers, of which $\mathrm{fs}_S(\tau, a) > 0$ $(a \leq \varrho)$ are the *frame scales*; inc_S embeds

the finite adelic points of the unitary group of the standard form on κ into G_S^{lat} as $\mathcal{H}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$. For $\gamma \in G_S(E)$, $Z_S(\gamma) = Z_{T_S^{\mathrm{rat}}(E)}(\gamma)$, $\mathsf{g}_S(\gamma)$ is the *term discriminant*, equal to $d\bar{d}\,(t\bar{t} - N(S)^2)$ when $Z_S(\gamma)$ is diagonal with entries $p_0, \dots, p_{\varrho-1}, t$ and $d = p_0 \cdots p_{\varrho-1}$; $\mathrm{link}(\gamma)$ is the *link* of γ, a subspace attached to γ in the set-up, and for a link Λ, $\mathrm{Tm}_S^{\mathrm{lk}}(\Lambda) = \{\gamma \in G_S(E) : \ \mathsf{g}_S(\gamma) \neq 0,\ \mathrm{link}(\gamma) = \Lambda\}$.

For a finite idele u, $\hat{K}_S(u)$ is the group of the $x \in G_S^{\mathrm{lat}}$ that have entries in $\hat{O}_E$, as has x^{-1}, and for which $u^{-1}(x-1)$ has entries in $\hat{O}_E$. Then $K(S) = \hat{K}_S(1)$, and with $\hat{g}(S)$, the integer $g(S)$ as a finite idele, $K_g(S) = \hat{K}_S(\hat{g}(S))$ is the kernel of the reduction $\mathrm{red}_S \colon K(S) \to \mathsf{G}(S)$ modulo the chosen places, where $\mathsf{G}(S) = \prod_i \prod_{w \in \mathrm{Ch}_i(S)} \mathrm{GL}_{I(\mathfrak{s})}(\mathbb{F}_w)$; for a subgroup B of G_S^{lat}, $\overline{B}^S = \mathrm{red}_S(B \cap K(S))$. The character $\psi_S(x) = \psi_f(-x/\varpi(S))$ of $\mathbb{A}_{E,f}$ is self-dual: $\psi_S(ra) = 1$ for all $r \in \hat{O}_E$ exactly if $a \in \hat{O}_E$; and $\psi_S^{\mathcal{U}}(u) = \psi_S(\hat{g}(S)^{-1}\,\mathrm{sd}(u))$ for $u \in G_S^{\mathrm{lat}}$. For a finite set T of finite places, $\psi_{S,\mathsf{T}}(y) = \psi_S(\tilde{y})$ is the character of E_{T} of the set-up. $\mathrm{Cl}(S)$ is the finite set of the double cosets of the finite adelic unitary group of κ by its rational points and $K^{\mathrm{col}}(S)$; each class c has a chosen representative y_c, and $h_S(c) = \mathrm{inc}_S(y_c)$. For classes c, c' and $\gamma \in G_S(E)$,

$$\mathrm{pos}_S(c, c'; \gamma) = h_S(c)^{-1}\, \mathrm{r}_S^f(\gamma)\, h_S(c') \in G_S^{\mathrm{lat}},$$

and for $s \in \mathsf{G}(S)$ the set of *terms over* (c, s, c') is $\mathrm{Tm}_S(c, s, c') = \{\gamma \in G_S(E) : \mathrm{pos}_S(c, c'; \gamma) \in K(S),\ \mathrm{red}_S(\mathrm{pos}_S(c, c'; \gamma)) = s\}$. $\mathcal{U}_{S,c}$ is the group of the rational $u \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$ with $\mathrm{pos}_S(c, c; u) \in K_g(S)$.

Amplitudes. $\mathrm{Bc}(S)$ is the set of the families $(i_\tau)_\tau$ of indices of the Bessel systems. For a side σ, a point y of $\prod_\tau G_1(\mathfrak{s})$ and $i \in \mathrm{Bc}(S)$, $W_S^\sigma(y)_i$ is the product over τ of the i_τ-th coordinates at y_τ of the full or the cut Bessel function of $\mathfrak{B}_\tau(S)$; $\mathrm{bk}_S^\sigma(\gamma, y) \in \ell^2(\mathrm{Bc}(S) \times \mathrm{Bc}(S))$, the *Bessel kernel*, has the coordinates $W_S^\sigma(y)_i\, \overline{W_S^\sigma(\mathrm{r}_S^\infty(\gamma)^{-1} y)_{i'}}$. With the cutoffs $\beta_{S,c} \geq 0$ of the set-up, functions on $\prod_\tau G_1(\mathfrak{s})$, the *amplitude* of $\gamma \in G_S(E)$ over c, c' and the *amplitude sum* of a set $\Xi \subseteq G_S(E)$ are

$$\begin{aligned} a_S^\sigma(c, c'; \gamma) &= \int \beta_{S,c}(y)\, \beta_{S,c'}(\mathrm{r}_S^\infty(\gamma)^{-1} y)\, \mathrm{bk}_S^\sigma(\gamma, y)\, dy, \\ A_S^\sigma(\Xi; c, s, c') &= \sum_{\gamma \in \Xi \cap \mathrm{Tm}_S(c, s, c')} a_S^\sigma(c, c'; \gamma), \end{aligned}$$

with dy the Haar measure of the set-up; $|a|_S^\sigma(c, c'; \gamma)$ is the same integral with the norm of the Bessel kernel. The *order* is $\mathrm{uo}(S) = g(S)^{f(E)\varrho(\mathfrak{s})(2k(\mathfrak{s})+2\varrho(\mathfrak{s})+1)}$. S is *admissible* if the standing hypotheses of the set-up hold: $K^{\mathrm{col}}(S) = \mathrm{inc}_S^{-1}(K_g(S))$, the reduction red_S is surjective, and the inequalities listed in the proof of Theorem D.

The local objects of Note T. For a finite set T of finite places and a unit u of E_{T}, $K_{S,\mathsf{T}}(u)$ is the group of the $y \in G(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$ that have entries in O_{T}, as has y^{-1}, and for which $u^{-1}(y-1)$ has entries in O_{T}. For a level $\mathcal{L}$ of $T_S^{\mathrm{lat}}(E_{\mathsf{T}})$: $\mathrm{K}^\circ(\mathcal{L}) = K_{S,\mathsf{T}}(g_{\mathcal{L}})$; $\chi_g(y)$ is the finite Bessel function; $\mathrm{Int}(\mathcal{L})$ the set of the intermediate groups Z, with $\mathrm{K}^\circ(\mathcal{L}) \leq Z \leq K_{S,\mathsf{T}}(1)$; $\mathrm{Cos}(Z)$ the set of the cosets of $\mathrm{K}^\circ(\mathcal{L})$ in Z, each coset c with a representative a_c; $W_Z(y)$ the finite Bessel vector, with the coordinate $\chi_{g_{\mathcal{L}}}(y\, a_c^{-1})$ at c; $\mathsf{v}(\mathcal{L}) = \prod_{l<\varrho} \mathsf{d}_l(\mathcal{L}; g_{\mathcal{L}})$ the unipotent mass, where $\mathsf{d}_l(\mathcal{L}; g) = \mathrm{mod}_{\mathsf{T}}(g)^{k(\mathfrak{s})+1+2(\varrho-l-1)+1/2}$; $\mathsf{h}_\varrho(\mathcal{L}; p, 1)$ the head density;

and for a sequence p of units of E_{T} and $l \in G(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$, the correlation is $\mathcal{J}(Z;p,l) = \mathsf{h}_{\varrho}(\mathcal{L};p,1)\cdot \mathrm{J}(\mathsf{M}(\mathcal{L}), W_Z(\cdot); w_{\varrho}(p)\,l^{-1})$, where for a measure μ, a vector-valued function W and a group element γ the *Bessel correlation* $\mathrm{J}(\mu, W;\gamma)$ is the tensor with the coordinates $\int W(y)_i\,\overline{W(\gamma^{-1}y)_{i'}}\,d\mu(y)$. $\eta_{<j}$ is the convolution of the first j Heisenberg measures.

Definition 2.2 (integrality off T and the components at T)**.** In Setting 2.1, let T be a finite set of finite places of E.

(a) For $x \in G_S^{\mathrm{lat}}$ we write $x \in \mathrm{Int}^{\mathsf{T}}(S)$, and say that x is *integral off* T, if the entries of x and of x^{-1} are integral off T.

(b) If T is stable under complex conjugation, $\mathrm{loc}_{S,\mathsf{T}}\colon G_S^{\mathrm{lat}} \to G(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$ is the homomorphism that takes the components at T of every entry.

LEAN `Filling.IsIntegralOff` · `Filling.toPlaces`

Definition 2.3 (filling levels)**.** In Setting 2.1, let T be a finite set of finite places of E that is stable under complex conjugation, suppose that E_{T} carries an integral place structure, and let

$$\mathcal{L} \in \mathrm{Lev}(\{1,\dots,k(\mathfrak{s}) - 6\varrho(\mathfrak{s}) - 13\}, T_S^{\mathrm{lat}}(E_{\mathsf{T}}), 3\varrho(\mathfrak{s}) + 6)$$

be a level, with $3\varrho(\mathfrak{s}) + 6$ further hyperbolic pairs and an orthonormal part of $k(\mathfrak{s}) - 6\varrho(\mathfrak{s}) - 13$ vectors. Then $\mathcal{L} \in \mathrm{FL}(S,\mathsf{T})$ ($\mathcal{L}$ *is a filling level for* S *at* T) means: $\mathcal{L}$ is a standard level, $\mathcal{L} \in \mathrm{StdLev}$; the character of the place structure is $\psi_{\mathsf{T}} = \psi_{S,\mathsf{T}}$; and $g_{\mathcal{L}} = \hat{g}(S)_{\mathsf{T}}$.

LEAN `Filling.IsFillingLevel`

Setting 2.4 (a filling level)**.** In Setting 2.1, with E, $\mathfrak{s}$ and S as there: T is a finite set of finite places of E that is stable under complex conjugation; E_{T} carries an integral place structure; and $\mathcal{L}$ is a level of $T_S^{\mathrm{lat}}(E_{\mathsf{T}})$ with $\mathcal{L} \in \mathrm{FL}(S,\mathsf{T})$, a filling level.

Definition 2.5 (coset amplitudes)**.** In Setting 2.1, let T be a finite set of finite places of E, σ a side, c, c' classes, $s \in \mathsf{G}(S)$ and $\gamma \in G_S(E)$. The *coset amplitude* is

$$a^{\sigma}_{S,\mathsf{T}}(c,s,c';\gamma) = \sum_{(u,u')} a^{\sigma}_S(c,c';u\gamma u'^{-1}) \;\in\; \ell^2(\mathrm{Bc}(S)\times\mathrm{Bc}(S)),$$

the sum over the pairs (u,u') of elements of $\mathcal{U}(T_S^{\mathrm{rat}}(E))$ with $\mathrm{pos}_S(c,c;u) \in \mathrm{Int}^{\mathsf{T}}(S)$ and $u\gamma u'^{-1} \in \mathrm{Tm}_S(c,s,c')$. The side is an instance argument of the formal definition; it is the superscript σ of the notation, and every statement of this note says which side it is.

LEAN `Filling.cosetAmplitude`

Setting 2.6 (the admissible data)**.** $P = (N_0, d)$ are the parameters of the proof and D is admissible data for P (Chapter III of the handout). Attached to D are the CM field $E(D)$ with its CM type, which carries Weil data; the ranks $m(D)$, $r(D)$, $k(D)$ with $r(D)+1 = k(D)+m(D)$ and $m(D) = 2\varrho(D)+2$; the number $f(D)$ of the infinite places of $E(D)$; the third scale $b(D)$; the level $N(D)$, a prime, and a finite idele $\hat{\pi}(D)$ with $\hat{\pi}(D)\overline{\hat{\pi}(D)} = N(D)$, the principal idele of the integer $\pi_N(D)$ of $E(D)$ of the overview and of Note F; the primes $g_0(D)$, $g_1(D)$ with their chosen places; the generator $\varpi(P)$ of the different; from the units on the columns and on

the seesaw, the index set $X_0(D)$ of the columns, with $k(D)$ elements (Lemma 7.2), the level $K_0^{\mathrm{col}}(D)$ of the columns, the shape $\mathfrak{s}(D) = (\varrho(D), X_0(D), k_0)$ with k_0 the last index, the finite group $\mathsf{G}(D)$, and for every infinite place τ the doubled row parameter $2\varkappa_D(\tau,\cdot)$ and the column weight $\mathsf{w}_D^{\mathrm{col}}(\tau)$, a weight of the unitary group of the columns.

3 What is taken for granted

The following statements are used and not proved here. Inputs 1 to 3 belong to the set-up of this unit (the notes *The filling data*, *Box partitions of arithmetic unipotent groups* and *The energy of amplitude sums*) and, for the big cell, to the background library; Input 4 to Note T and to the note *Poincaré series of cut Bessel functions* of this unit; Input 5 to the background library; Input 6 to Chapter III and to the units on the columns, the rows, the real place and the discrete series. They are stated from the dossier's readings of their quoted Lean statements. Where a classical theorem stands behind an input, its reference is to be confirmed, except where another note is cited for it.

Input 1 (levels and the unipotent character; set-up of this unit). In Setting 2.1, with $\mathcal{U} = \mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ and $\mathcal{H} = \mathcal{H}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$:

(a) The kernel of red_S is $\hat{K}_S(\hat{g}(S))$, as said in §2; $\hat{g}(S) \in \hat{O}_E$; the split Gram matrix of $T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})$ and its inverse have entries in $\hat{O}_E$; and the components of $T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})$ at T form $T_S^{\mathrm{lat}}(E_{\mathsf{T}})$, the lattice form for the integer $N(S)$ of E_{T}.

(b) $\psi_S(ra) = 1$ for all $r \in \hat{O}_E$ exactly if $a \in \hat{O}_E$; and $\psi_{S,\mathsf{T}}(x_{\mathsf{T}}) = \psi_S(x)$ for every finite adele x that is integral off T.

(c) For a rational $u \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$ and a class c: $\mathrm{r}_S^f(u) \in \mathcal{U}$ and $\mathrm{pos}_S(c,c;u) \in \mathcal{U}$. If $u, v \in \mathcal{U}$ and h is in the image of inc_S with $u^{-1}h^{-1}vh \in K_g(S)$, then $\psi_S^{\mathcal{U}}(v) = \psi_S^{\mathcal{U}}(u)$.

(d) *(Rational approximation.)* For a class c and $u \in \mathcal{U}$ there is a rational $u_0 \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$ with $u^{-1}\mathrm{pos}_S(c,c;u_0) \in K_g(S)$ and $\psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u_0)) = \psi_S^{\mathcal{U}}(u)$.

(e) For a form T as in Note T over $\mathbb{A}_{E,f}$ or over E_{T}, a unit g, $u \in \mathcal{U}(T)$ and $h \in \mathcal{H}(T)$: if uh lies in the congruence subgroup modulo g, so does h. Every element of $\mathcal{S}(T)$ is such a product uh.

Input 2 (amplitudes; set-up of this unit). In Setting 2.1, let σ be a side and c, c' classes.

(a) *(Partition of unity.)* If E is stably unramified and S admissible, then $\sum_{u \in \mathcal{U}_{S,c}} \beta_{S,c}(\mathrm{r}_S^\infty(u)\, y) = 1$ for every point y.

(b) *(The kernel under unipotent elements.)* For rational $u, u' \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$, $\gamma \in G_S(E)$ and a point y: $\mathrm{bk}_S^\sigma(u\gamma u'^{-1}, \mathrm{r}_S^\infty(u)\, y) = \psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u))^{-1}\, \psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u')) \cdot \mathrm{bk}_S^\sigma(\gamma, y)$.

(c) *(Summability.)* If E is stably unramified and S admissible, then for every $\Xi \subseteq G_S(E)$ and $s \in \mathsf{G}(S)$ the family of the amplitudes $a_S^\sigma(c,c';\gamma)$, $\gamma \in \Xi \cap \mathrm{Tm}_S(c,s,c')$, is summable, and $\sum_{\gamma \in \mathrm{Tm}_S(c,s,c')} |a|_S^\sigma(c,c';\gamma) < \infty$.

Input 3 (the big cell and links; background library and set-up). Let T be an $\mathfrak{s}$-form over a field with involution whose split Gram matrix is hermitian and invertible; here $T = T_S^{\mathrm{rat}}(E)$.

(a) If all leading minors of $Z_T(\gamma)$ are non-zero, there are $u, u' \in \mathcal{U}(T)$, a sequence p of units and $\gamma_1 \in \mathcal{H}_\varrho(T)$ with $\gamma = u^{-1}\, w_\varrho(p)\gamma_1\, u'$.

(b) If $v_1\, w_\varrho(p)\gamma_1\, v_2^{-1} = v_1'\, w_\varrho(q)\gamma_1\, v_2'^{-1}$ with $v_1, v_2, v_1', v_2' \in \mathcal{U}(T)$ and $\gamma_1 \in \mathcal{H}_\varrho(T)$, then $p_a = q_a$ for $a < \varrho$, $v_1 = v_1'$ and $v_2 = v_2'$.

(c) For $\gamma_1 \in \mathcal{H}_\varrho(T)$, $Z_T(w_\varrho(p)\gamma_1)$ is the diagonal matrix with the entries $p_0, \dots, p_{\varrho-1}$ and $h(z, \gamma_1 z)$. For $u, u' \in \mathcal{U}(T)$, the leading minors of $Z_T(u^{-1}\gamma u')$ are those of $Z_T(\gamma)$, the term discriminants of $u^{-1}\gamma u'$ and γ agree, and $\mathrm{link}(u^{-1}\gamma u') = \mathrm{link}(\gamma)$. For $\gamma_1, \gamma_2 \in \mathcal{H}_\varrho(T)$ and sequences p, q of units: $\mathrm{link}(w_\varrho(p)\gamma_1) = \mathrm{link}(w_\varrho(q)\gamma_2)$ exactly if $\gamma_1 = \gamma_2$.

Input 4 (the local theory; Note T and the note on standard levels). Let A be a semilocal field with an involution and an integral place structure, T a form and $\mathcal{L}$ a level as in Note T, with integers O_A, character ψ_A and modulus mod_A.

(a) *(Note T, Lemma on the finite Bessel function.)* For a unit g and $y \in G(T)$: if $\chi_g(y) \neq 0$ there is $u \in \mathcal{U}(T)$ with $u^{-1}y$ in the congruence subgroup modulo g; for every such u, $\chi_g(y) = \psi_A(g^{-1}\,\mathrm{sd}(u))$; $\chi_g(yk) = \chi_g(y)$ for k in that subgroup; and $\chi_g(uy) = \psi_A(g^{-1}\,\mathrm{sd}(u))\chi_g(y)$ for $u \in \mathcal{U}(T)$. Moreover $\mathrm{sd}(h) = 0$ for $h \in \mathcal{H}(T)$, and $\mathrm{sd}(uu') = \mathrm{sd}(u) + \mathrm{sd}(u')$ for $u, u' \in \mathcal{S}(T)$.

(b) *(Standard levels.)* If $\mathcal{L} \in \mathrm{StdLev}$: $\mathsf{M}(\mathcal{L})$ is a Haar measure, invariant under left and right translations and finite on the group of the integral isometries; $\eta_{<\varrho}$ is carried by $\mathcal{U}(T)$ and invariant under right translation by its elements; and for a unit $g \in O_A$ with $\bar{g} = g$, the congruence subgroup modulo g has $\eta_{<\varrho}$-measure $\prod_{l<\varrho} \mathsf{d}_l(\mathcal{L}; g)$.

(c) *(Lattice levels.)* Let $N \in O_A$ with $\bar{N} = N$ be such that the split Gram matrix of the lattice form of the shape $\mathfrak{s}$ for N is a hermitian unit, g a unit of A with $g \in O_A$, $\bar{g} = g$, and ι a finite set, m' a natural number with $k(\mathfrak{s}) = \#\iota + 2m' + 1$. Then that form has a level in $\mathrm{Lev}(\iota, \cdot, m')$ with unit g that is a *lattice level.* A level in $\mathrm{Lev}(\iota, \cdot, m+1)$ is standard exactly if it is a lattice level, $\zeta_A(m)^{n(\mathfrak{s})} \leq 1 + 2^{-n(\mathfrak{s})}$ for the ideal zeta function of O_A, $m < 9n(\mathfrak{s})$, $13n(\mathfrak{s}) \leq \#\iota$, and the unitary group has a Haar measure that is right invariant.

Input 5 (adeles; background library). Let T be a finite set of finite places of E.

(a) *(Extension by zero.)* The map $y \mapsto \tilde{y}$ from E_{T} to $\mathbb{A}_{E,f}$ is additive and multiplicative, $(\tilde{y})_{\mathsf{T}} = y$, $\tilde{y}$ is integral off T, and $\bar{\tilde{y}} = \tilde{\bar{y}}$ if T is stable under complex conjugation. A finite adele lies in $\hat{O}_E$ exactly if it is integral off T and its components at T are integral.

(b) *(Indices and archimedean absolute values.)* With the generalised index $[\cdot]$ of E_{T}: for $x \in E^{\times}$ integral off T, $[x_{\mathsf{T}}]^{1/2} \leq \prod_{\tau} |x_{\tau}|$.

(c) *(Zeta functions.)* The ideal zeta function of O_{T} is at most that of $\hat{O}_E$; and if $n\,[E : \mathbb{Q}] \leq 2^{v}$, then $\zeta_{\hat{O}_E}(u+v+4)^{n} \leq 1 + 2^{-u}$.

(d) *(Unimodular lattices.)* A hermitian matrix b over $\mathbb{A}_{E,f}$ that has, with its inverse, entries in $\hat{O}_E$ is $k^{*}bk = 1$ for some matrix k with k, k^{-1} integral. The image of a Haar measure of a locally compact group, restricted to an open subgroup, under a continuous homomorphism that is surjective on that subgroup is a Haar measure of the image; and E_{T} has a unit δ with $\bar{\delta} = -\delta$, $\delta, \delta^{-1} \in O_{\mathsf{T}}$, and an element $\tau_0 \in O_{\mathsf{T}}$ with $\tau_0 + \bar{\tau}_0 = 1$.

(e) *(Finite groups.)* The stabiliser of a non-zero vector of k^{n} in $\mathrm{GL}_n(k)$, for a finite field k and $n \geq 1$, has index $\#k^{n} - 1$.

Input 6 (the data; Chapter III and other units). In Setting 2.6:

(a) *(Chapter III.)* The logarithmic bounds of the admissible data, in particular $b(D)^{19} \le g_0(D)g_1(D) \le b(D)^{23}$, the window for $\log N(D)$, $k(D) \le a(D)$, $\pi \le b(D)$, and the inequalities of Setting 2.1 for $\mathfrak{s}(D)$ and $b(D)$; the primes $g_i(D)$ are different from $N(D)$, an integral adele is divisible by $g_0(D)g_1(D)$ exactly if it and its conjugate vanish modulo all chosen places, the conjugate of a chosen place is not chosen, and there are $f(D)$ chosen places above each $g_i(D)$; $\varpi(P)$ generates the different and $\overline{\varpi(P)} = -\varpi(P)$.

(b) *(Columns.)* The class automorphism groups of $K_0^{\mathrm{col}}(D)$ are trivial, so that the number of its classes is a class sum, which for a subgroup of K_f is its index times the class sum of K_f; the class sum of K_f is $\mathrm{mass}(E(D), k(D))$; $K_0^{\mathrm{col}}(D)$ is the group of the elements of K_f that reduce to 1 at all chosen places and whose row at the last index is congruent to that of the identity modulo every place dividing $\hat{\pi}(D)$; there are $f(D)$ such places, each with $N(D)$ elements in its residue field, none conjugate to another.

(c) *(The filling data fit.)* The filling data $S(D)$ of Definition 7.1 fit the space $Y_0(D)$ of the rows in the sense of the note on Poincaré series, and every family of residue matrices at the chosen places is the reduction of an integral element of the finite adelic unitary group; consequently $\mathrm{red}_{S(D)}$ is surjective. Under inc_S, an element y lies in $\mathrm{inc}_S^{-1}(K(S))$ exactly if it is integral and its row at k_0 is congruent to that of the identity modulo every place dividing $\bar{\pi}(S)$, and the reduction of $\mathrm{inc}_S(y)$ at a chosen place is 1 exactly if that of y is (set-up of the unit).

(d) *(Bessel functions; units on the discrete series, on the real place and on the exchange.)* For a Harish-Chandra parameter with a large gap: the formal degree is positive and is given by Weyl's dimension polynomial (the reference and its relation to the formal statement are given in Note F, Input 8(g)); the kernel of the discrete series in a frame is idempotent; there is a Bessel function of the type of an irreducible unitary representation of the unitary group of the columns whose matrix coefficient is the coefficient of the lowest type, with the bounds of the Laplace method for steep parameters; a Bessel function with these bounds has a cut pair; the full function of a cut pair satisfies Schur bounds; and a cut Bessel function has band limits.

(e) *(Sizes of the counts.)* A bound for $\#\mathrm{Bc}(S(D))$, the logarithm of the degree $\deg(S(D))$ within an explicit interval, and $|\log(\mathrm{mass}(E(D), i+1)/\mathrm{mass}(E(D), i)) - \cdots|$ bounded for $k(D) \le i < r(D)$, as the formal text states them (the note on the degree bounds of this unit and the library on masses).

(f) *(Line coordinates; unit on the rows.)* With the notation of the note *The height of a ratio of lift volumes*: the polynomial of the line coordinates $\mathrm{lc}_D(j, i, Y)$ in the weight of the first columns is the product Δ of the stacked determinants of the matrices $\mathrm{dr}_D(j, i, Y, \tau)$ along $\mathrm{pe}_D(j)$, read as a polynomial in the first columns with coefficients polynomials in the other columns.

4 The dictionary between adelic and local objects

Lemma 4.1 (lifting from the places of T)**.** *In Setting 2.1, let T be a finite set of finite places of E that is stable under complex conjugation. For every $y \in G(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$ there is $x \in G_S^{\mathrm{lat}}$ with $\mathrm{loc}_{S,\mathsf{T}}(x) = y$ and $x \in \mathrm{Int}^{\mathsf{T}}(S)$, such that $x \in \mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ if $y \in \mathcal{U}(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$, and $x \in \mathcal{H}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ if $y \in \mathcal{H}(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$.*

Proof. Let $e = y - 1$ and $x = 1 + \tilde{e}$, the extension by zero taken entry by entry. Let J be the split Gram matrix of $T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})$ and J_{T} that of $T_S^{\mathrm{lat}}(E_{\mathsf{T}})$, the matrix of the components of J. Extension by zero is multiplicative and commutes with conjugation (Input 3(a)), and $\tilde{a}\,b = \widetilde{a\,b_{\mathsf{T}}}$ for a finite adele b. So

$$x^* J x - J = \tilde{e}^* J + J\tilde{e} + \tilde{e}^* J \tilde{e} = e^* J_{\mathsf{T}} + \widetilde{J_{\mathsf{T}} e} + e^* J_{\mathsf{T}} e = y^* \widetilde{J_{\mathsf{T}} y} - J_{\mathsf{T}} = 0,$$

and $x \in G_S^{\mathrm{lat}}$. Its components at T are those of y; off T it is the identity, and its inverse is $1 + \widetilde{y^{-1} - 1}$, so $x \in \mathrm{Int}^{\mathsf{T}}(S)$. The groups $\mathcal{U}$ and $\mathcal{H}$ are defined by conditions that certain entries are 0 or 1; these pass from y to x. □

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Lemma 4.2 (level groups at T)**.** *In Setting 2.1, let T be a finite set of finite places of E that is stable under complex conjugation, and $x \in G_S^{\mathrm{lat}}$.*

(a) *For a finite idele u such that u^{-1} is integral off T: $x \in \hat{K}_S(u)$ exactly if $x \in \mathrm{Int}^{\mathsf{T}}(S)$ and $\mathrm{loc}_{S,\mathsf{T}}(x) \in K_{S,\mathsf{T}}(u_{\mathsf{T}})$.*
(b) *$x \in K(S)$ exactly if $x \in \mathrm{Int}^{\mathsf{T}}(S)$ and $\mathrm{loc}_{S,\mathsf{T}}(x) \in K_{S,\mathsf{T}}(1)$.*

Suppose moreover that we are in Setting 2.4, with a filling level $\mathcal{L}$, and that $\hat{g}(S)^{-1}$ is integral off T.

(c) *$x \in K_g(S)$ exactly if $x \in \mathrm{Int}^{\mathsf{T}}(S)$ and $\mathrm{loc}_{S,\mathsf{T}}(x) \in \mathrm{K}^\circ(\mathcal{L})$.*
(d) *For $k, l \in K(S)$: $\mathrm{red}_S(k) = \mathrm{red}_S(l)$ exactly if $\mathrm{loc}_{S,\mathsf{T}}(k)^{-1}\,\mathrm{loc}_{S,\mathsf{T}}(l) \in \mathrm{K}^\circ(\mathcal{L})$.*

Proof. (a) A finite adele is integral exactly if it is integral off T and its components at T are integral (Input 3(a)). If the entries of x are integral off T, so are those of $u^{-1}(x-1)$, because u^{-1} is. So the three conditions that define $\hat{K}_S(u)$ split into $x \in \mathrm{Int}^{\mathsf{T}}(S)$ and the three conditions at T, which define $K_{S,\mathsf{T}}(u_{\mathsf{T}})$. (b) is (a) for $u = 1$. (c) is (a) for $u = \hat{g}(S)$, as $K_g(S) = \hat{K}_S(\hat{g}(S))$ (Input 3(a)) and $\mathrm{K}^\circ(\mathcal{L}) = K_{S,\mathsf{T}}(g_{\mathcal{L}})$ with $g_{\mathcal{L}} = \hat{g}(S)_{\mathsf{T}}$. (d) The reductions agree exactly if $k^{-1}l \in K_g(S)$; as $k^{-1}l \in K(S)$ is integral off T by (b), this is the condition of (c). □

LEAN `Filling.mem_unitaryLevel_iff` · `Filling.mem_latStabilizer_iff` · `Filling.IsFillingLevel.mem_principalLevel_iff` · `Filling.IsFillingLevel.reduction_eq_iff`

Lemma 4.3 (the unipotent character at T)**.** *In Setting 2.4, suppose that $\hat{g}(S)^{-1}$ is integral off T, and let $x \in G_S^{\mathrm{lat}}$ with $x \in \mathrm{Int}^{\mathsf{T}}(S)$. Then $\psi_{\mathsf{T}}(g_{\mathcal{L}}^{-1}\,\mathrm{sd}(\mathrm{loc}_{S,\mathsf{T}}(x))) = \psi_S^{\mathcal{U}}(x)$.*

Proof. The left side is $\psi_{S,\mathsf{T}}(a_{\mathsf{T}})$ for the finite adele $a = \hat{g}(S)^{-1}\,\mathrm{sd}(x)$, because $\psi_{\mathsf{T}} = \psi_{S,\mathsf{T}}$ and $g_{\mathcal{L}} = \hat{g}(S)_{\mathsf{T}}$. As a is integral off T, this is $\psi_S(a) = \psi_S^{\mathcal{U}}(x)$ by Input 3(b). □

LEAN `Filling.IsFillingLevel.psi_simpleRootSum_eq_unipotentChar`

Lemma 4.4 (moduli)**.** *In Setting 2.1, let T be a finite set of finite places of E that is stable under complex conjugation, and suppose that E_{T} carries an integral place structure.*

(a) *For a unit x of E such that x and x^{-1} are integral off T:* $\mathrm{mod}_{\mathsf{T}}(x_{\mathsf{T}})^{1/2} = \left(\prod_\tau |x_\tau|\right)^{-1}$.

Suppose moreover that we are in Setting 2.4, with a filling level $\mathcal{L}$, and that $\hat{g}(S)^{-1}$ is integral off T.

(b) $\mathrm{mod}_{\mathsf{T}}(g_{\mathcal{L}}) = g(S)^{-2f(E)}$.
(c) $\mathrm{uo}(S) \cdot \mathsf{v}(\mathcal{L}) = 1$.

Proof. (a) By Input 3(b) for x and for x^{-1}, $[x_{\mathsf{T}}]^{1/2} \le \prod_\tau |x_\tau|$ and $[x_{\mathsf{T}}]^{-1/2} \le \prod_\tau |x_\tau|^{-1}$; so $[x_{\mathsf{T}}]^{1/2} = \prod_\tau |x_\tau|$, and $\mathrm{mod}_{\mathsf{T}}(x_{\mathsf{T}}) = [x_{\mathsf{T}}]^{-1}$. (b) is (a) for the integer $x = g(S)$, whose absolute value at every infinite place is $g(S)$; its inverse is integral off T by hypothesis. (c) By (b),

$$\mathsf{v}(\mathcal{L}) = \prod_{l<\varrho} g(S)^{-2f(E)\,(k(\mathfrak{s})+1+2(\varrho-l-1)+1/2)} = g(S)^{-f(E)\,\varrho\,(2k(\mathfrak{s})+2\varrho+1)},$$

since $\sum_{l<\varrho}(k(\mathfrak{s}) + \frac{3}{2} + 2(\varrho - l - 1)) = \varrho(k(\mathfrak{s}) + \varrho + \frac{1}{2})$; this is $\mathrm{uo}(S)^{-1}$. □

LEAN `IntegralPlace.md_evalPlaces_rpow_half` · `Filling.IsFillingLevel.md_g` · `Filling.IsFillingLevel.g_pow_mul_unipotentMass`

Proof of Theorem C. (iii) A finite adele is integral at all but finitely many places; so are the finitely many entries of x and of x^{-1}.

(ii) Given a finite set T' of finite places, the finitely many finite adeles in question are integral off every sufficiently large set, by (iii); choose such a set that contains T', and add the conjugates of its places.

(i) Write $A = E_{\mathsf{T}}$. *The place structure.* By Input 3(d) there are a unit δ with $\bar{\delta} = -\delta$ and an element τ_0 with $\tau_0 + \bar{\tau}_0 = 1$, integral with integral inverse of δ. The data of the structure are: δ and τ_0; the Haar measure of the skew elements that gives mass 1 to the integral ones; the modulus $\mathrm{mod}_{\mathsf{T}}(u) = [u]^{-1}$; the character $\psi_{S,\mathsf{T}}$; and the Plancherel constant 1. The character is self-dual: for $a \in A$, $\psi_{S,\mathsf{T}}(ra) = 1$ for all $r \in O_{\mathsf{T}}$ exactly if $\psi_S(r'\tilde{a}) = 1$ for all $r' \in \hat{O}_E$, that is, exactly if $\tilde{a} \in \hat{O}_E$ (Input 3(b)), that is, $a \in O_{\mathsf{T}}$ (Input 3(a)). The formal text verifies the further axioms of a place structure for these data: the decomposition of the Haar measure of A along the map $(\beta, \beta') \mapsto \tau_0\delta\beta + \beta'$ from pairs of skew elements, the continuity of the modulus, a locally integrable majorant of $[x]^{1/2}$, and Plancherel's inequality, whose kernel is the indicator function of O_{T} by the self-duality.

The level. By Input 3(a), $T_S^{\mathrm{lat}}(E_{\mathsf{T}})$ is the lattice form for $N(S)$, and its split Gram matrix is a hermitian unit. Put $\iota = \{1, \dots, k(\mathfrak{s}) - 6\varrho - 13\}$ and $m' = 3\varrho + 6$; then $k(\mathfrak{s}) = \#\iota + 2m' + 1$, as $6\varrho + 13 \le k(\mathfrak{s})$. The unit $g = \hat{g}(S)_{\mathsf{T}}$ is integral and fixed by the involution. By Input 3(c) there is a lattice level $\mathcal{L}$ with $g_{\mathcal{L}} = g$. It is standard, with $m = 3\varrho + 5$: by Input 3(c), for $u = n(\mathfrak{s})$, $v = 2\varrho$ and $n = n(\mathfrak{s})$, and by $n(\mathfrak{s})[E : \mathbb{Q}] \le 2^{2\varrho}$ (Setting 2.1), $\zeta_A(m)^{n(\mathfrak{s})} \le \zeta_{\hat{O}_E}(m)^{n(\mathfrak{s})} \le 1 + 2^{-n(\mathfrak{s})}$; $m < 9n(\mathfrak{s})$; and $13n(\mathfrak{s}) \le k(\mathfrak{s}) - 6\varrho - 13$ because $1000(2\varrho + 2) \le k(\mathfrak{s})$. Finally the unitary group $G(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$ has a Haar measure invariant on both sides: by Input 3(d) the lattice has an orthonormal basis over $\hat{O}_E$, which identifies G_S^{lat} with the finite adelic unitary group of the standard form, a group with a Haar measure invariant on both sides; the elements that are integral off T form an open subgroup, on which $\mathrm{loc}_{S,\mathsf{T}}$ is surjective by Lemma 4.1; and the image of the restricted measure is a Haar measure of the image. So $\mathcal{L} \in \mathrm{FL}(S, \mathsf{T})$. □

Proposition 4.5 (counting reductions by a measure)**.** *In Setting 2.4, suppose that $\hat{g}(S)^{-1}$ is integral off T. Let B be a subgroup of G_S^{lat} and μ an s-finite measure on $G(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$ such that μ-almost every z is $\mathrm{loc}_{S,\mathsf{T}}(x)$ for some $x \in B$ with $x \in \mathrm{Int}^{\mathsf{T}}(S)$, and such that μ is invariant under right translation by $\mathrm{loc}_{S,\mathsf{T}}(x)$ for every $x \in B$. Then $\#\overline{B}^S \cdot \mu(\mathrm{K}^\circ(\mathcal{L})) = \mu(K_{S,\mathsf{T}}(1))$.*

Proof. For every $s \in \overline{B}^S$ choose $k_s \in B \cap K(S)$ with $\mathrm{red}_S(k_s) = s$, and let $C_s = \{z : z\,\mathrm{loc}_{S,\mathsf{T}}(k_s) \in \mathrm{K}^\circ(\mathcal{L})\}$. By the invariance, $\mu(C_s) = \mu(\mathrm{K}^\circ(\mathcal{L}))$. The sets C_s are pairwise disjoint: if $z \in C_s \cap C_t$, then $\mathrm{loc}_{S,\mathsf{T}}(k_s)^{-1}\mathrm{loc}_{S,\mathsf{T}}(k_t) \in \mathrm{K}^\circ(\mathcal{L})$, so $s = t$ by Lemma 4.2(d). Their union is $K_{S,\mathsf{T}}(1)$ up to a μ-null set. Indeed, each C_s lies in $K_{S,\mathsf{T}}(1)$, because $\mathrm{K}^\circ(\mathcal{L})$ and $\mathrm{loc}_{S,\mathsf{T}}(k_s)$ do (Lemma 4.2(b)). Conversely let $z = \mathrm{loc}_{S,\mathsf{T}}(x) \in K_{S,\mathsf{T}}(1)$ with $x \in B$, $x \in \mathrm{Int}^{\mathsf{T}}(S)$. Then $x \in K(S)$ by Lemma 4.2(b), $s = \mathrm{red}_S(x^{-1})$ lies in $\overline{B}^S$, and $xk_s \in K_g(S)$, so $z\,\mathrm{loc}_{S,\mathsf{T}}(k_s) \in \mathrm{K}^\circ(\mathcal{L})$ by Lemma 4.2(c). Summing the measures gives the claim. □

LEAN `Filling.IsFillingLevel.card_reducedImage_mul`

Corollary 4.6 (the reduction of the Bessel subgroup)**.** *In Setting 2.1, suppose that E carries Weil data, and write $\mathcal{S} = \mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$, $\mathcal{H} = \mathcal{H}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$. Then $\#\overline{\mathcal{S}}^S = \#\overline{\mathcal{H}}^S \cdot \mathrm{uo}(S)$.*

Proof. By Theorem C(ii), for $\mathsf{T}_0 = \emptyset$, and Theorem C(i) there are a set T off which $\hat{g}(S)^{-1}$ is integral and a filling level $\mathcal{L}$ for S at T. Apply Proposition 4.5 to $B = \mathcal{U} = \mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ and $\mu = \eta_{<\varrho}$: this measure is carried by the unipotent group at T, every element of which is the image of an element of $\mathcal{U}$ that is integral off T (Lemma 4.1), and it is invariant under right translation by that group (Input 3(b)). By the same input $\mu(\mathrm{K}^\circ(\mathcal{L})) = \mathsf{v}(\mathcal{L})$ and $\mu(K_{S,\mathsf{T}}(1)) = 1$. Hence $\#\overline{\mathcal{U}}^S \cdot \mathsf{v}(\mathcal{L}) = 1$, and $\#\overline{\mathcal{U}}^S = \mathrm{uo}(S)$ by Lemma 4.4(c). It remains to see that $(\bar{u}, \bar{h}) \mapsto \bar{u}\bar{h}$ is a bijection from $\overline{\mathcal{U}}^S \times \overline{\mathcal{H}}^S$ to $\overline{\mathcal{S}}^S$. It is injective because the two images meet in 1: if $u \in \mathcal{U} \cap K(S)$ and $h \in \mathcal{H} \cap K(S)$ have the same reduction, then $u^{-1}h \in K_g(S)$, so $h \in K_g(S)$ by Input 3(e). It is surjective because an element of $\mathcal{S} \cap K(S)$ is a product uh with $u \in \mathcal{U}$, $h \in \mathcal{H}$, and then $h \in K(S)$ by the same input, hence also $u \in K(S)$. □

LEAN `Filling.card_reducedImage_besselS`

5 The floor group, Weyl elements, and blocks as limits

Lemma 5.1 (positions of the elements of the floor group)**.** *In Setting 2.1, let $\gamma_1 \in G_S(E)$ with $\gamma_1 \in \mathcal{H}_\varrho(T_S^{\mathrm{rat}}(E))$ and let c, c' be classes. Put $l = \mathrm{pos}_S(c, c'; \gamma_1)$, $t = Z_S(\gamma_1)_{\varrho\varrho}$, and let $\omega \in E^\kappa$ be the vector of the coefficients of $\gamma_1 z$ at the basis vectors of κ. Then:*

(i) $l \in \mathcal{H}_\varrho(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$;
(ii) $l\,z = \mathrm{tv}(\bar{\pi}(S), t, y_c^{-1}\omega)$, *with t and ω read in $\mathbb{A}_{E,f}$;*
(iii) $N(S) \cdot \omega^*\omega = t\bar{t} - N(S)^2$;
(iv) *for every sequence p of units of E, $\mathrm{pos}_S(c, c'; w_\varrho(p)\gamma_1) = w_\varrho(p) \cdot l$, where on the right p is read as a sequence of finite ideles.*

Proof. As γ_1 fixes all e_a and e_a^*, it preserves the span of κ and z; write $\gamma_1 z = \sum_j \omega_j x_j + \zeta z$. The form is the identity on κ and $-N(S)$ at z, so $t = h(z, \gamma_1 z) = -N(S)\zeta$, and $h(\gamma_1 z, \gamma_1 z) = h(z, z)$ reads $\omega^*\omega - N(S)\bar{\zeta}\zeta = -N(S)$; this gives (iii). (i) The elements

$h_S(c)$, $h_S(c')$ lie in $\mathcal{H}$ and $\mathrm{r}^f_S(\gamma_1)$ fixes all e_a, e^*_a, since the change to the lattice basis only concerns the middle. (ii) $h_S(c')$ fixes z, so $l\,z = h_S(c)^{-1}\mathrm{r}^f_S(\gamma_1)z$; in the rational basis this is the vector with the κ-part $y_c^{-1}\omega$ and the coefficient $\zeta = -(\bar\pi(S)\overline{\bar\pi(S)})^{-1}t$ at z, and written in the lattice basis it is the tail vector. (iv) The Weyl element only moves the vectors e_a, e^*_a; it commutes with the change of basis and with the elements that fix all e_a and e^*_a, such as $h_S(c)$ and $h_S(c')$. □

LEAN `Filling.termPosition_floorGroup_spec`

Lemma 5.2 (Weyl elements at infinity; tail vectors at T)**.** *In Setting 2.1:*

(a) *For $\gamma \in G_S(E)$, a sequence p of units of E and an infinite place τ of E, $\mathrm{r}^\infty_S(w_\varrho(p)\gamma)_\tau = w_\varrho(p^{(\tau)}) \cdot \mathrm{r}^\infty_S(\gamma)_\tau$, where $p^{(\tau)}$ is the sequence $a \mapsto \mathrm{fs}_S(\tau, a) \cdot (p_a)_\tau$, read as 1 at the indices where this number is not a unit.*

(b) *For a finite set T of finite places of E that is stable under complex conjugation, a finite adele t and $x \in \mathbb{A}^\kappa_{E,f}$: $\mathrm{tv}(\bar\pi(S)_\mathsf{T}, t_\mathsf{T}, x_\mathsf{T}) = \big(\mathrm{tv}(\bar\pi(S), t, x)\big)_\mathsf{T}$.*

Proof. (a) r^∞_S is a homomorphism, and at the place τ it conjugates the image of a matrix by the diagonal matrix of the basis scales. The Weyl element has the entries p_a and $\bar p_a^{-1}$ between e_a and e^*_a; conjugation multiplies p_a by the quotient of the two scales, which is the frame scale $\mathrm{fs}_S(\tau, a)$, a positive real number, and $\bar p_a^{-1}$ by its inverse. (b) Taking components at T is a homomorphism of rings with involution, and the tail vector is given by ring operations. □

LEAN `Filling.ratToInf_weylElem_mul` · `Filling.tailVector_evalPlaces`

Proof of Theorem B. Write $T = T^{\mathrm{rat}}_S(E)$ and $\mathcal{U} = \mathcal{U}(T)$, and let

$$\mathcal{B} = \mathrm{Tm}^{\mathrm{lk}}_S(\Lambda) \cap \mathrm{Tm}_S(c, s, c').$$

As the amplitude sum over $\mathrm{Tm}^{\mathrm{lk}}_S(\Lambda)$ is not zero, there is an element γ_0 of $\mathcal{B}$. By hypothesis its leading minors do not vanish, so by Input 3(a) we can write $\gamma_0 = v^{-1}\, w_\varrho(r)\gamma_1\, v'$ with $v, v' \in \mathcal{U}$, a sequence r of units and $\gamma_1 \in \mathcal{H}_\varrho(T)$. Let $t = h(z, \gamma_1 z) = Z_S(\gamma_1)_{\varrho\varrho}$. By Input 3(c) the last leading minor of $Z_S(\gamma_0)$ is $r_0 \cdots r_{\varrho-1} t$, so t is a unit; the term discriminant of γ_0 is $d\bar d(t\bar t - N(S)^2) \neq 0$; and $\Lambda = \mathrm{link}(\gamma_0) = \mathrm{link}(w_\varrho(p)\gamma_1)$ for every sequence p.

(i) By Lemma 5.1(iii), at an infinite place τ, $|t_\tau|^2 - N(S)^2 = N(S)\sum_j |(\omega_j)_\tau|^2 \geq 0$, and it is not zero because $t\bar t \neq N(S)^2$ in E. The element $k = \mathrm{pos}_S(c, c'; \gamma_0)$ lies in $K(S)$ and has the reduction s, as $\gamma_0 \in \mathrm{Tm}_S(c, s, c')$.

(ii) For $p\colon \{0, \dots, \varrho - 1\} \to E^\times$ and $u, u' \in \mathcal{U}$ put $\gamma_{p,u,u'} = u\, w_\varrho((p \mid t))\gamma_1\, u'^{-1}$. The map $(p, u, u') \mapsto \gamma_{p,u,u'}$ is injective by Input 3(b). Its values lie in $\mathrm{Tm}^{\mathrm{lk}}_S(\Lambda)$: the link is Λ, and the term discriminant is $d\bar d(t\bar t - N(S)^2) \neq 0$ with $d = p_0 \cdots p_{\varrho-1}$ (Input 3(c)). Every $\gamma \in \mathcal{B}$ is a value: by Input 3(a), $\gamma = w^{-1} w_\varrho(q)\gamma_2 w'$ with $\gamma_2 \in \mathcal{H}_\varrho(T)$, and $\mathrm{link}(w_\varrho(q)\gamma_2) = \Lambda = \mathrm{link}(w_\varrho(1)\gamma_1)$ forces $\gamma_2 = \gamma_1$. Therefore the family $\mathbf{1}[\gamma_{p,u,u'} \in \mathrm{Tm}_S(c, s, c')] \cdot a^\sigma_S(c, c'; \gamma_{p,u,u'})$, indexed by the triples, is a reindexing of the summable family of the amplitudes over $\mathcal{B}$ (Input 3(c)), completed by zeros. Its sum is the amplitude sum of the theorem, and summing first over (u, u') gives a summable family

$$L_p = \sum_{(u,u')} \mathbf{1}\big[\gamma_{p,u,u'} \in \mathrm{Tm}_S(c, s, c')\big] \cdot a^\sigma_S(c, c'; \gamma_{p,u,u'})$$

with the same sum. The coset amplitude $a^\sigma_{S,\mathsf{T}}(c, s, c'; w_\varrho((p \mid t))\gamma_1)$ is the same sum with the further factor $\mathbf{1}[\mathrm{pos}_S(c, c; u) \in \mathrm{Int}^{\mathsf{T}}(S)]$. For each (u, u') this factor is 1 for

all sufficiently large T (Theorem C(iii)), and the terms are dominated by the norms of a summable family; by dominated convergence for series the coset amplitudes tend to L_p. $\square$

6 The unfolding

Proof of Theorem A. Write $G_{\mathsf{T}} = G(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$, $g = g_{\mathcal{L}}$, $\mathcal{U} = \mathcal{U}(T_S^{\mathrm{rat}}(E))$, $s = \mathrm{red}_S(k)$ and $P = \mathrm{pos}_S(c, c'; \gamma)$, which is integral off T; for $u, u' \in \mathcal{U}$ put $\gamma_{u,u'} = u\gamma u'^{-1}$.

Step 1: the left side. By Lemma 5.1(iv), $\mathrm{loc}_{S,\mathsf{T}}(P) = w_{\varrho}(p_{\mathsf{T}}) \cdot l_0$ with $l_0 = \mathrm{loc}_{S,\mathsf{T}}(\mathrm{pos}_S(c, c'; \gamma_1))$, so that $\mathcal{J}(Z; p_{\mathsf{T}}, l_0^{-1}) = \mathsf{h}_{\varrho}(\mathcal{L}; p_{\mathsf{T}}, 1) \cdot \mathrm{J}(\mathsf{M}(\mathcal{L}), W_Z(\cdot); \mathrm{loc}_{S,\mathsf{T}}(P))$. Its coordinate at (a, b) is the head density times

$$\int \chi_g(z\, a_a^{-1})\, \overline{\chi_g(\mathrm{loc}_{S,\mathsf{T}}(P)^{-1} z\, a_b^{-1})}\, d\mathsf{M}(\mathcal{L})(z) = \int \chi_g(z)\, \overline{\chi_g(\mathrm{loc}_{S,\mathsf{T}}(P)^{-1} z\, \mathrm{loc}_{S,\mathsf{T}}(k))}\, d\mathsf{M}(\mathcal{L})(z),$$

by the right invariance of $\mathsf{M}(\mathcal{L})$ (substitute $z\, a_a$ for z), the hypothesis $a_a a_b^{-1} \in \mathrm{loc}_{S,\mathsf{T}}(k)\, \mathrm{K}^\circ(\mathcal{L})$ and Input 3(a). The archimedean factor is $\mathrm{J}(dy, W^\sigma; \mathrm{r}_S^\infty(\gamma)) = \int \mathrm{bk}_S^\sigma(\gamma, y)\, dy$. So the left side of the theorem is $\mathsf{h}_{\varrho}(\mathcal{L}; p_{\mathsf{T}}, 1)$ times the integral over (z, y) of

$$\mathsf{r}(z, y) = \chi_g(z) \cdot \overline{\chi_g(\mathrm{loc}_{S,\mathsf{T}}(P)^{-1} z\, \mathrm{loc}_{S,\mathsf{T}}(k))} \cdot \mathrm{bk}_S^\sigma(\gamma, y). \tag{1}$$

Step 2: the right side. For (u, u') let $Q(u, u')$ be the condition of Definition 2.5: $\mathrm{pos}_S(c, c; u) \in \mathrm{Int}^{\mathsf{T}}(S)$ and $\gamma_{u,u'} \in \mathrm{Tm}_S(c, s, c')$. By the left invariance of $\mathsf{M}(\mathcal{L})$ and of dy, for every (u, u'),

$$\mathsf{M}(\mathcal{L})(\mathrm{K}^\circ(\mathcal{L})) \cdot a_S^\sigma(c, c'; \gamma_{u,u'}) = \int\!\!\int \mathbf{1}\big[\mathrm{loc}_{S,\mathsf{T}}(\mathrm{pos}_S(c, c; u))\, z \in \mathrm{K}^\circ(\mathcal{L})\big] \cdot \mathsf{a}_{u,u'}(\mathrm{r}_S^\infty(u)\, y)\, d\mathsf{M}(\mathcal{L})(z)\, dy,$$

where $\mathsf{a}_{u,u'}(y) = \beta_{S,c}(y)\, \beta_{S,c'}(\mathrm{r}_S^\infty(\gamma_{u,u'})^{-1} y)\, \mathrm{bk}_S^\sigma(\gamma_{u,u'}, y)$ is the integrand of the amplitude. The sum over the (u, u') with $Q(u, u')$ may be taken under the integral: the integral of the norm of the term of (u, u') is at most $\mathsf{M}(\mathcal{L})(K_{S,\mathsf{T}}(1)) \cdot |a|_S^\sigma(c, c'; \gamma_{u,u'})$, the map $(u, u') \mapsto \gamma_{u,u'}$ is injective (Input 3(b)), and the sum of these numbers over $\mathrm{Tm}_S(c, s, c')$ is finite (Input 3(c); the measure is finite by Input 3(b)). So it suffices to show, for all z and y,

$$\sum_{(u,u')} \mathbf{1}[Q(u, u')] \cdot \mathbf{1}\big[\mathrm{loc}_{S,\mathsf{T}}(\mathrm{pos}_S(c, c; u))\, z \in \mathrm{K}^\circ(\mathcal{L})\big] \cdot \mathsf{a}_{u,u'}(\mathrm{r}_S^\infty(u)\, y) = \mathsf{r}(z, y). \tag{2}$$

Step 3: the two conditions. Fix z and y. By Lemma 4.1, $z = \mathrm{loc}_{S,\mathsf{T}}(x)$ with $x \in \mathrm{Int}^{\mathsf{T}}(S)$. Put $\xi = P^{-1} x k$; it is integral off T (Lemma 4.2(b) for k), and $\mathrm{loc}_{S,\mathsf{T}}(\xi) = \mathrm{loc}_{S,\mathsf{T}}(P)^{-1} z\, \mathrm{loc}_{S,\mathsf{T}}(k)$. We claim that, for $u, u' \in \mathcal{U}$,

$$\begin{aligned} & Q(u, u') \text{ and } \mathrm{loc}_{S,\mathsf{T}}(\mathrm{pos}_S(c, c; u))\, z \in \mathrm{K}^\circ(\mathcal{L}) \\ \iff\quad & \mathrm{pos}_S(c, c; u)\, x \in K_g(S) \text{ and } \mathrm{pos}_S(c', c'; u')\, \xi \in K_g(S). \end{aligned} \tag{3}$$

By Lemma 4.2(c), and since x is integral off T, the element $l_1 = \mathrm{pos}_S(c, c; u)\, x$ lies in $K_g(S)$ exactly if $\mathrm{pos}_S(c, c; u)$ is integral off T and $\mathrm{loc}_{S,\mathsf{T}}(\mathrm{pos}_S(c, c; u))\, z \in \mathrm{K}^\circ(\mathcal{L})$. Suppose this. From $P = x k \xi^{-1}$ we get

$$\mathrm{pos}_S(c, c'; \gamma_{u,u'}) = \mathrm{pos}_S(c, c; u) \cdot P \cdot \mathrm{pos}_S(c', c'; u')^{-1} = l_1 \cdot k \cdot \big(\mathrm{pos}_S(c', c'; u')\, \xi\big)^{-1}.$$

As $l_1 \in K_g(S)$ and $k \in K(S)$ has the reduction s, the left side lies in $K(S)$ with the reduction s exactly if $\mathrm{pos}_S(c', c'; u')\,\xi \in K_g(S)$. This proves (3).

Step 4: phases. Let $u \in \mathcal{U}$ with $\mathrm{pos}_S(c, c; u)\,x \in K_g(S)$. Then

$$\chi_g(z) = \psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u))^{-1}. \tag{4}$$

Indeed $u_1 = \mathrm{pos}_S(c, c; u)$ lies in $\mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ and is integral off T (Input 3(c), Step 3); by Lemma 4.2(c), $\mathrm{loc}_{S,\mathsf{T}}(u_1)\,z \in \mathrm{K}^\circ(\mathcal{L})$, so by Input 3(a) and Lemma 4.3 $\chi_g(z) = \psi_{\mathsf{T}}(g^{-1}\,\mathrm{sd}(\mathrm{loc}_{S,\mathsf{T}}(u_1)^{-1})) = \psi_S^{\mathcal{U}}(u_1)^{-1}$; and $\psi_S^{\mathcal{U}}(u_1) = \psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u))$ by Input 3(c), because $u_1 = h_S(c)^{-1}\mathrm{r}_S^f(u)h_S(c)$. The same holds for u', c' and ξ. With Input 3(b) and the definition of the cutoffs' arguments, for (u, u') satisfying (3) the term of (2) is

$$\begin{aligned}&\beta_{S,c}(\mathrm{r}_S^\infty(u)y)\cdot\beta_{S,c'}(\mathrm{r}_S^\infty(u')\mathrm{r}_S^\infty(\gamma)^{-1}y)\cdot\psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u))^{-1}\psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u'))\cdot\mathrm{bk}_S^\sigma(\gamma,y)\\ &\quad=\beta_{S,c}(\mathrm{r}_S^\infty(u)y)\cdot\beta_{S,c'}(\mathrm{r}_S^\infty(u')\mathrm{r}_S^\infty(\gamma)^{-1}y)\cdot\mathsf{r}(z,y),\end{aligned}$$

by (4) for (u, x) and for (u', ξ), the values of $\psi_S^{\mathcal{U}}(\cdot)$ being of modulus 1.

Step 5: the cutoffs sum to one. So the left side of (2) is $\mathsf{r}(z, y)$ times the product of

$$\Sigma_c(x, y) = \sum_{u \in \mathcal{U}} \mathbf{1}[\mathrm{pos}_S(c, c; u)\,x \in K_g(S)] \cdot \beta_{S,c}(\mathrm{r}_S^\infty(u)\,y)$$

and of $\Sigma_{c'}(\xi, \mathrm{r}_S^\infty(\gamma)^{-1}y)$. If $\mathsf{r}(z, y) = 0$ we are done. Otherwise $\chi_g(z) \neq 0$ and $\chi_g(\mathrm{loc}_{S,\mathsf{T}}(\xi)) \neq 0$, and it suffices to show: if x is integral off T and $\chi_g(\mathrm{loc}_{S,\mathsf{T}}(x)) \neq 0$, then $\Sigma_c(x, y) = 1$ for all y. By Input 3(a) and Lemma 4.1 there is $u_2 \in \mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$, integral off T, with $\mathrm{loc}_{S,\mathsf{T}}(u_2^{-1}x) \in \mathrm{K}^\circ(\mathcal{L})$, hence $u_2^{-1}x \in K_g(S)$ (Lemma 4.2(c)). By Input 3(d) there is a rational $v \in \mathcal{U}$ with $u_2^{-1}\mathrm{pos}_S(c, c; v) \in K_g(S)$. Then, for $u \in \mathcal{U}$, $\mathrm{pos}_S(c, c; u)\,x \in K_g(S)$ exactly if $\mathrm{pos}_S(c, c; uv) \in K_g(S)$, that is, $uv \in \mathcal{U}_{S,c}$; so with $u = wv^{-1}$,

$$\Sigma_c(x, y) = \sum_{w \in \mathcal{U}_{S,c}} \beta_{S,c}(\mathrm{r}_S^\infty(w)\,\mathrm{r}_S^\infty(v)^{-1}y) = 1$$

by the partition of unity of Input 3(a). This proves (2) and the theorem. □

7 The filling data of the admissible data

Definition 7.1 (the filling data of D)**.** In Setting 2.6, the *filling data of* D, $S(D)$, are the filling data over $E(D)$ of the shape $\mathfrak{s}(D)$ with the following components: the number $b(D)$; the prime $N(D)$ and the idele $\hat{\pi}(D)$; the primes $g_0(D)$, $g_1(D)$ with the chosen places of D above them; the generator $\varpi(P)$ of the different; the column level $K_0^{\mathrm{col}}(D)$; and at every infinite place τ a Bessel system chosen among those of Proposition 7.3. The conditions on filling data (Setting 2.1) hold by Input 3(a), (b). In particular $b(S(D)) = b(D)$, $N(S(D)) = N(D)$, $g(S(D)) = g_0(D)g_1(D)$, and for the shape $\mathfrak{s} = \mathfrak{s}(D)$: $\varrho(\mathfrak{s}) = \varrho(D)$, $k(\mathfrak{s}) = k(D)$ and $r(\mathfrak{s}) = 2\varrho(D) + k(D) + 1 = r(D)$.

LEAN `Data.filling`

Lemma 7.2 (three small facts)**.** *In Setting 2.6: the index set $X_0(D)$ of the columns has $k(D)$ elements. Moreover the formal text fixes, as instances, a unitary matrix that diagonalises the real form of the shape $\mathfrak{s}(D)$ with the signs of the space of the rows (a* frame*), and records the bound $b(D)^{10} \leq 2^{\varrho(D)}$.*

Proof. The first is the count of the index set in the unit on the seesaw. The frame exists because the form has as many negative signs as the standard form of these signs, and one is chosen. The bound is one of the logarithmic bounds of Chapter III: $10 \log b(D) \le \varrho(D) \log 2$. □

LEAN `Data.card_X₀` · `Data.unitarySignFrameY` · `Data.instLbHPowRealNatOfNatgHMulLb`

Proposition 7.3 (Bessel systems for the data)**.** *In Setting 2.6, let τ be an infinite place of $E(D)$. Then there is a Bessel system $\mathfrak{B}$ on the real group $G_1(\mathfrak{s}(D))$, for the bound $b(D)$ and the number $f(D)$ in the sense of the set-up, of the type $\mathfrak{B} \in \mathrm{Bes}(2\varkappa_D(\tau,\cdot), \mathsf{w}_D^{\mathrm{col}}(\tau))$, which means:*

(i) *the degree of $\mathfrak{B}$ is the value of Weyl's dimension polynomial at the doubled row parameter $2\varkappa_D(\tau,\cdot)$;*

(ii) *there is an invertible matrix A such that, for every h in the unitary group of the columns, embedded as the subgroup that fixes the frame and its dual, and every point y, the full Bessel function satisfies $A\,W^{\mathrm{full}}(hy) = \mathrm{Sch}(h)\,A\,W^{\mathrm{full}}(y)$, where* Sch *is the transpose of the Schur representation of weight $\mathsf{w}_D^{\mathrm{col}}(\tau)$ at the transposed element; and*

(iii) *W^{full} is fixed by right convolution with the kernel of the discrete series of the parameter $2\varkappa_D(\tau,\cdot)$, read in the frame of Lemma 7.2.*

Proof. An assembly of Input 3(d); its content lies there. The steps, in the order of the formal text: (1) the doubled row parameter λ has entries of absolute value at most $3a(D)$ and a gap of size $r(D)^7$ (Chapter III), and its lowest type for the maximal compact subgroup is the column weight at one position and 0 at the others. (2) The formal degree of λ is positive and equals Weyl's dimension polynomial; the kernel of its discrete series in the frame is idempotent; and the parameter is steep in the sense of the unit on the exchange. (3) The Gram matrix of the Weyl module of the column weight is positive definite, of the form B^*B; conjugating the Schur representation by B gives a unitary representation ρ, and a matrix that commutes with it is scalar. (4) There is a Bessel function of type ρ whose matrix coefficient is that of the lowest type, with the bounds of the Laplace method. (5) It has a cut pair, whose full function satisfies the Schur bounds. (6) The cut function has band limits; the constant that this requires is bounded as follows: each factor $|\lambda_i - \lambda_j|/2$ of Weyl's polynomial is at most $3a(D)$, since both entries are at most $3a(D)$ in absolute value, and the remaining factorials and powers are compared with powers of $b(D)$ by the bounds of Chapter III. These data form a Bessel system; (i) holds by (2), (ii) with $A = B^*$ by (3), and (iii) by the convolution identity of the cut pair and the idempotent kernel. □

LEAN `Data.exists_besselSystem`

Proof of Theorem D. Write $S = S(D)$, $N = N(D)$, $k = k(D)$, $f = f(D)$.

(i) By Input 3(c), $\mathrm{inc}_S^{-1}(K_g(S))$ is the group of the $y \in K_f$ whose row at k_0 is congruent to that of the identity modulo every place dividing $\hat{\pi}(D)$, and whose reduction at every chosen place is 1. By Input 3(b) this is the description of $K_0^{\mathrm{col}}(D)$.

(ii) By Input 3(c), $\mathrm{inc}_S^{-1}(K(S)) \cap K_f$ is the group of the $y \in K_f$ such that, for each of the f places w dividing $\hat{\pi}(D)$, the inverse transpose of the reduction of y at w fixes the basis vector of k_0 in $\mathbb{F}_w^k$. No two of these places are conjugate, so the reduction

maps K_f onto the product of the groups $\mathrm{GL}_k(\mathbb{F}_w)$, and the index is the product of the indices of the stabilisers of a non-zero vector, each equal to N^k-1 (Input 3(e)).

(iii) The class automorphism groups of the column level are trivial, so $\#\mathrm{Cl}(S)$ is its class sum, which is $[K_f : K_0^{\mathrm{col}}(D)] \cdot \mathrm{mass}(E(D),k)$ (Input 3(b)). By (i) the index is

$$[K_f : \mathrm{inc}_S^{-1}(K(S)) \cap K_f] \cdot [\mathrm{inc}_S^{-1}(K(S)) : \mathrm{inc}_S^{-1}(K_g(S))] = (N^k-1)^f \cdot \#\overline{\mathcal{H}}^S,$$

by (ii), and because inc_S maps onto $\mathcal{H}$, so that the second index is the order of the image of $\mathcal{H} \cap K(S)$ under red_S.

(iv) The standing hypotheses are: (1) $K^{\mathrm{col}}(S) = \mathrm{inc}_S^{-1}(K_g(S))$; (2) red_S is surjective; (3) $|\varpi(S)_\tau| \le b(S)$ and $1 \le |\kappa_\tau(S)| \le b(S)^{23}$ for all τ, and $N(S) \le b(S)^{60n(\mathfrak{s})}$; (4) the band inequality $\prod_\tau c_{\mathrm{band}}(\mathfrak{s})\, b(S)^{7k(\mathfrak{s})} \left(|\kappa_\tau(S)|^{2\varrho(\mathfrak{s})+1}/(g(S)N(S))\right)^{k(\mathfrak{s})-n(\mathfrak{s})^7} \le \deg(S)^{-1}$; (5) $\mathrm{tl}(S)^{44}\mathtt{d}_1(S) \le \mathtt{d}(S)$, $\mathrm{tl}(S)^{18}\mathrm{cf}(S)\mathtt{d}(S) \le \mathtt{d}_1(S)$ and $\mathrm{cf}(S) \le 1$; (6) $\#\mathrm{Bc}(S) \le b(S)^{2k(\mathfrak{s})f(E)}$; (7) $(N(S)^{k(\mathfrak{s})}/b(S))^{f(E)}$ is at most the index of (ii); and (8) the bound for the packet constant $a_*(S)$ stated in Note F. Here $\kappa_\tau(S)$, $c_{\mathrm{band}}(\mathfrak{s})$, $\deg(S)$, $\mathrm{tl}(S)$, $\mathtt{d}_1(S)$, $\mathtt{d}(S)$, $\mathrm{cf}(S)$ are the constants of the set-up, as recalled in Note F. (1) is (i), and (2) is Input 3(c). The others are inequalities between positive real numbers. The formal proof makes every quantity explicit: $\#\overline{\mathcal{S}}^S = \#\overline{\mathcal{H}}^S \cdot \mathrm{uo}(S)$ (Corollary 4.6), $\#\mathrm{Cl}(S)$ by (iii), the index by (ii), $\prod_\tau |\varpi(S)_\tau|$ as a power of the root discriminant, the volumes and the covolume by their definitions, and $r(\mathfrak{s}) = k + 2\varrho(D) + 1$. It then takes logarithms; with the bounds for $\#\mathrm{Bc}(S)$, for $\log\deg(S)$ and for the ratios of the masses of the ranks from k to $r(D)$ (Input 3(e)), with $N \le N^k$ and $k\log N - 2/N \le \log(N^k-1)$, each inequality becomes a linear inequality between the logarithmic sizes of the data, which follows from the windows of Chapter III (Input 3(a)). This linear arithmetic is not reproduced here. □

Lemma 7.4 (an expansion of a product of stacked determinants)**.** *In Setting 2.6, with the notation of the note* The height of a ratio of lift volumes*: let j be a lift index, i a line choice with its short rows Y_i, and let Δ be the product of the stacked determinants of the matrices $\mathrm{dr}_D(j,i,Y_i,\tau)$ along $\mathrm{pe}_D(j)$, a polynomial in variables $x_{\tau,a,b}$ with τ an infinite place, $a < 2m(D)$, and b either a number below $k(D)+j-1$ or an index of the space L^+. Let n be a set, f a map from the row indices to n and g a map from the indices of L^+ to n. Let $\Delta^{f,g}$ be the image of Δ under the substitution $x_{\tau,a,b} \mapsto x_{\tau,a,f(\mathrm{bv}_D(b))}$ for the numbers b and $x_{\tau,a,b} \mapsto x_{\tau,a,g(b)}$ for the indices of L^+. Then*

$$\Delta^{f,g} = \sum_l f_*(P_l) \cdot g_*(\mathrm{lc}_D(j,i,Y_i)_l),$$

where l runs over the index set of the basis of the Weyl module of the weight of the first columns, P_l is the letter polynomial *of the l-th basis vector for the positive letters (its polynomial, with the variable of (τ,a) and of the number h replaced by the sum of the variables $x_{\tau,a,i'}$ over the row indices i' of positive sign and of order h, for the order function of the untwisted auxiliary pair), and f_*, g_* rename the last index of the variables.*

Proof. By Input 3(f) and the linearity of the polynomial of a vector, Δ, read as a polynomial in the first columns with coefficients in the variables of L^+, is $\sum_l p_l \cdot \mathrm{lc}_D(j,i,Y_i)_l$ with p_l the polynomial of the l-th basis vector. The substitution of the statement and the map that sends p_l to $f_*(P_l)$ and renames the coefficients by g are homomorphisms of algebras, so it suffices to compare them on the variables. On

a variable of L^{+} both are g_{*}. On the variable of (τ, a) and a number h, the second gives the sum of the $x_{\tau,a,f(i')}$ over the row indices i' of positive sign and order h; there is exactly one such index, $i' = \mathrm{bv}_{D}(h)$, because the order of such an index is its number, the numbering of the row indices is a bijection, and the sign is positive exactly for the numbers below $k(D) + j - 1$. □

LEAN `Data.stdClassSum_carrierXi_eq_lineMultiplier`

8 Where the results are used

Note F (*Translates of Poincaré series fill the rows*) quotes Theorem C as its Input 4(a); Lemma 4.1, Lemma 4.2, Lemma 4.3 and Lemma 4.4 as Input 4(b); Proposition 4.5 and Corollary 4.6 as Input 4(c); Lemma 5.1 and Lemma 5.2 as Input 4(d); Theorem B as Input 4(e); Theorem A as Input 4(f); and Definition 7.1, Proposition 7.3 and Theorem D as Input 9(a). Outside this unit, Proposition 7.3 is used in the unit on the seesaw, where the filling data are shown to fit the space of the rows (the module on generic Bessel functions), and Theorem D(i) in the seesaw theorem. Lemma 7.4 is used in the unit on the rows, in the module on the separation of the lines. Theorem D(ii), (iii) and Corollary 4.6 are used in the proof of Theorem D(iv).

9 Notes on the formal text

1. *Arguments that the census reads differently from Lean; to be settled by* `#check` *on a build.* In the three statements of Theorem C(ii), (iii) the set T is bound by "for all sufficiently large" or "there are arbitrarily large"; the census lists it as an argument, with its stability under conjugation, because a variable of the file has this name. The same holds for the stability of T in Theorem B and Corollary 4.6, where T is bound or absent. For Lemma 4.4(c) the census lists no hypothesis although the statement names the level; the note states the setting that Lean takes.
2. *Statements cut by the census.* The statements of Lemma 5.1 and of Theorem A contain local definitions at which the census ends the statement; they were read to their true end, with the second reader's table of corrected spans.
3. *The side.* The side is an instance argument of the coset amplitude and of Theorems A and B, with the full side as default instance; the note writes it as the superscript σ.
4. *Forms of the formal statements.* In Theorem A the two sides are written with scalar multiplications, and the measure of the congruence subgroup is taken as a real number. In Theorem B the family is indexed by the maps from $\{0, \dots, \varrho - 1\}$, the element k is given as an element of $K(S)$, and the limit is along all finite sets of finite places, not only the stable ones. In Definition 2.3 the index set of the orthonormal part and the number of further pairs are fixed by the type of the level. In Lemma 4.4(b) the right side is written as the square of the inverse of $g(S)^{f(E)}$.
5. *A hypothesis stronger than its use.* Proposition 4.5 assumes the invariance of μ under the images of all elements of B; the proof uses it only for the representatives $k_{s} \in B \cap K(S)$, which are integral. A statement with invariance only under the images of the elements of B that are integral off T is true by the same proof, but

it is not the formal one. Note F quotes the formal hypothesis in its Input 4(c), and its one application (to $B = \mathcal{H}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ and to the measure of the level on the group $\mathcal{H}(T_S^{\mathrm{lat}}(E_{\mathsf{T}}))$) satisfies it.

6. *Definition 7.1.* The formal definition gives several components of the filling data only through the proofs of their properties; they were read off these proofs. The identification of the group $\mathsf{G}(S(D))$ with $\mathsf{G}(D)$, used in Note F, is not a statement of the modules of this note.
7. *Lemma 7.4.* Its formal name speaks of a class sum and a line multiplier; the statement is the polynomial identity given here, and the class sums appear where it is used.
8. *Points where the dossier's first reading was corrected by the second reader.* In Lemma 4.4(a) the summary of the proof gave the wrong power of the index; the statement is as above. In the proof of Theorem C(i) the self-duality is the statement about the annihilator of O_{T}, with the quantifier over r. In Proposition 7.3 the bound $3a(D)$ holds for each entry of the parameter (so $6a(D)$ for a difference, before the division by 2). In Theorem D(iv) the elementary fact used is $N \le N^k$.
9. *What was not re-derived.* The linear arithmetic of Theorem D(iv); the constructions behind Proposition 7.3; the axioms of a place structure in Theorem C(i) other than the self-duality; in Lemma 7.4, that the order of a row index of positive sign is its number, which is taken from the last step of the formal proof. The measurability and summability conditions of the interchange of sum and integral in Theorem A are verified in the formal text; the text above gives the bound that is used.

How this note was written. From the structured proofs of the three modules in the third chunk of the unit's dossier, all of whose statements and claims a second reader had re-read; her remarks were taken into account. I read the three modules in the Lean text line by line and compared every statement of this note with it argument by argument; I read in the Lean text the definitions of the filling data, of admissibility, of Bessel systems and their type, of the character at T, and of the notions of the unit on the rows that Lemma 7.4 needs. The inputs are stated from the dossier's readings of their quoted statements. Arguments that are mine and shorter than the formal ones: the proof of Lemma 4.1 by the identity for $x^{*}Jx - J$, the proof of Lemma 5.1 in coordinates, and the organisation of the proof of Theorem A in five steps. Nothing was compiled for this note. The program check of the unit's dossier covered 1371 leaf steps of 179 proofs.

Appendix A. Concordance with the Lean artifact

The formal names of the objects and the letters of this note: `IsIntegralOff`, `toPlaces`, `IsFillingLevel`, `cosetAmplitude` are $x \in \mathrm{Int}^{\mathsf{T}}(S)$, $\mathrm{loc}_{S,\mathsf{T}}$, $\mathcal{L} \in \mathrm{FL}(S,\mathsf{T})$, $a^{\sigma}_{S,\mathsf{T}}(c,s,c';\gamma)$; `Data.filling` is $S(D)$; `LocProd`, `evalPlaces`, `adeleOfLocal`, `integralOff` are E_{T}, $x \mapsto x_{\mathsf{T}}$, $y \mapsto \tilde{y}$, integrality off T; `unitaryLevel` is $\hat{K}_S(u)$ over the finite adeles and $K_{S,\mathsf{T}}(u)$ over E_{T}; `latStabilizer`, `principalLevel`, `reduction`, `reducedImage` are $K(S)$, $K_g(S)$, red_S, $\overline{B}^{S}$; `termPosition`, `termsOver`, `termsOfLink` are $\mathrm{pos}_S(c,c';\gamma)$, $\mathrm{Tm}_S(c,s,c')$, $\mathrm{Tm}^{\mathrm{lk}}_S(\Lambda)$; `selfDualChar`, `charOnPlaces`, `unipotentChar` are ψ_S, $\psi_{S,\mathsf{T}}$, $\psi^{\mathcal{U}}_S(u)$; `besselKernel`, `classCutoff`, `amplitude`, `amplitudeSum` are

$\mathrm{bk}^{\sigma}_{S}(\gamma, y)$, $\beta_{S,c}$, $a^{\sigma}_{S}(c, c'; \gamma)$, $A^{\sigma}_{S}(\Xi; c, s, c')$; `unipOrder`, `unipotentMass`, `frameScaleSq` are $\mathrm{uo}(S)$, $\mathsf{v}(\mathcal{L})$, $\mathrm{fs}_{S}(\tau, a)$; `padTuple`, `tailVector`, `weylElem` are $(p \,|\, t)$, $\mathrm{tv}(\pi, t, x)$, $w_{\varrho}(p)$; the element `σ` of the formal statements is γ_1, the set `T` is T, the level `F` is $\mathcal{L}$; `paramW` and `Data.κ` are the doubled row parameter and the column weight. Two statements as the artifact has them:

```
lemma IsFillingLevel.correlation_smul_eq_cosetAmplitude [S.IsAdmissible] [Side] {σ p c c'} {Z :
  F.IntermediateGroup} {k : S.latStabilizer} (υ : σ ∈ GT 𝔰.ϱ) (o : S.IsIntegralOff T
  (S.termPosition c c' (weylElem 𝔰.ϱ p * σ))) {a b : Z.Cosets} (e : (S.toPlaces T k)⁻¹ * a.out.1 *
  b.out.1⁻¹ ∈ F.congGroup) : let q := Units.map (ratToPlaces T) ∘ p; let γ := weylElem 𝔰.ϱ p * σ
    F.correlation Z q (S.toPlaces T (S.termPosition c c' σ))⁻¹ (a, b) • besselCorrelation volume
      (.toLp 2 <| S.besselFn ·) (S.ratToInf γ) = (F.headDensity 𝔰.ϱ q 1).toReal •
      F.groupMeasure.real F.congGroup • S.cosetAmplitude T c (S.reduction k) c' γ

lemma exists_hasSum_cosetAmplitude [WeilData E] [S.IsAdmissible] {π : Side} {c s c' Λ} (h :
  S.amplitudeSum (S.termsOfLink Λ) c s c' ≠ 0) (hs : ∀ γ ∈ S.termsOver c s c', ∀ l, leadingMinor
  (S.frameProducts γ) l ≠ 0) : ∃ (t : Eˣ) (σ : S.G E), ∃ k, (σ ∈ GT 𝔰.ϱ ∧ S.frameProducts σ 𝔰.top
  𝔰.top = t ∧ (∀ v, S.N ^ 2 < ‖Φ v t.1‖ ^ 2) ∧ S.reduction k = s ∧ Λ = link (weylElem 𝔰.ϱ 1 * σ)) ∧
  ∃ L, HasSum L (S.amplitudeSum (S.termsOfLink Λ) c s c') ∧ ∀ p : Fin 𝔰.ϱ → _, Filter.Tendsto
  (fun T ↦ S.cosetAmplitude T c s c' (weylElem 𝔰.ϱ (padTuple p t) * σ)) .atTop (nhds (L p))
```

In this note	Lean declaration	File	Lines
Theorem D	`Data.relIndex_latStabilizer`	FillingAdmissible.lean	16–28
Theorem D	`Data.card_classes_filling`	FillingAdmissible.lean	30–37
Lemma 7.4	`Data.stdClassSum_carrierXi_eq_lineMultiplier`	FillingData.lean	26–36
Lemma 7.2	`Data.card_X₀`	FillingData.lean	42–42
Lemma 7.2	`Data.unitarySignFrameY`	FillingData.lean	45–49
Proposition 7.3	`Data.exists_besselSystem`	FillingData.lean	56–118
Lemma 7.2	`Data.instLbHPowRealNatOfNatϱHMulLb`	FillingData.lean	122–123
Definition 7.1	`Data.filling`	FillingData.lean	127–140
Theorem D	`Data.columnLevel_zero_eq_comap`	FillingData.lean	144–149
Definition 2.2	`Filling.IsIntegralOff`	GlobalToLocal.lean	28–28
Definition 2.5	`Filling.cosetAmplitude`	GlobalToLocal.lean	32–34
Definition 2.2	`Filling.toPlaces`	GlobalToLocal.lean	41–42
Definition 2.3	`Filling.IsFillingLevel`	GlobalToLocal.lean	46–49
Theorem C	`Filling.eventually_mem_integralOff`	GlobalToLocal.lean	52–53
Theorem C	`Filling.eventually_isIntegralOff`	GlobalToLocal.lean	57–58
Theorem C	`Filling.exists_places_integralOff`	GlobalToLocal.lean	63–67
Lemma 5.2	`Filling.ratToInf_weylElem_mul`	GlobalToLocal.lean	71–74
Lemma 4.1	`Filling.exists_toPlaces_eq`	GlobalToLocal.lean	78–89
Lemma 4.2	`Filling.mem_unitaryLevel_iff`	GlobalToLocal.lean	91–94
Lemma 4.2	`Filling.mem_latStabilizer_iff`	GlobalToLocal.lean	96–97
Theorem C	`Filling.exists_isFillingLevel`	GlobalToLocal.lean	102–155
Lemma 5.1	`Filling.termPosition_floorGroup_spec`	GlobalToLocal.lean	159–173
Theorem B	`Filling.exists_hasSum_cosetAmplitude`	GlobalToLocal.lean	178–212
Lemma 5.2	`Filling.tailVector_evalPlaces`	GlobalToLocal.lean	214–217
Lemma 4.4	`IntegralPlace.md_evalPlaces_rpow_half`	GlobalToLocal.lean	222–229
Lemma 4.2	`Filling.IsFillingLevel.mem_principalLevel_iff`	GlobalToLocal.lean	231–233
Lemma 4.2	`Filling.IsFillingLevel.reduction_eq_iff`	GlobalToLocal.lean	235–238
Lemma 4.3	`Filling.IsFillingLevel.psi_simpleRootSum_eq_unipotentChar`	GlobalToLocal.lean	240–243
Lemma 4.4	`Filling.IsFillingLevel.md_g`	GlobalToLocal.lean	245–247
Lemma 4.4	`Filling.IsFillingLevel.g_pow_mul_unipotentMass`	GlobalToLocal.lean	249–253
Proposition 4.5	`Filling.IsFillingLevel.card_reducedImage_mul`	GlobalToLocal.lean	258–269
Corollary 4.6	`Filling.card_reducedImage_besselS`	GlobalToLocal.lean	275–296
Theorem A	`Filling.IsFillingLevel.correlation_smul_eq_cosetAmplitude`	GlobalToLocal.lean	302–376

In this note	Lean declaration	File	Lines
Theorem D	`instance : D.filling.IsAdmissible`	FillingAdmissible.lean	40–81

Notation

E, $x \mapsto \bar{x}$, τ, v, x_τ, $f(E)$ a CM field with its CM type, conjugation, infinite places, the image at a place, their number

w, $\bar{w}$, $\mathbb{F}_w$, E_w, O_w, $\mathbb{A}_{E,f}$, $\hat{O}_E$, ψ_f a finite place, its conjugate, residue field, completion, integers; finite adeles, integral adeles, the standard character

T, T_0, T', E_T, O_T, x_T, $\tilde{y}$ finite sets of places (of units for T_0), the product of completions, its integers, components, extension by zero

ψ_T, mod_T, $[u]$, δ, τ_0, β, A character, modulus, generalised index, a skew unit, a trace-one element and skew elements of the place structure on $A = E_\mathsf{T}$

$\mathfrak{s} = (\varrho, \kappa, k_0)$, $\varrho(\mathfrak{s})$, $k(\mathfrak{s})$, $n(\mathfrak{s})$, $r(\mathfrak{s})$ a shape and its numbers; in proofs ϱ abbreviates $\varrho(\mathfrak{s})$

$I(\mathfrak{s})$, e_a, e_a^*, z, x_j, $\mathfrak{f}_a$, h index set, basis vectors, the frame, the hermitian form

T, $J(T)$, $G(T)$, $Z_T(x)$, $\mathcal{U}(T)$, $\mathcal{H}(T)$, $\mathcal{S}(T)$, $\mathcal{H}_\varrho(T)$ a form, split Gram matrix, unitary group, frame Gram matrix, Bessel unipotent subgroup, the subgroup fixing the frame and its dual, Bessel subgroup, floor group; $\mathcal{U}$, $\mathcal{H}$, $\mathcal{S}$ in proofs

$\mathrm{sd}(u)$, $w_\varrho(p)$, $(p \,|\, t)$, $\mathrm{tv}(\pi, t, x)$ sum above the diagonal, Weyl element, padded sequence, tail vector

S, $b(S)$, $N(S)$, $\bar{\pi}(S)$, $q_i(S)$, $g(S)$, $\mathrm{Ch}_i(S)$, $\varpi(S)$, $K^{\mathrm{col}}(S)$, $\mathfrak{B}_\tau(S)$ filling data and their components

$T_S^{\mathrm{rat}}(R)$, $T_S^{\mathrm{lat}}(R)$, $G_S(R)$, G_S^{lat}, $\Lambda(S)$, r_S^f, r_S^∞, inc_S, $\mathrm{fs}_S(\tau, a)$ forms, groups, the lattice basis, the maps to the finite adeles, to infinity and from the unitary group of the columns, frame scales

$\hat{K}_S(u)$, $K(S)$, $K_g(S)$, $\hat{g}(S)$, red_S, $\mathsf{G}(S)$, $\overline{B}^S$, $K_{S,\mathsf{T}}(u)$ level groups, the idele of g, reduction, the finite group, reduced image, level groups at T

ψ_S, $\psi_{S,\mathsf{T}}$, $\psi_S^{\mathcal{U}}(u)$, $\mathrm{uo}(S)$ characters, unipotent character, the order

$\mathrm{Cl}(S)$, c, c', y_c, $h_S(c)$, $\mathrm{pos}_S(c, c'; \gamma)$, $\mathrm{Tm}_S(c, s, c')$, $\mathcal{U}_{S,c}$ classes, representatives, positions, terms, the unipotent lattice of a class

$Z_S(\gamma)$, $\mathfrak{g}_S(\gamma)$, $\mathrm{link}(\gamma)$, Λ, $\mathrm{Tm}_S^{\mathrm{lk}}(\Lambda)$ frame Gram matrix, term discriminant, links, the terms of a link

$\mathrm{Bc}(S)$, σ, $W_S^\sigma(y)_i$, W^σ, $\mathrm{bk}_S^\sigma(\gamma, y)$, $\beta_{S,c}$, $a_S^\sigma(c, c'; \gamma)$, $|a|_S^\sigma(c, c'; \gamma)$, $A_S^\sigma(\Xi; c, s, c')$ Bessel coordinates, a side, Bessel functions, kernel, cutoffs, amplitudes, amplitude sums

$x \in \mathrm{Int}^{\mathsf{T}}(S)$, $\mathrm{loc}_{S,\mathsf{T}}$, $\mathcal{L} \in \mathrm{FL}(S, \mathsf{T})$, $a_{S,\mathsf{T}}^\sigma(c, s, c'; \gamma)$ integral off T, components at T, filling levels, coset amplitude

$\mathcal{L}$, $g_\mathcal{L}$, $\mathrm{K}^\circ(\mathcal{L})$, $\mathcal{L} \in \mathrm{StdLev}$, $\mathsf{M}(\mathcal{L})$, $\eta_{<j}$, $\mathsf{d}_l(\mathcal{L}; g)$, $\mathsf{v}(\mathcal{L})$, $\mathsf{h}_\varrho(\mathcal{L}; p, 1)$ a level, its unit, congruence subgroup, standard levels, group measure, Heisenberg measures, floor modulus, unipotent mass, head density

$\chi_g(y)$, $\mathrm{Int}(\mathcal{L})$, Z, $\mathrm{Cos}(Z)$, a_c, $W_Z(y)$, $\mathcal{J}(Z; p, l)$, $\mathrm{J}(\mu, W; \gamma)$ finite Bessel function, intermediate groups, cosets, representatives, finite Bessel vector, correlation, Bessel correlation

γ, γ_0, γ_1, γ_2, $\gamma_{u,u'}$, $\gamma_{p,u,u'}$, p, q, r, t, ω, ζ, d rational elements, sequences of units, the corner entry, the vector and coefficient of $\gamma_1 z$, a product

k, l, l_0, l_1, x, ξ, P, u, u', u_0, u_1, u_2, v', w', s, a, b, y elements of the level groups and adelic elements of the proofs, cosets, a point at infinity

L_p, $\mathcal{B}$, $Q(u, u')$, r, $\mathsf{a}_{u,u'}$, Σ_c, C_s, k_s, B, μ, J, e, G_T the limits, a block, the condition, integrands, sums of cutoffs, sets, representatives, a subgroup, a measure, a Gram matrix, a difference, the group at T

P, D, $E(D)$, $m(D)$, $r(D)$, $k(D)$, $\varrho(D)$, $f(D)$, $b(D)$, $a(D)$, $N(D)$, $\hat{\pi}(D)$, $\pi_N(D)$, $g_i(D)$, $\varpi(P)$, N_0 parameters, admissible data and their numbers

$X_0(D)$, $K_0^{\mathrm{col}}(D)$, $\mathfrak{s}(D)$, $\mathsf{G}(D)$, $S(D)$, K_f, $\varkappa_D$, $\mathsf{w}_D^{\mathrm{col}}$, λ index set and level of the columns, the shape, the finite group, the filling data, integral points, row parameter, column weight, the doubled parameter

$\mathfrak{B}$, $\mathfrak{B} \in \mathrm{Bes}(l, w)$, W^{full}, A, Sch, ρ, $G_1(\mathfrak{s})$ a Bessel system, its type, the full Bessel function, an intertwiner, the Schur representation, the real group

$\kappa_\tau(S)$, c_{band}, $\deg(S)$, $\mathrm{tl}(S)$, $\mathsf{d}_1(S)$, $\mathsf{d}(S)$, $\mathrm{cf}(S)$, $a_*(S)$, $\mathrm{mass}(E, r)$ constants of the set-up, the mass

Δ, $\Delta^{f,g}$, $x_{\tau,a,b}$, $\mathrm{bv}_D(b)$, $\mathrm{pe}_D(j)$, $\mathrm{dr}_D(j, i, Y, \tau)$, $\mathrm{lc}_D(j, i, Y)$, Y_i, L^+, P_l, p_l, f, g, n, j, i, i' the objects of Lemma 7.4

m, m', ι, n, ζ_A, $\zeta_{\hat{O}_E}$, $\ell^2(X)$, X, R, Ξ, N, K, F, O numbers of a level, zeta functions, $\mathbb{C}^X$, a ring, a set of rational elements, and letters in names

READBACK NOTE 39

Links of pairs of terms: integral lifts and the discriminant gain

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. In the second moment of the Poincaré series, two rational terms γ, γ' over the same classes and the same element of the level group interact through their *links*, two Lagrangian subspaces of a split hermitian space of dimension $2(k(\mathfrak{s})+1)$. This note proves the arithmetic input for the pairs whose links differ: the product of the invariants of the pair, which measure the lattice of the common part of the links, is smaller than the product of the gauges of the two terms by a power of $2/g(S)$ for each dimension in which the links differ. The proof computes the discriminant of the lattice of integral lifts of a link by a bordered determinant and Jacobi's identity, splits this lattice orthogonally along the second link, and applies an inequality of Liouville type to the pairing of the two complements, which is divisible by $g(S)$ because the two terms are congruent.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles and that Lean's kernel accepted the proofs; an independent replay is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note (Note P of its unit) treats 9 of the 497 declarations of the unit *Translates of Poincaré series fill the rows*: the module on link pairings. Proved here from the inputs: Theorems A and B and the statements of §4; where a step is a verification of matrix entries, the note says so and gives the identity that is verified. Taken for granted and stated as Inputs 1 to 4: the properties of links, levels and class coordinates from the set-up of the unit, an inequality of Liouville type for Gram determinants, the bound of an index by agreeing characters, and Jacobi's identity for discriminants of complements from the background library. Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 9 declarations.

1 The statements

The objects are recalled in §2. In brief: $\mathrm{link}(\gamma)$ is the link of a rational element, $\mathrm{link}^{\circ}(\gamma,\gamma')$ the common part of two links inside E^{κ}; $\mathrm{pi}_S(c,c';\gamma,\gamma')$ is the invariant of the pair and $\mathrm{cg}_S(j;c,c';\gamma)$ the gauge of a term of level j.

Theorem A (the discriminant gain). *In Setting 2.1, suppose that E is stably unramified. Let c, c' be classes of S, $s \in \mathsf{G}(S)$ an element that agrees, x a natural number such that s fixes norm at least x, j a level, and $\gamma, \gamma' \in \mathrm{Tm}_S(c, s, c')$ two terms with $\gamma \in \mathrm{Lv}_j(S)$ and $\gamma' \in \mathrm{Lv}_j(S)$. Then there is a natural number i such that*

(i) $k(\mathfrak{s}) \le \dim_E \mathrm{link}^\circ(\gamma, \gamma') + i + (j - \varrho(\mathfrak{s}))_+$;
(ii) *if $i = 0$, then $\mathrm{link}(\gamma) = \mathrm{link}(\gamma')$; and*
(iii) *with $x' = x$ if $\varrho(\mathfrak{s}) < j$ and $i = 0$, and $x' = 1$ otherwise,*

$$\mathrm{pi}_S(c, c'; \gamma, \gamma') \cdot \mathrm{pi}_S(c, c'; \gamma', \gamma) \cdot x'^2 \le \Big(\frac{2}{g(S)}\Big)^{2\,i\,f(E)} \cdot \mathrm{cg}_S(j; c, c'; \gamma) \cdot \mathrm{cg}_S(j; c, c'; \gamma').$$

LEAN `Filling.pairInvariant_mul_le`

Theorem B (splitting a link along a second one). *In Setting 2.1, suppose that E is stably unramified. Let c, c' be classes of S, $s \in \mathsf{G}(S)$ an element that agrees, x a natural number such that s fixes norm at least x, j a level, $\gamma \in \mathrm{Tm}_S(c, s, c')$ a term with $\gamma \in \mathrm{Lv}_j(S)$, and $\gamma' \in G_S(E)$. Put $k_\gamma = \mathrm{pos}_S(c, c'; \gamma)$ and $L = \mathrm{link}(\gamma) \cap \mathrm{link}(\gamma')$. Then*

(i) $0 < \mathrm{disc}_{S,c}(\mathrm{link}^\circ(\gamma, \gamma'))$ *and* $1 \le \mathrm{o}_{S,c}(k_\gamma; 1, \mathrm{link}^\circ(\gamma, \gamma'))$;
(ii) *there are a natural number i, a finite index set $\mathcal{C}$, a matrix m over E with columns indexed by $\mathcal{C}$, a matrix y over E with i columns, $v \in \mathrm{GL}_i(\mathbb{A}_{E,f})$ and a matrix z over $\mathbb{A}_{E,f}$ with rows $\mathcal{C}$ and i columns, such that: $i = k(\mathfrak{s}) + 1 - \dim L$; $k(\mathfrak{s}) \le \dim \mathrm{link}^\circ(\gamma, \gamma') + i + (j - \varrho(\mathfrak{s}))_+$; if $i = 0$ then $L = \mathrm{link}(\gamma)$; the columns of m span L; the columns of y lie in $\mathrm{link}(\gamma)$; $yt \in L$ only for $t = 0$; the columns of $(m \mid y)$ span $\mathrm{link}(\gamma)$; $\tilde{m}z + \tilde{y}v \in \mathrm{IL}(c; \gamma)$; and, with x' as in Theorem A,*

$$\big(\mathrm{pi}_S(c, c'; \gamma, \gamma') \cdot x'\big)^2 \cdot \mathrm{ld}_S(y, v) \le \mathrm{cg}_S(j; c, c'; \gamma)^2.$$

LEAN `Filling.exists_link_split_le_classGauge`

Theorem A is used in the crude estimate of the unit (the note on the energy of amplitude sums): there the pairs of terms of one level over one (c, s, c') are summed with the weights that the invariants of the pair control, and the factor $(2/g(S))^{2if(E)}$ pays for the number of links at distance i.

2 The setting

Conventions. The natural numbers are $0, 1, 2, \dots$, and $(a)_+$ is a if $a \ge 0$ and 0 otherwise. E is a CM field with its CM type: a number field that is a CM field, with complex conjugation $x \mapsto \bar{x}$ and one embedding at each infinite place τ; x_τ is the image of x, $f(E)$ the number of infinite places, $N_{E/\mathbb{Q}}$ the norm. $\mathbb{A}_{E,f}$ is the ring of finite adeles, $\hat{O}_E$ its subring of integral adeles, $\tilde{a}$ the image in $\mathbb{A}_{E,f}$ of a matrix a over E, and $[\xi]$ the generalised index of $\xi \in \mathbb{A}_{E,f}$: the index of $\xi\hat{O}_E$ in $\hat{O}_E$ for

integral ξ, extended multiplicatively to the units and 0 on non-units; for $\xi \in E^\times$ it is $|N_{E/\mathbb{Q}}(\xi)|$. E is *stably unramified* if it contains a primitive root of unity of order 5040. A *shape* is a triple $\mathfrak{s} = (\varrho, \kappa, k_0)$; $\varrho(\mathfrak{s})$, $k(\mathfrak{s}) = \#\kappa$; a *level* is an element j of $\{0, \dots, \varrho(\mathfrak{s}) + 1\}$. For a matrix X over a ring with involution, X^* is the conjugate transpose; a matrix *has a left inverse* if its columns are linearly independent, and its *column span* is the space spanned by its columns. Index sets written $\mathcal{C}$, $\mathcal{C}'$ are finite index sets of columns of matrices; where the formal text lists such a set, its finiteness or a matrix among the arguments of a statement that does not use them, this is said in §6. As in Note F, and unlike in the handout's overview, c, c' are classes of the filling data and s, t elements of the finite group $\mathsf{G}(S)$; the complex variable and the unit class of the overview do not occur.

Setting 2.1 (filling data)**.** E is a CM field with its CM type, $\mathfrak{s}$ is a shape, and S is filling data over E of shape $\mathfrak{s}$, in the sense of the set-up of this unit, with its prime $N(S)$, the number $g(S)$, the finite group $\mathsf{G}(S)$, the classes $\mathrm{Cl}(S)$, the groups $G_S(E)$ and G_S^{lat}, the integral subgroup $K(S)$ with its reduction red_S and kernel $K_g(S)$, the positions $\mathrm{pos}_S(c, c'; \gamma)$ and the sets of terms $\mathrm{Tm}_S(c, s, c')$, as recalled in the setting "filling data" of Note F of this unit (*Translates of Poincaré series fill the rows*).

Recalled definitions. All the notions below belong to the set-up of the unit; they are recalled with every argument of their formal definitions. The index set $I(\mathfrak{s})$ of the whole space consists of $e_0, \dots, e_{\varrho(\mathfrak{s})-1}$, of κ, of z and of $e_0^*, \dots, e_{\varrho(\mathfrak{s})-1}^*$; $M(\mathfrak{s}) = \kappa \sqcup \{z\}$ indexes the *middle*. X is the span of the e_a; the frame is $\mathfrak{f}_a = e_a$ $(a < \varrho(\mathfrak{s}))$, $\mathfrak{f}_{\varrho(\mathfrak{s})} = z$. In Setting 2.1: $T = T_S^{\mathrm{rat}}(E)$ is the diagonal matrix on $M(\mathfrak{s})$ with entries 1 on κ and $-N(S)$ at z; J_S is its split Gram matrix on $I(\mathfrak{s})$, the Gram matrix of the hermitian form h; $T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})$ and its split Gram matrix J_f are the same in the lattice basis over the finite adeles.

(a) For $\gamma \in G_S(E)$: $Z_S(\gamma) = (h(\mathfrak{f}_a, \gamma\mathfrak{f}_{a'}))_{a,a'}$ is the frame Gram matrix, $D_l(Z)$ the leading minor of order l of a matrix Z, and the *term discriminant* is

$$\mathfrak{g}_S(\gamma) = |D_{\varrho+1}(Z_S(\gamma))|^2 - N(S)^2 |D_\varrho(Z_S(\gamma))|^2,$$

with $|\xi|^2 = \xi\bar{\xi}$; $\|\mathfrak{g}\|_S(\gamma) = \prod_\tau |\mathfrak{g}_S(\gamma)_\tau|$.

(b) $[\mathfrak{f} \mid \gamma\mathfrak{f}] = (\mathfrak{f} \,|\, \gamma\mathfrak{f})$ is the matrix of the frame and its image, $[\mathfrak{e} \mid \gamma\mathfrak{e}] = (e_a \,|\, \gamma e_a)_{a<\varrho(\mathfrak{s})}$ its part on X. The *link matrix* $\mathrm{L}(\gamma)$ sends a vector t to the pair of the middle parts of t and of $\gamma^{-1}t$, a vector indexed by $M(\mathfrak{s}) \sqcup M(\mathfrak{s})$; and $\mathrm{link}(\gamma) = \mathrm{L}(\gamma)(\ker([\mathfrak{e} \mid \gamma\mathfrak{e}]^* J_S))$, the *link*: the pairs of the classes of t and of $\gamma^{-1}t$ for t orthogonal to X and γX.

(c) $\mathrm{L}_z(T)$ is the matrix with two rows that takes from a pair of middle vectors the two products with the row of T at z; $Q_S^{\pm} = \mathrm{diag}(T, -T)$ is the *link form* and

$$Q_S^{\mathrm{link}} = \tfrac{1}{2}\,\mathrm{diag}(T, T) + N(S)^{-1}\,\mathrm{L}_z(T)^* \mathrm{L}_z(T)$$

the *link metric*, both on pairs of middle vectors. For γ, γ', the subspace $\mathrm{link}^\circ(\gamma, \gamma')$ of E^κ is the space of the κ-parts of the first components of the pairs in $\mathrm{link}(\gamma) \cap \mathrm{link}(\gamma') \cap \ker \mathrm{L}_z(T)$.

(d) For a matrix v over E on pairs of middle vectors with columns $\mathcal{C}$ and $e \in \mathrm{GL}_{\mathcal{C}}(\mathbb{A}_{E,f})$: $\mathrm{ld}_S(v, e) = |N_{E/\mathbb{Q}} \det(v^* Q_S^{\mathrm{link}} v)| \cdot [\det e]^2$.

(e) For a class c, $\mathrm{C}_S(c)$ is the matrix over $\mathbb{A}_{E,f}$ that takes rational coordinates on the whole space to coordinates in the basis of the lattice of c, and $\mathrm{C}^\circ_S(c)$ the matrix whose columns are the basis vectors of E^κ in the basis of the middle part of that lattice. For classes c, c', $\gamma \in G_S(E)$ and $k = \mathrm{pos}_S(c, c'; \gamma)$: $\mathrm{C}_S(c)\,[\widetilde{\mathfrak{f} \mid \gamma\mathfrak{f}}] = [\mathfrak{f} \mid k\mathfrak{f}]$.

(f) For a class c, $\gamma \in G_S(E)$ and a matrix m over $\mathbb{A}_{E,f}$ on pairs of middle vectors with columns $\mathcal{C}$: $m \in \mathrm{IL}(c; \gamma)$ (*m has integral lifts*) if there are a finite index set $\mathcal{C}'$, a matrix t over E with rows $I(\mathfrak{s})$ and columns $\mathcal{C}'$ and a matrix w over $\mathbb{A}_{E,f}$ with rows $\mathcal{C}'$ and columns $\mathcal{C}$ such that $[\mathfrak{e} \mid \gamma\mathfrak{e}]^* J_S\, t = 0$, the matrix $\mathrm{C}_S(c)\,\tilde{t}\,w$ is integral, and $\widetilde{\mathrm{L}(\gamma)t}\,w = m$.

(g) For $k \in G_S^{\mathrm{lat}}$ with frame Gram matrix $Z(k)$ for $T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})$ and a level j: $\mathrm{i}_{S,j}(k)$ is the index of the $\hat{O}_E$-span of the conjugated first $\min(j, \varrho(\mathfrak{s}))$ rows of $Z(k)$ in that of its conjugated first $\varrho(\mathfrak{s})$ rows, and $\mathfrak{d}_S(k)$ the index of the $\hat{O}_E$-span of the columns of $[\mathfrak{e} \mid k\mathfrak{e}]$ in the lattice of the integral vectors of their span; both are 0 if infinite.

(h) For a class c and a subspace V of E^κ: $\mathrm{disc}_{S,c}(V)$ is the discriminant of the lattice of the adelic points of V that are integral in the lattice of c; for $k \in G_S^{\mathrm{lat}}$ and a unit a of $\mathbb{A}_{E,f}$, $\mathrm{o}_{S,c}(k; a, V)$ is the index in the lattice of the values $(h(k\mathfrak{f}_l, y))_{l<\varrho(\mathfrak{s})}$, for y in that lattice of V, of its intersection with a times the lattice spanned by the conjugated first $\varrho(\mathfrak{s})$ rows of $Z(k)$ (0 if infinite). The *invariant of a pair* is

$$\begin{aligned}\mathrm{pi}_S(c, c'; \gamma, \gamma') = {} & \mathrm{disc}_{S,c}(\mathrm{link}^\circ(\gamma, \gamma')) \cdot \mathrm{o}_{S,c}(\mathrm{pos}_S(c, c'; \gamma); 1, \mathrm{link}^\circ(\gamma, \gamma')) \\ & \cdot \mathfrak{d}_S(\mathrm{pos}_S(c, c'; \gamma)).\end{aligned}$$

(i) For a level j, $\gamma \in G_S(E)$ and classes c, c', with $k = \mathrm{pos}_S(c, c'; \gamma)$, the *minor norm* and the *gauge* are

$$\mathrm{mn}_S(j; \gamma) = |N_{E/\mathbb{Q}}(D_j(Z_S(\gamma)))| \cdot N(S)^{-2(j-\varrho(\mathfrak{s}))_+ f(E)},$$

$$\mathrm{cg}_S(j; c, c'; \gamma) = \frac{\max(\|\mathbf{g}\|_S(\gamma),\ N(S)^{f(E)}\mathrm{mn}_S(j; \gamma))}{\mathrm{i}_{S,j}(k) \cdot \mathrm{i}_{S,j}(k^{-1})}.$$

(j) $\gamma \in \mathrm{Lv}_j(S)$ (γ *has level* j) if γ is in normal form of index j, $D_j(Z_S(\gamma)) \neq 0$, and $\varrho(\mathfrak{s}) < j$ holds exactly if $\mathbf{g}_S(\gamma) \neq 0$. Here γ is in *normal form of index* j if there is $v \in E$ with $v\bar{v} = 1$ such that for every frame index $a \geq j$ there are coefficient vectors x, y supported on the indices $\leq a$ with $x_a = v$, $y_a = 1$ and $\sum_{a'} x_{a'}\mathfrak{f}_{a'} = \gamma \sum_{a'} y_{a'}\mathfrak{f}_{a'}$.

(k) $s \in \mathsf{G}(S)$ *agrees*, in symbols $s \in \mathrm{Agr}(S)$, if for all $u, u' \in K(S)$ in the Bessel unipotent group with $\mathrm{red}_S(u)\, s = s\, \mathrm{red}_S(u')$ the unipotent characters of u and u' are equal. For a natural number x, s *fixes norm at least* x, in symbols $\mathrm{fix}(s) \geq x$, in the sense of the definition of fixing sets recalled in Note F: there is a finite set of triples of an index $i \in \{0, 1\}$, a chosen place above the i-th prime of S and a Boolean, the product of whose primes is at least x, such that for each of them s, or its inverse adjoint if the Boolean is true, fixes at that place a non-zero vector of the span of the frame.

The symbols of (a) to (k) are those of the notes on the set-up of this unit, which state the definitions; the predicates of (k) are written there with their symbols and here in words.

3 What is taken for granted

Input 1 (links, levels, class coordinates; set-up of this unit). In Setting 2.1, for $\gamma, \gamma' \in G_S(E)$:

(a) $\dim_E \mathrm{link}(\gamma) = k(\mathfrak{s}) + 1$, and $\mathrm{link}(\gamma)$ is Lagrangian for the link form: a pair p lies in $\mathrm{link}(\gamma)$ exactly if $q^* Q_S^{\pm} p = 0$ for all $q \in \mathrm{link}(\gamma)$. In particular $v^* Q_S^{\pm} u = 0$ for matrices u, v with columns in $\mathrm{link}(\gamma)$.

(b) If the columns of t are orthogonal to $X + \gamma X$ and those of t' to $X + \gamma' X$, then for $a, a' \in E$: $(\mathrm{L}(\gamma)t)^* \,\mathrm{diag}(aT, -a'T)(\mathrm{L}(\gamma')t') = a\, t^* J_S t' - a'\, t^* J_S (\gamma\gamma'^{-1}) t'$.

(c) The link metric is positive definite at every infinite place, and $\det(v^* Q_S^{\mathrm{link}} v) \neq 0$ for every matrix v with a left inverse.

(d) For matrices t, t' over E with rows $I(\mathfrak{s})$: $(\mathrm{C}_S(c)\tilde{t})^* J_f (\mathrm{C}_S(c)\tilde{t}') = \widetilde{t^* J_S t'}$; J_f is unimodular; and the frame Gram matrix of $\mathrm{pos}_S(c, c'; \gamma)$ is the image of $Z_S(\gamma)$.

(e) If $\gamma \in \mathrm{Lv}_j(S)$, then $\gamma^{-1} \in \mathrm{Lv}_j(S)$ and $\mathrm{mn}_S(j; \gamma) > 0$. If γ is in normal form of index j, then for every vector y of frame coordinates there are x, y' with $y'_a = y_a$ for $a \geq j$ and $\sum_a x_a \mathfrak{f}_a = \gamma \sum_a y'_a \mathfrak{f}_a$; in such a relation the top coordinates satisfy $\bar{x}_\varrho x_\varrho = \bar{y}'_\varrho y'_\varrho$. If $\gamma \in \mathrm{Lv}_j(S)$ and $b \geq j$ is a frame index, the combination of the vectors $\gamma \mathfrak{f}_a$ with the coefficients of column b of the inverse of the matrix $\mathrm{pl}_j(Z_S(\gamma))$, whose rows $a < j$ are those of $Z_S(\gamma)$ and whose other rows are those of the identity, lies in the span of the frame.

(f) If $Z_S(\gamma)\, y = D\, e_\varrho$ for a coefficient vector y and $D \in E$, and $t = \sum_a y_a \gamma \mathfrak{f}_a$, then the coordinates of t at the e_a^* vanish, $-N(S)\, t_z = D$, and $N(S) \sum_{i \in \kappa} |t_i|^2 = |D|^2 - N(S)^2 |y_\varrho|^2$.

(g) If $\gamma \in \mathrm{Tm}_S(c, s, c')$ and $\gamma' \in \mathrm{Tm}_S(c', s', c'')$, then $\gamma\gamma' \in \mathrm{Tm}_S(c, ss', c'')$; the inverse of a term over (c, s, c') is a term over (c', s^{-1}, c); and the position of a term over $(c, 1, c)$ lies in $K_g(S)$, so that it is congruent to 1 modulo $g(S)$.

(h) If the columns of n are orthogonal to $X + kX$ for $k \in G(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ and $\mathrm{L}_z(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))\mathrm{L}(k)n = 0$, then $\mathrm{fm}(k)^*\, T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})\, n_M = -Z_X(k)^*\, n_X$, where n_X is the part of n on the e_a, n_M its middle part, $\mathrm{fm}(k)$ the matrix of the middle parts of the $k\mathfrak{f}_l$, $l < \varrho(\mathfrak{s})$, and $Z_X(k)$ the matrix of the first $\varrho(\mathfrak{s})$ rows of $Z(k)$.

Input 2 (Liouville's inequality for Gram determinants; set-up of this unit). Let E be stably unramified, Q a square matrix over E that is positive definite at every infinite place, a, a' matrices over E with the rows of Q and columns indexed by a finite set $\mathcal{C}$ with $\det(a^* Q a') \neq 0$, $v, v' \in \mathrm{GL}_{\mathcal{C}}(\mathbb{A}_{E,f})$, and $x_0 \geq 0$ a real number such that $x_0 \leq [\delta]$ whenever $\delta = \det(v^* \widetilde{a^* Q a'}\, v')$ is a unit of $\mathbb{A}_{E,f}$. Then $x_0^2 \leq [\det v]^2\, |N_{E/\mathbb{Q}} \det(a^* Q a)| \cdot |N_{E/\mathbb{Q}} \det(a'^* Q a')|\, [\det v']^2$.

Input 3 (lattices; set-up of this unit and background library). Let E be stably unramified.

(a) *(Primitive completion.)* If a matrix over $\mathbb{A}_{E,f}$ with columns $\mathcal{C} \sqcup \mathcal{C}'$ has a left inverse, it becomes primitive (its columns extend to a basis of the lattice of integral vectors) after multiplication on the right by a block matrix with diagonal blocks $a \in \mathrm{GL}_{\mathcal{C}}(\mathbb{A}_{E,f})$, $e \in \mathrm{GL}_{\mathcal{C}'}(\mathbb{A}_{E,f})$, an upper right block z and lower left block 0. A square primitive matrix is integral with integral inverse, so its determinant has generalised index 1.

(b) *(Jacobi's identity for discriminants of complements; reference to be confirmed.)* Let J be a unimodular hermitian matrix over $\mathbb{A}_{E,f}$, F an integral matrix, n a primitive matrix with $F^* J n = 0$, of complementary sizes, p a matrix of bordering columns and c_0 a unit. Then the solutions u' of $n^* J u' = 0$ are the combinations of the columns of F, and

$$[\det(n^*(J + c_0^{-1} J p p^* J) n)] \cdot \mathfrak{i}(F)^2 \cdot [c_0]^{d_p} = [\det M],$$

where M is the Gram matrix of $(F \,|\, p)$ for J with c_0 added on the diagonal block of p, d_p is the number of columns of p, and $\mathfrak{i}(F)$ the index of the lattice of integral vectors in the lattice of the y with Fy integral.

(c) *(Extension of a basis modulo a kernel.)* Let A and x be matrices over E with $\det(Ax) \neq 0$, and L a subspace containing the columns of x. Then there is a matrix y with $\dim(L \cap \ker A)$ columns such that $(x \,|\, y)$ has a left inverse and its columns span L, the columns of y lie in L, $Ay = 0$, and yt lies in the column span of x only for $t = 0$.

(d) *(Discriminant of a sublattice.)* Let c be a class, V a subspace of E^κ of dimension d, p a matrix over E with d columns in V, $a \in \mathrm{GL}_d(\mathbb{A}_{E,f})$ with $D = [\det(p^* p)]\,[\det a]^2 \neq 0$, and k in G_S^{lat}. If $\mathrm{C}_S^\circ(c)\,\tilde{p}\,a$ is integral and its values $(h(k\mathfrak{f}_l, \cdot))_l$ lie in the lattice spanned by the conjugated first $\varrho(\mathfrak{s})$ rows of $Z(k)$, then $(\mathrm{disc}_{S,c}(V) \cdot \mathrm{o}_{S,c}(k; 1, V))^2 \leq D$, $\mathrm{disc}_{S,c}(V) > 0$ and $\mathrm{o}_{S,c}(k; 1, V) \geq 1$.

Input 4 (agreeing characters force a large index; set-up of this unit). In Setting 2.1, let E be stably unramified, $k \in K(S)$ with $\mathrm{red}_S(k)$ agreeing, x a natural number such that $\mathrm{red}_S(k)$ fixes norm at least x, and n an integral matrix with rows $I(\mathfrak{s})$ and two columns whose *pairings* with the e_a and the $k e_a$ vanish, that is $[\mathfrak{e} \mid k\mathfrak{e}]^* J_f\, n = 0$; here the pairing of a column v of n with a vector w is $w^* J_f v = h(w, v)$. If the determinant δ of the 2×2 matrix of the pairings of the columns of n with z and kz is a unit of $\mathbb{A}_{E,f}$, then $x \leq [\delta]$. (The formal statement is about an integral matrix t and the plain products $[\mathfrak{f} \mid k\mathfrak{f}]^* t$; it is used for $t = J_f n$, which is integral because n and J_f are.)

4 Proofs

Throughout, in Setting 2.1, $N = N(S)$, $\varrho = \varrho(\mathfrak{s})$, $T = T_S^{\mathrm{rat}}(E)$, and $[N] = N^{2f(E)}$ is the generalised index of the integer N, the degree of E being $2f(E)$.

The bordered determinant.

Lemma 4.1 (a bordered Gram determinant)**.** *In Setting 2.1, let j be a level and $\gamma \in G_S(E)$ with $\gamma \in \mathrm{Lv}_j(S)$. Let P be the matrix over E with rows $I(\mathfrak{s})$ whose columns are: the vectors e_a for $a < \varrho(\mathfrak{s})$; the vectors γe_a for $a < \varrho(\mathfrak{s})$ with $a < j$; the basis vectors e_a^* for $a < \varrho(\mathfrak{s})$ with $a \geq j$; and z and γz. Let M be the sum of $P^* J_S P$ and of the diagonal matrix with entry $N(S)$ at the columns e_a^*, z, γz and 0 at the others. Then*

$$[\det \tilde{M}] = \left(N(S)^{2f(E)} \cdot \mathrm{mn}_S(j;\gamma)\right)^2.$$

Proof. Let $Z' = Z_S(\gamma^{-1})$, the conjugate transpose of $Z_S(\gamma)$, and let A be the square matrix of size $\varrho + 1$ whose rows $a < j$, $a < \varrho$, are those of Z', whose rows $j \leq a < \varrho$ are those of the identity, and whose top row is that of Z'.

Order the columns of P as the frame $\mathfrak{f}_0, \dots, \mathfrak{f}_\varrho$ (the e_a, $a < \varrho$, and z), followed by the $\varrho + 1$ columns γe_b ($b < \min(j, \varrho)$), e_b^* ($j \leq b < \varrho$) and γz, which we index by $b \leq \varrho$, the column γz having the index ϱ. On the frame, M is the matrix $(h(\mathfrak{f}_a, \mathfrak{f}_{a'}))_{a,a'} + \mathrm{diag}(0, \dots, 0, N)$, which is zero: the e_a are isotropic and orthogonal to z, and $h(z,z) = -N$. The block of the frame against the other columns has the entries

$$h(\mathfrak{f}_a, \gamma e_b) = Z_S(\gamma)_{ab}, \qquad h(\mathfrak{f}_a, e_b^*) = \delta_{ab}, \qquad h(\mathfrak{f}_a, \gamma z) = Z_S(\gamma)_{a\varrho};$$

its columns $b < \min(j, \varrho)$ and $b = \varrho$ are those of $Z_S(\gamma) = Z'^*$ and its columns $j \leq b < \varrho$ are those of the identity, so it is A^*. As M is hermitian,

$$M = \begin{pmatrix} 0 & A^* \\ A & * \end{pmatrix}, \qquad \det M = \pm \det A \cdot \overline{\det A},$$

and as $\det M \in E$, $[\det \tilde{M}] = |N_{E/\mathbb{Q}}(\det A)|^2$. It remains to show $|N_{E/\mathbb{Q}}(\det A)| = N^{2f(E)} \mathrm{mn}_S(j;\gamma)$.

If $j = \varrho + 1$, then $A = Z'$, $|N_{E/\mathbb{Q}} \det Z'| = |N_{E/\mathbb{Q}} D_{\varrho+1}(Z_S(\gamma))|$, and the claim is the definition of $\mathrm{mn}_S(\varrho + 1; \gamma)$, in which $(j - \varrho)_+ = 1$.

Let $j \leq \varrho$. The element γ^{-1} has level j (Input 1(e)), so for the frame index $\varrho \geq j$ the combination of the $\gamma^{-1}\mathfrak{f}_a$ with the coefficients y of the top column of $\mathrm{pl}_j(Z')^{-1}$ lies in the span of the frame: $\sum_a y_a \gamma^{-1} \mathfrak{f}_a = \sum_a x_a \mathfrak{f}_a$. The matrix A differs from $\mathrm{pl}_j(Z')$ only in the top row, so by Cramer's rule $\det A = \det \mathrm{pl}_j(Z') \cdot (Z'y)_\varrho$, and $\det \mathrm{pl}_j(Z') = D_j(Z') \neq 0$. Now $(Z'y)_\varrho = h(z, \sum_a y_a \gamma^{-1} \mathfrak{f}_a) = h(z, \sum_a x_a \mathfrak{f}_a) = -N x_\varrho$, and $\bar{x}_\varrho x_\varrho = \bar{y}_\varrho y_\varrho = 1$, because the top diagonal entry of $\mathrm{pl}_j(Z') \, \mathrm{pl}_j(Z')^{-1}$ is 1 and the top row of $\mathrm{pl}_j(Z')$ is that of the identity. So $\det A = -N x_\varrho \, D_j(Z')$ with x_ϱ of norm one at every place, and

$$|N_{E/\mathbb{Q}}(\det A)| = N^{2f(E)} \, |N_{E/\mathbb{Q}} D_j(Z_S(\gamma))| = N^{2f(E)} \mathrm{mn}_S(j;\gamma),$$

the leading minors of Z' being the conjugates of those of $Z_S(\gamma)$. □

LEAN `Filling.HasLevel.abs_norm_bordered_det`

Integral lifts of a basis of the link.

Proposition 4.2 (lifts with small discriminant)**.** *In Setting 2.1, suppose that E is stably unramified. Let c, c' be classes of S, $t \in \mathsf{G}(S)$, j a level, $\gamma \in \mathrm{Tm}_S(c,t,c')$ with $\gamma \in \mathrm{Lv}_j(S)$, $\mathcal{C}$ a finite index set and v a matrix over E with rows $M(\mathfrak{s}) \sqcup M(\mathfrak{s})$ and columns $\mathcal{C}$ that has a left inverse and whose column span is $\mathrm{link}(\gamma)$. Put $k = \mathrm{pos}_S(c,c';\gamma)$ and $w_0 = \mathrm{i}_{S,j}(k) \cdot \mathrm{i}_{S,j}(k^{-1}) \cdot \mathfrak{d}_S(k)$. Then there is $e \in \mathrm{GL}_{\mathcal{C}}(\mathbb{A}_{E,f})$ such that*

$$\tilde{v}\, e \in \mathrm{IL}(c;\gamma), \qquad \mathrm{ld}_S(v,e) \cdot w_0^2 \le \mathrm{mn}_S(j;\gamma)^2, \qquad w_0 \neq 0.$$

Proof. Let B be the set of the $a < \varrho$ with $a \ge j$; it is empty if $j \ge \varrho$.

Step 1: a rational lift. There is a matrix u over E with rows $I(\mathfrak{s})$ and columns $B \sqcup \mathcal{C}$ such that the columns of u are orthogonal to $X + \gamma X$; $\mathrm{L}(\gamma)u = (0 \,|\, v)$; and the e_a-coordinates of the columns of u, for $a \in B$, form the matrix $(1 \,|\, 0)$. Indeed, a column v_b of v lies in the link, so $v_b = \mathrm{L}(\gamma)t_b$ with t_b orthogonal to $X + \gamma X$. By the normal form of γ^{-1} (Input 1(e)), for every prescription of coordinates at the indices $a \in B$ there is a vector $w' = \sum_a y'_a e_a$ with these coordinates, with $y'_\varrho = 0$, such that $\gamma^{-1}w' = \sum_a x_a \mathfrak{f}_a$ with $x_\varrho = 0$. Then w' lies in $X \cap \gamma X$; it is orthogonal to $X + \gamma X$, both spaces being isotropic, and $\mathrm{L}(\gamma)w' = 0$, the middle parts of w' and of $\gamma^{-1}w'$ being zero. Adding such vectors to the t_b, and taking them alone for the columns of B, gives u.

Step 2: completion. The matrix $\mathrm{C}_S(c)\tilde{u}$ has a left inverse. By Input 3(a) there are $a \in \mathrm{GL}_B(\mathbb{A}_{E,f})$, $e \in \mathrm{GL}_{\mathcal{C}}(\mathbb{A}_{E,f})$ and z_1 such that $n = \mathrm{C}_S(c)\,\tilde{u}\,\left(\begin{smallmatrix} a & z_1 \\ 0 & e \end{smallmatrix}\right)$ is primitive. The columns of n indexed by $\mathcal{C}$ are $\mathrm{C}_S(c)\tilde{u}\binom{z_1}{e}$; they are integral, and $\widetilde{\mathrm{L}(\gamma)u}\binom{z_1}{e} = \tilde{v}\, e$. With Step 1 this shows $\tilde{v}\, e \in \mathrm{IL}(c;\gamma)$.

Step 3: Jacobi's identity. Let F and p' be the images under $\mathrm{C}_S(c)$ of the first columns (e_a for $a < \varrho$; γe_a for $a < j$, $a < \varrho$) and of the last columns (e_a^* for $a \in B$; z; γz) of the matrix P of Lemma 4.1; by the last sentence of §2(e), F consists of the vectors e_a and ke_a. The matrix F is integral because $k \in K(S)$; $F^* J_f n = 0$ by Step 1 and Input 1(d); and the sizes are complementary because $\#\mathcal{C} = k(\mathfrak{s}) + 1$ (Input 1(a)). Jacobi's identity (Input 3(b)), for $J = J_f$, F, n, p' and $c_0 = N$, gives

$$\left[\det(n^*(J_f + N^{-1}J_f p' p'^* J_f)n)\right] \cdot \mathfrak{i}(F)^2 \cdot [N]^{\#B+2} = [\det \tilde{M}] = \left(N^{2f(E)}\mathrm{mn}_S(j;\gamma)\right)^2,$$

the Gram matrix in the coordinates of the lattice being the rational one (Input 1(d)) and the last equality being Lemma 4.1. On the left, with Q the matrix of the products of the columns of u with e_a^* ($a \in B$), z and γz,

$$n^*(J_f + N^{-1}J_f p' p'^* J_f)n = \left(\begin{smallmatrix} a & z_1 \\ 0 & e \end{smallmatrix}\right)^* \left(u^* J_S u + \widetilde{N^{-1}Q^*Q}\right) \left(\begin{smallmatrix} a & z_1 \\ 0 & e \end{smallmatrix}\right),$$

$$u^* J_S u + N^{-1}Q^*Q = \begin{pmatrix} N^{-1} \cdot 1_B & 0 \\ 0 & v^* Q_S^{\mathrm{link}} v \end{pmatrix}.$$

The second identity follows from Step 1. A column of u indexed by B lies in $X \cap \gamma X$: its products with z, γz and with the other columns vanish, and its products with the e_a^*, $a \in B$, are its e_a-coordinates; this gives the block $N^{-1} \cdot 1_B$ and the zero blocks. For the columns indexed by $\mathcal{C}$, the products with the e_a^*, $a \in B$, vanish, the products with z and γz are the values of $\mathrm{L}_z(T)$ on the link images, which gives the term

$N^{-1}\mathrm{L}_z(T)^*\mathrm{L}_z(T)$ of the link metric, and Input 1(b), with $\gamma' = \gamma$ and the coefficients $\frac{1}{2}$, $-\frac{1}{2}$, gives $u^* J_S u = v^* \frac{1}{2}\operatorname{diag}(T,T)\, v$ on them. Taking generalised indices,

$$[\det a]^2\,[\det e]^2 \cdot [N]^{-\#B} \cdot |N_{E/\mathbb{Q}} \det(v^* Q_S^{\mathrm{link}} v)| \cdot \mathfrak{i}(F)^2 \cdot [N]^{\#B+2} = [N]^2\, \mathrm{mn}_S(j;\gamma)^2,$$

that is

$$\mathrm{ld}_S(v,e) \cdot [\det a]^2 \cdot \mathfrak{i}(F)^2 = \mathrm{mn}_S(j;\gamma)^2. \tag{1}$$

Step 4: the indices. The formal text shows two divisibilities. First, the matrix a is integral, the index of its column lattice in the integral vectors is $[\det a] \neq 0$, and by the relation of Input 1(h), applied to the columns of n indexed by B, and the description of the frame Gram matrix of k (Input 1(d)), the lattice of the conjugated first ϱ rows of $Z(k)$ is generated by that of its first $\min(j,\varrho)$ rows and the image of the column lattice of a; so $\mathrm{i}_{S,j}(k)$ divides $[\det a]$. Second, $\mathrm{i}_{S,j}(k^{-1})$ divides the index of the column lattice of $[\mathfrak{e} \mid k\mathfrak{e}]$ in that of F, $\mathfrak{d}_S(k)$ is the index of the column lattice of $[\mathfrak{e} \mid k\mathfrak{e}]$ in the lattice of the integral vectors of its span, and by the first assertion of Jacobi's identity that lattice lies in the lattice of the y with Fy integral; as indices multiply, $\mathrm{i}_{S,j}(k^{-1}) \cdot \mathfrak{d}_S(k)$ divides $\mathfrak{i}(F)$, which is non-zero by (1) and $\mathrm{mn}_S(j;\gamma) > 0$. Hence $w_0 \neq 0$, $w_0 \leq [\det a]\,\mathfrak{i}(F)$, and (1) gives $\mathrm{ld}_S(v,e)\, w_0^2 \leq \mathrm{mn}_S(j;\gamma)^2$. □

LEAN `Filling.exists_lifts_linkDiscSq_le`

Corollary 4.3 (positivity)**.** *In Setting 2.1, suppose that E is stably unramified. Let c, c' be classes of S, $s \in \mathsf{G}(S)$, j a level and $\gamma \in \mathrm{Tm}_S(c,s,c')$ with $\gamma \in \mathrm{Lv}_j(S)$, and $k = \mathrm{pos}_S(c,c';\gamma)$. Then $1 \leq \mathrm{i}_{S,j}(k)$, $1 \leq \mathfrak{d}_S(k)$ and $0 < \mathrm{cg}_S(j;c,c';\gamma)$.*

Proof. Apply Proposition 4.2 to a basis of the link: $w_0 \neq 0$, so its three factors, which are natural numbers, are at least 1; and the numerator of the gauge is at least $N^{f(E)}\mathrm{mn}_S(j;\gamma) > 0$ (Input 1(e)). □

LEAN `Filling.one_le_xRowsIndex_and_saturationIndex`

Lemma 4.4 (an integral matrix from lifts)**.** *In Setting 2.1, let c be a class of S, $\gamma \in G_S(E)$, $\mathcal{C}$ an index set, m a matrix over $\mathbb{A}_{E,f}$ with rows $M(\mathfrak{s}) \sqcup M(\mathfrak{s})$ and columns $\mathcal{C}$, and c' a class of S. Suppose that $m \in \mathrm{IL}(c;\gamma)$. Then, with $k = \mathrm{pos}_S(c,c';\gamma)$, there is an integral matrix n with rows $I(\mathfrak{s})$ and columns $\mathcal{C}$ such that*

$$\mathrm{L}_z(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))\,\mathrm{L}(k)\,n = \widetilde{\mathrm{L}_z(T)}\,m, \qquad [\mathfrak{e} \mid k\mathfrak{e}]^* J_f n = 0,$$

and such that, for every matrix p with rows κ: if the first half of m is p on the rows κ and 0 on the row z, then the middle rows of n are $\mathrm{C}_S^\circ(c)\,p$.

Proof. Let $(\mathcal{C}', t, w)$ be as in the definition of integral lifts and put $n = \mathrm{C}_S(c)\tilde{t}w$. It is integral. As $\mathrm{C}_S(c)[\mathfrak{f} \mid \widetilde{\gamma\mathfrak{f}}] = [\mathfrak{f} \mid k\mathfrak{f}]$, Input 1(d) turns the pairings of n with the columns of $[\mathfrak{f} \mid k\mathfrak{f}]$ into those of $\tilde{t}w$ with the columns of $[\mathfrak{f} \mid \gamma\mathfrak{f}]$: those with the e_a, γe_a vanish, and those with z, γz, which are the values of $\mathrm{L}_z(\cdot)$ on the link images, are $\widetilde{\mathrm{L}_z(T)}\,\widetilde{\mathrm{L}(\gamma)}t\,w = \widetilde{\mathrm{L}_z(T)}\,m$. The last assertion is read off from the block form of $\mathrm{C}_S(c)$ on the middle coordinates. □

LEAN `Filling.HasIntegralLifts.exists_isIntMatrix`

Lemma 4.5 (orthogonal splitting of lifts)**.** *In Setting 2.1, suppose that E is stably unramified. Let c be a class of S, $\gamma \in G_S(E)$, $\mathcal{C}$ a finite index set, m a matrix over E with rows $M(\mathfrak{s}) \sqcup M(\mathfrak{s})$ and columns $\mathcal{C}$, d a natural number, y a matrix over E with the same rows and d columns, and f an invertible matrix over $\mathbb{A}_{E,f}$ with rows and*

columns indexed by $\mathcal{C}$ and the d columns of y. Suppose that $m^{*}Q_S^{\mathrm{link}}\,y = 0$ and that $\widetilde{(m \mid y)}f \in \mathrm{IL}(c;\gamma)$. Then there are $a \in \mathrm{GL}_{\mathcal{C}}(\mathbb{A}_{E,f})$, $v \in \mathrm{GL}_d(\mathbb{A}_{E,f})$ and a matrix z_1 over $\mathbb{A}_{E,f}$ with rows $\mathcal{C}$ and d columns such that

$$\tilde{m}a \in \mathrm{IL}(c;\gamma), \qquad \tilde{m}z_1 + \tilde{y}v \in \mathrm{IL}(c;\gamma), \qquad \mathrm{ld}_S(m,a)\cdot \mathrm{ld}_S(y,v) = \mathrm{ld}_S((m \mid y), f).$$

Proof. By Input 3(a), applied to f^{-1}, there are a, v, z_1 such that $f^{-1}\left(\begin{smallmatrix} a & z_1 \\ 0 & v \end{smallmatrix}\right)$ is primitive, in particular integral with $[\det a]\,[\det v] = [\det f]$. Let $(\mathcal{C}', t, w)$ be lift data for $\widetilde{(m \mid y)}f$. Then $(\tilde{m}a \mid \tilde{m}z_1 + \tilde{y}v) = \widetilde{\mathrm{L}(\gamma)t}\cdot w\, f^{-1}\left(\begin{smallmatrix} a & z_1 \\ 0 & v \end{smallmatrix}\right)$, and $\mathrm{C}_S(c)\tilde{t}\cdot w\, f^{-1}\left(\begin{smallmatrix} a & z_1 \\ 0 & v \end{smallmatrix}\right)$ is integral; taking the two blocks of columns gives the two lifts. Since $m^{*}Q_S^{\mathrm{link}}y = 0$, the Gram matrix of $(m \mid y)$ for the link metric is block diagonal, so its determinant is the product of those of m and of y; with $[\det f]^2 = [\det a]^2[\det v]^2$ this is the identity of the discriminants. □

LEAN `Filling.HasIntegralLifts.exists_orthogonal_split`

The vector that carries the term discriminant.

Lemma 4.6. *In Setting 2.1, let j be a level and $\gamma \in G_S(E)$ with $\gamma \in \mathrm{Lv}_j(S)$. Then there are $a \in E^{\kappa}$ and $y \in E$ such that the pair $(a - z,\ yz)$ of middle vectors lies in* $\mathrm{link}(\gamma)$*, and*

$$\mathrm{g}_S(\gamma) = D_j(Z_S(\gamma))\cdot \overline{D_j(Z_S(\gamma))}\cdot(1 - \bar{y}y).$$

Proof. It suffices to find a coefficient vector x such that $t = \sum_b x_b\gamma\mathfrak{f}_b$ has middle part $a - z$ and such that $\gamma^{-1}t = \sum_b x_b\mathfrak{f}_b$ has middle part yz, that is $y = x_{\varrho}$, with t orthogonal to X and γX: then the pair is the link image of t.

If $j = \varrho + 1$, then $D = \det Z_S(\gamma) = D_j(Z_S(\gamma)) \neq 0$; let x' be the solution of $Z_S(\gamma)x' = D\,e_{\varrho}$ and $x = (N/D)\,x'$, so that $Z_S(\gamma)x = N e_{\varrho}$. By Input 1(f) the vector $t = \sum_b x_b\gamma\mathfrak{f}_b$ has no coordinates at the e_a^{*} and has $t_z = -1$; so its middle part is $a - z$ with $a \in E^{\kappa}$, and it is orthogonal to X. It is orthogonal to γX because $h(\gamma e_a, t) = h(e_a, \sum_b x_b\mathfrak{f}_b) = 0$. By Cramer's rule $x'_{\varrho} = D_{\varrho}(Z_S(\gamma))$, the cofactor of the top entry; so $y = x_{\varrho} = N D_{\varrho}(Z_S(\gamma))/D$, and

$$\mathrm{g}_S(\gamma) = D\bar{D} - N^2|D_{\varrho}(Z_S(\gamma))|^2 = D\bar{D}(1 - \bar{y}y).$$

If $j \le \varrho$, the normal form of γ^{-1} for the prescription -1 at the top (Input 1(e)) gives x, y' with $y'_{\varrho} = -1$ and $\sum_b x_b\mathfrak{f}_b = \gamma^{-1}\sum_b y'_b\mathfrak{f}_b$; take $t = \sum_b y'_b\mathfrak{f}_b$, whose middle part is $-z$ (so $a = 0$), and $y = x_{\varrho}$, for which $\bar{y}y = \bar{y}'_{\varrho}y'_{\varrho} = 1$. Both t and $\gamma^{-1}t$ lie in the span of the frame, so t is orthogonal to X and to γX. As $j \le \varrho$, $\mathrm{g}_S(\gamma) = 0$ by the definition of the level, and $1 - \bar{y}y = 0$. □

LEAN `Filling.HasLevel.exists_mem_link_termDisc_eq`

Comparison of the minor with a partial discriminant.

Proposition 4.7. *In Setting 2.1, suppose that E is stably unramified. Let c, c' be classes of S, $s \in \mathsf{G}(S)$ an element that agrees, x a natural number such that s fixes norm at least x, j a level and $\gamma \in \mathrm{Tm}_S(c, s, c')$ with $\gamma \in \mathrm{Lv}_j(S)$. Let r and i be natural numbers, w a matrix over $\mathbb{A}_{E,f}$ with rows $M(\mathfrak{s}) \sqcup M(\mathfrak{s})$ and r columns, y a matrix over E with the same rows and r columns, and $v \in \mathrm{GL}_r(\mathbb{A}_{E,f})$. Suppose that*

(a) *if $i = 0$, then* $\dim(\mathrm{link}(\gamma) \cap \ker \mathrm{L}_z(T)) + r = \dim \mathrm{link}(\gamma)$*;*

(b) *the columns of y lie in* $\mathrm{link}(\gamma)$*, and* $\mathrm{L}_z(T)\,y$ *has a left inverse;*
(c) $w \in \mathrm{IL}(c;\gamma)$*, and* $\widetilde{\mathrm{L}_z(T)}\,w = \widetilde{\mathrm{L}_z(T)}\,y\,v$.

Then $r \le 1 + (j - \varrho(\mathfrak{s}))_+$ *and, with* $x' = x$ *if* $\varrho(\mathfrak{s}) < j$ *and* $i = 0$*, and* $x' = 1$ *otherwise,*

$$x'^2 \cdot \mathrm{mn}_S(j;\gamma)^2 \;\le\; \mathrm{ld}_S(y,v) \cdot \max\big(\|\mathfrak{g}\|_S(\gamma),\; N(S)^{f(E)}\mathrm{mn}_S(j;\gamma)\big)^2.$$

Proof. Let $k = \mathrm{pos}_S(c,c';\gamma) \in K(S)$ and let n be the integral matrix of Lemma 4.4 for w. By Lemma 4.6 for γ and for γ^{-1} there are pairs $\varepsilon_1 = (a - z,\; uz)$ in $\mathrm{link}(\gamma)$ with $\mathfrak{g}_S(\gamma) = D_j\bar{D}_j(1-\bar{u}u)$, $D_j = D_j(Z_S(\gamma))$, and $(a' - z,\; u'z)$ in $\mathrm{link}(\gamma^{-1})$; exchanging the two components of the latter gives a second pair $\varepsilon_2 = (u'z,\; a' - z)$ in $\mathrm{link}(\gamma)$. Let ε be the matrix with the columns ε_1, ε_2.

Two identities. For every matrix w' with columns in $\mathrm{link}(\gamma)$,

$$\varepsilon^*\, Q_S^{\mathrm{link}}\, w' = \mathrm{L}_z(T)\, w', \qquad \mathrm{L}_z(T)\,\varepsilon = N\begin{pmatrix} 1 & -\bar{u} \\ -u & 1 \end{pmatrix}. \tag{2}$$

Indeed the value of $\mathrm{L}_z(T)$ on a pair with coordinates s_1, s_2 at z is $(-Ns_1, -Ns_2)$, which gives $\mathrm{L}_z(T)\varepsilon$ with the entry $-Nu'$ in the upper right corner; and $\varepsilon_1^* Q_S^{\pm}\varepsilon_2 = 0$ (Input 1(a)) reads $Nu' - N\bar{u} = 0$, so $u' = \bar{u}$. For the first identity write $Q_S^{\mathrm{link}} = \frac{1}{2}Q_S^{\pm} + \mathrm{diag}(0,T) + N^{-1}\mathrm{L}_z(T)^*\mathrm{L}_z(T)$; the link form vanishes on the link, and the two other terms, computed from the coordinates of ε_1, ε_2 and using once more that $\varepsilon_2^* Q_S^{\pm} w' = 0$, add up to $\mathrm{L}_z(T)w'$.

A minor. As $\mathrm{L}_z(T)y$ has two rows and a left inverse, $r \le 2$, and there is an injective map ς_0 from the r columns to the two rows such that the corresponding minor of $\mathrm{L}_z(T)y$ is non-zero. Let $\varepsilon_{\varsigma_0}$ be the matrix of the columns of ε selected by ς_0; by (2) the matrix $\varepsilon_{\varsigma_0}^* Q_S^{\mathrm{link}} y$ is that minor's matrix, so its determinant is non-zero.

Liouville's inequality (Input 2, with $Q = Q_S^{\mathrm{link}}$, which is positive definite at the infinite places, $a = \varepsilon_{\varsigma_0}$, $a' = y$, the identity and v) gives

$$x'^2 \;\le\; |N_{E/\mathbb{Q}} \det(\varepsilon_{\varsigma_0}^* Q_S^{\mathrm{link}} \varepsilon_{\varsigma_0})| \cdot \mathrm{ld}_S(y,v), \tag{3}$$

provided $x' \le [\delta]$ whenever $\delta = \det\big((\widetilde{\mathrm{L}_z(T)}\,y\,v)_{\varsigma_0}\big) = \det\big((\widetilde{\mathrm{L}_z(T)}w)_{\varsigma_0}\big)$ is a unit. By Lemma 4.4, $\widetilde{\mathrm{L}_z(T)}w$ is the matrix of the pairings of the columns of the integral matrix n with z and kz, and the pairings of n with the e_a, ke_a vanish; these are the pairings $w^* J_f v$ of Input 4. If $\varrho < j$ and $i = 0$, then $\mathfrak{g}_S(\gamma) \neq 0$, so $1 - \bar{u}u \neq 0$ and $\mathrm{L}_z(T)\varepsilon$ is invertible; hence $\dim \mathrm{link}(\gamma) = 2 + \dim(\mathrm{link}(\gamma) \cap \ker \mathrm{L}_z(T))$, hypothesis (a) gives $r = 2$, the minor is the full determinant, and Input 4 gives $x \le [\delta]$; otherwise $x' = 1 \le [\delta]$ because δ is a unit that is the determinant of an integral matrix.

The determinant of the selected block. By (2), $\varepsilon^* Q_S^{\mathrm{link}} \varepsilon = \mathrm{L}_z(T)\varepsilon$. If $r = 0$, the determinant in (3) is 1. If $r = 1$, it is a diagonal entry, equal to N, of absolute norm $N^{2f(E)}$. If $r = 2$, it is $N^2(1 - \bar{u}u)$, and it is non-zero: $\mathrm{L}_z(T)y = \varepsilon^* Q_S^{\mathrm{link}} y$ is invertible, so ε has rank 2 and the link metric is positive definite. Hence $\mathfrak{g}_S(\gamma) = D_j\bar{D}_j(1-\bar{u}u) \neq 0$ and $j = \varrho + 1$ by the definition of the level; so $r \le 1 + (j - \varrho)_+$ in all cases. Moreover, as $|N_{E/\mathbb{Q}}(\xi)| = \prod_\tau |\xi_\tau|^2$ and $\mathrm{mn}_S(\varrho+1;\gamma) = |N_{E/\mathbb{Q}}(D_j)|\, N^{-2f(E)}$, $|N_{E/\mathbb{Q}}(N^2(1-\bar{u}u))| \cdot \mathrm{mn}_S(j;\gamma)^2 = |N_{E/\mathbb{Q}}(D_j)|^2\, |N_{E/\mathbb{Q}}(1-\bar{u}u)| = \|\mathfrak{g}\|_S(\gamma)^2$. Multiplying (3) by $\mathrm{mn}_S(j;\gamma)^2$: for $r \le 1$ the right side is at most $\mathrm{ld}_S(y,v)\,(N^{f(E)}\mathrm{mn}_S(j;\gamma))^2$, and for $r = 2$ it is $\mathrm{ld}_S(y,v)\,\|\mathfrak{g}\|_S(\gamma)^2$. Both are bounded by the maximum. □

LEAN `Filling.minorNorm_sq_le_linkDiscSq_mul`

Proof of Theorem B.

Proof of Theorem B. Write $k_0 = k(\mathfrak{s})$, $k_\gamma = \mathrm{pos}_S(c, c'; \gamma)$, $L_0 = L \cap \ker \mathrm{L}_z(T)$ and $T^\circ = \mathrm{link}^\circ(\gamma, \gamma')$.

Bases. Let m be a matrix whose columns, indexed by a finite set $\mathcal{C}$, are a basis of L_0, and p its part on the rows κ of the first components. The rows z of m vanish, and the link form vanishes on L_0; so $m^* Q_S^{\mathrm{link}} m = p^* p$, and T° is the column span of p, of dimension $\#\mathcal{C}$. By Input 3(c) (with $A = m^* Q_S^{\mathrm{link}}$, whose product with m is invertible by Input 1(c)), extend m by a matrix u to a basis $(m \,|\, u)$ of L with $m^* Q_S^{\mathrm{link}} u = 0$, and then $(m \,|\, u)$ by a matrix y with i columns to a basis of $\mathrm{link}(\gamma)$ with $(m \,|\, u)^* Q_S^{\mathrm{link}} y = 0$ and $yt \in L$ only for $t = 0$. Then $i = k_0 + 1 - \dim L$ by Input 1(a), and if $i = 0$ then $L = \mathrm{link}(\gamma)$. Moreover $\mathrm{L}_z(T) u$ has a left inverse, the columns of u being independent modulo L_0.

Lifts. By Proposition 4.2 there is e with $(\widetilde{m \,|\, u} \,|\, y)e \in \mathrm{IL}(c; \gamma)$, $\mathrm{ld}_S((m \,|\, u \,|\, y), e)\, w_0^2 \le \mathrm{mn}_S(j; \gamma)^2$ and $w_0 = \mathrm{i}_{S,j}(k_\gamma)\mathrm{i}_{S,j}(k_\gamma^{-1})\mathfrak{d}_S(k_\gamma) \neq 0$. Lemma 4.5, applied to $m_1 = (m \,|\, u)$ and y, gives a', v, z with integral lifts of $\tilde{m}_1 a'$ and of $\tilde{m}_1 z + \tilde{y} v$; applied to m and u and the lift $\tilde{m}_1 a'$, it gives a, g_0, z' with integral lifts of $\tilde{m}a$ and of $\tilde{m}z' + \tilde{u}g_0$; and the discriminants multiply:

$$\mathrm{ld}_S(m, a) \cdot \mathrm{ld}_S(u, g_0) \cdot \mathrm{ld}_S(y, v) = \mathrm{ld}_S((m \,|\, u \,|\, y), e). \tag{4}$$

The matrix of the statement of Theorem B is m_1, whose columns span L, with the index set $\mathcal{C} \sqcup \{1, \dots, r\}$, r the number of columns of u.

The part in L_0. Let n be the integral matrix of Lemma 4.4 for $\tilde{m}a$ and c'. Its middle rows are $\mathrm{C}_S^\circ(c)\, \tilde{p}\, a$, its link image is killed by $\mathrm{L}_z(\cdot)$, and its columns are orthogonal to $X + k_\gamma X$; so by Input 1(h) the values $(h(k_\gamma \mathfrak{f}_l, \cdot))_{l < \varrho}$ of its middle part lie in the lattice of the conjugated first ϱ rows of $Z(k_\gamma)$. Input 3(d), with $V = T^\circ$, p, a and $D = [\det(p^* p)]\,[\det a]^2 = \mathrm{ld}_S(m, a) \neq 0$, gives (i) and

$$\big(\mathrm{disc}_{S,c}(T^\circ) \cdot \mathrm{o}_{S,c}(k_\gamma; 1, T^\circ)\big)^2 \;\le\; \mathrm{ld}_S(m, a). \tag{5}$$

The part in L off L_0. Proposition 4.7, for the matrix u, the lift $\tilde{m}z' + \tilde{u}g_0$ (whose image under $\widetilde{\mathrm{L}_z(T)}$ is that of $\tilde{u}g_0$) and the number i (if $i = 0$ then $L = \mathrm{link}(\gamma)$ and hypothesis (a) holds), gives $r \le 1 + (j - \varrho)_+$ and

$$x'^2\, \mathrm{mn}_S(j; \gamma)^2 \;\le\; \mathrm{ld}_S(u, g_0) \cdot \max\big(\|\mathrm{g}\|_S(\gamma), N^{f(E)} \mathrm{mn}_S(j; \gamma)\big)^2. \tag{6}$$

Since $k_0 + 1 = \dim \mathrm{link}(\gamma) = \#\mathcal{C} + r + i$, we get $k_0 \le \dim T^\circ + i + (j - \varrho)_+$.

Assembly. Multiplying (5), (6) and $\mathrm{ld}_S(y, v)$, then using (4) and the bound for the full lift,

$$\begin{aligned}\big(\mathrm{disc}_{S,c}(T^\circ)\mathrm{o}_{S,c}(k_\gamma; 1, T^\circ)\big)^2 x'^2\, \mathrm{mn}_S(j; \gamma)^2\, \mathrm{ld}_S(y, v) &\le \max(\cdots)^2\, \mathrm{ld}_S((m \,|\, u \,|\, y), e) \\ &\le \max(\cdots)^2\, \mathrm{mn}_S(j; \gamma)^2\, w_0^{-2}.\end{aligned}$$

Dividing by $\mathrm{mn}_S(j; \gamma)^2 > 0$ and multiplying by $\mathfrak{d}_S(k_\gamma)^2$, the left side becomes

$$\big(\mathrm{pi}_S(c, c'; \gamma, \gamma')\, x'\big)^2\, \mathrm{ld}_S(y, v),$$

and the right side becomes

$$\max(\cdots)^2 \cdot \big(\mathrm{i}_{S,j}(k_\gamma)\, \mathrm{i}_{S,j}(k_\gamma^{-1})\big)^{-2} = \mathrm{cg}_S(j; c, c'; \gamma)^2. \qquad \square$$

Proof of Theorem A.

Proof of Theorem A. Apply Theorem B to (γ,γ') and to (γ',γ); the space L is the same, so both give the same number $i = k(\mathfrak{s}) + 1 - \dim L$ and the same x'. This gives (i) and (ii) of Theorem A, matrices m, y, v, z for γ and m', y', v', z' for γ', with i columns in y and in y', and

$$\begin{aligned} \big(\mathrm{pi}_S(c,c';\gamma,\gamma')x'\big)^2 \mathrm{ld}_S(y,v) &\le \mathrm{cg}_S(j;c,c';\gamma)^2, \\ \big(\mathrm{pi}_S(c,c';\gamma',\gamma)x'\big)^2 \mathrm{ld}_S(y',v') &\le \mathrm{cg}_S(j;c,c';\gamma')^2. \end{aligned} \tag{7}$$

The pairing of the complements. Let Q_0 be the diagonal matrix with $Q_S^{\mathrm{link}} Q_0 = Q_S^{\pm}$; it satisfies $Q_0^* Q_S^{\pm} = 4\, Q_S^{\mathrm{link}}$ (both are checked on the diagonal entries). We apply Input 2 with $Q = Q_S^{\mathrm{link}}$, $a = g(S)^{-1}y$, $a' = Q_0 y'$, v, v' and $x_0 = 1$. Here

$$a^* Q a' = g(S)^{-1} y^* Q_S^{\pm} y', \qquad a^* Q a = g(S)^{-2} y^* Q_S^{\mathrm{link}} y, \qquad a'^* Q a' = 4\, y'^* Q_S^{\mathrm{link}} y',$$

and the determinants of the last two have the absolute norms

$$g(S)^{-4if(E)}\, |N_{E/\mathbb{Q}} \det(y^* Q_S^{\mathrm{link}} y)| \qquad \text{and} \qquad 2^{4if(E)}\, |N_{E/\mathbb{Q}} \det(y'^* Q_S^{\mathrm{link}} y')|.$$

So, once its hypotheses are verified, Input 2 gives

$$1 \;\le\; \Big(\frac{2}{g(S)}\Big)^{4if(E)} \mathrm{ld}_S(y,v) \cdot \mathrm{ld}_S(y',v'). \tag{8}$$

Invertibility. Suppose $y^* Q_S^{\pm} y'\, t = 0$. The vector $y't$ lies in $\mathrm{link}(\gamma')$, which is isotropic for the link form and contains L; so $y't$ is orthogonal to the columns of m and of y, which span $\mathrm{link}(\gamma)$. As the link is Lagrangian (Input 1(a)), $y't \in \mathrm{link}(\gamma)$, hence $y't \in L$ and $t = 0$ by the property of y'.

Integrality. Let (t_1, w_1) and (t_2, w_2) be lift data for $\tilde{m}z + \tilde{y}v$ and for $\tilde{m}'z' + \tilde{y}'v'$. The columns of m, m' lie in L, which is orthogonal for the link form to $\mathrm{link}(\gamma)$ and to $\mathrm{link}(\gamma')$; so

$$v^*\, \widetilde{y^* Q_S^{\pm} y'}\, v' = (\tilde{m}z + \tilde{y}v)^*\, \widetilde{Q_S^{\pm}}\, (\tilde{m}'z' + \tilde{y}'v') = w_1^*\, t_1^*\, J_S(1 \widetilde{-\gamma\gamma'^{-1}}) t_2\, w_2,$$

the last step by Input 1(b) with the coefficients 1, 1. In the coordinates of the lattice of c this is $(\mathrm{C}_S(c)\tilde{t}_1 w_1)^* J_f (1-\kappa_0)(\mathrm{C}_S(c)\tilde{t}_2 w_2)$ with $\kappa_0 = \mathrm{pos}_S(c,c;\gamma\gamma'^{-1})$; as $\gamma\gamma'^{-1}$ is a term over $(c,1,c)$, $g(S)^{-1}(1-\kappa_0)$ is integral (Input 1(g)), and the other factors are integral. Hence the matrix $v^*\, \widetilde{a^* Q a'}\, v'$ is integral, and if its determinant δ is a unit then $1 \le [\delta]$.

Conclusion. Multiplying the two inequalities (7) and using (8),

$$\big(\mathrm{pi}_S(c,c';\gamma,\gamma')\mathrm{pi}_S(c,c';\gamma',\gamma)\, x'^2\big)^2 \le \Big(\frac{2}{g(S)}\Big)^{4if(E)} \big(\mathrm{cg}_S(j;c,c';\gamma)\mathrm{cg}_S(j;c,c';\gamma')\big)^2,$$

and all the quantities are non-negative; taking square roots gives (iii). □

5 Where the results are used

Theorems A and B (the latter for its part (i)) and Corollary 4.3 are used in the crude estimate of the note on the energy of amplitude sums of this unit, in the estimate of the pairs of terms of one level (Input 3 of Note F); Corollary 4.3 is used also in the modules on thin classes and on the block bound. Proposition 4.2, Lemmas 4.1, 4.4, 4.5, 4.6 and Proposition 4.7 are used only inside this module. Nothing of this note is used directly by the filling theorem.

6 Notes on the formal text

1. *Arguments that the census reads differently from Lean; to be settled by* `#check` *on a build.* In Lemma 4.4 the formal statement names the index set of the columns of m, a variable of the file, so Lean takes it as an argument although the census does not list it; it is the set $\mathcal{C}$ of the statement here. For Theorem B the census lists a variable of the file for a matrix, its index set and the finiteness of that set; the statement introduces its own matrix m after "there are", and Lean takes none of the three. They have no counterpart in the statement of Theorem B.
2. *Statements cut by the census.* The statements of Lemma 4.1, Proposition 4.2, Lemma 4.4 and Theorem B contain local definitions at which the census ends the statement; they were read to their true end, with the second reader's table of corrected spans.
3. *The columns of the bordered matrix.* In Lemma 4.1 the columns are γe_a for $a < j$ and e_a^* for $a \geq j$, as the code selects them; the comment of the source says $a \leq j$ and $a > j$. The second reader of the dossier had noted this.
4. *The matrix m of Theorem B* spans $L = \mathrm{link}(\gamma) \cap \mathrm{link}(\gamma')$, not its part in $\ker \mathrm{L}_z(T)$; the latter appears only in the proof.
5. *Truncated differences.* $(j - \varrho(\mathfrak{s}))_+$ and $k(\mathfrak{s}) + 1 - \dim L$ are differences of natural numbers in the formal text; the first is 0 or 1, and the second is not truncated since $\dim L \leq k(\mathfrak{s}) + 1$.
6. *Exponents computed here.* In Proposition 4.2 the powers of N in Jacobi's identity cancel exactly: $[N]^{\#B+2} \cdot [N]^{-\#B} = [N]^2 = N^{4f(E)}$, the square of the factor in Lemma 4.1. In Theorem A the factor $(2/g(S))^{2if(E)}$ is the square root of $g(S)^{-4if(E)} \cdot 2^{4if(E)}$, the absolute norms of $g(S)^{-2i}$ and of 4^i. In Proposition 4.7 the case $r = 2$ uses $|N_{E/\mathbb{Q}}(N^2)| = N^{4f(E)}$, which cancels the normalisation of $\mathrm{mn}_S(\varrho + 1; \gamma)$.
7. *What was not re-derived.* One passage is stated as the formal text has it and was not derived again for this note: the two divisibilities of Step 4 of Proposition 4.2. All other steps are proved in the text from the inputs. In Lemma 4.1 the formal text reorders the rows and columns of the bordered matrix and verifies its block form entry by entry; the note gives the three lines that this verification amounts to.
8. *The threshold x* of "fixes norm at least x" is a natural number in the formal text; in the inequalities of Theorems A and B and of Proposition 4.7 it is read as a real number.

9. *Input 4.* The formal statement is about an integral matrix and its plain products with the frame and its image; the note states it for the pairings of the hermitian form, as it is applied.

How this note was written. From the structured proofs of the module on link pairings in the fourth chunk of the unit's dossier, which a second reader had re-read completely. I read the module in the Lean text line by line and compared every statement of this note with it argument by argument; I read in the Lean text the definitions recalled in §2. The inputs are stated from the dossier's readings of their quoted statements; of them I read in the Lean text only the definitions they involve. The block form of the bordered matrix in Lemma 4.1 is written as the referee of this note gave it. Arguments that are mine and replace matrix computations of the formal text: the evaluation of $\det A$ in Lemma 4.1 by a rank-one update, the block identity in Step 3 of Proposition 4.2, the two identities (2), the exclusion of $r = 2$ for $j \le \varrho$, and the computation of the pairing in Theorem A. Nothing was compiled for this note. The program check of the unit's dossier covered 1371 leaf steps of 179 proofs.

Appendix A. Concordance with the Lean artifact

The formal names of the objects and the letters of this note: `link`, `commonLink` and `linkMatrix` are $\mathrm{link}(\gamma)$, $\mathrm{link}^{\circ}(\gamma,\gamma')$ and $\mathrm{L}(\gamma)$; `framePair` and `isotropicFramePair` are $[\mathfrak{f} \mid \gamma\mathfrak{f}]$ and $[\mathfrak{e} \mid \gamma\mathfrak{e}]$; `linkForm`, `linkMetric`, `zFormPair` are $Q_S^{\pm}$, Q_S^{link}, $\mathrm{L}_z(T)$; `linkDiscSq`, `minorNorm`, `classGauge`, `pairInvariant`, `termDiscNorm` are $\mathrm{ld}_S(v,e)$, $\mathrm{mn}_S(j;\gamma)$, $\mathrm{cg}_S(j;c,c';\gamma)$, $\mathrm{pi}_S(c,c';\gamma,\gamma')$, $\|\mathrm{g}\|_S(\gamma)$; `xRowsIndex`, `saturationIndex`, `latticeDisc`, `frameValueIndex` are $\mathrm{i}_{S,j}(k)$, $\mathfrak{d}_S(k)$, $\mathrm{disc}_{S,c}(V)$, $\mathrm{o}_{S,c}(k;a,V)$; `HasIntegralLifts`, `HasLevel` are $m \in \mathrm{IL}(c;\gamma)$, $\gamma \in \mathrm{Lv}_j(S)$; `classCoord`, `classCoordX₀` are $\mathrm{C}_S(c)$, $\mathrm{C}^{\circ}_S(c)$; `genIndex` is $[\xi]$. Two statements as the artifact has them:

```
lemma pairInvariant_mul_le (h' : γ' ∈ S.termsOver c s c') (hd : S.HasLevel j γ') : ∃ i, ϑ.k ≤
  finrank E (commonLink γ γ') + i + (j - ϑ.ϱ) ∧ (i = 0 → link γ = link γ') ∧
  S.pairInvariant c c' γ γ' * S.pairInvariant c c' γ' γ * (if ϑ.ϱ < j ∧ i = 0 then x else 1) ^ 2 ≤
  (2 / S.g) ^ (2 * i * numInfPlaces E) * (S.classGauge j c c' γ * S.classGauge j c c' γ')

lemma exists_link_split_le_classGauge γ' : (0 < S.latticeDisc c (commonLink γ γ') ∧ 1 ≤
  S.frameValueIndex c (S.termPosition c c' γ) 1 (commonLink γ γ')) ∧ ∃ᵉ (i) (ρ) (_ : Fintype ρ) (m :
  Matrix _ ρ E) (y : Matrix _ (Fin i) E) (v) (z : Matrix ρ _ (𝔸f E)), let L := link γ ⊓ link γ'; i =
    ϑ.k + 1 - finrank E L ∧ ϑ.k ≤ finrank E (commonLink γ γ') + i + (j - ϑ.ϱ) ∧ (i = 0 → L =
    link γ) ∧ colSpan m = L ∧ colSpan y ≤ link γ ∧ (∀ t, y *ᵥ t ∈ L → t = 0) ∧ colSpan (fromCols
  m y) = link γ ∧ HasIntegralLifts c γ (finAdeleMat m * z + finAdeleMat y * v.1) ∧ (S.pairInvariant
    c c' γ γ' * ite (ϑ.ϱ < j ∧ i = 0) x 1) ^ 2 * S.linkDiscSq γ v ≤ S.classGauge j c c' γ ^ 2
```

In this note	Lean declaration	File	Lines
Lemma 4.1	`Filling.HasLevel.abs_norm_bordered_det`	LinkPairing.lean	25–50
Proposition 4.2	`Filling.exists_lifts_linkDiscSq_le`	LinkPairing.lean	55–122
Lemma 4.4	`Filling.HasIntegralLifts.exists_isIntMatrix`	LinkPairing.lean	130–138
Lemma 4.5	`Filling.HasIntegralLifts.exists_orthogonal_split`	LinkPairing.lean	151–162
Lemma 4.6	`Filling.HasLevel.exists_mem_link_termDisc_eq`	LinkPairing.lean	167–179
Corollary 4.3	`Filling.one_le_xRowsIndex_and_saturationIndex`	LinkPairing.lean	181–184
Proposition 4.7	`Filling.minorNorm_sq_le_linkDiscSq_mul`	LinkPairing.lean	193–232
Theorem B	`Filling.exists_link_split_le_classGauge`	LinkPairing.lean	239–278
Theorem A	`Filling.pairInvariant_mul_le`	LinkPairing.lean	284–315

Notation

E, τ, x_τ, $f(E)$, $N_{E/\mathbb{Q}}$ a CM field with its CM type, an infinite place, the image at it, the number of infinite places, the norm

$\mathbb{A}_{E,f}$, $\hat{O}_E$, $\tilde{a}$, $[\xi]$ finite adeles, integral adeles, the image of a matrix over E, the generalised index

$\mathfrak{s} = (\varrho, \kappa, k_0)$ a shape; $\varrho(\mathfrak{s})$, $k(\mathfrak{s})$, $I(\mathfrak{s})$, $M(\mathfrak{s})$; j a level; in proofs ϱ, k_0 abbreviate $\varrho(\mathfrak{s})$, $k(\mathfrak{s})$

e_a, e_a^*, z, $\mathfrak{f}_a$, X basis vectors, the frame, the span of the e_a; h the hermitian form

S filling data: $N(S)$ (also N), $g(S)$, $\mathsf{G}(S)$, $\mathrm{Cl}(S)$, $G_S(E)$, G_S^{lat}, $K(S)$, red_S, $K_g(S)$

T, $T_S^{\mathrm{rat}}(E)$, $T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})$, J_S, J_f the middle form and the split Gram matrices, rational and in the lattice basis

c, c', s, t, γ, γ' classes, elements of $\mathsf{G}(S)$, rational elements

$\mathrm{pos}_S(c, c'; \gamma)$, $\mathrm{Tm}_S(c, s, c')$, k, k_γ, κ_0 position, terms; positions in proofs

$Z_S(\gamma)$, $Z(k)$, $Z_X(k)$, $D_l(Z)$, D_j, D frame Gram matrices, their first rows, leading minors

$\mathfrak{g}_S(\gamma)$, $\|\mathfrak{g}\|_S(\gamma)$ term discriminant and the product of its absolute values

$[\mathfrak{f} \mid \gamma\mathfrak{f}]$, $[\mathfrak{e} \mid \gamma\mathfrak{e}]$, $\mathrm{L}(\gamma)$, $\mathrm{link}(\gamma)$, $\mathrm{link}^\circ(\gamma, \gamma')$ frame pair, isotropic frame pair, link matrix, link, common link; the italic L, L_0 and T° are subspaces in the proof of Theorem B (the upright L belongs to the link matrix and to $\mathrm{L}_z(T)$)

$\mathrm{L}_z(T)$, $Q_S^{\pm}$, Q_S^{link}, Q_0 the z-pairings, link form, link metric, a diagonal matrix

$\mathrm{ld}_S(v, e)$, $\mathrm{mn}_S(j; \gamma)$, $\mathrm{cg}_S(j; c, c'; \gamma)$, $\mathrm{pi}_S(c, c'; \gamma, \gamma')$ squared discriminant, minor norm, gauge, invariant of a pair

$\mathrm{C}_S(c)$, $\mathrm{C}_S^\circ(c)$, $m \in \mathrm{IL}(c; \gamma)$ class coordinates; integral lifts, for the filling data S of the setting

$\mathrm{i}_{S,j}(k)$, $\mathfrak{d}_S(k)$, $\mathrm{disc}_{S,c}(V)$, $\mathrm{o}_{S,c}(k; a, V)$, $\mathrm{i}(F)$, w_0 indices and discriminants of lattices

$\gamma \in \mathrm{Lv}_j(S)$, $\mathrm{pl}_j(Z)$, υ level; the padded leading matrix; the scalar of a normal form; $s \in \mathrm{Agr}(S)$, $\mathrm{fix}(s) \geq x$ the predicates "agrees" and "fixes norm at least x"

P, M, A, Z', B, F, p', Q, u, n, a, e, z_1 matrices of Lemma 4.1 and Proposition 4.2; here P is a matrix, not the parameters

$\mathcal{C}$, $\mathcal{C}'$, m, y, v, z, f, d, r, i, w, x, x', x_0 index sets, matrices and numbers of the statements

ε, ε_1, ε_2, u', a', ς_0, δ the two link vectors, a selection of rows, a determinant, in Proposition 4.7

p, g_0, z', m_1, m', y', v', t_1, t_2, w_1, w_2 in the proofs of Theorems B and A

J, c_0, d_p, V, q, l, b, D_0, O, G, K, U, R, H, I, $\mathrm{fm}(k)$, n_X, n_M in the inputs and as stated where they occur; GL, $\mathbb{Q}$

READBACK NOTE 41

The filling data: Bessel subgroups, links of frames, Siegel's lemma and strong approximation

One unit of the Lean artifact grh-compact, read back into mathematics: Note S of the unit on the filling theorem

Summary. The filling theorem behind Theorem B of the write-up (*enough forms*) is a second moment of Poincaré series on a unitary group of signature $(\varrho + k, \varrho + 1)$ at every infinite place of a CM field E, built from Bessel functions along the subgroup $\mathcal{S} = \mathcal{U}\mathcal{H}$ that is unitriangular on a frame of ϱ isotropic vectors and one negative vector. This note sets the stage. It defines the record of the *filling data* with its standing hypotheses, the groups, double cosets, terms, cutoffs and amplitudes of the second moment, and the abstract inputs from the real places (Bessel systems, sieve systems). It then proves the tools of the later notes: the dictionary between rational, finite adelic and real coordinates; that the *link* of two frames is a Lagrangian subspace and an invariant of the double coset of $\mathcal{U}$; Siegel's lemma for hermitian lattices, with the convergence of a height zeta function of subspaces; strong approximation for $\mathcal{U}$; and that a term in the Bessel subgroup with trivial reduction is unipotent, by the freeness of the classes.

What this note is. The artifact grh-compact is a body of Lean code whose final theorems state the Riemann hypothesis and its analogue for Dirichlet L-functions. That is a claim: its authors report that Lean's kernel accepts the code, a replay of that check by the owner of this write-up is recorded elsewhere, and nothing was compiled for this note; no mathematician has read the proof through. This note treats the first part of one unit (the folder of the filling theorem): 232 of its 497 declarations, the content of nine files. Three further notes of the same writer treat the rest of the first half of the unit: Note B (*Box partitions of arithmetic unipotent groups*), Note E (*The energy of amplitude sums*) and Note U (*Poincaré series of cut Bessel functions*); the second half has four notes of another writer: Note F (the filling theorem), Note P (the pairing of links), Note T (the top floor at a finite place) and Note G (from amplitude sums to local correlations). This note is Note S; the eight notes cite each other by these letters. Every statement was compared, argument by argument, with the Lean text. Proofs are complete at the level of a research paper, with these exceptions, each said where it occurs: results of other units and of the background library are quoted as *Inputs*; identities between explicit matrices that the formal text checks entry by entry are stated and not recomputed; and the argument lists that the census of the artifact reads from the text differ from Lean's for some declarations (Section 8). The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 232 declarations.

1 The statements

The notions are defined in Section 2; here is what they are in a few words. A *shape* $\mathfrak{s} = (\varrho, \kappa, k_0)$ fixes a hermitian space with basis $e_0, \dots, e_{\varrho-1}$; x_i $(i \in \kappa)$; z; $e^*_{\varrho-1}, \dots, e^*_0$, in which e_a and e^*_a are dual isotropic vectors and the middle part, spanned by the x_i and z, carries a form T. The *frame* is $(\mathfrak{f}_0, \dots, \mathfrak{f}_\varrho) = (e_0, \dots, e_{\varrho-1}, z)$. The group $\mathcal{U}(T)$ is the unipotent radical of the stabilizer of the flag of the e_a, the group $\mathcal{H}(T)$ fixes all e_a, e^*_a and z, and $\mathcal{S}(T) = \mathcal{U}(T)\mathcal{H}(T)$ is the Bessel subgroup. Filling data S over a CM field E single out the form $T^{\mathrm{rat}}_S(E) = \mathrm{diag}(1, \dots, 1, -N(S))$, a lattice, two primes $q_0(S), q_1(S)$ with product $g(S)$ and the principal congruence subgroup $K_g(S)$ of level $g(S)$ in the finite adelic group G^{lat}_S.

Theorem A (strong approximation for the unipotent group). *(a)* Let E be a number field that is a CM field, $\mathfrak{s}$ a shape, T an $\mathfrak{s}$-form over E that is hermitian and invertible, and T' an $\mathfrak{s}$-form over the finite adeles $\mathbb{A}_{E,f}$ such that $J(T')$ is the image of $J(T)$ under $E \to \mathbb{A}_{E,f}$. Then $\mathcal{U}(T')$ is contained in the closure, in $G(T')$, of the image of $\mathcal{U}(T)$ under the homomorphism $G(T) \to G(T')$ that embeds every entry.
(b) Let E be a CM field with a CM type, $\mathfrak{s}$ a shape and $S \in \mathrm{Fill}(E, \mathfrak{s})$. For every $x \in G^{\mathrm{lat}}_S$ with $x \in \mathcal{U}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))$ and every $y \in G^{\mathrm{lat}}_S$ there is a rational element $u \in \mathcal{U}(T^{\mathrm{rat}}_S(E))$ such that $y^{-1}\,(x^{-1}\,\mathrm{r}^f_S(u))\,y \in K_g(S)$.

Theorem B (the link of two frames). *(a)* Let F be a commutative ring with involution, $\mathfrak{s}$ a shape, T an $\mathfrak{s}$-form over F and $\gamma, u, u' \in G(T)$ with $u \in \mathcal{U}(T)$ and $u' \in \mathcal{U}(T)$. Then $\mathrm{link}(u^{-1}\gamma u') = \mathrm{link}(\gamma)$.
(b) Let E be a CM field with a CM type, $\mathfrak{s}$ a shape, $S \in \mathrm{Fill}(E, \mathfrak{s})$, $\gamma \in G_S(E)$ and $p \in E^{M(\mathfrak{s})} \times E^{M(\mathfrak{s})}$. Then $p \in \mathrm{link}(\gamma)$ if and only if $q^*\,Q^{\pm}_S\,p = 0$ for every $q \in \mathrm{link}(\gamma)$: the link is its own orthogonal complement for the form $Q^{\pm}_S$ of signature zero.

Theorem C (Siegel's lemma for hermitian lattices; a height zeta function). Let E be a CM field with a CM type that is stably unramified, κ a finite set, and $u \in G_f(E,\kappa)$ an element of the finite adelic unitary group of the standard form. Write $\|X\| = \big(\sum_\tau \sum_{i\in\kappa} |(X_i)_\tau|^2\big)^{1/2}$ for $X \in E^\kappa$, the sum over the infinite places τ of E, and $\mathrm{disc}_{u^{-1}}(V)$ for the discriminant of the lattice of the adelic points of a subspace $V \subseteq E^\kappa$ that lie in $u\,\hat{O}_E^\kappa$ (Definition 2.28).
(a) Let $V \subseteq E^\kappa$ be a subspace and s a natural number with $\dim_E V + s = \#\kappa$. Then there are vectors $X_1, \dots, X_s \in E^\kappa$, all non-zero and in $u\,\hat{O}_E^\kappa$, such that V is the set of the $v \in E^\kappa$ with $\sum_{i\in\kappa} \overline{(X_j)_i}\, v_i = 0$ for $j = 1, \dots, s$, and

$$\Big(\prod_{j=1}^{s} \|X_j\|\Big)^{2f(E)} \;\le\; \big(8\, f(E)\, s\ \mathrm{rd}_E\big)^{f(E)s} \cdot \mathrm{disc}_{u^{-1}}(V).$$

(b) For every natural number s,

$$\sum_{\substack{V\subseteq E^\kappa \\ \dim_E V + s = \#\kappa}} \mathrm{disc}_{u^{-1}}(V)^{-(\#\kappa+1)} \;\le\; \big(200\, f(E)\, \#\kappa\ \mathrm{rd}_E\big)^{f(E)(\#\kappa+1)s}.$$

Theorem D (terms in the Bessel subgroup). Let E be a CM field with a CM type, $\mathfrak{s}$ a shape and $S \in \mathrm{Fill}(E,\mathfrak{s})$ admissible filling data.
(a) Let $c, d \in \mathrm{Cl}(S)$ and $\gamma \in G_S(E)$ with $\gamma \in \mathcal{S}(T_S^{\mathrm{rat}}(E))$ and $\mathrm{pos}_S(c,d;\gamma) \in K_g(S)$. Then $c = d$ and $\gamma \in \mathcal{U}_{S,c}$.
(b) Let $c \in \mathrm{Cl}(S)$ and $s \in \overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{\,S}$. Then there are a class $d \in \mathrm{Cl}(S)$ and an element $\gamma \in \mathcal{S}(T_S^{\mathrm{rat}}(E))$ with $\gamma \in \mathrm{Tm}_S(c,s,d)$.
(c) Let $c, c', d \in \mathrm{Cl}(S)$, $s \in \mathsf{G}(S)$ and $\gamma, u \in G_S(E)$ with $\gamma \in \mathrm{Tm}_S(c,s,c')$, $u \in \mathrm{Tm}_S(c,s,d)$ and $\gamma \mathfrak{f}_a = u \mathfrak{f}_a$ for all frame vectors. Then $c' = d$ and $\gamma^{-1}u \in \mathcal{U}^0_{S,c'}$.

Theorem E (the dictionary). Let E be a CM field with a CM type, $\mathfrak{s}$ a shape and $S \in \mathrm{Fill}(E,\mathfrak{s})$.
(a) $K_g(S)$ is the group of the $k \in G_S^{\mathrm{lat}}$ such that k and k^{-1} have entries in $\hat{O}_E$ and all entries of $\hat{g}(S)^{-1}(k-1)$ lie in $\hat{O}_E$.
(b) Let $k \in G_f(E,\kappa)$, where κ is the index set of $\mathfrak{s}$. Then $\mathrm{inc}_S(k) \in K(S)$ if and only if k and k^{-1} have entries in $\hat{O}_E$ and the row k_0 of $k-1$ has entries in $\bar{\pi}(S)\,\hat{O}_E$.
(c) Let σ be a side, $x \in G_S(\mathbb{A}_E)$ with $x \in \mathcal{U}(T_S^{\mathrm{rat}}(\mathbb{A}_E))$, and $y \in G_\infty(E,\mathfrak{s})$. Then, as vectors indexed by $\mathrm{Bc}(S)$,

$$W_S^\sigma(\mathrm{a}_S^\infty(x)\, y)_{\cdot} = \frac{\psi_E\big(-(g(S)\,\varpi(S))^{-1}\,\mathrm{sd}(x)\big)}{\psi_S^{\mathcal{U}}(\mathrm{a}_S^f(x))}\, W_S^\sigma(y)_{\cdot},$$

where ψ_E is the standard additive character of $\mathbb{A}_E$.

Theorem A is the classical strong approximation theorem for a unipotent group, in the form needed here (the artifact names Platonov and Rapinchuk [2] for it); part (b) feeds Theorem D and the construction of Poincaré series in Note U. Theorem B

is used in Note E to sort the terms of the second moment by their links. Theorem C counts subspaces by the discriminants of their lattices; the sum in (b) is what makes the contribution of degenerate terms small. Theorem D identifies the terms that lie in the Bessel subgroup, and Theorem E translates between the three systems of coordinates in which the data are written.

2 The setting

Conventions. Rings are commutative with 1. A *ring with involution* is a commutative ring with a ring automorphism $x \mapsto \overline{x}$ of order at most two; letters R, R', F, A, C denote such rings (each statement says which structure it uses), and M^* is the conjugate transpose of a matrix M, M^{t} its transpose. All index sets κ, Ω, P, m, n, I of matrices are finite. A *shape* is denoted $\mathfrak{s}$, an $\mathfrak{s}$-form T or T', and $\gamma, \gamma', \delta, u, u', s, s', x, y, \sigma_0, \sigma_1$ are elements of the groups named in the statements; p, q are sequences of units; j, l, a, b, i are indices and c, c', c'', d classes or ring elements, as each statement says. E is a number field that is a CM field: a totally complex quadratic extension of a totally real field, with complex conjugation $x \mapsto \overline{x}$. A *CM type* of E is the choice, for every infinite place τ of E, of an embedding $E \to \mathbb{C}$ inducing τ; the image of $x \in E$ is written x_τ. When E has a CM type we say *CM field with a CM type*. S always denotes filling data over E for $\mathfrak{s}$ (Definition 2.15); in statements about S the field E and the shape $\mathfrak{s}$ are those of S. A *side* is one of the two words full, cut, and σ denotes a side. Measures on locally compact groups are Haar measures, fixed by the formal text; integrals of functions with values in $[0, \infty]$ are Lebesgue integrals with values in $[0, \infty]$. The letters S, s, σ, q, ψ, ζ have in the notes of this unit the meanings given here, not those of the overview of the write-up (there: a set of columns, the complex variable, a field automorphism, a rational number, a column, Riemann's function). For the filling data of admissible data (Note G), the primes $q_0(S)$, $q_1(S)$ are the primes that the overview calls g_0, g_1, and $g(S)$ is their product.

Adeles. For a CM field E with ring of integers O_E: $\mathbb{A}_E$ is the ring of adeles, $\mathbb{A}_{E,f}$ the ring of finite adeles, $\hat{O}_E \subset \mathbb{A}_{E,f}$ the closure of O_E, a compact open subring. Complex conjugation acts on $\mathbb{A}_{E,f}$ and on the set of finite places $w \mapsto \overline{w}$; $\mathbb{F}_w$ is the residue field at w, q_w its number of elements, and $\hat{O}_E \to \mathbb{F}_w$ the reduction modulo w. The number of infinite places is $f(E)$, the discriminant d_E, the root discriminant $\mathrm{rd}_E = |d_E|^{1/(2f(E))}$, and $\mathrm{vol}(\hat{O}_E) = 2^{f(E)}/\sqrt{|d_E|}$. The standard additive character ψ_f of $\mathbb{A}_{E,f}$ is $\psi_f(x) = e^{2\pi i \,\mathrm{Tr}_{E/\mathbb{Q}}(a)}$ for any $a \in E$ with $x - a \in \hat{O}_E$, and $\psi_E(x) = e^{-2\pi i \sum_\tau 2\,\mathrm{Re}\, x_\tau}\, \psi_f(x_f)$ is the standard additive character of $\mathbb{A}_E$. For a unit a of $\mathbb{A}_{E,f}$ (or of a finite product of local fields), $[a]$ is the *generalized index*: the inverse of the factor by which multiplication by a scales Haar measure, equal to the index $[\hat{O}_E : a\hat{O}_E]$ when $a \in \hat{O}_E$; it is 0 at non-units. E is *stably unramified* if it contains a primitive root of unity of order 5040.

Setting 2.1 (shapes, forms, groups; background)**.** A *shape* $\mathfrak{s} = (\varrho, \kappa, k_0)$ is a natural number ϱ, a finite set κ and an element $k_0 \in \kappa$. Put $k(\mathfrak{s}) = \#\kappa$, $n(\mathfrak{s}) = \varrho + 1$, $M(\mathfrak{s}) = \kappa \sqcup \{z\}$ (the *middle indices*), and let $I(\mathfrak{s})$ be the disjoint union of $\{0, \dots, \varrho - 1\}$,

$M(\mathfrak{s})$ and a second copy of $\{0,\dots,\varrho-1\}$; its $r(\mathfrak{s}) = 2\varrho + k(\mathfrak{s}) + 1$ elements index a basis written

$$e_0,\dots,e_{\varrho-1};\qquad x_i\ (i\in\kappa),\ z;\qquad e^*_{\varrho-1},\dots,e^*_0.$$

The *frame* is $\mathfrak{f}_a = e_a$ for $a<\varrho$ and $\mathfrak{f}_\varrho = z$. The *level* of a basis vector is $\mathrm{lev}(e_a)=a$, $\mathrm{lev}(x_i)=\mathrm{lev}(z)=\varrho$, $\mathrm{lev}(e^*_a)=2\varrho-a$, and its *floor* is $\mathrm{fl}(e_a)=\mathrm{fl}(e^*_a)=a$, $\mathrm{fl}(z)=\varrho$, $\mathrm{fl}(x_i)=\varrho+1$.

Let R be a ring. An $\mathfrak{s}$-*form* over R is a square matrix T over R indexed by $M(\mathfrak{s})$; for $N\in R$, $T^{\mathrm{rat}}(\mathfrak{s};N)$ is the diagonal $\mathfrak{s}$-form with entries 1 on κ and $-N$ at z. The *split Gram matrix* $J(T)$, indexed by $I(\mathfrak{s})$, has the entries 1 at (e_a,e^*_a) and (e^*_a,e_a), the block T on $M(\mathfrak{s})$, and 0 elsewhere. If R has an involution, $G(T)$ is the group of the invertible matrices g over R indexed by $I(\mathfrak{s})$ with $g^*\,J(T)\,g = J(T)$, and $h_T(v,w) = v^*\,J(T)\,w$. For the form $T^{\mathrm{rat}}(\mathfrak{s};1)$ over $\mathbb{C}$ the group is $G_1(\mathfrak{s})$, a unitary group of signature $(\varrho+k(\mathfrak{s}),\varrho+1)$. Subgroups of $G(T)$:

- $\mathcal{U}(T)$: the g with $g_{ij}=\delta_{ij}$ whenever $\mathrm{lev}(j)\le\mathrm{lev}(i)$ (unitriangular for the blocks of equal level);
- $\mathcal{U}^0(T)$: the $g\in\mathcal{U}(T)$ with $g\mathfrak{f}_a=\mathfrak{f}_a$ for all $a\le\varrho$;
- $\mathcal{H}_l(T)$, for a natural number l: the g whose column j is that of 1 whenever $\mathrm{fl}(j)<l$; and $\mathcal{H}(T)=\mathcal{H}_{\varrho+1}(T)$, which fixes all e_a, e^*_a and z;
- $\mathcal{S}(T)$, the *Bessel subgroup*: the g with $g_{i\,\mathfrak{f}_a}=\delta_{i\,\mathfrak{f}_a}$ for all $a\le\varrho$ and all i with $\mathrm{lev}(i)\ge\mathrm{lev}(\mathfrak{f}_a)$, that is, $g\mathfrak{f}_a-\mathfrak{f}_a\in\mathrm{span}(e_b : b<a)$.

For a matrix γ indexed by $I(\mathfrak{s})$, the *frame Gram matrix* is the square matrix of size $\varrho+1$

$$Z_T(\gamma) = \big((J(T)\gamma)_{\mathfrak{f}_a,\mathfrak{f}_b}\big)_{a,b\le\varrho} = \big(h_T(\mathfrak{f}_a,\gamma\mathfrak{f}_b)\big)_{a,b},$$

and for $y\in G_1(\mathfrak{s})$ we write $Z(y)$ for the frame Gram matrix of y for $T^{\mathrm{rat}}(\mathfrak{s};1)$. For a square matrix Z and l at most its size, $\mathrm{D}_l(Z)$ is the determinant of its upper left $l\times l$ block. For $s\in G(T)$, $\mathrm{sd}(s)=\sum_{a<\varrho} s_{\mathfrak{f}_a,\mathfrak{f}_{a+1}}$. For $v\in R^{\varrho+1}$ we write $\mathfrak{f}v=\sum_a v_a\mathfrak{f}_a$ and $(\gamma\mathfrak{f})v=\gamma(\mathfrak{f}v)$. For a sequence p of units of R and $l\le\varrho$, the *Weyl element* $w_l(p)\in G(T)$ maps $e_a\mapsto p_a e^*_a$ and $e^*_a\mapsto \overline{p_a}^{-1}e_a$ for $a<l$ and fixes the other basis vectors.

The following facts about these groups are used throughout; the formal text proves them in its background library.

Inputs (groups). In Setting 2.1, let R be a ring with involution and T an $\mathfrak{s}$-form over R. (G1), unit 112: if d is an invertible matrix that is block upper triangular for the levels and u is unitriangular for the levels, then $d^{-1}ud$ is unitriangular for the levels. (G2), unit 1: $\mathcal{S}(T)$ normalizes $\mathcal{U}(T)$; $\mathrm{sd}(hu) = \mathrm{sd}(h) + \mathrm{sd}(u)$ for $h, u \in \mathcal{S}(T)$; $\mathrm{sd}(h) = 0$ for $h \in \mathcal{H}(T)$. (G3), units 119 and 1: $\mathcal{U}(T) \subseteq \mathcal{S}(T)$ and $\mathcal{H}(T) \subseteq \mathcal{S}(T)$; if $J(T)$ is hermitian and invertible, then $s \in \mathcal{S}(T)$ if and only if $s = uh$ with $u \in \mathcal{U}(T)$ and $h \in \mathcal{H}(T)$. For $T = T^{\mathrm{rat}}(\mathfrak{s}; N)$ and g in the unitary group $\mathrm{U}_\kappa(R)$ of the identity matrix on κ, let $\iota_N(g) \in G(T)$ act as g on the x_i and fix the other basis vectors; then $\iota_N(g) \in \mathcal{H}(T)$, and if N is a unit every $h \in \mathcal{H}(T)$ is $\iota_N(g)$ for some g; if moreover $J(T)$ is hermitian and invertible and R has an element t with $t + \bar{t} = 1$, every $s \in \mathcal{S}(T)$ is $u\,\iota_N(g)$ with $u \in \mathcal{U}(T)$ and $g \in \mathrm{U}_\kappa(R)$. (G4), unit 119: for $u, u' \in \mathcal{U}(T)$ and $\gamma \in G(T)$, every leading minor satisfies $\mathrm{D}_l(Z_T(u^{-1}\gamma u')) = \mathrm{D}_l(Z_T(\gamma))$; and $Z_T(\gamma) = \mathfrak{f}^*\, J(T)\, \gamma \mathfrak{f}$. (G5), unit 120: for $u \in \mathcal{U}(T)$, $u\mathfrak{f}_b - \mathfrak{f}_b \in \mathrm{span}(\mathfrak{f}_a : a < b)$ for all $b \le \varrho$ (the matrix of u on the frame is upper unitriangular), and the entries of u in the rows of $M(\mathfrak{s})$ and of the e_a^* and the columns of the e_a and of $M(\mathfrak{s})$ are those of 1: u preserves $X = \mathrm{span}(e_a)$ and acts trivially on $X^\perp/X$. (G6), unit 29: for $g \in G(T)$, $J(T)\, g^{-1} = g^*\, J(T)$ and $h_T(gv, gw) = h_T(v, w)$.

The lattice form and the lattice basis.

Definition 2.2 (lattice form, lattice basis)**.** Let $\mathfrak{s} = (\varrho, \kappa, k_0)$ be a shape and R a ring.

(i) For $N \in R$, $T^{\mathrm{lat}}(\mathfrak{s}; N)$ is the $\mathfrak{s}$-form with diagonal entries 1 at the $i \in \kappa$, $i \neq k_0$, 0 at k_0 and $-N$ at z, with entries -1 at (k_0, z) and (z, k_0), and 0 elsewhere.

(ii) Let R have an involution and let p be a unit of R. The matrix $\Lambda^\circ(\mathfrak{s}; p)$, with rows indexed by $M(\mathfrak{s})$ and columns by κ, has the rows $i \in \kappa$ of the diagonal matrix with $\overline{p}$ at k_0 and 1 elsewhere, and its row z has the entry $-p^{-1}$ in column k_0 and 0 elsewhere.

(iii) With R and p as in (ii), $\Lambda(\mathfrak{s}; p)$ is the invertible matrix indexed by $I(\mathfrak{s})$ that is the identity on the e_a and e_a^* and whose middle block has the columns x_i $(i \neq k_0)$, $v = \overline{p}^{-1} x_{k_0} + (p\overline{p})^{-1} z$ in the place of x_{k_0}, and z. Its inverse has the middle block $(\Lambda^\circ(\mathfrak{s}; p) \mid z)$: in the basis x_i $(i \neq k_0)$, v, z one has $x_{k_0} = \overline{p}\, v - p^{-1} z$.

LEAN `latMiddleGram` · `middleBaseChange` · `besselLatticeBasis`

So $T^{\mathrm{lat}}(\mathfrak{s}; N)$ is the Gram matrix of the form $T^{\mathrm{rat}}(\mathfrak{s}; N)$ in the basis x_i $(i \neq k_0)$, v, z when $N = p\overline{p}$: the vector v is isotropic, $h(v, z) = -1$ and $h(z, z) = -N$.

Lemma 2.3. *Let $\mathfrak{s}$ be a shape.* (a) *Let R be a ring with involution, p a unit of R and $N \in R$ with $p\,\overline{p} = N$. Then*

$$\Lambda(\mathfrak{s}; p)^*\ J(T^{\mathrm{rat}}(\mathfrak{s}; N))\ \Lambda(\mathfrak{s}; p) = J(T^{\mathrm{lat}}(\mathfrak{s}; N)).$$

(b) *Let A be a ring with involution, $R \subseteq A$ a subring and $N \in A$ with $N \in R$ and $\overline{N} = N$. Then $T^{\mathrm{lat}}(\mathfrak{s}; N)$ is* unimodular over *R: it is hermitian, and it and its inverse have entries in R.*

Proof. (a) Both sides are the identity pairing on the e_a, e_a^*, so only the middle block matters. There the columns of $\Lambda(\mathfrak{s}; p)$ are x_i $(i \neq k_0)$, v, z, and $h(x_i, x_j) = \delta_{ij}$,

$h(x_i, v) = 0$ for $i \neq k_0$, $h(x_i, z) = 0$, $h(v, v) = (p\overline{p})^{-1} - N(p\overline{p})^{-2} = 0$, $h(v, z) = -N(p\overline{p})^{-1} = -1$, $h(z, z) = -N$. (b) The form is symmetric with entries $0, \pm 1, -N$ fixed by the involution. Its inverse is the matrix with 1 at the diagonal places $i \neq k_0$, with N at (k_0, k_0), -1 at (k_0, z) and (z, k_0), and 0 at (z, z), that is $T^{\mathrm{lat}}(\mathfrak{s}; 0)$ plus N at (k_0, k_0): on the plane of k_0 and z the product of $\left(\begin{smallmatrix} 0 & -1 \\ -1 & -N \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} N & -1 \\ -1 & 0 \end{smallmatrix}\right)$ is 1. Both have entries in R. □

LEAN `gram_besselLatticeBasis` · `isUnimodular_latMiddleGram`

The frame discriminant, frame pairs and the link.

Definition 2.4 (frame discriminant)**.** Let R be a ring with involution, $\mathfrak{s}$ a shape, $N \in R$ and Z a square matrix over R of size $\varrho + 1$. Then

$$\mathrm{g}(N; Z) = \det Z \cdot \overline{\det Z} - N^2 \, \mathrm{D}_\varrho(Z) \cdot \overline{\mathrm{D}_\varrho(Z)}.$$

LEAN `frameDisc`

Definition 2.5 (frame pairs, link)**.** Let R be a ring with involution, $\mathfrak{s}$ a shape, T an $\mathfrak{s}$-form over R and $\gamma \in G(T)$.

(i) $[\mathfrak{f} \mid \gamma\mathfrak{f}]$ is the matrix with $2(\varrho + 1)$ columns $\mathfrak{f}_0, \dots, \mathfrak{f}_\varrho, \gamma\mathfrak{f}_0, \dots, \gamma\mathfrak{f}_\varrho$, and $[\mathfrak{e} \mid \gamma\mathfrak{e}]$ the matrix with the 2ϱ columns $e_0, \dots, e_{\varrho-1}, \gamma e_0, \dots, \gamma e_{\varrho-1}$.

(ii) $\mathrm{L}(\gamma)$ is the matrix of the linear map $R^{I(\mathfrak{s})} \to R^{M(\mathfrak{s})} \times R^{M(\mathfrak{s})}$, $t \mapsto (t_{\mathrm{mid}}, (\gamma^{-1}t)_{\mathrm{mid}})$, where t_{mid} denotes the coordinates of t at the middle indices.

(iii) The *link* of γ is the submodule

$$\mathrm{link}(\gamma) = \left\{ \mathrm{L}(\gamma)\, t \;:\; t \in R^{I(\mathfrak{s})},\ [\mathfrak{e} \mid \gamma\mathfrak{e}]^* \, J(T)\, t = 0 \right\} \;\subseteq\; R^{M(\mathfrak{s})} \times R^{M(\mathfrak{s})}.$$

LEAN `framePair` · `isotropicFramePair` · `linkMatrix` · `link`

With $X = \mathrm{span}(e_0, \dots, e_{\varrho-1})$, the condition on t says $t \perp X$ and $t \perp \gamma X$, and for such t both t and $\gamma^{-1}t$ lie in $X^\perp$, whose quotient by X is the middle space. So $\mathrm{link}(\gamma)$ is the set of the pairs (class of t, class of $\gamma^{-1}t$) in $(X^\perp/X)^2$, for $t \perp X + \gamma X$. Three unfoldings are used below without mention: $\mathfrak{f}v$ has the e-coordinates $v_0, \dots, v_{\varrho-1}$, the z-coordinate v_ϱ and no other; $(\gamma\mathfrak{f})v = \gamma(\mathfrak{f}v)$; and for a hermitian invertible matrix J every row vector l is v^*J for $v = (lJ^{-1})^*$, so every linear form is $h(v, \cdot)$.

LEAN `frameImage_one_mulVec` · `frameImage_mulVec` · `exists_star_vecMul_eq`

Sieve systems. The local theory at the real places and at the finite places produces, floor by floor, a family of positive kernels. Their common shape is the following; it is used in Note U and in the notes of the second half.

Definition 2.6 (damped powers, head weights)**.** *(i)* For a natural number k and a real number r, $\mathrm{dp}_k(r) = r^k \cdot \min(1, r)^6$.

(ii) For natural numbers K, i and a sequence $p = (p_0, p_1, \dots)$ of non-zero complex numbers, $\Lambda_K^{\mathbb{C}}(i; p) = \prod_{l<i} \mathrm{dp}_K(|p_0 p_1 \cdots p_l|^{-1})$.

(iii) Let A be a finite product of non-archimedean local fields with its ring of integers O_A (in the formal text: a ring satisfying the axioms of such a product). For natural numbers K, i and a sequence p of units of A, with $D_l = p_0 p_1 \cdots p_l$,

$$\Lambda_K(i; p) = \prod_{l<i} \begin{cases} \sqrt{[D_l]}^{\,K} & \text{if } D_l \in O_A, \\ 0 & \text{otherwise.} \end{cases}$$

LEAN dampedPower · headWeight · finHeadWeight

For a sequence p, an index i and an element x we write $p[i \mapsto x]$ for the sequence p with its entry of index i replaced by x.

Definition 2.7 (sieve system)**.** Let C be a ring with a measure and U a topological space. Let $u\colon C \to U$ and $\iota\colon U \to C$ be maps, n a natural number, $J = (J_i)_{i\in\mathbb{N}}$ a family of functions $J_i\colon U^{\mathbb{N}} \times U^{\mathbb{N}} \to \mathbb{C}$, $\psi\colon C \to \mathbb{C}$ a function, En a real number, $\Lambda = (\Lambda_i)_{i\in\mathbb{N}}$ a family of functions $\Lambda_i\colon U^{\mathbb{N}} \to \mathbb{R}$ and $\beta\colon C \to \mathbb{R}$ a function. We write $\mathrm{Sieve}(u, \iota, n, J, \psi, En, \Lambda, \beta)$, and say that these data are a *sieve system* on C, if:

(S1) $\psi(0) = 1$;

(S2) for all $i \le n$, all $p, q \in U^{\mathbb{N}}$ and all $\xi \in C$, the function $f(c) = \psi(\xi c)\, J_{i+1}(p[i \mapsto u(c)], q[i \mapsto u(c)])$ on C is integrable; if $\xi = 0$ then $\int_C f = J_i(p, q)$; and if ξ is a unit of C then $|\int_C f| \le \beta(\xi)\, \Lambda_i(p)\, \Lambda_i(q)$;

(S3) for every finite set Ω, every family $(a_k)_{k\in\Omega}$ in $U^{\mathbb{N}}$ and every $t \in U$ with $(a_k)_n = t$ for all k, the matrix $\big(J_{n+1}(a_k, a_l)\big)_{k,l\in\Omega}$ is positive semidefinite;

(S4) J_{n+1} is continuous on $U^{\mathbb{N}} \times U^{\mathbb{N}}$;

(S5) $|J_0(p, q)| \le En$ for all p, q;

(S6) for all $i < n$ and all p, q such that $\xi = \iota(p_i) - \iota(q_i)$ is a unit of C, $|J_{i+1}(p, q)| \le \beta(\xi)\, \Lambda_{i+1}(p)\, \Lambda_{i+1}(q)$.

LEAN SieveSystem

So J_i is obtained from J_{i+1} by integrating over a common entry of index i (S2, $\xi = 0$), the top form is a positive kernel in the remaining variables (S3), and non-trivial frequencies and unequal entries are small (S2, S6). On $\mathbb{C}$ the standard choices are $u(c) = c$ for $c \neq 0$ and $u(0) = 1$, with values in $\mathbb{C}^\times$, which we write $u_{\mathbb{C}}$, and $\psi(c) = e^{c-\bar{c}}$, which we write $\mathbf{e}$.

Proposition 2.8 (products of sieve systems)**.** *Let C be a ring with an s-finite measure, U a topological space, Ω a finite set, and let K be a sieve system $\mathrm{Sieve}(u, \iota, n, J, \psi, En, \Lambda, \beta)$ on C. Let $\iota'\colon \mathbb{C}^\times \to \mathbb{C}$ be a map, $\Lambda' = (\Lambda'_i)_i$ a family of functions $(\mathbb{C}^\times)^{\mathbb{N}} \to \mathbb{R}$, and for every $v \in \Omega$ let J'_v, En'_v and $\beta'_v\colon \mathbb{C} \to \mathbb{R}$ be such that H_v is a sieve system $\mathrm{Sieve}(u_{\mathbb{C}}, \iota', n, J'_v, \mathbf{e}, En'_v, \Lambda', \beta'_v)$ on $\mathbb{C}$. Then*

$$\mathrm{Sieve}(u^+, \iota^+, n, J^+, \psi^+, En^+, \Lambda^+, \beta^+)$$

holds on the ring $\mathbb{C}^\Omega \times C$ with the product measure, where, for $x = ((x_v)_v, x_C)$ and for sequences p in $(\mathbb{C}^\times)^\Omega \times U$ with components $p^{(v)}$ and $p^{(C)}$,

$$u^+(x) = \big((u_{\mathbb{C}}(x_v))_v, u(x_C)\big), \qquad \iota^+(y) = \big((\iota'(y_v))_v, \iota(y_U)\big),$$
$$J_i^+(p, q) = \prod_v J'_{v,i}(p^{(v)}, q^{(v)}) \cdot J_i(p^{(C)}, q^{(C)}), \qquad \psi^+(x) = \prod_v \mathbf{e}(x_v) \cdot \psi(x_C),$$
$$\Lambda_i^+(p) = \prod_v \Lambda'_i(p^{(v)}) \cdot \Lambda_i(p^{(C)}), \qquad \beta^+(x) = \prod_v \beta'_v(x_v) \cdot \beta(x_C),$$

and $En^+ = \prod_v En'_v \cdot En$.

Proof. (S1) holds since $\mathbf{e}(0) = 1$ and $\psi(0) = 1$. A unit of $\mathbb{C}^\Omega \times C$ is a tuple of units. For (S2), the function attached to (i, p, q, ξ) is the product, in independent variables, of the functions attached to $(i, p^{(v)}, q^{(v)}, \xi_v)$ for H_v and to $(i, p^{(C)}, q^{(C)}, \xi_C)$ for K.

Each factor is integrable, hence so is the product, and by Fubini's theorem (the measures are s-finite) its integral is the product of the integrals of the factors. For $\xi = 0$ this is $J_i^+(p,q)$; for a unit ξ every factor is bounded by its own (S2), and the product of the bounds is $\beta^+(\xi)\Lambda_i^+(p)\Lambda_i^+(q)$. For (S3), the matrix $(J_{n+1}^+(a_k,a_l))_{k,l}$ is the entrywise product of the matrices of the factors, which are positive semidefinite because the components of the a_k share their entries of index n; by Schur's product theorem, applied $\#\Omega$ times, it is positive semidefinite. (S4) holds because products of continuous functions composed with projections are continuous. (S5) and (S6) follow by multiplying the bounds of the factors, all of them non-negative; in (S6), $\iota^+(p_i) - \iota^+(q_i)$ is a unit exactly when all its components are. □

LEAN `SieveSystem.pi`

Lemma 2.9 (one step, summed)**.** *Let C be a ring with a measure, U a topological space, Ω a finite set, and let K be a sieve system* $\mathrm{Sieve}(u,\iota,n,J,\psi,En,\Lambda,\beta)$ *on C. Let $i \le n$ and let $(a_p)_{p\in\Omega}$ be a family in $U^{\mathbb{N}}$. Put $F(c) = \sum_{p,q\in\Omega} J_{i+1}(a_p[i \mapsto u(c)], a_q[i \mapsto u(c)])$. Then F is integrable, $\int_C F = \sum_{p,q} J_i(a_p,a_q)$, and for every unit ξ of C, $|\int_C \psi(\xi c)F(c)\,dc| \le \beta(\xi)\big(\sum_p \Lambda_i(a_p)\big)^2$.*

Proof. Sum (S2) over the pairs (p,q), once with $\xi = 0$, where $\psi(0) = 1$ by (S1), and once with the given unit; note $\sum_{p,q}\Lambda_i(a_p)\Lambda_i(a_q) = \big(\sum_p \Lambda_i(a_p)\big)^2$. □

LEAN `SieveSystem.step_sum`

Proposition 2.10 (positivity and semicontinuity on every floor)**.** *Let C be a ring with a measure and a first countable topology, U a topological space, Ω a finite set, and let K be a sieve system* $\mathrm{Sieve}(u,\iota,n,J,\psi,En,\Lambda,\beta)$ *on C. Let $i \le n$, let $b\colon C \to (U^{\mathbb{N}})^\Omega$ and $c\colon C \to U$ be maps such that $(b(x)_p)_i = c(x)$ for all $x \in C$ and $p \in \Omega$, and put $F(x) = \sum_{p,q\in\Omega} J_{i+1}\big(b(x)_p, b(x)_q\big)$. Then $F(x)$ is a non-negative real number for every x, and F is lower semicontinuous at every point x at which all the maps $x' \mapsto b(x')_p$ are continuous.*

Proof. Downward induction on i, for all b and c at once. For $i = n$, $F(x)$ is the value at the vector $(1,\dots,1)$ of the hermitian form with matrix $(J_{n+1}(b(x)_p, b(x)_q))_{p,q}$, which is positive semidefinite by (S3), the sequences sharing the entry $c(x)$ of index n; and F is continuous at x by (S4), hence lower semicontinuous there. Let $i < n$ and suppose the claim for $i+1$ and all families. For $z \in C$ put $b^z(x)_p = b(x)_p[i+1 \mapsto u(z)]$. These sequences share their entry $u(z)$ of index $i+1$, and $x' \mapsto b^z(x')_p$ is continuous where $x' \mapsto b(x')_p$ is. By the induction hypothesis $F^z(x) = \sum_{p,q} J_{i+2}\big(b^z(x)_p, b^z(x)_q\big)$ is non-negative for all x, and lower semicontinuous in x at the points in question. By Lemma 2.9 with the index $i+1 \le n$ and the family $(b(x)_p)_p$, the function $z \mapsto F^z(x)$ is integrable and $\int_C F^z(x)\,dz = F(x)$. So $F(x) \ge 0$. If $x_k \to x$, then by Fatou's lemma and the semicontinuity of the F^z,

$$\begin{aligned} F(x) = \int_C F^z(x)\,dz &\le \int_C \liminf_k F^z(x_k)\,dz \\ &\le \liminf_k \int_C F^z(x_k)\,dz = \liminf_k F(x_k); \end{aligned}$$

since the topology of C is first countable, this is lower semicontinuity at x. The hypothesis on the common entry of index i is used only for $i = n$. □

LEAN `SieveSystem.nonneg_and_lowerSemicontinuous`

Eichler maps in the unipotent group. For a hermitian matrix J with form $h(v,w) = v^* J w$, vectors x, v and a scalar ς, the *Eichler map* is $w \mapsto w + x\,h(v,w) - (v+\varsigma x)\,h(x,w)$.

> **Input (E1)**, unit 119. If J is hermitian, $h(x,x) = 0$, $h(v,x) = 0$ and $\varsigma + \bar{\varsigma} = h(v,v)$, then the Eichler map of (x, v, ς) is an isometry of h.

Proposition 2.11 (Eichler maps along the frame)**.** *Let A be a ring with involution that has an element t with $t + \bar{t} = 1$, $\mathfrak{s}$ a shape and T an $\mathfrak{s}$-form over A such that $J(T)$ is hermitian and invertible. Let $a \le \varrho$ be a frame index, $j \in A^{\varrho+1}$ with $j_b = 0$ for all $b \ge a$, and $r \in A$. Let $v \in A^{I(\mathfrak{s})}$ and let $m = v^* J(T)$ be the linear form $h_T(v,\cdot)$, and suppose $m(\mathfrak{f}_b) = 0$ for all $b < a$. Then there is $u \in \mathcal{U}(T)$ with*

$$\mathrm{sd}(u) = r \sum_{b=1}^{\varrho} m(\mathfrak{f}_b)\, j_{b-1},$$

whose matrix is the Eichler map of $(r\,\mathfrak{f} j,\ v,\ t\,h_T(v,v))$.

Proof. Put $x = r\,\mathfrak{f} j = r\sum_{b<a} j_b \mathfrak{f}_b$ and $\varsigma = t\,h_T(v,v)$. Since $a \le \varrho$, x lies in $\mathrm{span}(e_b : b < a)$, which is isotropic, and $h_T(v,x) = m(x) = r\sum_{b<a} j_b m(\mathfrak{f}_b) = 0$. As $J(T)$ is hermitian, $h_T(v,v)$ is fixed by the involution, so $\varsigma + \bar{\varsigma} = (t+\bar{t})h_T(v,v) = h_T(v,v)$. By (E1) the Eichler map u lies in $G(T)$. The form $h_T(x,\cdot)$ vanishes on all basis vectors except the e_b^* with $b < a$, and the $e_{b'}^*$-coordinate of v is $h_T(e_{b'}, v) = \overline{m(\mathfrak{f}_{b'})} = 0$ for $b' < a$. So the matrix $(v+\varsigma x)\,h_T(x,\cdot)$ has non-zero entries only in columns e_b^*, $b < a$, and in rows other than the $e_{b'}^*$ with $b' < a$: these rows have smaller level than the column. The matrix $x\,m$ has non-zero entries only in rows e_c, $c < a$, and its entry at a column of level $\le c$, a vector $e_{c'}$ with $c' \le c$, is $x_c\, m(\mathfrak{f}_{c'}) = 0$. Hence $u \in \mathcal{U}(T)$. Finally $h_T(x, \mathfrak{f}_{b+1}) = 0$ for all $b < \varrho$, so $\mathrm{sd}(u) = \sum_{b<\varrho} x_{\mathfrak{f}_b}\, m(\mathfrak{f}_{b+1}) = r\sum_{b<\varrho} j_b\, m(\mathfrak{f}_{b+1})$. □

LEAN `exists_mem_besselU_eq_eichlerMap`

Bessel systems. The inputs from the real places are packaged in a record. In this paragraph $\mathfrak{s}$ is a shape, $n = n(\mathfrak{s})$, $k = k(\mathfrak{s})$, $r = r(\mathfrak{s})$, and $G_1(\mathfrak{s})$ carries its Haar measure. The group $\mathrm{U}_\kappa(\mathbb{C})$ is embedded in $G_1(\mathfrak{s})$ by ι_1 (Input (G3)). For a finite set I and a homomorphism rep from $\mathrm{U}_\kappa(\mathbb{C})$ to the unitary group of $\mathbb{C}^I$, a *Bessel function for* rep is a bounded measurable function $\Phi\colon G_1(\mathfrak{s}) \to \mathbb{C}^I$ with $\Phi(uy) = \mathbf{e}(\mathrm{sd}(u))\,\Phi(y)$ for $u \in \mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s};1))$ and $\Phi(\iota_1(h)y) = \mathrm{rep}(h)\Phi(y)$ for $h \in \mathrm{U}_\kappa(\mathbb{C})$ (background, unit 117). The measure $\mu_{\mathcal{U}\backslash G}$ on $G_1(\mathfrak{s})$ is Haar measure times a chosen cutoff for the left action of $\mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s};1))$ with its Haar measure, so that it integrates left $\mathcal{U}$-invariant functions over the quotient (background, unit 118). The numbers $\mathrm{vol}\,\mathrm{U}(j) = \prod_{i<j} 2\pi^{i+1}/i!$ are the volumes of the compact unitary groups.

Definition 2.12 (constants, sizes, boxes)**.** Let $\mathfrak{s}$ be a shape, with n, k, r as above.

(i) $c_G(\mathfrak{s}) = \dfrac{\mathrm{vol}\,\mathrm{U}(k)}{\mathrm{vol}\,\mathrm{U}(r)} \cdot \dfrac{2\pi^k}{(k-1)!}$ and $c_{\mathrm{band}}(\mathfrak{s}) = \dfrac{\mathrm{vol}\,\mathrm{U}(k)}{\mathrm{vol}\,\mathrm{U}(r)} \cdot \pi^{(2\varrho+1)k}$.

(ii) For $y \in G_1(\mathfrak{s})$, $\mathrm{bs}(y) = \inf_s(\|sy\| + 1)$, the infimum over $s \in \mathcal{S}(T^{\mathrm{rat}}(\mathfrak{s};1))$ of the operator norms of the matrices sy. For a real number b, $y \in \mathrm{Cut}(b)$ means $\mathrm{bs}(y) \le b^{6n+2}$.

(iii) For a real number ρ, $\mathrm{FBox}(\rho)$ is the set of the $y \in G_1(\mathfrak{s})$ with $\|y^{-1}\mathfrak{f}_a\| \le \rho$ for all $a \le \varrho$ (Euclidean norms in $\mathbb{C}^{I(\mathfrak{s})}$).

(iv) For a complex square matrix Z of size $\varrho+1$, $\mathfrak{g}^{\mathbb{R}}(Z) = \mathrm{Re}\,\mathfrak{g}(1;Z) = |\det Z|^2 - |\mathrm{D}_\varrho(Z)|^2$.

(v) For $y \in G_1(\mathfrak{s})$ and real numbers $\deg$, ρ, d, with $\mathfrak{g} = \mathfrak{g}^{\mathbb{R}}(Z(y))$,

$$\mathrm{cm}(y;\deg,\rho,d) = \Big(\frac{\rho^{9n}}{\mathfrak{g}}\Big)^{10^5 n^4} \cdot \frac{d^2}{\deg \cdot c_G(\mathfrak{s}) \cdot (\mathfrak{g}/2)^{k-1}}.$$

LEAN `Shape.openCellConst` · `Shape.bandConst` · `besselSize` · `InCut` · `frameBox` · `gramDiscReal` · `cellMass` · `Shape.openCellConst_pos`

The constant $c_G(\mathfrak{s})$ is positive, being a quotient of positive numbers. For $k \ge 1$ its second factor is $\mathrm{vol}\,\mathrm{U}(k)/\mathrm{vol}\,\mathrm{U}(k-1)$.

Definition 2.13 (correlations, Schur bounds, band limits)**.** Let $\mathfrak{s}$ be a shape, I a finite set and $W\colon G_1(\mathfrak{s}) \to \mathbb{C}^I$ a function.

(i) For a group G with a measure μ, a function $W\colon G \to \mathbb{C}^I$ and $\gamma \in G$, $\mathrm{J}(\mu, W;\gamma) \in \mathbb{C}^{I\times I}$ is the vector with the coordinates $\int_G W(y)_i\,\overline{W(\gamma^{-1}y)_{i'}}\,d\mu(y)$, $(i,i') \in I \times I$.

(ii) For $D \in [0,\infty]$, $\mathrm{Sch}(W) \le D$ means: for every integrable $\varphi\colon G_1(\mathfrak{s}) \to \mathbb{C}$,

$$\int \Big\| \int_{G_1(\mathfrak{s})} \varphi(y)\,W(\gamma y)\,dy \Big\|^2 d\mu_{\mathcal{U}\backslash G}(\gamma) \;\le\; D \int_{G_1(\mathfrak{s})} |\varphi(y)|^2\,dy.$$

(iii) For a homomorphism rep from $\mathrm{U}_\kappa(\mathbb{C})$ to the unitary group of $\mathbb{C}^I$, a real number B and $D \in [0,\infty]$, $\mathrm{SchB}(W,\mathrm{rep};B,D)$ means: there are continuous functions $V_w\colon G_1(\mathfrak{s}) \to \mathbb{C}^I$, one for each finite list w of pairs of elements of $I(\mathfrak{s})$, with $V_\emptyset = W$, such that for every w: V_w is a Bessel function for rep; for every curve $g\colon \mathbb{R} \to G_1(\mathfrak{s})$, every matrix X and every real t with $g'(t) = g(t)X$, the function $V_w \circ g$ has at t the derivative $\sum_{a,b\in I(\mathfrak{s})} X_{ab}\,V_{(a,b)w}(g(t))$; and if the length $|w|$ is at most $2n^2$, then $\mathrm{Sch}(V_w) \le B^{2|w|}D$.

(iv) For $y \in G_1(\mathfrak{s})$ and a sequence p of non-zero complex numbers,

$$\mathrm{J}^\circ(y;W,p) = \frac{\sqrt{c_G(\mathfrak{s})}\,\sqrt{|p_\varrho|^2-1}^{\,k-1}}{\#I} \prod_{a<\varrho} |p_a|^{k+1+2(\varrho-a-1)} \cdot \mathrm{J}(\mathrm{vol}, W; w_\varrho(p)\,y),$$

with vol the Haar measure of $G_1(\mathfrak{s})$.

(v) For real numbers b, $\deg$ and a family $J = (J_i)_i$ of functions on pairs of sequences of non-zero complex numbers, $\mathrm{BL}(b,W,\deg,J)$ means: $\mathrm{Sieve}(u_{\mathbb{C}},\, x \mapsto x^{-1},\, \varrho, J, \mathbf{e}, (1+b^{-10})/\deg, \Lambda, \beta)$ holds on $\mathbb{C}$ with $\Lambda_i(p) = \Lambda^{\mathbb{C}}_{n^6}(i;p)$ and $\beta(\xi) = c_{\mathrm{band}}(\mathfrak{s})\,b^{2000n^9}\,\mathrm{dp}_{k-n^7}(|\xi|^{-1})$; and for every $\sigma_0 \in \mathcal{H}_\varrho(T^{\mathrm{rat}}(\mathfrak{s};1))$ and all p, q with $p_\varrho = Z(\sigma_0)_{\varrho\varrho}$ and $q_\varrho = p_\varrho$,

$$J_n(p,q) = \big\langle \mathrm{J}^\circ(\sigma_0;W,p),\ \mathrm{J}^\circ(\sigma_0;W,q)\big\rangle.$$

LEAN `besselCorrelation` · `SchurLe` · `HasSchurBound` · `normCorrelation` · `HasBandLimits`

In (iii) the function $V_{(a,b)w}$ is the derivative of V_w along the elementary matrix E_{ab} acting on the right; in (v), $Z(\sigma_0)_{\varrho\varrho} = h(z,\sigma_0 z)$, and the inner product is that of $\mathbb{C}^{I\times I}$, conjugate linear in the first variable.

Definition 2.14 (sides, cut pairs, Bessel systems)**.** Let $\mathfrak{s}$ be a shape.

(i) A *side* is one of the two values full and cut. In the formal text a side is an ambient argument, with full as its default value.

(ii) Let I be a finite set, b a real number, f a natural number, rep a homomorphism from $\mathrm{U}_\kappa(\mathbb{C})$ to the unitary group of $\mathbb{C}^I$, and $W = (W^{\mathrm{full}}, W^{\mathrm{cut}})$ a pair of continuous functions $G_1(\mathfrak{s}) \to \mathbb{C}^I$. Then $\mathrm{CP}(b, f, \mathrm{rep}, W)$ is the conjunction of: W^{full} and W^{cut} are Bessel functions for rep; W^{full} is integrable for $\mu_{\mathcal{U}\backslash G}$; $\int \|W^{\mathrm{full}}\|^2 d\mu_{\mathcal{U}\backslash G} = \#I$; $\int \|W^{\mathrm{cut}}\|^2 d\mu_{\mathcal{U}\backslash G} \le \#I$; $W^{\mathrm{cut}}(y) \neq 0$ implies $y \in \mathrm{Cut}(b)$; $\int \|W^{\mathrm{cut}} - W^{\mathrm{full}}\|^2 d\mu_{\mathcal{U}\backslash G} \le (b^{-5r^2})^2\, \#I$; and there are a measurable function $A \colon G_1(\mathfrak{s}) \to [0,\infty]$, invariant under left translation by $\mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s}; 1))$, and a finite measure m on $G_1(\mathfrak{s})$ with

$$\epsilon(y)\,\|W^\sigma(y)\| \le A(y) \le \mathsf{m}\{g : yg \in \mathrm{Cut}(b)\} \quad \text{for all } y \text{ and both sides } \sigma,$$
$$\mathsf{m}(G_1(\mathfrak{s}))\ \int A\, d\mu_{\mathcal{U}\backslash G} \le b^{2r^3}\, \#I,$$

where $\epsilon(y) = 1$ if $W^{\mathrm{full}}(y) = W^{\mathrm{cut}}(y)$ and $\epsilon(y) = b^{3r^3 f}$ otherwise.

(iii) For a real number b and a natural number f, a *Bessel system* $\mathfrak{B} \in \mathrm{BSys}(\mathfrak{s}, b, f)$ is a record of: a non-empty finite set I; a pair $W = (W^{\mathrm{full}}, W^{\mathrm{cut}})$ of continuous functions $G_1(\mathfrak{s}) \to \mathbb{C}^I$; a real number $\deg > 0$; a homomorphism rep from $\mathrm{U}_\kappa(\mathbb{C})$ to the unitary group of $\mathbb{C}^I$; and a family J as in Definition 2.13(v); such that $\mathrm{CP}(b, f, \mathrm{rep}, W)$ holds; every matrix that commutes with all $\mathrm{rep}(h)$ is scalar; $\mathrm{SchB}(W^{\mathrm{full}}, \mathrm{rep}; P, \deg^{-1}\#I)$ for every real $P \ge b^5$; $8\,r \le b$; and $\mathrm{BL}(b, W^{\mathrm{cut}}, \deg, J)$.

LEAN `Side` · `IsCutPair` · `BesselSystem`

Bessel systems are constructed, from discrete series representations, in the notes on the real place (unit 132) and used here only through this record.

The filling data. For a CM field E and a finite set κ: $G_f(E,\kappa)$ is the unitary group of the identity matrix on κ over $\mathbb{A}_{E,f}$ (the matrices g with $g^* g = 1$), $K_f(E,\kappa)$ its subgroup of the matrices that have entries in $\hat{O}_E$ together with their inverses, and $\Gamma_f(E,\kappa)$ the image of $\mathrm{U}_\kappa(E)$ in $G_f(E,\kappa)$. A *level* is an open subgroup of $G_f(E,\kappa)$ contained in $K_f(E,\kappa)$. For a shape $\mathfrak{s}$, $G_\infty(E,\mathfrak{s})$ is the product of copies of $G_1(\mathfrak{s})$ over the infinite places of E, with the product of the Haar measures.

Definition 2.15 (filling data)**.** Let E be a CM field with a CM type and $\mathfrak{s} = (\varrho, \kappa, k_0)$ a shape; write $k = k(\mathfrak{s})$, $n = n(\mathfrak{s})$, $f = f(E)$. *Filling data* $S \in \mathrm{Fill}(E, \mathfrak{s})$ are a record of

- a natural number $N(S)$; two natural numbers $q_0(S)$, $q_1(S)$; a unit $\bar{\pi}(S)$ of $\mathbb{A}_{E,f}$; an integer $\varpi(S) \in O_E$; a real number $b(S)$;
- two finite sets $\mathrm{Ch}_0(S)$, $\mathrm{Ch}_1(S)$ of finite places of E, the *chosen places*;
- a level $K^{\mathrm{col}}(S) \subseteq G_f(E,\kappa)$, the *column level*;
- for every infinite place τ of E a Bessel system $\mathfrak{B}_\tau(S) \in \mathrm{BSys}(\mathfrak{s}, b(S), f)$, whose components we write $I_\tau(S)$, $W^\sigma_{S,\tau}$, $\deg_\tau(S)$, $\mathrm{rep}_{S,\tau}$;

subject to the following conditions, in which $b = b(S)$, $N = N(S)$, $q_i = q_i(S)$, $\bar{\pi} = \bar{\pi}(S)$:

(F1) $\Gamma_f(E,\kappa) \cap g\, K^{\mathrm{col}}(S)\, g^{-1} = 1$ for every $g \in G_f(E,\kappa)$;

(F2) $\bar{\pi}\cdot\overline{\bar{\pi}} = N$ in $\mathbb{A}_{E,f}$; N is a prime number; $\bar{\pi}\in\hat{O}_E$; and for every finite place w, either $\bar{\pi}$ is a unit at w, or $q_w = N$ and $\bar{\pi}$ is a unit at $\overline{w}$;

(F3) q_0 and q_1 are prime numbers different from N, with $b^{10}\le q_i\le b^{11}$;

(F4) every $w\in \mathrm{Ch}_i(S)$ has $q_w = q_i$; the sets $\mathrm{Ch}_0(S)$ and $\mathrm{Ch}_1(S)$ are disjoint; if w is chosen then $\overline{w}$ is not; and each $\mathrm{Ch}_i(S)$ has f elements;

(F5) for every $x\in\hat{O}_E$: q_0q_1 divides x in $\hat{O}_E$ if and only if x and $\overline{x}$ reduce to 0 modulo every chosen place;

(F6) $\overline{\varpi(S)} = -\varpi(S)$, and the different of E is the ideal generated by $\varpi(S)$;

(F7) $10^5\le\varrho$; $2^{-\varrho}\le b^{-6}$; $(\varrho+1)[E:\mathbb{Q}]\le 2^{2\varrho}$; $1000(2\varrho+2)^9\le k$; $1000(2\varrho+2)\le k$; $k\le f$; $10^{10}n^4\le n^6$, $3n^6+12\le n^7$ and $n^7\le k$;

(F8) $(200\,f\,k\,\mathrm{rd}_E)^{1000}\le b$; $f\le b/1000$; $k\le b$; $\mathrm{rd}_E\le b$; $\varrho+1\le b$; $1000\le b$; $1\le b$; $b^{19}\le q_0q_1\le b^{23}$; $5\le N$.

LEAN Filling

Definition 2.16 (forms and groups of the data)**.** Let $S\in\mathrm{Fill}(E,\mathfrak{s})$ and let R be a ring.

(i) $g(S) = q_0(S)\cdot q_1(S)$.

(ii) $T_S^{\mathrm{rat}}(R) = T^{\mathrm{rat}}(\mathfrak{s};N(S))$ and $T_S^{\mathrm{lat}}(R) = T^{\mathrm{lat}}(\mathfrak{s};N(S))$, with $N(S)$ read in R.

(iii) If R has an involution, $G_S(R) = G(T_S^{\mathrm{rat}}(R))$. Further $G_S^{\mathrm{lat}} = G(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$, the finite adelic group *in the lattice basis*.

(iv) $\mathsf{G}(S) = \prod_{i\in\{0,1\}}\prod_{w\in\mathrm{Ch}_i(S)}\mathrm{GL}_{I(\mathfrak{s})}(\mathbb{F}_w)$, a finite group.

(v) $\mathrm{Bc}(S) = \prod_\tau I_\tau(S)$, the product over the infinite places.

(vi) $\mathrm{Cl}(S) = \Gamma_f(E,\kappa)\backslash G_f(E,\kappa)/K^{\mathrm{col}}(S)$, the set of *classes*. It is finite and non-empty; every class c has a chosen representative $\dot{c}\in G_f(E,\kappa)$.

(vii) For $\gamma\in G_S(E)$, $Z_S(\gamma) = Z_{T_S^{\mathrm{rat}}(E)}(\gamma)$ and $\mathfrak{g}_S(\gamma) = \mathfrak{g}(N(S);Z_S(\gamma))$.

LEAN Filling.g · Filling.ratMiddle · Filling.latMiddle · Filling.G · Filling.GFin · Filling.GRes · Filling.BesselCoord · Filling.Classes · Filling.frameProducts · Filling.termDisc

So $Z_S(\gamma) = (h(\mathfrak{f}_a,\gamma\mathfrak{f}_b))_{a,b}$ for the form h of signature $(\varrho+k,\varrho+1)$ at every infinite place, and $\mathfrak{g}_S(\gamma) = |\mathrm{D}_{\varrho+1}(Z)|^2 - N(S)^2|\mathrm{D}_\varrho(Z)|^2$ with $Z = Z_S(\gamma)$ and $|x|^2 = x\overline{x}$.

Definition 2.17 (the integral subgroup and its reduction)**.** Let $S\in\mathrm{Fill}(E,\mathfrak{s})$.

(i) $K(S)$ is the subgroup of the $k\in G_S^{\mathrm{lat}}$ that have entries in $\hat{O}_E$ together with their inverses.

(ii) $\mathrm{red}_S\colon K(S)\to\mathsf{G}(S)$ is the homomorphism that reduces every entry modulo each chosen place.

(iii) $K_g(S)\subseteq G_S^{\mathrm{lat}}$ is the kernel of red_S, the *principal level*.

(iv) For a subgroup $A\subseteq G_S^{\mathrm{lat}}$, $\overline{A}^{\,S} = \mathrm{red}_S(A\cap K(S))$.

(v) $\hat{g}(S)$ is the integer $g(S)$ as a unit of $\mathbb{A}_{E,f}$.

LEAN Filling.latStabilizer · Filling.reduction · Filling.principalLevel · Filling.reducedImage · Filling.gIdele

Definition 2.18 (scales at the infinite places)**.** Let $S \in \mathrm{Fill}(E, \mathfrak{s})$, τ an infinite place of E, a a natural number and $i \in I(\mathfrak{s})$.

(i) $\kappa_\tau(S) = \mathrm{Re}(g(S)\,\varpi(S)_\tau/(2\pi i))$.

(ii) With $s_a = \kappa_\tau(S)^{\varrho-a}/\sqrt{N(S)}$: $\mathrm{sc}_{S,\tau}(i)$ is s_a for $i = e_a$, 1 for $i = x_j$, $s_\varrho = 1/\sqrt{N(S)}$ for $i = z$, and s_a^{-1} for $i = e_a^*$; and $\mathrm{Sc}_{S,\tau}$ is the invertible diagonal complex matrix with the entries $\mathrm{sc}_{S,\tau}(i)$.

(iii) $\mathrm{fs}_S(\tau, a) = \kappa_\tau(S)^{2(\varrho-a)}/N(S)$, and $\mathrm{fs}(S) = \prod_\tau \prod_{a=0}^{\varrho} \mathrm{fs}_S(\tau, a)^{\,k+2(\varrho-a)}$, with $k = k(\mathfrak{s})$.

LEAN Filling.kappa · Filling.basisScale · Filling.basisScaleGL · Filling.frameScaleSq · Filling.frameScaleProd

Definition 2.19 (coordinates)**.** Let $S \in \mathrm{Fill}(E, \mathfrak{s})$.

(i) $\Lambda(S) = \Lambda(\mathfrak{s}; \bar\pi(S))$, an invertible matrix over $\mathbb{A}_{E,f}$, and $\mathrm{lb}_S \colon G_S(\mathbb{A}_{E,f}) \to G_S^{\mathrm{lat}}$ is the isomorphism $x \mapsto \Lambda(S)^{-1} x\, \Lambda(S)$.

(ii) $\mathrm{r}_S^{\mathbb{A}} \colon G_S(E) \to G_S(\mathbb{A}_E)$ embeds every entry diagonally. For $x \in G_S(\mathbb{A}_E)$, $\mathrm{a}_S^\infty(x) \in G_\infty(E, \mathfrak{s})$ has the component $\mathrm{Sc}_{S,\tau}^{-1}\, x_\tau\, \mathrm{Sc}_{S,\tau}$ at τ, where x_τ is the component of x at τ; and $\mathrm{a}_S^f(x) = \mathrm{lb}_S(x_f)$ for the finite part x_f of x. All three are homomorphisms.

(iii) $\mathrm{r}_S^\infty = \mathrm{a}_S^\infty \circ \mathrm{r}_S^{\mathbb{A}}$ and $\mathrm{r}_S^f = \mathrm{a}_S^f \circ \mathrm{r}_S^{\mathbb{A}}$.

(iv) $\mathrm{inc}_S \colon G_f(E, \kappa) \to G_S^{\mathrm{lat}}$ is $\mathrm{lb}_S \circ \iota_{N(S)}$, with $\iota_{N(S)}$ the embedding of Input (G3) over $\mathbb{A}_{E,f}$; and $h_S(c) = \mathrm{inc}_S(\dot c)$ for a class $c \in \mathrm{Cl}(S)$.

(v) ψ_S is the additive character $x \mapsto \psi_f(-x/\varpi(S))$ of $\mathbb{A}_{E,f}$, with values in $\mathbb{C}$, and for $u \in G_S^{\mathrm{lat}}$, $\psi_S^{\mathcal{U}}(u) = \psi_S(\hat g(S)^{-1}\,\mathrm{sd}(u))$.

LEAN Filling.latBasis · Filling.toLatBasis · Filling.ratToAdelic · Filling.adelicToInf · Filling.adelicToFin · Filling.ratToInf · Filling.ratToFin · Filling.inclHFin · Filling.h · Filling.selfDualChar · Filling.unipotentChar

Definition 2.20 (terms and arithmetic unipotent groups)**.** Let $S \in \mathrm{Fill}(E, \mathfrak{s})$, $c, c' \in \mathrm{Cl}(S)$, $s \in \mathsf{G}(S)$ and $\gamma \in G_S(E)$.

(i) $\mathrm{pos}_S(c, c'; \gamma) = h_S(c)^{-1}\, \mathrm{r}_S^f(\gamma)\, h_S(c') \in G_S^{\mathrm{lat}}$, the *position* of γ.

(ii) $\mathrm{Tm}_S(c, s, c')$ is the set of the $\gamma \in G_S(E)$ whose position $\mathrm{pos}_S(c, c'; \gamma)$ lies in $K(S)$ and has reduction s; its elements are the *terms over* (c, s, c').

(iii) $\mathcal{U}_{S,c}$ is the group of the $u \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$ with $\mathrm{pos}_S(c, c; u) \in K_g(S)$, and $\mathcal{U}_{S,c}^0 = \mathcal{U}_{S,c} \cap \mathcal{U}^0(T_S^{\mathrm{rat}}(E))$.

(iv) $s \in \mathrm{Agr}(S)$ means: for all $u, u' \in K(S)$ that lie in $\mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ and satisfy $\mathrm{red}_S(u)\, s = s\, \mathrm{red}_S(u')$, one has $\psi_S^{\mathcal{U}}(u) = \psi_S^{\mathcal{U}}(u')$.

(v) A set $\Xi \subseteq G_S(E)$ is *stable*, $\Xi \in \mathrm{Stab}(S)$, if for all $u, u' \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$ and all $\gamma \in G_S(E)$: $u^{-1}\gamma u' \in \Xi$ if and only if $\gamma \in \Xi$.

(vi) For a submodule $\Lambda \subseteq E^{M(\mathfrak{s})} \times E^{M(\mathfrak{s})}$, $\mathrm{Tm}_S^{\mathrm{lk}}(\Lambda)$ is the set of the $\gamma \in G_S(E)$ with $\mathfrak{g}_S(\gamma) \neq 0$ and $\mathrm{link}(\gamma) = \Lambda$.

LEAN Filling.termPosition · Filling.termsOver · Filling.classUnip · Filling.classUnip₀ · Filling.Agrees · Filling.IsStable · Filling.termsOfLink

Definition 2.21 (Bessel functions, boxes, cutoffs, amplitudes)**.** Let $S \in \mathrm{Fill}(E,\mathfrak{s})$, σ a side, $c, c' \in \mathrm{Cl}(S)$, $s \in \mathsf{G}(S)$, $\gamma \in G_S(E)$, $y \in G_\infty(E,\mathfrak{s})$ and $i \in \mathrm{Bc}(S)$; write $n = n(\mathfrak{s})$.

(i) $W_S^\sigma(y)_i = \prod_\tau W_{S,\tau}^\sigma(y_\tau)_{i_\tau}$, and $\mathcal{W}_S^\sigma(\gamma;y) \in \mathbb{C}^{\mathrm{Bc}(S)\times\mathrm{Bc}(S)}$ is the vector with the coordinates $W_S^\sigma(y)_i \cdot \overline{W_S^\sigma(\mathrm{r}_S^\infty(\gamma)^{-1}y)_{i'}}$.

(ii) $\mathrm{bx}(S) = b(S)^{2n^2}$.

(iii) For a function $\omega\colon G_\infty(E,\mathfrak{s}) \to [0,\infty]$, $\omega \in \mathrm{BP}(S,c)$ means: ω is measurable; $\omega \le 1$; $\omega(ty) = \omega(y)$ for all y and all $t \in G_\infty(E,\mathfrak{s})$ with $t_\tau \in \mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1)) \cdot \mathcal{H}(T^{\mathrm{rat}}(\mathfrak{s};1))$ for every τ (more exactly, in the subgroup generated by the two); for all y and τ, if $\omega(y) \neq 0$ and $y_\tau \in \mathrm{Cut}(b(S))$ then $y_\tau \in \mathrm{FBox}(\mathrm{bx}(S))$; and for all y and every function $\alpha\colon \mathcal{U}_{S,c} \to [0,\infty]$ with $\sum_{u\in\mathcal{U}_{S,c}^0} \alpha(u\gamma_0) = 1$ for all $\gamma_0 \in \mathcal{U}_{S,c}$, one has $\sum_{\gamma_0\in\mathcal{U}_{S,c}} \alpha(\gamma_0)\,\omega(\mathrm{r}_S^\infty(\gamma_0)y) = 1$.

(iv) $\beta_{S,c}\colon G_\infty(E,\mathfrak{s}) \to [0,\infty]$ is the product $\beta^0 \cdot \omega$ of a chosen measurable function β^0 with $\sum_{u\in\mathcal{U}_{S,c}^0} \beta^0(\mathrm{r}_S^\infty(u)y) = 1$ for all y (it exists by Lemma 3.3) and a chosen function ω with $\omega \in \mathrm{BP}(S,c)$ if there is one, an arbitrary function otherwise.

(v) The *amplitude* and its majorant are

$$a_S^\sigma(c,c';\gamma) = \int_{G_\infty(E,\mathfrak{s})} \beta_{S,c}(y)\,\beta_{S,c'}(\mathrm{r}_S^\infty(\gamma)^{-1}y)\ \mathcal{W}_S^\sigma(\gamma;y)\,dy \ \in \mathbb{C}^{\mathrm{Bc}(S)\times\mathrm{Bc}(S)},$$

$$|a|_S^\sigma(c,c';\gamma) = \int_{G_\infty(E,\mathfrak{s})} \beta_{S,c}(y)\,\beta_{S,c'}(\mathrm{r}_S^\infty(\gamma)^{-1}y)\ \|\mathcal{W}_S^\sigma(\gamma;y)\|\,dy \ \in [0,\infty],$$

where in the first integral the product of the two cutoffs is read as a real number (0 where it is infinite) and the integral is 0 if the integrand is not integrable.

(vi) For a set $\Xi \subseteq G_S(E)$, the *amplitude sum* is

$$A_S^\sigma(\Xi;c,s,c') = \sum_{\gamma\in\Xi\cap\mathrm{Tm}_S(c,s,c')} a_S^\sigma(c,c';\gamma),$$

read as 0 if the family is not summable.

LEAN `Filling.besselFn` · `Filling.besselKernel` · `Filling.boxSize` · `Filling.IsBoxPartition` · `Filling.classCutoff` · `Filling.amplitude` · `Filling.amplitudeAbs` · `Filling.amplitudeSum`

Note B shows that under the standing hypotheses, for E stably unramified, a function with $\omega \in \mathrm{BP}(S,c)$ exists, so that $\beta_{S,c}$ is a cutoff for $\mathcal{U}_{S,c}$ acting through r_S^∞, supported, where the cut Bessel function is non-zero, in a box.

Definition 2.22 (constants)**.** Let $S \in \mathrm{Fill}(E,\mathfrak{s})$; write $k = k(\mathfrak{s})$, $n = n(\mathfrak{s})$, $r = r(\mathfrak{s})$, $f = f(E)$, $b = b(S)$, $g = g(S)$, $N = N(S)$.

(i) $\mathrm{cd}(S) = \#\mathrm{Cl}(S) \cdot \#\mathrm{Bc}(S)$ and $\deg(S) = \prod_\tau \deg_\tau(S)$.

(ii) $\mathsf{d}(S) = \mathrm{cd}(S)^2/(\deg(S)\cdot\mathrm{mass}(E,r))$, where $\mathrm{mass}(E,r)$ is the mass of the genus of the standard hermitian lattice of rank r over E (background).

(iii) $\mathrm{uo}(S) = g^{f\varrho(2k+2\varrho+1)}$ and $\mathrm{tl}(S) = b^{kf/4}$.

(iv) $c_*(S) = \left(\pi^k/((k-1)!\,N^k)\right)^f \cdot |d_E|^{1/4} \cdot \mathrm{vol}(\hat{O}_E)^k$.

(v) $a_*(S) = \overline{\#\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{\,S} \cdot \mathrm{uo}(S) \cdot \#\mathrm{Cl}(S) \cdot c_*(S)$ and $\mathsf{d}_1(S) = \overline{\#\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{\,S} \cdot \mathrm{cd}(S)$.

(vi) $\mathrm{cf}(S) = (b/g)^{kf} \cdot 2\,\mathrm{tl}(S)^2$.

(vii) With $d = \sum_{a<\varrho}(2a+k+1) = \varrho(\varrho+k)$,

$$\nu(S) = \Big(\frac{g^{2d+\varrho}}{N^k}\Big)^f \cdot \mathrm{vol}(\hat{O}_E)^{-d} \cdot |d_E|^{\varrho/4} \cdot \mathrm{fs}(S)^{-1}.$$

LEAN `Filling.columnDim` · `Filling.besselDegree` · `Filling.mainEnergy` · `Filling.unipOrder` · `Filling.tailScale` · `Filling.siegelConst` · `Filling.packetConst` · `Filling.besselEnergy` · `Filling.codimFactor` · `Filling.unipCovolume`

Since $\mathrm{vol}(\hat{O}_E) = 2^f|d_E|^{-1/2}$ and $|d_E| = \mathrm{rd}_E^{2f}$, one has $c_*(S) = \big((2\pi)^k\,\mathrm{rd}_E^{1/2-k}/((k-1)!N^k)\big)^f$. Note B proves that $\nu(S)$ is the covolume of the image of $\mathcal{U}_{S,c}$ in the product of the groups $\mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s};1))$ over the infinite places.

Definition 2.23 (standing hypotheses)**.** Let $S \in \mathrm{Fill}(E,\mathfrak{s})$, with the abbreviations of Definition 2.22. We say that S is admissible if:

(A1) $K^{\mathrm{col}}(S) = \mathrm{inc}_S^{-1}(K_g(S))$, and $\mathrm{red}_S\colon K(S) \to \mathsf{G}(S)$ is surjective;

(A2) for every infinite place τ: $|\varpi(S)_\tau| \le b$ and $1 \le |\kappa_\tau(S)| \le b^{23}$;

(A3) $N \le b^{60n}$;

(A4) $\prod_\tau c_{\mathrm{band}}(\mathfrak{s})\, b^{7k}\Big(\dfrac{|\kappa_\tau(S)|^{2\varrho+1}}{g\,N}\Big)^{k-n^7} \le \deg(S)^{-1}$;

(A5) $\mathrm{tl}(S)^{44}\,\mathsf{d}_1(S) \le \mathsf{d}(S)$, $\mathrm{tl}(S)^{18}\,\mathrm{cf}(S)\,\mathsf{d}(S) \le \mathsf{d}_1(S)$ and $\mathrm{cf}(S) \le 1$;

(A6) $\#\mathrm{Bc}(S) \le b^{2kf}$;

(A7) $(N^k/b)^f \le [K_f(E,\kappa) : K_f(E,\kappa) \cap \mathrm{inc}_S^{-1}(K(S))]$;

(A8) $a_*(S) \le (1+b^{-9})\,\mathsf{d}(S)\,\deg(S)\,c_G(\mathfrak{s})^f\,\mathrm{vol}(\hat{O}_E)^{-n}\,\big(\nu(S)\,\mathrm{fs}(S)/\#\mathrm{Bc}(S)\big)^2$.

LEAN `Filling.IsAdmissible`

Lattices of classes, levels of terms, near terms.

Definition 2.24 (class bases)**.** Let $S \in \mathrm{Fill}(E,\mathfrak{s})$ and $c \in \mathrm{Cl}(S)$.

(i) $\mathrm{B}_S(c) = \iota_{N(S)}(\dot{c}) \cdot \Lambda(S)$, an invertible matrix over $\mathbb{A}_{E,f}$ indexed by $I(\mathfrak{s})$. Its columns are a basis of the lattice of the class c in rational coordinates.

(ii) $\mathrm{C}_S(c) = \mathrm{B}_S(c)^{-1}$, as a matrix.

(iii) $\mathrm{C}^\circ_S(c) = \Lambda^\circ(\mathfrak{s};\bar{\pi}(S)) \cdot \dot{c}^{-1}$, a matrix with rows indexed by $M(\mathfrak{s})$ and columns by κ.

LEAN `Filling.classBasis` · `Filling.classCoord` · `Filling.classCoordX₀`

Definition 2.25 (reduction above g, fixers)**.** Let $S \in \mathrm{Fill}(E,\mathfrak{s})$.

(i) For a ring F and a truth value ϵ, adj_S^ϵ is the automorphism of the group of invertible matrices over F indexed by $I(\mathfrak{s})$ that is the identity if ϵ is false, and $s \mapsto J^{-1}\,(s^{-1})^{\mathrm{t}}\,J$ with $J = J(T_S^{\mathrm{lat}}(F))$ if ϵ is true.

(ii) For $x = ((i,w),\epsilon)$, with $i \in \{0,1\}$, $w \in \mathrm{Ch}_i(S)$ and a truth value ϵ, $\mathrm{red}_{S,x}\colon \hat{O}_E \to \mathbb{F}_w$ is the reduction modulo w, preceded by complex conjugation if ϵ is true.

(iii) For $w \in \mathrm{Ch}_i(S)$ and a pair $d = (\epsilon, v)$ of a truth value and a non-zero vector $v \in \mathbb{F}_w^{\varrho+1}$, $\mathrm{Fix}_{S,w}(d)$ is the subgroup of the $s \in \mathsf{G}(S)$ such that $\mathrm{adj}_S^\epsilon(s_{i,w})$ fixes the vector $\mathfrak{f}v \in \mathbb{F}_w^{I(\mathfrak{s})}$.

(iv) For $s \in \mathsf{G}(S)$ and a natural number x, $\mathrm{fix}(s) \geq x$ means: there is a finite set P of pairs $((i,w),\epsilon)$ as in (ii) with $x \leq \prod_{((i,w),\epsilon)\in P} q_i(S)$, such that for each of them $s \in \mathrm{Fix}_{S,w}((\epsilon, v))$ for some non-zero v.

(v) For a finite set $\Sigma \subseteq \mathrm{Ch}_0(S)$ and a family $d = (d_w)_w$ of pairs as in (iii), $\mathrm{Fix}_S(\Sigma, d)$ is the set of the $s \in \mathsf{G}(S)$ with $s \in \mathrm{Fix}_{S,w}(d_w)$ for all $w \in \Sigma$ and such that either $\Sigma = \mathrm{Ch}_0(S)$, or there is $w \in \mathrm{Ch}_0(S)$ with $\mathrm{D}_l(Z_{T_S^{\mathrm{lat}}(\mathbb{F}_w)}(s_{0,w})) \neq 0$ for all $l \leq \varrho + 1$.

LEAN `Filling.idOrInvAdjoint` · `Filling.reduceAboveG` · `Filling.vectorFixer` · `Filling.FixesNormAtLeast` · `Filling.fixingSet`

The truth value records whether one looks at w or at $\overline{w}$: by Lemma 3.6 the reduction modulo $\overline{w}$ of an element of $K(S)$ is the inverse adjoint of its reduction modulo w. An entry of $\mathrm{red}_S(k)$ depends only on the corresponding entry of k.

LEAN `Filling.reduction_apply_congr`

Definition 2.26 (common links, flag vectors, normal forms)**.** Let R be a ring with involution, $\mathfrak{s}$ a shape, T an $\mathfrak{s}$-form over R and $\gamma, \gamma' \in G(T)$.

(i) $\mathrm{L}_z(T)$ is the matrix with two rows and columns indexed by $M(\mathfrak{s}) \sqcup M(\mathfrak{s})$ that is block diagonal with both blocks equal to the row of T of index z (for this R need not have an involution).

(ii) $\mathrm{link}^\circ(\gamma, \gamma') \subseteq R^\kappa$ is the set of the restrictions $p_1|_\kappa$ of the pairs $(p_1, p_2) \in \mathrm{link}(\gamma) \cap \mathrm{link}(\gamma')$ with $\mathrm{L}_z(T)\,(p_1, p_2) = 0$, that is, with $(Tp_1)_z = (Tp_2)_z = 0$.

(iii) For $c, c' \in R$ and a frame index $a \leq \varrho$, $\mathrm{CF}_a(\gamma; c, c')$ means: there are $x, y \in R^{\varrho+1}$ with $\mathfrak{f}x = (\gamma\mathfrak{f})y$, $x_a = c$, $y_a = c'$, and $x_b = y_b = 0$ for all $b > a$.

(iv) For a natural number j, $\gamma \in \mathrm{NF}_j$ means: there is $c \in R$ with $c\,\overline{c} = 1$ such that $\mathrm{CF}_a(\gamma; c, 1)$ for every frame index $a \geq j$. We then say that γ is in *normal form from* j.

LEAN `Filling.zFormPair` · `commonLink` · `HasCommonFlagVector` · `IsNormalForm`

In words: $\mathrm{CF}_a(\gamma; c, c')$ says that the flags of the frames $\mathfrak{f}$ and $\gamma\mathfrak{f}$ share a vector of the a-th steps with leading coefficients c and c'.

Definition 2.27 (levels, congruent terms, near terms)**.** Let $S \in \mathrm{Fill}(E, \mathfrak{s})$, $c, c' \in \mathrm{Cl}(S)$, $\gamma, \gamma' \in G_S(E)$, $j \in \{0, \dots, \varrho + 1\}$ and i a natural number; write $k = k(\mathfrak{s})$, $f = f(E)$.

(i) $J_S = J(T_S^{\mathrm{rat}}(E))$, and $Q_S^{\pm}$ is the block diagonal matrix with the blocks $T_S^{\mathrm{rat}}(E)$ and $-T_S^{\mathrm{rat}}(E)$, indexed by $M(\mathfrak{s}) \sqcup M(\mathfrak{s})$.

(ii) $\mathrm{cw}_i(S) = \#\overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{\,S} \cdot \#\mathrm{Cl}(S) \cdot N(S)^{-kf} \cdot \left(2\,\mathrm{tl}(S)^2\right)^i$.

(iii) $\gamma \in \mathrm{Lv}_j(S)$, in words γ *has level* j, means: $\gamma \in \mathrm{NF}_j$, $\mathrm{D}_j(Z_S(\gamma)) \neq 0$, and $\mathfrak{g}_S(\gamma) \neq 0$ if and only if $j = \varrho + 1$.

(iv) $\gamma \sim_{S;c,c'} \gamma'$, in words γ *and* γ' *are congruent terms*, means: there is $s \in \mathsf{G}(S)$ with $\gamma \in \mathrm{Tm}_S(c, s, c')$ and $\gamma' \in \mathrm{Tm}_S(c, s, c')$.

(v) $\|\mathfrak{g}\|_S(\gamma) = \prod_\tau |\mathfrak{g}_S(\gamma)_\tau|$.

(vi) $\mathrm{BxP}(S)$ is the set of the $y \in G_\infty(E, \mathfrak{s})$ with $y_\tau \in \mathrm{FBox}(\mathrm{bx}(S))$ and $y_\tau \in \mathrm{Cut}(b(S))$ for all τ; and $\gamma \in \mathrm{Near}(S)$, in words γ *is near*, means that there is $y \in \mathrm{BxP}(S)$ with $\mathrm{r}_S^\infty(\gamma)^{-1} y \in \mathrm{BxP}(S)$.

(vii) $\gamma \in \mathrm{ChA}(S)$, in words *the characters agree at* γ, means: for all $u, u' \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$ with $u\gamma = \gamma u'$, $\psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u)) = \psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u'))$.

LEAN `Filling.ratGram` · `Filling.linkForm` · `Filling.countWeight` · `Filling.HasLevel` · `Filling.CongruentTerms` · `Filling.termDiscNorm` · `Filling.boxedPoints` · `Filling.IsNear` · `Filling.CharsAgree`

Lattices in subspaces. For a ring A with a subring O and a matrix M over A: M is *integral* if its entries lie in O, *primitive* if it is integral and has an integral left inverse; the *column lattice* of M is the O-module spanned by its columns, its *column space* the A-module they span; the *index* $[L' : L]$ of two O-submodules of A^m is the index of $L \cap L'$ in L', and 0 if that is infinite. Over $\mathbb{A}_{E,f}$ the subring is $\hat{O}_E$.

Definition 2.28 (lattice bases and discriminants of subspaces)**.** Let E be a CM field (a number field for (i)), Ω and P finite sets.

(i) For a matrix M over E, $(M)_{\mathbb{A}_f}$ is the matrix over $\mathbb{A}_{E,f}$ with the same entries.

(ii) Let E be stably unramified, let H be a $P \times \Omega$ matrix over $\mathbb{A}_{E,f}$ with a left inverse L, $LH = 1$, and let $V \subseteq E^\Omega$ be a subspace of dimension s. Then $\mathrm{m}_H(V)$ is a chosen $\Omega \times s$ matrix m over $\mathbb{A}_{E,f}$ such that $H\mathrm{m}$ is primitive over $\hat{O}_E$ and $\mathrm{m} = (y)_{\mathbb{A}_f}\, a$ for a matrix y over E with a left inverse whose columns span V, and an invertible $s \times s$ matrix a over $\mathbb{A}_{E,f}$. (It exists by the Inputs (La1) and (La2) of Section 5.)

(iii) In the situation of (ii), $\mathrm{disc}_H(V) = \sqrt{[\det(\mathrm{m}^*\mathrm{m})]}$ for $\mathrm{m} = \mathrm{m}_H(V)$.

LEAN `Filling.finAdeleMat` · `Filling.sublatticeBasis` · `Filling.sublatticeDisc`

The columns of $\mathrm{m}_H(V)$ are a basis of the lattice of the adelic points v of V with Hv integral; for $H = u^{-1}$ with $u \in G_f(E, \Omega)$ this is $V(\mathbb{A}_{E,f}) \cap u\,\hat{O}_E^\Omega$, the lattice of Theorem C. Two elementary compatibilities are used without mention: $(M^*)_{\mathbb{A}_f} = (M)^*_{\mathbb{A}_f}$; $(aM)_{\mathbb{A}_f} = a(M)_{\mathbb{A}_f}$ for $a \in E$; and for an embedding $\varphi\colon E \to \mathbb{C}$, which commutes with the conjugations because E is a CM field, the matrix of the images of M^* is the conjugate transpose of that of M.

LEAN `Filling.finAdeleMat_conjTranspose` · `Filling.finAdeleMat_smul` · `Filling.map_conjTranspose`

3 First properties and the dictionary

Throughout this section E is a CM field with a CM type, $\mathfrak{s} = (\varrho, \kappa, k_0)$ a shape and $S \in \mathrm{Fill}(E, \mathfrak{s})$; we abbreviate $N = N(S)$, $g = g(S)$, $b = b(S)$, $k = k(\mathfrak{s})$, $n = n(\mathfrak{s})$, $f = f(E)$ in proofs.

Lemma 3.1 (scales)**.** *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$, τ an infinite place of E, a a natural number, $b \le \varrho$ a frame index and $i \in I(\mathfrak{s})$.*

(a) $\varpi(S) \neq 0$*;* $\kappa_\tau(S) = g(S)\,\varpi(S)_\tau/(2\pi i)$*, so this complex number is real;* $|\kappa_\tau(S)| = g(S)\,|\varpi(S)_\tau|/(2\pi)$*; and* $\kappa_\tau(S) \neq 0$.

(b) $\mathrm{sc}_{S,\tau}(\mathfrak{f}_b) = \kappa_\tau(S)^{\varrho-b}/\sqrt{N(S)}$*;* $\mathrm{sc}_{S,\tau}(i) \neq 0$*;* $\mathrm{fs}_S(\tau, a) > 0$*;* $\mathrm{fs}(S) > 0$*;* $\deg(S) > 0$*; and* $\nu(S) > 0$.

(c) $\mathrm{Sc}^*_{S,\tau}\ J(T_S^{\mathrm{rat}}(\mathbb{C}))\ \mathrm{Sc}_{S,\tau} = J(T^{\mathrm{rat}}(\mathfrak{s}; 1))$.

Proof. (a) The different is a non-zero ideal, so its generator $\varpi(S)$ is not zero (F6). The embedding at τ commutes with complex conjugation, so $\overline{\varpi(S)_\tau} = -\varpi(S)_\tau$ by (F6), and $\varpi(S)_\tau/(2\pi i)$ is real; hence the real part in the definition of $\kappa_\tau(S)$ is the

number itself. The rest follows, $g(S)$ being a product of two primes. (b) The first formula is the definition, and all scales are powers of the non-zero real number $\kappa_\tau(S)$ divided or multiplied by $\sqrt{N(S)} > 0$. Further $\mathrm{fs}_S(\tau, a)$ is the square of a non-zero real number divided by $N(S)$, the degrees of Bessel systems are positive by definition, and $\nu(S)$ is a quotient of positive numbers. (c) The scales are real, with product 1 on each pair e_a, e_a^*, equal to 1 on the x_j and to $1/\sqrt{N(S)}$ on z, where the form has the value $-N(S)$. □

LEAN `Filling.varpi_ne_zero` · `Filling.coe_kappa` · `Filling.abs_kappa_eq` · `Filling.kappa_ne_zero` · `Filling.basisScale_besselFrame` · `Filling.basisScale_ne_zero` · `Filling.frameScaleSq_pos` · `Filling.frameScaleProd_pos` · `Filling.besselDegree_pos` · `Filling.unipCovolume_pos` · `Filling.gram_basisScaleGL`

Part (c) is what makes a_S^∞ take values in $G_\infty(E, \mathfrak{s})$. That $\mathrm{r}_S^{\mathbb{A}}$, a_S^f and the reduction maps are defined uses that a ring homomorphism $R \to R'$ carries $J(T_S^{\mathrm{rat}}(R))$ to $J(T_S^{\mathrm{rat}}(R'))$ and $J(T_S^{\mathrm{lat}}(R))$ to $J(T_S^{\mathrm{lat}}(R'))$, and that it maps a matrix that is the identity at a set of positions to a matrix with the same property; also $T_S^{\mathrm{rat}}(E)$ is hermitian and invertible, and $G_S(E)$ is countable.

LEAN `Filling.map_splitGram_ratMiddle` · `Filling.map_splitGram_latMiddle` · `Membership.mem.map_mem_unitriangularOn` · `Membership.mem.ratToInf_mem_unitriangularOn`

Lemma 3.2 (coordinates)**.** *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$, τ an infinite place, $i, j \in I(\mathfrak{s})$, and $a, b \le \varrho$ frame indices.*

(a) *For $x \in G_S(\mathbb{A}_E)$: $\left(\mathrm{a}_S^\infty(x)_\tau\right)_{ij} = (x_{ij})_\tau \cdot \mathrm{sc}_{S,\tau}(j)/\mathrm{sc}_{S,\tau}(i)$. For $\gamma \in G_S(E)$: $\left(\mathrm{r}_S^\infty(\gamma)_\tau\right)_{ij} = (\gamma_{ij})_\tau \cdot \mathrm{sc}_{S,\tau}(j)/\mathrm{sc}_{S,\tau}(i)$ and $\mathrm{r}_S^f(\gamma) = \Lambda(S)^{-1}\,\gamma\,\Lambda(S)$, with γ read over $\mathbb{A}_{E,f}$.*

(b) *For $\gamma \in G_S(E)$: $Z(\mathrm{r}_S^\infty(\gamma)_\tau)_{ab} = \dfrac{\kappa_\tau(S)^{\varrho-a}\,\kappa_\tau(S)^{\varrho-b}}{N(S)} \cdot \left(Z_S(\gamma)_{ab}\right)_\tau$; and for a finite set m and a map e from m to the frame indices,*

$$\det\left(Z(\mathrm{r}_S^\infty(\gamma)_\tau)_{e(a'),e(b')}\right)_{a',b'\in m} = \prod_{a'\in m} \mathrm{fs}_S(\tau, e(a')) \cdot \left(\det\left(Z_S(\gamma)_{e(a'),e(b')}\right)_{a',b'\in m}\right)_\tau.$$

(c) *For $u \in G_S(\mathbb{A}_E)$: $\mathrm{sd}(\mathrm{a}_S^f(u)) = \mathrm{sd}(u)_f$, the finite part; for $u \in G_S(E)$: $\mathrm{sd}(\mathrm{r}_S^{\mathbb{A}}(u)) = \mathrm{sd}(u)$, embedded diagonally.*

Proof. (a) is the definition of a_S^∞ as conjugation by the diagonal matrix $\mathrm{Sc}_{S,\tau}$, and of lb_S. (b) By Lemma 3.1(c), $h_1(v, \mathrm{Sc}_{S,\tau}^{-1} w) = h_N(\mathrm{Sc}_{S,\tau} v, w)$ for the forms h_1, h_N of $T^{\mathrm{rat}}(\mathfrak{s}; 1)$ and $T_S^{\mathrm{rat}}(\mathbb{C})$. So the (a, b) entry is $h_N(\mathrm{Sc}_{S,\tau}\mathfrak{f}_a, \gamma\,\mathrm{Sc}_{S,\tau}\mathfrak{f}_b) = \mathrm{sc}_{S,\tau}(\mathfrak{f}_a)\,\mathrm{sc}_{S,\tau}(\mathfrak{f}_b)\,h_N(\mathfrak{f}_a, \gamma\mathfrak{f}_b)_\tau$, the scales being real, and Lemma 3.1(b) gives the factor. In a minor, the row and the column of index $e(a')$ each carry the factor $\kappa_\tau(S)^{\varrho-e(a')}/\sqrt{N(S)}$, whose square is $\mathrm{fs}_S(\tau, e(a'))$. (c) The entries of $\mathrm{sd}(\cdot)$ are at the positions $(\mathfrak{f}_a, \mathfrak{f}_{a+1})$; the matrix $\Lambda(S)$ is the identity on the e_a and its column z is z, so conjugation by it does not change these entries. □

LEAN `Filling.adelicToInf_apply` · `Filling.ratToInf_apply` · `Filling.coe_ratToFin` · `Filling.realFrameGram_ratToInf` · `Filling.det_submatrix_realFrameGram_ratToInf` · `Filling.simpleRootSum_adelicToFin` · `Filling.simpleRootSum_ratToAdelic`

Inputs (adeles). Let E be a CM field. (Ad1), unit 123: E, embedded diagonally, is a lattice of covolume one in $\mathbb{A}_E$; in particular a compact subset of $\mathbb{A}_E$ contains only finitely many elements of E. $\hat{O}_E$ is compact and open in $\mathbb{A}_{E,f}$ and stable under conjugation, and E is dense in $\mathbb{A}_{E,f}$. (Ad2), unit 122: the group of the invertible matrices over $\hat{O}_E$ is compact and open in the group of invertible matrices over $\mathbb{A}_{E,f}$. (Ad3), unit 114: a countable discrete subgroup Γ of a locally compact second countable group has a *cutoff*: a measurable function β with values in $[0,\infty]$ and $\sum_{\gamma\in\Gamma}\beta(\gamma y)=1$ for all y. (Ad4), unit 123: for $x\in\mathbb{A}_{E,f}$, $\psi_f(ux)=1$ for all $u\in\hat{O}_E$ if and only if $dx\in\hat{O}_E$ for every d in the different of E; and ψ_E is trivial on E.

Lemma 3.3 (discreteness and cutoffs)**.** *Let $S\in\mathrm{Fill}(E,\mathfrak{s})$ and $c\in\mathrm{Cl}(S)$.*

(a) *Let m be a finite set, ϕ a map from m to $I(\mathfrak{s})$, and C a compact set of families $(M_\tau)_\tau$ of complex $m\times m$ matrices indexed by the infinite places τ. Then only finitely many points of C are of the form $\left((\mathrm{r}_S^\infty(u)_\tau)_{\phi(i),\phi(j)}\right)_{\tau;\,i,j\in m}$ with $u\in\mathcal{U}_{S,c}$.*

(b) *Let K be a subgroup of $\mathcal{U}_{S,c}$ and $\Gamma=\mathrm{r}_S^\infty(K)\subseteq G_\infty(E,\mathfrak{s})$. Then Γ is discrete, and there is a measurable function $\beta\colon G_\infty(E,\mathfrak{s})\to[0,\infty]$ with $\sum_{u\in K}\beta(\mathrm{r}_S^\infty(u)y)=1$ for all y; it is a cutoff for Γ with the counting measure.*

Proof. (a) Let $u\in\mathcal{U}_{S,c}$ and $k=\mathrm{pos}_S(c,c;u)\in K_g(S)$, a matrix with entries in $\hat{O}_E$ together with its inverse. By Lemma 3.2(a), $u=(\Lambda(S)h_S(c))\,k\,(\Lambda(S)h_S(c))^{-1}$ over $\mathbb{A}_{E,f}$. So the finite adele of an entry $x=u_{\phi(i)\phi(j)}\in E$ lies in a compact set that does not depend on u, the image of the compact group of (Ad2) under a continuous map. If the point attached to u lies in C, then by Lemma 3.2(a) the archimedean components of x are those of a point of C times fixed scales, and lie in a compact set too. By (Ad1) only finitely many $x\in E$ are possible for each pair (i,j), and the point is determined by these entries. (b) The map $u\mapsto\mathrm{r}_S^\infty(u)$ is injective, the embeddings $E\to\mathbb{C}$ being injective. By (a) with ϕ the identity, every compact subset of $G_\infty(E,\mathfrak{s})$ contains only finitely many elements of Γ, so Γ is discrete. It is countable, and (Ad3) gives a cutoff β for Γ; the sum over K is the sum over Γ. □

LEAN `Filling.finite_classUnip_inter_compact` · `Filling.exists_cutoff_classUnip`

Lemma 3.4 (the groups $\mathcal{U}$ and $\mathcal{H}$ in the three systems of coordinates)**.** *Let $S\in\mathrm{Fill}(E,\mathfrak{s})$.*

(a) *For $x\in G_S(\mathbb{A}_{E,f})$: $\mathrm{lb}_S(x)\in\mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ if and only if $x\in\mathcal{U}(T_S^{\mathrm{rat}}(\mathbb{A}_{E,f}))$. For $\gamma\in G_S(E)$: $\mathrm{r}_S^f(\gamma)\in\mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ if and only if $\gamma\in\mathcal{U}(T_S^{\mathrm{rat}}(E))$.*

(b) *For every $k\in G_f(E,\kappa)$, $\mathrm{inc}_S(k)\in\mathcal{H}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$; and the image of inc_S is $\mathcal{H}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$. For $\gamma_0\in\mathrm{U}_\kappa(E)$, $\mathrm{r}_S^f(\iota_{N(S)}(\gamma_0))=\mathrm{inc}_S(\gamma_0)$, where on the right γ_0 is read in $G_f(E,\kappa)$.*

(c) *For $c,c'\in\mathrm{Cl}(S)$ and $\gamma\in G_S(E)$: $\mathrm{pos}_S(c,c';\gamma)=\mathrm{B}_S(c)^{-1}\,\gamma\,\mathrm{B}_S(c')$ as matrices. For $c\in\mathrm{Cl}(S)$ and $\gamma\in G_S(E)$ with $\gamma\in\mathcal{U}(T_S^{\mathrm{rat}}(E))$: $\mathrm{pos}_S(c,c;\gamma)\in\mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$.*

(d) *Let $c,c',c''\in\mathrm{Cl}(S)$, $s,t\in\mathsf{G}(S)$ and $\gamma,\delta\in G_S(E)$. If $\gamma\in\mathrm{Tm}_S(c,s,c')$ and $\delta\in\mathrm{Tm}_S(c',t,c'')$, then $\gamma\delta\in\mathrm{Tm}_S(c,st,c'')$. If $\gamma\in\mathrm{Tm}_S(c,s,c')$, then $\gamma^{-1}\in\mathrm{Tm}_S(c',s^{-1},c)$.*

Proof. (a) The matrix $\Lambda(S)$ and its inverse are block diagonal for the levels: they are the identity on the e_a and e_a^* and mix only the middle block, of level ϱ. By (G1), conjugation by them preserves unitriangularity for the levels, in both directions. The second claim follows, since an entry of γ is 0 or 1 in E exactly when it is so in $\mathbb{A}_{E,f}$. (b) $\iota_N(k)$ lies in $\mathcal{H}(T_S^{\mathrm{rat}}(\mathbb{A}_{E,f}))$ by (G3), and conjugation by $\Lambda(S)$ keeps the columns of the e_a, e_a^* and z, since $\Lambda(S)$ fixes these vectors. Conversely let $h \in \mathcal{H}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$. Then $\Lambda(S)\, h\, \Lambda(S)^{-1} \in \mathcal{H}(T_S^{\mathrm{rat}}(\mathbb{A}_{E,f}))$, and it is $\iota_N(k)$ for some k by (G3), N being a unit of $\mathbb{A}_{E,f}$ by (F2); so $h = \mathrm{inc}_S(k)$. The last formula is the definition. (c) By definition $h_S(c) = \Lambda(S)^{-1}\iota_N(\dot{c})\Lambda(S)$ and $\mathrm{r}_S^f(\gamma) = \Lambda(S)^{-1}\gamma\Lambda(S)$, which gives the formula. For the second claim, $\mathrm{r}_S^f(\gamma) \in \mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ by (a), and $h_S(c)^{-1}$ lies in $\mathcal{H}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})) \subseteq \mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$, which normalizes $\mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ by (G2). (d) Positions multiply, $\mathrm{pos}_S(c, c''; \gamma\delta) = \mathrm{pos}_S(c, c'; \gamma) \cdot \mathrm{pos}_S(c', c''; \delta)$, and $\mathrm{pos}_S(c', c; \gamma^{-1}) = \mathrm{pos}_S(c, c'; \gamma)^{-1}$; $K(S)$ is a group and red_S a homomorphism. □

LEAN `Filling.toLatBasis_mem_besselU` · `Filling.ratToFin_mem_besselU` · `Filling.inclHFin_mem_besselH` · `Filling.range_inclHFin` · `Filling.ratToFin_toBesselH` · `Filling.coe_termPosition` · `Membership.mem.termPosition_mem_besselU` · `Filling.termsOver.mul` · `Filling.termsOver.inv`

Lemma 3.5 (the character)**.** *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$.*

(a) $|\psi_S(x)| = 1$ *for all* $x \in \mathbb{A}_{E,f}$*. For* $a \in \mathbb{A}_{E,f}$*:* $\psi_S(ra) = 1$ *for all* $r \in \hat{O}_E$ *if and only if* $a \in \hat{O}_E$*. In particular* $\psi_S(x) = 1$ *for every* $x \in \mathbb{A}_{E,f}$ *with* $x \in \hat{O}_E$*.*

(b) $\hat{g}(S)$ *has the value* $g(S)$ *in* $\mathbb{A}_{E,f}$ *and lies in* $\hat{O}_E$*; and* $\psi_S^{\mathcal{U}}(1) = 1$*.*

(c) *For* $r \in \hat{O}_E$*:* $\hat{g}(S)^{-1}r \in \hat{O}_E$ *if and only if* $\mathrm{red}_{S,x}(r) = 0$ *for all* $x = ((i, w), \epsilon)$ *as in Definition 2.25(ii).*

(d) *For every family* $(t_x)_x$ *of elements* $t_x \in \mathbb{F}_w$*, indexed by the* $x = ((i, w), \epsilon)$*, there is* $y \in \hat{O}_E$ *with* $\mathrm{red}_{S,x}(y) = t_x$ *for all* x*.*

Proof. (a) $\psi_S(ra) = \psi_f(r \cdot (-a/\varpi(S)))$. By (Ad4) this is 1 for all $r \in \hat{O}_E$ if and only if $-da/\varpi(S) \in \hat{O}_E$ for all d in the different, which is generated by $\varpi(S)$ (F6); that is, if and only if $a \in \hat{O}_E$. Take $a = 1$ for the last claim. (b) is clear; $\mathrm{sd}(1) = 0$. (c) is (F5), with the two reductions of r and of $\overline{r}$ written as the four maps $\mathrm{red}_{S,x}$. (d) Lift t_x to $\tilde{t}_x \in O_E$ and let $\varepsilon_w \in \hat{O}_E$ be the idempotent that is 1 at w and 0 at the other places. Put $y = \sum_x \tilde{t}_x \varepsilon_w$ over the x with ϵ false, plus $\sum_x \overline{\tilde{t}_x \varepsilon_w}$ over the x with ϵ true. Since $\overline{\varepsilon_w} = \varepsilon_{\overline{w}}$, the chosen places are distinct and no conjugate of a chosen place is chosen (F4), exactly one term survives each $\mathrm{red}_{S,x}$, and it gives t_x. □

LEAN `Filling.norm_selfDualChar` · `Filling.forall_selfDualChar_mul_eq_one_iff` · `Filling.selfDualChar_eq_one_of_mem` · `Filling.gIdele_val` · `Filling.gIdele_mem` · `Filling.unipotentChar_one` · `Filling.inv_gIdele_smul_mem_iff` · `Filling.exists_reduceAboveG_eq`

Lemma 3.6 (reduction modulo a conjugate place)**.** *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$ and let F be a ring.*

(a) $J(T_S^{\mathrm{lat}}(F))$ *is a symmetric matrix. If F has an involution, then $T_S^{\mathrm{lat}}(F)$ is unimodular over every subring R of F; in particular $J(T_S^{\mathrm{lat}}(F))$ is hermitian and invertible.*

(b) *Let* $k \in K(S)$*,* $i \in \{0, 1\}$*,* $w \in \mathrm{Ch}_i(S)$ *and* ϵ *a truth value. Applying* $\mathrm{red}_{S,((i,w),\epsilon)}$ *to every entry of* k *gives the matrix* $\mathrm{adj}_S^{\epsilon}(\mathrm{red}_S(k)_{i,w})$*.*

Proof. (a) The matrix has the entries 0, ± 1, $-N(S)$ and is symmetric; the image of the natural number $N(S)$ lies in every subring and is fixed by every involution, so

Lemma 2.3(b) applies. (b) For ϵ false this is the definition of red_S. Let ϵ be true. The matrix k is an isometry of $J = J(T_S^{\mathrm{lat}}(\hat{O}_E))$ over $\hat{O}_E$, so $Jk^{-1} = k^* J$ by (G6), and $\overline{k} = (k^*)^{\mathrm{t}} = (Jk^{-1}J^{-1})^{\mathrm{t}} = J^{-1}\,(k^{-1})^{\mathrm{t}}\,J$, J being symmetric. Reducing modulo w gives the claim. □

LEAN `Filling.isUnimodular_latMiddle` · `Filling.transpose_splitGram` · `Filling.intGL_map_reduceAboveG`

Proof of Theorem E(a). Let $k \in K_g(S)$, that is $k \in K(S)$ with $\mathrm{red}_S(k) = 1$. By Lemma 3.6(b), for every $x = ((i,w),\epsilon)$ the map $\mathrm{red}_{S,x}$ sends k to $\mathrm{adj}_S^{\epsilon}(1) = 1$, so it kills every entry of $k-1$; by Lemma 3.5(c) the entries of $\hat{g}(S)^{-1}(k-1)$ lie in $\hat{O}_E$. Conversely, if $k \in G_S^{\mathrm{lat}}$ has entries in $\hat{O}_E$ with its inverse and $\hat{g}(S)^{-1}(k-1)$ is integral, then by Lemma 3.5(c) with ϵ false every entry of $k-1$ vanishes modulo every chosen place, so $\mathrm{red}_S(k) = 1$. □

LEAN `Filling.principalLevel_eq_unitaryLevel`

Proof of Theorem E(c). Put $\mathtt{z} = -(g\,\varpi(S))^{-1}\,\mathrm{sd}(x) \in \mathbb{A}_E$. By Lemma 3.2(c), $\psi_S^{\mathcal{U}}(\mathrm{a}_S^f(x)) = \psi_f(-\hat{g}(S)^{-1}\,\mathrm{sd}(x)_f/\varpi(S)) = \psi_f(\mathtt{z}_f)$. Let τ be an infinite place and $x'_\tau = \mathrm{a}_S^\infty(x)_\tau$. It lies in $\mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s};1))$, a conjugate of a unitriangular matrix by a diagonal one (G1), and by Lemma 3.2(a) and Lemma 3.1,

$$\mathrm{sd}(x'_\tau) = \sum_{a<\varrho}(x_{\mathfrak{f}_a\mathfrak{f}_{a+1}})_\tau\,\frac{\mathrm{sc}_{S,\tau}(\mathfrak{f}_{a+1})}{\mathrm{sc}_{S,\tau}(\mathfrak{f}_a)} = \kappa_\tau(S)^{-1}\,\mathrm{sd}(x)_\tau = \frac{2\pi i}{g\,\varpi(S)_\tau}\,\mathrm{sd}(x)_\tau = -2\pi i\,\mathtt{z}_\tau.$$

Since $W_{S,\tau}^\sigma$ is a Bessel function, $W_{S,\tau}^\sigma(x'_\tau y_\tau) = \mathbf{e}(-2\pi i \mathtt{z}_\tau)\,W_{S,\tau}^\sigma(y_\tau)$ and $\mathbf{e}(-2\pi i\mathtt{z}_\tau) = e^{-2\pi i\cdot 2\,\mathrm{Re}\,\mathtt{z}_\tau}$. The product over τ of these factors is the archimedean part of $\psi_E(\mathtt{z})$, that is $\psi_E(\mathtt{z})/\psi_f(\mathtt{z}_f)$. □

LEAN `Filling.besselFn_adelicToInf_mul`

Corollary 3.7. *Let $S \in \mathrm{Fill}(E,\mathfrak{s})$, σ a side, $\gamma \in G_S(E)$ with $\gamma \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$, and $y \in G_\infty(E,\mathfrak{s})$. Then $W_S^\sigma(\mathrm{r}_S^\infty(\gamma)y)_\cdot = \psi_S^{\mathcal{U}}(\mathrm{r}_S^f(\gamma))^{-1}\,W_S^\sigma(y)_\cdot$.*

Proof. Theorem E(c) for $x = \mathrm{r}_S^{\mathbb{A}}(\gamma)$: here $\mathtt{z} \in E$, and ψ_E is trivial on E (Ad4). □

LEAN `Filling.besselFn_ratToInf_mul`

Lemma 3.8 (norms of kernels and amplitudes)**.** *Let $S \in \mathrm{Fill}(E,\mathfrak{s})$, σ a side, $\gamma \in G_S(E)$, $c,c' \in \mathrm{Cl}(S)$, $y \in G_\infty(E,\mathfrak{s})$ and $i \in \mathrm{Bc}(S)$. Then $y' \mapsto W_S^\sigma(y')_i$ and $y' \mapsto \mathcal{W}_S^\sigma(\gamma;y')$ are continuous,*

$$\|\mathcal{W}_S^\sigma(\gamma;y)\| = \prod_\tau \|W_{S,\tau}^\sigma(y_\tau)\| \cdot \prod_\tau \|W_{S,\tau}^\sigma((\mathrm{r}_S^\infty(\gamma)^{-1}y)_\tau)\|,$$

and $\|a_S^\sigma(c,c';\gamma)\| \le |a|_S^\sigma(c,c';\gamma)$.

Proof. The functions of a Bessel system are continuous. The squared norm of $\mathcal{W}_S^\sigma(\gamma;y)$ is $\sum_{i,i'}|W_S^\sigma(y)_i|^2|W_S^\sigma(\mathrm{r}_S^\infty(\gamma)^{-1}y)_{i'}|^2$, and a sum over tuples of a product is the product of the sums. The last inequality is the triangle inequality for the integral. □

LEAN `Filling.continuous_besselFn` · `Filling.continuous_besselKernel` · `Filling.enorm_besselKernel` · `Filling.enorm_amplitude_le`

The group $\mathcal{H}$ and the integral subgroup.

Lemma 3.9. *Let $S \in \mathrm{Fill}(E,\mathfrak{s})$ and write $\bar\pi = \bar\pi(S)$; for a unit p of $\mathbb{A}_{E,f}$ let D_p be the diagonal matrix indexed by κ with p at k_0 and 1 elsewhere.*

(a) *Let m be a finite set and l a matrix over $\mathbb{A}_{E,f}$ with rows indexed by κ and columns by m. If $D_{\bar\pi^{-1}}\, l$ is integral, then $\Lambda^\circ(\mathfrak{s};\bar\pi)\, l$ is integral. And $\Lambda^\circ(\mathfrak{s};\bar\pi)\, l$ is integral if and only if l is integral and its row k_0 has entries in $\bar\pi\hat{O}_E$.*

(b) *Let κ' be a finite set, p a unit of $\mathbb{A}_{E,f}$ such that at every finite place w either p is a unit at w or p is a unit at $\overline{w}$, $o \in \kappa'$, and $k \in G_f(E,\kappa')$ such that k and k^{-1} are integral and the row o of $k-1$ has entries in $p\hat{O}_E$. Then $D_{p^{-1}}\,(k-1)\,D_{\overline{p}^{-1}}$ is integral, with D now indexed by κ' and o in the place of k_0.*

(c) *Let $k \in G_f(E,\kappa)$ and let Π be a property of elements of $\mathbb{A}_{E,f}$ that holds for 0. Then all entries of $\mathrm{inc}_S(k)-1$ have Π if and only if all entries of $\Lambda^\circ(\mathfrak{s};\bar\pi)\,(k-1)\,D_{\overline{\bar\pi}^{-1}}$ have Π.*

Proof. (a) The rows of $\Lambda^\circ(\mathfrak{s};\bar\pi)\, l$ are the rows l_i for $i \neq k_0$, the row $\overline{\bar\pi}\, l_{k_0}$ and the row $-\bar\pi^{-1} l_{k_0}$. Both $\bar\pi$ and $\overline{\bar\pi}$ lie in $\hat{O}_E$ by (F2) and (Ad1), and $\overline{\bar\pi}\, l_{k_0} = N \cdot \bar\pi^{-1} l_{k_0}$; both claims follow. (b) The row o of $k-1$ lies in $p\hat{O}_E$. So does the row o of $k^{-1}-1 = -(k-1)\,k^{-1}$, because k^{-1} is integral. As $k^{-1} = k^*$, this row is the conjugate of the column o of $k-1$, which therefore lies in $\overline{p}\hat{O}_E$. The entries of the scaled matrix are $(k-1)_{ij}$, $p^{-1}(k-1)_{oj}$, $(k-1)_{io}\overline{p}^{-1}$ and $p^{-1}(k-1)_{oo}\overline{p}^{-1}$. At a place w where p is a unit, division by p is harmless and the column condition handles $\overline{p}$; at a place w where p is a unit at $\overline{w}$, $\overline{p}$ is a unit at w and the row condition handles p. (c) $\mathrm{inc}_S(k) - 1 = \Lambda(S)^{-1}(\iota_N(k)-1)\Lambda(S)$. The matrix $\iota_N(k)-1$ is zero outside the block of κ, where it is $k-1$; the columns of $\Lambda(S)^{-1}$ at κ form the matrix $\Lambda^\circ(\mathfrak{s};\bar\pi)$ in the middle rows, and the rows of $\Lambda(S)$ at κ form $D_{\overline{\bar\pi}^{-1}}$ in the columns of κ and vanish elsewhere. So the entries of $\mathrm{inc}_S(k)-1$ are those of the product and zeros. □

LEAN `Filling.isIntMatrix_middleBaseChange_mul` · `Filling.isIntMatrix_scaleAt_sub_one` · `Filling.forall_inclHFin_sub_one_iff`

Proof of Theorem E(b). An isometry x of the unimodular matrix $J = J(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ lies in $K(S)$ if and only if $x-1$ is integral, since then $x^{-1} = J^{-1}x^*J$ is integral too (Lemma 3.6(a)). By Lemma 3.9(c), with Π integrality, $\mathrm{inc}_S(k) \in K(S)$ if and only if $M = \Lambda^\circ(\mathfrak{s};\bar\pi)\,(k-1)\,D_{\overline{\bar\pi}^{-1}}$ is integral. If M is integral, so is $M\,D_{\overline{\bar\pi}} = \Lambda^\circ(\mathfrak{s};\bar\pi)(k-1)$, and Lemma 3.9(a) shows that $k-1$ is integral with row k_0 in $\bar\pi\hat{O}_E$; then k is integral, and so is $k^{-1} = k^*$. Conversely, if k satisfies these conditions, Lemma 3.9(b) with $p = \bar\pi$ and $o = k_0$, whose hypothesis is (F2), shows that $D_{\bar\pi^{-1}}(k-1)D_{\overline{\bar\pi}^{-1}}$ is integral, and Lemma 3.9(a) with $l = (k-1)D_{\overline{\bar\pi}^{-1}}$ that M is. □

LEAN `Filling.inclHFin_mem_latStabilizer`

Lemma 3.10. *Let $S \in \mathrm{Fill}(E,\mathfrak{s})$.*

(a) *Let $k \in G_f(E,\kappa)$ with $\mathrm{inc}_S(k) \in K(S)$, $i \in \{0,1\}$ and $w \in \mathrm{Ch}_i(S)$. Then $\mathrm{red}_S(\mathrm{inc}_S(k))_{i,w} = 1$ if and only if k, which is integral by Theorem E(b), reduces to 1 modulo w.*

(b) *For every real function ϕ on $G_f(E,\kappa)$, the class sum $\sum_c \phi(\dot c)/\#(\Gamma_f(E,\kappa) \cap \dot c\, K^{\mathrm{col}}(S)\, \dot c^{-1})$ over $c \in \mathrm{Cl}(S)$ equals $\sum_{c\in\mathrm{Cl}(S)} \phi(\dot c)$.*

(c) *For $i \in \{0,1\}$ and $w \in \mathrm{Ch}_i(S)$, the residue field $\mathbb{F}_w$ has $q_i(S)$ elements.*

Proof. (a) Both conditions say that every entry of the matrix minus 1 lies in the maximal ideal at w. By Lemma 3.9(c) with this property, the condition for $\mathrm{inc}_S(k)$ is the condition for $\Lambda^\circ(\mathfrak{s};\bar\pi)(k-1)D_{\overline{\bar\pi}^{-1}}$. A chosen place has $q_w = q_i(S) \neq N$ by (F3) and (F4), so $\bar\pi$ is a unit at w by (F2), and so is $\overline{\bar\pi}$, as $\bar\pi\overline{\bar\pi} = N$ is a unit at w. Multiplying rows and columns by units at w does not change the condition, and the row z of $\Lambda^\circ(\mathfrak{s};\bar\pi)(k-1)$ is a unit multiple of the row k_0 of $k-1$. (b) The stabilizers are trivial by (F1). (c) is (F4). □

LEAN `Filling.reduction_inclHFin_eq_one_iff` · `Filling.classSum_eq_sum` · `Filling.card_resField_eq_levelPrime`

4 Links and normal forms

In this section F is a ring with involution, $\mathfrak{s}$ a shape and T an $\mathfrak{s}$-form over F; vectors of $F^{I(\mathfrak{s})}$ are written (a,p,a') with e-coordinates a, middle part p and e^*-coordinates a'. By the shape of $J(T)$, $J(T)\,(a,p,a') = (a', Tp, a)$; hence a vector is orthogonal to $X = \mathrm{span}(e_0,\dots,e_{\varrho-1})$ if and only if its e^*-part vanishes, and $[\mathfrak{e} \mid \gamma\mathfrak{e}]^* J(T)\,t = 0$ if and only if the e^*-parts of t and of $\gamma^{-1}t$ vanish; and $\mathrm{L}(\gamma)t = (t_{\mathrm{mid}}, (\gamma^{-1}t)_{\mathrm{mid}})$.

LEAN `splitGram_mulVec` · `isotropicFramePair_mulVec_eq_zero_iff` · `linkMatrix_mulVec`

Lemma 4.1 (the link in coordinates)**.** *Let F be a ring with involution, $\mathfrak{s}$ a shape, T an $\mathfrak{s}$-form over F, $\gamma \in G(T)$ and $p = (p_1,p_2) \in F^{M(\mathfrak{s})} \times F^{M(\mathfrak{s})}$. Then $p \in \mathrm{link}(\gamma)$ if and only if there is $a \in F^\varrho$ such that $\gamma\,(a,p_2,0)$ has middle part p_1 and vanishing e^*-part.*

Proof. Every t is γw for a unique w. The conditions on t in Definition 2.5 say that γw and w have vanishing e^*-parts, and then $\mathrm{L}(\gamma)t = ((\gamma w)_{\mathrm{mid}}, w_{\mathrm{mid}})$. □

LEAN `mem_link`

Proof of Theorem B(a). Let $u,w \in \mathcal{U}(T)$. We show $\mathrm{link}(\gamma) \subseteq \mathrm{link}(u\gamma w^{-1})$. Let $t \perp X + \gamma X$. By (G5), u and w preserve X, hence $X^\perp$, and act trivially on $X^\perp/X$. So $ut \perp X$, $(u\gamma w^{-1})^{-1}(ut) = w\gamma^{-1}t \perp X$, and the middle parts of ut and of $w\gamma^{-1}t$ are those of t and of $\gamma^{-1}t$. Thus $\mathrm{L}(\gamma)t \in \mathrm{link}(u\gamma w^{-1})$. Applying the inclusion to $(u^{-1}, u'^{-1}, \gamma)$ and to $(u, u', u^{-1}\gamma u')$ gives the two inclusions of the theorem. □

LEAN `Filling.link_besselU_mul`

Lemma 4.2 (normal forms)**.** *Let F be a ring with involution, $\mathfrak{s}$ a shape, T an $\mathfrak{s}$-form over F, $\gamma, u, u' \in G(T)$, and j a natural number.*

(a) *If $\gamma \in \mathrm{NF}_j$, then $\gamma^{-1} \in \mathrm{NF}_j$.*

(b) *If $u \in \mathcal{U}(T)$ and $\gamma \in \mathrm{NF}_j$, then $u\gamma \in \mathrm{NF}_j$.*

(c) *Let $j \le \varrho + 1$ and suppose $\gamma \in \mathrm{NF}_j$. Then for every $y \in F^{\varrho+1}$ there are $x, y' \in F^{\varrho+1}$ with $y'_a = y_a$ for all frame indices $a \ge j$ and $\mathfrak{f}x = (\gamma\mathfrak{f})y'$.*

(d) *If $u \in \mathcal{U}(T)$ and $u' \in \mathcal{U}(T)$, then for every $N \in F$, $\mathfrak{g}(N; Z_T(u^{-1}\gamma u')) = \mathfrak{g}(N; Z_T(\gamma))$.*

Proof. (a) Let c with $c\bar c = 1$ be as in Definition 2.26, and for $a \ge j$ let x, y be the vectors with $\mathfrak{f}x = (\gamma\mathfrak{f})y$, $x_a = c$, $y_a = 1$. Then $\mathfrak{f}(\bar c y) = (\gamma^{-1}\mathfrak{f})(\bar c x)$, the coefficients at a are $\bar c$ and $\bar c c = 1$, and those above a vanish; so $\bar c$ serves for γ^{-1}. (b) Keep c and y, and replace x by the vector x' with $\mathfrak{f}x' = u\,\mathfrak{f}x$. By (G5), $u\mathfrak{f}_b - \mathfrak{f}_b \in \mathrm{span}(\mathfrak{f}_{b'} : b' < b)$, so $x'_a = x_a$ and $x'_b = 0$ for $b > a$; and $\mathfrak{f}x' = u(\gamma\mathfrak{f})y = ((u\gamma)\mathfrak{f})y$. (c) Downward

induction on j. For $j = \varrho + 1$ take $x = y' = 0$. Let $j \le \varrho$ and let x, y'' solve the problem for $j+1$, which is possible because normal form from j implies normal form from $j+1$. With X_j, Y_j the vectors of Definition 2.26 for $a = j$, so that $(Y_j)_j = 1$ and both vanish above j, put $x' = x + (y_j - y''_j)X_j$ and $y' = y'' + (y_j - y''_j)Y_j$. (d) By (G4) the determinant of $Z_T(\gamma)$, which is its leading minor of order $\varrho + 1$, and its leading minor of order ϱ do not change. □

LEAN `IsNormalForm.inv` · `IsNormalForm.besselU_mul` · `Filling.exists_frameImage_eq_of_isNormalForm` · `Filling.frameDisc_frameGram_besselU_mul`

Corollary 4.3. *Let E be a CM field with a CM type, $\mathfrak{s}$ a shape, $S \in \mathrm{Fill}(E, \mathfrak{s})$ and $j \in \{0, \dots, \varrho + 1\}$. The set of the $\gamma \in G_S(E)$ with $\gamma \in \mathrm{Lv}_j(S)$ is stable: $\{\gamma : \gamma \in \mathrm{Lv}_j(S)\} \in \mathrm{Stab}(S)$.*

Proof. Let $u, u' \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$. From $\gamma \in \mathrm{NF}_j$ one gets normal form from j of $u^{-1}\gamma$, of $\gamma^{-1}u$, of $u'^{-1}\gamma^{-1}u$ and of $u^{-1}\gamma u'$, by Lemma 4.2(b), (a), (b), (a); the converse is the same with u^{-1}, u'^{-1}. The minor $\mathrm{D}_j(Z_S(\gamma))$ and $\mathfrak{g}_S(\gamma)$ are unchanged by (G4) and Lemma 4.2(d). □

LEAN `Filling.isStable_hasLevel`

> **Input (L1)**, unit 112. Let P, P' be matrices over a field with the same finite index sets, s, s' sets of column indices, and x^0, y^0 column vectors. If every row vector l with $(lP)_b = 0$ for $b \in s$ and $(lP')_b = 0$ for $b \in s'$ satisfies $lPx^0 = lP'y^0$, then there are x, y with $Px = P'y$, $x_b = x^0_b$ for $b \notin s$ and $y_b = y^0_b$ for $b \notin s'$.

Proof of Theorem B(b). Write $T = T_S^{\mathrm{rat}}(E)$, $h = h_T$, and $p = (p_1, p_2)$. For vectors s, t with vanishing e^*-parts, $h(s,t) = s^*_{\mathrm{mid}}\, T\, t_{\mathrm{mid}}$. Let $p = \mathrm{L}(\gamma)t$ and $q = \mathrm{L}(\gamma)s$ be in the link. Then by (G6)

$$q^*\, Q_S^{\pm}\, p = h(s,t) - h(\gamma^{-1}s, \gamma^{-1}t) = 0.$$

Conversely let p be orthogonal to the link. By Lemma 4.1 we need $x = (a', p_1, 0)$ and $y = (a, p_2, 0)$ with $x = \gamma y$. Apply (L1) to $P = 1$, $P' = \gamma$, the sets $s = s'$ of the indices of the e_a, and $x^0 = (0, p_1, 0)$, $y^0 = (0, p_2, 0)$. A row vector l is $h(t, \cdot)$ for a vector t, since J_S is hermitian and invertible. The conditions on l say $h(t, e_a) = 0$ and $h(t, \gamma e_a) = 0$ for all a, that is, $t \perp X + \gamma X$; then $q = \mathrm{L}(\gamma)t$ lies in the link, and

$$lx^0 - l\gamma y^0 = h(t, x^0) - h(\gamma^{-1}t, y^0) = t^*_{\mathrm{mid}} T p_1 - (\gamma^{-1}t)^*_{\mathrm{mid}} T p_2 = q^*\, Q_S^{\pm}\, p = 0.$$

So (L1) applies. □

LEAN `mem_link_iff_orthogonal`

> **Input (L2)**, unit 119. Let A be a field with involution, T an $\mathfrak{s}$-form over A, p a sequence of units of A and $\sigma_0 \in \mathcal{H}_\varrho(T)$. The matrix of $w_\varrho(p)\,\sigma_0$ maps $(a, m, a') \mapsto ((\overline{p_b}^{-1} a'_b)_b,\ \sigma_0^{\mathrm{mid}} m,\ (p_b a_b)_b)$, where σ_0^{mid} is the middle block of σ_0; and $Z_T(w_\varrho(p)\,\sigma_0)$ is the diagonal matrix with the entries $p_0, \dots, p_{\varrho-1}$ and $Z_T(\sigma_0)_{\varrho\varrho} = h_T(z, \sigma_0 z)$.

Proposition 4.4 (the big cell)**.** *Let A be a field with involution, $\mathfrak{s}$ a shape, T an $\mathfrak{s}$-form over A, p, q sequences of units of A and $\sigma_0, \sigma_1 \in G(T)$ with $\sigma_0 \in \mathcal{H}_\varrho(T)$ and $\sigma_1 \in \mathcal{H}_\varrho(T)$.*

(a) *For $N \in A$, with $t = Z_T(\sigma_0)_{\varrho\varrho}$ and $d = \prod_{a<\varrho} p_a$:* $\mathsf{g}(N; Z_T(w_\varrho(p)\,\sigma_0)) = d\,\overline{d}\,(t\,\overline{t} - N^2)$.

(b) $\mathrm{link}(w_\varrho(p)\,\sigma_0) = \mathrm{link}(w_\varrho(q)\,\sigma_1)$ *if and only if* $\sigma_0 = \sigma_1$.

Proof. (a) By (L2) the frame Gram matrix is diagonal with determinant dt and leading minor d of order ϱ. (b) By Lemma 4.1 and (L2), $(p_1, p_2) \in \mathrm{link}(w_\varrho(p)\sigma_0)$ if and only if there is a such that $w_\varrho(p)\sigma_0(a, p_2, 0) = (0, \sigma_0^{\mathrm{mid}} p_2, (p_b a_b)_b)$ has middle part p_1 and no e^*-part: this forces $a = 0$ and $p_1 = \sigma_0^{\mathrm{mid}} p_2$. So the link is the graph of the middle block of σ_0, whatever p is, and two elements of $\mathcal{H}_\varrho(T)$ with the same middle block are equal. □

LEAN `Filling.frameDisc_weylElem_mul` · `Filling.link_big_cell_eq_iff`

5 Lattices in subspaces and Siegel's lemma

Lemma 5.1 (a basis adapted to a kernel)**.** *Let K_0 be a field, P and Ω finite sets, A an $\Omega \times P$ matrix and x a $P \times \Omega$ matrix over K_0 with $\det(Ax) \neq 0$, and $L \subseteq K_0^P$ a subspace that contains the columns of x. Then there are a natural number s and a matrix y with s columns such that: $s = \dim(L \cap \ker A)$; $(x \mid y)$ has a left inverse; the columns of $(x \mid y)$ span L; the columns of y lie in L; $Ay = 0$; and if yt lies in the span of the columns of x for a vector t, then $t = 0$.*

Proof. Let y be a matrix whose columns are a basis of $L \cap \ker A$, with a left inverse l, and put $C = (Ax)^{-1}A$. Then $Cx = 1$ and $Cy = 0$, since $Ay = 0$. The matrix with the rows C and $l - lxC$ is a left inverse of $(x \mid y)$. Every $v \in L$ is $x(Cv) + (v - xCv)$ with $v - xCv \in L \cap \ker A$. If $yt = xw$, then $w = Cxw = Cyt = 0$, so $yt = 0$ and $t = l(yt) = 0$. □

LEAN `Filling.exists_basis_extend_ker` · `Filling.mul_eq_zero_of_range_le_ker`

Inputs (lattices). Let E be a stably unramified CM field. (La1), unit 120: every subspace of K_0^P, K_0 a field, is the column space of a matrix with a left inverse. (La2), unit 57: for a matrix M over $\mathbb{A}_{E,f}$ with a left inverse there is an invertible matrix a over $\mathbb{A}_{E,f}$ such that Ma is primitive over $\hat{O}_E$ and the columns of a span the lattice of the v with Mv integral. With (La1) this gives the matrix $\mathrm{m}_H(V)$ of Definition 2.28: take a matrix y over E with a left inverse l whose columns span V, and apply (La2) to $M = H(y)_{\mathbb{A}_f}$, which has the left inverse $(l)_{\mathbb{A}_f} L$. (La3), unit 57: $[\det(J^*MJ)] = [\det M]\,[\det J]^2$ for square matrices over $\mathbb{A}_{E,f}$; for an integral invertible matrix M, the index of its column lattice in $\hat{O}_E^s$ is $[\det M]$; indices of lattices are multiplicative in towers, and the index of the images of two lattices under a linear map divides their index; and (units 57 and 123) $[x] = |N_{E/\mathbb{Q}}(x)| = \prod_\tau |x_\tau|^2$ for $x \in E^\times$. (La4), unit 57, *Jacobi's identity for complements*: let J be a hermitian matrix that is unimodular over $\hat{O}_E$, F an integral matrix and n a primitive matrix with $F^*J\mathsf{n} = 0$ whose numbers of columns add up to the size of J, and suppose F has a left inverse. Then $[\det(\mathsf{n}^*J\mathsf{n})] \cdot i^2 = [\det(F^*JF)]$, where i is the index of $\hat{O}_E^s \cap \{v : Fv \text{ integral}\}$ in $\{v : Fv \text{ integral}\}$.

Proposition 5.2 (discriminants and indices)**.** *Let E be a stably unramified CM field, P, Ω, Ω' finite sets, H a $P \times \Omega$ matrix over $\mathbb{A}_{E,f}$ with a left inverse L, $LH = 1$,*

$V \subseteq E^{\Omega}$ a subspace, s a natural number with $s = \dim_E V$, x an $\Omega \times s$ matrix over E whose columns lie in V, w an invertible $s \times s$ matrix over $\mathbb{A}_{E,f}$, and D a real number with

$$[\det(x^* x)_{\mathbb{A}_f}] \cdot [\det w]^2 = D \qquad \text{and} \qquad D \neq 0.$$

Let Y be an $\Omega' \times P$ matrix over $\mathbb{A}_{E,f}$ and K an $\hat{O}_E$-submodule of $\mathbb{A}_{E,f}^{\Omega'}$. Put $\mathsf{n} = H(x)_{\mathbb{A}_f} w$. If n is integral and every column of $Y\mathsf{n}$ lies in K, then, with i the index of $K \cap \Lambda$ in the column lattice Λ of $YH\,\mathrm{m}_H(V)$,

$$\left(\mathrm{disc}_H(V) \cdot i\right)^2 \leq D, \qquad \mathrm{disc}_H(V) > 0, \qquad i \geq 1.$$

Proof. Let $\mathrm{m} = \mathrm{m}_H(V) = (y)_{\mathbb{A}_f} a$ with y, a as in Definition 2.28, and let l be an integral left inverse of $H\mathrm{m}$. The columns of x lie in V, the column space of y, so $x = yB$ for a matrix B over E, and $(x)_{\mathbb{A}_f} w = \mathrm{m}\, A$ with $A = a^{-1}(B)_{\mathbb{A}_f} w$. By (La3),

$$D = [\det(w^* (x^* x)_{\mathbb{A}_f} w)] = [\det(A^* \mathrm{m}^* \mathrm{m}\, A)] = \mathrm{disc}_H(V)^2 \cdot [\det A]^2.$$

As $D \neq 0$, $[\det A] \neq 0$ and $\mathrm{disc}_H(V) \neq 0$, so A is invertible and $\mathrm{disc}_H(V) > 0$. Moreover $A = l(H\mathrm{m}A) = l\,\mathsf{n}$ is integral, so by (La3) $[\det A]$ is the index of the column lattice of A in $\hat{O}_E^s$. The map $v \mapsto YH\mathrm{m}v$ carries $\hat{O}_E^s$ onto Λ and the column lattice of A into $K \cap \Lambda$, the columns of $Y\mathsf{n} = YH\mathrm{m}A$ lying in K. Hence i divides the index of the image of the column lattice of A in Λ, which divides $[\det A]$; so $1 \leq i \leq [\det A]$ and $(\mathrm{disc}_H(V)\, i)^2 \leq D$. □

LEAN `Filling.sublatticeDisc_mul_index_sq_le`

Inputs (geometry of numbers). Let E be a CM field with $f = f(E)$. (M1), unit 121, *Minkowski's second theorem in adelic form*: let s be a natural number, r_τ an invertible complex $s \times s$ matrix for each infinite place τ and a an invertible $s \times s$ matrix over $\mathbb{A}_{E,f}$. Then there are a basis $B_1, \dots, B_s$ of E^s and integers $z_1, \dots, z_s$ such that $a^{-1}B_j \in \hat{O}_E^s$, $\sum_\tau \|r_\tau^{-1}(B_j)_\tau\|^2 \leq 4^{z_j}$ for all j, and $\prod_\tau |\det r_\tau|^2 \cdot [\det a]^{-1} \cdot 4^{f\sum_j z_j} \leq (8fs\ \mathrm{rd}_E)^{fs}$. The second reader of this write-up compared it with Bombieri and Vaaler [1], Theorem 3, of which it is a weaker form. (M2), unit 120, Gram–Schmidt: a complex matrix X with a left inverse has Xr with orthonormal columns for some invertible r. (M3), unit 121: for $u \in G_f(E, \kappa)$, every non-zero $x \in E^\kappa \cap u\hat{O}_E^\kappa$ has $\|x\| \geq 1$. (M4), unit 114: if points of a real normed space of dimension d have norms and mutual distances at least 1, then for every integer $\nu \geq d + 2$ the sum of their norms to the power $-\nu$ is at most 5^ν.

Proof of Theorem C(a). Let $\mathrm{m} = \mathrm{m}_{u^{-1}}(V) = (x)_{\mathbb{A}_f} b$, with x a matrix over E with a left inverse l_0 whose columns span V, and b invertible over $\mathbb{A}_{E,f}$. The orthogonal complement $V^\perp = \ker x^*$ has dimension s, because x^* is surjective, with right inverse l_0^*. Let $\mathrm{m}' = \mathrm{m}_{u^{-1}}(V^\perp) = (y)_{\mathbb{A}_f} a$, with y a matrix over E with a left inverse l whose columns span $V^\perp$, and a invertible. Then $y^* x = 0$, and since $\ker x^*$ lies in the column space of y there are matrices f_0, t_0 with $yf_0 + t_0 x^* = 1$; hence a vector v with $y^* v = 0$ is $v = x(t_0^* v) \in V$, and $V = \ker y^*$.

The two discriminants agree. Apply (La4) with $J = 1$, $F = u^{-1}\mathrm{m}$ and $\mathsf{n} = u^{-1}\mathrm{m}'$: both are primitive, their numbers of columns add up to $\#\kappa$, and $F^*\mathsf{n} = \mathrm{m}^*\mathrm{m}' = b^* (x^* y)_{\mathbb{A}_f} a = 0$ because u is unitary. Since F is primitive, Fv is integral only for integral v, so the index in (La4) is 1 and

$$\mathrm{disc}_{u^{-1}}(V^\perp)^2 = [\det(\mathrm{m}'^* \mathrm{m}')] = [\det(\mathrm{m}^* \mathrm{m})] = \mathrm{disc}_{u^{-1}}(V)^2. \tag{1}$$

Short vectors in $V^\perp$. By (M2) choose for each τ an invertible r_τ such that $y_\tau r_\tau$ has orthonormal columns; then $|\det r_\tau|^{-2} = \det(y^*y)_\tau$. Apply (M1) to $(r_\tau)_\tau$ and a, and put $X_j = yB_j$. These vectors span $V^\perp$, so $V = \ker y^*$ is the set of the v with $X_j^* v = 0$ for all j. They are non-zero, as y has a left inverse, and $u^{-1}X_j = (u^{-1}\mathrm{m}')(a^{-1}B_j)$ is integral. Since $y_\tau r_\tau$ is an isometry,

$$\|X_j\|^2 = \sum_\tau \|y_\tau r_\tau \cdot r_\tau^{-1}(B_j)_\tau\|^2 = \sum_\tau \|r_\tau^{-1}(B_j)_\tau\|^2 \le 4^{z_j}.$$

Finally, by (La3) and (1),

$$\mathrm{disc}_{u^{-1}}(V)^2 = [\det(a^*\,(y^*y)_{\mathbb{A}_f}\,a)] = \prod_\tau |\det(y^*y)_\tau|^2 \cdot [\det a]^2 = \Big(\prod_\tau |\det r_\tau|^{-2}\,[\det a]\Big)^2.$$

So

$$\Big(\prod_j \|X_j\|\Big)^{2f} \le 4^{f\sum_j z_j} \le (8fs\ \mathrm{rd}_E)^{fs} \prod_\tau |\det r_\tau|^{-2}[\det a] = (8fs\ \mathrm{rd}_E)^{fs}\ \mathrm{disc}_{u^{-1}}(V).$$

□

LEAN `Filling.siegel_lemma`

Proof of Theorem C(b). Write $k = \#\kappa$ and $\Lambda_u = E^\kappa \cap u\hat{O}_E^\kappa$. If $s > k$, no subspace V has $\dim V + s = k$, the sum is empty, and the bound holds because its right side is at least 1. Let $s \le k$. For each V with $\dim V + s = k$ choose $X(V) = (X_1, \dots, X_s)$ as in (a). The subspace V is determined by $X(V)$, and $\prod_j \|X_j\| \ge 1$ by (M3). Raising the bound of (a) to the power $k+1$ and using $(8fs\ \mathrm{rd}_E)^{fs(k+1)} \le (8fk\ \mathrm{rd}_E)^{f(k+1)s}$, which holds as $s \le k$,

$$\mathrm{disc}_{u^{-1}}(V)^{-(k+1)} \le (8fk\ \mathrm{rd}_E)^{f(k+1)s} \prod_{j=1}^{s} \|X_j\|^{-2f(k+1)}.$$

Summing over V, and then over all s-tuples of non-zero vectors of Λ_u, the right side is at most $(8fk\ \mathrm{rd}_E)^{f(k+1)s}\big(\sum_x \|x\|^{-2f(k+1)}\big)^s$, the sum over the non-zero $x \in \Lambda_u$. These points of the real vector space $\mathbb{C}^{f\kappa}$ of dimension $2fk$ have norms and mutual distances at least 1 by (M3), and $2f(k+1) \ge 2fk+2$; by (M4) the inner sum is at most $5^{2f(k+1)} = 25^{f(k+1)}$. As $25 \cdot 8 = 200$, the claim follows. □

LEAN `Filling.tsum_sublatticeDisc_inv_le`

6 Strong approximation and terms in the Bessel subgroup

Inputs (the unipotent group, floor by floor). Let A be a ring with involution with an element t, $t+\overline{t}=1$, and T an $\mathfrak{s}$-form over A with $J(T)$ hermitian and invertible. For $l<\varrho$ and a pair (v,β) of a vector $v\in A^{I(\mathfrak{s})}$ and an element $\beta\in A$ with $\overline{\beta}=-\beta$, the *Heisenberg element* $\nu_l(v,\beta)\in G(T)$ is an Eichler map along e_l whose entries are polynomials in the coordinates of v, their conjugates, β and t. (U1), unit 119: $\nu_l(v,\beta)\in\mathcal{U}(T)$, and $\mathcal{U}(T)$ is generated by these elements: if a subset of $G(T)$ contains 1 and is stable under left multiplication by all $\nu_l(v,\beta)$, $l<\varrho$, then it contains $\mathcal{U}(T)$. (U2), unit 29: if $J(T)$ is invertible over a Hausdorff topological ring with continuous involution, the topology of $G(T)$ is induced by the entries. (U3), unit 1: for a subring $Q\subseteq A$, a unit g of A, and $k\in G(T)$ with entries in Q together with its inverse and $g^{-1}(k-1)$ with entries in Q: $g^{-1}\operatorname{sd}(k)\in Q$; and if $k=uh$ with $u\in\mathcal{U}(T)$ and $h\in\mathcal{H}(T)$, then h has the same two properties. (U4), unit 112: let $\Gamma\to G$ be a homomorphism, $K\subseteq G$ a subgroup and $(r_c)_c$ representatives of the double cosets $\Gamma\backslash G/K$. Every $g\in G$ is $\gamma r_c k$ with $\gamma\in\Gamma$, a class c and $k\in K$. If $r_d^{-1}\gamma r_d\in K$ only for $\gamma=1$, then $r_c^{-1}\gamma r_d\in K$ implies $c=d$ and $\gamma=1$.

Proof of Theorem A(a). $J(T')$ is hermitian and invertible, as the image of $J(T)$. Let $\mathcal{C}$ be the closure in $G(T')$ of the image of $\mathcal{U}(T)$; it is a subgroup. By (U1) for T', with $t=\frac{1}{2}$, it suffices to show $\nu_l(v,\beta)\in\mathcal{C}$ for all $l<\varrho$, all v over $\mathbb{A}_{E,f}$ and all skew $\beta\in\mathbb{A}_{E,f}$. A skew β is $\beta'-\overline{\beta'}$ with $\beta'=\frac{1}{2}\beta$. The map $(v,\beta')\mapsto\nu_l(v,\beta'-\overline{\beta'})$ is continuous by (U2), its entries being polynomials. The pairs with entries in E are dense by (Ad1), and for such a pair $\nu_l(v,\beta'-\overline{\beta'})$ is the image of the Heisenberg element over E (an identity of matrices that the formal text checks entry by entry), which lies in $\mathcal{U}(T)$ by (U1). Hence all $\nu_l(v,\beta)$ lie in $\mathcal{C}$. □

LEAN `besselU_dense`

Proof of Theorem A(b). By Theorem E(a) and (Ad1), (Ad2), $K_g(S)$ is open in G_S^{lat}: integrality of a matrix and of $\hat{g}(S)^{-1}(k-1)$ are open conditions. The map $\zeta\mapsto y^{-1}x^{-1}\operatorname{lb}_S(\zeta)\,y$ from $G_S(\mathbb{A}_{E,f})$ to G_S^{lat} is continuous, so the preimage of $K_g(S)$ is open; it contains $\zeta_0=\operatorname{lb}_S^{-1}(x)$, which lies in $\mathcal{U}(T_S^{\mathrm{rat}}(\mathbb{A}_{E,f}))$ by Lemma 3.4(a). By Theorem A(a) for $T=T_S^{\mathrm{rat}}(E)$ and $T'=T_S^{\mathrm{rat}}(\mathbb{A}_{E,f})$, the point ζ_0 is a limit of images of elements of $\mathcal{U}(T_S^{\mathrm{rat}}(E))$; one of them, u, has its image in the open set, and lb_S of this image is $\mathrm{r}_S^f(u)$. □

LEAN `Filling.exists_besselU_approx`

Lemma 6.1 (characters and the principal level)**.** *Let $S\in\operatorname{Fill}(E,\mathfrak{s})$.*

(a) *Let $u,v\in G_S^{\mathrm{lat}}$ with $u\in\mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ and $v\in\mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$, and $g_0\in G_f(E,\kappa)$ with $u^{-1}(\operatorname{inc}_S(g_0)^{-1}\,v\operatorname{inc}_S(g_0))\in K_g(S)$. Then $\psi_S^{\mathcal{U}}(v)=\psi_S^{\mathcal{U}}(u)$.*

(b) *Let $c\in\operatorname{Cl}(S)$ and $u\in G_S^{\mathrm{lat}}$ with $u\in\mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$. Then there is $u_0\in\mathcal{U}(T_S^{\mathrm{rat}}(E))$ with $u^{-1}\operatorname{pos}_S(c,c;u_0)\in K_g(S)$ and $\psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u_0))=\psi_S^{\mathcal{U}}(u)$.*

Proof. (a) Put $h=\operatorname{inc}_S(g_0)\in\mathcal{H}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ and $k=u^{-1}h^{-1}vh\in K_g(S)$. By Theorem E(a) and (U3), $\hat{g}(S)^{-1}\operatorname{sd}(k)\in\hat{O}_E$, so $\psi_S(\hat{g}(S)^{-1}\operatorname{sd}(k))=1$ by Lemma 3.5(a). All elements involved lie in $\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$, where $\operatorname{sd}(\cdot)$ is additive and vanishes on $\mathcal{H}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ by (G2). Hence $\operatorname{sd}(v)=\operatorname{sd}(h^{-1}vh)=\operatorname{sd}(uk)=\operatorname{sd}(u)+\operatorname{sd}(k)$

and $\psi^{\mathcal{U}}_S(v) = \psi_S(\hat{g}(S)^{-1}\,\mathrm{sd}(u))\,\psi_S(\hat{g}(S)^{-1}\,\mathrm{sd}(k)) = \psi^{\mathcal{U}}_S(u)$. (b) The element $x = h_S(c)\,u\,h_S(c)^{-1}$ lies in $\mathcal{U}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))$ by (G2). Theorem A(b), with $y = h_S(c)$, gives $u_0 \in \mathcal{U}(T^{\mathrm{rat}}_S(E))$ with $h_S(c)^{-1}x^{-1}\mathrm{r}^f_S(u_0)h_S(c) = u^{-1}\mathrm{pos}_S(c,c;u_0) \in K_g(S)$. Now (a) applies to u, $v = \mathrm{r}^f_S(u_0)$ and $g_0 = \dot{c}$. □

LEAN `Filling.unipotentChar_eq_of_mem_principalLevel` · `Filling.exists_rat_approx_unipotentChar_eq`

Proof of Theorem D(a). By (G3) over E, with $t = \frac{1}{2}$, write $\gamma = u\,\iota_N(\eta)$ with $u \in \mathcal{U}(T^{\mathrm{rat}}_S(E))$ and $\eta \in \mathrm{U}_\kappa(E)$. By Lemma 3.4(b),

$$\mathrm{pos}_S(c,d;\gamma) = (h_S(c)^{-1}\mathrm{r}^f_S(u)h_S(c)) \cdot \mathrm{inc}_S(\dot{c}^{-1}\eta\,\dot{d}),$$

the first factor in $\mathcal{U}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))$ by Lemma 3.4(c), the second in $\mathcal{H}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))$. The product lies in $K_g(S)$; by Theorem E(a) and (U3) so does the second factor. By (A1), $\dot{c}^{-1}\eta\,\dot{d} \in K^{\mathrm{col}}(S)$. By (F1) and the injectivity of $\mathrm{U}_\kappa(E) \to G_f(E,\kappa)$, the hypothesis of the second part of (U4) holds for $\Gamma = \mathrm{U}_\kappa(E)$, $K = K^{\mathrm{col}}(S)$; hence $c = d$ and $\eta = 1$. So $\gamma = u$, and $\mathrm{pos}_S(c,c;\gamma) \in K_g(S)$ says $\gamma \in \mathcal{U}_{S,c}$. □

LEAN `Filling.eq_and_mem_classUnip_of_besselS`

Proof of Theorem D(b). Let $s = \mathrm{red}_S(k)$ with $k \in K(S)$, $k \in \mathcal{S}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))$. Then $h_S(c)k \in \mathcal{S}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))$, so by (G3) and Lemma 3.4(b), $h_S(c)k = u\,\mathrm{inc}_S(g_0)$ with $u \in \mathcal{U}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))$ and $g_0 \in G_f(E,\kappa)$. Theorem A(b), with $x = u$ and $y = \mathrm{inc}_S(g_0)$, gives $x_0 \in \mathcal{U}(T^{\mathrm{rat}}_S(E))$ with $k' = \mathrm{inc}_S(g_0)^{-1}u^{-1}\mathrm{r}^f_S(x_0)\mathrm{inc}_S(g_0) \in K_g(S)$. By the first part of (U4), $g_0 = \eta\,\dot{d}\,a$ with $\eta \in \mathrm{U}_\kappa(E)$, a class d and $a \in K^{\mathrm{col}}(S)$. Put $\gamma = x_0\,\iota_N(\eta)$; it lies in $\mathcal{S}(T^{\mathrm{rat}}_S(E))$. Using $\mathrm{r}^f_S(x_0) = u\,\mathrm{inc}_S(g_0)\,k'\,\mathrm{inc}_S(g_0)^{-1}$, $h_S(c)^{-1}u\,\mathrm{inc}_S(g_0) = k$ and $\mathrm{inc}_S(g_0)^{-1}\mathrm{inc}_S(\eta)h_S(d) = \mathrm{inc}_S(a)^{-1}$,

$$\mathrm{pos}_S(c,d;\gamma) = h_S(c)^{-1}\mathrm{r}^f_S(x_0)\,\mathrm{inc}_S(\eta)\,h_S(d) = k\,k'\,\mathrm{inc}_S(a)^{-1}.$$

Here $k' \in K_g(S)$ and $\mathrm{inc}_S(a) \in K_g(S)$ by (A1). So the position lies in $K(S)$ and has the reduction $\mathrm{red}_S(k) = s$. □

LEAN `Filling.exists_besselS_mem_termsOver`

Input (N1), unit 50, neatness; proved below. Let E be a CM field with a CM type and at least two infinite places, Ω a finite set, $w_1, \dots, w_\nu$ distinct finite places of E with $(2\#\Omega)^{[E:\mathbb{Q}]} < \prod_j q_{w_j}$, and $K \subseteq K_f(E,\Omega)$ a subgroup of elements that reduce to 1 modulo every w_j. Then $\Gamma_f(E,\Omega) \cap gKg^{-1} = 1$ for every $g \in G_f(E,\Omega)$. This is the statement of the background library. The catalogue of the write-up names Serre [3], appendix, for the method (place not confirmed); by the second reader's verdict Serre's statement, on automorphisms of finite order that are congruent to 1 modulo an integer at least 3, is a different one, and the relation is unresolved. No reference is needed: the statement has the short proof that follows, which is the argument of the formal proof (a trace, its norm, the eigenvalues at the infinite places).

Proof of (N1). Let $\gamma \in \Gamma_f(E,\Omega) \cap gKg^{-1}$ be the image of $\gamma_0 \in \mathrm{U}_\Omega(E)$, and put $t = \mathrm{tr}\,\gamma_0 - \#\Omega \in E$. The matrix $g^{-1}\gamma g$ lies in $K \subseteq K_f(E,\Omega)$ and has the trace of γ, so $t \in E \cap \hat{O}_E = O_E$; and $t = \mathrm{tr}(g^{-1}\gamma g - 1)$ lies in the prime ideal of every w_j, since $g^{-1}\gamma g$ reduces to 1 modulo w_j. As the w_j are distinct, t lies in the product of these prime ideals, so $\prod_j q_{w_j} \le |N_{E/\mathbb{Q}}(t)|$ if $t \neq 0$. At an infinite place τ the

complex matrix $(\gamma_0)_\tau$ is unitary, because the embedding of the CM type commutes with the complex conjugations; so its eigenvalues $\lambda_1, \dots, \lambda_{\#\Omega}$ have absolute value 1, and $|t_\tau| = |\sum_i \lambda_i - \#\Omega| \le 2\#\Omega$. Hence $|N_{E/\mathbb{Q}}(t)| = \prod_\tau |t_\tau|^2 \le (2\#\Omega)^{[E:\mathbb{Q}]}$, and the hypothesis forces $t = 0$. Then $\sum_i \operatorname{Re}\lambda_i = \#\Omega$ with $\operatorname{Re}\lambda_i \le 1$, so every λ_i is 1; a unitary matrix all of whose eigenvalues are 1 is the identity, so $(\gamma_0)_\tau = 1$, and $\gamma_0 = 1$. The second infinite place was not used; the formal proof needs it because it passes through a lemma on forms that are definite at the places other than a fixed one. □

Proposition 6.2 (freeness)**.** *Let E be a stably unramified CM field with a CM type, $\mathfrak{s}$ a shape, $S \in \mathrm{Fill}(E, \mathfrak{s})$, Ω a finite set with $\#\Omega \le k(\mathfrak{s})$, and $i \in \{0, 1\}$. Let $\gamma \in \mathrm{U}_\Omega(E)$ and $g_0 \in G_f(E, \Omega)$ be such that $g_0^{-1}\gamma g_0 \in K_f(E, \Omega)$ and that $g_0^{-1}\gamma g_0$ reduces to 1 modulo every $w \in \mathrm{Ch}_i(S)$. Then $\gamma = 1$.*

Proof. Apply (N1) to the $f = f(E)$ places of $\mathrm{Ch}_i(S)$, all of norm $q_i(S)$ by (F4), and the group K of all elements of $K_f(E, \Omega)$ that reduce to 1 modulo them. Since $[E : \mathbb{Q}] = 2f$, the condition is $(2\#\Omega)^2 < q_i(S)$, and $(2\#\Omega)^2 \le 4k^2 \le 4b^2 < b^{10} \le q_i(S)$ by (F3), (F8). There are at least two infinite places, as $f \ge k \ge 1000$ by (F7); the formal statement of (N1) asks for this. The image of γ lies in $\Gamma_f(E, \Omega) \cap g_0 K g_0^{-1}$, so it is 1, and $\mathrm{U}_\Omega(E) \to G_f(E, \Omega)$ is injective. □

LEAN `Filling.eq_one_of_reduceAt_eq_one`

7 Frames of terms

In this section $S \in \mathrm{Fill}(E, \mathfrak{s})$, and N, b, n abbreviate $N(S)$, $b(S)$, $n(\mathfrak{s})$. Three remarks on frames are used below. For $s, s' \in G(T)$ over any ring with involution: $(ss')\mathfrak{f} = s\,(s'\mathfrak{f})$; for $u \in \mathcal{U}^0(T)$, $(\gamma u)\mathfrak{f} = \gamma\mathfrak{f}$; and $(s^{-1}s')\mathfrak{f} = \mathfrak{f}$ if and only if $s\mathfrak{f} = s'\mathfrak{f}$. Also $s \in \mathcal{S}(T)$ if and only if $(s\mathfrak{f})_{i,a} = \mathfrak{f}_{i,a}$ for all frame indices a and all $i \in I(\mathfrak{s})$ with $\mathrm{lev}(i) \ge a$, which is the definition in Setting 2.1.

LEAN `Filling.frameImage_mul` · `Filling.frameImage_mul_of_mem_besselU0` · `Filling.frameImage_inv_mul_eq_iff` · `Filling.mem_besselS_iff_frameImage`

Lemma 7.1 (a common vector of two frames)**.** *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$, $\gamma \in G_S(E)$ and $x, y \in E^{\varrho+1}$ with $\mathfrak{f}x = (\gamma\mathfrak{f})y$. Then $\overline{x_\varrho}\,x_\varrho = \overline{y_\varrho}\,y_\varrho$, and $Z_S(\gamma)\,y$ is the vector with the entry $-N(S)\,x_\varrho$ at the index ϱ and 0 elsewhere. Moreover, if $v \in E^{M(\mathfrak{s})}$ has $v_z = 0$ and $v^*\,T_S^{\mathrm{rat}}(E)\,v = 0$, then $v = 0$.*

Proof. For the form h of $T_S^{\mathrm{rat}}(E)$, $h(\mathfrak{f}x, \mathfrak{f}x) = -N\overline{x_\varrho}x_\varrho$, the e_a being isotropic and orthogonal to z; and $h((\gamma\mathfrak{f})y, (\gamma\mathfrak{f})y) = h(\mathfrak{f}y, \mathfrak{f}y)$ by (G6). As $N \ne 0$, the first claim follows. The entry of index a of $Z_S(\gamma)\,y$ is $h(\mathfrak{f}_a, (\gamma\mathfrak{f})y) = h(\mathfrak{f}_a, \mathfrak{f}x)$, which is 0 for $a < \varrho$ and $-Nx_\varrho$ for $a = \varrho$. In the last claim $\sum_{i\in\kappa}|(v_i)_\tau|^2 = 0$ at any infinite place. □

LEAN `Filling.star_mul_last_eq_of_frameImage_eq` · `Filling.frameProducts_mulVec_of_frameImage_eq` · `Filling.eq_zero_of_ratMiddle_form_eq_zero`

Proposition 7.2 (the norm identity)**.** *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$, $\gamma \in G_S(E)$, $y \in E^{\varrho+1}$, $D \in E$ and $t \in E^{I(\mathfrak{s})}$ such that $Z_S(\gamma)\,y$ is the vector with the entry D at the index ϱ and 0 elsewhere, and $(\gamma\mathfrak{f})y = t$. Then the e^*-coordinates of t vanish, $-N(S)\,t_z = D$, and for every infinite place τ,*

$$N(S)\sum_{i\in\kappa}|(t_{x_i})_\tau|^2 = |D_\tau|^2 - \left(N(S)\,|(y_\varrho)_\tau|\right)^2;$$

if this number is ≤ 0 for some τ, then $t = \mathfrak{f}x$ for some $x \in E^{\varrho+1}$.

Proof. The e_a^*-coordinate of t is $h(e_a, t) = (Z_S(\gamma)y)_a = 0$, and $h(z,t) = -Nt_z$ is the entry of index ϱ, which is D. As t has no e^*-part, $h(t,t) = \sum_i \overline{t_{x_i}} t_{x_i} - N\overline{t_z} t_z$; on the other hand $h(t,t) = h(\mathfrak{f}y, \mathfrak{f}y) = -N\overline{y_\varrho} y_\varrho$ by (G6). Substituting $t_z = -D/N$, multiplying by N and applying the embedding at τ gives the identity. If the left side is ≤ 0 at one place, it is 0 there, so all t_{x_i} vanish and t is a combination of the e_a and z. □

LEAN `Filling.norm_sq_middle_part_eq`

Corollary 7.3. *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$, $\gamma \in G_S(E)$ and τ an infinite place of E.*

(a) *$\mathfrak{g}_S(\gamma)_\tau$ is a non-negative real number: $\mathfrak{g}_S(\gamma)_\tau = |\mathfrak{g}_S(\gamma)_\tau|$.*

(b) *If $\mathfrak{g}_S(\gamma) \neq 0$, then $\gamma \in \mathrm{Lv}_{\varrho+1}(S)$.*

(c) $\mathfrak{g}^{\mathbb{R}}(Z(\mathrm{r}_S^\infty(\gamma)_\tau)) = \left(\prod_{a \le \varrho} \mathrm{fs}_S(\tau, a)\right)^2 \cdot |\mathfrak{g}_S(\gamma)_\tau|$.

Proof. (a) Let $Z = Z_S(\gamma)$ and let y be the last column of the adjugate of Z, so that Zy has the entry $\det Z$ at ϱ and 0 elsewhere, and $y_\varrho = \mathrm{D}_\varrho(Z)$. Proposition 7.2 with $D = \det Z$ shows that $\mathfrak{g}_S(\gamma)_\tau = |\det Z_\tau|^2 - N^2 |\mathrm{D}_\varrho(Z)_\tau|^2$ is N times a sum of squares. (b) Normal form from $\varrho + 1$ is an empty condition. If $\det Z = 0$, then $\mathfrak{g}_S(\gamma)_\tau \le 0$, hence $= 0$ by (a), against the hypothesis; so $\mathrm{D}_{\varrho+1}(Z) \neq 0$. (c) By Lemma 3.2(b) the determinant of $Z(\mathrm{r}_S^\infty(\gamma)_\tau)$ is $\prod_{a \le \varrho} \mathrm{fs}_S(\tau, a) \cdot \det Z_\tau$ and its leading minor of order ϱ is $\prod_{a < \varrho} \mathrm{fs}_S(\tau, a) \cdot \mathrm{D}_\varrho(Z)_\tau$; and $\mathrm{fs}_S(\tau, \varrho) = 1/N$. With (a) this gives the formula. □

LEAN `Filling.map_termDisc_eq_norm` · `Filling.hasLevel_last` · `Filling.gramDiscReal_realFrameGram_ratToInf`

Inputs (the real place), unit 132. For $x, y \in G_1(\mathfrak{s})$, frame indices a, b, and $t \in G_1(\mathfrak{s})$: (R1) $|Z(x)_{ab}| \le \|y^{-1}\mathfrak{f}_a\| \cdot \|(x^{-1}y)^{-1}\mathfrak{f}_b\|$; (R2) $\mathrm{bs}(ty) = \mathrm{bs}(y)$ if $t \in \mathcal{S}(T^{\mathrm{rat}}(\mathfrak{s}; 1))$; (R3) $(ty)^{-1}\mathfrak{f}_a = y^{-1}\mathfrak{f}_a$ if t lies in the group generated by $\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s}; 1))$ and $\mathcal{H}(T^{\mathrm{rat}}(\mathfrak{s}; 1))$.

Lemma 7.4 (near terms and the archimedean coordinates)**.** *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$, $\gamma, u, u' \in G_S(E)$ and τ an infinite place of E.*

(a) *If $u \in \mathcal{U}^0(T_S^{\mathrm{rat}}(E))$, then $\mathrm{r}_S^\infty(u)_\tau \in \mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s}; 1))$.*

(b) *Let $y, t \in G_\infty(E, \mathfrak{s})$ with $t_{\tau'} \in \mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s}; 1))$ for every infinite place τ'. Then $ty \in \mathrm{BxP}(S)$ if and only if $y \in \mathrm{BxP}(S)$.*

(c) *If $\gamma \in \mathrm{Near}(S)$, $u \in \mathcal{U}^0(T_S^{\mathrm{rat}}(E))$ and $u' \in \mathcal{U}^0(T_S^{\mathrm{rat}}(E))$, then $u\gamma u' \in \mathrm{Near}(S)$.*

(d) *For a natural number j: if $\gamma \in \mathrm{NF}_j$, then $\mathrm{r}_S^\infty(\gamma)_\tau \in \mathrm{NF}_j$.*

Proof. (a) Conjugation by the diagonal matrix $\mathrm{Sc}_{S,\tau}$ and the embedding at τ preserve the entries equal to 0 or 1 that define $\mathcal{U}^0(\cdot)$. (b) By (R2), (R3), since $\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s}; 1)) \subseteq \mathcal{S}(T^{\mathrm{rat}}(\mathfrak{s}; 1))$. (c) If y and $\mathrm{r}_S^\infty(\gamma)^{-1}y$ lie in $\mathrm{BxP}(S)$, then so do $\mathrm{r}_S^\infty(u)y$ and $\mathrm{r}_S^\infty(u\gamma u')^{-1}\mathrm{r}_S^\infty(u)y = \mathrm{r}_S^\infty(u')^{-1}\mathrm{r}_S^\infty(\gamma)^{-1}y$, by (a) and (b). (d) Let c be the constant of the normal form; then $|c_\tau|^2 = 1$. For a frame index $a \ge j$ and x, y with $\mathfrak{f}x = (\gamma\mathfrak{f})y$, put $d_b = \mathrm{sc}_{S,\tau}(\mathfrak{f}_a)/\mathrm{sc}_{S,\tau}(\mathfrak{f}_b)$ and $x'_b = d_b(x_b)_\tau$, $y'_b = d_b(y_b)_\tau$. Applying the embedding and the diagonal matrix $\mathrm{sc}_{S,\tau}(\mathfrak{f}_a)\,\mathrm{Sc}_{S,\tau}^{-1}$ to the equation gives $\mathfrak{f}x' = (\mathrm{r}_S^\infty(\gamma)_\tau\mathfrak{f})y'$; the supports are unchanged, and $x'_a = c_\tau$, $y'_a = 1$. □

LEAN `Filling.ratToInf_mem_besselU0` · `Filling.mul_mem_boxedPoints_iff` · `Filling.IsNear.mul_of_mem_besselU0` · `IsNormalForm.ratToInf`

Lemma 7.5 (terms in the lattice basis)**.** *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$, $c, c' \in \mathrm{Cl}(S)$, $s \in \mathsf{G}(S)$ and $\gamma \in G_S(E)$.*

(a) $\mathrm{C}_S(c)^*\ J(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))\ \mathrm{C}_S(c) = J(T_S^{\mathrm{rat}}(\mathbb{A}_{E,f}))$.

(b) $\mathrm{pos}_S(c, c'; \gamma)\, \mathfrak{f} = \mathrm{C}_S(c) \cdot (\gamma \mathfrak{f})_{\mathbb{A}_f}$, *and* $Z_{T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})}(\mathrm{pos}_S(c, c'; \gamma)) = (Z_S(\gamma))_{\mathbb{A}_f}$.

(c) *If $\gamma \in \mathrm{Tm}_S(c, s, c')$, then $\mathrm{pos}_S(c, c'; \gamma)$ has entries in $\hat{O}_E$, and for all frame indices a, b the entry $Z_S(\gamma)_{ab}$ is an algebraic integer.*

Proof. (a) By Lemma 2.3(a) and (F2), $\Lambda(S)^*\, J(T_S^{\mathrm{rat}}(\mathbb{A}_{E,f}))\, \Lambda(S) = J(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$, and $\iota_N(\dot{c})$ is an isometry of $T_S^{\mathrm{rat}}(\mathbb{A}_{E,f})$; so $\mathrm{B}_S(c)^* J(T_S^{\mathrm{rat}}(\mathbb{A}_{E,f}))\mathrm{B}_S(c) = J(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$, which is (a) for the inverse matrix. (b) By Lemma 3.4(c), $\mathrm{pos}_S(c, c'; \gamma) = \mathrm{C}_S(c)\, \gamma\, \mathrm{B}_S(c')$, and $\mathrm{B}_S(c')$ fixes the frame vectors: $\iota_N(\dot{c}')$ acts on the x_i only and $\Lambda(S)$ fixes the e_a and z. This is the first formula. By (G4) and (a),

$$\begin{aligned} Z_{T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})}(\mathrm{pos}_S(c, c'; \gamma)) &= \mathfrak{f}^*\, J(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))\, \mathrm{C}_S(c)\, \gamma \mathfrak{f} \\ &= (\mathrm{B}_S(c)\mathfrak{f})^*\, J(T_S^{\mathrm{rat}}(\mathbb{A}_{E,f}))\, \gamma \mathfrak{f} = \mathfrak{f}^* J(T_S^{\mathrm{rat}}(\mathbb{A}_{E,f}))\gamma \mathfrak{f}. \end{aligned}$$

(c) The position lies in $K(S)$. By (b) the matrix $(Z_S(\gamma))_{\mathbb{A}_f}$ is a block of the product of the integral matrices $J(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ and $\mathrm{pos}_S(c, c'; \gamma)$; an element of E that is integral at all finite places is an algebraic integer. □

LEAN `Filling.classCoord_conj_splitGram` · `Filling.frameImage_termPosition` · `Filling.frameGram_termPosition` · `Filling.termsOver.isIntMatrix_termPosition` · `Filling.isIntegral_frameProducts`

Proof of Theorem D(c). The element $\delta = \gamma^{-1} u$ fixes the frame, so it lies in $\mathcal{S}(T_S^{\mathrm{rat}}(E))$. By Lemma 3.4(d) it is a term over $(c', s^{-1}s, d) = (c', 1, d)$, that is, $\mathrm{pos}_S(c', d; \delta)$ lies in $K_g(S)$. Theorem D(a) gives $c' = d$ and $\delta \in \mathcal{U}_{S,c'}$; as δ fixes the frame, $\delta \in \mathcal{U}^0_{S,c'}$. □

LEAN `Filling.termsOver.eq_and_mem_classUnip₀`

Lemma 7.6 (sizes)**.** *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$ and e a natural number.*

(a) *If $e \le 10^5\, n(\mathfrak{s})^9\, f(E)$, then $b(S)^e \le \mathrm{tl}(S)$.*

(b) $\mathrm{tl}(S)^4 = b(S)^{k(\mathfrak{s})\, f(E)}$.

(c) $2 \le g(S)$; $2 \le \mathrm{tl}(S)$; $1000 \le b(S)$; $0 \le \mathsf{d}_1(S)$; $0 \le \mathsf{d}(S)$; $0 < N(S)$; $0 < \nu(S)$; *and* $\overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{S}$ *is not empty. (In the formal text: a statement p holds as soon as it follows from these facts together with $2 \le b(S)$, $0 < b(S)$ and $1 \le b(S)$.)*

Proof. (a) $\mathrm{tl}(S) = b^{kf/4}$ with $b \ge 1000 > 1$, and $e \le 10^5 n^9 f \le 1000(2n)^9 f/4 \le kf/4$ by (F7). (b) is the definition. (c) g is a product of two primes; $\mathrm{tl}(S) \ge b \ge 2$ by (a) with $e = 1$; $\mathsf{d}(S)$ is a square divided by the product of the positive numbers $\deg(S)$ and $\mathrm{mass}(E, r)$ (the mass is positive: background, unit 99); N is a prime; $\nu(S) > 0$ by Lemma 3.1; and the reduced image is a group. □

LEAN `Filling.pow_le_tailScale` · `Filling.tailScale_pow_four` · `Filling.of_size_bounds`

Proposition 7.7 (frame products of near terms are small)**.** *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$ be admissible filling data, $\gamma \in G_S(E)$ with $\gamma \in \mathrm{Near}(S)$, τ an infinite place of E, $a, b \le \varrho$ frame indices and $l \le \varrho + 1$. Then, with $n = n(\mathfrak{s})$,*

$$|(Z_S(\gamma)_{ab})_\tau| \le b(S)^{5n^2} \qquad \text{and} \qquad |\mathrm{D}_l(Z_S(\gamma))_\tau|^2 \le b(S)^{11n^3}.$$

Proof. Let $y \in \mathrm{BxP}(S)$ with $y' = \mathrm{r}_S^\infty(\gamma)^{-1} y \in \mathrm{BxP}(S)$. By (R1) for $x = \mathrm{r}_S^\infty(\gamma)_\tau$, $|Z(\mathrm{r}_S^\infty(\gamma)_\tau)_{ab}| \le \|y_\tau^{-1}\mathfrak{f}_a\| \, \|y_\tau'^{-1}\mathfrak{f}_b\| \le \mathrm{bx}(S)^2 = b^{4n^2}$. By Lemma 3.2(b) the left side is $|\kappa_\tau(S)|^{2\varrho-a-b} N^{-1} |(Z_S(\gamma)_{ab})_\tau|$, and $|\kappa_\tau(S)| \ge 1$, $N \le b^{60n}$ by (A2), (A3). So $|(Z_S(\gamma)_{ab})_\tau| \le b^{60n+4n^2} \le b^{5n^2}$, since $n \ge 10^5$ by (F7). A determinant of size l with entries of absolute value at most $B = b^{5n^2}$ is at most $l!\, B^l \le (lB)^l$ in absolute value, and $l \le n \le b$ by (F8); so its square is at most $(b^{1+5n^2})^{2n} = b^{2n+10n^3} \le b^{11n^3}$. □

LEAN `Filling.IsNear.norm_frameProducts_le` · `Filling.IsNear.norm_map_leadingMinor_le`

8 Notes on the formal text

Arguments as the census reads them. The census of the artifact reads the arguments of a declaration from its text, without running Lean, and for some declarations of this note the reading is evidently not Lean's signature: a variable of the file is counted because the declaration binds a variable of the same name, or because a name such as that of $b(S)$ contains its letter; or the reading stops at a definition inside a statement. The reader of the unit recorded these cases, with Lean's presumable signature and the evidence for it; nothing was compiled. The definitions concerned are written here with Lean's arguments: the reduction red_S and the maps $\mathrm{r}_S^{\mathbb{A}}$, a_S^∞, a_S^f (the census reads the source of the homomorphism as an argument); the scales $\mathrm{sc}_{S,\tau}(i)$, the numbers $\mathrm{fs}(S)$, $\nu(S)$, $\deg(S)$; the sets $\mathrm{Tm}_S(c,s,c')$, $\mathrm{Tm}_S^{\mathrm{lk}}(\Lambda)$, $\mathrm{FBox}(\rho)$ and the properties $\omega \in \mathrm{BP}(S,c)$, $\Xi \in \mathrm{Stab}(S)$, $\gamma \in \mathrm{NF}_j$, S is admissible (a bound variable is counted); the notions $\mathrm{J}(\mu, W; \gamma)$, $\mathrm{Sch}(W) \le D$, $\mathrm{SchB}(W, \mathrm{rep}; B, D)$, $\mathrm{CP}(b, f, \mathrm{rep}, W)$, $\mathrm{BSys}(\mathfrak{s}, b, f)$ (variables of the file are counted, for a Bessel system the components I, W, deg of the record itself); the matrix $\mathrm{m}_H(V)$ (the hypothesis $LH = 1$ is missed). The Bessel function $W_S^\sigma(y)_i$, the kernel $\mathcal{W}_S^\sigma(\gamma; y)$, the amplitudes and the amplitude sum take the side as an ambient argument in Lean, which the census does not list; the notation shows it. The notations $\mathrm{B}_S(c)$, $\mathrm{Fix}_{S,w}(d)$, $\mathrm{Fix}_S(\Sigma, d)$ show the data S, which Lean takes implicitly. For theorems the same happens in the statements of Propositions 2.11, 5.2, 7.2, 7.7, of Lemmas 3.2(b), 3.3, 5.1, 7.4(b), 7.6 and of Corollary 4.3; in each case the statement here has Lean's arguments, as far as the text shows them, and the account of the census's extra arguments points to the words of the statement that name the bound variable. In Proposition 4.4(a) the census misses the sequence p, which the statement uses. For Proposition 7.7 the census reads the proof of the first bound as part of its statement, and lists as an argument a hypothesis (γ is a term) that the statement does not have; the reader gave the argument in a note, and the proof here follows the Lean text.

Where the formal text differs from the statements above. (1) Theorem C(b) is, in the formal text, a sum over all subspaces of a term that is 0 unless the dimension condition holds, in $[0, \infty]$. (2) The conclusion of Proposition 2.10 is stated there for the function $x \mapsto \max(\mathrm{Re}\, F(x), 0)$ with values in $[0, \infty]$; it equals F by the first assertion. Its hypothesis on the common entry of index i is used only for $i = n$. (3) Lemma 3.3(b) gives the cutoff in both forms, as a sum over K and as an integral for the counting measure of Γ. (4) The cutoff $\beta_{S,c}$ is defined by two choices; the second is a choice among the functions ω with $\omega \in \mathrm{BP}(S,c)$ and is arbitrary when there is none. Nothing in this note depends on it; Note B proves that such functions exist under the standing hypotheses when E is stably unramified. (5) The record

of filling data lists some consequences of its own conditions: $1 \le b$ beside $1000 \le b$, and $1000(2\varrho+2) \le k$ beside $1000(2\varrho+2)^9 \le k$. (6) The comment of the formal text on $c_*(S)$ gives the second formula of the remark after Definition 2.22; the definition is the first. (7) Lemma 7.6(c) is stated in the formal text as a rule: a proposition holds if it follows from the listed facts. (8) In Lemma 3.6(a) and in the facts that $T_S^{\mathrm{rat}}(E)$ and $T_S^{\mathrm{lat}}(F)$ are hermitian and invertible, that $\frac{1}{2} \in \mathbb{A}_{E,f}$ has trace 1, that $\mathrm{Cl}(S)$ is finite and $G_S(E)$ countable, the formal text has instances without names; they are not in the concordance. (9) Theorem B(b) is the precise form of "the link is Lagrangian"; the dimension of the link is computed in Note B. (10) The Inputs (U1), (La2), (La4) and (M1) are stated here in the special cases that are used: (U1) combines an induction principle over the floors with the description of the elements of one floor; (La4) is the case of the formal lemma in which its additional block is absent.

What this note takes for granted. From the background library: the groups of Setting 2.1 and the Inputs (G1)–(G6), (E1), (L1), (L2), (U1)–(U4) (units 1, 29, 112, 119, 120); adeles, the standard characters and cutoffs (Ad1)–(Ad4) (units 114, 122, 123); lattices and indices (La1)–(La4) (units 57, 120); Minkowski's second theorem, Gram–Schmidt, the first minimum and a packing bound (M1)–(M4) (units 114, 120, 121); the positivity of the mass (unit 99); and neatness (N1) (unit 50), which is not taken for granted any more: since the revision after the referee's report a proof follows its statement. From the notes on the real place (unit 132): (R1)–(R3). The notions of Bessel function and of the measure $\mu_{\mathcal{U}\backslash G}$ are background (units 117, 118).

How this note was written. From the dossier of the unit: the texts of the members of the nine files treated here and their structured proofs. I read the nine Lean files completely, and compared every statement of this note with the Lean statement, argument by argument; for the statements that contain a local definition I read the full Lean text and not the census's span. For Proposition 7.7, which has no structured proof, I read the proof in the Lean text. The proofs follow the formal ones in their structure. Written out here where the formal text calls a decision procedure or a simplifier: the verification that the Eichler map of Proposition 2.11 is unitriangular; the inverse of the lattice form in Lemma 2.3; the bookkeeping of Lemma 3.9; the exponents of Proposition 7.7 and of Proposition 6.2. In the proof of Theorem C(a) I use that the index in Jacobi's identity is 1, so that the discriminants of a subspace and of its orthogonal complement agree; the formal text uses only that it is at least 1. Identities between explicit matrices that the formal text checks entry by entry (the image of a rational Heisenberg element, the matrix of the lattice basis and its inverse) are stated and not recomputed. Of the Inputs I read the Lean statements of (La2), (La4), (M1), (M4), (N1), (L1), (R1)–(R3), (G3) and (U3) and the dossier's readings of the others; I did not open their proofs. The references are those of the second reader's verdicts and of the catalogue of the write-up; I did not open the cited works. The program check of the unit's dossier covered 1371 leaf steps of 179 proofs.

Appendix A. Concordance with the Lean artifact

The formal names of the objects are these. The fields `S.N`, `S.levelPrime i`, `S.piBar`, `S.ϖ`, `S.b`, `S.chosenPlaces i`, `S.columnLevel`, `S.besselSys v` are $N(S)$, $q_i(S)$, $\bar{\pi}(S)$, $\varpi(S)$, $b(S)$, $\mathrm{Ch}_i(S)$, $K^{\mathrm{col}}(S)$, $\mathfrak{B}_\tau(S)$; the Lean variable `v` for an infinite place is τ and `𝔓 v x` is x_τ; `besselU`, `besselU0`, `besselH`, `besselS`, `GT l` are $\mathcal{U}(T)$, $\mathcal{U}^0(T)$, $\mathcal{H}(T)$, $\mathcal{S}(T)$, $\mathcal{H}_l(T)$; `frameGram`, `realFrameGram`, `leadingMinor`, `simpleRootSum`, `frameImage` are $Z_T(\gamma)$, $Z(y)$, $\mathrm{D}_l(Z)$, $\mathrm{sd}(u)$, $\gamma\mathfrak{f}$; `genIndex` is $[\cdot]$; `toBesselH` is ι_N; a set `T` of rational elements is Ξ, the subspace `T` of Siegel's lemma is V, and the finite index sets `τ`, `σ` are Ω. Four statements as the artifact has them:

```
lemma Filling.siegel_lemma {E} [Field E] [CMType E] [StablyUnram E] {κ} [Fintype κ] υ {T : Submodule
  E _} {s} (hs : finrank E T + s = card κ) : ∃ X : Matrix (Fin s) κ E, Xᵀᴴ.mulVecLin.ker = T ∧
  (∀ i, X i ∈ ((stdSpace ..).move (1, υ)).baseLattice ∧ X i ≠ 0) ∧
  (∏ i, ‖((stdSpace ..).move (1, υ)).archCoords (X i)‖) ^ (2 * numInfPlaces E) ≤
  (8 * (numInfPlaces E * s) * rootDiscr E) ^ (numInfPlaces E * s) * sublatticeDisc υ.1.3 T

lemma mem_link_iff_orthogonal {γ : S.G E} {p} : p ∈ link γ ↔ ∀ q ∈ link γ,
  star q ⬝ᵥ S.linkForm *ᵥ p = 0

lemma Filling.exists_besselU_approx {x} (hx : x ∈ besselU) γ : ∃ υ ∈ besselU,
  γ⁻¹ * (x⁻¹ * S.ratToFin υ) * γ ∈
  S.principalLevel

lemma Filling.eq_and_mem_classUnip_of_besselS [ok : S.IsAdmissible] {c d γ} (h : γ ∈ besselS) (o :
  S.termPosition c d γ ∈ S.principalLevel) : c = d ∧ γ ∈ S.classUnip c
```

In this note	Lean declaration	File	Lines
Definition 2.12	`Shape.openCellConst`	BesselSystem.lean	18–18
Definition 2.12	`Shape.openCellConst_pos`	BesselSystem.lean	20–20
Definition 2.12	`Shape.bandConst`	BesselSystem.lean	24–24
Definition 2.12	`besselSize`	BesselSystem.lean	29–29
Definition 2.12	`InCut`	BesselSystem.lean	33–33
Definition 2.12	`frameBox`	BesselSystem.lean	36–36
Definition 2.12	`gramDiscReal`	BesselSystem.lean	40–40
Definition 2.12	`cellMass`	BesselSystem.lean	44–45
Definition 2.13	`besselCorrelation`	BesselSystem.lean	49–50
Definition 2.13	`SchurLe`	BesselSystem.lean	54–55
Definition 2.13	`HasSchurBound`	BesselSystem.lean	59–62
Definition 2.13	`normCorrelation`	BesselSystem.lean	67–69
Definition 2.13	`HasBandLimits`	BesselSystem.lean	74–77
Definition 2.14	`Side`	BesselSystem.lean	80–80
Definition 2.14	`IsCutPair`	BesselSystem.lean	85–96
Definition 2.14	`BesselSystem`	BesselSystem.lean	102–109
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Definition 2.25	`Filling.vectorFixer`	CrudeBoundDefs.lean	27–29
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Definition 2.24	`Filling.classCoord`	CrudeBoundDefs.lean	76–76
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In this note	Lean declaration	File	Lines
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Definition 2.27	`Filling.termDiscNorm`	CrudeBoundDefs.lean	102–102
Definition 2.27	`Filling.boxedPoints`	CrudeBoundDefs.lean	106–107
Definition 2.27	`Filling.IsNear`	CrudeBoundDefs.lean	111–111
Definition 2.27	`Filling.CharsAgree`	CrudeBoundDefs.lean	115–116
Definition 2.2	`latMiddleGram`	EichlerFlag.lean	17–19
Definition 2.2	`middleBaseChange`	EichlerFlag.lean	29–29
Definition 2.2	`besselLatticeBasis`	EichlerFlag.lean	33–37
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Definition 2.5	`framePair`	EichlerFlag.lean	52–52
Definition 2.5	`isotropicFramePair`	EichlerFlag.lean	55–55
Definition 2.5	`linkMatrix`	EichlerFlag.lean	59–59
Definition 2.5	`link`	EichlerFlag.lean	63–63
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Definition 2.5	`frameImage_mulVec`	EichlerFlag.lean	155–156
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Notation

E, O_E, $\overline{x}$ a CM field, its integers, complex conjugation or the involution of a ring

τ, x_τ an infinite place of E; the image of $x \in E$ under the embedding of the CM type

w, $\overline{w}$, $\mathbb{F}_w$, q_w a finite place, its conjugate, the residue field, its number of elements

$\mathbb{A}_E$, $\mathbb{A}_{E,f}$, $\hat{O}_E$ adeles, finite adeles, integral finite adeles

$f(E)$, d_E, rd_E, $\mathrm{vol}(\hat{O}_E)$ number of infinite places, discriminant, root discriminant, $2^f/\sqrt{|d_E|}$

ψ_f, ψ_E, $\mathbf{e}$ standard additive characters; $\mathbf{e}(c) = e^{c-\overline{c}}$

π, ε_w, ϵ the number π; the idempotent of $\hat{O}_E$ at w; a truth value, or the factor of Definition 2.14

$[a]$ generalized index of a unit a

$\mathfrak{s} = (\varrho, \kappa, k_0)$ a shape; $k(\mathfrak{s})$, $n(\mathfrak{s})$, $r(\mathfrak{s})$, $M(\mathfrak{s})$, $I(\mathfrak{s})$: Setting 2.1

e_a, x_i, z, e_a^*; $\mathfrak{f}_a$, $\mathfrak{f}$ the basis; the frame

lev, fl level and floor of a basis vector

R, R', F, A, C, K_0 rings; K_0 a field

T, T', $T^{\mathrm{rat}}(\mathfrak{s}; N)$, $J(T)$, h_T, h forms, the diagonal form, the split Gram matrix, the hermitian form

$G(T)$, $G_1(\mathfrak{s})$, $G_\infty(E, \mathfrak{s})$ unitary groups; the product over the infinite places

$\mathcal{U}(T)$, $\mathcal{U}^0(T)$, $\mathcal{H}(T)$, $\mathcal{H}_l(T)$, $\mathcal{S}(T)$ unipotent group, its frame-fixing part, the groups $\mathcal{H}$, the Bessel subgroup

$Z_T(\gamma)$, $Z(y)$, Z, $\mathrm{D}_l(Z)$ frame Gram matrices; leading principal minors

$\mathrm{sd}(s)$, $w_l(p)$, ι_N, $\nu_l(v,\beta)$ sum of the superdiagonal frame entries, Weyl element, embedding of U_κ, Heisenberg element

M^*, M^{t}, $(M)_{\mathbb{A}_f}$ conjugate transpose, transpose, the same matrix over $\mathbb{A}_{E,f}$

$T^{\mathrm{lat}}(\mathfrak{s};N)$, $\Lambda^\circ(\mathfrak{s};p)$, $\Lambda(\mathfrak{s};p)$ lattice form, base change, lattice basis: Definition 2.2

$\mathfrak{g}(N;Z)$, $\mathfrak{g}^{\mathbb{R}}(Z)$ frame discriminants: Definitions 2.4, 2.12

$[\mathfrak{f} \mid \gamma\mathfrak{f}]$, $[\mathfrak{e} \mid \gamma\mathfrak{e}]$, $\mathrm{L}(\gamma)$, $\mathrm{link}(\gamma)$ frame pairs, link matrix, link: Definition 2.5

X, t_{mid} the span of the e_a; middle part of a vector

$\mathrm{dp}_k(r)$, $\Lambda^{\mathbb{C}}_K(i;p)$, $\Lambda_K(i;p)$ damped power, head weights: Definition 2.6

$\mathrm{Sieve}(u,\iota,n,J,\psi,En,\Lambda,\beta)$ sieve system: Definition 2.7; U, ξ, Ω, $u_{\mathbb{C}}$

$c_G(\mathfrak{s})$, $c_{\mathrm{band}}(\mathfrak{s})$, $\mathrm{bs}(y)$, $y \in \mathrm{Cut}(b)$, $\mathrm{FBox}(\rho)$, $\mathrm{cm}(y;\deg,\rho,d)$ Definition 2.12

$\mathrm{J}(\mu,W;\gamma)$, $\mathrm{Sch}(W) \le D$, $\mathrm{SchB}(W,\mathrm{rep};B,D)$, $\mathrm{J}^\circ(y;W,p)$, $\mathrm{BL}(b,W,\deg,J)$ Definition 2.13; φ, V_w, X, B, D

σ, full, cut; $\mathrm{CP}(b,f,\mathrm{rep},W)$, $\mathrm{BSys}(\mathfrak{s},b,f)$ sides; cut pairs, Bessel systems: Definition 2.14; m, ϵ, P, $\mathfrak{B}$, Φ, g

$\mu_{\mathcal{U}\backslash G}$, vol the measure for $\mathcal{U}\backslash G_1$; volumes

$S \in \mathrm{Fill}(E,\mathfrak{s})$ filling data: Definition 2.15; $N(S)$, $q_i(S)$, $\bar{\pi}(S)$, $\varpi(S)$, $b(S)$, $\mathrm{Ch}_i(S)$, $K^{\mathrm{col}}(S)$, $\mathfrak{B}_\tau(S)$

$I_\tau(S)$, $W^\sigma_{S,\tau}$, $\deg_\tau(S)$, $\mathrm{rep}_{S,\tau}$ the components of the Bessel system at τ

$G_f(E,\kappa)$, $K_f(E,\kappa)$, $\Gamma_f(E,\kappa)$, U_κ finite adelic unitary group, its integral subgroup, rational points

$g(S)$, $T^{\mathrm{rat}}_S(R)$, $T^{\mathrm{lat}}_S(R)$, $G_S(R)$, G^{lat}_S, $\mathsf{G}(S)$, $\mathrm{Bc}(S)$, $\mathrm{Cl}(S)$, $\dot{c}$, $Z_S(\gamma)$, $\mathfrak{g}_S(\gamma)$ Definition 2.16

$K(S)$, red_S, $K_g(S)$, $\overline{A}^S$, $\hat{g}(S)$ Definition 2.17

$\kappa_\tau(S)$, $\mathrm{sc}_{S,\tau}(i)$, $\mathrm{Sc}_{S,\tau}$, $\mathrm{fs}_S(\tau,a)$, $\mathrm{fs}(S)$ Definition 2.18

$\Lambda(S)$, lb_S, $\mathrm{r}^{\mathbb{A}}_S$, a^∞_S, a^f_S, r^∞_S, r^f_S, inc_S, $h_S(c)$, ψ_S, $\psi^{\mathcal{U}}_S(u)$ Definition 2.19

$\mathrm{pos}_S(c,c';\gamma)$, $\mathrm{Tm}_S(c,s,c')$, $\mathcal{U}_{S,c}$, $\mathcal{U}^0_{S,c}$, $s \in \mathrm{Agr}(S)$, $\Xi \in \mathrm{Stab}(S)$, $\mathrm{Tm}^{\mathrm{lk}}_S(\Lambda)$ Definition 2.20

$W^\sigma_S(y)_i$, $\mathcal{W}^\sigma_S(\gamma;y)$, $\mathrm{bx}(S)$, $\omega \in \mathrm{BP}(S,c)$, $\beta_{S,c}$, $a^\sigma_S(c,c';\gamma)$, $|a|^\sigma_S(c,c';\gamma)$, $A^\sigma_S(\Xi;c,s,c')$ Definition 2.21; α, β^0

$\mathrm{cd}(S)$, $\deg(S)$, $\mathfrak{d}(S)$, $\mathrm{uo}(S)$, $\mathrm{tl}(S)$, $c_*(S)$, $a_*(S)$, $\mathfrak{d}_1(S)$, $\mathrm{cf}(S)$, $\nu(S)$, $\mathrm{mass}(E,r)$ Definition 2.22

S is admissible the standing hypotheses (A1)–(A8): Definition 2.23

$\mathrm{B}_S(c)$, $\mathrm{C}_S(c)$, $\mathrm{C}^\circ_S(c)$ Definition 2.24

adj^ϵ_S, $\mathrm{red}_{S,x}$, $\mathrm{Fix}_{S,w}(d)$, $\mathrm{fix}(s) \ge x$, $\mathrm{Fix}_S(\Sigma,d)$ Definition 2.25

$\mathrm{L}_z(T)$, $\mathrm{link}^\circ(\gamma,\gamma')$, $\mathrm{CF}_a(\gamma;c,c')$, $\gamma \in \mathrm{NF}_j$ Definition 2.26

J_S, $Q^\pm_S$, $\mathrm{cw}_i(S)$, $\gamma \in \mathrm{Lv}_j(S)$, $\gamma \sim_{S;c,c'} \gamma'$, $\|\mathfrak{g}\|_S(\gamma)$, $\mathrm{BxP}(S)$, $\gamma \in \mathrm{Near}(S)$, $\gamma \in \mathrm{ChA}(S)$ Definition 2.27

$\mathrm{m}_H(V)$, $\mathrm{disc}_H(V)$, H, L, V, Y, K, $\mathfrak{n}$, m, w Definition 2.28, Proposition 5.2

$\|X\|$, $|x|$ Euclidean norms over all infinite places; absolute values

c, c', c'', d classes; also ring elements in Definition 2.26; d also a product of units and a real number

$\gamma, \gamma', \delta, \gamma_0$; u, u', u_0; s, t rational elements; unipotent elements; elements of $\mathsf{G}(S)$ or of groups, natural numbers

y, y', x, x' points of $G_\infty(E,\mathfrak{s})$, vectors, group elements

a, b, j, l, i, i'; m, n, k, r, f; e indices; sizes ($n = n(\mathfrak{s})$, $k = k(\mathfrak{s})$, $r = r(\mathfrak{s})$, $f = f(E)$, $b = b(S)$, $g = g(S)$, $N = N(S)$ in proofs); an exponent or a map of indices

p, q; ξ; ρ; ζ, ζ_0; η; ϕ; ν; ς; z sequences of units; a frequency; a radius; points of $G_S(\mathbb{A}_{E,f})$; an element of $\mathrm{U}_\kappa(E)$; a map or a function; an exponent; the scalar of an Eichler map; an adele

σ_0, σ_1; o; v; z_j, B_j, r_τ; x^0, y^0 elements of $\mathcal{H}_\varrho(T)$; an index; a vector or an index; the data of Minkowski's theorem; vectors of Input (L1)

γ_0, t, λ_i, w_j, $N_{E/\mathbb{Q}}$ in the proof of (N1): a rational unitary matrix, its trace minus $\#\Omega$, its eigenvalues at an infinite place, the places of the level, the norm

M, J, O, O_A, P, Q, G, I, U, W, Π, Γ, Λ, Σ, Ξ, $\mathcal{C}$ matrices; a ring of integers; index sets; a subring; groups; a finite set; a space; Bessel functions; a property; discrete groups; a submodule or head bounds; a set of places; a set of rational elements; a closure

References

References

[1] E. Bombieri, J. Vaaler, On Siegel's lemma, Invent. Math. 73 (1983). Theorem 3 (adelic successive minima), as recorded in the second reader's comparison; the formal statement (M1) is a weaker form. The artifact names the same paper for Theorem C(a). Not opened.

[2] V. Platonov, A. Rapinchuk, *Algebraic groups and number theory*, 1994, §7.1. As cited by the artifact for strong approximation in unipotent groups (Theorem A); place not confirmed, not opened.

[3] J.-P. Serre, Rigidité du foncteur de Jacobi d'échelon $n \geq 3$, Séminaire Henri Cartan 13 (1960/61), appendix. As recorded in the catalogue of the write-up for (N1); place not confirmed, relation unresolved by the second reader. Not used: (N1) is proved in Section 6.

READBACK NOTE 42

Box partitions of arithmetic unipotent groups: frame counts, the covolume, amplitude sums

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The filling theorem of the artifact (Theorem B of the write-up, "enough forms") rests on a second moment of Poincaré series along a Bessel subgroup of a unitary group. Unfolded, this moment is a sum of *amplitudes* over rational elements, weighted by cutoffs for the arithmetic unipotent groups $\mathcal{U}_{S,c}$. This note treats the three files that build the first layer of estimates: the congruence that agreeing unipotent characters force on pairings of frames (Theorem A); cutoffs supported in boxes and the resulting count of frames (Theorem B); the exact covolume of $\mathcal{U}_{S,c}$ (Theorem C); and the amplitude sums themselves: convergence, comparison of full and cut Bessel functions, change of weight, vanishing for characters that do not agree, and the identity term (Theorems D, E, F).

What this note is. The artifact grh-compact is a body of Lean code whose final theorems state the Riemann hypothesis and its analogue for Dirichlet L-functions. That is a claim: its authors report that Lean's kernel accepts the code, a replay of that check by the owner of this write-up is recorded elsewhere, and nothing was compiled for this note; no mathematician has read the proof through. This note treats the second part of the set-up of one unit (the folder of the filling theorem): 68 of its 497 declarations, those of three files. The objects of the first part, the note *The filling data: Bessel subgroups, links of frames, Siegel's lemma and strong approximation* (Note S below), are recalled in Section 2, and its results are quoted as *Inputs*, like the results of other units and of the background library. Every statement of this note was compared, argument by argument, with the Lean text, and the proofs are complete relative to the Inputs. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 68 declarations.

The notes of this unit are cited by letters: S (the filling data), B (box partitions; this note), U (Poincaré series and the sieve), E (the energy of amplitude sums), F (the filling theorem), P (the pairing of links), T (the top floor at a finite place) and G (from amplitude sums to local correlations).

1 The statements

The notation is that of Note S and is recalled in Section 2. In a few words: E is a CM field, $\mathfrak{s} = (\varrho, \kappa, k_0)$ a shape, and S filling data; V is the hermitian space with basis $e_0, \dots, e_{\varrho-1}$, x_i $(i \in \kappa)$, z, $e^*_{\varrho-1}, \dots, e^*_0$, the e_a and e^*_a in hyperbolic pairs, the x_i orthonormal and $h(z,z) = -N(S)$; the *frame* is $\mathfrak{f}_0, \dots, \mathfrak{f}_\varrho = e_0, \dots, e_{\varrho-1}, z$; $\mathcal{U}(T)$ is the unipotent group of the flag of the e_a, and $\mathcal{S}(T) \supseteq \mathcal{U}(T)$ the Bessel subgroup; $\mathrm{Cl}(S)$ is a finite set of classes, $\mathsf{G}(S)$ a finite group (a product of general linear groups over the residue fields of the chosen places above $g(S) = q_0(S)q_1(S)$), and $\mathcal{U}_{S,c}$ the

group of the rational $u \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$ whose position at the class c is congruent to 1 modulo $g(S)$.

Setting 1.1. E is a CM field with a CM type, $\mathfrak{s}$ is a shape and $S \in \mathrm{Fill}(E,\mathfrak{s})$ are filling data (Section 2). Moreover E is *stably unramified*: it contains a primitive root of unity of order 5040.

Setting 1.2. E is a CM field with a CM type that is stably unramified (it contains a primitive root of unity of order 5040), $\mathfrak{s}$ is a shape, and $S \in \mathrm{Fill}(E,\mathfrak{s})$ are filling data that are *admissible*: the standing hypotheses of Note S hold for S; they are recalled in (R11).

Theorem A (agreeing characters force congruences). In Setting 1.1, let $k \in K(S)$ be such that $\mathrm{red}_S(k) \in \mathrm{Agr}(S)$, and let x be a natural number. Let Ω be a finite set with a bijection $\rho\colon \Omega \to \{1,2\}$, let t be a matrix over $\mathbb{A}_{E,f}$ with rows $I(\mathfrak{s})$ and columns Ω that is integral, and put $C = [\mathfrak{f} \mid k\mathfrak{f}]^* t$; its rows are indexed by the pairs (i,a), $i \in \{1,2\}$, $0 \le a \le \varrho$, the row $(1,a)$ belonging to $\mathfrak{f}_a$ and the row $(2,a)$ to $k\mathfrak{f}_a$. Suppose that $C_{(i,a),l} = 0$ for all $i \in \{1,2\}$, $a < \varrho$ and $l \in \Omega$, and let

$$\delta = \det\left(C_{(\rho(l),\varrho),\,l'}\right)_{l,l' \in \Omega}.$$

If δ is a unit of $\mathbb{A}_{E,f}$ and $\mathrm{fix}(\mathrm{red}_S(k)) \ge x$, then $x \le [\delta]$.

Here $[\delta]$ is the generalized index (for $\delta \in \hat{O}_E$ a unit of $\mathbb{A}_{E,f}$ it is $\#\hat{O}_E/\delta\hat{O}_E$), and $\mathrm{fix}(s) \ge x$ says that s, or its inverse adjoint, fixes a non-zero vector of the span of the frame modulo chosen places whose norms have product at least x. The columns of t are two integral linear forms $v \mapsto t_l^* v$ that vanish on $X = \mathrm{span}(e_a)$ and on kX; $\overline{\delta}$ is the determinant of their values at z and kz (and $[\overline{\delta}] = [\delta]$). So wherever the frames of 1 and of k meet modulo a chosen place, that place divides δ.

Theorem B (frame count). In Setting 1.2, let $c \in \mathrm{Cl}(S)$, $s \in \mathsf{G}(S)$, let P be a set, $y, y' \in \prod_\tau G_1(\mathfrak{s})$ and $x \in [0,\infty]$, and for every $p \in \mathsf{P}$ let $d_p \in \mathrm{Cl}(S)$, $\gamma_p, \delta_p \in G_S(E)$ and $\mathsf{w}_p \in [0,\infty]$ be given. Suppose that

(i) for $p, q \in \mathsf{P}$, if $d_p = d_q$ and $\delta_p = \delta_q$ then $p = q$;

(ii) for every p with $\mathsf{w}_p \neq 0$: $\gamma_p \in \mathrm{Tm}_S(c,s,d_p)$, $\delta_p^{-1}\gamma_p \in \mathcal{U}_{S,d_p}$, $\mathrm{r}_S^\infty(\gamma_p)^{-1} y \in \mathrm{BxP}(S)$, and $\mathsf{w}_p \le \beta_{S,d_p}\left(\mathrm{r}_S^\infty(\delta_p)^{-1} y'\right) \cdot x$;

(iii) V is a real subspace of the space of families $(v_{\tau,a})$ of vectors of $\mathbb{C}^{I(\mathfrak{s})}$, indexed by the infinite places τ of E and $0 \le a \le \varrho$, and $F_p - F_q \in V$ for all p, q with $\mathsf{w}_p \neq 0 \neq \mathsf{w}_q$, where $F_p = \left((\mathrm{r}_S^\infty(\gamma_p)_\tau^{-1} y_\tau)^{-1}\mathfrak{f}_a\right)_{\tau,a}$.

Then $\sum_{p \in \mathsf{P}} \mathsf{w}_p \;\le\; b(S)^{\,6\,n(\mathfrak{s})^2 \dim_{\mathbb{R}} V} \cdot x$.

Theorem C (the covolume). In Setting 1.2, for every $c \in \mathrm{Cl}(S)$ and every $y \in \prod_\tau G_1(\mathfrak{s})$,

$$\int_{\prod_\tau \mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s};1))} \beta_{S,c}\big((u_\tau y_\tau)_\tau\big)\, d\Big(\prod_\tau \mu_{\mathfrak{s}}^{\mathcal{U}}\Big)(u) \;=\; \nu(S),$$

the integral over the product, over the infinite places τ of E, of the unipotent groups of $G_1(\mathfrak{s})$ with their Haar measures $\mu_{\mathfrak{s}}^{\mathcal{U}}$ of (R3); $\nu(S)$ is the explicit number of (R10).

Since $\sum_{u \in \mathcal{U}_{S,c}} \beta_{S,c}(\mathrm{r}_S^\infty(u)y) = 1$, Theorem C says that the image of $\mathcal{U}_{S,c}$ is a lattice of covolume $\nu(S)$ in $\prod_\tau \mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s};1))$, whatever the class c.

Theorem D (convergence; full against cut). In Setting 1.2, let $\sigma \in \{\mathrm{full}, \mathrm{cut}\}$ be a side, $\Xi \subseteq G_S(E)$ a set, $c \in \mathrm{Cl}(S)$ and $s \in \mathsf{G}(S)$, and put $X(d) = \Xi \cap \mathrm{Tm}_S(c,s,d)$ for $d \in \mathrm{Cl}(S)$. Then:

(1) for every $c' \in \mathrm{Cl}(S)$ there is a real number $M \geq 0$ such that for all $y \in \prod_\tau G_1(\mathfrak{s})$

$$\sum_{\gamma \in X(c')} \beta_{S,c'}(\mathrm{r}_S^\infty(\gamma)^{-1}y) \prod_\tau \big\| W^\sigma_{S,\tau}\big((\mathrm{r}_S^\infty(\gamma)^{-1}y)_\tau\big)\big\| \;\leq\; M,$$

and moreover $\sum_{\gamma \in X(c')} |a|_S^\sigma(c,c';\gamma) < \infty$;

(2) with $f = f(E)$ and $k = k(\mathfrak{s})$,

$$\sum_{d \in \mathrm{Cl}(S)} \sum_{\gamma \in X(d)} \int \beta_{S,c}(y)\, \beta_{S,d}(\mathrm{r}_S^\infty(\gamma)^{-1}y)\, \big\|\mathcal{W}_S^\sigma(\gamma;y) - \mathcal{W}_S^{\mathrm{cut}}(\gamma;y)\big\|\, dy \leq \frac{\nu(S)}{2\, b(S)^{\,k^3 f}}.$$

Theorem E (change of weight). In Setting 1.2, let σ be a side, $\Xi \subseteq G_S(E)$ a set with $\Xi \in \mathrm{Stab}(S)$, let $c, c' \in \mathrm{Cl}(S)$ and $s \in \mathsf{G}(S)$, and put $X = \Xi \cap \mathrm{Tm}_S(c,s,c')$. Let $\phi\colon G_S(E) \times \prod_\tau G_1(\mathfrak{s}) \to [0,\infty]$ be such that every $\phi(\gamma,\cdot)$ is measurable and

$$\sum_{(u,u') \in \mathcal{U}_{S,c} \times \mathcal{U}_{S,c'}} \phi(u\gamma u'^{-1},\ \mathrm{r}_S^\infty(u)\, y) = 1 \qquad \text{for all } \gamma \in X \text{ and all } y.$$

Then for every $\gamma \in X$ the function $y \mapsto \phi(\gamma,y)\, \mathcal{W}_S^\sigma(\gamma;y)$ is integrable, the family of its integrals is summable with

$$\sum_{\gamma \in X} \int \phi(\gamma,y)\, \mathcal{W}_S^\sigma(\gamma;y)\, dy \;=\; A_S^\sigma(\Xi;c,s,c'),$$

and $$\sum_{\gamma \in X} \int \phi(\gamma,y)\, \big\|\mathcal{W}_S^\sigma(\gamma;y) - \mathcal{W}_S^{\mathrm{cut}}(\gamma;y)\big\|\, dy \;\leq\; \frac{\nu(S)}{2\, b(S)^{\,k(\mathfrak{s})^3 f(E)}}.$$

Theorem F (characters that do not agree; the identity term). In Setting 1.2, let $c, c' \in \mathrm{Cl}(S)$.

(i) Let σ be a side, $\Xi \subseteq G_S(E)$ a set with $\Xi \in \mathrm{Stab}(S)$, and $s \in \mathsf{G}(S)$. If s does not agree, that is, if $s \in \mathrm{Agr}(S)$ fails, then $A^{\sigma}_S(\Xi; c, s, c') = 0$.

(ii) For $\Xi = \mathcal{S}(T^{\mathrm{rat}}_S(E))$, the group of rational points of the Bessel subgroup, for the full Bessel functions, and for all $i, j \in \mathrm{Bc}(S)$, the coordinate (i, j) of $A^{\mathrm{full}}_S(\mathcal{S}(T^{\mathrm{rat}}_S(E)); c, 1, c')$ is $\nu(S)$ if $c = c'$ and $i = j$, and 0 otherwise.

Theorems A to C are proved in Sections 4, 7 and 8; Theorems D, E, F in Section 9. Sections 3, 5 and 6 hold the tools: a positive form on links, integral Eichler elements, and covering lemmas, together with the definitions of the indices, discriminants and weights that the later estimates of the unit use (Note E, *The energy of amplitude sums*, and Note P, on the pairing of links).

2 The setting

Conventions. Throughout, E is a CM field with a fixed CM type: a number field that is a totally imaginary quadratic extension of a totally real field, with its complex conjugation $x \mapsto \overline{x}$ and, for every infinite place τ, a chosen embedding $x \mapsto x_\tau$ of E into $\mathbb{C}$; $f(E)$ is the number of its infinite places, d_E its discriminant, $\mathbb{A}_{E,f}$ its ring of finite adeles with the involution induced by conjugation, and $\hat{O}_E \subset \mathbb{A}_{E,f}$ the closure of the integers. The letter $\mathfrak{s}$ denotes a shape, and S, S', S_1 denote filling data in $\mathrm{Fill}(E, \mathfrak{s})$ (R4). E' is a number field. R and A are commutative rings with an involution, T is an $\mathfrak{s}$-form over R, and O a subring. Index sets written Ω, Ω' are finite sets; P is an arbitrary set with a subset A; a matrix has its rows and columns indexed by the sets that are named, and M^* is its conjugate transpose. An element k, γ, γ', γ_2 of a unitary group is identified with its matrix. A side is an element σ of $\{\mathrm{full}, \mathrm{cut}\}$. Indices of frames and levels are counted from 0, as in the formal text; j, j' denote levels and c, c', d classes. A sum of numbers in $[0, \infty]$ over an arbitrary set is the supremum of its finite partial sums. A matrix over a ring containing $\hat{O}_E$ is *integral* if its entries lie in $\hat{O}_E$ (or in the subring O that is named). The letters S, s, σ, q, θ, Θ, ψ, ζ have in the notes of this unit the meanings given here, not those of the overview of the write-up (there: a set of columns, the complex variable, a field automorphism, a rational number, a family of twists, a chain of lifts, a column, Riemann's function). For the filling data of admissible data (Note G), the primes $q_0(S)$, $q_1(S)$ are the primes that the overview calls g_0, g_1, and $g(S)$ is their product.

Recollections from Note S. The following objects are defined in Note S (and, for the first three items, in the background library); they are recalled without their formal names, which are in the concordance of that note.

(R1) *Shapes, forms, groups.* A shape $\mathfrak{s} = (\varrho, \kappa, k_0)$ is a natural number ϱ, a finite set κ and an element $k_0 \in \kappa$; $k(\mathfrak{s}) = \#\kappa$, $n(\mathfrak{s}) = \varrho + 1$. Its index set $I(\mathfrak{s})$ labels the vectors $e_0, \dots, e_{\varrho-1}$; x_i $(i \in \kappa)$; z; $e^*_{\varrho-1}, \dots, e^*_0$, so $r(\mathfrak{s}) = \#I(\mathfrak{s}) = 2\varrho + k(\mathfrak{s}) + 1$; the middle index set $M(\mathfrak{s}) = \kappa \sqcup \{z\}$ labels the x_i and z. An $\mathfrak{s}$-form over R is a matrix T on $M(\mathfrak{s})$; $J(T)$ is the matrix on $I(\mathfrak{s})$ that pairs e_a with e^*_a by 1, is T on the middle block and 0 elsewhere; $h_T(v, w) = v^* J(T) w$, and $G(T)$ is the

group of the invertible g with $g^{*}J(T)g = J(T)$. For $N \in R$, $T^{\mathrm{rat}}(\mathfrak{s}; N)$ is the diagonal form with entries 1 on κ and $-N$ at z, and $G_1(\mathfrak{s}) = G(T^{\mathrm{rat}}(\mathfrak{s}; 1))$ over $\mathbb{C}$. The *level* of an index is $\mathrm{lev}(e_a) = a$, $\mathrm{lev}(x_i) = \mathrm{lev}(z) = \varrho$, $\mathrm{lev}(e_a^{*}) = 2\varrho - a$. The *frame* is $\mathfrak{f}_a = e_a$ for $a < \varrho$ and $\mathfrak{f}_\varrho = z$; we write $\mathfrak{f}$ for the matrix with columns $\mathfrak{f}_0, \dots, \mathfrak{f}_\varrho$, so that $g\mathfrak{f}$ has the columns $g\mathfrak{f}_a$ and $\mathfrak{f}x = \sum_a x_a \mathfrak{f}_a$, and $X = \mathrm{span}(e_0, \dots, e_{\varrho-1})$. The subgroups of $G(T)$ are: $\mathcal{U}(T)$, the g with $g_{il} = \delta_{il}$ whenever $\mathrm{lev}(l) \le \mathrm{lev}(i)$; $\mathcal{U}^0(T)$, the $g \in \mathcal{U}(T)$ with $g\mathfrak{f} = \mathfrak{f}$; $\mathcal{H}(T)$, the g fixing all e_a, e_a^{*} and z; and the Bessel subgroup $\mathcal{S}(T)$, the g with $g(i, \mathfrak{f}_a) = \delta_{i,\mathfrak{f}_a}$ whenever $\mathrm{lev}(i) \ge a$. Then $\mathcal{U}(T)$ and $\mathcal{H}(T)$ lie in $\mathcal{S}(T)$, and $\mathcal{S}(T)$ normalizes $\mathcal{U}(T)$. For $g \in G(T)$, $\mathrm{sd}(g) = \sum_{a<\varrho} g(\mathfrak{f}_a, \mathfrak{f}_{a+1})$, the frame matrix is $\mathrm{fm}(g) = (g(\mathfrak{f}_a, \mathfrak{f}_b))_{a,b}$, and $Z_T(g) = (h_T(\mathfrak{f}_a, g\mathfrak{f}_b))_{a,b}$; $\mathrm{D}_l(Z)$ is the leading principal minor of order l of a square matrix Z. On $\mathcal{U}(T)$ the map $\mathrm{fm}(\cdot)$ is multiplicative with upper unitriangular values, and $\mathrm{fm}(g) = 1$ exactly for $g \in \mathcal{U}^0(T)$.

(R2) *At the complex numbers.* For $y \in G_1(\mathfrak{s})$, $\|y\|$ is the operator norm of its matrix on the Euclidean space $\mathbb{C}^{I(\mathfrak{s})}$, the vectors $y^{-1}\mathfrak{f}_a$ are its *frame vectors*, $\mathrm{bs}(y) = \inf_{s \in \mathcal{S}(T^{\mathrm{rat}}(\mathfrak{s};1))}(\|sy\| + 1)$, and $Z(y) = Z_{T^{\mathrm{rat}}(\mathfrak{s};1)}(y)$. For real b, $y \in \mathrm{Cut}(b)$ means $\mathrm{bs}(y) \le b^{6n(\mathfrak{s})+2}$, and for real ρ the set $\mathrm{FBox}(\rho)$ consists of the y with $\|y^{-1}\mathfrak{f}_a\| \le \rho$ for all a.

(R3) *Haar measures.* dy is the Haar measure of $G_1(\mathfrak{s})$ fixed by the background library, and also its product on $\prod_\tau G_1(\mathfrak{s})$. On the unipotent group $\mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s}; 1))$ of $G_1(\mathfrak{s})$ the library fixes a measure $\mu_{\mathfrak{s}}^{\mathcal{U}}$ that is invariant under left and right translations and under inversion, and $\mu_{\mathfrak{s}}^{\mathcal{U}\backslash} = \beta\, dy$ for a chosen measurable $\beta \ge 0$ with $\int \beta(uy)\, d\mu_{\mathfrak{s}}^{\mathcal{U}}(u) = 1$ for all y. The map $(x, u) \mapsto ux$ carries $\mu_{\mathfrak{s}}^{\mathcal{U}\backslash} \times \mu_{\mathfrak{s}}^{\mathcal{U}}$ to dy; so $\int F\, d\mu_{\mathfrak{s}}^{\mathcal{U}\backslash}$ is the integral over the quotient by the unipotent group of a function $F \ge 0$ that is invariant under it.

(R4) *Filling data.* A record $S \in \mathrm{Fill}(E, \mathfrak{s})$ has, among its components: a prime number $N(S)$; two prime numbers $q_0(S), q_1(S) \ne N(S)$, with product $g(S)$; a finite idele $\bar{\pi}(S) \in \hat{O}_E$ with $\bar{\pi}(S)\overline{\bar{\pi}(S)} = N(S)$; an integer $\varpi(S)$ of E that generates the different, with $\overline{\varpi(S)} = -\varpi(S)$; a real number $b(S)$; for $i \in \{0, 1\}$ a set $\mathrm{Ch}_i(S)$ of $f(E)$ finite places of E of norm $q_i(S)$, the *chosen places*, pairwise distinct, none the conjugate of another; and for every infinite place τ a Bessel system (R8). For $r \in \hat{O}_E$, $g(S)$ divides r if and only if r and $\bar{r}$ vanish modulo every chosen place. Among the conditions: $\varrho \ge 10^5$; $1000\,(2\varrho + 2)^9 \le k(\mathfrak{s}) \le f(E)$; $f(E) \le b(S)/1000$; $\varrho + 1 \le b(S)$; and $b(S)^{10} \le q_i(S) \le b(S)^{11}$.

(R5) *Groups and coordinates.* $T_S^{\mathrm{rat}}(R) = T^{\mathrm{rat}}(\mathfrak{s}; N(S))$ over R and $G_S(R) = G(T_S^{\mathrm{rat}}(R))$. The lattice basis $\Lambda(S)$ is the matrix over $\mathbb{A}_{E,f}$ that is the identity on the e_a and e_a^{*} and has the columns x_i $(i \ne k_0)$, $x_{k_0}/\overline{\bar{\pi}(S)} + z/N(S)$ and z on the middle block; in this basis the form has the Gram matrix $J(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$, where $T_S^{\mathrm{lat}}(R)$ has diagonal entries 1 at $i \ne k_0$, 0 at k_0, $-N(S)$ at z, and entries -1 at (k_0, z) and (z, k_0). $G_S^{\mathrm{lat}} = G(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$; $K(S)$ is its subgroup of the integral elements with integral inverse; $\mathsf{G}(S) = \prod_i \prod_{w \in \mathrm{Ch}_i(S)} \mathrm{GL}_{I(\mathfrak{s})}(\mathbb{F}_w)$, with $\mathbb{F}_w$ the residue field at w; $\mathrm{red}_S \colon K(S) \to \mathsf{G}(S)$ is the reduction of entries modulo the chosen places, $K_g(S)$ its kernel, and $\overline{A}^{S} = \mathrm{red}_S(A \cap K(S))$ for a subgroup A of G_S^{lat}. The maps $\mathrm{r}_S^f \colon G_S(E) \to G_S^{\mathrm{lat}}$, $\gamma \mapsto \Lambda(S)^{-1}\gamma\Lambda(S)$, and $\mathrm{r}_S^\infty \colon G_S(E) \to \prod_\tau G_1(\mathfrak{s})$, $(\mathrm{r}_S^\infty(\gamma)_\tau)_{il} = (\gamma_{il})_\tau \mathrm{sc}_{S,\tau}(l)/\mathrm{sc}_{S,\tau}(i)$, are homo-

morphisms, with the scales of (R10); r_S^∞ maps $\mathcal{U}(T_S^{\mathrm{rat}}(E))$, $\mathcal{U}^0(T_S^{\mathrm{rat}}(E))$, $\mathcal{S}(T_S^{\mathrm{rat}}(E))$ into the groups of the same name of $G_1(\mathfrak{s})$. $\mathrm{Cl}(S)$ is the finite set of double cosets of the finite adelic points of the unitary group of $X_0 = \mathrm{span}(x_i)$ by its rational points and the level of the record; every class c has a fixed representative $\tilde{c}$ (a unitary matrix on $\mathbb{A}_{E,f}^\kappa$), and $\mathrm{B}_S(c) = \mathrm{diag}(1, \tilde{c}, 1, 1)\,\Lambda(S)$ (blocks e, κ, z, e^*), $\mathrm{C}_S(c) = \mathrm{B}_S(c)^{-1}$, $\mathrm{C}_S^\circ(c) = \Lambda^\circ(\mathfrak{s}; \bar{\pi}(S))\,\tilde{c}^{-1}$, where $\Lambda^\circ(\mathfrak{s}; \bar{\pi}(S))$ is the matrix with rows $M(\mathfrak{s})$ and columns κ that expresses the x_i in the middle part of the lattice basis; $h_S(c) \in G_S^{\mathrm{lat}}$ is the image of $\tilde{c}$. For $c, c' \in \mathrm{Cl}(S)$ and $\gamma \in G_S(E)$ the *position* is $\mathrm{pos}_S(c, c'; \gamma) = h_S(c)^{-1}\mathrm{r}_S^f(\gamma)h_S(c') = \mathrm{B}_S(c)^{-1}\gamma\,\mathrm{B}_S(c') \in G_S^{\mathrm{lat}}$; for $s \in \mathsf{G}(S)$, $\mathrm{Tm}_S(c, s, c')$ is the set of the γ with $\mathrm{pos}_S(c, c'; \gamma) \in K(S)$ of reduction s (the *terms over* (c, s, c')); $\mathcal{U}_{S,c} = \{u \in \mathcal{U}(T_S^{\mathrm{rat}}(E)) : \mathrm{pos}_S(c, c; u) \in K_g(S)\}$ and $\mathcal{U}_{S,c}^0 = \mathcal{U}_{S,c} \cap \mathcal{U}^0(T_S^{\mathrm{rat}}(E))$. $G_S(E)$ is countable.

(R6) *Characters and reductions.* $\hat{g}(S)$ is the idele of $g(S)$; ψ_S is the additive character $x \mapsto \psi_f(-x/\varpi(S))$ of $\mathbb{A}_{E,f}$, ψ_f the standard character; and $\psi_S^{\mathcal{U}}(u) = \psi_S(\hat{g}(S)^{-1}\,\mathrm{sd}(u))$ for $u \in G_S^{\mathrm{lat}}$. For a chosen place w and $\epsilon \in \{0, 1\}$, $\mathrm{red}_{S,(w,\epsilon)} \colon \hat{O}_E \to \mathbb{F}_w$ is the reduction modulo w, preceded by the conjugation if $\epsilon = 1$; adj_S^ϵ is the automorphism of $\mathrm{GL}_{I(\mathfrak{s})}(\mathbb{F}_w)$ that is the identity if $\epsilon = 0$ and $s \mapsto J^{-1}(s^{-1})^{\mathsf{T}}J$, $J = J(T_S^{\mathrm{lat}}(\mathbb{F}_w))$, if $\epsilon = 1$. For $w \in \mathrm{Ch}_i(S)$ and $\mathbf{d} = (\epsilon, v)$ with $0 \neq v \in \mathbb{F}_w^{\varrho+1}$, $\mathrm{Fix}_{S,w}(\mathbf{d})$ is the subgroup of the $s \in \mathsf{G}(S)$ such that $\mathrm{adj}_S^\epsilon(s_{i,w})$ fixes $\mathfrak{f}v$. For a natural number x, $\mathrm{fix}(s) \geq x$ means: there is a finite set A of pairs (w, ϵ) such that x is at most the product of the norms of the places w over $(w, \epsilon) \in \mathsf{A}$, and for each of them some $v \neq 0$ with $s \in \mathrm{Fix}_{S,w}((\epsilon, v))$. An element $s \in \mathsf{G}(S)$ *agrees*, $s \in \mathrm{Agr}(S)$, if $\psi_S^{\mathcal{U}}(u) = \psi_S^{\mathcal{U}}(u')$ for all $u, u' \in K(S) \cap \mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ with $\mathrm{red}_S(u)\,s = s\,\mathrm{red}_S(u')$. A set $\Xi \subseteq G_S(E)$ is *stable*, $\Xi \in \mathrm{Stab}(S)$, if $u^{-1}\gamma u' \in \Xi \Leftrightarrow \gamma \in \Xi$ for all $u, u' \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$ and all γ.

(R7) *Frames and links.* For $\gamma \in G(T)$: $[\mathfrak{f} \mid \gamma\mathfrak{f}]$ is the matrix with the columns $\mathfrak{f}_a$ and $\gamma\mathfrak{f}_a$ ($0 \leq a \leq \varrho$, indexed by $(1, a)$ and $(2, a)$), $[\mathfrak{e} \mid \gamma\mathfrak{e}]$ its submatrix of the columns with $a < \varrho$; $\mathrm{L}(\gamma)$ is the matrix of $t \mapsto (\mathrm{mid}\,t, \mathrm{mid}\,\gamma^{-1}t)$, where $\mathrm{mid}\,t \in R^{M(\mathfrak{s})}$ is the middle part of $t \in R^{I(\mathfrak{s})}$; the *link* is $\mathrm{link}(\gamma) = \mathrm{L}(\gamma)\{t : [\mathfrak{e} \mid \gamma\mathfrak{e}]^* J(T)t = 0\} \subseteq R^{M(\mathfrak{s}) \sqcup M(\mathfrak{s})}$; $\mathrm{L}_z(T)$ is the matrix with two rows that sends (p, p') to $((Tp)_z, (Tp')_z)$; and $\mathrm{link}^\circ(\gamma, \gamma') \subseteq R^\kappa$ consists of the κ-parts of the first components of the $(p, p') \in \mathrm{link}(\gamma) \cap \mathrm{link}(\gamma')$ with $\mathrm{L}_z(T)(p, p') = 0$. For filling data, $J_S = J(T_S^{\mathrm{rat}}(E))$, $Q_S^\pm = \mathrm{diag}(T_S^{\mathrm{rat}}(E), -T_S^{\mathrm{rat}}(E))$, $Z_S(\gamma) = Z_{T_S^{\mathrm{rat}}(E)}(\gamma)$, $\mathrm{g}_S(\gamma) = \mathrm{g}(N(S); Z_S(\gamma))$ with $\mathrm{g}(N; Z) = \det Z \overline{\det Z} - N^2\,\mathrm{D}_\varrho(Z)\,\overline{\mathrm{D}_\varrho(Z)}$, and $\|\mathrm{g}\|_S(\gamma) = \prod_\tau |\mathrm{g}_S(\gamma)_\tau|$. For $j \leq \varrho + 1$, $\gamma \in \mathrm{NF}_j$ (*normal form from* j, as in Note S) means: there is $c_0 \in R$ with $c_0\overline{c_0} = 1$ such that for every a with $j \leq a \leq \varrho$ there are $x, y \in R^{\varrho+1}$ with $\mathfrak{f}x = \gamma\mathfrak{f}y$, $x_a = c_0$, $y_a = 1$ and $x_b = y_b = 0$ for $b > a$; and $\gamma \in \mathrm{Lv}_j(S)$ means $\gamma \in \mathrm{NF}_j$, $\mathrm{D}_j(Z_S(\gamma)) \neq 0$, and $j = \varrho + 1$ exactly if $\mathrm{g}_S(\gamma) \neq 0$. $\gamma \sim_{S;c,c'} \gamma'$ means that $\gamma, \gamma' \in \mathrm{Tm}_S(c, s, c')$ for one s. For a matrix M over E, $(M)_{\mathbb{A}_f}$ is the same matrix over $\mathbb{A}_{E,f}$. For matrices L, H over $\mathbb{A}_{E,f}$ with $LH = 1$ and a subspace V of the space E^Ω on which H acts, $\mathrm{m}_H(V)$ is a chosen matrix $(y)_{\mathbb{A}_f}\,a$, y a basis matrix of V and a an invertible matrix over $\mathbb{A}_{E,f}$, such that $H\,\mathrm{m}_H(V)$ is integral with

an integral left inverse, and $\mathrm{disc}_H(V) = \sqrt{[\det(\mathrm{m}_H(V)^{*}\mathrm{m}_H(V))]}$. Finally $\mathrm{cw}_i(S) = \#\overline{\mathcal{S}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))}^{S} \cdot \#\mathrm{Cl}(S) \cdot N(S)^{-k(\mathfrak{s})f(E)}\,(2\,\mathrm{tl}(S)^2)^i$.

(R8) *Bessel systems.* The Bessel system of S at τ provides a finite non-empty set $I_\tau(S)$, a real number $\deg_\tau(S) > 0$ and continuous functions $W^{\mathrm{full}}_{S,\tau}, W^{\mathrm{cut}}_{S,\tau} \colon G_1(\mathfrak{s}) \to \mathbb{C}^{I_\tau(S)}$, with properties of which we use those of Input (S7). $\mathrm{Bc}(S) = \prod_\tau I_\tau(S)$; for $y \in \prod_\tau G_1(\mathfrak{s})$, $i \in \mathrm{Bc}(S)$ and a side σ, $W^\sigma_S(y)_i = \prod_\tau W^\sigma_{S,\tau}(y_\tau)_{i_\tau}$, and for $\gamma \in G_S(E)$ the *kernel* $\mathcal{W}^\sigma_S(\gamma; y) \in \mathbb{C}^{\mathrm{Bc}(S)\times\mathrm{Bc}(S)}$ has the coordinates $W^\sigma_S(y)_i\,\overline{W^\sigma_S(\mathrm{r}^\infty_S(\gamma)^{-1}y)_j}$, with the Euclidean norm.

(R9) *Boxes, cutoffs, amplitudes.* $\mathrm{bx}(S) = b(S)^{2n(\mathfrak{s})^2}$; $\mathrm{BxP}(S)$ is the set of the $y \in \prod_\tau G_1(\mathfrak{s})$ with $y_\tau \in \mathrm{FBox}(\mathrm{bx}(S))$ and $y_\tau \in \mathrm{Cut}(b(S))$ for all τ; and $\gamma \in \mathrm{Near}(S)$ if some $y \in \mathrm{BxP}(S)$ has $\mathrm{r}^\infty_S(\gamma)^{-1}y \in \mathrm{BxP}(S)$. A function $\omega \colon \prod_\tau G_1(\mathfrak{s}) \to [0,\infty]$ is a *box partition* for $c \in \mathrm{Cl}(S)$, $\omega \in \mathrm{BP}(S,c)$, if: (i) ω is measurable; (ii) if $\omega(y) \neq 0$ and $y_\tau \in \mathrm{Cut}(b(S))$ then $y_\tau \in \mathrm{FBox}(\mathrm{bx}(S))$; (iii) for every y and every $\alpha \colon \mathcal{U}_{S,c} \to [0,\infty]$ with $\sum_{u\in\mathcal{U}^0_{S,c}} \alpha(u\gamma) = 1$ for all γ, one has $\sum_{\gamma\in\mathcal{U}_{S,c}} \alpha(\gamma)\,\omega(\mathrm{r}^\infty_S(\gamma)y) = 1$; (iv) $\omega \le 1$; (v) $\omega(sy) = \omega(y)$ whenever every s_τ lies in the group generated by $\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))$ and $\mathcal{H}(T^{\mathrm{rat}}(\mathfrak{s};1))$. The cutoff is $\beta_{S,c} = \beta^\flat_{S,c} \cdot \omega_{S,c}$, where $\beta^\flat_{S,c}$ is a chosen measurable function with $\sum_{u\in\mathcal{U}^0_{S,c}} \beta^\flat_{S,c}(\mathrm{r}^\infty_S(u)y) = 1$ for all y (Input (S6)) and $\omega_{S,c}$ is a chosen box partition for c if there is one (and an arbitrary function otherwise). The amplitudes are

$$a^\sigma_S(c,c';\gamma) = \int \beta_{S,c}(y)\,\beta_{S,c'}(\mathrm{r}^\infty_S(\gamma)^{-1}y)\,\mathcal{W}^\sigma_S(\gamma;y)\,dy,$$

$$A^\sigma_S(\Xi;c,s,c') = \sum_{\gamma\in\Xi\cap\mathrm{Tm}_S(c,s,c')} a^\sigma_S(c,c';\gamma),$$

and $|a|^\sigma_S(c,c';\gamma) \in [0,\infty]$ is the first integral with the kernel replaced by its norm. In the first integral a value ∞ of the weight is read as 0; by Lemma 7.3(a) the weight is at most 1.

(R10) *Constants.* $\kappa_\tau(S) = g(S)\,\varpi(S)_\tau/(2\pi i)$, a non-zero real number. The scales are $\mathrm{sc}_{S,\tau}(e_a) = \kappa_\tau(S)^{\varrho-a}/\sqrt{N(S)}$, $\mathrm{sc}_{S,\tau}(x_i) = 1$, $\mathrm{sc}_{S,\tau}(z) = 1/\sqrt{N(S)}$, $\mathrm{sc}_{S,\tau}(e^*_a) = \mathrm{sc}_{S,\tau}(e_a)^{-1}$; so $\mathrm{sc}_{S,\tau}(\mathfrak{f}_a) = \kappa_\tau(S)^{\varrho-a}/\sqrt{N(S)}$ for $a \le \varrho$. Further $\mathrm{fs}_S(\tau,a) = \kappa_\tau(S)^{2(\varrho-a)}/N(S)$, $\mathrm{fs}(S) = \prod_\tau \prod_{a\le\varrho} \mathrm{fs}_S(\tau,a)^{k(\mathfrak{s})+2(\varrho-a)}$, $\mathrm{tl}(S) = b(S)^{k(\mathfrak{s})f(E)/4}$, and, with $k = k(\mathfrak{s})$, $f = f(E)$ and $\mathrm{vol}(\hat{O}_E) = 2^f/\sqrt{|d_E|}$,

$$\nu(S) = \Big(\frac{g(S)^{\varrho(2\varrho+2k+1)}}{N(S)^k}\Big)^f \cdot \frac{|d_E|^{\varrho/4}}{\mathrm{vol}(\hat{O}_E)^{\varrho(\varrho+k)}\ \mathrm{fs}(S)} \ > 0.$$

(R11) *Admissibility.* The filling data S are admissible if a list of conditions of Note S holds, of which this note uses: $|\varpi(S)_\tau| \le b(S)$ and $1 \le |\kappa_\tau(S)| \le b(S)^{23}$ for all τ; $N(S) \le b(S)^{60n(\mathfrak{s})}$; $\#\mathrm{Bc}(S) \le b(S)^{2k(\mathfrak{s})f(E)}$; and that the level of the record is the preimage of $K_g(S)$, which enters through Input (S3).

Inputs from Note S (of this unit; *The filling data: Bessel subgroups, links of frames, Siegel's lemma and strong approximation*). $S \in \mathrm{Fill}(E, \mathfrak{s})$ throughout; $c, c', d \in \mathrm{Cl}(S)$.

(S1) *Levels and characters.* $K_g(S)$ is the set of the $k \in K(S)$ with $\hat{g}(S)^{-1}(k-1)$ integral. $J(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ is hermitian, integral with integral inverse. For $a \in \mathbb{A}_{E,f}$: $\psi_S(ra) = 1$ for all $r \in \hat{O}_E$ if and only if $a \in \hat{O}_E$. For $r \in \hat{O}_E$: $\hat{g}(S)^{-1} r \in \hat{O}_E$ if and only if $\mathrm{red}_{S,(w,\epsilon)}(r) = 0$ for all chosen w and both ϵ. Every family of residues $(t_{w,\epsilon})$, $t_{w,\epsilon} \in \mathbb{F}_w$, is the family of the $\mathrm{red}_{S,(w,\epsilon)}(r)$ of some $r \in \hat{O}_E$. For $k \in K(S)$, a chosen place $w \in \mathrm{Ch}_i(S)$ and $\epsilon \in \{0,1\}$: the entrywise image of k under $\mathrm{red}_{S,(w,\epsilon)}$ is $\mathrm{adj}_S^{\epsilon}(\mathrm{red}_S(k)_{i,w})$. The residue field of $w \in \mathrm{Ch}_i(S)$ has $q_i(S)$ elements.

(S2) *Eichler elements.* Let A carry an element τ_0 with $\tau_0 + \overline{\tau_0} = 1$, and let T be an $\mathfrak{s}$-form over A with $J(T)$ hermitian and invertible. Every linear form on $A^{I(\mathfrak{s})}$ is $h_T(v, \cdot)$ for some v. Let $a \le \varrho$, $j \in A^{\varrho+1}$ with $j_b = 0$ for $b \ge a$, $r \in A$, and let $\mu = h_T(v, \cdot)$ vanish at $\mathfrak{f}_b$ for all $b < a$. Then there is $u \in \mathcal{U}(T)$ with $\mathrm{sd}(u) = r \sum_{b=1}^{\varrho} \mu(\mathfrak{f}_b)\, j_{b-1}$ whose matrix is $1 + \xi\, v^* J(T) - (v + \varsigma\xi)\, \xi^* J(T)$, with $\xi = r\,\mathfrak{f} j$ and $\varsigma = \tau_0\, h_T(v, v)$.

(S3) *Positions and terms.* $\mathrm{C}_S(c)^*\, J(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))\, \mathrm{C}_S(c) = (J_S)_{\mathbb{A}_f}$, and $\mathrm{pos}_S(c, c'; \gamma)\,\mathfrak{f} = \mathrm{C}_S(c)\,(\gamma\mathfrak{f})_{\mathbb{A}_f}$. If $\gamma \in \mathrm{Tm}_S(c, s, c')$ and $\gamma' \in \mathrm{Tm}_S(c', s', d)$ then $\gamma\gamma' \in \mathrm{Tm}_S(c, ss', d)$ and $\gamma^{-1} \in \mathrm{Tm}_S(c', s^{-1}, c)$; for such γ the matrix $\mathrm{pos}_S(c, c'; \gamma)$ is integral and all entries of $Z_S(\gamma)$ are integral over $\mathbb{Z}$. Let S be admissible. If $\gamma \in \mathcal{S}(T_S^{\mathrm{rat}}(E))$ and $\mathrm{pos}_S(c, d; \gamma) \in K_g(S)$, then $c = d$ and $\gamma \in \mathcal{U}_{S,c}$. If $\gamma \in \mathrm{Tm}_S(c, s, c')$, $\gamma' \in \mathrm{Tm}_S(c, s, d)$ and $\gamma\mathfrak{f} = \gamma'\mathfrak{f}$, then $c' = d$ and $\gamma^{-1}\gamma' \in \mathcal{U}^0_{S,c'}$.

(S4) *Characters of the unipotent group.* For $\gamma \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$, every side σ and all y, i: $W_S^{\sigma}(\mathrm{r}_S^{\infty}(\gamma) y)_i = \psi_S^{\mathcal{U}}(\mathrm{r}_S^{f}(\gamma))^{-1}\, W_S^{\sigma}(y)_i$. If $u, v \in \mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ and $u^{-1} h_S(c)^{-1} v\, h_S(c) \in K_g(S)$, then $\psi_S^{\mathcal{U}}(v) = \psi_S^{\mathcal{U}}(u)$; this holds with any element of the image of the finite adelic unitary group of X_0 in the place of $h_S(c)$. For every $u \in \mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ there is $u_0 \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$ with $u^{-1}\mathrm{pos}_S(c, c; u_0) \in K_g(S)$ and $\psi_S^{\mathcal{U}}(\mathrm{r}_S^{f}(u_0)) = \psi_S^{\mathcal{U}}(u)$.

Inputs from Note S, continued. **(S5)** *Frames.* For $\gamma \in G(T)$ and $t \in R^{I(\mathfrak{s})}$: $[\mathfrak{e} \mid \gamma\mathfrak{e}]^* J(T) t = 0$ if and only if the e^*-coordinates of t and of $\gamma^{-1}t$ vanish. For $\gamma \in G_S(E)$ and $p \in E^{M(\mathfrak{s}) \sqcup M(\mathfrak{s})}$: $p \in \mathrm{link}(\gamma)$ if and only if $q^* Q_S^{\pm}\, p = 0$ for all $q \in \mathrm{link}(\gamma)$. If $\gamma \in \mathrm{NF}_j$ then $\gamma^{-1} \in \mathrm{NF}_j$, and for every $y \in R^{\varrho+1}$ there are x, y' with $\mathfrak{f} x = \gamma \mathfrak{f} y'$ and $y'_a = y_a$ for $a \geq j$. If $\gamma \in G_S(E)$ and $\mathfrak{f} x = \gamma \mathfrak{f} y$, then $Z_S(\gamma)\, y = -N(S)\, x_\varrho\, \epsilon_\varrho$ with ϵ_ϱ the last unit vector. If $v \in E^{M(\mathfrak{s})}$ has $v_z = 0$ and $v^* T_S^{\mathrm{rat}}(E)\, v = 0$, then $v = 0$. For $s, s' \in G(T)$: $(s^{-1}s')\mathfrak{f} = \mathfrak{f}$ if and only if $s\mathfrak{f} = s'\mathfrak{f}$. And $Z(\mathrm{r}_S^\infty(\gamma)_\tau)_{ab} = \kappa_\tau(S)^{\varrho-a} \kappa_\tau(S)^{\varrho-b}\, N(S)^{-1}\, (Z_S(\gamma)_{ab})_\tau$.

(S6) *Discreteness.* For a finite family of indices and a compact set $\mathcal{C}$ of families of complex matrices, only finitely many points of $\mathcal{C}$ are of the form (the submatrix of $\mathrm{r}_S^\infty(u)_\tau$ on these indices)$_\tau$ with $u \in \mathcal{U}_{S,c}$. For every subgroup Γ of $\mathcal{U}_{S,c}$ there is a measurable $\beta\colon \prod_\tau G_1(\mathfrak{s}) \to [0,\infty]$ with $\sum_{u \in \Gamma} \beta(\mathrm{r}_S^\infty(u) y) = 1$ for all y.

(S7) *Bessel systems.* For both sides t, all $n \in \mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s}; 1))$ and $y \in G_1(\mathfrak{s})$: $W^t_{S,\tau}(ny) = e^{\mathrm{sd}(n) - \overline{\mathrm{sd}(n)}}\, W^t_{S,\tau}(y)$. There are a measurable $A_\tau\colon G_1(\mathfrak{s}) \to [0,\infty]$, invariant under left translation by $\mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s}; 1))$, and a finite measure m_τ on $G_1(\mathfrak{s})$ such that, with $\theta_\tau(y) = 1$ if $W^{\mathrm{full}}_{S,\tau}(y) = W^{\mathrm{cut}}_{S,\tau}(y)$ and $\theta_\tau(y) = b(S)^{3r(\mathfrak{s})^3 f(E)}$ otherwise: $\theta_\tau(y)\, \|W^t_{S,\tau}(y)\| \leq A_\tau(y) \leq m_\tau\{g : yg \in \mathrm{Cut}(b(S))\}$ for both sides t and all y, and $m_\tau(G_1(\mathfrak{s})) \int A_\tau\, d\mu_{\mathfrak{s}}^{\mathcal{U}\backslash} \leq b(S)^{2r(\mathfrak{s})^3}\, \#I_\tau(S)$. Moreover $\|\mathcal{W}^\sigma_S(\gamma; y)\| = \prod_\tau \|W^\sigma_{S,\tau}(y_\tau)\| \cdot \prod_\tau \|W^\sigma_{S,\tau}((\mathrm{r}_S^\infty(\gamma)^{-1} y)_\tau)\|$, $y \mapsto \mathcal{W}^\sigma_S(\gamma; y)$ is continuous, and $\|a^\sigma_S(c, c'; \gamma)\| \leq |a|^\sigma_S(c, c'; \gamma)$.

Inputs from other units. (J1), unit 132 (the real place). The maps $y \mapsto y^{-1}\mathfrak{f}_a$ and $y \mapsto \mathrm{bs}(y)$ are continuous on $G_1(\mathfrak{s})$. For $y \in G_1(\mathfrak{s})$: $\|y^{-1}\mathfrak{f}_a\| \le \|y\|$; if s lies in the group generated by $\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))$ and $\mathcal{H}(T^{\mathrm{rat}}(\mathfrak{s};1))$ then $(sy)^{-1}\mathfrak{f}_a = y^{-1}\mathfrak{f}_a$; if $s \in \mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s};1))$ then $y^{-1}\mathfrak{f}_a = \sum_b \mathrm{fm}(s)_{ba}\,(sy)^{-1}\mathfrak{f}_b$; if $s \in \mathcal{S}(T^{\mathrm{rat}}(\mathfrak{s};1))$ then $\mathrm{bs}(sy) = \mathrm{bs}(y)$; there is $u \in \mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s};1))$ with $\|uy\| \le \mathrm{bs}(y)$; for complex numbers c_a, an index $i \in I(\mathfrak{s})$ and $s \in G_1(\mathfrak{s})$, $\left|\sum_a c_a\, s(i,\mathfrak{f}_a)\right| \le \|sy\| \cdot \left\|\sum_a c_a\, y^{-1}\mathfrak{f}_a\right\|$, and likewise for the sum over $a < \varrho$ only; $|\langle v, J(T^{\mathrm{rat}}(\mathfrak{s};1))w\rangle| \le \|v\|\,\|w\|$; and for $x, y' \in G_1(\mathfrak{s})$, $Z(x)_{ab} = \langle y'^{-1}\mathfrak{f}_a,\, J(T^{\mathrm{rat}}(\mathfrak{s};1))\,(x^{-1}y')^{-1}\mathfrak{f}_b\rangle$. The coordinates of $W^{\mathrm{full}}_{S,\tau}$ are orthonormal: $\int W^{\mathrm{full}}_{S,\tau}(y)_i\, \overline{W^{\mathrm{full}}_{S,\tau}(y)_j}\, d\mu_{\mathfrak{s}}^{\mathcal{U}\backslash}(y) = \delta_{ij}$.
(J2), units 113 and 57 (generalized index). $[\cdot]$ is multiplicative on $\mathbb{A}_{E,f}$, zero on non-units, positive on units and invariant under conjugation; $[a] = \#\hat{O}_E/a\hat{O}_E$ for a unit $a \in \hat{O}_E$; $[x] = |\,\mathrm{N}_{E/\mathbb{Q}}(x)| = \prod_\tau |x_\tau|^2$ for $x \in E$; in particular $[n] = n^{2f(E)}$ for a natural number n. For an invertible integral matrix M over $\mathbb{A}_{E,f}$ on a finite set Ω, the index of $M\hat{O}_E^\Omega$ in $\hat{O}_E^\Omega$ is $[\det M]$ (unit 57).
(J3), unit 114 (packing). In a real normed space V of finite dimension, a set of points that are pairwise at distance at least $m > 0$ and of norm at most $r \ge 0$ has at most $(1 + 2r/m)^{\dim V}$ elements.
(J4), unit 121. An algebraic integer x of a number field with $|x_\tau| < 1$ at all infinite places τ is 0.
(J5), unit 112 (duality). Over a field F, let P, P' be matrices with the same finite set of rows and the same finite set of columns, $\mathfrak{a}, \mathfrak{a}'$ sets of columns and X, Y vectors. If every linear form m that kills the columns of P in $\mathfrak{a}$ and the columns of P' in $\mathfrak{a}'$ satisfies $m(PX) = m(P'Y)$, then $Px = P'y$ for some x equal to X outside $\mathfrak{a}$ and some y equal to Y outside $\mathfrak{a}'$.
(J6), unit 123. If E is stably unramified, the integers of E contain ω_0 with $\omega_0 + \overline{\omega_0} = 1$.
The catalogue of the write-up classes (J2) to (J6) as elementary and gives no reference for them.

Definitions of this note. They are stated where they are first needed: the link metric and the link discriminant (Definition 3.1), integral lifts (Definition 3.2), and the indices, gauges and weights of the later estimates (Definitions 6.5 to 6.9).

3 A positive form on links; class coordinates

The link of γ is a Lagrangian subspace for the indefinite form $Q_S^{\pm}$ (Input (S5)). To measure lattices in links one needs a positive form; it is obtained by changing the sign of $Q_S^{\pm}$ on the second copy and correcting the two z-directions.

Definition 3.1 (link metric, link discriminant)**.** Let $S \in \mathrm{Fill}(E, \mathfrak{s})$ and write $T = T_S^{\mathrm{rat}}(E)$. The *link metric* is the matrix over E on $M(\mathfrak{s}) \sqcup M(\mathfrak{s})$

$$Q_S^{\mathrm{link}} = \tfrac{1}{2}\,\mathrm{diag}(T, T) + N(S)^{-1}\,\mathrm{L}_z(T)^*\,\mathrm{L}_z(T).$$

For a finite set Ω, a matrix v over E with rows $M(\mathfrak{s}) \sqcup M(\mathfrak{s})$ and columns Ω, and an invertible matrix e over $\mathbb{A}_{E,f}$ on Ω, the *link discriminant* is

$$\mathrm{ld}_S(v, e) = \left|\mathrm{N}_{E/\mathbb{Q}} \det(v^*\, Q_S^{\mathrm{link}}\, v)\right| \cdot [\det e]^2,$$

the squared discriminant, for the link metric, of the lattice with the adelic basis $(v)_{\mathbb{A}_f}\, e$.

LEAN `Filling.linkMetric` · `Filling.linkDiscSq`

Definition 3.2 (integral lifts)**.** Let $S \in \mathrm{Fill}(E, \mathfrak{s})$, $c \in \mathrm{Cl}(S)$, $\gamma \in G_S(E)$, let Ω' be a set and m a matrix over $\mathbb{A}_{E,f}$ with rows $M(\mathfrak{s}) \sqcup M(\mathfrak{s})$ and columns Ω'. We write $m \in \mathrm{IL}(c;\gamma)$, and say that m *has integral lifts* for c and γ, if there are a finite set Ω, a matrix t over E with rows $I(\mathfrak{s})$ and columns Ω, and a matrix w over $\mathbb{A}_{E,f}$ with rows Ω and columns Ω', such that

$$[\mathfrak{e} \mid \gamma\mathfrak{e}]^*\, J_S\, t = 0, \qquad \mathrm{C}_S(c)\,(t)_{\mathbb{A}_f}\, w \ \text{ is integral}, \qquad (\mathrm{L}(\gamma)\, t)_{\mathbb{A}_f}\, w = m.$$

So the columns of m are the images in the link of vectors of the lattice of the class c that are adelic combinations of rational vectors orthogonal to X and to γX.

LEAN `Filling.HasIntegralLifts`

Lemma 3.3 (class coordinates and frames)**.** *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$, $c, c' \in \mathrm{Cl}(S)$ and $\gamma \in G_S(E)$.*

(a) For $k \in G(T)$, the two rows $(1, \varrho)$ and $(2, \varrho)$ of $[\mathfrak{f} \mid k\mathfrak{f}]^* J(T)$ form the matrix $\mathrm{L}_z(T)\,\mathrm{L}(k)$.

(b) $\mathrm{C}_S(c)$ is block diagonal for the blocks e, middle, e^*: the identity on the outer blocks, and on the middle block the matrix whose columns at κ are those of $\mathrm{C}^\circ_S(c)$ and whose column at z is the unit vector at z.

(c) $\mathrm{C}^\circ_S(c)^*\; T^{\mathrm{lat}}_S(\mathbb{A}_{E,f})\; \mathrm{C}^\circ_S(c) = 1$.

(d) For matrices t, t' over E with rows $I(\mathfrak{s})$ and any sets of columns,

$$(\mathrm{C}_S(c)(t)_{\mathbb{A}_f})^*\, J(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))\, (\mathrm{C}_S(c)(t')_{\mathbb{A}_f}) = (t^*\, J_S\, t')_{\mathbb{A}_f}.$$

(e) $\mathrm{C}_S(c)\,([\mathfrak{f} \mid \gamma\mathfrak{f}])_{\mathbb{A}_f} = [\mathfrak{f} \mid k\mathfrak{f}]$ and $\mathrm{C}_S(c)\,([\mathfrak{e} \mid \gamma\mathfrak{e}])_{\mathbb{A}_f} = [\mathfrak{e} \mid k\mathfrak{e}]$ for $k = \mathrm{pos}_S(c, c'; \gamma)$.

(f) If $k \in G(T)$ is integral over the subring O of R, then so is $[\mathfrak{f} \mid k\mathfrak{f}]$.

(g) If a matrix v over a field, with a finite set Ω' of rows and a finite set Ω of columns, has a left inverse, then $\#\Omega$ is the dimension of the span of its columns.

Proof. (a) The row of $[\mathfrak{f} \mid k\mathfrak{f}]^* J(T)$ at $(1, \varrho)$ is the linear form $t \mapsto h_T(z, t) = (T \operatorname{mid} t)_z$, and the row at $(2, \varrho)$ is $t \mapsto h_T(kz, t) = h_T(z, k^{-1}t) = (T \operatorname{mid} k^{-1}t)_z$, because k is unitary. These are the two rows of $\mathrm{L}_z(T)\mathrm{L}(k)$. (b) By (R5), $\mathrm{B}_S(c)$ is the product of two matrices that are the identity on the outer blocks, with middle blocks $\mathrm{diag}(\tilde{c}, 1)$ and the middle block of $\Lambda(S)$. The inverse of the latter has the columns $\Lambda^\circ(\mathfrak{s}; \bar{\pi}(S))$ at κ and the unit vector at z, since z belongs to the lattice basis. Hence the middle block of $\mathrm{C}_S(c) = \mathrm{B}_S(c)^{-1}$ has the columns $\Lambda^\circ(\mathfrak{s}; \bar{\pi}(S))\tilde{c}^{-1} = \mathrm{C}^\circ_S(c)$ at κ and the unit vector at z. (c) Restrict the first identity of Input (S3) to the rows and columns of κ and use (b): the block of J_S on κ is 1. (d) is the first identity of (S3) multiplied by $(t)^*_{\mathbb{A}_f}$ and $(t')_{\mathbb{A}_f}$. (e) By (S3), $k\mathfrak{f} = \mathrm{C}_S(c)(\gamma\mathfrak{f})_{\mathbb{A}_f}$, and $\mathrm{C}_S(c)\mathfrak{f} = \mathfrak{f}$ by (b); omitting the columns of z and kz gives the second identity. (f) The entries of $[\mathfrak{f} \mid k\mathfrak{f}]$ are entries of 1 and of k. (g) The matrix is injective, so its rank is $\#\Omega$. □

LEAN `Filling.framePair_last_eq_zFormPair_mul` · `Filling.classCoord_eq_fromBlocks` · `Filling.gram_classCoordX₀` · `Filling.classCoord_gram` · `Filling.classCoord_mul_framePair` · `Filling.classCoord_mul_isotropicFramePair` · `Filling.isIntMatrix_framePair` · `Filling.card_eq_finrank_colSpan`

Lemma 3.4 (the link is Lagrangian)**.** *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$ and $\gamma \in G_S(E)$.*

(a) $\dim_E \mathrm{link}(\gamma) = k(\mathfrak{s}) + 1$.

(b) If v and u are matrices over E with rows $M(\mathfrak{s}) \sqcup M(\mathfrak{s})$ and finite sets of columns Ω' and Ω, whose columns all lie in $\mathrm{link}(\gamma)$*, then $v^* \, Q_S^{\pm} \, u = 0$.*

Proof. By Input (S5) the link is its own orthogonal for the hermitian form $Q_S^{\pm}$ on $E^{M(\mathfrak{s}) \sqcup M(\mathfrak{s})}$; this is (b). The form is non-degenerate, because $T_S^{\mathrm{rat}}(E) = \mathrm{diag}(1, \dots, 1, -N(S))$ is invertible. So the orthogonal of the link has dimension $2(k(\mathfrak{s})+1) - \dim \mathrm{link}(\gamma)$, and (a) follows. (The formal proof applies the rank theorem to $p \mapsto v^* Q_S^{\pm} p$ for a basis matrix v of the link.) □

LEAN `Filling.finrank_link` · `Filling.gram_linkForm_eq_zero_of_le_link`

Lemma 3.5 (positivity of the link metric)**.** *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$.*

(a) For every infinite place τ of E the image of Q_S^{link} under $x \mapsto x_\tau$ is positive definite.

(b) If a matrix v over E with rows $M(\mathfrak{s}) \sqcup M(\mathfrak{s})$ and a finite set Ω' of columns has a left inverse, then $\det(v^* \, Q_S^{\mathrm{link}} \, v) \neq 0$.

Proof. (a) With $T = T_S^{\mathrm{rat}}(E)$, the matrix $\mathrm{L}_z(T)$ has the entry $-N(S)$ at the two positions (first row, first z) and (second row, second z) and zeros elsewhere, so $\mathrm{L}_z(T)^* \mathrm{L}_z(T)$ is diagonal with entries $N(S)^2$ at the two z and 0 elsewhere. Hence Q_S^{link} is the diagonal matrix with entries $\frac{1}{2}$ at the indices of κ and $-\frac{1}{2}N(S) + N(S) = \frac{1}{2}N(S)$ at the two z: rational and positive. (b) Under $x \mapsto x_\tau$ the matrix v stays injective, so $v_\tau^* (Q_S^{\mathrm{link}})_\tau v_\tau$ is positive definite by (a), and its determinant $\det(v^* Q_S^{\mathrm{link}} v)_\tau$ is positive. □

LEAN `Filling.linkMetric_posDef` · `Filling.det_gram_linkMetric_ne_zero`

Lemma 3.6 (products in two links)**.** *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$, $\gamma, \gamma' \in G_S(E)$, and let t, t' be matrices over E with rows $I(\mathfrak{s})$ and sets of columns Ω, Ω' such that $[\mathfrak{e} \mid \gamma\mathfrak{e}]^* \, J_S \, t = 0$ and $[\mathfrak{e} \mid \gamma'\mathfrak{e}]^* \, J_S \, t' = 0$. Then for all $\alpha, \alpha' \in E$, with $T = T_S^{\mathrm{rat}}(E)$,*

$$(\mathrm{L}(\gamma)t)^* \, \mathrm{diag}(\alpha T, -\alpha' T) \, (\mathrm{L}(\gamma')t') = \alpha \, t^* \, J_S \, t' - \alpha' \, t^* \, J_S \, (\gamma\gamma'^{-1}) \, t'.$$

Proof. Let t_i and t'_l be columns of t and t'. By Input (S5) the e^*-coordinates of t_i, $\gamma^{-1}t_i$, t'_l and $\gamma'^{-1}t'_l$ vanish, and for two vectors v, w without e^*-coordinates $h_T(v, w) = (\mathrm{mid}\, v)^* T \, \mathrm{mid}\, w$. Since $\mathrm{L}(\gamma)t_i = (\mathrm{mid}\, t_i, \mathrm{mid}\, \gamma^{-1}t_i)$, the entry (i, l) of the left side is $\alpha \, h_T(t_i, t'_l) - \alpha' \, h_T(\gamma^{-1}t_i, \gamma'^{-1}t'_l)$, and $h_T(\gamma^{-1}t_i, \gamma'^{-1}t'_l) = h_T(t_i, \gamma\gamma'^{-1}t'_l)$ because γ is unitary. □

LEAN `Filling.gram_linkMatrix_mul`

Lemma 3.7 (a Liouville inequality for Gram determinants)**.** *Let E be stably unramified, let Ω, Ω' be finite sets, Q a matrix over E on Ω' whose image Q_τ is positive definite for every infinite place τ, let a, b be matrices over E with rows Ω' and columns Ω such that $\det(a^* Q b) \neq 0$, and v, v' invertible matrices over $\mathbb{A}_{E,f}$ on Ω. Let $x \geq 0$ be a real number such that $x \leq [\delta]$ whenever $\delta = \det(v^* \, (a^* Q b)_{\mathbb{A}_f} \, v')$ is a unit of $\mathbb{A}_{E,f}$. Then*

$$x^2 \;\leq\; [\det v]^2 \cdot \left|\mathrm{N}_{E/\mathbb{Q}} \det(a^* Q a)\right| \cdot \left|\mathrm{N}_{E/\mathbb{Q}} \det(b^* Q b)\right| \cdot [\det v']^2.$$

Proof. $\delta = \overline{\det v} \cdot \det(a^* Q b) \cdot \det v'$ is a product of units, so $x \leq [\delta]$, and by (J2) $[\delta] = [\det v] \, |\mathrm{N}_{E/\mathbb{Q}} \det(a^* Q b)| \, [\det v']$ with $|\mathrm{N}_{E/\mathbb{Q}} \det(a^* Q b)| = \prod_\tau |\det(a^* Q b)_\tau|^2$.

At every τ the block matrix with blocks $a_\tau^* Q_\tau a_\tau$, $a_\tau^* Q_\tau b_\tau$, $b_\tau^* Q_\tau a_\tau$, $b_\tau^* Q_\tau b_\tau$ is positive semidefinite, being $(a_\tau\ b_\tau)^* Q_\tau (a_\tau\ b_\tau)$. For such a matrix with square blocks P_1, A_1, A_1^*, P_2 one has $|\det A_1|^2 \le \det P_1 \cdot \det P_2$ (an elementary input from unit 112; the artifact calls it the Cauchy–Schwarz inequality for determinants). Hence $|\det(a^*Qb)_\tau|^4 \le |\det(a^*Qa)_\tau|^2\,|\det(b^*Qb)_\tau|^2$, and the product over τ gives $|\mathrm{N}_{E/\mathbb{Q}} \det(a^*Qb)|^2 \le |\mathrm{N}_{E/\mathbb{Q}} \det(a^*Qa)| \cdot |\mathrm{N}_{E/\mathbb{Q}} \det(b^*Qb)|$. Squaring the bound for x finishes the proof. □

LEAN `Filling.liouville_gram`

4 Agreeing characters: proof of Theorem A

Proof of Theorem A. Step 1: reduction to congruences. By (R6), the hypothesis $\mathrm{fix}(\mathrm{red}_S(k)) \ge x$ gives a finite set A of pairs $p = (w_p, \epsilon_p)$ of a chosen place and $\epsilon_p \in \{0,1\}$, with $x \le \prod_{p\in\mathsf{A}} \#\mathbb{F}_{w_p}$ (the norm of a chosen place is the number of elements of its residue field, (S1)), and for every $p \in \mathsf{A}$ a vector $v_p \ne 0$ in $\mathbb{F}_{w_p}^{\varrho+1}$ such that $\mathrm{adj}_S^{\epsilon_p}(\mathrm{red}_S(k)_{w_p})$ fixes $\mathfrak{f} v_p$.

Put $Q = t^*[\mathfrak{f} \mid k\mathfrak{f}] = C^*$. Its entries lie in $\hat{O}_E$, because t is integral, $\hat{O}_E$ is stable under the conjugation, and $k \in K(S)$ is integral. Its columns are indexed by the pairs (i,a); those with $a < \varrho$ vanish. Let Q° be the matrix on Ω whose column l is the column $(\rho(l), \varrho)$ of Q, and $\delta' = \det Q^\circ$. Then $\delta' = \overline{\delta}$, so $\delta' \in \hat{O}_E$ is a unit of $\mathbb{A}_{E,f}$ and, by (J2), $[\delta] = [\delta'] = \#\hat{O}_E/\delta'\hat{O}_E$. By (S1) the ring homomorphism $\hat{O}_E \to \prod_{p\in\mathsf{A}} \mathbb{F}_{w_p}$, $r \mapsto (\mathrm{red}_{S,p}(r))_p$, is surjective; let $\mathfrak{a}$ be its kernel, so that $\#\hat{O}_E/\mathfrak{a} = \prod_{p\in\mathsf{A}} \#\mathbb{F}_{w_p} \ge x$. If $\mathrm{red}_{S,p}(\delta') = 0$ for every $p \in \mathsf{A}$, then $\delta'\hat{O}_E \subseteq \mathfrak{a}$ and $[\delta] = \#\hat{O}_E/\delta'\hat{O}_E \ge \#\hat{O}_E/\mathfrak{a} \ge x$. So it suffices to prove that $\mathrm{red}_{S,p}(\delta') = 0$ for every $p \in \mathsf{A}$.

Step 2: the frames do not meet where δ' is a unit. Fix $p = (w,\epsilon) \in \mathsf{A}$, write $\mathbb{F} = \mathbb{F}_w$, and let a subscript $\mathbb{F}$ denote the image under $\mathrm{red}_{S,p}$, entry by entry: for a matrix M over $\hat{O}_E$, $M_\mathbb{F}$ is its entrywise image, and $(MM')_\mathbb{F} = M_\mathbb{F} M'_\mathbb{F}$. The field $\mathbb{F}$ has no involution, so a conjugate transpose is always taken over $\hat{O}_E$, before the image. Suppose that $\delta'_\mathbb{F} = \det Q^\circ_\mathbb{F} \ne 0$ in $\mathbb{F}$. By (S1), $k_\mathbb{F} \in \mathrm{GL}_{I(\mathfrak{s})}(\mathbb{F})$ equals $\mathrm{adj}_S^\epsilon(\mathrm{red}_S(k)_w)$, so $k_\mathbb{F}\,\mathfrak{f} v_p = \mathfrak{f} v_p$. We claim:

$$\text{if } \xi, \eta \in \mathbb{F}^{\varrho+1} \text{ satisfy } \mathfrak{f}\xi = k_\mathbb{F}\,\mathfrak{f}\eta, \text{ then } \xi = 0. \tag{$*$}$$

For $\xi = \eta = v_p$ this contradicts $v_p \ne 0$, and the theorem follows. We prove by descending induction on $a \in \{\varrho, \dots, 0\}$ that $\xi_a = 0$ for all pairs (ξ,η) as in $(*)$.

The case $a = \varrho$. For such a pair, $Q_\mathbb{F}(\xi, -\eta) = (t^*)_\mathbb{F}(\mathfrak{f}\xi - k_\mathbb{F}\,\mathfrak{f}\eta) = 0$, where $(\xi,-\eta)$ has the coordinate ξ_a at $(1,a)$ and $-\eta_a$ at $(2,a)$. The columns (i,a) of $Q_\mathbb{F}$ with $a < \varrho$ vanish, so ξ_ϱ times the column $(1,\varrho)$ equals η_ϱ times the column $(2,\varrho)$. These two columns are those of $Q^\circ_\mathbb{F}$, which is invertible. Hence $\xi_\varrho = \eta_\varrho = 0$.

The step. Let $a < \varrho$, suppose the assertion holds for all indices above a, and let $\mathfrak{f}\xi = k_\mathbb{F}\,\mathfrak{f}\eta$. Then $\xi_b = 0$ for $b > a$ and $\eta_\varrho = 0$. For $\zeta \in \mathbb{F}^{\varrho+1}$ write $\zeta^+ = (0, \zeta_0, \dots, \zeta_{\varrho-1})$. We shall prove that

$$m(\mathfrak{f}\xi^+) = m(k_\mathbb{F}\,\mathfrak{f}\eta^+) \tag{$**$}$$

for every linear form m on $\mathbb{F}^{I(\mathfrak{s})}$ with $m(\mathfrak{f}_b) = 0$ for $b \le a$ and $m(k_\mathbb{F}\mathfrak{f}_b) = 0$ for $b < \varrho$. By the duality (J5), applied to the matrices $\mathfrak{f}$ and $k_\mathbb{F}\,\mathfrak{f}$, the sets of columns

$\{b \le a\}$ and $\{b < \varrho\}$ and the vectors ξ^+, η^+, there are then ξ', η' with $\mathfrak{f}\xi' = k_{\mathbb{F}}\,\mathfrak{f}\eta'$ and $\xi'_b = \xi^+_b$ for $b > a$. By the induction hypothesis at $a+1$, $\xi_a = \xi^+_{a+1} = \xi'_{a+1} = 0$.

Proof of $(**)$: integral lifts. By (S1) there is a map $o\colon \mathbb{F} \to \hat{O}_E$ with $o(0) = 0$, $\mathrm{red}_{S,p}(o(\theta)) = \theta$, and $\mathrm{red}_{S,p'}(o(\theta)) = 0$ for every pair $p' \neq p$. Then, for every chosen place w', $\mathrm{red}_{S,(w',\epsilon)}(\overline{o(\theta)}) = \mathrm{red}_{S,(w',1-\epsilon)}(o(\theta)) = 0$. For a matrix M over $\mathbb{F}$ on $I(\mathfrak{s})$ and $\zeta \in \mathbb{F}^{\varrho+1}$ put

$$\mathsf{A}(M,\zeta) = \sum_{b=1}^{\varrho} o(m(M\mathfrak{f}_b))\, o(\zeta_{b-1}) \in \hat{O}_E, \quad \text{so that} \quad \mathrm{red}_{S,p}(\mathsf{A}(M,\zeta)) = m(M\mathfrak{f}\zeta^+).$$

Thus $(**)$ says that $\mathsf{A}(1,\xi) - \mathsf{A}(k_{\mathbb{F}},\eta)$ vanishes under $\mathrm{red}_{S,p}$. By the two equivalences of (S1) it suffices to show that

$$\psi_S\Big(r\,\hat{g}(S)^{-1}\big(\mathsf{A}(1,\xi) - \mathsf{A}(k_{\mathbb{F}},\eta)\big)\Big) = 1 \qquad \text{for every } r \in \hat{O}_E.$$

Eichler elements over $\hat{O}_E$. Fix $r \in \hat{O}_E$. Let $a_0 \le \varrho$, let $\zeta \in \mathbb{F}^{\varrho+1}$ have $\zeta_b = 0$ for $b \ge a_0$, and let M be such that $m(M\mathfrak{f}_b) = 0$ for $b < a_0$. We claim that there is $u \in K(S) \cap \mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ with $\mathrm{sd}(u) = r\,\mathsf{A}(M,\zeta)$ whose entrywise image under $\mathrm{red}_{S,(w',\epsilon)}$ is 1 for every chosen place $w' \neq w$, and is $1 + r_{\mathbb{F}}(\mathfrak{f}\zeta)(mM)$ for $w' = w$; here $(\mathfrak{f}\zeta)(mM)$ is the product of a column and a row. Indeed, apply (S2) to the ring $\hat{O}_E$ with its conjugation and the form $T_S^{\mathrm{lat}}(\hat{O}_E)$: its Gram matrix J is hermitian and invertible over $\hat{O}_E$ by (S1), and $\hat{O}_E$ contains ω_0 with $\omega_0 + \overline{\omega_0} = 1$ by (J6). The linear form $\mu = o \circ (mM)$ on $\hat{O}_E^{I(\mathfrak{s})}$ is v^*J for some v, it vanishes at $\mathfrak{f}_b$ for $b < a_0$ since $o(0) = 0$, and $j = o \circ \zeta$ vanishes at $b \ge a_0$. So (S2) gives u unitary over $\hat{O}_E$, in the unipotent group, with $\mathrm{sd}(u) = r\sum_{b\ge1} \mu(\mathfrak{f}_b)\, j_{b-1} = r\,\mathsf{A}(M,\zeta)$ and with matrix $1 + \xi_0\,\mu - (v + \varsigma\xi_0)\,\xi_0^* J$, where $\xi_0 = r\,\mathfrak{f}j$. It is integral with integral inverse, so it lies in $K(S)$. Under $\mathrm{red}_{S,(w',\epsilon)}$ the entries $\overline{r}\,\overline{o(\zeta_b)}$ of ξ_0^* go to 0, while ξ_0 goes to $r_{\mathbb{F}}\,\mathfrak{f}\zeta$ and μ to mM if $w' = w$, and ξ_0 goes to 0 if $w' \neq w$.

Conclusion. Let u be the element of the claim for $(a_0,\zeta,M) = (a+1,\xi,1)$ and u' the one for $(\varrho,\eta,k_{\mathbb{F}})$; the hypotheses hold by the properties of ξ, η and m. We compare $\mathrm{red}_S(u)\,\mathrm{red}_S(k)$ and $\mathrm{red}_S(k)\,\mathrm{red}_S(u')$ in $\mathsf{G}(S)$. At the component of a chosen place w', after the automorphism $\mathrm{adj}_S^{\epsilon}$, these are by (S1) the entrywise images of uk and ku' under $\mathrm{red}_{S,(w',\epsilon)}$. For $w' \neq w$ both are the image of k. For $w' = w$ they are

$$(1 + r_{\mathbb{F}}(\mathfrak{f}\xi)\,m)k_{\mathbb{F}} = k_{\mathbb{F}} + r_{\mathbb{F}}(\mathfrak{f}\xi)(mk_{\mathbb{F}}) \quad \text{and}$$
$$k_{\mathbb{F}}(1 + r_{\mathbb{F}}(\mathfrak{f}\eta)(mk_{\mathbb{F}})) = k_{\mathbb{F}} + r_{\mathbb{F}}(k_{\mathbb{F}}\,\mathfrak{f}\eta)(mk_{\mathbb{F}}),$$

which agree because $\mathfrak{f}\xi = k_{\mathbb{F}}\,\mathfrak{f}\eta$. Hence $\mathrm{red}_S(u)\,\mathrm{red}_S(k) = \mathrm{red}_S(k)\,\mathrm{red}_S(u')$, and since $\mathrm{red}_S(k)$ agrees, $\psi_S^{\mathcal{U}}(u) = \psi_S^{\mathcal{U}}(u')$, that is $\psi_S(\hat{g}(S)^{-1} r\,\mathsf{A}(1,\xi)) = \psi_S(\hat{g}(S)^{-1} r\,\mathsf{A}(k_{\mathbb{F}},\eta))$. As ψ_S is an additive character, this is the required identity. □

LEAN `Filling.le_genIndex_det_of_fixesNormAtLeast`

5 Integral Eichler elements of the arithmetic unipotent group

Rational elements of $\mathcal{U}_{S,c}$ with prescribed entries are produced by Eichler transformations. A position (a,l), $a<\varrho$, $l\in I(\mathfrak{s})$, is called *free* if $l=e_b$ or $l=e_b^*$ with $b>a$, or if l is a middle index (this is a notion of unit 118); and $[P]$ denotes 1 if the statement P holds and 0 otherwise.

Lemma 5.1 (Eichler elements with several columns)**.** *In Setting 1.1, let $c\in\mathrm{Cl}(S)$, let Ω be a finite set, $p\colon\Omega\to\{0,\dots,\varrho-1\}$ a map, and x the matrix with rows $I(\mathfrak{s})$ and columns Ω whose column b is the unit vector $e_{p(b)}$. Let u be a matrix over E with rows $I(\mathfrak{s})$ and columns Ω, and s a matrix over E on Ω, such that*

(i) $u(e_a^*,b)=0$ *for all a, b, b' with $a\le p(b')$;*
(ii) $s^*=-s$;
(iii) the matrices $\mathrm{C}_S(c)\,(u)_{\mathbb{A}_f}$ and $(s)_{\mathbb{A}_f}$ are integral.

Then there are $\gamma\in\mathcal{U}_{S,c}$ and $\lambda\in E$ with

$$\gamma=1+g(S)\Big(x\,u^*J_S-\big(u+x\,(\lambda\,u^*J_S\,u-s)\big)\,x^*J_S\Big),$$

and if $(u^*J_S)(b,\mathfrak{f}_a)=0$ for all $b\in\Omega$ and all $a\le\varrho$, then $\gamma\in\mathcal{U}^0_{S,c}$.

Proof. Write $J=J_S$ and $g=g(S)$, take ω_0 from (J6), and put $\lambda=g\omega_0$, $B=\lambda\,u^*Ju-s$ and $N=x\,u^*J-(u+xB)\,x^*J$, so that the matrix in question is $1+gN$. The columns of x are isotropic and mutually orthogonal, and by (i) they are orthogonal to the columns of u: $x^*Jx=0$ and $x^*Ju=0$. Moreover x^*J vanishes at all frame vectors, and $\mathrm{C}_S(c)\,(x)_{\mathbb{A}_f}=(x)_{\mathbb{A}_f}$ by Lemma 3.3(b).

Unitarity. $(1+gN)^*J(1+gN)=J$ means $N^*J+JN+g\,N^*JN=0$. Now $JN+N^*J=-Jx\,(B+B^*)\,x^*J$, the terms Jxu^*J and Jux^*J cancelling; and in N^*JN every product contains x^*Jx, x^*Ju or u^*Jx, except $Jx\,(u^*Ju)\,x^*J$. Since $s^*=-s$ and u^*Ju is hermitian, $B+B^*=(\lambda+\overline{\lambda})\,u^*Ju=g\,u^*Ju$. So the sum vanishes, and $\gamma=1+gN\in G_S(E)$.

γ lies in the unipotent group. The row b of u^*J has the entry $\overline{u(e_a^*,b)}$ at e_a, which by (i) vanishes unless $a>p(b')$ for all b'. So the entries of $x\,u^*J$ sit in rows $e_{p(b')}$ and in columns of level greater than $p(b')$. The matrix x^*J has its entries in the columns $e^*_{p(b)}$, of level $2\varrho-p(b)\ge\varrho+1$, and the rows of $u+xB$ that do not vanish are rows e_a, middle rows, or rows e_a^* with $a>p(b)$ for all b, all of level less than $2\varrho-p(b)$. Hence N is strictly upper triangular for the levels and $\gamma\in\mathcal{U}(T_S^{\mathrm{rat}}(E))$.

The position. By (R5), $\mathrm{pos}_S(c,c;\gamma)=1+\hat g(S)\,\mathrm{C}_S(c)(N)_{\mathbb{A}_f}\mathrm{C}_S(c)^{-1}$, and by (S3), $(J)_{\mathbb{A}_f}\mathrm{C}_S(c)^{-1}=\mathrm{C}_S(c)^*J'$ with $J'=J(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$. Therefore, writing $\hat x=\mathrm{C}_S(c)(x)_{\mathbb{A}_f}$ and $\hat u=\mathrm{C}_S(c)(u)_{\mathbb{A}_f}$,

$$\mathrm{C}_S(c)\,(N)_{\mathbb{A}_f}\,\mathrm{C}_S(c)^{-1}=\big(\hat x\,\hat u^*-(\hat u+\hat x\,(B)_{\mathbb{A}_f})\,\hat x^*\big)\,J',\qquad (B)_{\mathbb{A}_f}=(\lambda)_{\mathbb{A}_f}\,\hat u^*J'\hat u-(s)_{\mathbb{A}_f},$$

the last by Lemma 3.3(d). All matrices on the right are integral, by (iii) and (S1). So $\hat g(S)^{-1}(\mathrm{pos}_S(c,c;\gamma)-1)$ is integral; then $\mathrm{pos}_S(c,c;\gamma)$ is integral, and so is its inverse $J'^{-1}\mathrm{pos}_S(c,c;\gamma)^*J'$. By (S1) the position lies in $K_g(S)$, that is $\gamma\in\mathcal{U}_{S,c}$.

The frame. If u^*J vanishes at the frame vectors, then $N\mathfrak{f}_a=0$ for all a, so $\gamma\mathfrak{f}=\mathfrak{f}$ and $\gamma\in\mathcal{U}^0_{S,c}$. □

LEAN `Filling.exists_mem_classUnip`

Lemma 5.2 (Eichler elements in one row)**.** *In Setting 1.1, let $c \in \mathrm{Cl}(S)$ and $a < \varrho$, let l be a row vector over E indexed by $I(\mathfrak{s})$ with $l_j = 0$ unless (a, j) is a free position, such that the row vector $(l)_{\mathbb{A}_f} \mathrm{B}_S(c)$ is integral, and let s be an integer of E with $\overline{s} = -s$. Then there is $\gamma \in \mathcal{U}_{S,c}$ such that for all $b < \varrho$ and all $j \in I(\mathfrak{s})$ with $j \neq e_a^*$ or $l = 0$,*

$$\gamma(e_b, j) = \delta_{e_b,j} + g(S)\,[b = a]\,(l_j + [j = e_a^*]\,s).$$

Proof. Since $J = J_S$ is invertible there is a column vector v with $v^* J = l$. We apply Lemma 5.1 with Ω a single point, $p = a$, $x = e_a$, the column $u = v$ and the matrix (s). Condition (i): $\overline{v(e_{a'}^*)} = l_{e_{a'}} = 0$ for $a' \leq a$, because $(a, e_{a'})$ is not free. Condition (ii) is $\overline{s} = -s$. Condition (iii): $v = J^{-1} l^*$, and from (S3), $\mathrm{C}_S(c)(J)^{-1}_{\mathbb{A}_f} = J'^{-1}\mathrm{B}_S(c)^*$ with $J' = J(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$; so $\mathrm{C}_S(c)(v)_{\mathbb{A}_f} = J'^{-1}((l)_{\mathbb{A}_f}\mathrm{B}_S(c))^*$, which is integral by hypothesis and (S1). The lemma gives $\gamma \in \mathcal{U}_{S,c}$ and λ with $\gamma = 1 + g(S)(e_a\, l - (v + e_a(\lambda\, v^* J v - s))\, e_a^* J)$. The row vector $e_a^* J$ is the unit row at e_a^*. So for $j \neq e_a^*$ the entry $\gamma(e_b, j)$ is $\delta_{e_b,j} + g(S)[b = a]\, l_j$; and if $l = 0$ then $v = 0$ and $\gamma(e_b, e_a^*) = g(S)[b = a]\, s$. □

LEAN `Filling.exists_eichler_classUnip`

6 Covering lemmas; indices, gauges and weights

The first four lemmas are elementary counting tools; of them this note uses only the first. The definitions that follow, and the two lemmas after them, prepare the later estimates of the unit.

Lemma 6.1 (colouring)**.** *Let P and C be sets, $\mathsf{A} \subseteq \mathsf{P}$, let $\bowtie$ be a relation on P, $m, x \in [0, \infty]$, and $\mathsf{w} \colon \mathsf{P} \to [0, \infty]$. Let $\mathrm{col} \colon \mathsf{A} \to \mathsf{C}$ be a map whose image has at most m elements and such that $\mathrm{col}(p) = \mathrm{col}(q)$ implies $p \bowtie q$. Suppose that $\sum_{p \in \mathsf{B}} \mathsf{w}_p \leq x$ for every non-empty $\mathsf{B} \subseteq \mathsf{A}$ with $p \bowtie q$ for all $p, q \in \mathsf{B}$. Then $\sum_{p \in \mathsf{A}} \mathsf{w}_p \leq m \cdot x$.*

Proof. The sum over A is the sum, over the colours in the image of col, of the sums over the fibres. Each fibre is non-empty and consists of pairwise related elements, so its sum is at most x; and there are at most m colours. □

LEAN `Filling.tsum_le_of_encard_range_le`

Lemma 6.2 (colouring by cosets)**.** *Let P be a set, $\mathsf{A} \subseteq \mathsf{P}$, Ω a finite set, M an abelian group with subgroups L, L' such that the index of $\mathsf{L} \cap \mathsf{L}'$ in L' is finite, and $f \colon \mathsf{P} \to \mathsf{M}^\Omega$ a map with $f(p)_i \in \mathsf{L}'$ for all $p \in \mathsf{A}$ and $i \in \Omega$. Let $\mathsf{w} \colon \mathsf{P} \to [0, \infty]$ and $x \in [0, \infty]$, and suppose that $\sum_{p \in \mathsf{B}} \mathsf{w}_p \leq x$ for every non-empty $\mathsf{B} \subseteq \mathsf{A}$ such that $f(p)_i - f(q)_i \in \mathsf{L}$ for all $p, q \in \mathsf{B}$ and $i \in \Omega$. Then*

$$\sum_{p \in \mathsf{A}} \mathsf{w}_p \;\leq\; [\mathsf{L}' : \mathsf{L} \cap \mathsf{L}']^{\#\Omega} \cdot x.$$

Proof. Colour $p \in \mathsf{A}$ by the family of the classes of the $f(p)_i$ in $\mathsf{L}'/(\mathsf{L} \cap \mathsf{L}')$. There are at most $[\mathsf{L}' : \mathsf{L} \cap \mathsf{L}']^{\#\Omega}$ colours, and two elements of one colour have $f(p)_i - f(q)_i \in \mathsf{L}$ for all i. Lemma 6.1 applies. □

LEAN `Filling.tsum_le_of_colouring`

Lemma 6.3 (bounded integral vectors)**.** *Let E' be a number field with a chosen complex embedding $x \mapsto x_\tau$ at each of its infinite places τ, let P be a set, $\mathsf{A} \subseteq \mathsf{P}$, Ω a finite set, $r \geq 0$ a real number, and $F \colon \mathsf{P} \to {E'}^\Omega$ a map that is injective on A and*

such that, for all $p \in \mathsf{A}$ and $i \in \Omega$, $F(p)_i$ is integral over $\mathbb{Z}$ and $|(F(p)_i)_\tau| \le r$ for all τ. Then $\#\mathsf{A} \le (1+2r)^{2\,\#\Omega\, f(E')}$, where $f(E')$ is the number of infinite places of E'.

Proof. Map $p \in \mathsf{A}$ to the point $((F(p)_i)_\tau)_{i,\tau}$ of the real vector space $\mathbb{C}^{\Omega\times\{\tau\}}$, of dimension $2\,\#\Omega\, f(E')$, with the supremum norm. All these points have norm at most r. Two of them at distance less than 1 belong to p, q for which every $F(p)_i - F(q)_i$ is an algebraic integer with all absolute values less than 1, hence 0 by (J4), so $p = q$ by injectivity. The packing bound (J3) with $m = 1$ gives the claim. □

LEAN `Filling.encard_integral_le`

Lemma 6.4 (index of a scaled lattice)**.** *Let E' be a CM field that is stably unramified, Ω a finite set, and c a finite idele of E' that lies in the closure $\hat{O}_{E'}$ of the integers. Then the index of $c\,\hat{O}^\Omega_{E'}$ in $\hat{O}^\Omega_{E'}$ is $[c]^{\#\Omega}$; in particular it is finite and not zero.*

Proof. This is (J2) for the invertible integral matrix $c \cdot 1$, of determinant $c^{\#\Omega}$; and $[c] > 0$. □

LEAN `Filling.submoduleIndex_smul_intVectors` · `Filling.submoduleIndex_smul_ne_zero`

Indices, gauges and weights. In the following definitions an index $[\mathsf{L}' : \mathsf{L} \cap \mathsf{L}']$ of subgroups of a group is read as 0 when it is infinite, as in the formal text, and $\langle M\rangle_{\hat{O}_E}$ denotes the $\hat{O}_E$-module generated by the columns of a matrix M over $\mathbb{A}_{E,f}$.

Definition 6.5 (rows, padded matrices, middle parts)**.** Let $j \le \varrho + 1$ be a level. $\mathrm{Ib}(j)$ is the set of the $a \in \{0, \dots, \varrho - 1\}$ with $a < j$. For a matrix Z with rows $0, \dots, \varrho$ and $\varrho + 1$ columns, Z^{x} is its submatrix of the rows $0, \dots, \varrho - 1$. For a square matrix Z of size $\varrho + 1$ over a commutative ring, $\mathrm{pad}_j(Z)$ is the matrix whose rows $a < j$ are those of Z and whose other rows are those of the identity. For a square matrix γ on $I(\mathfrak{s})$, $(\gamma\mathfrak{f})^{\mathrm{mid}}$ is its submatrix of the rows in $M(\mathfrak{s})$ and the columns $\mathfrak{f}_0, \dots, \mathfrak{f}_\varrho$: the middle parts of the vectors $\gamma\mathfrak{f}_a$.

LEAN `Filling.IndexBelow` · `Filling.xRows` · `Filling.paddedLeading` · `Filling.frameMiddle`

Lemma 6.6. *For a level $j \le \varrho + 1$ and a square matrix Z of size $\varrho + 1$ over a commutative ring R:* $\det \mathrm{pad}_j(Z) = \mathrm{D}_j(Z)$.

Proof. Expanding along the rows $a \ge j$, which are rows of the identity, leaves the determinant of the upper left block of size j. □

LEAN `Filling.det_paddedLeading`

Definition 6.7 (lattice discriminants and indices)**.** Let E be stably unramified, $S \in \mathrm{Fill}(E, \mathfrak{s})$ and $c \in \mathrm{Cl}(S)$; write $T' = T^{\mathrm{lat}}_S(\mathbb{A}_{E,f})$ and, for $k \in G^{\mathrm{lat}}_S$, $Z'(k) = Z_{T'}(k)$.

(a) $\mathrm{disc}_{S,c}$ is the function on the subspaces V of E^κ given by $\mathrm{disc}_{S,c}(V) = \mathrm{disc}_{\mathrm{C}^\circ_S(c)}(V)$, the discriminant of (R7) for the matrix $H = \mathrm{C}^\circ_S(c)$, which has the left inverse $\mathrm{C}^\circ_S(c)^* T'$ by Lemma 3.3(c). It is the discriminant of the lattice cut out by V in the X_0-part of the lattice of the class c.

(b) For $k \in G^{\mathrm{lat}}_S$, a finite idele a and a subspace V of E^κ,

$$\mathrm{o}_{S,c}(k; a, V) = [\mathsf{L}_2 : a\,\mathsf{L}_1 \cap \mathsf{L}_2],$$

where $\mathsf{L}_1 = \langle (Z'(k)^{\mathrm{x}})^* \rangle_{\hat{O}_E}$ and $\mathsf{L}_2 = \langle (k\mathfrak{f})^{\mathrm{mid}\,*}\, T'\, \mathrm{C}^\circ_S(c)\, \mathrm{m}_{\mathrm{C}^\circ_S(c)}(V) \rangle_{\hat{O}_E}$.

(c) For a level $j \le \varrho+1$ and $k \in G_S^{\mathrm{lat}}$, $\mathrm{i}_{S,j}(k) = [\mathsf{L}_1 : \mathsf{L}_1^{(j)} \cap \mathsf{L}_1]$, where L_1 is as in (b) and $\mathsf{L}_1^{(j)}$ is generated by the columns of the conjugate transpose of the submatrix of $Z'(k)$ of the rows $a \in \mathrm{Ib}(j)$.

(d) For $k \in G_S^{\mathrm{lat}}$, $\mathfrak{d}_S(k)$ is the index of $\langle [\mathfrak{e} \mid k\mathfrak{e}] \rangle_{\hat{O}_E}$ in the module of the vectors of $\hat{O}_E^{I(\mathfrak{s})}$ that lie in the $\mathbb{A}_{E,f}$-span of the columns of $[\mathfrak{e} \mid k\mathfrak{e}]$.

LEAN Filling.latticeDisc · Filling.frameValueIndex · Filling.xRowsIndex · Filling.saturationIndex

In (b), L_1 is spanned by the vectors $(\overline{h(e_i, k\mathfrak{f}_l)})_l$, $i < \varrho$, and L_2 by the vectors $(h(k\mathfrak{f}_l, y))_l$ for y in the basis of the lattice cut out by V; in (d), the columns of $[\mathfrak{e} \mid k\mathfrak{e}]$ are the e_a and ke_a.

Definition 6.8 (invariants of pairs, gauges, near representatives)**.** Let E be stably unramified, $S \in \mathrm{Fill}(E, \mathfrak{s})$, $c, c' \in \mathrm{Cl}(S)$, $\gamma, \gamma' \in G_S(E)$ and $j \le \varrho+1$ a level.

(a) For a pair $p = (c', \gamma)$: $\mathrm{el}_S(c; p) = \mathrm{pos}_S(c, c'; \gamma)$.

(b) $\mathrm{pi}_S(c, c'; \gamma, \gamma') = \mathrm{disc}_{S,c}(V) \cdot \mathrm{o}_{S,c}(k; 1, V) \cdot \mathfrak{d}_S(k)$ with $V = \mathrm{link}^\circ(\gamma, \gamma')$ and $k = \mathrm{pos}_S(c, c'; \gamma)$.

(c) $\mathrm{mn}_S(j; \gamma) = \left|\mathrm{N}_{E/\mathbb{Q}}\, \mathrm{D}_j(Z_S(\gamma))\right| \cdot N(S)^{-2\,[j=\varrho+1]\, f(E)}$.

(d) $\mathrm{cg}_S(j; c, c'; \gamma) = \max(\|\mathbf{g}\|_S(\gamma),\ N(S)^{f(E)}\, \mathrm{mn}_S(j; \gamma)) \big/ (\mathrm{i}_{S,j}(k) \cdot \mathrm{i}_{S,j}(k^{-1}))$ with $k = \mathrm{pos}_S(c, c'; \gamma)$.

(e) $\gamma \in \mathrm{NRep}_S(c, c')$ means that $\gamma \in \mathrm{Near}(S)$ and that γ is the chosen representative of its double coset $\mathcal{U}^0_{S,c}\, \gamma\, \mathcal{U}^0_{S,c'}$ (one representative is chosen in every double coset).

LEAN Filling.termElem · Filling.pairInvariant · Filling.minorNorm · Filling.classGauge · Filling.IsNearRep

Definition 6.9 (weights and labelled terms)**.** Let E be stably unramified, $S \in \mathrm{Fill}(E, \mathfrak{s})$, $c \in \mathrm{Cl}(S)$, $j \le \varrho+1$ a level, and write $k = k(\mathfrak{s})$.

(a) For a function $n \colon G_S^{\mathrm{lat}} \to \mathbb{N}$ and a pair $p = (c', \gamma) \in \mathrm{Cl}(S) \times G_S(E)$: $\mathrm{ipw}_S(c; n, p) = n(\mathrm{el}_S(c; p))^{-k} \in [0, \infty)$, read as 0 when $n(\mathrm{el}_S(c; p)) = 0$.

(b) For a natural number i, pairs $p, p' \in \mathrm{Cl}(S) \times G_S(E)$ and a subspace V of E^κ:

$$\mathrm{w}_S(j, i; c; p, p'; V) = \mathrm{cw}_i(S)^{-1} \cdot \mathrm{disc}_{S,c}(V)^{-(k+1)} \cdot \mathrm{ipw}_S(c; n_{j,V}, p) \cdot \mathrm{ipw}_S(c; \mathfrak{d}_S(\cdot), p'),$$

where $n_{j,V}(k') = [\hat{O}_E^{\mathrm{Ib}(j)} : \hat{g}(S)\hat{O}_E^{\mathrm{Ib}(j)}] \cdot \mathrm{o}_{S,c}(k'; 1, V)$ for $k' \in G_S^{\mathrm{lat}}$, and an inverse of 0 is read as 0.

(c) For $c' \in \mathrm{Cl}(S)$ and $\gamma, \gamma' \in G_S(E)$, with $V = \mathrm{link}^\circ(\gamma, \gamma')$:

$$\mathrm{pw}_S(j; c, c'; \gamma, \gamma') = \mathrm{w}_S(j, k - \dim V; c; (c', \gamma), (c', \gamma'); V)$$

if $\gamma \sim_{S;c,c'} \gamma'$, $\gamma \in \mathrm{NRep}_S(c, c')$, $\gamma \in \mathrm{Lv}_j(S)$, $\gamma' \in \mathrm{NRep}_S(c, c')$ and $\gamma' \in \mathrm{Lv}_j(S)$ all hold (the difference $k - \dim V$ being 0 if $\dim V > k$), and $\mathrm{pw}_S(j; c, c'; \gamma, \gamma') = 0$ otherwise.

(d) For $t \in \overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{S}$ and a pair $p = (c', \gamma)$: $p \in \mathrm{Lab}_S(j; c, t)$ means that $\gamma \in \mathrm{NRep}_S(c, c')$, $\gamma \in \mathrm{Lv}_j(S)$, and $\gamma \in \mathrm{Tm}_S(c, s, c')$ for some $s \in \mathsf{G}(S)$ whose component in the subgroup $\overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{S}$, for a fixed bijection of $\mathsf{G}(S)$ with the product of the set of cosets of this subgroup and the subgroup, is t.

LEAN Filling.invPowWeight · Filling.termWeight · Filling.pairWeight · Filling.IsLabelledTerm

By Lemma 6.4 and (J2), $[\hat{O}_E^{\mathrm{Ib}(j)} : \hat{g}(S)\hat{O}_E^{\mathrm{Ib}(j)}] = g(S)^{2f(E)\min(j,\varrho)}$.

Lemma 6.10 (terms of level j)**.** *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$, let $j \le \varrho + 1$ be a level and $\gamma \in G_S(E)$ with $\gamma \in \mathrm{Lv}_j(S)$. Then:*

(a) $\gamma^{-1} \in \mathrm{Lv}_j(S)$*;*
(b) $\mathrm{mn}_S(j;\gamma) > 0$*;*
(c) $\mathrm{pad}_j(Z_S(\gamma))$ *is invertible:* $\mathrm{pad}_j(Z_S(\gamma)) \cdot \mathrm{pad}_j(Z_S(\gamma))^{-1} = 1$*;*
(d) for every frame index b with $j \le b$ there is $w \in E^{\varrho+1}$ with

$$\mathfrak{f}w = \gamma\mathfrak{f} \cdot (\textit{column } b \textit{ of } \mathrm{pad}_j(Z_S(\gamma))^{-1}) :$$

this combination of the vectors $\gamma\mathfrak{f}_a$ lies in the span of the frame.

Proof. (b) and (c) hold because $\mathrm{D}_j(Z_S(\gamma)) \neq 0$ and by Lemma 6.6. (a) The normal form passes to γ^{-1} by (S5). Since γ is unitary, $Z_S(\gamma^{-1}) = Z_S(\gamma)^*$, so the leading minors of $Z_S(\gamma^{-1})$ are the conjugates of those of $Z_S(\gamma)$, and $\mathfrak{g}_S(\gamma^{-1}) = \mathfrak{g}_S(\gamma)$. (d) By (S5), applied to the unit vector ϵ_b, there are w and y' with $\mathfrak{f}w = \gamma\mathfrak{f}y'$ and $y'_a = \delta_{ab}$ for $a \ge j$. By (S5) again, $Z_S(\gamma)y'$ is a multiple of the last unit vector; as $b \le \varrho$ forces $j \le \varrho$, the rows $a < j$ of $Z_S(\gamma)y'$ vanish. So $\mathrm{pad}_j(Z_S(\gamma))\, y' = \epsilon_b$, and y' is the column b of the inverse. □

LEAN Filling.HasLevel.inv · Filling.HasLevel.minorNorm_pos · Filling.HasLevel.paddedLeading_mul_inv · Filling.HasLevel.exists_shared_frame_vector

Lemma 6.11. *Let $k \in G(T)$ and let n be a matrix over R with rows $I(\mathfrak{s})$ and a set Ω of columns. If $[\mathfrak{e} \mid k\mathfrak{e}]^*\, J(T)\, n = 0$ and $\mathrm{L}_z(T)\,\mathrm{L}(k)\, n = 0$, then*

$$(Z_T(k)^{\mathrm{x}})^* \cdot (-n_e) = (k\mathfrak{f})^{\mathrm{mid}^*}\, T n_{\mathrm{mid}},$$

where n_e and n_{mid} are the submatrices of n of the rows $e_0, \dots, e_{\varrho-1}$ and of the middle rows.

Proof. Let n_l be a column of n and $b \le \varrho$. The entries (b, l) of the two sides are

$$-\sum_{a<\varrho} h_T(k\mathfrak{f}_b, e_a)\, n_l(e_a) \qquad \text{and} \qquad (\mathrm{mid}\, k\mathfrak{f}_b)^*\, T\, \mathrm{mid}\, n_l.$$

By (S5) the e^*-coordinates of n_l vanish, so

$$h_T(k\mathfrak{f}_b, n_l) = \sum_{a<\varrho} h_T(k\mathfrak{f}_b, e_a)\, n_l(e_a) + (\mathrm{mid}\, k\mathfrak{f}_b)^* T\, \mathrm{mid}\, n_l.$$

Finally $h_T(k\mathfrak{f}_b, n_l) = 0$: for $b < \varrho$ by the first hypothesis, and for $b = \varrho$ by the second and Lemma 3.3(a). □

LEAN Filling.xRows_conjTranspose_mul_eq

7 Box partitions and the frame count

Throughout this section $\mathcal{U} = \mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s};1))$ is the unipotent group of $G_1(\mathfrak{s})$, and in Setting 1.2 we abbreviate $b = b(S)$, $n = n(\mathfrak{s})$, $N = N(S)$, $f = f(E)$; these are abbreviations of numbers, not of functions.

Lemma 7.1 (frame vectors of two elements)**.** *Let $S \in \mathrm{Fill}(E,\mathfrak{s})$ and let $\gamma,\gamma' \in G_S(E)$ be such that $\gamma\mathfrak{f}$ and $\gamma'\mathfrak{f}$ have the same rows outside the rows $e_0,\dots,e_{\varrho-1}$. Then for every $y \in G_1(\mathfrak{s})$, every infinite place τ of E and every frame index l,*

$$\left(\mathrm{r}_S^\infty(\gamma)_\tau^{-1}y\right)^{-1}\mathfrak{f}_l - \left(\mathrm{r}_S^\infty(\gamma')_\tau^{-1}y\right)^{-1}\mathfrak{f}_l = \sum_{a<\varrho}\left((\gamma-\gamma')(e_a,\mathfrak{f}_l)\right)_\tau \frac{\mathrm{sc}_{S,\tau}(\mathfrak{f}_l)}{\mathrm{sc}_{S,\tau}(e_a)}\; y^{-1}\mathfrak{f}_a.$$

Proof. $(\mathrm{r}_S^\infty(\gamma)_\tau^{-1}y)^{-1}\mathfrak{f}_l = y^{-1}\mathrm{r}_S^\infty(\gamma)_\tau\mathfrak{f}_l = \sum_{i\in I(\mathfrak{s})}\mathrm{r}_S^\infty(\gamma)_\tau(i,\mathfrak{f}_l)\,y^{-1}e_i$, with e_i the unit vectors; by (R5) the coefficient is $(\gamma(i,\mathfrak{f}_l))_\tau\,\mathrm{sc}_{S,\tau}(\mathfrak{f}_l)/\mathrm{sc}_{S,\tau}(i)$. In the difference only the rows $i = e_a$ remain, and $y^{-1}e_a = y^{-1}\mathfrak{f}_a$. □

LEAN `Filling.frameVector_sub_eq_sum`

> **Input. (J7)**, unit 99 (classed as elementary by the catalogue, without a reference). Let $d \neq 0$ lie in the different of E. For every family (z_τ) of complex numbers there is an integer x of E with $|z_\tau - x_\tau| \le (2\,f(E))^2\,|d_\tau|$ for all infinite places τ. (The formal statement also prescribes x modulo $\hat{O}_E$ by a finite adele; it is used with the adele 0.)

Proposition 7.2 (box partitions exist)**.** *In Setting 1.2, for every $c \in \mathrm{Cl}(S)$ the chosen function $\omega_{S,c}$ of (R9) is a box partition for c: $\omega_{S,c} \in \mathrm{BP}(S,c)$.*

Proof. It suffices to construct one box partition for c; the choice then picks one. By (R5), $\mathrm{r}_S^\infty(u)_\tau \in \mathcal{U}$ for $u \in \mathcal{U}_{S,c}$.

Step 1: reduction of frame matrices. We claim: for every family (t_τ) of elements of $\mathcal{U}$ there is $u \in \mathcal{U}_{S,c}$ such that all entries of the frame matrices $\mathrm{fm}(t_\tau\mathrm{r}_S^\infty(u)_\tau)$ have absolute value at most $b^{60\varrho+91}$. The entries (k,l) with $k \ge l$ are 0 or 1 (R1). We treat the pairs (k,l), $k<l$, one after the other, the rows k in decreasing order. Suppose that $u_0 \in \mathcal{U}_{S,c}$ satisfies the bound at the pairs already treated, and let (k,l) be the next pair; write $F_\tau = \mathrm{fm}(t_\tau\mathrm{r}_S^\infty(u_0)_\tau)$ and

$$r_\tau = g(S)\,N\,\mathrm{sc}_{S,\tau}(\mathfrak{f}_l)/\mathrm{sc}_{S,\tau}(e_k) = g(S)\,N\,\kappa_\tau(S)^{\,k-l}.$$

By (J7), with $d = \varpi(S)$, there is an integer x of E with $|F_{\tau,kl}/r_\tau - x_\tau| \le (2f)^2|\varpi(S)_\tau|$ for all τ. Let λ be the row vector with the entry $-xN$ at $\mathfrak{f}_l$ and 0 elsewhere. The position $(k,\mathfrak{f}_l)$ is free because $l>k$, and $(\lambda)_{\mathbb{A}_f}\,\mathrm{B}_S(c)$ is integral: the row e_l of $\mathrm{B}_S(c)$ is a unit row, and its row z is that of $\Lambda(S)$, with the entries $1/N$ at k_0 and 1 at z (R5). Lemma 5.2, with $s=0$, gives $u_1 \in \mathcal{U}_{S,c}$ with $u_1(e_{k'},i) = \delta_{e_{k'},i} + g(S)[k'=k]\,\lambda_i$ for $i \neq e_k^*$. By the formula for r_S^∞, and because the last row of the frame matrix of an element of $\mathcal{U}$ is a unit row, $\mathrm{fm}(\mathrm{r}_S^\infty(u_1)_\tau) = 1 - r_\tau x_\tau\,E_{kl}$ with E_{kl} the elementary matrix. As $\mathrm{fm}(\cdot)$ is multiplicative on $\mathcal{U}$,

$$\mathrm{fm}(t_\tau\mathrm{r}_S^\infty(u_0u_1)_\tau) = F_\tau\,(1 - r_\tau x_\tau E_{kl}) :$$

the column l of F_τ changes by $-r_\tau x_\tau$ times its column k, which is 0 below the row k and 1 in the row k. So the rows below k and the other entries of the row k are unchanged, and the entry (k,l) becomes $r_\tau(F_{\tau,kl}/r_\tau - x_\tau)$, of absolute value at most

$$|r_\tau|\,(2f)^2\,|\varpi(S)_\tau| \;\le\; b^{22}\cdot b^{60n}\cdot b^2\cdot b \;\le\; b^{60\varrho+91},$$

by $|\kappa_\tau(S)| \ge 1$ and $l > k$, $g(S) \le b^{22}$, $N \le b^{60n}$, $2f \le b$ and $|\varpi(S)_\tau| \le b$ (R4) and (R11). After the last pair the claim is proved.

Step 2: the function. The map $\theta\colon u \mapsto (\mathrm{fm}(\mathrm{r}_S^\infty(u)_\tau))_\tau$ is a homomorphism on $\mathcal{U}_{S,c}$ with kernel $\mathcal{U}_{S,c}^0$ (R1). For $y \in \prod_\tau G_1(\mathfrak{s})$ and a coset $q = u\,\mathcal{U}_{S,c}^0$ let $\mathsf{I}(y,q) = 1$ if

$$\Big\| \sum_a \theta(u)_{\tau,al}\; y_\tau^{-1} \mathfrak{f}_a \Big\| \;\le\; b^{60\varrho+92}\; \mathrm{bs}(y_\tau) \qquad \text{for all } \tau \text{ and all } l \le \varrho,$$

and $\mathsf{I}(y,q) = 0$ otherwise. By (J1) the vector in the norm is $(\mathrm{r}_S^\infty(u)_\tau^{-1} y_\tau)^{-1}\mathfrak{f}_l$. Put $\Sigma(y) = \sum_q \mathsf{I}(y,q)$ and $\omega(y) = \mathsf{I}(y,1)/\Sigma(y)$. For $u \in \mathcal{U}_{S,c}$, (J1) gives $\mathrm{bs}(\mathrm{r}_S^\infty(u)_\tau y_\tau) = \mathrm{bs}(y_\tau)$, hence $\mathsf{I}(\mathrm{r}_S^\infty(u)y, q) = \mathsf{I}(y, u^{-1}q)$ and $\Sigma(\mathrm{r}_S^\infty(u)y) = \Sigma(y)$.

$\Sigma(y) \ge 1$. By (J1) choose $w_\tau \in \mathcal{U}$ with $\|w_\tau y_\tau\| \le \mathrm{bs}(y_\tau)$, and let u be as in Step 1 for $t = w$. By (J1) for $w_\tau \mathrm{r}_S^\infty(u)_\tau \in \mathcal{U}$, $(\mathrm{r}_S^\infty(u)_\tau^{-1} y_\tau)^{-1}\mathfrak{f}_l = \sum_a \mathrm{fm}(w_\tau \mathrm{r}_S^\infty(u)_\tau)_{al}\,(w_\tau y_\tau)^{-1}\mathfrak{f}_a$, of norm at most $(\varrho+1)\,b^{60\varrho+91}\,\|w_\tau y_\tau\| \le b^{60\varrho+92}\,\mathrm{bs}(y_\tau)$. So $\mathsf{I}(y, u\,\mathcal{U}_{S,c}^0) = 1$.

$\Sigma(y) < \infty$. If $\mathsf{I}(y, u\,\mathcal{U}_{S,c}^0) = 1$ then, by the last inequality of (J1) with $s = 1$ and the index $\mathfrak{f}_a$, $|\theta(u)_{\tau,al}| \le \|y_\tau\|\, b^{60\varrho+92}\,\mathrm{bs}(y_\tau)$ for all a, l. So $\theta(u)$ lies in a compact set that depends only on y, and by (S6) only finitely many values $\theta(u)$ occur there; θ is injective on cosets.

The five properties. (i) Every $\mathsf{I}(\cdot, q)$ is the indicator of a set defined by inequalities between continuous functions, and there are countably many q. (ii) If $\omega(y) \ne 0$ then $\mathsf{I}(y,1) = 1$, so $\|y_\tau^{-1}\mathfrak{f}_l\| \le b^{60\varrho+92}\,\mathrm{bs}(y_\tau)$; if also $y_\tau \in \mathrm{Cut}(b)$ this is at most $b^{60\varrho+92+6n+2} \le b^{2n^2} = \mathrm{bx}(S)$, as $\varrho \ge 10^5$. (iii) Let α be as in (R9). Then

$$\sum_{\gamma \in \mathcal{U}_{S,c}} \alpha(\gamma)\,\omega(\mathrm{r}_S^\infty(\gamma)y) = \frac{1}{\Sigma(y)} \sum_q \mathsf{I}(y, q^{-1}) \sum_{\gamma \in q} \alpha(\gamma) = \frac{1}{\Sigma(y)} \sum_q \mathsf{I}(y, q^{-1}) = 1,$$

because the sum of α over a coset of $\mathcal{U}_{S,c}^0$ is 1 and $0 < \Sigma(y) < \infty$. (iv) $\mathsf{I}(y,1) \le \Sigma(y)$. (v) If every s_τ lies in the group generated by $\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))$ and $\mathcal{H}(T^{\mathrm{rat}}(\mathfrak{s};1))$, which is contained in $\mathcal{S}(T^{\mathrm{rat}}(\mathfrak{s};1))$, then by (J1) the frame vectors and the numbers $\mathrm{bs}(\cdot)$ of sy and y agree, so $\mathsf{I}(sy, q) = \mathsf{I}(y, q)$ for all q. □

LEAN `Filling.isBoxPartition_epsilon`

Lemma 7.3 (the cutoffs)**.** *In Setting 1.2, let $c \in \mathrm{Cl}(S)$.*

(a) $\sum_{u \in \mathcal{U}_{S,c}} \beta_{S,c}(\mathrm{r}_S^\infty(u)\,y) = 1$ *for every* $y \in \prod_\tau G_1(\mathfrak{s})$.
(b) $\beta_{S,c}$ *is measurable.*
(c) If $y \in \prod_\tau G_1(\mathfrak{s})$ *satisfies* $y_\tau \in \mathrm{Cut}(b(S))$ *for all* τ*, then there is* $u \in \mathcal{U}_{S,c}$ *with* $\mathrm{r}_S^\infty(u)\,y \in \mathrm{BxP}(S)$.

Proof. (a) Apply property (iii) of the box partition $\omega_{S,c}$ (Proposition 7.2) to $\alpha(\gamma) = \beta^\flat_{S,c}(\mathrm{r}_S^\infty(\gamma)y)$, whose sums over the cosets $\mathcal{U}_{S,c}^0\gamma$ are 1 by the choice of $\beta^\flat_{S,c}$. (b) Both factors are measurable. (c) By (a) some $u \in \mathcal{U}_{S,c}$ has $\beta_{S,c}(\mathrm{r}_S^\infty(u)y) \ne 0$, hence $\omega_{S,c}(\mathrm{r}_S^\infty(u)y) \ne 0$. Since $\mathrm{r}_S^\infty(u)_\tau \in \mathcal{U}$, (J1) gives $\mathrm{r}_S^\infty(u)_\tau y_\tau \in \mathrm{Cut}(b(S))$, and property (ii) puts $\mathrm{r}_S^\infty(u)_\tau y_\tau$ in $\mathrm{FBox}(\mathrm{bx}(S))$. □

LEAN `Filling.tsum_classCutoff_eq_one` · `Filling.measurable_classCutoff` · `Filling.exists_classUnip_mul_mem_boxedPoints`

Proof of Theorem B. Let $\mathsf{A} = \{p : \mathsf{w}_p \ne 0\}$, and give the space of families $(v_{\tau,a})$ the norm $\sup_{\tau,a}\|v_{\tau,a}\|$. Since $\mathrm{r}_S^\infty(\gamma_p)^{-1}y \in \mathrm{BxP}(S)$, we have $\|F_p\| \le \mathrm{bx}(S)$ for $p \in \mathsf{A}$.

Step 1: close frames are equal. Let $p, q \in \mathsf{A}$ with $\varepsilon = \|F_p - F_q\| < (N\,\mathrm{bx}(S))^{-1}$. We show $\gamma_p \mathfrak{f} = \gamma_q \mathfrak{f}$. Put $g = \gamma_p^{-1}\gamma_q$ and $y^\circ = \mathrm{r}_S^\infty(\gamma_p)^{-1} y \in \mathrm{BxP}(S)$. By (S3), $g \in \mathrm{Tm}_S(d_p, 1, d_q)$, and $1 \in \mathrm{Tm}_S(d_p, 1, d_p)$. For $u \in G_S(E)$ let $F'(u)$ be the family of the vectors $(\mathrm{r}_S^\infty(u)_\tau^{-1} y_\tau^\circ)^{-1}\mathfrak{f}_a$; then $F'(1) = F_p$ and $F'(g) = F_q$.

(a) $Z_S(g) = Z_S(1)$. By (J1), $Z(\mathrm{r}_S^\infty(u)_\tau)_{la}$ is the hermitian product of $(y_\tau^\circ)^{-1}\mathfrak{f}_l$ with the image under $J(T^{\mathrm{rat}}(\mathfrak{s}; 1))$ of the component (τ, a) of $F'(u)$. So, by (J1) again, $|Z(\mathrm{r}_S^\infty(g)_\tau)_{la} - Z(1)_{la}| \le \mathrm{bx}(S)\,\varepsilon$, and by the last formula of (S5) and $|\kappa_\tau(S)| \ge 1$, $|(Z_S(g)_{la} - Z_S(1)_{la})_\tau| \le N\,\mathrm{bx}(S)\,\varepsilon < 1$ for all τ. The entries of $Z_S(g)$ and $Z_S(1)$ are algebraic integers (S3); by (J4) they agree.

(b) *The rows of $g\mathfrak{f}$ and $\mathfrak{f}$ outside the rows e_a agree.* $Z_S(1)$ is the diagonal matrix with the entries $0, \dots, 0, -N$. The entry (a, l) of $Z_S(g)$ is $g(e_a^*, \mathfrak{f}_l)$ for $a < \varrho$ and $-N\,g(z, \mathfrak{f}_l)$ for $a = \varrho$. So the rows e_a^* and z of $g\mathfrak{f}$ are those of $\mathfrak{f}$. Let v_l be the middle part of $(g-1)\mathfrak{f}_l$; its z-coordinate is 0. The vectors $g\mathfrak{f}_l$ and $\mathfrak{f}_l$ have no e^*-coordinates, so $h(g\mathfrak{f}_l, g\mathfrak{f}_l) = h(\mathfrak{f}_l, \mathfrak{f}_l)$ reads $(v_l + \mathrm{mid}\,\mathfrak{f}_l)^* T (v_l + \mathrm{mid}\,\mathfrak{f}_l) = (\mathrm{mid}\,\mathfrak{f}_l)^* T\,\mathrm{mid}\,\mathfrak{f}_l$ with $T = T_S^{\mathrm{rat}}(E)$; as T is diagonal and $\mathrm{mid}\,\mathfrak{f}_l$ is 0 or the unit vector at z, this gives $v_l^* T v_l = 0$, and $v_l = 0$ by (S5).

(c) *The rows e_a.* We show $(g-1)(e_a, \mathfrak{f}_l) = 0$ for all l, by descending induction on $a < \varrho$. By (b) and Lemma 7.1 (for g and 1), the component (τ, l) of $F'(g) - F'(1)$ is $\sum_{a'<\varrho} c_{a'}\,(y_\tau^\circ)^{-1}\mathfrak{f}_{a'}$ with $c_{a'} = ((g-1)(e_{a'}, \mathfrak{f}_l))_\tau\,\mathrm{sc}_{S,\tau}(\mathfrak{f}_l)/\mathrm{sc}_{S,\tau}(e_{a'})$, and $c_{a'} = 0$ for $a' > a$ by the induction hypothesis. By (J1) there is $w \in \mathcal{U}$ with $\|w y_\tau^\circ\| \le \mathrm{bs}(y_\tau^\circ) \le b^{6n+2}$, and

$$\Big|\sum_{a'<\varrho} c_{a'}\, w(e_a, e_{a'})\Big| \;\le\; \|w y_\tau^\circ\| \cdot \Big\|\sum_{a'<\varrho} c_{a'}\,(y_\tau^\circ)^{-1}\mathfrak{f}_{a'}\Big\| \;\le\; b^{6n+2}\,\varepsilon.$$

As $w(e_a, e_{a'})$ is 0 for $a' < a$ and 1 for $a' = a$, the left side is $|c_a|$. Hence

$$\big|((g-1)(e_a, \mathfrak{f}_l))_\tau\big| \;\le\; |\kappa_\tau(S)|^{\,l-a}\, b^{6n+2}\,\varepsilon \;\le\; b^{23\varrho+6n+2}\,\varepsilon \;\le\; \mathrm{bx}(S)\,\varepsilon < 1,$$

using $1 \le |\kappa_\tau(S)| \le b^{23}$. The numbers $g(e_a, \mathfrak{f}_l)$ are algebraic integers: by (R5) the matrices $\mathrm{B}_S(\cdot)$ are the identity on the rows e_a and fix the frame vectors, so $g(e_a, \mathfrak{f}_l)$ is an entry of the position $\mathrm{pos}_S(d_p, d_q; g)$, which is integral by (S3). By (J4), $(g-1)(e_a, \mathfrak{f}_l) = 0$.

So $g\mathfrak{f} = \mathfrak{f}$, that is $\gamma_p\mathfrak{f} = \gamma_q\mathfrak{f}$ by (S5).

Step 2: the count. Let $\mathsf{A}_0 \subseteq \mathsf{A}$ contain exactly one p for every matrix $\gamma_p\mathfrak{f}$, $p \in \mathsf{A}$. By Step 1 the points F_p, $p \in \mathsf{A}_0$, are pairwise at distance at least $(N\,\mathrm{bx}(S))^{-1}$. Fix $p_0 \in \mathsf{A}_0$; the points $F_p - F_{p_0}$ lie in V, by (iii), and have norm at most $2\,\mathrm{bx}(S)$. By (J3),

$$\#\mathsf{A}_0 \;\le\; \big(1 + 4N\,\mathrm{bx}(S)^2\big)^{\dim_{\mathbb{R}} V} \;\le\; b^{\,6n^2 \dim_{\mathbb{R}} V},$$

since $1 + 4N\,\mathrm{bx}(S)^2 \le b^{3+60n+4n^2} \le b^{6n^2}$.

Colour $p \in \mathsf{A}$ by the matrix $\gamma_p\mathfrak{f}$; there are $\#\mathsf{A}_0$ colours. Call p, q related if $d_p = d_q$ and $\gamma_p^{-1}\gamma_q \in \mathcal{U}_{S,d_p}^0$. Two elements of one colour are related, by (S3) and (ii). Let $\mathsf{B} \subseteq \mathsf{A}$ be a set of pairwise related elements and $p \in \mathsf{B}$; write $d = d_p$. For $q \in \mathsf{B}$ put $u_q = \delta_q^{-1}\gamma_p = (\delta_q^{-1}\gamma_q)(\gamma_q^{-1}\gamma_p) \in \mathcal{U}_{S,d}$; by (i) the map $q \mapsto u_q$ is injective on B. By (ii) and Lemma 7.3(a),

$$\sum_{q\in\mathsf{B}} \mathsf{w}_q \;\le\; x \sum_{q\in\mathsf{B}} \beta_{S,d}\big(\mathrm{r}_S^\infty(u_q)\,\mathrm{r}_S^\infty(\gamma_p)^{-1} y'\big) \;\le\; x.$$

Lemma 6.1 gives $\sum_{p\in\mathsf{A}} \mathsf{w}_p \le \#\mathsf{A}_0 \cdot x$, which is the assertion. □

8 The covolume of the arithmetic unipotent group

As before $\mathcal{U} = \mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s}; 1))$. For $a < \varrho$ let $\mathrm{Fr}(a)$ be the set of the $l \in I(\mathfrak{s})$ such that (a, l) is a free position (Section 5): the e_b and e_b^* with $b > a$ and the middle indices; it has $2(\varrho - 1 - a) + k(\mathfrak{s}) + 1$ elements.

Inputs. (J8), unit 118 (coordinates of the unipotent group). The *coordinates* of $u \in \mathcal{U}$ are the complex numbers $u(e_a, l)$, $a < \varrho$, $l \in \mathrm{Fr}(a)$, and the real numbers $\operatorname{Im} u(e_a, e_a^*)$, $a < \varrho$. The coordinate map is injective on $\mathcal{U}$ and carries the measure $\mu_{\mathfrak{s}}^{\mathcal{U}}$ to Lebesgue measure.
(J9), unit 106 (fundamental domains). (i) Let Ω be a finite set, $(t_{l,\tau})$ non-zero complex numbers indexed by $l \in \Omega$ and the infinite places τ of E, and M an invertible matrix over $\mathbb{A}_{E,f}$ on Ω. The group of the families $(t_{l,\tau}\, x_{l,\tau})_{l,\tau}$, for $x \in E^{\Omega}$ with $M(x)_{\mathbb{A}_f}$ integral, has a measurable set $F \subseteq \mathbb{C}^{\Omega \times \{\tau\}}$ of volume $\prod_\tau \left|\prod_l t_{l,\tau}\right|^2 / ([\det M]\ \mathrm{vol}(\hat{O}_E)^{\#\Omega})$ that contains exactly one translate of every point by the group. (ii) Let E be stably unramified and (t_τ) non-zero real numbers. The group of the families $(t_\tau \operatorname{Im} s_\tau)_\tau$, for integers s of E with $\overline{s} = -s$, has a measurable set $F \subseteq \mathbb{R}^{\{\tau\}}$ of volume $\prod_\tau |t_\tau| \cdot |d_E|^{1/4}$ with the same property. (iii) Let G be a group, Γ a subgroup, $(V_i)_{i \in \iota}$ a finite family of abelian groups with a degree $\deg \colon \iota \to \mathbb{N}$, $\mathsf{c} = (\mathsf{c}_i) \colon G \to \prod_i V_i$ an injective map, and for every i a submonoid $L_i \subseteq V_i$ and a set $F_i \subseteq V_i$ that contains exactly one translate of every point by L_i. Suppose that (1) for every i and every $l \in L_i$ there is $\gamma \in \Gamma$ with $\mathsf{c}_i(\gamma) = l$ and $\mathsf{c}_{i'}(\gamma) = \mathsf{c}_{i'}(1)$ for all $i' \neq i$ with $\deg i' \leq \deg i$; and (2) for every i and every $\gamma \in \Gamma$ with $\mathsf{c}_{i'}(\gamma) = \mathsf{c}_{i'}(1)$ for all i' with $\deg i' < \deg i$, one has $\mathsf{c}_i(\gamma) \in L_i$ and $\mathsf{c}_i(x\gamma) = \mathsf{c}_i(\gamma) + \mathsf{c}_i(x)$ for all $x \in G$. Then for every $x \in G$ there is exactly one $\gamma \in \Gamma$ with $\mathsf{c}(x\gamma) \in \prod_i F_i$.
The catalogue classes these inputs as elementary and gives no reference.

Lemma 8.1 (additivity of the coordinates)**.** *Let $u \in \mathcal{U}$ and $a < \varrho$. Suppose that the coordinates of u in the rows e_b, $b > a$, vanish: the entries $u(e_b, l)$, $l \in \mathrm{Fr}(b)$, and the numbers $\operatorname{Im} u(e_b, e_b^*)$.*

(a) Then $(xu)(e_a, l) = x(e_a, l) + u(e_a, l)$ for all $x \in \mathcal{U}$ and all $l \in \mathrm{Fr}(a)$.
(b) If moreover $u(e_a, l) = 0$ for all $l \in \mathrm{Fr}(a)$, then $\operatorname{Re} u(e_a, e_a^*) = 0$ and $(xu)(e_a, e_a^*) = x(e_a, e_a^*) + u(e_a, e_a^*)$ for all $x \in \mathcal{U}$.

Proof. Since $(xu)(e_a, l) = x(e_a, l) + u(e_a, l) + \sum_i (x-1)(e_a, i)\,(u-1)(i, l)$ and $(x-1)(e_a, i) = 0$ unless $\mathrm{lev}(i) > a$, it suffices to show that $(u-1)(i, l) = 0$ for all i with $\mathrm{lev}(i) > a$, that is for $i = e_b$ with $b > a$, i a middle index, or $i = e_b^*$; and for $l \in \mathrm{Fr}(a)$ in (a), $l = e_a^*$ in (b).

Let $J = J(T^{\mathrm{rat}}(\mathfrak{s}; 1))$; then $J^2 = 1$, so $uJu^* = J$: writing ρ_i for the row i of u and $\langle \rho, \rho' \rangle = \rho J \rho'^*$, we have $\langle \rho_i, \rho_{i'} \rangle = J(i, i')$. Let $c > a$, or $c = a$ in case (b). Outside the columns e_d^* the row ρ_{e_c} is the unit row at e_c, because $u \in \mathcal{U}$ and its free entries vanish. Hence $\langle \rho, \rho_{e_c} \rangle = \rho(e_c^*) + \sum_d \rho(e_d)\, \overline{u(e_c, e_d^*)}$ for every row vector ρ.

For i a middle index or $i = e_b^*$ we have $\rho_i(e_d) = 0$, so $u(i, e_c^*) = J(i, e_c) = \delta_{i, e_c^*}$. Now let $i = e_b$ with $b > a$, or $b = a$ in case (b). Then $\rho_{e_b}(e_d) = \delta_{bd}$, so $u(e_b, e_c^*) + \overline{u(e_c, e_b^*)} =$

0. If $b < c$ the first term is a free entry of the row e_b, hence 0, and so is the second; by symmetry both terms vanish whenever $b \neq c$. If $b = c$ the relation says $\operatorname{Re} u(e_b, e_b^*) = 0$; for $b > a$ also $\operatorname{Im} u(e_b, e_b^*) = 0$ by hypothesis, and for $b = a$ this is the first assertion of (b). This proves $(u-1)(i, e_c^*) = 0$ for all i of level greater than a and all such c.

It remains to treat, in (a), the columns $l = e_c$ with $c > a$ and the middle columns l. There $u(i,l) = \delta_{il}$ if $\operatorname{lev}(i) \geq \operatorname{lev}(l)$, and otherwise $i = e_b$ with $a < b$, and (b, l) is a free position. □

Proof of Theorem C. Write $g = g(S)$, $N = N(S)$, $k = k(\mathfrak{s})$, $f = f(E)$. Let $G = \prod_\tau \mathcal{U}$ with the measure $\mu = \prod_\tau \mu_{\mathfrak{s}}^{\mathcal{U}}$, let $\varphi\colon \mathcal{U}_{S,c} \to G$, $u \mapsto (\mathrm{r}_S^\infty(u)_\tau)_\tau$, an injective homomorphism (R5), and Γ its image.

The class basis. By (R5), $\mathrm{B}_S(c)$ is block diagonal: the identity on the e_a and e_a^*, and a matrix m_c on $M(\mathfrak{s})$, the product of $\operatorname{diag}(\tilde{c}, 1)$ and the middle block of $\Lambda(S)$, whose determinant is $\overline{\bar{\pi}(S)}^{-1}$. As $\tilde{c}$ is unitary, $[\det \tilde{c}]^2 = 1$; and $[\bar{\pi}(S)]^2 = [N] = N^{2f}$ by (J2). Hence $[\det m_c] = N^{-f}$. For $a < \varrho$ let $m(a)$ be the transpose of the block of $\mathrm{B}_S(c)$ on $\mathrm{Fr}(a)$: for a row vector x supported on $\mathrm{Fr}(a)$, $m(a)x$ is the restriction of $x\,\mathrm{B}_S(c)$ to $\mathrm{Fr}(a)$, and $x\,\mathrm{B}_S(c)$ vanishes outside $\mathrm{Fr}(a)$. Its determinant is $\det m_c$.

Graded coordinates. The index set ι consists of the pairs (a, fr) and (a, sk), $a < \varrho$, with $\deg(a, \mathrm{fr}) = 2(\varrho - 1 - a)$ and $\deg(a, \mathrm{sk}) = 2(\varrho - 1 - a) + 1$. For $u = (u_\tau) \in G$ put $\mathsf{c}_{a,\mathrm{fr}}(u) = (u_\tau(e_a, l))_{l \in \mathrm{Fr}(a), \tau}$ and $\mathsf{c}_{a,\mathrm{sk}}(u) = (\operatorname{Im} u_\tau(e_a, e_a^*))_\tau$. By (J8), c is injective and carries μ to Lebesgue measure. Let

$$t_{l,\tau} = g\,\mathrm{sc}_{S,\tau}(l)/\mathrm{sc}_{S,\tau}(e_a) \quad (l \in \mathrm{Fr}(a)), \qquad t'_\tau = g/\mathrm{fs}_S(\tau, a),$$

let $L_{a,\mathrm{fr}}$ be the group of the families $(t_{l,\tau} x_{l,\tau})$, $x \in E^{\mathrm{Fr}(a)}$ with $m(a)(x)_{\mathbb{A}_f}$ integral, and $L_{a,\mathrm{sk}}$ the group of the families $(t'_\tau \operatorname{Im} s_\tau)$, s an integer of E with $\overline{s} = -s$. Let $F_{a,\mathrm{fr}}$, $F_{a,\mathrm{sk}}$ be the sets given by (J9)(i),(ii).

Condition (1) of (J9)(iii). Let $(t_{l,\tau} x_{l,\tau}) \in L_{a,\mathrm{fr}}$, and extend x by 0 to a row vector on $I(\mathfrak{s})$. Then $(x)_{\mathbb{A}_f} \mathrm{B}_S(c)$ is integral, and Lemma 5.2 with $s = 0$ gives $u \in \mathcal{U}_{S,c}$ with $u(e_b, l) = \delta_{e_b,l} + g\,[b = a]\,x_l$ for all $b < \varrho$ and $l \neq e_a^*$. By the formula for r_S^∞, $\mathsf{c}_{a,\mathrm{fr}}(\varphi(u)) = (t_{l,\tau} x_{l,\tau})$, and all other coordinates of $\varphi(u)$ vanish, except possibly $\mathsf{c}_{a,\mathrm{sk}}$, whose degree is larger. Let $(t'_\tau \operatorname{Im} s_\tau) \in L_{a,\mathrm{sk}}$. Lemma 5.2 with $l = 0$ gives $u \in \mathcal{U}_{S,c}$ with $u(e_b, j) = \delta_{e_b,j} + g\,[b = a]\,[j = e_a^*]\,s$ for all b and j. Since $\mathrm{sc}_{S,\tau}(e_a^*)/\mathrm{sc}_{S,\tau}(e_a) = 1/\mathrm{fs}_S(\tau, a)$ is real, $\mathsf{c}_{a,\mathrm{sk}}(\varphi(u)) = (t'_\tau \operatorname{Im} s_\tau)$, and all other coordinates vanish.

Condition (2). Let $u \in \mathcal{U}_{S,c}$, let $i = (a, \mathrm{fr})$ or $i = (a, \mathrm{sk})$, and suppose that the coordinates of $\varphi(u)$ of degree less than $\deg i$ vanish. The additivity $\mathsf{c}_i(x\varphi(u)) = \mathsf{c}_i(\varphi(u)) + \mathsf{c}_i(x)$ is Lemma 8.1, applied at every τ. For the membership in L_i note first that the row e_a of $\hat{g}(S)^{-1}(u-1)_{\mathbb{A}_f} \mathrm{B}_S(c)$ is integral: this matrix equals $\mathrm{B}_S(c)\,\hat{g}(S)^{-1}(\mathrm{pos}_S(c, c; u) - 1)$, the row e_a of $\mathrm{B}_S(c)$ is a unit row, and $\hat{g}(S)^{-1}(\mathrm{pos}_S(c, c; u) - 1)$ is integral by (S1). If $i = (a, \mathrm{fr})$, put $x_l = g^{-1} u(e_a, l)$ for $l \in \mathrm{Fr}(a)$; then $\mathsf{c}_i(\varphi(u)) = (t_{l,\tau} x_{l,\tau})$, and $m(a)(x)_{\mathbb{A}_f}$ is the restriction to $\mathrm{Fr}(a)$ of that integral row, because the entries of the row e_a of $u - 1$ outside $\mathrm{Fr}(a)$ sit in columns e_d^*, $d \leq a$, on which $\mathrm{B}_S(c)$ is the identity. If $i = (a, \mathrm{sk})$, put $s = g^{-1} u(e_a, e_a^*)$. It is integral at all finite places, hence an integer of E; and $\operatorname{Re} \mathrm{r}_S^\infty(u)_\tau(e_a, e_a^*) = 0$ by Lemma 8.1(b), that is $s_\tau + \overline{s_\tau} = 0$ for all τ, so $\overline{s} = -s$. Then $\mathsf{c}_i(\varphi(u)) = (t'_\tau \operatorname{Im} s_\tau) \in L_{a,\mathrm{sk}}$.

Unfolding. Let $\mathfrak{B} = \prod_i F_i$ and $D = \mathfrak{c}^{-1}(\mathfrak{B}) \subseteq G$. By (J9)(iii), for every $x \in G$ there is exactly one $\gamma \in \Gamma$ with $x\gamma \in D$; applied to x^{-1}: exactly one $\gamma \in \Gamma$ with $\gamma x \in D^{-1}$. So G is the disjoint union of the sets $\gamma^{-1}D^{-1}$, and by the left invariance of μ and Lemma 7.3(a),

$$\int_G \beta_{S,c}(uy)\,d\mu(u) = \int_{D^{-1}} \sum_{\gamma \in \mathcal{U}_{S,c}} \beta_{S,c}(\mathrm{r}_S^{\infty}(\gamma)\,uy)\,d\mu(u) = \mu(D^{-1}) = \mu(D),$$

the last by the invariance of μ under inversion (R3). By (J8), $\mu(D)$ is the Lebesgue measure of $\mathfrak{B}$, the product of the volumes of the F_i.

The volume. The scales of the e_b and e_b^* are inverse to each other, so

$$\prod_{l \in \mathrm{Fr}(a)} |\mathrm{sc}_{S,\tau}(l)|^2 = |\mathrm{sc}_{S,\tau}(z)|^2 = N^{-1} \qquad \text{and} \qquad |\mathrm{sc}_{S,\tau}(e_a)|^{-2} = \mathrm{fs}_S(\tau,a)^{-1}.$$

With $n_a = \#\mathrm{Fr}(a)$, (J9) and $[\det m(a)] = N^{-f}$ give

$$\mathrm{vol}(F_{a,\mathrm{fr}}) = \frac{g^{2n_a f}}{\mathrm{vol}(\hat{O}_E)^{\,n_a}} \prod_\tau \mathrm{fs}_S(\tau,a)^{-n_a}, \qquad \mathrm{vol}(F_{a,\mathrm{sk}}) = g^f\,|d_E|^{1/4} \prod_\tau \mathrm{fs}_S(\tau,a)^{-1}.$$

Now $n_a + 1 = k + 2(\varrho - a)$ and $\sum_{a<\varrho} n_a = \varrho(\varrho + k)$, so the product over $a < \varrho$ of these volumes is

$$g^{\varrho(2\varrho+2k+1)f}\ \mathrm{vol}(\hat{O}_E)^{-\varrho(\varrho+k)}\ |d_E|^{\varrho/4} \prod_\tau \prod_{a<\varrho} \mathrm{fs}_S(\tau,a)^{-(k+2(\varrho-a))}.$$

This is $\nu(S)$ (R10): the factor $a = \varrho$ of $\mathrm{fs}(S)$ is $\prod_\tau \mathrm{fs}_S(\tau,\varrho)^k = N^{-kf}$, which cancels the factor N^{-kf} of $\nu(S)$. □

LEAN `Filling.lintegral_classCutoff_eq_unipCovolume`

Lemma 8.2 (unfolding in products)**.** *Let A be a subgroup of $G_1(\mathfrak{s})$ and μ_A a measure on A that is a unimodular Haar measure with a cutoff: a σ-finite Haar measure of A, invariant under right translations and inversion, for which there is a measurable $\beta \geq 0$ on $G_1(\mathfrak{s})$ with $\int_A \beta(ay)\,d\mu_A(a) = 1$ for all y; let $\mu_{A\backslash} = \beta\,dy$ for the chosen such β. Then for every finite set Ω the map $((x_p)_p, (a_p)_p) \mapsto (a_p x_p)_p$ carries the measure $(\prod_{p\in\Omega} \mu_{A\backslash}) \times (\prod_{p\in\Omega} \mu_A)$ to the Haar measure $\prod_{p\in\Omega} dy$ of $G_1(\mathfrak{s})^\Omega$.*

Proof. For one factor this is the unfolding of the background library of unit 114, recalled in (R3). A product of measure preserving maps preserves the product measures, and regrouping the factors does. □

LEAN `Filling.measurePreserving_pi_orbitMeasure`

The formal text records here that $\mu_{A\backslash}$ is σ-finite, the chosen β being finite almost everywhere, and later that $\mathsf{G}(S)$ is finite; both are unnamed instances.

Lemma 8.3 (integrals against a cutoff)**.** *In Setting 1.2, let $c \in \mathrm{Cl}(S)$ and, for every infinite place τ of E, let $F_\tau \colon G_1(\mathfrak{s}) \to [0,\infty]$ be measurable and invariant under left translation by $\mathcal{U}$. Then*

$$\int \beta_{S,c}(y) \prod_\tau F_\tau(y_\tau)\,dy = \nu(S) \cdot \prod_\tau \int F_\tau\,d\mu_{\mathfrak{s}}^{\mathcal{U}\backslash}.$$

Proof. By Lemma 8.2 for $A = \mathcal{U}$ and the set of the infinite places, the left side is $\iint \beta_{S,c}((u_\tau x_\tau)_\tau) \prod_\tau F_\tau(u_\tau x_\tau)\,d\mu(u)\,d(\prod_\tau \mu_{\mathfrak{s}}^{\mathcal{U}\backslash})(x)$. The factors $F_\tau(u_\tau x_\tau) = F_\tau(x_\tau)$ do not depend on u, the integral of the cutoff over u is $\nu(S)$ by Theorem C, and the remaining integral over x is the product of the integrals of the F_τ. □

LEAN `Filling.lintegral_classCutoff_mul_prod`

9 Amplitude sums

In this section $\mathcal{U} = \mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s};1))$, and for $\gamma \in G_S(E)$ and $y \in \prod_\tau G_1(\mathfrak{s})$ we write $\gamma^{-1}y$ for $\mathrm{r}_S^\infty(\gamma)^{-1}y$ and γy for $\mathrm{r}_S^\infty(\gamma)y$. All integrals over y are over $\prod_\tau G_1(\mathfrak{s})$ for the Haar measure dy, which is invariant under these translations.

Lemma 9.1. *For $S \in \mathrm{Fill}(E,\mathfrak{s})$, the group $\mathcal{S}(T_S^{\mathrm{rat}}(E))$ of the rational points of the Bessel subgroup is stable: $\mathcal{S}(T_S^{\mathrm{rat}}(E)) \in \mathrm{Stab}(S)$.*

Proof. It is a group that contains $\mathcal{U}(T_S^{\mathrm{rat}}(E))$ (R1). □

LEAN `Filling.isStable_besselS`

Lemma 9.2 (coordinates of a sum of integrals)**.** *Let (Y,μ) be a measure space, P a countable set, I a finite set, and for $p \in \mathsf{P}$ let $g_p\colon \mathsf{Y} \to \mathbb{C}^I$ be measurable, with $\sum_{p\in\mathsf{P}} \int \|g_p\|\,d\mu < \infty$. Then for every $i \in I$,*

$$\Big(\sum_{p\in\mathsf{P}} \int g_p\,d\mu\Big)_i = \int \sum_{p\in\mathsf{P}} g_p(x)_i\,d\mu(x).$$

Proof. Each g_p is integrable and its integral has the coordinates $\int (g_p)_i\,d\mu$; the sum of the integrals converges absolutely, so its coordinate i is $\sum_p \int (g_p)_i\,d\mu$; and since $\sum_p \int |(g_p)_i|\,d\mu < \infty$, sum and integral may be exchanged. □

LEAN `Filling.tsum_integral_apply`

Lemma 9.3 (invariance of the Bessel functions and of the kernel)**.** *Let $S \in \mathrm{Fill}(E,\mathfrak{s})$ and let σ be a side.*

(a) For $c \in \mathrm{Cl}(S)$, $u \in \mathcal{U}_{S,c}$, $y \in \prod_\tau G_1(\mathfrak{s})$ and $i \in \mathrm{Bc}(S)$: $W_S^\sigma(\mathrm{r}_S^\infty(u)\,y)_i = W_S^\sigma(y)_i$.
(b) For $u,u' \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$, $\gamma \in G_S(E)$ and $y \in \prod_\tau G_1(\mathfrak{s})$:

$$\mathcal{W}_S^\sigma(u\gamma u'^{-1};\mathrm{r}_S^\infty(u)\,y) = \psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u))^{-1}\,\psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u'))\cdot \mathcal{W}_S^\sigma(\gamma;y).$$

Proof. (a) By (S4), $W_S^\sigma(\mathrm{r}_S^\infty(u)y)_i = \psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u))^{-1} W_S^\sigma(y)_i$. The position of u, which is $\mathrm{pos}_S(c,c;u) = h_S(c)^{-1}\mathrm{r}_S^f(u)h_S(c)$, lies in $K_g(S)$, so the second statement of (S4), for the pair $(1,\mathrm{r}_S^f(u))$, gives $\psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u)) = \psi_S^{\mathcal{U}}(1) = 1$. (b) The coordinate (i,j) of the left side is $W_S^\sigma(\mathrm{r}_S^\infty(u)y)_i$ times the conjugate of $W_S^\sigma(\mathrm{r}_S^\infty(u')\mathrm{r}_S^\infty(\gamma)^{-1}y)_j$. By (S4) these are $\psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u))^{-1}W_S^\sigma(y)_i$ and the conjugate of $\psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u'))^{-1}W_S^\sigma(\mathrm{r}_S^\infty(\gamma)^{-1}y)_j$, and the values of $\psi_S^{\mathcal{U}}(\cdot)$ have absolute value 1. □

LEAN `Filling.besselFn_mul_of_mem_classUnip` · `Filling.besselKernel_besselU_mul`

Proof of Theorem D. Write $b = b(S)$, $n = n(\mathfrak{s})$, $r = r(\mathfrak{s})$, $k = k(\mathfrak{s})$, $f = f(E)$. Take A_τ, m_τ and θ_τ from (S7) and put, for $y \in \prod_\tau G_1(\mathfrak{s})$ and a side t,

$$\mathbf{A}(y) = \prod_\tau A_\tau(y_\tau), \qquad \mathbf{W}^t(y) = \prod_\tau \|W_{S,\tau}^t(y_\tau)\|, \qquad \Theta(y) = \prod_\tau \theta_\tau(y_\tau) \;\ge 1,$$

so that $\Theta(y)\,\mathbf{W}^t(y) \le \mathbf{A}(y)$, and $\|\mathcal{W}_S^t(\gamma;y)\| = \mathbf{W}^t(y)\,\mathbf{W}^t(\gamma^{-1}y)$.

Step 1: the frame count. We claim that for every y, with $C_0 = b^{12n^3rf}$,

$$\sum_{d\in\mathrm{Cl}(S)} \sum_{\gamma\in X(d)} \beta_{S,d}(\gamma^{-1}y)\,\mathbf{A}(\gamma^{-1}y) \;\le\; C_0 \prod_\tau m_\tau(G_1(\mathfrak{s})). \tag{1}$$

Let $m = \prod_\tau m_\tau$, a finite measure on $\prod_\tau G_1(\mathfrak{s})$, and for a point v let $Q(v)$ be the set of the g with $v_\tau g_\tau \in \mathrm{Cut}(b)$ for all τ. By (S7), $\mathbf{A}(v) \le m(Q(v))$. So the left side of (1) is at most the integral over g, for m, of

$$\sum_d \sum_{\gamma \in X(d)} \beta_{S,d}(\gamma^{-1}y)\, 1_{Q(\gamma^{-1}y)}(g),$$

and it suffices to bound this sum by C_0 for every g. Fix g and apply Theorem B to the following data: the index set P of the pairs $p = (d, \gamma)$ with $\gamma \in X(d)$; $d_p = d$, $\delta_p = \gamma$; the weight $\mathsf{w}_p = \beta_{S,d}(\gamma^{-1}y)\, 1_{Q(\gamma^{-1}y)}(g)$; $x = 1$; the point y in the place of y' and the point $yg = (y_\tau g_\tau)_\tau$ in the place of y; and V the whole space of families, of real dimension $2rnf$. For p with $\mathsf{w}_p \neq 0$ we have $g \in Q(\gamma^{-1}y)$, so by Lemma 7.3(c) there is $u_p \in \mathcal{U}_{S,d}$ with $\mathrm{r}_S^\infty(u_p)(\gamma^{-1}y)g \in \mathrm{BxP}(S)$; put $\gamma_p = \gamma u_p^{-1}$. Then hypothesis (i) of Theorem B is clear; for (ii), $\gamma_p \in \mathrm{Tm}_S(c, s, d)$ by (S3), $\delta_p^{-1}\gamma_p = u_p^{-1} \in \mathcal{U}_{S,d}$, $\gamma_p^{-1}(yg) = \mathrm{r}_S^\infty(u_p)(\gamma^{-1}y)g \in \mathrm{BxP}(S)$, and $\mathsf{w}_p \le \beta_{S,d}(\delta_p^{-1}y)$; and (iii) is empty. Theorem B bounds the sum by $b^{6n^2 \cdot 2rnf} = C_0$.

Step 2: integration. Multiply (1) by $\beta_{S,c}(y)\, \mathbf{A}(y)$ and integrate over y. By Lemma 8.3 and (S7), $\int \beta_{S,c}(y)\mathbf{A}(y)\, dy = \nu(S) \prod_\tau \int A_\tau \, d\mu_{\mathfrak{s}}^{\mathcal{U}\backslash}$ and $\prod_\tau m_\tau(G_1(\mathfrak{s})) \int A_\tau \, d\mu_{\mathfrak{s}}^{\mathcal{U}\backslash} \le b^{2r^3 f}\, \#\mathrm{Bc}(S)$. Hence

$$\sum_{d \in \mathrm{Cl}(S)} \sum_{\gamma \in X(d)} \int \beta_{S,c}(y)\, \beta_{S,d}(\gamma^{-1}y)\, \mathbf{A}(y)\, \mathbf{A}(\gamma^{-1}y)\, dy \;\le\; C_0\, b^{2r^3 f}\, \#\mathrm{Bc}(S) \cdot \nu(S). \quad (2)$$

Assertion (1) of the theorem. Since $\mathbf{W}^\sigma \le \mathbf{A}$, the first sum is at most the left side of (1), so $M = C_0 \prod_\tau m_\tau(G_1(\mathfrak{s}))$ will do. By (S7) and the definition, $|a|_S^\sigma(c, c'; \gamma) = \int \beta_{S,c}(y)\beta_{S,c'}(\gamma^{-1}y)\mathbf{W}^\sigma(y)\mathbf{W}^\sigma(\gamma^{-1}y)\, dy$, and the sum of these over $\gamma \in X(c')$ is at most the left side of (2), which is finite.

Assertion (2). We claim that for all γ and y

$$\left\| \mathcal{W}_S^\sigma(\gamma; y) - \mathcal{W}_S^{\mathrm{cut}}(\gamma; y) \right\| \;\le\; 2\, b^{-3r^3 f}\, \mathbf{A}(y)\, \mathbf{A}(\gamma^{-1}y).$$

If $\Theta(y) = \Theta(\gamma^{-1}y) = 1$, the full and the cut Bessel functions agree at all y_τ and all $(\gamma^{-1}y)_\tau$, and the left side is 0. Otherwise $\Theta(y)\,\Theta(\gamma^{-1}y) \ge b^{3r^3 f}$, and for both sides t, $\|\mathcal{W}_S^t(\gamma; y)\| = \mathbf{W}^t(y)\mathbf{W}^t(\gamma^{-1}y) \le b^{-3r^3 f}\mathbf{A}(y)\mathbf{A}(\gamma^{-1}y)$; the claim follows by the triangle inequality. With (2) and $\#\mathrm{Bc}(S) \le b^{2kf}$ (R11), the left side of assertion (2) is at most

$$2\, b^{-3r^3 f} \cdot C_0\, b^{2r^3 f}\, b^{2kf} \cdot \nu(S) = 2\, b^{(12n^3 r + 2k - r^3)f}\, \nu(S).$$

Now $r = k + 2n - 1$ with $2n - 1 \le k$ (R4), so $r \le 2k$ and $r^3 - k^3 \ge 3k^2$, while $12n^3 r + 2k \le 24n^3 k + 2k \le 2k^2$ because $k \ge 1000(2n)^9$. Hence the exponent is at most $-(k^3 + k^2)f$, and $2\, b^{-k^2 f} \le \frac{1}{2}$. □

LEAN `Filling.amplitudeAbs_lt_top_and_cut_le`

Corollary 9.4. *In Setting 1.2, for every side σ, every set $\Xi \subseteq G_S(E)$, all $c, c' \in \mathrm{Cl}(S)$ and $s \in \mathsf{G}(S)$, the family of the amplitudes $a_S^\sigma(c, c'; \gamma)$, $\gamma \in \Xi \cap \mathrm{Tm}_S(c, s, c')$, is absolutely summable.*

Proof. $\|a_S^\sigma(c, c'; \gamma)\| \le |a|_S^\sigma(c, c'; \gamma)$ by (S7), and Theorem D(1). □

LEAN `Filling.summable_amplitude`

Proof of Theorem E. The countable group $\Delta = \mathcal{U}_{S,c} \times \mathcal{U}_{S,c'}$ acts on $\mathsf{P} = X \times \prod_\tau G_1(\mathfrak{s})$ by $(u,u') \cdot (\gamma, y) = (u\gamma u'^{-1}, \mathrm{r}_S^\infty(u)y)$: indeed $u\gamma u'^{-1} \in \Xi$ because Ξ is stable, and $u\gamma u'^{-1} \in \mathrm{Tm}_S(c,s,c')$ by (S3), as $u \in \mathrm{Tm}_S(c,1,c)$ and $u'^{-1} \in \mathrm{Tm}_S(c',1,c')$. The action preserves the measure μ_P on P that is the product of the counting measure on X and dy. By Lemma 9.3(a) the functions $\mathsf{K}^t(\gamma, y) = \mathcal{W}_S^t(\gamma; y)$ are invariant under Δ, for both sides t. Put $\beta_0(\gamma, y) = \beta_{S,c}(y)\, \beta_{S,c'}(\gamma^{-1} y)$. By Lemma 7.3(a),

$$\sum_{(u,u') \in \Delta} \beta_0((u,u') \cdot (\gamma, y)) = \sum_u \beta_{S,c}(\mathrm{r}_S^\infty(u)y) \cdot \sum_{u'} \beta_{S,c'}(\mathrm{r}_S^\infty(u')\gamma^{-1} y) = 1,$$

and ϕ has the same property by hypothesis; in particular $\phi \le 1$ on P. For a measurable Δ-invariant $F \colon \mathsf{P} \to [0,\infty]$ we get, inserting the sum of β_0 and then moving δ to the other side by the invariance of μ_P and of F,

$$\begin{aligned} \int \phi F \, d\mu_\mathsf{P} &= \sum_{\delta \in \Delta} \int \phi(p) F(p) \beta_0(\delta p)\, d\mu_\mathsf{P}(p) \\ &= \sum_{\delta \in \Delta} \int \phi(\delta^{-1} p) F(p) \beta_0(p)\, d\mu_\mathsf{P}(p) = \int \beta_0 F \, d\mu_\mathsf{P}. \end{aligned} \tag{3}$$

For $F = \|\mathsf{K}^\sigma\|$ the right side is $\sum_{\gamma \in X} |a|_S^\sigma(c, c'; \gamma)$, finite by Theorem D(1). So every $y \mapsto \phi(\gamma, y) \mathcal{W}_S^\sigma(\gamma; y)$ is integrable, the sums below converge absolutely, and the computation (3), now with the vector valued K^σ in the place of F and Fubini's theorem, gives

$$\sum_{\gamma \in X} \int \phi(\gamma, y) \mathcal{W}_S^\sigma(\gamma; y)\, dy = \sum_{\gamma \in X} \int \beta_0(\gamma, y) \mathcal{W}_S^\sigma(\gamma; y)\, dy = A_S^\sigma(\Xi; c, s, c').$$

Finally (3) for $F = \|\mathsf{K}^\sigma - \mathsf{K}^{\mathrm{cut}}\|$ bounds the last sum of the theorem by the term $d = c'$ of Theorem D(2). □

LEAN `Filling.hasSum_amplitudeSum_change_of_weight`

Proposition 9.5 (moves)**.** *In Setting 1.2, let σ be a side and $\Xi \subseteq G_S(E)$ a set with $\Xi \in \mathrm{Stab}(S)$.*

(a) Let $c, c_1, c', c_1' \in \mathrm{Cl}(S)$, $t, t' \in \mathsf{G}(S)$, and let $\eta, \eta' \in \mathcal{S}(T_S^{\mathrm{rat}}(E))$ be such that $\eta \in \mathrm{Tm}_S(c, t, c_1)$ and $\eta' \in \mathrm{Tm}_S(c', t', c_1')$, and that $\gamma \in \Xi \Leftrightarrow \eta\gamma\eta'^{-1} \in \Xi$ for all $\gamma \in G_S(E)$. Let L be a continuous $\mathbb{C}$-linear map of $\mathbb{C}^{\mathrm{Bc}(S) \times \mathrm{Bc}(S)}$ to itself such that $\mathcal{W}_S^\sigma(\eta\gamma\eta'^{-1}; \mathrm{r}_S^\infty(\eta)\, y) = L(\mathcal{W}_S^\sigma(\gamma; y))$ for all γ and y. Then for every $s \in \mathsf{G}(S)$,

$$A_S^\sigma(\Xi; c, t\, s\, t'^{-1}, c') = L(A_S^\sigma(\Xi; c_1, s, c_1')).$$

(b) Let $u, u' \in K(S) \cap \mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$, $c, c' \in \mathrm{Cl}(S)$ and $s \in \mathsf{G}(S)$. Then

$$A_S^\sigma(\Xi; c, \mathrm{red}_S(u)\, s\, \mathrm{red}_S(u')^{-1}, c') = \psi_S^\mathcal{U}(u)^{-1}\, \psi_S^\mathcal{U}(u') \cdot A_S^\sigma(\Xi; c, s, c').$$

Proof. (a) By (S3), $\gamma \in \mathrm{Tm}_S(c_1, s, c_1')$ if and only if $\eta\gamma\eta'^{-1} \in \mathrm{Tm}_S(c, t\, s\, t'^{-1}, c')$. For $\eta' = \eta$ and $s = 1$ this shows, together with the fact that η normalizes $\mathcal{U}(T_S^{\mathrm{rat}}(E))$, that $u \mapsto \eta u \eta^{-1}$ is a bijection of $\mathcal{U}_{S,c_1}$ onto $\mathcal{U}_{S,c}$; likewise for η'. Put $\phi(\gamma, y) = \beta_{S,c}(\eta y)\, \beta_{S,c'}(\eta' \gamma^{-1} y)$. For $\gamma \in X_1 = \Xi \cap \mathrm{Tm}_S(c_1, s, c_1')$,

$$\begin{aligned} &\sum_{(u,u') \in \mathcal{U}_{S,c_1} \times \mathcal{U}_{S,c_1'}} \phi(u\gamma u'^{-1}, uy) \\ &\qquad = \sum_u \beta_{S,c}((\eta u \eta^{-1})\, \eta y) \cdot \sum_{u'} \beta_{S,c'}((\eta' u' \eta'^{-1})\, \eta' \gamma^{-1} y) = 1 \end{aligned}$$

by the two bijections and Lemma 7.3(a). Theorem E, for (c_1, s, c_1'), gives $A_S^\sigma(\Xi; c_1, s, c_1') = \sum_{\gamma \in X_1} \int \phi(\gamma, y) \mathcal{W}_S^\sigma(\gamma; y)\, dy$. A continuous linear map commutes with these sums and integrals, so by the hypothesis on L and the substitution $y \mapsto \eta^{-1} y$,

$$\begin{aligned} L(A_S^\sigma(\Xi; c_1, s, c_1')) &= \sum_{\gamma \in X_1} \int \beta_{S,c}(y)\, \beta_{S,c'}((\eta\gamma\eta'^{-1})^{-1} y)\, \mathcal{W}_S^\sigma(\eta\gamma\eta'^{-1}; y)\, dy \\ &= \sum_{\gamma \in X_1} a_S^\sigma(c, c'; \eta\gamma\eta'^{-1}). \end{aligned}$$

As $\gamma \mapsto \eta\gamma\eta'^{-1}$ maps X_1 bijectively onto $\Xi \cap \mathrm{Tm}_S(c, t\, s\, t'^{-1}, c')$, this is the left side of the assertion.

(b) By (S4) there is $x \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$ with $u^{-1}\mathrm{pos}_S(c, c; x) \in K_g(S)$ and $\psi_S^{\mathcal{U}}(\mathrm{r}_S^f(x)) = \psi_S^{\mathcal{U}}(u)$; then $\mathrm{pos}_S(c, c; x) \in u\, K_g(S)$, so $x \in \mathrm{Tm}_S(c, \mathrm{red}_S(u), c)$. Let x' be the same for u' and c'. Apply (a) with $\eta = x$, $\eta' = x'$, $c_1 = c$, $c_1' = c'$, $t = \mathrm{red}_S(u)$, $t' = \mathrm{red}_S(u')$: the condition on Ξ holds by stability, and by Lemma 9.3(b) the multiplication by $\psi_S^{\mathcal{U}}(u)^{-1}\psi_S^{\mathcal{U}}(u')$ can be taken for L. □

LEAN `Filling.amplitudeSum_move` · `Filling.amplitudeSum_besselU_mul`

Proof of Theorem F. (i) Suppose $A_S^\sigma(\Xi; c, s, c') \neq 0$, and let $u, u' \in K(S) \cap \mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ with $\mathrm{red}_S(u)\, s = s\, \mathrm{red}_S(u')$. Then $\mathrm{red}_S(u)\, s\, \mathrm{red}_S(u')^{-1} = s$, and Proposition 9.5(b) gives $A_S^\sigma(\Xi; c, s, c') = \psi_S^{\mathcal{U}}(u)^{-1}\psi_S^{\mathcal{U}}(u')\, A_S^\sigma(\Xi; c, s, c')$, hence $\psi_S^{\mathcal{U}}(u) = \psi_S^{\mathcal{U}}(u')$. So s agrees.

(ii) Write $\Xi = \mathcal{S}(T_S^{\mathrm{rat}}(E))$; it is stable by Lemma 9.1. A term $\gamma \in \Xi$ over $(c, 1, c')$ has $\mathrm{pos}_S(c, c'; \gamma) \in K_g(S)$, so by (S3) $c = c'$ and $\gamma \in \mathcal{U}_{S,c}$. Thus for $c \neq c'$ the sum is empty. Let $c = c'$, and let $\phi(\gamma, y) = \beta_{S,c}(y)$ if $\gamma = 1$ and $\phi(\gamma, y) = 0$ otherwise. For $\gamma \in \Xi \cap \mathrm{Tm}_S(c, 1, c) \subseteq \mathcal{U}_{S,c}$ the condition $u\gamma u'^{-1} = 1$ determines $u' = u\gamma$, so $\sum_{(u,u')} \phi(u\gamma u'^{-1}, uy) = \sum_{u \in \mathcal{U}_{S,c}} \beta_{S,c}(uy) = 1$. By Theorem E, with $\mathcal{W}_S^{\mathrm{full}}(1; y)$ of coordinates $W_S^{\mathrm{full}}(y)_i\, \overline{W_S^{\mathrm{full}}(y)_j}$,

$$A_S^{\mathrm{full}}(\Xi; c, 1, c)_{(i,j)} = \int \beta_{S,c}(y)\, W_S^{\mathrm{full}}(y)_i\, \overline{W_S^{\mathrm{full}}(y)_j}\, dy.$$

By (S7), $W_{S,\tau}^{\mathrm{full}}(ny)_{i_\tau}\, \overline{W_{S,\tau}^{\mathrm{full}}(ny)_{j_\tau}} = W_{S,\tau}^{\mathrm{full}}(y)_{i_\tau}\, \overline{W_{S,\tau}^{\mathrm{full}}(y)_{j_\tau}}$ for $n \in \mathcal{U}$, the factor $e^{\mathrm{sd}(n) - \overline{\mathrm{sd}(n)}}$ having absolute value 1. So we may unfold as in Lemma 8.3 (for integrable complex functions, by Lemma 8.2 and Fubini's theorem): the integral over $\prod_\tau \mathcal{U}$ of the cutoff is $\nu(S)$ by Theorem C, and there remains

$$\nu(S) \cdot \prod_\tau \int W_{S,\tau}^{\mathrm{full}}(y)_{i_\tau}\, \overline{W_{S,\tau}^{\mathrm{full}}(y)_{j_\tau}}\, d\mu_{\mathfrak{s}}^{\mathcal{U}\backslash}(y) = \nu(S) \prod_\tau \delta_{i_\tau j_\tau}$$

by the orthonormality (J1). □

LEAN `Filling.amplitudeSum_eq_zero_of_not_agrees` · `Filling.amplitudeSum_besselS_one`

10 Notes on the formal text

Arguments that the census reads differently from Lean. The notations of this note show the explicit arguments of the Lean definitions. For four definitions the census of the artifact, which reads the arguments from the text, lists other arguments than Lean takes, and the notation follows Lean: integral lifts (Definition 3.2; Lean takes the class, γ and m, the census adds the bound matrix t), and the three weights of Definition 6.9(a)–(c) (the census adds a variable k of the file, because the definitions contain $k(\mathfrak{s})$). Likewise the census lists a matrix variable of the file (and, for the first of them, instances of the file's field) among the arguments of Lemma 6.3, Lemma 7.3(c) and Theorem B, and a type among those of Lemma 5.1, where the statements only bind variables of the same name; and it stops at the first `let` of the statements of Theorems A, D and E and of Lemma 3.7, whose remaining arguments were read in the Lean text. The notation $\mathrm{disc}_{S,c}$ is a function of the subspace, as in Lean. The discriminant $\mathrm{disc}_H(V)$ of Note S takes in Lean the proof that H has the left inverse L; we show H.

The side. In Lean the side is an instance argument with the default value "full". Theorems D and E compare the kernel of the side σ with the cut kernel; for $\sigma = \mathrm{cut}$ their second assertions are empty. Theorem F(ii) is stated for the full side only, as in Lean.

Where the formal statements are more general or more special. Theorem A needs no admissibility, and its number x is a natural number. In Lemma 3.7 the hypothesis on x is conditional on δ being a unit, although under the other hypotheses δ always is one. In Theorem B the element s only serves to make the γ_p terms over a common s. Proposition 7.2 is stated in Lean for the function selected by a choice operator, as here. Lemma 3.3(g) holds over any field. In Lemma 7.1 the second element enters only through its matrix. Lemma 6.10(c) is stated in Lean as the identity $\mathrm{pad}_j(Z)\,\mathrm{pad}_j(Z)^{-1} = 1$ for the matrix inverse of the library, which is 0 for singular matrices.

What the three files do not use themselves. Lemmas 3.6, 3.7, 6.2, 6.3, 6.4, 6.10, 6.11, 9.2, Lemma 3.4, Lemma 3.5 and the definitions of Sections 3 and 6 are used only by later files of the unit (the pairing of links and the energy estimates). Theorem A is used in the estimate of pairs of terms, Theorems B to F throughout the second moment.

Proofs that differ from the formal ones. Lemma 3.4(a): the formal proof applies the rank theorem to a basis of the link, we use that the link is its own orthogonal. Theorem B: the formal proof takes a subset on which the colouring is injective with the same image, and measures distances from a chosen element; this is Step 2. Theorem C: Lemma 8.1 is inside the formal proof, where the vanishing of the entries is found by a decision procedure from the same relations between rows. Theorem D: the comparison of the exponents at the end is done by a decision procedure in Lean and by hand here. Theorem E: the formal proof uses the library's measure on a quotient and the fact that the distribution of an invariant function does not depend on the cutoff (unit 114); identity (3) is the same fact, proved directly. In

Proposition 7.2 the formal induction adds the pairs with the row index decreasing, as here.

Related work named by the authors. The comments at the head of the first file name, as related work on the sorting of double cosets by unipotent characters and on second moments, papers of Friedberg (1987), Y. Liu (2014), Furusawa and Shalika (2003), Michel, Ramakrishnan and Yang (2023), Kowalski, Saha and Tsimerman (2012) and Blomer (2023). These are the authors' pointers; they were not opened for this note, and the formal proofs do not depend on them.

How this note was written. From the dossier of the unit: the texts of the members of the three files and their structured proofs. I read the three Lean files completely. The proofs of Theorems A, B, C and D and of Proposition 7.2, which the readers of the dossier marked as their least certain, I read line by line in the Lean text, and the proofs given here follow those lines; the full statements of Theorems A, B, D, E and of Lemma 3.7 were read in the source, beyond the point where the census cuts them. Every statement of this note was compared with its Lean statement, argument by argument. Of the Inputs I read the Lean statements (for Note S in its files, for the other units in theirs), and not the proofs; the description of the Inputs (J2) to (J9) as elementary is the catalogue's. Two remarks on the dossier: its reading of the induction in Proposition 7.2 says that the pairs are added with increasing row index, while the Lean text adds them with decreasing row index; and the statement of Theorem D is complete in its text although the census lists only the side among its arguments. Nothing was compiled for this note. The program check of the unit's dossier covered 1371 leaf steps of 179 proofs.

Appendix A. Concordance with the Lean artifact

The formal names of the objects: `S.b`, `S.N`, `S.g`, `S.piBar`, `S.ϖ` are $b(S)$, $N(S)$, $g(S)$, $\bar{\pi}(S)$, $\varpi(S)$; `𝔰.n`, `𝔰.k`, `𝔰.r` are $n(\mathfrak{s})$, $k(\mathfrak{s})$, $r(\mathfrak{s})$; an infinite place `v` is τ and `Φ v x` is x_τ; `numInfPlaces E` is $f(E)$; `besselU`, `besselU0`, `besselH`, `besselS` are $\mathcal{U}(T)$, $\mathcal{U}^0(T)$, $\mathcal{H}(T)$, $\mathcal{S}(T)$; `classUnip`, `classUnip₀`, `classCutoff`, `unipCovolume` are $\mathcal{U}_{S,c}$, $\mathcal{U}^0_{S,c}$, $\beta_{S,c}$, $\nu(S)$; `termsOver`, `termPosition`, `ratToInf`, `ratToFin` are $\mathrm{Tm}_S(c,s,c')$, $\mathrm{pos}_S(c,c';\gamma)$, r^∞_S, r^f_S; `genIndex` is $[\cdot]$; `frameVector y a` is $y^{-1}\mathfrak{f}_a$ and `frameImage γ` is $\gamma\mathfrak{f}$; a set `T` of rational elements is Ξ; the elements `σ`, `σ'` of Proposition 9.5 are η, η'; the types `σ`, `τ`, `τ'`, `ι`, `m` of indices are Ω, Ω'; the type `α` is `P`. Three statements, as the artifact has them:

```
lemma tsum_classCutoff_box_le {c s α y y' x} {k γ δ w : α → _} (hi : ∀ {p q}, k p = k q → δ p = δ
  q → p = q) (H : ∀ p ∈ w.support, γ p ∈ S.termsOver c s (k p) ∧ (δ p)⁻¹ * γ p ∈ S.classUnip (k p) ∧
  (S.ratToInf (γ p))⁻¹ * y ∈ S.boxedPoints ∧ w p ≤ S.classCutoff (k p) ((S.ratToInf (δ p))⁻¹ * y') *
  x) {V : Submodule ℝ _} (hV : let F p v := frameVector ((S.ratToInf (γ p) v)⁻¹ * y v); ∀ᵉ
    (p ∈ w.support) (q ∈ w.support), F p - F q ∈ V) :
  ∑' p, w p ≤ .ofReal (S.b ^ (6 * 𝔰.n ^ 2 * finrank ℝ V)) * x

lemma Filling.lintegral_classCutoff_eq_unipCovolume {E} [Field E] [CMType E] [StablyUnram E] {𝔰}
  (S : Filling E 𝔰) c [S.IsAdmissible] (γ : _ → _) :
  ∫⁻ υ, S.classCutoff c fun v ↦ υ v * γ v ∂(.pi fun _ ↦ 𝔰.besselHaar) = .ofReal S.unipCovolume

lemma amplitudeSum_besselS_one i j : S.amplitudeSum (σ := .full) (S.ratMiddle E).besselS c 1 c' (i,
  j) = ite (c = c' ∧ i = j) S.unipCovolume 0
```

In this note	Lean declaration	File	Lines
Lemma 9.1	`Filling.isStable_besselS`	AmplitudeSums.lean	16–17
Lemma 8.2	`Filling.measurePreserving_pi_orbitMeasure`	AmplitudeSums.lean	25–28
Lemma 9.2	`Filling.tsum_integral_apply`	AmplitudeSums.lean	30–35
Lemma 9.3	`Filling.besselFn_mul_of_mem_classUnip`	AmplitudeSums.lean	41–44
Lemma 9.3	`Filling.besselKernel_besselU_mul`	AmplitudeSums.lean	46–49
Lemma 8.3	`Filling.lintegral_classCutoff_mul_prod`	AmplitudeSums.lean	54–58
Theorem D	`Filling.amplitudeAbs_lt_top_and_cut_le`	AmplitudeSums.lean	64–106
Corollary 9.4	`Filling.summable_amplitude`	AmplitudeSums.lean	110–111
Theorem E	`Filling.hasSum_amplitudeSum_change_of_weight`	AmplitudeSums.lean	119–158
Proposition 9.5	`Filling.amplitudeSum_move`	AmplitudeSums.lean	163–179
Proposition 9.5	`Filling.amplitudeSum_besselU_mul`	AmplitudeSums.lean	181–187
Theorem F	`Filling.amplitudeSum_eq_zero_of_not_agrees`	AmplitudeSums.lean	192–194
Theorem F	`Filling.amplitudeSum_besselS_one`	AmplitudeSums.lean	201–220
Lemma 3.7	`Filling.liouville_gram`	BoxPartition.lean	45–55
Definition 3.1	`Filling.linkMetric`	BoxPartition.lean	59–60
Definition 3.1	`Filling.linkDiscSq`	BoxPartition.lean	64–65
Lemma 3.3	`Filling.framePair_last_eq_zFormPair_mul`	BoxPartition.lean	72–74
Lemma 3.3	`Filling.classCoord_eq_fromBlocks`	BoxPartition.lean	76–78
Definition 3.2	`Filling.HasIntegralLifts`	BoxPartition.lean	86–88
Lemma 3.4	`Filling.finrank_link`	BoxPartition.lean	90–96
Lemma 3.3	`Filling.gram_classCoordX₀`	BoxPartition.lean	98–100
Lemma 3.3	`Filling.classCoord_gram`	BoxPartition.lean	102–104
Lemma 3.5	`Filling.linkMetric_posDef`	BoxPartition.lean	106–108
Lemma 3.4	`Filling.gram_linkForm_eq_zero_of_le_link`	BoxPartition.lean	110–113
Lemma 3.5	`Filling.det_gram_linkMetric_ne_zero`	BoxPartition.lean	115–118
Lemma 3.3	`Filling.card_eq_finrank_colSpan`	BoxPartition.lean	121–123
Lemma 3.6	`Filling.gram_linkMatrix_mul`	BoxPartition.lean	125–134
Lemma 3.3	`Filling.classCoord_mul_framePair`	BoxPartition.lean	136–138
Lemma 3.3	`Filling.classCoord_mul_isotropicFramePair`	BoxPartition.lean	142–143
Lemma 3.3	`Filling.isIntMatrix_framePair`	BoxPartition.lean	145–146
Theorem A	`Filling.le_genIndex_det_of_fixesNormAtLeast`	BoxPartition.lean	158–212
Lemma 5.1	`Filling.exists_mem_classUnip`	BoxPartition.lean	227–250
Lemma 5.2	`Filling.exists_eichler_classUnip`	BoxPartition.lean	252–263
Lemma 6.1	`Filling.tsum_le_of_encard_range_le`	BoxPartition.lean	270–274
Lemma 6.2	`Filling.tsum_le_of_colouring`	BoxPartition.lean	276–282
Lemma 6.3	`Filling.encard_integral_le`	BoxPartition.lean	287–292
Lemma 6.4	`Filling.submoduleIndex_smul_intVectors`	BoxPartition.lean	296–298
Lemma 6.4	`Filling.submoduleIndex_smul_ne_zero`	BoxPartition.lean	300–302
Definition 6.5	`Filling.IndexBelow`	BoxPartition.lean	305–305
Definition 6.5	`Filling.xRows`	BoxPartition.lean	313–314
Definition 6.5	`Filling.paddedLeading`	BoxPartition.lean	318–318
Lemma 6.6	`Filling.det_paddedLeading`	BoxPartition.lean	320–323
Definition 6.5	`Filling.frameMiddle`	BoxPartition.lean	327–327
Definition 6.7	`Filling.latticeDisc`	BoxPartition.lean	331–331
Definition 6.7	`Filling.frameValueIndex`	BoxPartition.lean	335–337
Definition 6.7	`Filling.xRowsIndex`	BoxPartition.lean	341–342
Definition 6.7	`Filling.saturationIndex`	BoxPartition.lean	348–349
Definition 6.8	`Filling.pairInvariant`	BoxPartition.lean	353–354
Definition 6.8	`Filling.termElem`	BoxPartition.lean	357–357
Definition 6.8	`Filling.minorNorm`	BoxPartition.lean	361–362
Definition 6.8	`Filling.classGauge`	BoxPartition.lean	366–367
Definition 6.8	`Filling.IsNearRep`	BoxPartition.lean	371–371
Definition 6.9	`Filling.invPowWeight`	BoxPartition.lean	375–375
Definition 6.9	`Filling.termWeight`	BoxPartition.lean	380–383
Definition 6.9	`Filling.pairWeight`	BoxPartition.lean	387–389

In this note	Lean declaration	File	Lines
Definition 6.9	`Filling.IsLabelledTerm`	BoxPartition.lean	393–395
Lemma 6.10	`Filling.HasLevel.minorNorm_pos`	BoxPartition.lean	401–402
Lemma 6.10	`Filling.HasLevel.inv`	BoxPartition.lean	404–405
Lemma 6.10	`Filling.HasLevel.paddedLeading_mul_inv`	BoxPartition.lean	407–408
Lemma 6.10	`Filling.HasLevel.exists_shared_frame_vector`	BoxPartition.lean	413–418
Lemma 6.11	`Filling.xRows_conjTranspose_mul_eq`	BoxPartition.lean	425–429
Lemma 7.1	`Filling.frameVector_sub_eq_sum`	BoxPartition.lean	443–447
Proposition 7.2	`Filling.isBoxPartition_epsilon`	BoxPartition.lean	453–505
Lemma 7.3	`Filling.tsum_classCutoff_eq_one`	BoxPartition.lean	507–508
Theorem B	`Filling.tsum_classCutoff_box_le`	BoxPartition.lean	513–556
Lemma 7.3	`Filling.exists_classUnip_mul_mem_boxedPoints`	BoxPartition.lean	558–561
Lemma 7.3	`Filling.measurable_classCutoff`	BoxPartition.lean	563–564
Theorem C	`Filling.lintegral_classCutoff_eq_unipCovolume`	UnipotentCovolume.lean	24–107

Notation

E, $x \mapsto \overline{x}$, x_τ a CM field with a CM type, its conjugation, the embedding at the infinite place τ

$f(E)$, d_E, $\mathrm{vol}(\hat{O}_E)$ number of infinite places; discriminant; $2^{f(E)}/\sqrt{|d_E|}$

$\mathbb{A}_{E,f}$, $\hat{O}_E$, $\mathbb{F}_w$ finite adeles; closure of the integers; residue field of a finite place w

$\mathfrak{s} = (\varrho, \kappa, k_0)$ a shape; $k(\mathfrak{s})$, $n(\mathfrak{s})$, $r(\mathfrak{s})$, $I(\mathfrak{s})$, $M(\mathfrak{s})$ as in (R1)

e_a, x_i, z, e_a^*; X, X_0 the basis of the space; the spans of the e_a and of the x_i

$\mathfrak{f}_a$, $\mathfrak{f}$, lev the frame, its matrix, the level of an index

T, $J(T)$, h_T, $G(T)$ an $\mathfrak{s}$-form over a ring R (or A), its Gram matrix, the form, the unitary group

$\mathcal{U}(T)$, $\mathcal{U}^0(T)$, $\mathcal{H}(T)$, $\mathcal{S}(T)$ unipotent group, its part fixing the frame, the group of X_0, the Bessel subgroup

$T^{\mathrm{rat}}(\mathfrak{s}; N)$, $G_1(\mathfrak{s})$, $\mathcal{U}$ the diagonal form; the group at $\mathbb{C}$; its unipotent group

$\mathrm{sd}(g)$, $\mathrm{fm}(g)$, $Z_T(g)$, $Z(y)$, $\mathrm{D}_l(Z)$ sum of the superdiagonal entries on the frame; frame matrix; frame Gram matrices; leading minor

$\mathrm{bs}(y)$, $y \in \mathrm{Cut}(b)$, $\mathrm{FBox}(\rho)$ size modulo the Bessel subgroup, the cut condition, the frame box

dy, $\mu_{\mathfrak{s}}^{\mathcal{U}}$, $\mu_{\mathfrak{s}}^{\mathcal{U}\backslash}$, μ Haar measures; the measure for the quotient by $\mathcal{U}$; in Section 8 the product measure

S, S', S_1 filling data; $N(S)$, $q_i(S)$, $g(S)$, $\bar{\pi}(S)$, $\varpi(S)$, $b(S)$, $\mathrm{Ch}_i(S)$ as in (R4)

$T_S^{\mathrm{rat}}(R)$, $T_S^{\mathrm{lat}}(R)$, $G_S(R)$, G_S^{lat} forms and groups of (R5)

$K(S)$, $K_g(S)$, $\mathsf{G}(S)$, red_S, $\overline{A}^S$ integral subgroup, principal level, finite group, reduction, reduced image

$\Lambda(S)$, r_S^f, r_S^∞, $\mathrm{sc}_{S,\tau}(i)$ lattice basis; finite and real coordinates; the scales

$\mathrm{Cl}(S)$, c, c', d, $\tilde{c}$, $h_S(c)$ classes, their representatives

$\mathrm{B}_S(c)$, $\mathrm{C}_S(c)$, $\mathrm{C}_S^\circ(c)$, $\Lambda^\circ(\mathfrak{s}; p)$, m_c, $m(a)$ class basis, class coordinates, their X_0-part; blocks of the class basis

$\mathrm{pos}_S(c, c'; \gamma)$, $\mathrm{Tm}_S(c, s, c')$ position; terms over (c, s, c')

$\mathcal{U}_{S,c}$, $\mathcal{U}_{S,c}^0$ arithmetic unipotent group of level $g(S)$, its part fixing the frame

$\hat{g}(S)$, ψ_S, ψ_f, $\psi_S^{\mathcal{U}}(u)$ idele of $g(S)$; additive characters; the unipotent character

$\mathrm{red}_{S,(w,\epsilon)}$, adj_S^ϵ, $\mathrm{Fix}_{S,w}(\mathbf{d})$ reductions, inverse adjoint, fixer of a vector of the frame

$\mathrm{fix}(s) \geq x$, $s \in \mathrm{Agr}(S)$, $\Xi \in \mathrm{Stab}(S)$ s fixes vectors at places of norm product $\geq x$; s agrees; Ξ is stable

$[\mathfrak{f} \mid \gamma\mathfrak{f}]$, $[\mathfrak{e} \mid \gamma\mathfrak{e}]$, $\mathrm{L}(\gamma)$, $\mathrm{link}(\gamma)$, $\mathrm{L}_z(T)$, $\mathrm{link}^\circ(\gamma, \gamma')$ frames, link matrix, link, the z-rows, common link

J_S, $Q_S^\pm$, $Z_S(\gamma)$, $\mathrm{g}_S(\gamma)$, $\mathrm{g}(N; Z)$, $\|\mathrm{g}\|_S(\gamma)$ Gram matrix, link form, frame products, discriminants

$\gamma \in \mathrm{NF}_j$, $\gamma \in \mathrm{Lv}_j(S)$, $\gamma \sim_{S;c,c'} \gamma'$ normal form, level j, congruent terms

$(M)_{\mathbb{A}_f}$, $\mathrm{m}_H(V)$, $\mathrm{disc}_H(V)$, $[a]$ a matrix over the adeles; lattice basis and discriminant of a subspace; generalized index

$W_{S,\tau}^\sigma$, $I_\tau(S)$, $\mathrm{Bc}(S)$, $\deg$ Bessel functions of the system at τ, their index sets, tuples of indices, the constant of the system

$W_S^\sigma(y)_i$, $\mathcal{W}_S^\sigma(\gamma; y)$ product Bessel function, kernel

$\mathrm{bx}(S)$, $\mathrm{BxP}(S)$, $\gamma \in \mathrm{Near}(S)$, $\omega \in \mathrm{BP}(S, c)$ box size, boxed points, near elements, box partitions

$\beta_{S,c}$, $\beta_{S,c}^\flat$, $\omega_{S,c}$ the cutoff of the class c and its two factors

$a_S^\sigma(c, c'; \gamma)$, $|a|_S^\sigma(c, c'; \gamma)$, $A_S^\sigma(\Xi; c, s, c')$ amplitude, its absolute version, amplitude sum

$\kappa_\tau(S)$, $\mathrm{fs}_S(\tau,a)$, $\mathrm{fs}(S)$, $\mathrm{tl}(S)$, $\nu(S)$, $\mathrm{cw}_i(S)$ constants of (R10) and (R7)

Q_S^{link}, $\mathrm{ld}_S(v,e)$, $m \in \mathrm{IL}(c;\gamma)$ link metric, link discriminant, integral lifts (Section 3)

$\mathrm{Ib}(j)$, Z^{x}, $\mathrm{pad}_j(Z)$, $(\gamma\mathfrak{f})^{\mathrm{mid}}$ Definition 6.5

$\mathrm{disc}_{S,c}$, $\mathrm{o}_{S,c}(k;a,V)$, $\mathrm{i}_{S,j}(k)$, $\mathfrak{d}_S(k)$ Definition 6.7; L_1, L_2, T', Z' there

$\mathrm{el}_S(c;p)$, $\mathrm{pi}_S(c,c';\gamma,\gamma')$, $\mathrm{mn}_S(j;\gamma)$, $\mathrm{cg}_S(j;c,c';\gamma)$, $\gamma \in \mathrm{NRep}_S(c,c')$ Definition 6.8

$\mathrm{ipw}_S(c;n,p)$, $\mathrm{w}_S(j,i;c;p,p';V)$, $\mathrm{pw}_S(j;c,c';\gamma,\gamma')$, $p \in \mathrm{Lab}_S(j;c,t)$ Definition 6.9

Ω, Ω', ι; P, A, B, C, w, col finite index sets; sets, colours and weights of the counting lemmas

σ; Ξ; $\gamma,\gamma',\gamma_2,\gamma_p,\delta_p$; u,u'; s,t,t' a side; a set of rational elements; rational elements; unipotent elements; elements of $\mathsf{G}(S)$ (also matrices where said)

y,y',y°; g; k; η,η' points at infinity; a group element, or $g(S)$ where said; an element of $K(S)$ or G_S^{lat}, or $k(\mathfrak{s})$ where said; elements of the Bessel subgroup, or vectors in Section 4

a,b,l,i,j,j'; p,q; r; n, N, f, M, m, x indices and levels; elements of P or pairs, p also a map in Lemma 5.1; a real number or an adele, $r(\mathfrak{s})$ where said; numbers, with the abbreviations said in each proof

C, Q, Q°, δ, δ', ρ, $\mathfrak{a}$, $\mathbb{F}$, o, A, ξ,ζ,θ, ξ_0, μ, v, ς, ω_0, τ_0, ϵ objects of Theorem A and its proof (Section 4) and of Input (S2)

B, λ, $\hat{x}$, $\hat{u}$, J, J', P matrices and scalars of Section 5; Gram matrices; a statement in $[P]$

α, α', β, β_0, ϕ, ε scalars; cutoffs and weights; a distance

θ, I, Σ, F_τ, E_{kl}, r_τ, w_τ, w objects of Section 7; w also a finite place

F_p, F', V, v_l, c_a, D, $\mathfrak{B}$, G, Γ, φ, c, L_i, F_i, $\mathrm{Fr}(a)$, $t_{l,\tau}$ frames and spaces of Theorem B; objects of Section 8

A_τ, m_τ, θ_τ, $\mathbf{A}$, $\mathbf{W}$, Θ, C_0, $X(d)$, X_1, Δ, K, Y, g_p, L, μ_{P}, M, L objects of Section 9 and of the counting lemmas

ψ, π, ν, ϖ, $\mathcal{C}$, H, O, R, A, U, W, Y, Z, K, I, e, h, z, d letters inside the notations above (ψ_f, $2\pi i$, $\nu(S)$, $\varpi(S)$, a compact set, a matrix, rings, and the symbols of the macros)

READBACK NOTE 43

The energy of amplitude sums: Schur's test on thin classes, block bounds, incoherent and degenerate pairs

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. In the proof of Theorem B of the write-up (enough forms), the second moment of translates of Poincaré series unfolds into sums of *amplitudes* over rational elements of a unitary group, collected in blocks $A_S^\sigma(\Xi; c, s, c')$. This note bounds the *energy* of such blocks, the sum of their squared norms: a Schur test bounds the total amplitude of one double coset of the arithmetic unipotent group (Theorem A); summing over the cosets gives bounds for blocks (Theorem B); passing from the Bessel functions to their cuts costs little (Theorem C); pairs of terms with different links or with degenerate frames are negligible, by lattice counts (Theorem D); and the result is the crude estimate (Theorem E) that the proof of the filling theorem uses.

What this note is. The artifact grh-compact is a body of Lean code whose final theorems state the Riemann hypothesis and its analogue for Dirichlet L-functions. That is a claim: its authors report that Lean's kernel accepts the code, a replay of that check by the owner of this write-up is recorded elsewhere, and nothing was compiled for this note; no mathematician has read the proof through. This note treats one part of one unit (the folder of the filling theorem): 31 of its 497 declarations, those of four files on the energy of amplitude sums. Every statement was compared, argument by argument, with the Lean text. The proofs are complete, with two exceptions that are said where they occur: results of the other notes of the unit, of the unit on the real place and of the background library are quoted as *Inputs*, with their owners, and the comparisons of exponents that the formal text leaves to a decision procedure are carried out by hand with cruder intermediate bounds (Section 10). The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 31 declarations.

The notes of this unit are cited by letters: S (the filling data), B (box partitions), U (Poincaré series and the sieve), E (the energy of amplitude sums; this note), F (the filling theorem), P (the pairing of links), T (the top floor at a finite place) and G (from amplitude sums to local correlations).

1 The statements

The objects are defined in Section 2; most of them belong to the set-up of the unit, which is the subject of Note S (*The filling data: Bessel subgroups, links of frames, Siegel's lemma and strong approximation*) and of Note B (*Box partitions of arithmetic unipotent groups: frame counts, the covolume, amplitude sums*). In a few words: S is the data of the filling theorem over a CM field E; $G_S(E)$ is the unitary group of a hermitian space with $\varrho(\mathfrak{s})$ hyperbolic planes; $\mathrm{Cl}(S)$ is a finite set of classes and $\mathsf{G}(S)$ a finite group; a rational element $\gamma \in G_S(E)$ lies over (c, s, c') if it carries the lattice of the class c' to that of c and reduces to s; its *amplitude* $a^{\sigma}_S(c, c'; \gamma)$ is an integral of a product of two Bessel functions, full ($\sigma = \mathrm{full}$) or cut ($\sigma = \mathrm{cut}$), against cutoffs for the arithmetic unipotent groups $\mathcal{U}_{S,c}$, $\mathcal{U}_{S,c'}$; and $A^{\sigma}_S(\Xi; c, s, c')$ is the sum of the amplitudes of the elements of a set Ξ that lie over (c, s, c'). The two standing hypotheses are collected in Setting 2.2, those of the energy estimates in Setting 2.6.

Theorem A (Schur's test on a thin class). In Setting 2.2, let $j \in \{0, \dots, \varrho(\mathfrak{s})+1\}$, let $c, c' \in \mathrm{Cl}(S)$, $s \in \mathsf{G}(S)$, and let $\gamma \in G_S(E)$ be such that $\gamma \in \mathrm{Tm}_S(c, s, c')$ and $\gamma \in \mathrm{Lv}_j(S)$. Let $\mathcal{D} = \mathcal{U}^0_{S,c}\, \gamma\, \mathcal{U}^0_{S,c'}$ be its double coset.
(a) With $j_0 = \min(\varrho(\mathfrak{s}), j)$,

$$\sum_{\delta \in \mathcal{D}} \mathrm{SI}_S(c, c'; \delta) \;\le\; \nu(S) \cdot \#\mathrm{Bc}(S) \cdot \mathrm{tl}(S)^2 \cdot \big(b(S)^4\, g(S)\big)^{-j_0\, k(\mathfrak{s})\, f(E)} \cdot \Big(\frac{\mathrm{i}_{S,j}(\mathrm{pos}_S(c, c'; \gamma))}{\sqrt{\mathrm{mn}_S(j; \gamma)}} \Big)^{k(\mathfrak{s})+1}.$$

(b) If moreover $\|\mathsf{g}\|_S(\gamma) \le N(S)^{f(E)}\, \mathrm{mn}_S(j; \gamma)$, then

$$\sum_{\delta \in \mathcal{D}} |a|^{\mathrm{cut}}_S(c, c'; \delta) \;\le\; \nu(S) \cdot \mathrm{cs}_S(j; \mathrm{cg}_S(j; c, c'; \gamma)).$$

Theorem B (block bounds). In Setting 2.2, let $\Xi \subseteq G_S(E)$ be stable, $\Xi \in \mathrm{Stab}(S)$, and let $c, c' \in \mathrm{Cl}(S)$ and $s \in \mathsf{G}(S)$.
(a) If $\mathsf{g}_S(\gamma) \neq 0$ for all $\gamma \in \Xi$, then

$$\big\| A^{\mathrm{cut}}_S(\Xi; c, s, c') \big\| \;\le\; \sum_{\substack{\gamma \in \Xi \cap \mathrm{Tm}_S(c,s,c') \\ \gamma \in \mathrm{NRep}_S(c,c')}} \nu(S) \cdot \mathrm{cs}_S(\varrho(\mathfrak{s}) + 1; \|\mathsf{g}\|_S(\gamma)).$$

(b) If $j \in \{0, \dots, \varrho(\mathfrak{s}) + 1\}$ and $\gamma \in \mathrm{Lv}_j(S)$ for all $\gamma \in \Xi$, then $A^{\mathrm{cut}}_S(\Xi; c, s, c') = 0$ unless $s \in \mathrm{Agr}(S)$, and if $s \in \mathrm{Agr}(S)$ then

$$\big\| A^{\mathrm{cut}}_S(\Xi; c, s, c') \big\| \;\le\; \sum_{\substack{\gamma \in \Xi \cap \mathrm{Tm}_S(c,s,c') \\ \gamma \in \mathrm{NRep}_S(c,c')}} \nu(S) \cdot \mathrm{cs}_S(j; \mathrm{cg}_S(j; c, c'; \gamma)).$$

Theorem C (full against cut). In Setting 2.2, for every set $\Xi \subseteq G_S(E)$ and every set $\Omega \subseteq \mathsf{G}(S)$,

$$\mathrm{En}_S(A^{\mathrm{full}}_S(\Xi; \cdot, \cdot, \cdot) - A^{\mathrm{cut}}_S(\Xi; \cdot, \cdot, \cdot); \Omega) \;\le\; b(S)^{-30} \cdot \mathsf{d}_1(S).$$

Theorem D (incoherent and degenerate pairs are negligible). In Setting 2.6, with its C, Ω, Ξ and X:
(a) Let $w \geq 0$ be a real number, $j \in \{0, \dots, \varrho(\mathfrak{s})+1\}$, and R a set of E-subspaces of $(E^{M(\mathfrak{s})})^2$. Suppose that $\gamma \in \mathrm{Lv}_j(S)$ for all $\gamma \in \Xi$; that $\mathrm{tl}(S)^{12} X^{-k(\mathfrak{s})}\, \mathsf{d}(S) \leq C$ in case $\varrho(\mathfrak{s}) < j$ and some subspace does not belong to R; and that $\mathrm{LE}_S^{\mathrm{cut}}(\Xi; \Omega, R) \leq w\, \nu(S)^2$. Then

$$\mathrm{En}_S(A_S^{\mathrm{cut}}(\Xi; \cdot, \cdot, \cdot); \Omega) \;\leq\; w + \frac{C}{b(S)^{30} \cdot 4^{\,\varrho(\mathfrak{s})+1-j}}.$$

(b) Let $l \in \{0, \dots, \varrho(\mathfrak{s}) + 1\}$ be such that every $\gamma \in \Xi$ with $\gamma \in \mathrm{ChA}(S)$ satisfies $\gamma \in \mathrm{Lv}_j(S)$ for some $j < l$. Then

$$\mathrm{En}_S(A_S^{\mathrm{cut}}(\Xi; \cdot, \cdot, \cdot); \Omega) \;\leq\; \frac{C}{b(S)^{30} \cdot 4^{\,\varrho(\mathfrak{s})+1-l}}.$$

Theorem E (the crude estimate). In Setting 2.6, with its C, Ω, Ξ and X, let $w \geq 0$ be a real number and R a set of E-subspaces of $(E^{M(\mathfrak{s})})^2$. Suppose that $\mathrm{tl}(S)^{12} X^{-k(\mathfrak{s})}\, \mathsf{d}(S) \leq C$ in case some subspace does not belong to R, and that

$$\mathrm{LE}_S^{\mathrm{cut}}(\Xi \cap \{\gamma : \mathfrak{g}_S(\gamma) \neq 0\}; \Omega, R) \;\leq\; w\, \nu(S)^2.$$

Then

$$\mathrm{En}_S(A_S^{\mathrm{full}}(\Xi; \cdot, \cdot, \cdot); \Omega) \;\leq\; \left(1 + b(S)^{-4}\right)^2 w + b(S)^{-12}\, C.$$

Theorem E is what the other notes of the unit use: the note on the filling theorem applies it twice, in its estimate of the main term and, with R empty and $w = 0$, for the terms outside the Bessel subgroup. Theorems A to D are its proof: E follows from C, D (a) at the top level $j = \varrho(\mathfrak{s}) + 1$ and D (b); D (a) follows from B (b), from the inequality of pair invariants of Note P (Input (P2)) and from the count of pairs (Proposition 7.5); B follows from A and from the open-cell bound of the real place (Input (R3)).

2 The setting

Conventions. The natural numbers are $0, 1, 2, \dots$; for natural numbers $j_0 \leq j$ the difference $j - j_0$ is a natural number. E is a CM field with its CM type: a number field that is a CM field, with complex conjugation $x \mapsto \overline{x}$, together with the choice of one embedding at each infinite place τ; $x_\tau \in \mathbb{C}$ is the image of $x \in E$ under it, and $f(E)$ is the number of infinite places, so that $[E : \mathbb{Q}] = 2\, f(E)$. $\mathrm{N}_{E/\mathbb{Q}}$ is the norm, rd_E the root discriminant and d_E the discriminant of E; $\mathbb{A}_{E,f}$ is the ring of finite adeles, $\hat{O}_E$ its subring of integral adeles, and $\psi_{E,f}$ the standard additive character of $\mathbb{A}_{E,f}$. For $a \in \mathbb{A}_{E,f}$, $[a]$ is the *generalized index*: for a unit a, the inverse of the factor by which multiplication by a scales a Haar measure of $\mathbb{A}_{E,f}$, and 0 for a non-unit; so $[a] = [\hat{O}_E : a\hat{O}_E]$ for a unit $a \in \hat{O}_E$, and $[x] = |\mathrm{N}_{E/\mathbb{Q}}(x)| = \prod_\tau |x_\tau|^2$ for $x \in E$. For subgroups M, M' of a group, $[M' : M]$ is the index of $M \cap M'$ in M', read as 0 when it is infinite. A matrix over $\mathbb{A}_{E,f}$ is *integral* if its entries lie

in $\hat{O}_E$; for a matrix M with rows indexed by a finite set I_1, $\hat{O}_E\langle M\rangle \subseteq \mathbb{A}_{E,f}^{I_1}$ is the $\hat{O}_E$-module spanned by its columns. The field E is *stably unramified* if it contains a primitive root of unity of order 5040. A *shape* is a triple $\mathfrak{s} = (\varrho, \kappa, k_0)$ of a natural number, a finite set and an element of it; we write $\varrho(\mathfrak{s})$, $k(\mathfrak{s}) = \#\kappa$, $n(\mathfrak{s}) = \varrho(\mathfrak{s}) + 1$ and $r(\mathfrak{s}) = 2\varrho(\mathfrak{s}) + k(\mathfrak{s}) + 1$. For a commutative ring with involution, M^* is the conjugate transpose of a matrix M. S is filling data over E of shape $\mathfrak{s}$ (Setting 2.1); it is *admissible* if it satisfies the standing hypotheses of Note S, of which Setting 2.2 lists those used here. A *side* σ is one of the two words full, cut. V is a normed additive group. Sums of non-negative terms over arbitrary index sets and integrals of functions with values in $[0,\infty]$ are taken in $[0,\infty]$; the norm of a vector is also read there. The letters S, s, σ, q, θ, Θ, ψ have in the notes of this unit the meanings given here, not those of the overview of the write-up (there: a set of columns, the complex variable, a field automorphism, a rational number, a family of twists, a chain of lifts, a column). For the filling data of admissible data (Note G), the primes $q_0(S)$, $q_1(S)$ are the primes that the overview calls g_0, g_1, and $g(S)$ is their product.

Setting 2.1 (filling data; from Note S)**.** E is a CM field with its CM type, $\mathfrak{s}$ a shape, and S is filling data over E of shape $\mathfrak{s}$, $S \in \mathrm{Fill}(E,\mathfrak{s})$. Of its components and conditions this note uses:

(i) a real number $b(S) \geq 1000$; a prime number $N(S) \geq 5$ and a finite idele $\bar{\pi}(S) \in \hat{O}_E$ with $\bar{\pi}(S)\,\overline{\bar{\pi}(S)} = N(S)$; two prime numbers $q_0(S)$, $q_1(S)$ with $b(S)^{10} \leq q_i(S) \leq b(S)^{11}$, and $g(S) = q_0(S)\,q_1(S)$, so that $b(S)^{19} \leq g(S) \leq b(S)^{23}$;

(ii) for $i = 0, 1$ a set $\mathrm{Ch}_i(S)$ of $f(E)$ finite places of E, the *chosen places* above $q_i(S)$, whose residue fields have $q_i(S)$ elements;

(iii) a generator $\varpi(S)$ of the different of E with $\overline{\varpi(S)} = -\varpi(S)$; an open compact subgroup $K^{\mathrm{col}}(S)$ of the unitary group $G_f(E,\kappa)$ of the standard hermitian form on $\mathbb{A}_{E,f}^{\kappa}$ (the *column level*), all of whose class automorphism groups are trivial; and for every infinite place τ a Bessel system $\mathfrak{B}_\tau(S)$ for $(\mathfrak{s}, b(S), f(E))$, described below;

(iv) the inequalities $10^5 \leq \varrho(\mathfrak{s})$, $1000\,(2\varrho(\mathfrak{s})+2)^9 \leq k(\mathfrak{s})$, $k(\mathfrak{s}) \leq f(E) \leq b(S)/1000$, $k(\mathfrak{s}) \leq b(S)$, $n(\mathfrak{s}) \leq b(S)$ and $(200\,f(E)\,k(\mathfrak{s})\ \mathrm{rd}_E)^{1000} \leq b(S)$.

Forms and groups. These notions belong to the background library and to Note S. The index set $I(\mathfrak{s})$ consists of $e_0, \dots, e_{\varrho(\mathfrak{s})-1}$, of the elements of κ, of z, and of $e_0^*, \dots, e_{\varrho(\mathfrak{s})-1}^*$; $M(\mathfrak{s}) = \kappa \sqcup \{z\}$ is the set of the *middle* indices. The *frame* is $\mathfrak{f}_a = e_a$ for $a < \varrho(\mathfrak{s})$ and $\mathfrak{f}_{\varrho(\mathfrak{s})} = z$. For a commutative ring A with involution, an *$\mathfrak{s}$-form* over A is a matrix T indexed by $M(\mathfrak{s})$; its *split Gram matrix* $J(T)$, indexed by $I(\mathfrak{s})$, pairs e_a with e_a^* to 1 and is T on the middle; $G(T)$ is the unitary group of $J(T)$ over A, acting on the column vectors $t \in A^{I(\mathfrak{s})}$, and $h_T(t,t') = t^* J(T)\,t'$. For $\gamma \in G(T)$ we write $\gamma\mathfrak{f}$ for the matrix with the columns $\gamma\mathfrak{f}_0, \dots, \gamma\mathfrak{f}_{\varrho(\mathfrak{s})}$, $\mathfrak{f}$ for this matrix at $\gamma = 1$, and $\gamma\mathfrak{f}\,v = \sum_a v_a\,\gamma\mathfrak{f}_a$ for $v \in A^{n(\mathfrak{s})}$. The *frame Gram matrix* is $Z_T(\gamma) = (h_T(\mathfrak{f}_a, \gamma\mathfrak{f}_{a'}))_{a,a' \leq \varrho(\mathfrak{s})}$. For a square matrix Z of size $n(\mathfrak{s})$ and $l \leq n(\mathfrak{s})$, $\mathrm{D}_l(Z)$ is its leading principal minor of order l, and for $N' \in A$

$$\mathrm{g}(N'; Z) = \det Z \cdot \overline{\det Z} - N'^2\,\mathrm{D}_{\varrho(\mathfrak{s})}(Z)\,\overline{\mathrm{D}_{\varrho(\mathfrak{s})}(Z)}.$$

In $G(T)$: $\mathcal{U}(T)$ is the *Bessel unipotent subgroup*, the elements that are upper triangular with identity diagonal blocks for the order $e_0, \dots, e_{\varrho(\mathfrak{s})-1}$, middle, $e_{\varrho(\mathfrak{s})-1}^*, \dots, e_0^*$; $\mathcal{U}^0(T)$

is its subgroup of the elements that fix every frame vector; $\mathcal{H}(T)$ is the subgroup of the elements that fix all e_a, all e_a^* and z; and $\mathcal{S}(T) = \mathcal{U}(T)\,\mathcal{H}(T)$ is the *Bessel subgroup*. For $u \in \mathcal{S}(T)$, $\mathrm{sd}(u) = \sum_{a<\varrho(\mathfrak{s})} u_{\mathsf{f}_a,\mathsf{f}_{a+1}}$ is the sum of its entries above the diagonal of the frame.

Over $\mathbb{C}$ we use the form $T^{\mathrm{rat}}(\mathfrak{s};1)$, diagonal with entries 1 on κ and -1 at z, and $G_1(\mathfrak{s}) = G(T^{\mathrm{rat}}(\mathfrak{s};1))$, with the Haar measure that the library fixes on it; $Z(y) = Z_{T^{\mathrm{rat}}(\mathfrak{s};1)}(y)$ for $y \in G_1(\mathfrak{s})$, and $\mathsf{g}^{\mathbb{R}}(Z) = \mathrm{Re}\,\mathsf{g}(1;Z) = |\det Z|^2 - |\mathrm{D}_{\varrho(\mathfrak{s})}(Z)|^2$. For $y \in G_1(\mathfrak{s})$, $\mathrm{bs}(y) = \inf_u \|u\,y\| + 1$, the infimum over $u \in \mathcal{S}(T^{\mathrm{rat}}(\mathfrak{s};1))$ of operator norms; for a real number b_0, $y \in \mathrm{Cut}(b_0)$ means $\mathrm{bs}(y) \le b_0^{6n(\mathfrak{s})+2}$; and for a real number ρ, $\mathrm{FBox}(\rho)$ is the set of the $y \in G_1(\mathfrak{s})$ with $\|y^{-1}\mathsf{f}_a\| \le \rho$ for all $a \le \varrho(\mathfrak{s})$ (Euclidean norms). The *cell mass* of $y \in G_1(\mathfrak{s})$ for real numbers $\deg_0$, ρ, d_0 is

$$\mathrm{cm}(y;\deg_0,\rho,d_0) = \Big(\frac{\rho^{9n(\mathfrak{s})}}{\mathsf{g}^{\mathbb{R}}(Z(y))}\Big)^{10^5 n(\mathfrak{s})^4} \cdot \frac{d_0^2}{\deg_0 \cdot c_G(\mathfrak{s}) \cdot (\mathsf{g}^{\mathbb{R}}(Z(y))/2)^{k(\mathfrak{s})-1}},$$

with a positive constant $c_G(\mathfrak{s})$ of the shape. We write $G_\infty = \prod_\tau G_1(\mathfrak{s})$, with the product measure; it depends on E and $\mathfrak{s}$.

The groups of the data. In Setting 2.1, for a commutative ring A, $T_S^{\mathrm{rat}}(A)$ is the $\mathfrak{s}$-form over A that is diagonal with 1 on κ and $-N(S)$ at z, and $G_S(A) = G(T_S^{\mathrm{rat}}(A))$ when A has an involution. For $\gamma \in G_S(E)$: $Z_S(\gamma) = Z_{T_S^{\mathrm{rat}}(E)}(\gamma)$, $\mathsf{g}_S(\gamma) = \mathsf{g}(N(S);Z_S(\gamma)) \in E$, and $\|\mathsf{g}\|_S(\gamma) = \prod_\tau |\mathsf{g}_S(\gamma)_\tau|$. $T_S^{\mathrm{lat}}(A)$ is the Gram matrix of the same hermitian space in the *lattice basis*, in which the basis vector of index k_0 is replaced by an isotropic vector; $G_S^{\mathrm{lat}} = G(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$, $K(S) \subseteq G_S^{\mathrm{lat}}$ is the subgroup of the integral matrices with integral inverse, $\mathsf{G}(S) = \prod_i \prod_{w\in \mathrm{Ch}_i(S)} \mathrm{GL}_{I(\mathfrak{s})}(\mathbb{F}_w)$ with the residue fields $\mathbb{F}_w$, and $\mathrm{red}_S\colon K(S) \to \mathsf{G}(S)$ is the reduction of the entries; its kernel is $K_g(S)$, and $\overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^S$ is the image of $K(S) \cap \mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$, a subgroup of $\mathsf{G}(S)$. There are homomorphisms $\mathrm{r}_S^f\colon G_S(E) \to G_S^{\mathrm{lat}}$ (the diagonal embedding, written in the lattice basis) and $\mathrm{r}_S^\infty\colon G_S(E) \to G_\infty$, where $\mathrm{r}_S^\infty(\gamma)_\tau$ is the matrix of γ at τ conjugated by the diagonal matrix of the *scales* $\mathrm{sc}_{S,\tau}(\cdot)$: $\mathrm{sc}_{S,\tau}(e_a) = \kappa_\tau(S)^{\varrho(\mathfrak{s})-a}/\sqrt{N(S)}$, $\mathrm{sc}_{S,\tau}(i) = 1$ on κ, $\mathrm{sc}_{S,\tau}(z) = 1/\sqrt{N(S)}$, $\mathrm{sc}_{S,\tau}(e_a^*) = \mathrm{sc}_{S,\tau}(e_a)^{-1}$. Here $\kappa_\tau(S)$ is a non-zero real number, the real part of $g(S)\,\varpi(S)_\tau/(2\pi i)$, and $\mathrm{fs}_S(\tau,a) = \kappa_\tau(S)^{2(\varrho(\mathfrak{s})-a)}/N(S)$ for $a \le \varrho(\mathfrak{s})$.

Classes and terms. $\mathrm{Cl}(S)$ is the finite set of the double cosets of $G_f(E,\kappa)$ by the rational points of the unitary group on the left and $K^{\mathrm{col}}(S)$ on the right; inc_S embeds $G_f(E,\kappa)$ in $\mathcal{H}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})) \subseteq G_S^{\mathrm{lat}}$, and $h_S(c)$ is the image of a fixed representative of the class c. For $c,c' \in \mathrm{Cl}(S)$ and $\gamma \in G_S(E)$,

$$\mathrm{pos}_S(c,c';\gamma) = h_S(c)^{-1}\,\mathrm{r}_S^f(\gamma)\,h_S(c') \in G_S^{\mathrm{lat}},$$

and for $s \in \mathsf{G}(S)$, $\mathrm{Tm}_S(c,s,c')$ is the set of the γ with $\mathrm{pos}_S(c,c';\gamma) \in K(S)$ and $\mathrm{red}_S(\mathrm{pos}_S(c,c';\gamma)) = s$: the *terms over* (c,s,c'). The arithmetic unipotent groups are $\mathcal{U}_{S,c} = \{u \in \mathcal{U}(T_S^{\mathrm{rat}}(E)) : \mathrm{pos}_S(c,c;u) \in K_g(S)\}$ and $\mathcal{U}_{S,c}^0 = \mathcal{U}_{S,c} \cap \mathcal{U}^0(T_S^{\mathrm{rat}}(E))$. The matrix $\mathrm{C}_S(c)$ over $\mathbb{A}_{E,f}$ takes rational coordinates to the coordinates in the lattice basis of the class c, so that the matrix of $\mathrm{pos}_S(c,c';\gamma)$ is $\mathrm{C}_S(c)\,(\gamma)_{\mathbb{A}_f}\,\mathrm{C}_S(c')^{-1}$, where $(M)_{\mathbb{A}_f}$ is a matrix M over E read over $\mathbb{A}_{E,f}$; $\mathrm{C}_S^\circ(c)$ is its part on κ, a matrix with rows in $M(\mathfrak{s})$ and columns in κ.

The real places. The Bessel system $\mathfrak{B}_\tau(S)$ consists of a non-empty finite set $I_\tau(S)$, a number $\deg_\tau(S) > 0$ and two continuous functions $W^{\mathrm{full}}_{S,\tau}, W^{\mathrm{cut}}_{S,\tau} \colon G_1(\mathfrak{s}) \to \mathbb{C}^{I_\tau(S)}$, the *full* and the *cut Bessel function*, with further properties of which this note uses three. (W1) For each side σ there is a character χ of $\mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s};1))$ with values of absolute value 1 such that $W^\sigma_{S,\tau}(u\,y) = \chi(u)\,W^\sigma_{S,\tau}(y)$ for $u \in \mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s};1))$. (W2) $\int \|W^{\mathrm{cut}}_{S,\tau}\|^2 \le \#I_\tau(S)$, the integral over the quotient of $G_1(\mathfrak{s})$ by $\mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s};1))$. (W3) If $W^{\mathrm{cut}}_{S,\tau}(y) \neq 0$ then $y \in \mathrm{Cut}(b(S))$. We put $\mathrm{Bc}(S) = \prod_\tau I_\tau(S)$ and $\deg(S) = \prod_\tau \deg_\tau(S)$. For a side σ, $y \in G_\infty$ and $\gamma \in G_S(E)$, the vector $\mathcal{W}^\sigma_S(\gamma;y) \in \mathbb{C}^{\mathrm{Bc}(S)\times\mathrm{Bc}(S)}$ is the tensor product over τ of the vectors $W^\sigma_{S,\tau}(y_\tau)$ and of the complex conjugates of the vectors $W^\sigma_{S,\tau}((\mathrm{r}^\infty_S(\gamma)^{-1}y)_\tau)$; its norm is the product of their norms.

The *box size* is $\mathrm{bx}(S) = b(S)^{2n(\mathfrak{s})^2}$, and $\mathrm{BxP}(S)$ is the set of the *boxed points* $y \in G_\infty$ with $y_\tau \in \mathrm{FBox}(\mathrm{bx}(S))$ and $y_\tau \in \mathrm{Cut}(b(S))$ for all τ. An element $\gamma \in G_S(E)$ is *near*, $\gamma \in \mathrm{Near}(S)$, if there is $y \in \mathrm{BxP}(S)$ with $\mathrm{r}^\infty_S(\gamma)^{-1}y \in \mathrm{BxP}(S)$; and $\gamma \in \mathrm{NRep}_S(c,c')$ means that γ is near and is the representative, fixed once and for all, of its double coset $\mathcal{U}^0_{S,c}\,\gamma\,\mathcal{U}^0_{S,c'}$. For a class c, $\beta_{S,c} \colon G_\infty \to [0,\infty]$ is the *cutoff* of Note B, a product $\beta_{S,c} = \theta_{S,c}\cdot\omega_{S,c}$ of two functions described in Inputs (S9) and (B1). The *amplitude* and the *absolute amplitude* of γ over c, c' are

$$a^\sigma_S(c,c';\gamma) = \int_{G_\infty} \beta_{S,c}(y)\,\beta_{S,c'}(\mathrm{r}^\infty_S(\gamma)^{-1}y)\;\mathcal{W}^\sigma_S(\gamma;y)\,dy,$$

$$|a|^\sigma_S(c,c';\gamma) = \int_{G_\infty} \beta_{S,c}(y)\,\beta_{S,c'}(\mathrm{r}^\infty_S(\gamma)^{-1}y)\;\big\|\mathcal{W}^\sigma_S(\gamma;y)\big\|\,dy,$$

and for a set $\Xi \subseteq G_S(E)$ the *amplitude sum* is

$$A^\sigma_S(\Xi;c,s,c') = \sum_{\gamma\in\Xi\cap\mathrm{Tm}_S(c,s,c')} a^\sigma_S(c,c';\gamma) \;\in \mathbb{C}^{\mathrm{Bc}(S)\times\mathrm{Bc}(S)}.$$

The formal text defines the integral and the sum by conventions that give 0 when they do not converge; under the hypotheses of Setting 2.2 they converge (Input (B4)).

Levels, links, characters. Let $\gamma \in G_S(E)$. For $a \le \varrho(\mathfrak{s})$ and $c_1, c_2 \in E$, $\mathrm{CF}_a(\gamma;c_1,c_2)$ means: there are $v, v' \in E^{n(\mathfrak{s})}$ with $\mathfrak{f}v = \gamma\mathfrak{f}v'$, $v_a = c_1$, $v'_a = c_2$ and $v_{a'} = v'_{a'} = 0$ for $a' > a$; the flags of the two frames share a vector with these leading coefficients. For a natural number j, $\gamma \in \mathrm{NF}_j$ (*normal form from* j, as in Note S) means: there is $c_1 \in E$ with $c_1\overline{c_1} = 1$ such that $\mathrm{CF}_a(\gamma;c_1,1)$ for all $a \le \varrho(\mathfrak{s})$ with $j \le a$. For $j \le \varrho(\mathfrak{s})+1$, γ *has level* j, $\gamma \in \mathrm{Lv}_j(S)$, if $\gamma \in \mathrm{NF}_j$, $\mathrm{D}_j(Z_S(\gamma)) \neq 0$, and $\varrho(\mathfrak{s}) < j$ holds exactly if $\mathfrak{g}_S(\gamma) \neq 0$. The first two notions are defined in the same way for an element of $G(T)$, for any $\mathfrak{s}$-form T over a commutative ring with involution; they are used below also for $G_1(\mathfrak{s})$.

The *link* $\mathrm{link}(\gamma)$ is the E-subspace of $(E^{M(\mathfrak{s})})^2$ of the pairs formed by the middle coordinates of t and of $\gamma^{-1}t$, for the vectors $t \in E^{I(\mathfrak{s})}$ orthogonal to all e_a and all γe_a, $a < \varrho(\mathfrak{s})$. For γ, γ', the *common link* $\mathrm{link}^\circ(\gamma,\gamma')$ is the subspace of E^κ of the vectors t_0 such that some pair in $\mathrm{link}(\gamma)\cap\mathrm{link}(\gamma')$, both of whose members have z-coordinate 0, has first member t_0 on κ.

For $u \in G^{\mathrm{lat}}_S$ put $\psi^{\mathcal{U}}_S(u) = \psi_{E,f}(-\mathrm{sd}(u)/(g(S)\,\varpi(S)))$; for $u \in \mathcal{U}(T^{\mathrm{rat}}_S(E))$ this gives $\psi^{\mathcal{U}}_S(\mathrm{r}^f_S(u)) = \psi_{E,f}(-\mathrm{sd}(u)/(g(S)\,\varpi(S)))$. $\gamma \in \mathrm{ChA}(S)$ (*the characters agree at* γ) means that $\psi^{\mathcal{U}}_S(\mathrm{r}^f_S(u)) = \psi^{\mathcal{U}}_S(\mathrm{r}^f_S(u'))$ for all $u, u' \in \mathcal{U}(T^{\mathrm{rat}}_S(E))$ with $u\gamma = \gamma u'$. An

element $s \in \mathsf{G}(S)$ *agrees*, $s \in \mathrm{Agr}(S)$, if $\psi_S^{\mathcal{U}}(u) = \psi_S^{\mathcal{U}}(u')$ for all $u, u' \in K(S) \cap \mathcal{U}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$ with $\mathrm{red}_S(u)\, s = s\, \mathrm{red}_S(u')$. A set $\Xi \subseteq G_S(E)$ is *stable*, $\Xi \in \mathrm{Stab}(S)$, if $u^{-1}\gamma u' \in \Xi$ is equivalent to $\gamma \in \Xi$ for all $u, u' \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$. For $s \in \mathsf{G}(S)$ and a natural number X, $\mathrm{fix}(s) \geq X$ means that there is a finite set of chosen places and of conjugates of chosen places, with product of residue cardinalities at least X, at each of which s, or its inverse adjoint at a conjugate, fixes a non-zero vector of the span of the frame (Note S).

Constants. With $\mathrm{cd}(S) = \#\mathrm{Cl}(S) \cdot \#\mathrm{Bc}(S)$:

$$\mathrm{tl}(S) = b(S)^{\,k(\mathfrak{s})f(E)/4}, \qquad \mathrm{cf}(S) = (b(S)/g(S))^{k(\mathfrak{s})f(E)} \cdot 2\,\mathrm{tl}(S)^2,$$

$$\mathsf{d}_1(S) = \#\overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{S} \cdot \mathrm{cd}(S), \qquad \mathsf{d}(S) = \frac{\mathrm{cd}(S)^2}{\deg(S) \cdot \mathrm{mass}(E, r(\mathfrak{s}))},$$

$$\mathrm{cw}_i(S) = \frac{\#\overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{S} \cdot \#\mathrm{Cl}(S)}{N(S)^{k(\mathfrak{s})f(E)}}\,(2\,\mathrm{tl}(S)^2)^i \quad (i \in \mathbb{N}),$$

where $\mathrm{mass}(E, r(\mathfrak{s})) > 0$ is the mass of the genus of the standard lattice of rank $r(\mathfrak{s})$; and $\nu(S) > 0$ is the *covolume* of Note B, given there by a formula in $g(S)$, $N(S)$, d_E and the $\kappa_\tau(S)$. The names in the formal text are: tail scale, codimension factor, Bessel energy, main energy, count weight. Three more constants occur in one inequality of admissibility:

$$\mathrm{fs}(S) = \prod_{\tau} \prod_{a \leq \varrho(\mathfrak{s})} \mathrm{fs}_S(\tau, a)^{\,k(\mathfrak{s})+2(\varrho(\mathfrak{s})-a)},$$

$$c_*(S) = \Big(\frac{\pi^{k(\mathfrak{s})}}{(k(\mathfrak{s})-1)!\, N(S)^{k(\mathfrak{s})}}\Big)^{f(E)} |d_E|^{1/4}\ \mathrm{vol}(\hat{O}_E)^{k(\mathfrak{s})},$$

$$a_*(S) = \#\overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{S} \cdot g(S)^{\,f(E)\varrho(\mathfrak{s})(2k(\mathfrak{s})+2\varrho(\mathfrak{s})+1)} \cdot \#\mathrm{Cl}(S) \cdot c_*(S),$$

with $\mathrm{vol}(\hat{O}_E) = (2/\,\mathrm{rd}_E)^{f(E)}$, and $|d_E| = \mathrm{rd}_E^{2f(E)}$.

Setting 2.2 (standing hypotheses)**.** In Setting 2.1, the field E is stably unramified and the filling data S is admissible. Of the conditions of admissibility (Note S) this note uses, for all infinite places τ:

- (A1) $K^{\mathrm{col}}(S)$ is the preimage of $K_g(S)$ under inc_S;
- (A2) $1 \leq |\kappa_\tau(S)| \leq b(S)^{23}$, and $N(S) \leq b(S)^{60n(\mathfrak{s})}$;
- (A3) $\mathrm{tl}(S)^{44}\,\mathsf{d}_1(S) \leq \mathsf{d}(S)$, $\mathrm{tl}(S)^{18}\,\mathrm{cf}(S)\,\mathsf{d}(S) \leq \mathsf{d}_1(S)$ and $\mathrm{cf}(S) \leq 1$;
- (A4) $\#\mathrm{Bc}(S) \leq b(S)^{2k(\mathfrak{s})f(E)}$;
- (A5) $(N(S)^{k(\mathfrak{s})}/b(S))^{f(E)} \leq [K_f(E,\kappa) : \mathrm{inc}_S^{-1}(K(S))]$, where $K_f(E,\kappa)$ is the group of the integral elements of $G_f(E,\kappa)$ with integral inverse;
- (A6) $a_*(S) \leq (1+b(S)^{-9}) \cdot \mathsf{d}(S) \cdot \deg(S) \cdot c_G(\mathfrak{s})^{f(E)} \cdot \mathrm{vol}(\hat{O}_E)^{-n(\mathfrak{s})} \cdot (\nu(S)\,\mathrm{fs}(S)/\#\mathrm{Bc}(S))^2$.

Lattices and weights (from Note B). Let $\mathsf{k} \in G_S^{\mathrm{lat}}$. Write $Z_{T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})}(\mathsf{k})$ for its frame Gram matrix, Z^{x} for the matrix of the first $\varrho(\mathfrak{s})$ rows of a matrix Z with $n(\mathfrak{s})$ rows, and, for $j \leq \varrho(\mathfrak{s}) + 1$, $\mathrm{Ib}(j)$ for the set of the $a < \varrho(\mathfrak{s})$ with $a < j$. Let $\mathcal{R}(\mathsf{k}) = \hat{O}_E\langle (Z_{T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})}(\mathsf{k})^{\mathrm{x}})^* \rangle$ be the lattice spanned by the conjugates of the first $\varrho(\mathfrak{s})$ rows of $Z_{T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})}(\mathsf{k})$, and $\mathcal{R}_j(\mathsf{k})$ the sublattice spanned by the conjugates of the rows of index in $\mathrm{Ib}(j)$. Then

$$\mathrm{i}_{S,j}(\mathsf{k}) = [\mathcal{R}(\mathsf{k}) : \mathcal{R}_j(\mathsf{k})], \qquad \mathfrak{d}_S(\mathsf{k}) = [\hat{O}_E^{I(\mathfrak{s})} \cap \mathrm{span}_{\mathbb{A}_{E,f}}(e_a, \mathsf{k}e_a) : \hat{O}_E\langle e_a, \mathsf{k}e_a \rangle],$$

the second with the vectors e_a, $\mathsf{k}e_a$, $a < \varrho(\mathfrak{s})$ (the *saturation index*). For $\gamma \in G_S(E)$, classes c, c' and $j \le \varrho(\mathfrak{s})+1$, with $j_1 = j - \min(\varrho(\mathfrak{s}), j) \in \{0,1\}$ and $\mathsf{k} = \mathrm{pos}_S(c,c';\gamma)$,

$$\mathrm{mn}_S(j;\gamma) = \frac{|\mathrm{N}_{E/\mathbb{Q}}(\mathrm{D}_j(Z_S(\gamma)))|}{N(S)^{2j_1 f(E)}}, \qquad \mathrm{cg}_S(j;c,c';\gamma) = \frac{\max(\|\mathbf{g}\|_S(\gamma),\ N(S)^{f(E)}\,\mathrm{mn}_S(j;\gamma))}{\mathrm{i}_{S,j}(\mathsf{k})\cdot \mathrm{i}_{S,j}(\mathsf{k}^{-1})}$$

(the *minor norm* and the *class gauge*). For a class c and an E-subspace $U \subseteq E^\kappa$, $\mathrm{disc}_{S,c}(U)$ is the discriminant of the lattice of the points t_0 of $U \otimes_E \mathbb{A}_{E,f}$ with $\mathrm{C}^\circ_S(c)\,t_0$ integral: it is $\sqrt{[\det(M_U^* M_U)]}$ for a matrix $M_U = \mathrm{m}_{\mathrm{C}^\circ_S(c)}(U)$ over $\mathbb{A}_{E,f}$, fixed by Note S, whose columns form an $\hat{O}_E$-basis of this lattice (more precisely, M_U is a basis matrix of U over E times an invertible adelic matrix, and $\mathrm{C}^\circ_S(c)\,M_U$ is integral with an integral left inverse). For a unit $a \in \mathbb{A}_{E,f}$ the *frame value index* is

$$\mathrm{o}_{S,c}(\mathsf{k};a,U) = [\Lambda_U(\mathsf{k}) : \Lambda_U(\mathsf{k}) \cap a\,\mathcal{R}(\mathsf{k})],$$
$$\Lambda_U(\mathsf{k}) = \hat{O}_E\langle((\mathsf{k}\mathfrak{f})^{\mathrm{mid}})^*\ T^{\mathrm{lat}}_S(\mathbb{A}_{E,f})\ \mathrm{C}^\circ_S(c)\,M_U\rangle,$$

where $(\mathsf{k}\mathfrak{f})^{\mathrm{mid}}$ is the matrix of the middle rows of $\mathsf{k}\mathfrak{f}$; so $\Lambda_U(\mathsf{k})$ is the lattice of the vectors $(h(\mathsf{k}\mathfrak{f}_a, t_0))_a$ for t_0 in the lattice of U and c. The *pair invariant* of γ, γ' is, with $U = \mathrm{link}^\circ(\gamma,\gamma')$ and $\mathsf{k} = \mathrm{pos}_S(c,c';\gamma)$,

$$\mathrm{pi}_S(c,c';\gamma,\gamma') = \mathrm{disc}_{S,c}(U)\cdot \mathrm{o}_{S,c}(\mathsf{k};1,U)\cdot \mathfrak{d}_S(\mathsf{k}).$$

For a pair $p = (c',\gamma) \in \mathrm{Cl}(S) \times G_S(E)$ put $\mathrm{el}_S(c;p) = \mathrm{pos}_S(c,c';\gamma)$, and for a function $\eta\colon G^{\mathrm{lat}}_S \to \mathbb{N}$,

$$\mathrm{ipw}_S(c;\eta,p) = \eta(\mathrm{el}_S(c;p))^{-k(\mathfrak{s})} \in [0,\infty],$$

read as 0 when $\eta(\mathrm{el}_S(c;p)) = 0$. For $\mathsf{t} \in \overline{\mathcal{S}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))}^S$, $p \in \mathrm{Lab}_S(j;c,\mathsf{t})$ (*p is a labelled term*) means that $\gamma \in \mathrm{NRep}_S(c,c')$, $\gamma \in \mathrm{Lv}_j(S)$, and $\gamma \in \mathrm{Tm}_S(c,s,c')$ for some s whose component in $\overline{\mathcal{S}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))}^S$ is t, for a fixed bijection of $\mathsf{G}(S)$ with $(\mathsf{G}(S)/\overline{\mathcal{S}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))}^S) \times \overline{\mathcal{S}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))}^S$. Two elements are *congruent terms*, $\gamma \sim_{S;c,c'} \gamma'$, if both lie in $\mathrm{Tm}_S(c,s,c')$ for one and the same s. Finally, for $i \in \mathbb{N}$, pairs p, p' and a subspace U, the *term weight* and the *pair weight* are

$$\mathrm{w}_S(j,i;c;p,p';U) = \mathrm{cw}_i(S)^{-1}\cdot \mathrm{disc}_{S,c}(U)^{-(k(\mathfrak{s})+1)}\cdot \mathrm{ipw}_S(c;\eta_{j,U},p)\cdot \mathrm{ipw}_S(c;\mathfrak{d}_S(\cdot),p'),$$
$$\eta_{j,U}(\mathsf{k}) = [\hat{g}(S)]^{\#\mathrm{Ib}(j)}\cdot \mathrm{o}_{S,c}(\mathsf{k};1,U), \qquad [\hat{g}(S)] = g(S)^{\,2f(E)},$$

where $\hat{g}(S)$ is $g(S)$ as a finite idele, and

$$\mathrm{pw}_S(j;c,c';\gamma,\gamma') = \mathrm{w}_S(j, k(\mathfrak{s}) - \dim_E U; c; (c',\gamma),(c',\gamma'); U), \qquad U = \mathrm{link}^\circ(\gamma,\gamma'),$$

if $\gamma \sim_{S;c,c'} \gamma'$, $\gamma \in \mathrm{NRep}_S(c,c')$, $\gamma \in \mathrm{Lv}_j(S)$, $\gamma' \in \mathrm{NRep}_S(c,c')$ and $\gamma' \in \mathrm{Lv}_j(S)$; otherwise the pair weight is 0. Finally, for a square matrix Z of size $n(\mathfrak{s})$, $\mathrm{pad}_j(Z)$ is the matrix whose rows of index $a < j$ are those of Z and whose other rows are those of the identity matrix; $\det \mathrm{pad}_j(Z) = \mathrm{D}_j(Z)$.

The notions of this note. The remaining definitions are those of the four files treated here.

Definition 2.3 (energy, link energy)**.** In Setting 2.1: (a) for a normed additive group V, a family $\mathcal{A}\colon \mathrm{Cl}(S)\times \mathsf{G}(S)\times \mathrm{Cl}(S)\to V$ and a set $\Omega\subseteq \mathsf{G}(S)$, the *energy* of $\mathcal{A}$ over Ω is

$$\mathrm{En}_S(\mathcal{A};\Omega)=\sum_{c\in \mathrm{Cl}(S)}\sum_{c'\in \mathrm{Cl}(S)}\sum_{s\in\Omega}\frac{\|\mathcal{A}(c,s,c')\|^2}{\nu(S)^2}.$$

(b) For a side σ, a set $\Xi\subseteq G_S(E)$, a set $\Omega\subseteq \mathsf{G}(S)$ and a set R of E-subspaces of $(E^{M(\mathfrak{s})})^2$, the *link energy* is

$$\mathrm{LE}^{\sigma}_S(\Xi;\Omega,R)=\sum_{c\in \mathrm{Cl}(S)}\sum_{c'\in \mathrm{Cl}(S)}\sum_{s\in\Omega}\sum_{\Lambda\in R}\left\|A^{\sigma}_S(\Xi\cap\{\gamma:\mathrm{link}(\gamma)=\Lambda\};c,s,c')\right\|^2\in[0,\infty].$$

LEAN `Filling.energy` · `Filling.linkEnergy`

The group $\mathsf{G}(S)$ is finite, so the energy is a finite sum of real numbers. For the family of the amplitude sums of a set Ξ we write $A^{\sigma}_S(\Xi;\cdot,\cdot,\cdot)$; here $V=\mathbb{C}^{\mathrm{Bc}(S)\times \mathrm{Bc}(S)}$ with its Euclidean norm.

Definition 2.4 (class size)**.** In Setting 2.1, for $j\in\{0,\dots,\varrho(\mathfrak{s})+1\}$ and a real number $x>0$,

$$\mathrm{cs}_S(j;x)=\mathrm{tl}(S)^2\cdot\sqrt{\frac{\mathsf{D}_j}{\mathrm{cw}_0(S)\cdot g(S)^{\,2\min(\varrho(\mathfrak{s}),j)\,k(\mathfrak{s})\,f(E)}\cdot x^{k(\mathfrak{s})+1}}},$$

where $\mathsf{D}_j=\mathsf{d}(S)$ if $\varrho(\mathfrak{s})<j$, and $\mathsf{D}_j=\mathrm{tl}(S)^{13}\,\mathrm{tl}(S)^{-32j}\,\mathsf{d}_1(S)$ if $j\le\varrho(\mathfrak{s})$.

LEAN `Filling.classSize`

Definition 2.5 (boxed cutoff, Schur weight, Schur integral)**.** In Setting 2.1, let $c,c'\in\mathrm{Cl}(S)$, $\gamma,\gamma'\in G_S(E)$ and $y\in G_\infty$. (a) The *boxed cutoff* $\beta^0_{S,c}\colon G_\infty\to[0,\infty]$ is equal to $\beta_{S,c}$ on the set $\mathrm{BxP}(S)$ of the boxed points and to 0 outside it. (b) With the frame volume Δ on $G_1(\mathfrak{s})$ (Input (R1)), the *Schur weight* is

$$\mathrm{sw}_S(c;\gamma,y)=\beta^0_{S,c}(\mathrm{r}^{\infty}_S(\gamma)^{-1}y)\cdot\prod_{\tau}\Big(\frac{\Delta(y_\tau)}{\Delta((\mathrm{r}^{\infty}_S(\gamma)^{-1}y)_\tau)}\Big)^{k(\mathfrak{s})+n(\mathfrak{s})}\in[0,\infty].$$

(c) The *Schur integral* is

$$\mathrm{SI}_S(c,c';\gamma')=\int_{G_\infty}\beta^0_{S,c}(y)\cdot\prod_{\tau}\left\|W^{\mathrm{cut}}_{S,\tau}(y_\tau)\right\|^2\cdot\mathrm{sw}_S(c';\gamma',y)\,dy\in[0,\infty].$$

LEAN `Filling.boxCutoff` · `Filling.schurWeight` · `Filling.schurIntegral`

Setting 2.6 (the hypotheses of the energy estimates)**.** The hypotheses of Setting 2.2 hold: E is stably unramified and S is admissible. Further, C is a real number with $0\le C$; $\Omega\subseteq\mathsf{G}(S)$ is a set; $\Xi\subseteq G_S(E)$ is a stable set, $\Xi\in\mathrm{Stab}(S)$; X is a natural number with $X\neq 0$ such that $\mathrm{fix}(s)\ge X$ for every $s\in\Omega$; $\mathsf{d}_1(S)\le C$; and $\mathrm{tl}(S)^{20}\cdot\mathsf{d}_1(S)\le C$ in case there are $\gamma\in\Xi$, classes $c,c'\in\mathrm{Cl}(S)$ and $s\in\Omega$ with $\gamma\in\mathrm{Tm}_S(c,s,c')$ and $\gamma\in\mathrm{NF}_0$.

3 Inputs

What follows is taken for granted. Each statement was compared with its Lean text; the owners are the two other notes on the set-up of this unit (Note S, Note B), the note on the pairing of links of the same unit (Note P), the unit on the real place (unit 132 of the reading, Chapter VI of the write-up), and the background library. For the last, the catalogue of the write-up classes all the statements used here as elementary, with one exception that is said in (B8). Throughout, S is as in Setting 2.1.

Inputs. *From Note S.* (S1) $\mathrm{tl}(S) \geq 2$, $\mathrm{tl}(S)^4 = b(S)^{k(\mathfrak{s})f(E)}$, and $b(S)^e \leq \mathrm{tl}(S)$ for every natural number $e \leq 10^5\, n(\mathfrak{s})^9\, f(E)$; $\nu(S) > 0$. (S2) $\|a^\sigma_S(c,c';\gamma)\| \leq |a|^\sigma_S(c,c';\gamma)$. (S3) r^∞_S maps $\mathcal{U}(T^{\mathrm{rat}}_S(E))$ into the elements u of G_∞ with all $u_\tau \in \mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s};1))$, and $\mathcal{U}^0(T^{\mathrm{rat}}_S(E))$ into those with all $u_\tau \in \mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))$. For such u with all $u_\tau \in \mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))$: $u\,y \in \mathrm{BxP}(S)$ exactly if $y \in \mathrm{BxP}(S)$. If $\gamma \in \mathrm{Near}(S)$ and $u, u' \in \mathcal{U}^0(T^{\mathrm{rat}}_S(E))$, then $u\gamma u' \in \mathrm{Near}(S)$. (S4) For $\gamma \in G_S(E)$, $a, a' \leq \varrho(\mathfrak{s})$ and $j \leq n(\mathfrak{s})$: $Z(\mathrm{r}^\infty_S(\gamma)_\tau)_{a,a'} = \kappa_\tau(S)^{\varrho(\mathfrak{s})-a}\kappa_\tau(S)^{\varrho(\mathfrak{s})-a'}N(S)^{-1}\,(Z_S(\gamma)_{a,a'})_\tau$; $\mathrm{D}_j(Z(\mathrm{r}^\infty_S(\gamma)_\tau)) = \prod_{a<j}\mathrm{fs}_S(\tau,a)\cdot \mathrm{D}_j(Z_S(\gamma))_\tau$; the number $\mathrm{g}_S(\gamma)_\tau$ is real and non-negative; and $\mathrm{g}^{\mathbb{R}}(Z(\mathrm{r}^\infty_S(\gamma)_\tau)) = \left(\prod_{a\leq\varrho(\mathfrak{s})}\mathrm{fs}_S(\tau,a)\right)^2 |\mathrm{g}_S(\gamma)_\tau|$. (S5) If S is admissible and $\gamma \in \mathrm{Near}(S)$, then $|(Z_S(\gamma)_{a,a'})_\tau| \leq b(S)^{5n(\mathfrak{s})^2}$ and $|\mathrm{D}_l(Z_S(\gamma))_\tau|^2 \leq b(S)^{11n(\mathfrak{s})^3}$ for all $a, a' \leq \varrho(\mathfrak{s})$, $l \leq n(\mathfrak{s})$ and all τ. (S6) If $\gamma \in \mathrm{Tm}_S(c,s,c')$, then the entries of $Z_S(\gamma)$ are algebraic integers, $\mathrm{pos}_S(c,c';\gamma)$ is integral, $\gamma^{-1} \in \mathrm{Tm}_S(c',s^{-1},c)$, and $\gamma\delta \in \mathrm{Tm}_S(c,s\,s',c'')$ for $\delta \in \mathrm{Tm}_S(c',s',c'')$. For all γ: $Z_{T^{\mathrm{lat}}_S(\mathbb{A}_{E,f})}(\mathrm{pos}_S(c,c';\gamma)) = (Z_S(\gamma))_{\mathbb{A}_f}$, and the matrix $\mathrm{pos}_S(c,c';\gamma)\,\mathfrak{f}$ is $\mathrm{C}_S(c)\,(\gamma\mathfrak{f})_{\mathbb{A}_f}$. (S7) $K_g(S)$ is the set of the $\mathsf{k} \in K(S)$ such that all entries of $\mathsf{k}-1$ lie in $g(S)\,\hat{O}_E$. (S8) With the representative $x_c \in G_f(E,\kappa)$ of the class c and the matrix $\Lambda^\circ(\mathfrak{s};\bar\pi(S))$ of Note S: $\mathrm{C}^\circ_S(c) = \Lambda^\circ(\mathfrak{s};\bar\pi(S))\,x_c^{-1}$; for a matrix M over $\mathbb{A}_{E,f}$ with rows in κ, $\Lambda^\circ(\mathfrak{s};\bar\pi(S))\,M$ is integral exactly if M is integral and its row k_0 has entries in $\bar\pi(S)\hat{O}_E$; and for $\mathsf{k} \in G_f(E,\kappa)$, $\mathrm{inc}_S(\mathsf{k}) \in K(S)$ exactly if $\mathsf{k} \in K_f(E,\kappa)$ and the row k_0 of $\mathsf{k}-1$ has entries in $\bar\pi(S)\hat{O}_E$. (S9) For every subgroup Γ of $\mathcal{U}_{S,c}$ there is a measurable $\theta\colon G_\infty \to [0,\infty]$ with $\sum_{u\in\Gamma}\theta(\mathrm{r}^\infty_S(u)\,y) = 1$ for all y; the image of Γ in G_∞ is discrete, and for every measurable function F_0 on G_∞ with values in a finite-dimensional complex vector space that is invariant under left translation by the image of Γ, the integral $\int \theta\, F_0$ is the integral of F_0 over the quotient of G_∞ by this image. $\theta_{S,c}$ is the function fixed in this way for $\Gamma = \mathcal{U}^0_{S,c}$.

Inputs. *From Note S, continued.* (S10) For $u, u' \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$: $\mathrm{link}(u^{-1}\gamma u') = \mathrm{link}(\gamma)$, $\mathrm{g}_S(u^{-1}\gamma u') = \mathrm{g}_S(\gamma)$, $\mathrm{D}_l(Z_S(u^{-1}\gamma u')) = \mathrm{D}_l(Z_S(\gamma))$, and $u^{-1}\gamma u' \in \mathrm{Lv}_j(S)$ exactly if $\gamma \in \mathrm{Lv}_j(S)$. If $u' \in \mathcal{U}^0(T_S^{\mathrm{rat}}(E))$ then $(\gamma u')\mathfrak{f} = \gamma\mathfrak{f}$. (S11) A pair (t_1, t_2) of vectors of $E^{M(\mathfrak{s})}$ lies in $\mathrm{link}(\gamma)$ exactly if there is a vector t with middle coordinates t_2 and no e^*-coordinates such that γt has middle coordinates t_1 and no e^*-coordinates. The link is Lagrangian for the form $T_S^{\mathrm{rat}}(E) \oplus (-T_S^{\mathrm{rat}}(E))$: a pair lies in $\mathrm{link}(\gamma)$ exactly if it is orthogonal to $\mathrm{link}(\gamma)$. If $t_1 \in E^{M(\mathfrak{s})}$ has z-coordinate 0 and $t_1^*\, T_S^{\mathrm{rat}}(E)\, t_1 = 0$, then $t_1 = 0$. (S12) If $\gamma \in \mathrm{NF}_j$ then $\gamma^{-1} \in \mathrm{NF}_j$ and $\mathrm{r}_S^{\infty}(\gamma)_\tau \in \mathrm{NF}_j$ for every τ; for an $\mathfrak{s}$-form T, $u \in \mathcal{U}(T)$ and $\gamma_1 \in G(T)$ with $\gamma_1 \in \mathrm{NF}_j$ also $u\gamma_1 \in \mathrm{NF}_j$; and for every $v_0 \in E^{n(\mathfrak{s})}$ there are $v, v' \in E^{n(\mathfrak{s})}$ with $\mathfrak{f}\, v = \gamma\mathfrak{f}\, v'$ and $v'_a = (v_0)_a$ for all $a \geq j$. If $\mathrm{g}_S(\gamma) \neq 0$ then $\gamma \in \mathrm{Lv}_{\varrho(\mathfrak{s})+1}(S)$. (S13) If $\mathfrak{f}\, v = \gamma\mathfrak{f}\, v'$ then $\overline{v_{\varrho(\mathfrak{s})}}\, v_{\varrho(\mathfrak{s})} = \overline{v'_{\varrho(\mathfrak{s})}}\, v'_{\varrho(\mathfrak{s})}$ and $Z_S(\gamma)\, v'$ is the vector with entry $-N(S)\, v_{\varrho(\mathfrak{s})}$ at $\varrho(\mathfrak{s})$ and 0 elsewhere. If $Z_S(\gamma)\, v'$ is the vector with entry $D_0 \in E$ at $\varrho(\mathfrak{s})$ and 0 elsewhere, then $t = \gamma\mathfrak{f}\, v'$ has no e^*-coordinates, z-coordinate $-D_0/N(S)$, and for every τ

$$N(S) \sum_{i \in \kappa} |(t_i)_\tau|^2 = |(D_0)_\tau|^2 - \left(N(S)\, |(v'_{\varrho(\mathfrak{s})})_\tau|\right)^2;$$

and if this number is ≤ 0 for some τ, then $t = \mathfrak{f}\, v$ for some $v \in E^{n(\mathfrak{s})}$. (S14) (Eichler elements.) Let $a \leq \varrho(\mathfrak{s})$, $v_0 \in E^{n(\mathfrak{s})}$ with $(v_0)_{a'} = 0$ for $a' \geq a$, $r_0 \in E$, and t a vector such that $h(t, \mathfrak{f}_{a'}) = 0$ for $a' < a$, with $h = h_{T_S^{\mathrm{rat}}(E)}$. Then the map $t' \mapsto t' + t_0\, h(t, t') - (t + \frac{1}{2} h(t, t)\, t_0)\, h(t_0, t')$, with $t_0 = r_0\, \mathfrak{f}\, v_0$, is an element u of $\mathcal{U}(T_S^{\mathrm{rat}}(E))$ with $\mathrm{sd}(u) = r_0 \sum_{1 \leq a' \leq \varrho(\mathfrak{s})} h(t, \mathfrak{f}_{a'})\, (v_0)_{a'-1}$. Every E-linear form on $E^{I(\mathfrak{s})}$ is $h(t, \cdot)$ for some t.

Inputs. *From Note S, continued.* (S15) (Lattice discriminants.) Let E be stably unramified. For $x \in G_f(E, \kappa)$ and an E-subspace $U \subseteq E^\kappa$ let $\mathrm{disc}_{x^{-1}}(U)$ be the discriminant of the lattice $(U \otimes \mathbb{A}_{E,f}) \cap x\hat{O}_E^\kappa$. Then for every $m \in \mathbb{N}$

$$\sum_{U\colon\, \dim_E U + m = k(\mathfrak{s})} \mathrm{disc}_{x^{-1}}(U)^{-(k(\mathfrak{s})+1)} \;\leq\; \left(200\, f(E)\, k(\mathfrak{s})\, \mathrm{rd}_E\right)^{f(E)(k(\mathfrak{s})+1)m}.$$

Further, let H be a matrix over $\mathbb{A}_{E,f}$ with columns in a finite set I_1, having a left inverse, $U \subseteq E^{I_1}$ an E-subspace of dimension d_1, x_0 a matrix over E with d_1 columns in U, $w_0 \in \mathrm{GL}_{d_1}(\mathbb{A}_{E,f})$, and $D_1 = [\det(x_0^* x_0)_{\mathbb{A}_f}] \cdot [\det w_0]^2 \neq 0$. If $H\, (x_0)_{\mathbb{A}_f}\, w_0$ is integral and all columns of $Y_0\, H\, (x_0)_{\mathbb{A}_f}\, w_0$ lie in an $\hat{O}_E$-module K_0, for a matrix Y_0 over $\mathbb{A}_{E,f}$, then $(\mathrm{disc}_H(U) \cdot i_0)^2 \leq D_1$, $\mathrm{disc}_H(U) > 0$ and $i_0 \geq 1$, where $i_0 = [\hat{O}_E\langle Y_0 H M_U\rangle : K_0]$ and $M_U = \mathrm{m}_H(U)$ is the basis matrix of the lattice of the points of $U \otimes \mathbb{A}_{E,f}$ that H makes integral.

Inputs. *From Note B.* In Setting 2.2 unless said otherwise. (B1) $\beta_{S,c} = \theta_{S,c} \cdot \omega_{S,c}$, where $\omega_{S,c} \colon G_\infty \to [0,\infty]$ is a *box partition*, $\omega_{S,c} \in \mathrm{BP}(S,c)$: it is measurable; $\omega_{S,c} \le 1$; if $\omega_{S,c}(y) \neq 0$ and $y_\tau \in \mathrm{Cut}(b(S))$ then $y_\tau \in \mathrm{FBox}(\mathrm{bx}(S))$; $\omega_{S,c}(u\,y) = \omega_{S,c}(y)$ whenever every u_τ lies in the subgroup generated by $\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))$ and $\mathcal{H}(T^{\mathrm{rat}}(\mathfrak{s};1))$; and $\sum_{\gamma \in \mathcal{U}_{S,c}} \alpha(\gamma)\, \omega_{S,c}(\mathrm{r}_S^\infty(\gamma) y) = 1$ for every y and every $\alpha \colon \mathcal{U}_{S,c} \to [0,\infty]$ with $\sum_{u \in \mathcal{U}^0_{S,c}} \alpha(u\gamma) = 1$ for all γ. Moreover $\beta_{S,c}$ is measurable and $\sum_{u \in \mathcal{U}_{S,c}} \beta_{S,c}(\mathrm{r}_S^\infty(u) y) = 1$ for all y. (B2) If $y_\tau \in \mathrm{Cut}(b(S))$ for all τ, there is $u \in \mathcal{U}_{S,c}$ with $\mathrm{r}_S^\infty(u)\, y \in \mathrm{BxP}(S)$. (B3) (Frame count.) Let $c \in \mathrm{Cl}(S)$, $s \in \mathsf{G}(S)$, $y, y' \in G_\infty$, $C_1 \in [0,\infty]$, and let $\mathcal{P}$ be a set with, for each $p \in \mathcal{P}$, a class c_p, elements $\gamma_p, \delta_p \in G_S(E)$ and a weight $\varphi_p \in [0,\infty]$, such that $p \mapsto (c_p, \delta_p)$ is injective. Suppose that for every p with $\varphi_p \neq 0$: $\gamma_p \in \mathrm{Tm}_S(c, s, c_p)$, $\delta_p^{-1}\gamma_p \in \mathcal{U}_{S,c_p}$, $\mathrm{r}_S^\infty(\gamma_p)^{-1} y \in \mathrm{BxP}(S)$ and $\varphi_p \le \beta_{S,c_p}(\mathrm{r}_S^\infty(\delta_p)^{-1} y') \cdot C_1$. Let $\mathcal{V}$ be a real subspace of $\prod_\tau \prod_{a \le \varrho(\mathfrak{s})} \mathbb{C}^{I(\mathfrak{s})}$ that contains the differences of the families $\big((\mathrm{r}_S^\infty(\gamma_p)_\tau^{-1} y_\tau)^{-1} \mathfrak{f}_a\big)_{\tau,a}$ for any two p with $\varphi_p \neq 0$. Then $\sum_p \varphi_p \le b(S)^{\,6n(\mathfrak{s})^2 \dim_{\mathbb{R}} \mathcal{V}} \cdot C_1$. (B4) For a side σ and a set $\Xi \subseteq G_S(E)$: $\sum_{\gamma \in \Xi \cap \mathrm{Tm}_S(c,s,c')} |a|_S^\sigma(c, c'; \gamma) < \infty$, so the amplitudes are absolutely summable; and for fixed c and s,

$$\sum_{c'} \sum_{\gamma \in \Xi \cap \mathrm{Tm}_S(c,s,c')} \int \beta_{S,c}(y)\, \beta_{S,c'}(\mathrm{r}_S^\infty(\gamma)^{-1} y)\, \big\| \mathcal{W}_S^\sigma(\gamma; y) - \mathcal{W}_S^{\mathrm{cut}}(\gamma; y) \big\|\, dy \le \tfrac{1}{2}\, \nu(S)\, b(S)^{-k(\mathfrak{s})^3 f(E)}.$$

(B5) (Change of weight.) Let σ be a side, Ξ stable, and $\varphi \colon G_S(E) \times G_\infty \to [0,\infty]$ measurable in the second variable with $\sum_{(u,u') \in \mathcal{U}_{S,c} \times \mathcal{U}_{S,c'}} \varphi(u\gamma u'^{-1}, \mathrm{r}_S^\infty(u) y) = 1$ for all $\gamma \in \Xi \cap \mathrm{Tm}_S(c,s,c')$ and all y. Then the functions $y \mapsto \varphi(\gamma, y)\, \mathcal{W}_S^\sigma(\gamma; y)$ are integrable, $A_S^\sigma(\Xi; c, s, c') = \sum_\gamma \int \varphi(\gamma, y)\, \mathcal{W}_S^\sigma(\gamma; y)\, dy$, and $\sum_\gamma \int \varphi(\gamma, y)\, \|\mathcal{W}_S^\sigma(\gamma; y) - \mathcal{W}_S^{\mathrm{cut}}(\gamma; y)\|\, dy \le \tfrac{1}{2}\nu(S) b(S)^{-k(\mathfrak{s})^3 f(E)}$; the sums are over $\gamma \in \Xi \cap \mathrm{Tm}_S(c,s,c')$.

Inputs. *From Note B, continued.* (B6) If Ξ is stable and s does not agree, then $A^{\sigma}_{S}(\Xi; c, s, c') = 0$. (B7) In Setting 2.1: for $u, u' \in \mathcal{U}(T^{\mathrm{rat}}_{S}(E))$, $\mathcal{W}^{\sigma}_{S}(u\gamma u'^{-1}; \mathrm{r}^{\infty}_{S}(u)y) = \psi^{\mathcal{U}}_{S}(\mathrm{r}^{f}_{S}(u))^{-1}\, \psi^{\mathcal{U}}_{S}(\mathrm{r}^{f}_{S}(u')) \cdot \mathcal{W}^{\sigma}_{S}(\gamma; y)$; and for $u \in \mathcal{U}_{S,c}$ the product over τ of the Bessel functions is invariant under left translation by $\mathrm{r}^{\infty}_{S}(u)$. (B8) Let η^{0} be the product over τ of the Haar measures of $\mathcal{U}^{0}(T^{\mathrm{rat}}(\mathfrak{s}; 1))$ fixed by the library; they are invariant under left and right translations. There is a measure μ^{0} on G_{∞}, product of measures on the factors, such that $(y, u) \mapsto u\, y$ carries $\mu^{0} \times \eta^{0}$ to the Haar measure of G_{∞} (the catalogue of the write-up refers for this to Weil's integration formula for a quotient; the second reader of the reading project could not confirm the cited place and derived the statement from Tonelli's theorem). And if, for each τ, $\phi_{\tau} \colon G_{1}(\mathfrak{s}) \to [0, \infty]$ is measurable and invariant under left translation by $\mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s}; 1))$, then $\int \beta_{S,c}(y) \prod_{\tau} \phi_{\tau}(y_{\tau})\, dy = \nu(S) \cdot \prod_{\tau} \int \phi_{\tau}$, the integrals on the right over the quotient of $G_{1}(\mathfrak{s})$ by $\mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s}; 1))$, as in (W2). (B9) (Colourings.) Let $\mathcal{P}$ be a set with weights $\varphi_{p} \in [0, \infty]$. (i) If a map from $\mathcal{P}$ to some set takes at most $N_{1} \in [0, \infty]$ values and every non-empty subset on which it is constant has total weight at most C_{1}, then $\sum_{p} \varphi_{p} \le N_{1}\, C_{1}$. (ii) Let I_{1} be a finite set, $K_{0} \subseteq K_{1}$ subgroups of an abelian group with $[K_{1} : K_{0}] \neq 0$, and $F_{1}(p) \in K_{1}^{I_{1}}$ for $p \in \mathcal{P}$; if every non-empty $\mathcal{B} \subseteq \mathcal{P}$ with $F_{1}(p) - F_{1}(p') \in K_{0}^{I_{1}}$ for all $p, p' \in \mathcal{B}$ has total weight at most C_{1}, then $\sum_{p} \varphi_{p} \le [K_{1} : K_{0}]^{\#I_{1}}\, C_{1}$. (iii) A set of vectors in $E^{I_{1}}$ whose entries are algebraic integers of absolute value at most $r_{1} \ge 0$ at every infinite place has at most $(1 + 2r_{1})^{2\,\#I_{1} f(E)}$ elements. (iv) For a unit $d_{0} \in \hat{O}_{E}$ of $\mathbb{A}_{E,f}$: $[\hat{O}_{E}^{I_{1}} : d_{0}\hat{O}_{E}^{I_{1}}] = [d_{0}]^{\#I_{1}} \neq 0$.

Inputs. *From Note B, continued.* (B10) (Unipotent elements with prescribed first rows.) Let E be stably unramified, c a class, I_1 a finite set, $p_1 \colon I_1 \to \{0, \dots, \varrho(\mathfrak{s}) - 1\}$, x_1 the matrix over E with the columns $e_{p_1(i)}$, U_1 a matrix over E with rows in $I(\mathfrak{s})$ and columns in I_1 whose e_a^*-rows vanish for all $a \le \max p_1$, and Ψ a square matrix over E indexed by I_1 with $\Psi^* = -\Psi$, such that $\mathrm{C}_S(c)\,(U_1)_{\mathbb{A}_f}$ and $(\Psi)_{\mathbb{A}_f}$ are integral. Then there are $\gamma_1 \in \mathcal{U}_{S,c}$ and $\vartheta_1 \in E$ with, for $J = J(T_S^{\mathrm{rat}}(E))$,

$$\gamma_1 = 1 + g(S) \cdot \left(x_1\, U_1^* J - \left(U_1 + x_1(\vartheta_1\, U_1^* J U_1 - \Psi)\right) x_1^* J\right),$$

and $\gamma_1 \in \mathcal{U}_{S,c}^0$ if the columns of U_1 are orthogonal to all frame vectors. (B11) (Shared vectors.) In Setting 2.1: if $\gamma \in \mathrm{Lv}_j(S)$ then $\gamma^{-1} \in \mathrm{Lv}_j(S)$ and $\mathrm{mn}_S(j;\gamma) > 0$; the matrix $Y = \mathrm{pad}_j(Z_S(\gamma))$ is invertible, and for every $a \le \varrho(\mathfrak{s})$ with $j \le a$ there is $v \in E^{n(\mathfrak{s})}$ with $\mathfrak{f}\, v = \gamma \mathfrak{f}\, (Y^{-1})_{\cdot,a}$, the column a of Y^{-1}. (B12) In Setting 2.1: if $\gamma\mathfrak{f}$ and $\gamma'\mathfrak{f}$ have the same middle rows and the same e^*-rows, then for $y \in G_1(\mathfrak{s})$, every τ and $a' \le \varrho(\mathfrak{s})$,

$$\begin{aligned}(\mathrm{r}_S^\infty(\gamma)_\tau^{-1} y)^{-1}\mathfrak{f}_{a'} - (\mathrm{r}_S^\infty(\gamma')_\tau^{-1} y)^{-1}\mathfrak{f}_{a'} \\ = \sum_{a < \varrho(\mathfrak{s})} \left((\gamma - \gamma')_{e_a, \mathfrak{f}_{a'}}\right)_\tau \frac{\mathrm{sc}_{S,\tau}(\mathfrak{f}_{a'})}{\mathrm{sc}_{S,\tau}(e_a)} \cdot y^{-1} e_a.\end{aligned}$$

For matrices t, t' over E with rows in $I(\mathfrak{s})$,

$$(\mathrm{C}_S(c)(t)_{\mathbb{A}_f})^*\, J(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))\, (\mathrm{C}_S(c)(t')_{\mathbb{A}_f}) = (t^* J(T_S^{\mathrm{rat}}(E))\, t')_{\mathbb{A}_f},$$

and $\mathrm{C}_S^\circ(c)^*\, T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})\, \mathrm{C}_S^\circ(c) = 1$. The matrix $\mathrm{C}_S(c)$ is the identity on the coordinates e_a and e_a^*, and on the middle coordinates it has the columns of $\mathrm{C}_S^\circ(c)$ on κ and the column of z; so it fixes the vectors e_a, and it maps γe_a, read over $\mathbb{A}_{E,f}$, to $\mathrm{pos}_S(c, c'; \gamma)\, e_a$, for all $a < \varrho(\mathfrak{s})$.

Inputs. *From Note P* (the note *link-pairing* of this unit). Let E be stably unramified, $j \le \varrho(\mathfrak{s}) + 1$, $j_1 = j - \min(\varrho(\mathfrak{s}), j)$, and $\gamma \in \mathrm{Tm}_S(c, s, c')$ with $\gamma \in \mathrm{Lv}_j(S)$. (P1) $\mathrm{i}_{S,j}(\mathrm{pos}_S(c, c'; \gamma)) \ge 1$, $\mathfrak{d}_S(\mathrm{pos}_S(c, c'; \gamma)) \ge 1$ and $\mathrm{cg}_S(j; c, c'; \gamma) > 0$. (P2) Let moreover $s \in \mathrm{Agr}(S)$, let X be a natural number with $\mathrm{fix}(s) \ge X$, and let $\gamma' \in \mathrm{Tm}_S(c, s, c')$ with $\gamma' \in \mathrm{Lv}_j(S)$. Then $\mathrm{disc}_{S,c}(\mathrm{link}^\circ(\gamma, \gamma')) > 0$ and $\mathrm{o}_{S,c}(\mathrm{pos}_S(c, c'; \gamma); 1, \mathrm{link}^\circ(\gamma, \gamma')) \ge 1$, and there is $i \in \mathbb{N}$ such that $k(\mathfrak{s}) \le \dim_E \mathrm{link}^\circ(\gamma, \gamma') + i + j_1$, such that $\mathrm{link}(\gamma) = \mathrm{link}(\gamma')$ if $i = 0$, and such that, with $X' = X$ if $\varrho(\mathfrak{s}) < j$ and $i = 0$, and $X' = 1$ otherwise,

$$\mathrm{pi}_S(c, c'; \gamma, \gamma') \cdot \mathrm{pi}_S(c, c'; \gamma', \gamma) \cdot X'^2 \;\le\; \left(\frac{2}{g(S)}\right)^{2if(E)} \cdot \mathrm{cg}_S(j; c, c'; \gamma) \cdot \mathrm{cg}_S(j; c, c'; \gamma').$$

Inputs. *From unit 132 (the real place).* (R1) The *frame volume* $\Delta\colon G_1(\mathfrak{s}) \to \mathbb{R}$ is continuous and positive, and $\Delta(u\,y) = \Delta(y)$ for $u \in \mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))$. (It is the absolute value of the determinant of the Euclidean products of the vectors $y^{-1}\mathfrak{f}_a$ with a basis of a maximal negative subspace.) (R2) (Volume kernel.) Let $y \in G_1(\mathfrak{s})$ and θ_0 a real number with $\mathrm{bs}(y) \le \theta_0$; let $j \le n(\mathfrak{s})$, Σ a finite set with an injection into $\{0,\dots,j-1\}$, R_0 an invertible complex matrix indexed by Σ, Z a complex square matrix of size $n(\mathfrak{s})$ whose leading block Z_j of size j has $\det Z_j \neq 0$, and V_0 a complex matrix with rows in κ and j columns. Then there is a measurable $\mathcal{K}\colon \mathrm{Mat}_{\kappa\times\Sigma}(\mathbb{C}) \to [0,\infty]$ with

$$\int \mathcal{K} \;\le\; \Big(\frac{(2\theta_0^2)^j}{|\det Z_j|}\Big)^{k(\mathfrak{s})+n(\mathfrak{s})} \cdot \Big(\frac{\pi^{\#\Sigma}}{(\#\Sigma)!\,(\theta_0^{\#\Sigma}\,|\det R_0|)^2}\Big)^{k(\mathfrak{s})}$$

(Lebesgue measure) such that $(\Delta(y)/\Delta(\gamma^{-1}y))^{k(\mathfrak{s})+n(\mathfrak{s})} \le \mathcal{K}(A)$ for all $\gamma \in G_1(\mathfrak{s})$ and $A \in \mathrm{Mat}_{\kappa\times\Sigma}(\mathbb{C})$ with: $Z(\gamma) = Z$; $\gamma \in \mathrm{NF}_j$; and the matrix of the κ-coordinates of $\gamma\mathfrak{f}_0,\dots,\gamma\mathfrak{f}_{j-1}$ is $V_0 + A\,R_0\,Z_\Sigma$, where Z_Σ consists of the rows of Z_j with index in the image of Σ. (R3) (Open-cell bound.) Let τ be an infinite place, $\gamma, y \in G_1(\mathfrak{s})$ and $\rho \ge b(S)$ a real number with $\mathfrak{g}^{\mathbb{R}}(Z(\gamma)) > 0$, $y \in \mathrm{FBox}(\rho)$ and $\gamma^{-1}y \in \mathrm{FBox}(\rho)$. Then there are a compact set $\mathcal{C} \subseteq G_1(\mathfrak{s})$, a finite measure n_1 on $G_1(\mathfrak{s})$ with total mass at most $\mathrm{cm}(\gamma;\deg_\tau(S),\rho,\#I_\tau(S))$, a probability space (Q_1,m_1) and a measurable map $t_1\colon Q_1 \to G_1(\mathfrak{s})$ such that: for m_1-almost every q, $t_1(q)$ lies in the subgroup generated by $\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))$ and $\mathcal{H}(T^{\mathrm{rat}}(\mathfrak{s};1))$, and every $y' \in \mathrm{FBox}(\rho)$ with $\mathrm{bs}(\gamma\,t_1(q)\,y') \le \rho$ lies in $\mathcal{C}$; and for every bounded measurable $\phi\colon G_1(\mathfrak{s}) \to \mathbb{C}$,

$$\int_{Q_1} \Big\| \int_{\mathcal{C}} \phi(y')\; W^{\mathrm{full}}_{S,\tau}(\gamma\,t_1(q)\,y') \otimes \overline{W^{\mathrm{full}}_{S,\tau}(t_1(q)\,y')}\,dy' \Big\|^2 dm_1(q) \;\le\; \int |\phi|^2\,dn_1.$$

(R4) If $\mathrm{bs}(y) \le \theta_0$, there is $u \in \mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s};1))$ with $\|u\,y\| \le \theta_0$.

Inputs. *From unit 132, continued.* (R5) (Congruence count.) Let E be stably unramified; let $d_0 \in \hat{O}_E$ be a unit of $\mathbb{A}_{E,f}$; I_1, I_2, I_3 finite sets with $\#I_2 = \#I_3 + m_1$; $\mathcal{P}$ a set with integral matrices Y_p (rows in I_1, columns in I_2) for $p \in \mathcal{P}$; M_0 an integral matrix with rows in I_2 and columns in I_3 that has an integral left inverse; $K_0 \supseteq d_0 \hat{O}_E^{I_1}$ an $\hat{O}_E$-module; $\eta_p \in \mathbb{N}$ with $[\hat{O}_E\langle Y_p M_0\rangle : K_0]$ dividing η_p; $l_1 \geq \#I_3$; and $C_1 \in [0,\infty]$ such that every non-empty $\mathcal{B} \subseteq \mathcal{P}$ with $\hat{O}_E\langle Y_p - Y_{p'}\rangle \subseteq K_0$ for all $p, p' \in \mathcal{B}$ has at most C_1 elements. Then $\sum_{p\in\mathcal{P}} \eta_p^{-l_1} \leq [d_0]^{\#I_1(\#I_1+m_1)} \cdot C_1$. (R6) If M is a matrix over E with rows in κ such that $\mathrm{C}^\circ_S(c)(M)_{\mathbb{A}_f}$ is integral, and a column of M has $\sum_{i\in\kappa} |(M_i)_\tau|^2 < 1$ at every τ for its entries M_i, then this column is zero. (R7) If S is admissible, $p = (c_1, \gamma_1)$ and $p' = (c_2, \gamma_2)$ satisfy $p \in \mathrm{Lab}_S(j; c, \mathfrak{t})$ and $p' \in \mathrm{Lab}_S(j; c, \mathfrak{t})$, and $(u\gamma_1)\mathfrak{f} = (u'\gamma_2)\mathfrak{f}$ for some $u, u' \in \mathcal{U}^0_{S,c}$, then $p = p'$. (R8) Let E be stably unramified, $\gamma \in G_S(E)$ and $\gamma' \in G_S(E)$ with $\gamma \in \mathrm{Lv}_j(S)$, $\gamma' \in \mathrm{Lv}_j(S)$ and $Z_S(\gamma') = Z_S(\gamma)$; let Σ be a set of indices $a < \varrho(\mathfrak{s})$ containing every $a < \min(j, \varrho(\mathfrak{s}))$, ξ the matrix of the rows of $Z_S(\gamma)$ with index in Σ, and $Y = \mathrm{pad}_j(Z_S(\gamma))$. Suppose that, for each middle index, the conjugate of the corresponding row of $(\mathrm{pos}_S(c, c''; \gamma')\mathfrak{f})^{\mathrm{mid}} - (\mathrm{pos}_S(c, c'; \gamma)\mathfrak{f})^{\mathrm{mid}}$ lies in $g(S)\,\hat{O}_E\langle(\xi)^*_{\mathbb{A}_f}\rangle + \{v : g(S)^{-1}(Y^{-1})^*_{\mathbb{A}_f} v \text{ is integral}\}$. Then there is a matrix ℓ over E with rows in κ and columns in Σ such that $\mathrm{C}^\circ_S(c)(\ell)_{\mathbb{A}_f}$ is integral and $\gamma'\mathfrak{f} - \gamma\mathfrak{f}$ has the rows $g(S)\cdot\ell\,\xi$ on κ and vanishes on the row of z and on the rows e^*_a.

Inputs. *From the background library.* (L1) (Cauchy–Schwarz.) For measurable $F_1, F_2 \geq 0$ and a function $F_0 \geq 0$ on a measure space with $F_0^2 \leq F_1 F_2$ pointwise: $(\int F_0)^2 \leq \int F_1 \cdot \int F_2$. (L2) (Class integrals.) Let Γ_0 be the group of the rational points in $G_f(E,\kappa)$. There is a functional $F_0 \mapsto \int_{\Gamma_0\backslash} F_0$ on the real functions on $G_f(E,\kappa)$ that are left Γ_0-invariant and right invariant under an open subgroup, with these properties: it is additive, and non-negative on non-negative functions; $\int_{\Gamma_0\backslash} F_0(\cdot\, x) = \int_{\Gamma_0\backslash} F_0$ for $x \in G_f(E,\kappa)$; and if F_0 is right invariant under $K^{\mathrm{col}}(S)$, then $\int_{\Gamma_0\backslash} F_0 = \sum_{c\in \mathrm{Cl}(S)} F_0(x_c)$, by the triviality of the class automorphism groups. (L3) Let Γ be a discrete subgroup of G_∞, and F_0 a continuous function on G_∞ with values in a finite-dimensional complex vector space, left invariant under Γ. If $F_0(u\,y) = c_0\,F_0(y)$ for all y, for some $u \in G_\infty$ and a complex number c_0 with $|c_0| = 1$, $c_0 \neq 1$, then the integral of F_0 over the quotient by Γ is 0. (L4) Let M, M' be matrices over a field with the same rows, J_0, J_0' sets of column indices of M, M', and v_0, v_0' coefficient vectors. If every linear form that vanishes on the columns of M in J_0 and on those of M' in J_0' takes the same value at Mv_0 and at $M'v_0'$, then $Mv = M'v'$ for some v, v' that agree with v_0, v_0' outside J_0, J_0'. (L5) If $a_1, a_2 \in \mathbb{A}_{E,f}$ satisfy $\psi_{E,f}(t\,a_1) = \psi_{E,f}(t\,a_2)$ for all $t \in E$, then $a_1 = a_2$ (the character is non-degenerate and E is dense in $\mathbb{A}_{E,f}$). (L6) A non-zero algebraic integer x of E has $\prod_\tau |x_\tau| \geq 1$; in particular an algebraic integer with $|x_\tau| < 1$ for all τ is 0. (L7) A finite commutative ring with q_1 elements has at most $q_1^{n_1^2}$ invertible matrices of size n_1. A complex square matrix of size n_1 with entries of absolute value at most r_1 has operator norm at most $n_1^2 r_1$, and its determinant is at most the n_1-th power of its operator norm. (L8) For $x, x' \in \mathcal{U}(T)$: $Z_T(x'^{-1}\gamma x) = M(x')^*\, Z_T(\gamma)\, M(x)$, where $M(x)$ is the block of x on the frame, upper triangular with diagonal entries 1, and equal to 1 for $x \in \mathcal{U}^0(T)$.

Inputs. *From the background library, continued.* (L9) The map $\alpha_1 \mapsto u(\alpha_1)$ that sends a complex matrix α_1 with rows $a < \varrho(\mathfrak{s})$ and columns in κ to the element of $\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))$ with $u(\alpha_1)e_a = e_a$, $u(\alpha_1)x_i = x_i + \sum_a (\alpha_1)_{a,i} e_a$ for the basis vectors x_i, $i \in \kappa$, and $u(\alpha_1)e_a^* = e_a^* - \sum_i \overline{(\alpha_1)_{a,i}}\, x_i - \frac{1}{2}\sum_{a'} (\alpha_1\alpha_1^*)_{a',a} e_{a'}$ is continuous; every element of $\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))$ is $u(\alpha_1)$ times an element that moves each e_a^* by a combination of the $e_{a'}$ and fixes the other basis vectors. (L10) An element x of $\mathcal{U}(T)$ that fixes $\varrho(\mathfrak{s})$ vectors whose matrix of e^*-coordinates is the identity is 1. (L11) For a linear map P_0 and submodules: if $D_2 \subseteq P_0^{-1}(K_0)$ and $K_1 \subseteq P_0(M_2) + K_0$, then $[K_1 : K_0]$ divides $[M_2 : D_2]$. (L12) (Products of mean-square bounds.) If for each τ a measurable kernel $k_\tau(q, y')$, integrable in y', satisfies $\int \sum_i |\int \phi(y') k_{\tau,i}(q,y')\, dy'|^2\, dm_\tau(q) \leq \int |\phi|^2 dn_\tau$ for all bounded measurable ϕ, with σ-finite measures, then the product kernels satisfy the same bound for the product measures.

4 Energies, and the passage to the cut functions

Lemma 4.1 (splitting)**.** *In Setting 2.2, let σ be a side and $\Xi, \Xi' \subseteq G_S(E)$. Then, for all $c, c' \in \mathrm{Cl}(S)$ and $s \in \mathsf{G}(S)$,*

$$A_S^\sigma(\Xi; c, s, c') = A_S^\sigma(\Xi \setminus \Xi'; c, s, c') + A_S^\sigma(\Xi \cap \Xi'; c, s, c').$$

Proof. By Input (B4) the amplitudes of the elements of $\Xi \cap \mathrm{Tm}_S(c, s, c')$ are absolutely summable, and a sum of an absolutely summable family splits over a partition of its index set. □

LEAN `Filling.amplitudeSum_split`

Lemma 4.2. *In Setting 2.1, let V be a normed additive group, $\mathcal{A}, \mathcal{A}' : \mathrm{Cl}(S) \times \mathsf{G}(S) \times \mathrm{Cl}(S) \to V$ two families, $\Omega \subseteq \mathsf{G}(S)$ and ε a real number with $\varepsilon > 0$. Then*

$$\mathrm{En}_S(\mathcal{A} + \mathcal{A}'; \Omega) \;\le\; (1+\varepsilon)\,\mathrm{En}_S(\mathcal{A}; \Omega) + (1+\varepsilon^{-1})\,\mathrm{En}_S(\mathcal{A}'; \Omega).$$

Proof. Term by term: for $v_1, v_2 \in V$, $\|v_1 + v_2\|^2 \le \|v_1\|^2 + 2\|v_1\|\|v_2\| + \|v_2\|^2$ and $2\|v_1\|\|v_2\| \le \varepsilon\|v_1\|^2 + \varepsilon^{-1}\|v_2\|^2$; divide by $\nu(S)^2$ and sum. □

LEAN `Filling.energy_add_le`

Proof of Theorem C. Fix c, s, c'. By Input (B4), for both sides, the amplitudes are absolutely summable and each amplitude is an absolutely convergent integral, so the block of the difference at (c, s, c') is

$$\sum_{\gamma \in \Xi \cap \mathrm{Tm}_S(c,s,c')} \int \beta_{S,c}(y)\,\beta_{S,c'}(\mathrm{r}_S^\infty(\gamma)^{-1} y)\,(\mathcal{W}_S^{\mathrm{full}}(\gamma; y) - \mathcal{W}_S^{\mathrm{cut}}(\gamma; y))\,dy,$$

and its norm is at most the number $\epsilon(c, s, c')$ obtained by putting the norm inside the integral. By the second statement of (B4), with $\sigma = \mathrm{full}$, $\sum_{c'} \epsilon(c, s, c') \le \frac{1}{2}\nu(S)\,b(S)^{-k(\mathfrak{s})^3 f(E)}$ for all c and s. Since $\sum_{c'} \epsilon(c, s, c')^2 \le (\sum_{c'} \epsilon(c, s, c'))^2$, summing over the classes c and over $s \in \Omega \subseteq \mathsf{G}(S)$ gives

$$\mathrm{En}_S(A_S^{\mathrm{full}}(\Xi; \cdot, \cdot, \cdot) - A_S^{\mathrm{cut}}(\Xi; \cdot, \cdot, \cdot); \Omega) \;\le\; \#\mathrm{Cl}(S) \cdot \#\mathsf{G}(S) \cdot \tfrac{1}{4}\,b(S)^{-2k(\mathfrak{s})^3 f(E)}.$$

The group $\mathsf{G}(S)$ is a product, over the $2f(E)$ chosen places, of general linear groups of size $r(\mathfrak{s})$ over fields with $q_0(S)$ or $q_1(S)$ elements, $f(E)$ of each kind (Setting 2.1); by (L7), $\#\mathsf{G}(S) \le (q_0(S)q_1(S))^{r(\mathfrak{s})^2 f(E)} = g(S)^{r(\mathfrak{s})^2 f(E)} \le b(S)^{23\,r(\mathfrak{s})^2 f(E)}$. Since $2\varrho(\mathfrak{s}) + 1 \le k(\mathfrak{s})$, we have $r(\mathfrak{s}) \le 2k(\mathfrak{s})$, hence $23\,r(\mathfrak{s})^2 f(E) + 30 \le 92\,k(\mathfrak{s})^2 f(E) + 30 \le 2\,k(\mathfrak{s})^3 f(E)$, the last because $k(\mathfrak{s}) \ge 61$. As $b(S) \ge 1$, $\#\mathsf{G}(S) \cdot b(S)^{-2k(\mathfrak{s})^3 f(E)} \le b(S)^{-30}$. Finally $\#\mathrm{Cl}(S) \le \overline{\#\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{\,S} \cdot \#\mathrm{Cl}(S) \cdot \#\mathrm{Bc}(S) = \mathsf{d}_1(S)$. □

LEAN `Filling.energy_sub_cut_le`

5 Class sizes and Schur weights

Lemma 5.1 (the open-cell mass fits under the class size)**.** *In Setting 2.1, let S be admissible, and let $\gamma \in G_S(E)$ be near, $\gamma \in \mathrm{Near}(S)$, with $\|\mathsf{g}\|_S(\gamma) \ge 1$. Put $x = \|\mathsf{g}\|_S(\gamma)$. Then*

$$b(S)^{-k(\mathfrak{s})^3 f(E)} \;\le\; \mathrm{cs}_S(\varrho(\mathfrak{s}) + 1; x) \qquad \text{and}$$

$$\prod_\tau \mathrm{cm}(\mathrm{r}_S^\infty(\gamma)_\tau; \deg_\tau(S), \mathrm{bx}(S), \#I_\tau(S)) \;\le\; \Big(\frac{\nu(S) \cdot \mathrm{cs}_S(\varrho(\mathfrak{s}) + 1; x)}{2}\Big)^2.$$

Proof. Write $\varkappa = \prod_\tau |\kappa_\tau(S)|$; by (A2), $\varkappa \geq 1$. At the level $\varrho(\mathfrak{s})+1$, Definition 2.4 and the formula for $\mathrm{cw}_0(S)$ give

$$\mathrm{cs}_S(\varrho(\mathfrak{s})+1;x)^2 = \frac{\mathrm{tl}(S)^4 \cdot \mathsf{d}(S) \cdot N(S)^{k(\mathfrak{s})f(E)}}{\#\overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{S} \cdot \#\mathrm{Cl}(S) \cdot g(S)^{\,2\varrho(\mathfrak{s})k(\mathfrak{s})f(E)} \cdot x^{k(\mathfrak{s})+1}}. \tag{1}$$

First inequality. By (A3), $\mathsf{d}(S) \geq \mathrm{tl}(S)^{44}\mathsf{d}_1(S) \geq \#\overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{S} \cdot \#\mathrm{Cl}(S)$, and $N(S) \geq 1$, $\mathrm{tl}(S) \geq 1$, $g(S) \leq b(S)^{23}$; so the right side of (1) is at least $b(S)^{-46\varrho(\mathfrak{s})k(\mathfrak{s})f(E)}x^{-(k(\mathfrak{s})+1)}$. By (S4) the number $\mathfrak{g}_S(\gamma)_\tau$ is real and non-negative; as it equals $|\mathrm{D}_{n(\mathfrak{s})}(Z_S(\gamma))_\tau|^2 - N(S)^2|\mathrm{D}_{\varrho(\mathfrak{s})}(Z_S(\gamma))_\tau|^2$, it is at most $|\mathrm{D}_{n(\mathfrak{s})}(Z_S(\gamma))_\tau|^2 \leq b(S)^{11n(\mathfrak{s})^3}$ by (S5). Hence $x \leq b(S)^{11n(\mathfrak{s})^3 f(E)}$ and

$$\mathrm{cs}_S(\varrho(\mathfrak{s})+1;x)^2 \;\geq\; b(S)^{-(46\varrho(\mathfrak{s})k(\mathfrak{s})+11n(\mathfrak{s})^3(k(\mathfrak{s})+1))f(E)} \;\geq\; b(S)^{-2k(\mathfrak{s})^3 f(E)},$$

because $46\varrho(\mathfrak{s}) \leq k(\mathfrak{s})^2$ and $11n(\mathfrak{s})^3(k(\mathfrak{s})+1) \leq 22n(\mathfrak{s})^3k(\mathfrak{s}) \leq k(\mathfrak{s})^2$, both by $k(\mathfrak{s}) \geq 1000(2n(\mathfrak{s}))^9$.

Second inequality. By (S4), and since $\prod_{a\leq\varrho(\mathfrak{s})} \mathrm{fs}_S(\tau,a) = |\kappa_\tau(S)|^{\varrho(\mathfrak{s})n(\mathfrak{s})}N(S)^{-n(\mathfrak{s})}$,

$$\prod_\tau \mathfrak{g}^{\mathbb{R}}(Z(\mathrm{r}_S^\infty(\gamma)_\tau)) = \varkappa^{2\varrho(\mathfrak{s})n(\mathfrak{s})}\, N(S)^{-2n(\mathfrak{s})f(E)}\, x.$$

With $\mathrm{bx}(S)^{\,9n(\mathfrak{s})\cdot 10^5 n(\mathfrak{s})^4} = b(S)^{18\cdot 10^5 n(\mathfrak{s})^7}$, $\prod_\tau \#I_\tau(S) = \#\mathrm{Bc}(S)$ and $\prod_\tau \deg_\tau(S) = \deg(S)$, the product of the cell masses is

$$L_0 = \frac{b(S)^{18\cdot 10^5 n(\mathfrak{s})^7 f(E)} \cdot 2^{(k(\mathfrak{s})-1)f(E)} \cdot (\#\mathrm{Bc}(S))^2}{\left(\varkappa^{2\varrho(\mathfrak{s})n(\mathfrak{s})}N(S)^{-2n(\mathfrak{s})f(E)}x\right)^{10^5 n(\mathfrak{s})^4+k(\mathfrak{s})-1} \cdot \deg(S) \cdot c_G(\mathfrak{s})^{f(E)}}.$$

On the other side, (A6) and the definition of $a_*(S)$ give

$$\frac{\nu(S)^2\,\mathsf{d}(S)}{\#\overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{S} \cdot \#\mathrm{Cl}(S)} \;\geq\; \frac{g(S)^{\,f(E)\varrho(\mathfrak{s})(2k(\mathfrak{s})+2\varrho(\mathfrak{s})+1)} \cdot c_*(S) \cdot \mathrm{vol}(\hat{O}_E)^{n(\mathfrak{s})} \cdot (\#\mathrm{Bc}(S))^2}{(1+b(S)^{-9}) \cdot \deg(S) \cdot c_G(\mathfrak{s})^{f(E)} \cdot \mathrm{fs}(S)^2},$$

so that, by (1), the right side $R_0 = (\nu(S)\ \mathrm{cs}_S(\varrho(\mathfrak{s})+1;x)/2)^2$ of the claim satisfies

$$R_0 \;\geq\; \frac{\mathrm{tl}(S)^4 \cdot g(S)^{\,f(E)\varrho(\mathfrak{s})(2\varrho(\mathfrak{s})+1)} \cdot N(S)^{k(\mathfrak{s})f(E)} c_*(S) \cdot \mathrm{vol}(\hat{O}_E)^{n(\mathfrak{s})} \cdot (\#\mathrm{Bc}(S))^2}{4\,(1+b(S)^{-9}) \cdot x^{k(\mathfrak{s})+1} \cdot \deg(S) \cdot c_G(\mathfrak{s})^{f(E)} \cdot \mathrm{fs}(S)^2}.$$

Since $\sum_{a\leq\varrho(\mathfrak{s})}(k(\mathfrak{s})+2(\varrho(\mathfrak{s})-a)) = n(\mathfrak{s})(k(\mathfrak{s})+\varrho(\mathfrak{s}))$ and $\sum_{a\leq\varrho(\mathfrak{s})} 2(\varrho(\mathfrak{s})-a)(k(\mathfrak{s})+2(\varrho(\mathfrak{s})-a)) = k(\mathfrak{s})\varrho(\mathfrak{s})n(\mathfrak{s}) + \frac{2}{3}\varrho(\mathfrak{s})n(\mathfrak{s})(2\varrho(\mathfrak{s})+1)$,

$$\mathrm{fs}(S) = \varkappa^{\,k(\mathfrak{s})\varrho(\mathfrak{s})n(\mathfrak{s})+\frac{2}{3}\varrho(\mathfrak{s})n(\mathfrak{s})(2\varrho(\mathfrak{s})+1)} \cdot N(S)^{-f(E)n(\mathfrak{s})(k(\mathfrak{s})+\varrho(\mathfrak{s}))}.$$

In the quotient L_0/R_0 the factors $(\#\mathrm{Bc}(S))^2$, $\deg(S)$ and $c_G(\mathfrak{s})^{f(E)}$ cancel, and the powers of $\varkappa$, x and $N(S)$ that remain are $\varkappa^{-\epsilon_1}$, $x^{-\epsilon_2}$ and $N(S)^{\epsilon_3 f(E)}$ with

$$\epsilon_1 = 2\varrho(\mathfrak{s})n(\mathfrak{s})(10^5 n(\mathfrak{s})^4 - 1) - \tfrac{4}{3}\varrho(\mathfrak{s})n(\mathfrak{s})(2\varrho(\mathfrak{s})+1) \geq 0,$$
$$\epsilon_2 = 10^5 n(\mathfrak{s})^4 - 2 \geq 0, \qquad \epsilon_3 = 2n(\mathfrak{s})(10^5 n(\mathfrak{s})^4 - 1 - \varrho(\mathfrak{s})) \leq 2\cdot 10^5 n(\mathfrak{s})^5.$$

Using $\varkappa \geq 1$, $x \geq 1$, $g(S) \geq 1$, $N(S) \leq b(S)^{60n(\mathfrak{s})}$ (A2), $\mathrm{tl}(S)^4 = b(S)^{k(\mathfrak{s})f(E)}$ (S1), $4(1+b(S)^{-9}) \leq 8$ and $N(S)^{k(\mathfrak{s})f(E)}c_*(S) = (\pi^{k(\mathfrak{s})}/(k(\mathfrak{s})-1)!)^{f(E)}|d_E|^{1/4}\,\mathrm{vol}(\hat{O}_E)^{k(\mathfrak{s})}$, we get

$$\frac{L_0}{R_0} \;\leq\; \frac{b(S)^{(18\cdot 10^5 n(\mathfrak{s})^7 + 12\cdot 10^6 n(\mathfrak{s})^6)f(E)} \cdot 2^{(k(\mathfrak{s})-1)f(E)+3}}{b(S)^{k(\mathfrak{s})f(E)} \cdot (\pi^{k(\mathfrak{s})}/(k(\mathfrak{s})-1)!)^{f(E)} \cdot |d_E|^{1/4} \cdot \mathrm{vol}(\hat{O}_E)^{k(\mathfrak{s})+n(\mathfrak{s})}}.$$

Now $|d_E|^{1/4}\operatorname{vol}(\hat{O}_E)^{k(\mathfrak{s})+n(\mathfrak{s})} = 2^{f(E)(k(\mathfrak{s})+n(\mathfrak{s}))}\operatorname{rd}_E^{-f(E)(k(\mathfrak{s})+n(\mathfrak{s})-1/2)}$ and $\pi^{k(\mathfrak{s})}/(k(\mathfrak{s})-1)! \geq 2^{k(\mathfrak{s})}k(\mathfrak{s})^{-k(\mathfrak{s})}$; the power of 2 that results is $2^{3-f(E)(k(\mathfrak{s})+n(\mathfrak{s})+1)} \leq 1$. Further $k(\mathfrak{s})^{k(\mathfrak{s})f(E)}\operatorname{rd}_E^{f(E)(k(\mathfrak{s})+n(\mathfrak{s})-1/2)} \leq (k(\mathfrak{s})\operatorname{rd}_E)^{f(E)(k(\mathfrak{s})+n(\mathfrak{s}))} \leq b(S)^{f(E)(k(\mathfrak{s})+n(\mathfrak{s}))/1000}$, by $\operatorname{rd}_E \geq 1$ and the last inequality of Setting 2.1. Altogether

$$\frac{L_0}{R_0} \leq b(S)^{f(E)\,(18\cdot 10^5 n(\mathfrak{s})^7 + 12\cdot 10^6 n(\mathfrak{s})^6 + (k(\mathfrak{s})+n(\mathfrak{s}))/1000 - k(\mathfrak{s}))} \leq 1,$$

since $18\cdot 10^5 n(\mathfrak{s})^7 + 12\cdot 10^6 n(\mathfrak{s})^6 \leq 3\cdot 10^6 n(\mathfrak{s})^7 \leq k(\mathfrak{s})/2$ and $(k(\mathfrak{s})+n(\mathfrak{s}))/1000 \leq k(\mathfrak{s})/2$, again by $k(\mathfrak{s}) \geq 1000(2n(\mathfrak{s}))^9$. □

LEAN `Filling.IsNear.le_classSize_and_cellMass_le`

Lemma 5.2. *In Setting 2.1, let $\gamma \in G_S(E)$ and $y \in G_\infty$ with $\mathcal{W}_S^{\mathrm{cut}}(\gamma; y) \neq 0$. Then, for every infinite place τ, $y_\tau \in \mathrm{Cut}(b(S))$ and $(\mathrm{r}_S^\infty(\gamma)^{-1}y)_\tau \in \mathrm{Cut}(b(S))$.*

Proof. The norm of the kernel is the product of the norms of the vectors $W_{S,\tau}^{\mathrm{cut}}(y_\tau)$ and $W_{S,\tau}^{\mathrm{cut}}((\mathrm{r}_S^\infty(\gamma)^{-1}y)_\tau)$; so all of them are non-zero, and (W3) applies. □

LEAN `Filling.inCut_of_besselKernel_ne_zero`

Lemma 5.3 (Cauchy–Schwarz with Schur weights)**.** *In Setting 2.2, for $c, c' \in \mathrm{Cl}(S)$ and $\gamma \in G_S(E)$,*

$$|a|_S^{\mathrm{cut}}(c, c'; \gamma)^2 \leq \mathrm{SI}_S(c, c'; \gamma)\cdot \mathrm{SI}_S(c', c; \gamma^{-1}).$$

Proof. The boxed cutoffs $\beta_{S,c}^0$ and the Schur weights $y \mapsto \mathrm{sw}_S(c; \gamma, y)$ are measurable: the cutoff is measurable (B1), the set $\mathrm{BxP}(S)$ is measurable, and Δ is continuous and positive (R1). In the second Schur integral substitute $y \mapsto \mathrm{r}_S^\infty(\gamma)^{-1}y$, which preserves the Haar measure; as $\mathrm{r}_S^\infty(\gamma^{-1})^{-1} = \mathrm{r}_S^\infty(\gamma)$, it becomes the integral of

$$F_2(y) = \beta_{S,c'}^0(\mathrm{r}_S^\infty(\gamma)^{-1}y)\prod_\tau \|W_{S,\tau}^{\mathrm{cut}}((\mathrm{r}_S^\infty(\gamma)^{-1}y)_\tau)\|^2 \cdot \beta_{S,c}^0(y)\prod_\tau\Big(\frac{\Delta((\mathrm{r}_S^\infty(\gamma)^{-1}y)_\tau)}{\Delta(y_\tau)}\Big)^{k(\mathfrak{s})+n(\mathfrak{s})},$$

while the first is the integral of

$$F_1(y) = \beta_{S,c}^0(y)\prod_\tau \|W_{S,\tau}^{\mathrm{cut}}(y_\tau)\|^2 \cdot \beta_{S,c'}^0(\mathrm{r}_S^\infty(\gamma)^{-1}y)\prod_\tau\Big(\frac{\Delta(y_\tau)}{\Delta((\mathrm{r}_S^\infty(\gamma)^{-1}y)_\tau)}\Big)^{k(\mathfrak{s})+n(\mathfrak{s})}.$$

Let $F_0(y) = \beta_{S,c}(y)\,\beta_{S,c'}(\mathrm{r}_S^\infty(\gamma)^{-1}y)\,\|\mathcal{W}_S^{\mathrm{cut}}(\gamma; y)\|$ be the integrand of the absolute amplitude. We claim $F_0^2 \leq F_1F_2$ pointwise; then (L1) gives the lemma. Let $F_0(y) \neq 0$. Then the kernel is non-zero, so by Lemma 5.2 all y_τ and all $(\mathrm{r}_S^\infty(\gamma)^{-1}y)_\tau$ are in the cut; and $\omega_{S,c}(y) \neq 0$, $\omega_{S,c'}(\mathrm{r}_S^\infty(\gamma)^{-1}y) \neq 0$, so by (B1) both points have all their components in $\mathrm{FBox}(\mathrm{bx}(S))$. Thus y and $\mathrm{r}_S^\infty(\gamma)^{-1}y$ are boxed points, the boxed cutoffs coincide there with the cutoffs, the frame volumes cancel, and $F_1(y)F_2(y) = F_0(y)^2$. □

LEAN `Filling.amplitudeAbs_sq_le` · `Filling.measurable_boxCutoff` · `Filling.measurable_schurWeight`

Proposition 5.4 (averaging over unipotent translates)**.** *In Setting 2.2, let $c \in \mathrm{Cl}(S)$; let μ be an s-finite measure on a measurable space Q, and $\phi\colon Q \to G_\infty$ a measurable map with $\phi(q)_\tau \in \mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s}; 1))$ for all $q \in Q$ and all τ; let $F\colon G_\infty \to [0,\infty]$ be measurable with $F(\mathrm{r}_S^\infty(u)\,y) = F(y)$ for all $u \in \mathcal{U}_{S,c}^0$ and $y \in G_\infty$; let $Q_0 \subseteq Q$ with $\mu(Q_0) \geq 1$; and let $C_0 \in [0,\infty]$ be such that*

$$\int_{Q_0} F(u\,\phi(q)\,y)\,d\mu(q) \leq C_0$$

for every $u \in G_\infty$ with all $u_\tau \in \mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))$ and every $y \in \mathrm{BxP}(S)$. Then

$$\int_{G_\infty} \beta^0_{S,c}(y)\cdot \prod_\tau \big\|W^{\mathrm{cut}}_{S,\tau}(y_\tau)\big\|^2 \cdot F(y)\,dy \;\le\; C_0\cdot \nu(S)\cdot \#\mathrm{Bc}(S).$$

Proof. Write $W(y)=\prod_\tau \|W^{\mathrm{cut}}_{S,\tau}(y_\tau)\|^2$ and $\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))^\infty$ for the group of the $u\in G_\infty$ with all $u_\tau\in\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))$, with its Haar measure η^0 of (B8). By (W1) the function W is invariant under $\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))^\infty$, even under all u with $u_\tau\in\mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s};1))$; so by (B8) and (W2)

$$\nu(S)\cdot\#\mathrm{Bc}(S) \;\ge\; \int \beta_{S,c}(y)\,W(y)\,dy = \int\!\!\int \omega_{S,c}(y)\,W(y)\,\theta_{S,c}(u\,y)\,d\eta^0(u)\,d\mu^0(y), \tag{2}$$

where we used the measure μ^0 of (B8), the factorization $\beta_{S,c}=\theta_{S,c}\omega_{S,c}$ and the invariance of $\omega_{S,c}$ under $\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))^\infty$ (B1). In the same way, because $u\,y$ is a boxed point exactly if y is (S3), the left side of the claim is

$$\int \mathbf{1}_{\mathrm{BxP}(S)}(y)\;\omega_{S,c}(y)\,W(y)\Big(\int \theta_{S,c}(u\,y)\,F(u\,y)\,d\eta^0(u)\Big)d\mu^0(y). \tag{3}$$

So it suffices to show, for every boxed point y,

$$\int \theta_{S,c}(u\,y)\,F(u\,y)\,d\eta^0(u) \;\le\; C_0\int \theta_{S,c}(u\,y)\,d\eta^0(u). \tag{4}$$

Fix y and $u_1\in\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))^\infty$. Since $\sum_{u'\in\mathcal{U}^0_{S,c}}\theta_{S,c}(\mathrm{r}^\infty_S(u')\,y')=1$ for all y' (S9), and $\mathrm{r}^\infty_S(u')\in\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))^\infty$ (S3),

$$\begin{aligned}
\int \theta_{S,c}(uy)\,F(u\,u_1y)\,d\eta^0(u) &= \sum_{u'}\int \theta_{S,c}(uy)\;\theta_{S,c}(\mathrm{r}^\infty_S(u')^{-1}u\,u_1y)\;F(\mathrm{r}^\infty_S(u')^{-1}u\,u_1y)\,d\eta^0(u)\\
&= \sum_{u'}\int \theta_{S,c}(\mathrm{r}^\infty_S(u')\,uy)\;\theta_{S,c}(u\,u_1y)\,F(u\,u_1y)\,d\eta^0(u)\\
&= \int \theta_{S,c}(u\,u_1y)\,F(u\,u_1y)\,d\eta^0(u) = \int \theta_{S,c}(uy)\,F(uy)\,d\eta^0(u).
\end{aligned}$$

In the first line we inserted the sum over u'^{-1} at the point $u\,u_1y$ and used the invariance of F under $\mathcal{U}^0_{S,c}$; in the second we substituted $u\mapsto \mathrm{r}^\infty_S(u')u$; in the third we summed over u' at the point uy and used the right invariance of η^0. Apply this with $u_1=\phi(q)$ and integrate over $q\in Q_0$: as $\mu(Q_0)\ge 1$, Tonelli's theorem and the hypothesis give

$$\int \theta_{S,c}(uy)\,F(uy)\,d\eta^0(u) \;\le\; \int \theta_{S,c}(uy)\Big(\int_{Q_0} F(u\,\phi(q)\,y)d\mu(q)\Big)d\eta^0(u) \;\le\; C_0\int \theta_{S,c}(uy)\,d\eta^0(u),$$

which is (4). Multiplying (4) by $\omega_{S,c}(y)W(y)$, integrating against μ^0 over the boxed points and comparing (3) with (2) proves the proposition. □

LEAN `Filling.lintegral_boxCutoff_le_of_average`

6 Thin classes: the proof of Theorem A

Lemma 6.1 (the terms of one class)**.** *In Setting 2.1, let $j \in \{0, \dots, \varrho(\mathfrak{s}) + 1\}$, $c, c' \in \mathrm{Cl}(S)$, $s \in \mathsf{G}(S)$, and $\gamma \in G_S(E)$ with $\gamma \in \mathrm{Tm}_S(c, s, c')$ and $\gamma \in \mathrm{Lv}_j(S)$. Let δ be an element of the double coset $\mathcal{U}^0_{S,c}\,\gamma\,\mathcal{U}^0_{S,c'}$. Then $\delta \in \mathrm{Lv}_j(S)$, $Z_S(\delta) = Z_S(\gamma)$, $\delta \in \mathrm{Tm}_S(c, s, c')$, and for every middle index the conjugate of the corresponding row of $(\mathrm{pos}_S(c, c'; \delta)\mathfrak{f})^{\mathrm{mid}} - (\mathrm{pos}_S(c, c'; \gamma)\mathfrak{f})^{\mathrm{mid}}$ lies in $g(S) \cdot \mathcal{R}(\mathrm{pos}_S(c, c'; \gamma))$.*

Proof. Write $\delta = u\gamma u'$ with $u \in \mathcal{U}^0_{S,c}$, $u' \in \mathcal{U}^0_{S,c'}$. The level is the same by (S10); the frame Gram matrix is the same by (L8), since u^{-1} and u' lie in $\mathcal{U}^0(T^{\mathrm{rat}}_S(E))$; and $\delta \in \mathrm{Tm}_S(c, 1 \cdot s \cdot 1, c')$ by (S6), since $u \in \mathrm{Tm}_S(c, 1, c)$ and $u' \in \mathrm{Tm}_S(c', 1, c')$. For the last statement put $\mathsf{k} = \mathrm{pos}_S(c, c'; \gamma)$ and $\mathsf{u} = \mathrm{pos}_S(c, c; u) \in K_g(S)$. Since u' fixes the frame and $h_S(c')$ lies in $\mathcal{H}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))$, the matrix $\mathrm{pos}_S(c, c'; \delta)\,\mathfrak{f}$ is $\mathsf{u}\,\mathsf{k}\,\mathfrak{f}$. The element u lies in $\mathcal{U}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))$, because $h_S(c) \in \mathcal{H}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))$ normalizes this group; so for every vector t the middle coordinates of $\mathsf{u}\,t$ are those of t plus $\sum_{a<\varrho(\mathfrak{s})} \mathsf{u}_{i,e^*_a}\, t_{e^*_a}$ at the middle index i. For $t = \mathsf{k}\mathfrak{f}_{a'}$ the coordinate $t_{e^*_a}$ is $h(e_a, \mathsf{k}\mathfrak{f}_{a'}) = Z_{T^{\mathrm{lat}}_S(\mathbb{A}_{E,f})}(\mathsf{k})_{a,a'}$. Hence the row i of $(\mathsf{u}\mathsf{k}\mathfrak{f})^{\mathrm{mid}} - (\mathsf{k}\mathfrak{f})^{\mathrm{mid}}$ is $\sum_{a<\varrho(\mathfrak{s})} \mathsf{u}_{i,e^*_a}$ times the row a of $Z_{T^{\mathrm{lat}}_S(\mathbb{A}_{E,f})}(\mathsf{k})$, and $\mathsf{u}_{i,e^*_a} \in g(S)\,\hat{O}_E$ by (S7). As $\hat{O}_E$ and $g(S)$ are stable under conjugation, the claim follows. □

LEAN `Filling.terms_of_class_spec`

Proof of Theorem A (a). Put $j_0 = \min(\varrho(\mathfrak{s}), j) = \#\mathrm{Ib}(j)$, $j_1 = j - j_0 \in \{0, 1\}$, $Z = Z_S(\gamma)$, $\mathsf{k} = \mathrm{pos}_S(c, c'; \gamma)$, and let ξ be the matrix of the rows of Z with index in $\mathrm{Ib}(j)$.

Step 1: averaging. Let $Q = \prod_\tau \mathrm{Mat}_{\kappa \times \mathrm{Ib}(j)}(\mathbb{C})$ with Lebesgue measure μ, and let Q_0 be the set of the $q \in Q$ all of whose entries have absolute value less than 1; then $\mu(Q_0) = \pi^{f(E)k(\mathfrak{s})j_0} \geq 1$. For each τ let R_τ be the diagonal matrix indexed by $\mathrm{Ib}(j)$ with the entries $\pi\, g(S)/(b(S) \cdot \mathrm{sc}_{S,\tau}(e_a))$, and for $q \in Q$ let $\phi(q)_\tau = u((q_\tau R_\tau)^*) \in \mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s}; 1))$ in the notation of (L9), the matrix $(q_\tau R_\tau)^*$ being completed by zero rows. The map ϕ is continuous. Let

$$F(y) = \sum_{\delta \in \mathcal{D}} \mathrm{sw}_S(c'; \delta, y).$$

By monotone convergence the left side of (a) is the integral of $\beta^0_{S,c}(y) \prod_\tau \|W^{\mathrm{cut}}_{S,\tau}(y_\tau)\|^2 F(y)$. The function F is measurable, and it is invariant under $\mathcal{U}^0_{S,c}$: for $u_0 \in \mathcal{U}^0_{S,c}$, $\Delta(\mathrm{r}^\infty_S(u_0)_\tau y_\tau) = \Delta(y_\tau)$ by (S3) and (R1), so $\mathrm{sw}_S(c'; \delta, \mathrm{r}^\infty_S(u_0)y) = \mathrm{sw}_S(c'; u_0^{-1}\delta, y)$, and $\delta \mapsto u_0^{-1}\delta$ permutes $\mathcal{D}$. By Proposition 5.4 it suffices to prove, for every $u \in G_\infty$ with all $u_\tau \in \mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s}; 1))$ and every $y \in \mathrm{BxP}(S)$,

$$\int_{Q_0} F(u\,\phi(q)\,y)\, d\mu(q) \;\leq\; \mathrm{tl}(S)^2 \cdot (b(S)^4 g(S))^{-j_0 k(\mathfrak{s}) f(E)} \cdot \Big(\frac{\mathrm{i}_{S,j}(\mathsf{k})}{\sqrt{\mathrm{mn}_S(j; \gamma)}}\Big)^{k(\mathfrak{s})+1}. \quad (5)$$

Fix such u and y.

Step 2: colouring. By Lemma 6.1, for $\delta \in \mathcal{D}$ the conjugates of the middle rows of $(\mathrm{pos}_S(c, c'; \delta)\mathfrak{f})^{\mathrm{mid}} - (\mathsf{k}\mathfrak{f})^{\mathrm{mid}}$ lie in $g(S)\,\mathcal{R}(\mathsf{k})$. We apply (B9) (ii) to the set $\mathcal{D}$, with $I_1 = M(\mathfrak{s})$, which has $k(\mathfrak{s}) + 1$ elements, $K_1 = g(S)\,\mathcal{R}(\mathsf{k})$, $K_0 = g(S)\,\mathcal{R}_j(\mathsf{k})$ and the weights $\int_{Q_0} \mathrm{sw}_S(c'; \delta, u\phi(q)y)\, d\mu(q)$. Here $[K_1 : K_0] = \mathrm{i}_{S,j}(\mathsf{k})$, which is not zero by (P1). So (5) follows if, for every non-empty $\mathcal{B} \subseteq \mathcal{D}$ such that for all $\delta, \delta' \in \mathcal{B}$

the conjugates of the middle rows of $(\mathrm{pos}_S(c,c';\delta)\mathfrak{f})^{\mathrm{mid}} - (\mathrm{pos}_S(c,c';\delta')\mathfrak{f})^{\mathrm{mid}}$ lie in $g(S)\,\mathcal{R}_j(\mathsf{k})$,

$$\sum_{\delta\in\mathcal{B}} \int_{Q_0} \mathrm{sw}_S(c';\delta,u\,\phi(q)\,y)\,d\mu(q) \;\le\; \mathrm{tl}(S)^2\cdot(b(S)^4 g(S))^{-j_0 k(\mathfrak{s}) f(E)}\cdot \mathrm{mn}_S(j;\gamma)^{-(k(\mathfrak{s})+1)/2}. \tag{6}$$

Fix such a set $\mathcal{B}$ and an element $\delta_0\in\mathcal{B}$.

Step 3: the terms of $\mathcal{B}$ are lattice translates. Let $\delta\in\mathcal{B}$. By Lemma 6.1, δ and δ_0 have level j and the frame Gram matrix Z, and $\mathcal{R}_j(\mathsf{k}) = \hat{O}_E\langle(\xi)^*_{\mathbb{A}_f}\rangle$ by (S6). So (R8), with $\Sigma=\mathrm{Ib}(j)$, gives a matrix ℓ_δ over E with rows in κ and columns in $\mathrm{Ib}(j)$ such that $\mathrm{C}^\circ_S(c)(\ell_\delta)_{\mathbb{A}_f}$ is integral and

$$\delta\mathfrak{f}-\delta_0\mathfrak{f} \text{ has the rows } g(S)\cdot\ell_\delta\,\xi \text{ on } \kappa \text{ and vanishes on the row of } z \text{ and on the rows } e^*_a. \tag{7}$$

Put $L_\delta = ((b(S)/\pi)\,(\ell_\delta)_\tau)_\tau \in Q$, the embeddings being applied to the entries.

Step 4: one kernel for all terms. For each τ let $Z^\tau = Z(\mathrm{r}^\infty_S(\gamma)_\tau)$. By (S4) its leading block Z^τ_j of size j has the determinant $\prod_{a<j}\mathrm{fs}_S(\tau,a)\cdot \mathrm{D}_j(Z)_\tau$, which is not zero because γ has level j. Since y is a boxed point, $\mathrm{bs}(y_\tau)\le b(S)^{6n(\mathfrak{s})+2}$. Let $\mathcal{K}_\tau$ be the kernel of (R2) for the point y_τ, the bound $\theta_0 = b(S)^{6n(\mathfrak{s})+2}$, the set $\Sigma=\mathrm{Ib}(j)\subseteq\{0,\dots,j-1\}$, the matrix $R_0=R_\tau$, the matrix Z^τ, and V_0 the matrix of the κ-coordinates of the vectors $u_\tau^{-1}\mathrm{r}^\infty_S(\delta_0)_\tau\mathfrak{f}_a$, $a<j$. We claim that for all $\delta\in\mathcal{B}$, $q\in Q$ and τ

$$\Big(\frac{\Delta((u\phi(q)y)_\tau)}{\Delta((\mathrm{r}^\infty_S(\delta)^{-1}u\phi(q)y)_\tau)}\Big)^{k(\mathfrak{s})+n(\mathfrak{s})} \;\le\; \mathcal{K}_\tau(q_\tau+(L_\delta)_\tau). \tag{8}$$

Indeed, let $u_1 = u_\tau\phi(q)_\tau\in\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))$ and $\gamma_1 = u_1^{-1}\mathrm{r}^\infty_S(\delta)_\tau$. By (R1) the left side of (8) is $(\Delta(y_\tau)/\Delta(\gamma_1^{-1}y_\tau))^{k(\mathfrak{s})+n(\mathfrak{s})}$. We check the three conditions of (R2) for γ_1 and $A = q_\tau+(L_\delta)_\tau$. First, $Z(\gamma_1) = Z(\mathrm{r}^\infty_S(\delta)_\tau) = Z^\tau$ by (L8) and (S4), since $Z_S(\delta)=Z$. Second, $\gamma_1\in\mathrm{NF}_j$ by (S12). Third, an element of $\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s};1))$ leaves the e^*-coordinates of a vector unchanged, because it is an isometry that fixes the vectors e_a, and it changes the κ-coordinates only by a combination of the e^*-coordinates; for $u(\alpha_1)^{-1}$ the change is α_1^* applied to the e^*-coordinates (L9). By (7) and the definition of r^∞_S, the vector $\mathrm{r}^\infty_S(\delta)_\tau\mathfrak{f}_a - \mathrm{r}^\infty_S(\delta_0)_\tau\mathfrak{f}_a$ has no e^*-coordinates and the κ-coordinates $g(S)\,((\ell_\delta\,\xi)_{\cdot,a})_\tau\,\mathrm{sc}_{S,\tau}(\mathfrak{f}_a)$; so the same holds after applying u_τ^{-1}. The $e^*_{a'}$-coordinate of $u_\tau^{-1}\mathrm{r}^\infty_S(\delta)_\tau\mathfrak{f}_a$ is $h(e_{a'},u_\tau^{-1}\mathrm{r}^\infty_S(\delta)_\tau\mathfrak{f}_a) = Z^\tau_{a',a}$, because u_τ fixes $e_{a'}$. Applying $\phi(q)_\tau^{-1} = u((q_\tau R_\tau)^*)^{-1}$ we find that the κ-coordinates of $\gamma_1\mathfrak{f}_a$, for $a<j$, are the column a of

$$V_0 + g(S)\,((\ell_\delta\,\xi)_{\cdot,a})_\tau\,\mathrm{sc}_{S,\tau}(\mathfrak{f}_a) + q_\tau R_\tau Z^\tau_\Sigma.$$

Finally $Z^\tau_{a',a} = \mathrm{sc}_{S,\tau}(e_{a'})\,\mathrm{sc}_{S,\tau}(\mathfrak{f}_a)\,(Z_{a',a})_\tau$ by (S4), so by the choice of R_τ the middle term is the column a of $(L_\delta)_\tau R_\tau Z^\tau_\Sigma$. This is the third condition, and (8) follows.

Step 5: translation. For $\delta\in\mathcal{B}$ and $q\in Q$ put

$$\Phi_\delta(q) = \mathbf{1}_{Q_0}(q-L_\delta)\cdot\mathrm{sw}_S(c';\delta,u\,\phi(q-L_\delta)\,y).$$

By the invariance of Lebesgue measure, $\int_{Q_0}\mathrm{sw}_S(c';\delta,u\phi(q)y)\,d\mu(q) = \int_Q\Phi_\delta\,d\mu$, and by (8) at the point $q-L_\delta$

$$\Phi_\delta(q) \;\le\; \mathbf{1}_{Q_0}(q-L_\delta)\cdot\beta^0_{S,c'}(\mathrm{r}^\infty_S(\delta)^{-1}u\,\phi(q-L_\delta)\,y)\cdot\prod_\tau\mathcal{K}_\tau(q_\tau). \tag{9}$$

Fix $q \in Q$. If $\Phi_\delta(q) \neq 0$ and $\Phi_{\delta'}(q) \neq 0$, then $q - L_\delta$ and $q - L_{\delta'}$ lie in Q_0, so every entry of $(\ell_\delta - \ell_{\delta'})_\tau$ has absolute value less than $2\pi/b(S) \le 8/b(S)$, and for every column the sum over κ of the squares of these absolute values is less than $64\,k(\mathfrak{s})/b(S)^2 \le 64/b(S) < 1$, by $k(\mathfrak{s}) \le b(S)$. As $\mathrm{C}^\circ_S(c)(\ell_\delta - \ell_{\delta'})_{\mathbb{A}_f}$ is integral, (R6) gives $\ell_\delta = \ell_{\delta'}$. So there is one matrix ℓ with $\ell_\delta = \ell$ for all δ with $\Phi_\delta(q) \neq 0$; let L be the corresponding point of Q and $y'' = u\,\phi(q - L)\,y$.

Step 6: the frame count. We apply (B3) to the index set $\mathcal{B}$, with the classes $c_\delta = c'$, the elements $\gamma_\delta = \delta_\delta = \delta$, the weights $\Phi_\delta(q)$, the points y'' and y'', and $C_1 = \prod_\tau \mathcal{K}_\tau(q_\tau)$. If $\Phi_\delta(q) \neq 0$, then $\delta \in \mathrm{Tm}_S(c, s, c')$ (Lemma 6.1), $\mathrm{r}^\infty_S(\delta)^{-1} y''$ is a boxed point because the boxed cutoff does not vanish there, and $\Phi_\delta(q) \le \beta_{S,c'}(\mathrm{r}^\infty_S(\delta)^{-1} y'') \cdot C_1$ by (9). For two such δ, δ', by (7) with $\ell_\delta = \ell_{\delta'}$, the matrices $\delta\mathfrak{f}$ and $\delta'\mathfrak{f}$ have the same middle rows and the same e^*-rows; so by (B12) the differences of the vectors $(\mathrm{r}^\infty_S(\delta)^{-1}_\tau y''_\tau)^{-1}\mathfrak{f}_a$ lie in the complex span of the $\varrho(\mathfrak{s})$ vectors $(y''_\tau)^{-1} e_{a'}$. These spans form a real subspace $\mathcal{V}$ of dimension at most $2\,\varrho(\mathfrak{s})\,n(\mathfrak{s})\,f(E)$. Hence, with (S1) and $12\,\varrho(\mathfrak{s})\,n(\mathfrak{s})^3 \le 10^5 n(\mathfrak{s})^9$,

$$\sum_{\delta \in \mathcal{B}} \Phi_\delta(q) \;\le\; b(S)^{12\varrho(\mathfrak{s})n(\mathfrak{s})^3 f(E)} \cdot \prod_\tau \mathcal{K}_\tau(q_\tau) \;\le\; \mathrm{tl}(S) \cdot \prod_\tau \mathcal{K}_\tau(q_\tau).$$

Integrating over q, the left side of (6) is at most $\mathrm{tl}(S) \cdot \prod_\tau \int \mathcal{K}_\tau$.

Step 7: the integral of the kernels. Let $P_\tau = \prod_{a<j_0} |\kappa_\tau(S)|^{2(\varrho(\mathfrak{s})-a)} \ge 1$ (A2). Then $|\det R_\tau|^2 = (\pi g(S)/b(S))^{2j_0}\, N(S)^{j_0}/P_\tau$ and $|\det Z^\tau_j| = P_\tau\, N(S)^{-j}\,|\mathrm{D}_j(Z)_\tau|$. With $\theta_0 = b(S)^{6n(\mathfrak{s})+2}$, the bound of (R2) is

$$\int \mathcal{K}_\tau \;\le\; \frac{(2\theta_0^2\, N(S))^{j(k(\mathfrak{s})+n(\mathfrak{s}))}}{P_\tau^{n(\mathfrak{s})}\,|\mathrm{D}_j(Z)_\tau|^{k(\mathfrak{s})+n(\mathfrak{s})}} \cdot \frac{b(S)^{2j_0 k(\mathfrak{s})}}{(j_0!)^{k(\mathfrak{s})}\,\theta_0^{2j_0k(\mathfrak{s})}\,\pi^{j_0k(\mathfrak{s})}\,g(S)^{2j_0k(\mathfrak{s})}\,N(S)^{j_0k(\mathfrak{s})}}.$$

The number $\mathrm{D}_j(Z)$ is a non-zero algebraic integer (S6); so $\Pi_0 = \prod_\tau |\mathrm{D}_j(Z)_\tau| \ge 1$ by (L6), and $\Pi_0^2 = |\mathrm{N}_{E/\mathbb{Q}}(\mathrm{D}_j(Z))| = \mathrm{mn}_S(j;\gamma)\,N(S)^{2j_1 f(E)}$. Hence $\Pi_0^{-(k(\mathfrak{s})+n(\mathfrak{s}))} \le \Pi_0^{-(k(\mathfrak{s})+1)} = \mathrm{mn}_S(j;\gamma)^{-(k(\mathfrak{s})+1)/2}\,N(S)^{-j_1 f(E)(k(\mathfrak{s})+1)}$. Using $P_\tau \ge 1$, $j_0! \ge 1$, $\pi \ge 2$ and $j = j_0 + j_1$, the product over τ is at most

$$\mathrm{mn}_S(j;\gamma)^{-(k(\mathfrak{s})+1)/2} \cdot (b(S)^4 g(S))^{-j_0 k(\mathfrak{s}) f(E)} \cdot \beta_1^{f(E)} \cdot \beta_2^{f(E)},$$
$$\beta_1 = (2\theta_0^2)^{j_1 k(\mathfrak{s})}\,(b(S)^6/g(S))^{j_0 k(\mathfrak{s})}, \qquad \beta_2 = (2\theta_0^2\,N(S))^{jn(\mathfrak{s})}.$$

The number β_1 is at most 1: for $j_1 = 0$ because $g(S) \ge b(S)^{19}$; for $j_1 = 1$, where $j_0 = \varrho(\mathfrak{s})$, because $2\theta_0^2\,(b(S)^6/g(S))^{\varrho(\mathfrak{s})} \le 2\,b(S)^{12n(\mathfrak{s})+4-13\varrho(\mathfrak{s})} = 2\,b(S)^{16-\varrho(\mathfrak{s})} \le 1$. The number β_2 is at most $(2\,b(S)^{12n(\mathfrak{s})+4}\,b(S)^{60n(\mathfrak{s})})^{n(\mathfrak{s})^2} \le b(S)^{(72n(\mathfrak{s})+5)n(\mathfrak{s})^2}$, by (A2), and $b(S)^{(72n(\mathfrak{s})+5)n(\mathfrak{s})^2 f(E)} \le \mathrm{tl}(S)$ by (S1). So $\prod_\tau \int \mathcal{K}_\tau \le \mathrm{tl}(S) \cdot (b(S)^4 g(S))^{-j_0 k(\mathfrak{s}) f(E)} \cdot \mathrm{mn}_S(j;\gamma)^{-(k(\mathfrak{s})+1)/2}$, and with Step 6 this is (6). □

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Proof of Theorem A (b). Put $\mathsf{k} = \mathrm{pos}_S(c, c'; \gamma)$, $j_0 = \min(\varrho(\mathfrak{s}), j)$ and $D_2 = \mathrm{mn}_S(j;\gamma)$, which is positive by (B11). By Lemma 5.3 and (L1) for the counting measure on $\mathcal{D}$,

$$\Big(\sum_{\delta \in \mathcal{D}} |a|^{\mathrm{cut}}_S(c, c'; \delta)\Big)^2 \;\le\; \sum_{\delta \in \mathcal{D}} \mathrm{SI}_S(c, c'; \delta) \cdot \sum_{\delta \in \mathcal{D}} \mathrm{SI}_S(c', c; \delta^{-1}).$$

The first factor is bounded by part (a). In the second, δ^{-1} runs through the double coset $\mathcal{U}^0_{S,c'}\,\gamma^{-1}\,\mathcal{U}^0_{S,c}$; here $\gamma^{-1} \in \mathrm{Tm}_S(c', s^{-1}, c)$ (S6), γ^{-1} has level j (B11),

$\mathrm{pos}_S(c', c; \gamma^{-1}) = \mathsf{k}^{-1}$, and $\mathrm{mn}_S(j; \gamma^{-1}) = D_2$ because $Z_S(\gamma^{-1}) = Z_S(\gamma)^*$. So part (a) applies to it as well, and

$$\Big(\sum_{\delta \in \mathcal{D}} |a|_S^{\mathrm{cut}}(c, c'; \delta)\Big)^2 \;\le\; \nu(S)^2\,(\#\mathrm{Bc}(S))^2\,\mathrm{tl}(S)^4\,(b(S)^4 g(S))^{-2 j_0 k(\mathfrak{s}) f(E)} \Big(\frac{\mathrm{i}_{S,j}(\mathsf{k})\,\mathrm{i}_{S,j}(\mathsf{k}^{-1})}{D_2}\Big)^{k(\mathfrak{s})+1}.$$

On the other side, by the hypothesis of (b) the class gauge is $N(S)^{f(E)} D_2 / (\mathrm{i}_{S,j}(\mathsf{k})\,\mathrm{i}_{S,j}(\mathsf{k}^{-1}))$, so that by Definition 2.4 and the formula for $\mathrm{cw}_0(S)$

$$\mathrm{cs}_S(j; \mathrm{cg}_S(j; c, c'; \gamma))^2 = \frac{\mathrm{tl}(S)^4\,\mathsf{D}_j}{\#\overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{S} \cdot \#\mathrm{Cl}(S) \cdot N(S)^{f(E)} \cdot g(S)^{2 j_0 k(\mathfrak{s}) f(E)} \cdot \Big(\frac{\mathrm{i}_{S,j}(\mathsf{k})\,\mathrm{i}_{S,j}(\mathsf{k}^{-1})}{D_2}\Big)^{k(\mathfrak{s})+1}}.$$

Comparing, it remains to see that $N(S)^{f(E)} \cdot \mathsf{d}_1(S) \cdot \#\mathrm{Bc}(S) \le \mathsf{D}_j \cdot b(S)^{8 j_0 k(\mathfrak{s}) f(E)}$. Now $N(S)^{f(E)} \le b(S)^{60 n(\mathfrak{s}) f(E)} \le \mathrm{tl}(S)$ by (A2) and (S1), $\#\mathrm{Bc}(S) \le b(S)^{2k(\mathfrak{s})f(E)} = \mathrm{tl}(S)^8$ by (A4), and $b(S)^{8 j_0 k(\mathfrak{s}) f(E)} = \mathrm{tl}(S)^{32 j_0}$. If $j \le \varrho(\mathfrak{s})$ then $j_0 = j$ and $\mathsf{D}_j\,\mathrm{tl}(S)^{32j} = \mathrm{tl}(S)^{13}\mathsf{d}_1(S) \ge \mathrm{tl}(S)^9 \mathsf{d}_1(S)$; if $j = \varrho(\mathfrak{s}) + 1$ then $\mathsf{D}_j = \mathsf{d}(S) \ge \mathrm{tl}(S)^{44}\mathsf{d}_1(S)$ by (A3). □

LEAN `Filling.tsum_amplitudeAbs_class_le`

7 Counting: discriminants, shared vectors, pairs

In this section a power x^{-l} of a real number $x \ge 0$, as a term of a sum in $[0, \infty]$, is read as 0 when $x = 0$, as in the formal text; this concerns the discriminants $\mathrm{disc}_{S,c}(U)$ and the indices.

Proposition 7.1 (inverse discriminants, summed over classes and subspaces)**.** *In Setting 2.2, for every $m \in \mathbb{N}$,*

$$\sum_{c \in \mathrm{Cl}(S)} \sum_{\substack{U \subseteq E^\kappa \\ \dim_E U + m = k(\mathfrak{s})}} \mathrm{disc}_{S,c}(U)^{-(k(\mathfrak{s})+1)} \;\le\; \#\mathrm{Cl}(S) \cdot N(S)^{-k(\mathfrak{s})f(E)} \cdot \mathrm{tl}(S)^{m+1},$$

the inner sum over the E-subspaces U of E^κ of codimension m.

Proof. Put $B_0 = \mathrm{tl}(S)^{m+1} N(S)^{-k(\mathfrak{s})f(E)}$ and, for a class c, let $\Phi(c) \in [0, \infty]$ be the inner sum. Let $K' = K_f(E, \kappa) \cap \mathrm{inc}_S^{-1}(K(S))$ and let $\mathfrak{A}$ be the set of the left cosets of K' in $K_f(E, \kappa)$. By (A5) the index $[K_f(E, \kappa) : K']$ is at least $(N(S)^{k(\mathfrak{s})}/b(S))^{f(E)} > 0$; as an infinite index is read as 0, the set $\mathfrak{A}$ is finite and $\#\mathfrak{A} \ge (N(S)^{k(\mathfrak{s})}/b(S))^{f(E)}$. Fix a representative $\tilde{a}$ of each $a \in \mathfrak{A}$, and write $[x] \in \mathrm{Cl}(S)$ for the class of $x \in G_f(E, \kappa)$.

Claim. For every $x \in G_f(E, \kappa)$: $\sum_{a \in \mathfrak{A}} \Phi([x\tilde{a}]) \le \#\mathfrak{A} \cdot B_0$.

Let $a \in \mathfrak{A}$. The representative of the class $[x\tilde{a}]$ is $\gamma_a\, x\, \tilde{a}\, \mathsf{k}_a$ with γ_a rational and $\mathsf{k}_a \in K^{\mathrm{col}}(S)$. Put $y_a = \tilde{a}\,\mathsf{k}_a$. Since $K^{\mathrm{col}}(S)$ is contained in $K_f(E, \kappa)$ and, by (A1) and $K_g(S) \subseteq K(S)$, in K', the element y_a lies in $K_f(E, \kappa)$ and represents a. Let r_a be the row k_0 of y_a^{-1}, an integral row vector. By (S8), $\mathrm{C}_S^\circ([x\tilde{a}]) = \Lambda^\circ(\mathfrak{s}; \bar{\pi}(S))\, y_a^{-1} x^{-1} \gamma_a^{-1}$.

Let U be a subspace of codimension m, and $M = (x_0)_{\mathbb{A}_f} w_0$ the basis matrix of its lattice for the class $[x\tilde{a}]$, with x_0 a basis matrix of U over E and w_0 invertible, so that $\mathrm{disc}_{S,[x\tilde{a}]}(U)^2 = D_1$ with $D_1 = [\det(x_0^* x_0)_{\mathbb{A}_f}]\,[\det w_0]^2$. If $D_1 = 0$ the term of U is 0. Otherwise, by (S8) the integrality of $\mathrm{C}_S^\circ([x\tilde{a}])M$ means that $y_a^{-1} x^{-1} \gamma_a^{-1} M$ is

integral and that $r_a\,x^{-1}\gamma_a^{-1}M$ has entries in $\bar\pi(S)\hat O_E$; as y_a is integral, also $x^{-1}\gamma_a^{-1}M$ is integral. We apply the second part of (S15) to $H=x^{-1}$, the subspace $U'=\gamma_a^{-1}U$, the matrix $\gamma_a^{-1}x_0$, which has the same Gram matrix $x_0^*x_0$ because γ_a is unitary, the matrix w_0, $Y_0=r_a$ and $K_0=\bar\pi(S)\hat O_E$. It gives

$$\begin{aligned}\operatorname{disc}_{S,[x\tilde a]}(U)^{-(k(\mathfrak s)+1)} \;&\le\; \big(\operatorname{disc}_{x^{-1}}(U')\cdot i_a(U')\big)^{-(k(\mathfrak s)+1)},\\ i_a(U')&=\big[\hat O_E\langle r_a\,x^{-1}M_{U'}\rangle:\bar\pi(S)\hat O_E\big]\ge 1,\end{aligned}$$

where $M_{U'}=\mathrm m_{x^{-1}}(U')$. Since $U\mapsto\gamma_a^{-1}U$ is injective and preserves the dimension, summing over U and then over a,

$$\sum_a\Phi([x\tilde a])\;\le\;\sum_{U'}\operatorname{disc}_{x^{-1}}(U')^{-(k(\mathfrak s)+1)}\cdot\sum_{a\in\mathfrak A}i_a(U')^{-(k(\mathfrak s)+1)},\tag{10}$$

the outer sum over the subspaces U' of codimension m. Fix U'. We apply the congruence count (R5) to the set $\mathfrak A$ with $d_0=\bar\pi(S)$, a set I_1 with one element, $I_2=\kappa$, I_3 the set of the $k(\mathfrak s)-m$ columns of $M_{U'}$, $m_1=m$, the integral matrices $Y_a=r_a$, the matrix $M_0=x^{-1}M_{U'}$, which is integral with an integral left inverse, $K_0=\bar\pi(S)\hat O_E$, $\eta_a=i_a(U')$ and $l_1=k(\mathfrak s)+1$. Here $C_1=1$ is admissible: if all entries of $r_a-r_{a'}$ lie in $\bar\pi(S)\hat O_E$, then so do those of $(r_a-r_{a'})\,y_{a'}$, which is the row k_0 of $y_a^{-1}y_{a'}-1$; by (S8) this means $y_a^{-1}y_{a'}\in K'$, that is $a=a'$. Hence $\sum_a i_a(U')^{-(k(\mathfrak s)+1)}\le[\bar\pi(S)]^{1+m}$, and $[\bar\pi(S)]=N(S)^{f(E)}$ because $[\bar\pi(S)]^2=[N(S)]=N(S)^{2f(E)}$. With the first part of (S15), (10) gives

$$\begin{aligned}\sum_a\Phi([x\tilde a])\;&\le\;N(S)^{f(E)(m+1)}\cdot\big(200\,f(E)\,k(\mathfrak s)\;\mathrm{rd}_E\big)^{f(E)(k(\mathfrak s)+1)m}\\&\le\;b(S)^{\,60n(\mathfrak s)f(E)(m+1)+f(E)(k(\mathfrak s)+1)m/1000},\end{aligned}$$

by (A2) and Setting 2.1. On the other hand

$$\#\mathfrak A\cdot B_0\;\ge\;b(S)^{-f(E)}\,\mathrm{tl}(S)^{m+1}=b(S)^{\,k(\mathfrak s)f(E)(m+1)/4-f(E)}\qquad\text{and}$$
$$60\,n(\mathfrak s)\,(m+1)+\frac{(k(\mathfrak s)+1)\,m}{1000}+1\;\le\;\frac{k(\mathfrak s)\,(m+1)}{4},$$

because $60n(\mathfrak s)+1\le k(\mathfrak s)/8$ and $(k(\mathfrak s)+1)/1000\le k(\mathfrak s)/8$. This proves the claim.

Averaging. Every class is $[x\tilde a]$ for suitable x and a, so the claim shows that Φ has finite values. Let $F_0(x)=B_0-\Phi([x])$, a real function on $G_f(E,\kappa)$ that is invariant under the rational points on the left and under $K^{\mathrm{col}}(S)$ on the right. By the claim, $\sum_a F_0(x\tilde a)\ge 0$ for all x. By the properties (L2) of the class integral,

$$0\;\le\;\int_{\Gamma_0\backslash}\sum_{a\in\mathfrak A}F_0(\,\cdot\,\tilde a)=\#\mathfrak A\cdot\int_{\Gamma_0\backslash}F_0=\#\mathfrak A\cdot\sum_{c\in\mathrm{Cl}(S)}(B_0-\Phi(c)),$$

which is the proposition. □

LEAN `Filling.sum_latticeDisc_inv_le`

Lemma 7.2 (shared vectors are short)**.** *In Setting 2.1, let S be admissible, $j\in\{0,\dots,\varrho(\mathfrak s)+1\}$, and $\gamma\in G_S(E)$ with $\gamma\in\mathrm{Near}(S)$ and $\gamma\in\mathrm{Lv}_j(S)$. Let $u\in\mathcal U^0(T_S^{\mathrm{rat}}(E))$, let $a'\le\varrho(\mathfrak s)$ with $j\le a'$, and $Y=\mathrm{pad}_j(Z_S(\gamma))$. Then for every infinite place τ and every $a<\varrho(\mathfrak s)$, the coordinate at e_a of the vector $t=(u\gamma)\mathfrak f\cdot\mathrm{adj}(Y)_{\cdot,a'}$, where $\mathrm{adj}(Y)_{\cdot,a'}=\det Y\cdot(Y^{-1})_{\cdot,a'}$ is the column a' of the adjugate, satisfies*

$$2\,|(t_{e_a})_\tau|\;<\;q_0(S)^{\,n(\mathfrak s)^4}.$$

Proof. By (B11) there is $v \in E^{n(\mathfrak{s})}$ with $\mathfrak{f} v = \gamma \mathfrak{f} (Y^{-1})_{\cdot,a'}$. As u fixes the frame, $t = \det Y \cdot u \, \mathfrak{f} v = \det Y \cdot \mathfrak{f} v$, and $t_{e_a} = \det Y \cdot v_a$. Fix τ.

Since γ is near, there is $y \in \mathrm{BxP}(S)$ with $\mathrm{r}_S^\infty(\gamma)^{-1} y \in \mathrm{BxP}(S)$. By (R4) there is $u_1 \in \mathcal{U}(T^{\mathrm{rat}}(\mathfrak{s}; 1))$ with $\|u_1 y_\tau\| \le b(S)^{6n(\mathfrak{s})+2}$. Let G' and G'' be the complex square matrices of size $n(\mathfrak{s})$ with the entries

$$G'_{l,a''} = \frac{\mathrm{sc}_{S,\tau}(\mathfrak{f}_l)}{\mathrm{sc}_{S,\tau}(\mathfrak{f}_{a''})} \, (u_1)_{\mathfrak{f}_l, \mathfrak{f}_{a''}}, \qquad G''_{l,a''} = \frac{\mathrm{sc}_{S,\tau}(\mathfrak{f}_l)}{\mathrm{sc}_{S,\tau}(\mathfrak{f}_{a''})} \, (u_1 \, \mathrm{r}_S^\infty(\gamma)_\tau)_{\mathfrak{f}_l, \mathfrak{f}_{a''}}.$$

The block of u_1 on the frame is upper triangular with diagonal entries 1 (L8), so $\det G' = 1$. All entries of G' and G'' have absolute value at most $b(S)^{5n(\mathfrak{s})^2}$: the entry of u_1 or $u_1 \mathrm{r}_S^\infty(\gamma)_\tau$ is a coordinate of $(u_1 y_\tau) \, y'^{-1} \mathfrak{f}_{a''}$ for $y' = y_\tau$ or $y' = (\mathrm{r}_S^\infty(\gamma)^{-1} y)_\tau$, hence at most $b(S)^{6n(\mathfrak{s})+2} \cdot \mathrm{bx}(S) = b(S)^{6n(\mathfrak{s})+2+2n(\mathfrak{s})^2}$ because both points are boxed, and the quotient of scales is a power of $\kappa_\tau(S)$ with exponent between $-\varrho(\mathfrak{s})$ and $\varrho(\mathfrak{s})$, at most $b(S)^{23\varrho(\mathfrak{s})}$ by (A2). Conjugating the relation $\mathfrak{f} v = \gamma \mathfrak{f} (Y^{-1})_{\cdot,a'}$ at τ by the scales, applying u_1 and reading the coordinates at the frame gives $G' \, v_\tau = G'' \, ((Y^{-1})_{\cdot,a'})_\tau$. Therefore the square matrix of size $2n(\mathfrak{s})$

$$\mathfrak{M} = \begin{pmatrix} G' & -G'' \\ 0 & Y_\tau \end{pmatrix}$$

maps the vector with the blocks v_τ and $((Y^{-1})_{\cdot,a'})_\tau$ to the vector with the blocks 0 and the unit vector of index a', and $\det \mathfrak{M} = \det Y_\tau$. By Cramer's rule, $(\det Y \cdot v_a)_\tau$ is the determinant of the matrix obtained from $\mathfrak{M}$ by replacing its column a with that vector. All entries of this matrix have absolute value at most $b(S)^{5n(\mathfrak{s})^2}$: those of Y_τ are 0, 1 or entries of $Z_S(\gamma)$ at τ, bounded by (S5). By (L7), with $2 \le b(S)$ and $n(\mathfrak{s}) \le b(S)$,

$$\begin{aligned} 2 \, |(\det Y \cdot v_a)_\tau| \;&\le\; 2 \left((2n(\mathfrak{s}))^2 \, b(S)^{5n(\mathfrak{s})^2} \right)^{2n(\mathfrak{s})} \;\le\; b(S)^{\,10n(\mathfrak{s})^3 + 8n(\mathfrak{s}) + 1} \\ &< \; b(S)^{10n(\mathfrak{s})^4} \;\le\; q_0(S)^{\,n(\mathfrak{s})^4}. \end{aligned}$$

□

LEAN `Filling.HasLevel.norm_shared_vector_lt`

Proposition 7.3 (a weighted count of labelled terms)**.** *In Setting 2.2, let* $c \in \mathrm{Cl}(S)$, $j \in \{0, \dots, \varrho(\mathfrak{s}) + 1\}$, $\mathtt{t} \in \overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{\,S}$ *and* $m \in \mathbb{N}$*; let* $\mathcal{P}$ *be a set of pairs* $p \in \mathrm{Cl}(S) \times G_S(E)$ *such that* $p \in \mathrm{Lab}_S(j; c, \mathtt{t})$ *for all* $p \in \mathcal{P}$*; let* $U \subseteq E^\kappa$ *be an* E*-subspace with* $k(\mathfrak{s}) \le \dim_E U + m$*; and let* $\eta \colon G_S^{\mathrm{lat}} \to \mathbb{N}$ *be a function such that* $\mathrm{o}_{S,c}(\mathrm{el}_S(c; p); \hat{g}(S), U)$ *divides* $\eta(\mathrm{el}_S(c; p))$ *for every* $p \in \mathcal{P}$*. Then*

$$\sum_{p \in \mathcal{P}} \mathrm{ipw}_S(c; \eta, p) \;\le\; b(S)^{\,2000 \, n(\mathfrak{s})^6 \, (m+1) \, f(E)}.$$

Proof. Write $m' = k(\mathfrak{s}) - \dim_E U \le m$. For $p = (c', \gamma) \in \mathcal{P}$ the element γ is near, has level j, and lies in $\mathrm{Tm}_S(c, s, c')$ for some s.

Step 1: few frame Gram matrices. By (S6) and (S5) the entries of $Z_S(\gamma)$, for $(c', \gamma) \in \mathcal{P}$, are algebraic integers of absolute value at most $b(S)^{5n(\mathfrak{s})^2}$ at every place; so by (B9) (iii) the map $(c', \gamma) \mapsto Z_S(\gamma)$ takes at most $N_1 = (1 + 2b(S)^{5n(\mathfrak{s})^2})^{2n(\mathfrak{s})^2 f(E)}$ values. By (B9) (i) it suffices to bound the sum over a non-empty subset $\mathcal{P}'$ on which $Z_S(\gamma)$ is a fixed matrix Z. Fix $\mathcal{P}'$ and Z.

Step 2: the modulus. Let $Y = \mathrm{pad}_j(Z)$; its entries are algebraic integers, and $\det Y = \mathrm{D}_j(Z) \neq 0$. Put $m_0 = q_0(S)^{\,n(\mathfrak{s})^4}$ and $d_0 = m_0 \cdot g(S) \cdot \det(YY^*)$, a non-zero algebraic integer of E with $\overline{d_0} = d_0$. By (S5), $|(d_0)_\tau| \leq b(S)^{11n(\mathfrak{s})^4} \cdot b(S)^{23} \cdot b(S)^{11n(\mathfrak{s})^3}$, so

$$[d_0] = \prod_\tau |(d_0)_\tau|^2 \;\leq\; b(S)^{(22n(\mathfrak{s})^4 + 22n(\mathfrak{s})^3 + 46) f(E)}. \tag{11}$$

Step 3: the congruence count. Let $\mathcal{R} = \hat{O}_E\langle (Z^{\mathrm{x}})^*_{\mathbb{A}_f}\rangle$; by (S6) it is $\mathcal{R}(\mathrm{el}_S(c;p))$ for every $p \in \mathcal{P}'$. We apply (R5) to $\mathcal{P}'$ with d_0; with $I_1 = \{0, \dots, \varrho(\mathfrak{s})\}$, $I_2 = M(\mathfrak{s})$, I_3 the set of the columns of the basis matrix M_U of the lattice of U and c, so that $m_1 = m' + 1$; with $Y_p = ((\mathrm{el}_S(c;p)\mathfrak{f})^{\mathrm{mid}})^*$, integral by (S6); $M_0 = T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})\, \mathrm{C}_S^\circ(c)\, M_U$, integral with an integral left inverse; $K_0 = g(S)\,\mathcal{R} + d_0 \hat{O}_E^{I_1}$; $\eta_p = \eta(\mathrm{el}_S(c;p))$; and $l_1 = k(\mathfrak{s})$. The divisibility holds: $\hat{O}_E\langle Y_p M_0\rangle$ is the lattice $\Lambda_U(\mathrm{el}_S(c;p))$, and its index relative to K_0 divides its index relative to the smaller module $g(S)\,\mathcal{R}$, which is the frame value index of the hypothesis. So

$$\sum_{p \in \mathcal{P}'} \mathrm{ipw}_S(c;\eta,p) \;\leq\; [d_0]^{\,n(\mathfrak{s})(n(\mathfrak{s})+m'+1)} \cdot C_2,$$

if every non-empty $\mathcal{B} \subseteq \mathcal{P}'$ such that, for $p, p' \in \mathcal{B}$, the conjugates of the middle rows of $(\mathrm{el}_S(c;p)\mathfrak{f})^{\mathrm{mid}} - (\mathrm{el}_S(c;p')\mathfrak{f})^{\mathrm{mid}}$ lie in K_0, has at most C_2 elements. We show this for $C_2 = [d_0]^{n(\mathfrak{s})\varrho(\mathfrak{s})}$. Fix such a $\mathcal{B}$ and an element $p_1 = (c_1, \gamma_1)$ of it.

Step 4: equal middle rows. Let $p = (c', \gamma) \in \mathcal{B}$. The module $d_0 \hat{O}_E^{I_1}$ consists of vectors v with $g(S)^{-1}(Y^{-1})^*_{\mathbb{A}_f} v$ integral, because $g(S)^{-1}(Y^{-1})^* d_0 = m_0 \det Y \cdot \mathrm{adj}(Y)^*$. So (R8) applies to the pair γ, γ_1, with Σ the set of all $a < \varrho(\mathfrak{s})$: there is a matrix ℓ_p over E with rows in κ and $\varrho(\mathfrak{s})$ columns such that $\mathrm{C}_S^\circ(c)(\ell_p)_{\mathbb{A}_f}$ is integral and $\gamma_1\mathfrak{f} - \gamma\mathfrak{f}$ has the rows $g(S)\,\ell_p\, Z^{\mathrm{x}}$ on κ and vanishes on the row of z and the rows e_a^*. Apply (B10) with $I_1 = \{0, \dots, \varrho(\mathfrak{s}) - 1\}$, p_1 the identity, $\Psi = 0$ and U_1 the matrix with the rows $-\ell_p$ on κ and zero rows elsewhere: its columns are orthogonal to the frame and $\mathrm{C}_S(c)(U_1)_{\mathbb{A}_f}$ is integral (B12). We get $u_p \in \mathcal{U}^0_{S,c}$ with $u_p\, t' = t' - g(S) \sum_{a < \varrho(\mathfrak{s})} (U_1)_{\cdot,a}\, h(e_a, t') +$ (a combination of the e_a) for all vectors t'. For $t' = \gamma\mathfrak{f}_{a'}$ we have $h(e_a, t') = Z_{a,a'}$; so $(u_p\gamma)\mathfrak{f}$ and $\gamma_1\mathfrak{f}$ have the same rows outside the rows e_a. Put $\gamma_p = u_p\gamma$; it lies over the same triple as γ (S6) and has the frame Gram matrix Z (L8).

Step 5: colouring by the first rows. By (S6) and (B12) the rows e_a of the integral matrix $\mathrm{pos}_S(c, c'; \gamma_p)\,\mathfrak{f}$ are those of $\gamma_p\mathfrak{f}$. We colour $\mathcal{B}$ by these $\varrho(\mathfrak{s})$ rows modulo $d_0\hat{O}_E^{I_1}$: by (B9) (ii) and (iv), $\#\mathcal{B} \leq [d_0]^{n(\mathfrak{s})\varrho(\mathfrak{s})}$, provided that two elements $p = (c', \gamma)$ and $p' = (c'', \gamma')$ of $\mathcal{B}$ are equal whenever the matrix Υ of the rows e_a of $\gamma_{p'}\mathfrak{f} - \gamma_p\mathfrak{f}$ has all its entries in $d_0\hat{O}_E$. Let p, p' be such, and $\Upsilon = d_0\, H$; the entries of H are algebraic integers.

Both $\gamma_p\mathfrak{f}$ and $\gamma_{p'}\mathfrak{f}$ have the Gram matrix of the frame, and they differ by Υ in the isotropic rows e_a only. Expanding, and using $h(e_a, \gamma_p\mathfrak{f}_{a'}) = Z_{a,a'}$, we get $\Upsilon^* Z^{\mathrm{x}} + (Z^{\mathrm{x}})^* \Upsilon = 0$, that is

$$H^*\, Z^{\mathrm{x}} = -(Z^{\mathrm{x}})^*\, H. \tag{12}$$

Let α be the matrix of the first $\varrho(\mathfrak{s})$ columns of $\mathrm{adj}(Y)$. For $a' \geq j$ the column a' of $H\,\mathrm{adj}(Y)$ vanishes: its entries, multiplied by d_0, are differences of two coordinates as in Lemma 7.2, applied to γ' with $u_{p'}$ and to γ with u_p, so their absolute values at every place are less than m_0; dividing by m_0 we see that $g(S) \det(YY^*)\, (H\,\mathrm{adj}(Y))_{a,a'}$

is an algebraic integer with all absolute values less than 1, hence zero by (L6). Consequently

$$H\alpha\, Z^{\mathrm{x}} = \det Y \cdot H \ \text{ if } j \le \varrho(\mathfrak{s}), \qquad Z^{\mathrm{x}}\,\alpha = \det Y \cdot 1 \ \text{ if } j = \varrho(\mathfrak{s})+1 : \tag{13}$$

in the first case, the columns of $H\,\mathrm{adj}(Y)$ of index $a \ge j$ vanish and the rows of index $a < j$ of Z^{x} are those of Y, so $H\alpha\, Z^{\mathrm{x}} = H\,\mathrm{adj}(Y)\,Y$; in the second, $Y = Z$. Put $\Theta = H\alpha$ and

$$\Psi = m_0 \cdot \big(\overline{\det Y}\cdot \Theta - \det Y \cdot \Theta^{*} - (Z^{\mathrm{x}}\alpha)^{*}\,\Theta\big).$$

Its entries are algebraic integers, and $\Psi + \Psi^{*} = -m_0\,\alpha^{*}\big((Z^{\mathrm{x}})^{*}H + H^{*}Z^{\mathrm{x}}\big)\alpha = 0$ by (12). Moreover $\Theta^{*}Z^{\mathrm{x}} = -(Z^{\mathrm{x}}\alpha)^{*}H$ by (12), and with (13) one finds in both cases $\Psi\, Z^{\mathrm{x}} = m_0 \det(YY^{*})\, H$, that is $g(S)\,\Psi\, Z^{\mathrm{x}} = \Upsilon$. Now apply (B10) with $U_1 = 0$ and this Ψ: the element $r = 1 + g(S)\,x_1 \Psi x_1^{*} J$ lies in $\mathcal{U}^{0}_{S,c}$, and it adds to a vector t' the combination $g(S)\sum_{a,a''} e_a \Psi_{a,a''} h(e_{a''}, t')$. For $t' = \gamma_p \mathfrak{f}_{a'}$ this is the column a' of $g(S)\,\Psi\, Z^{\mathrm{x}} = \Upsilon$, placed in the rows e_a. Hence $(r\,u_p\,\gamma)\mathfrak{f} = (u_{p'}\gamma')\mathfrak{f}$, and $p = p'$ by (R7).

Step 6: the exponents. Altogether the sum over $\mathcal{P}$ is at most $N_1 \cdot [d_0]^{n(\mathfrak{s})(n(\mathfrak{s})+m'+1)+n(\mathfrak{s})\varrho(\mathfrak{s})} = N_1 \cdot [d_0]^{n(\mathfrak{s})(2n(\mathfrak{s})+m')}$. Here $N_1 \le b(S)^{(5n(\mathfrak{s})^2+2)\,2n(\mathfrak{s})^2 f(E)} \le b(S)^{12 n(\mathfrak{s})^4 f(E)}$, and by (11), $m' \le m$ and $2n(\mathfrak{s}) + m \le 2n(\mathfrak{s})(m+1)$, the second factor is at most $b(S)^{23 n(\mathfrak{s})^4 \cdot 2n(\mathfrak{s})^2 (m+1) f(E)}$. The sum of the exponents is at most $2000\, n(\mathfrak{s})^6 (m+1) f(E)$. □

LEAN `Filling.tsum_invPowWeight_le_of_codim_le`

Lemma 7.4. *In Setting 2.1, let I_1 be a finite set and $M_1 \subseteq M_2$ two $\hat{O}_E$-submodules of $\mathbb{A}_{E,f}^{I_1}$ with $[M_2 : M_1] \ge 1$. Then $[M_2 : g(S)M_2] = [M_1 : g(S)M_1]$.*

Proof. Indices are multiplicative in the towers $g(S)M_1 \subseteq M_1 \subseteq M_2$ and $g(S)M_1 \subseteq g(S)M_2 \subseteq M_2$, and $[g(S)M_2 : g(S)M_1] = [M_2 : M_1]$ because multiplication by the idele $\hat{g}(S)$ is bijective. So $[M_2 : M_1]\cdot[M_1 : g(S)M_1] = [M_2 : g(S)M_2]\cdot[M_2 : M_1]$, and $[M_2 : M_1]$ can be cancelled. □

LEAN `Filling.submoduleIndex_gIdele_smul`

Proposition 7.5 (the pair weights are summable)**.** *In Setting 2.2, for every $j \in \{0, \dots, \varrho(\mathfrak{s})+1\}$,*

$$\sum_{c \in \mathrm{Cl}(S)} \sum_{c' \in \mathrm{Cl}(S)} \sum_{\gamma \in G_S(E)} \sum_{\gamma' \in G_S(E)} \mathrm{pw}_S(j; c, c'; \gamma, \gamma') \;\le\; 2\,\mathrm{tl}(S)^2.$$

Proof. Step 1: regrouping. Let $\mathrm{pw}_S(j; c, c'; \gamma, \gamma') \neq 0$. Then γ and γ' lie in $\mathrm{Tm}_S(c, s, c')$ for one s, are near representatives for (c, c') and have level j. Let $U = \mathrm{link}^{\circ}(\gamma, \gamma')$, $i = k(\mathfrak{s}) - \dim_E U$, and let $\mathfrak{t}$ be the component of s in $\overline{\mathcal{S}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))}^{S}$. Then $q = (c', \gamma)$ and $p = (c', \gamma')$ satisfy $q \in \mathrm{Lab}_S(j; c, \mathfrak{t})$ and $p \in \mathrm{Lab}_S(j; c, \mathfrak{t})$, and the pair weight is $\mathrm{w}_S(j, i; c; q, p; U)$. For a labelled $q = (c', \gamma)$ and a subspace U let $\mathcal{P}(q, U)$ be the set of the $p = (c'', \gamma')$ with $p \in \mathrm{Lab}_S(j; c, \mathfrak{t})$, $\gamma \sim_{S;c,c''} \gamma'$ and $\mathrm{link}^{\circ}(\gamma, \gamma') = U$. The left side of the proposition is then at most

$$\sum_{i \in \mathbb{N}} \sum_{\mathfrak{t} \in \overline{\mathcal{S}(T^{\mathrm{lat}}_S(\mathbb{A}_{E,f}))}^{S}} \sum_{c} \sum_{\dim_E U + i = k(\mathfrak{s})} \frac{\mathrm{disc}_{S,c}(U)^{-(k(\mathfrak{s})+1)}}{\mathrm{cw}_i(S)} \cdot \Sigma_1(i, \mathfrak{t}, c, U),$$

$$\Sigma_1(i, \mathfrak{t}, c, U) = \sum_{q} \Big(\mathrm{ipw}_S(c; \eta_{j,U}, q) \cdot \sum_{p \in \mathcal{P}(q,U)} \mathrm{ipw}_S(c; \mathfrak{d}_S(\cdot), p) \Big),$$

the sum over q running over the pairs with $q \in \mathrm{Lab}_S(j; c, \mathsf{t})$.

Step 2: the two inner sums. Fix i, t, c, U and $q = (c', \gamma)$. Let U' be the subspace of the $t_0 \in U$ that are orthogonal, for the standard form of E^κ, to the κ-parts of all vectors $\gamma \mathfrak{f}_a$, $a \le \varrho(\mathfrak{s})$. Then $\dim_E U' \ge \dim_E U - n(\mathfrak{s})$, so $k(\mathfrak{s}) \le \dim_E U' + i + \varrho(\mathfrak{s}) + 1$. We show in Step 3 that $\mathrm{o}_{S,c}(\mathrm{el}_S(c;p); \hat{g}(S), U')$ divides $\mathfrak{d}_S(\mathrm{el}_S(c;p))$ for all $p \in \mathcal{P}(q, U)$, and in Step 4 that $\mathrm{o}_{S,c}(\mathsf{k}; \hat{g}(S), U)$ divides $\eta_{j,U}(\mathsf{k})$ for $\mathsf{k} = \mathrm{el}_S(c; q)$. Proposition 7.3, applied to $\mathcal{P}(q, U)$ with U' and $m = i + \varrho(\mathfrak{s}) + 1$, and to the set of all labelled q with U and $m = i$, then gives

$$\Sigma_1(i, \mathsf{t}, c, U) \;\le\; b(S)^{\,2000\, n(\mathfrak{s})^6 (2i + \varrho(\mathfrak{s}) + 3) f(E)} \;\le\; \mathrm{tl}(S)^{\,i+1},$$

the last by (S1), since $2i + \varrho(\mathfrak{s}) + 3 \le (i+1)(\varrho(\mathfrak{s}) + 3)$ and $2000\, n(\mathfrak{s})^6 (\varrho(\mathfrak{s}) + 3) \le 10^5 n(\mathfrak{s})^9$. By Proposition 7.1, the sum over t, c and U of $\mathrm{disc}_{S,c}(U)^{-(k(\mathfrak{s})+1)}$ is at most $\#\overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^{\,S} \cdot \#\mathrm{Cl}(S) \cdot N(S)^{-k(\mathfrak{s}) f(E)} \cdot \mathrm{tl}(S)^{i+1} = \mathrm{cw}_i(S)\,(2\mathrm{tl}(S)^2)^{-i}\,\mathrm{tl}(S)^{i+1}$. So the terms with a given i contribute at most $(2\mathrm{tl}(S)^2)^{-i}\,\mathrm{tl}(S)^{2i+2} = 2^{-i}\mathrm{tl}(S)^2$, and the sum over i is at most $2\,\mathrm{tl}(S)^2$.

Step 3: the saturation index. Let $p = (c'', \gamma') \in \mathcal{P}(q, U)$; so $\gamma, \gamma' \in \mathrm{Tm}_S(c, s, c'')$ for some s, and $U = \mathrm{link}^\circ(\gamma, \gamma')$. Put $\mathsf{k} = \mathrm{pos}_S(c, c''; \gamma)$ and $\mathsf{k}' = \mathrm{pos}_S(c, c''; \gamma') = \mathrm{el}_S(c; p)$; then $\mathsf{k}'\mathsf{k}^{-1} \in K_g(S)$. Let $M_{U'} = (p_0)_{\mathbb{A}_f}\, a_0$ be the basis matrix of the lattice of U' and c, with p_0 a basis matrix of U' over E, and let P_0 be the matrix over E with the rows p_0 on κ and zero rows elsewhere.

(i) Every column of $\gamma'\gamma^{-1}P_0 - P_0$ is a combination of the vectors e_a and $\gamma' e_a$, $a < \varrho(\mathfrak{s})$. Indeed, let t be a column of P_0, with κ-part $t_0 \in U'$. As $t_0 \in \mathrm{link}^\circ(\gamma, \gamma')$, there is $t_2 \in E^\kappa$ such that the pair $((t_0, 0), (t_2, 0))$, the zeros being the z-coordinates, lies in $\mathrm{link}(\gamma) \cap \mathrm{link}(\gamma')$. By (S11) for γ', t is the sum of $\gamma'\tilde{t}_2$, where $\tilde{t}_2$ has the κ-part t_2 and no other coordinates, and of a combination of the e_a and the $\gamma' e_a$. On the other hand t is orthogonal to all $\gamma\mathfrak{f}_a$, by the definition of U'; so $\gamma^{-1}t$ is orthogonal to the frame, has no e^*-coordinates and no z-coordinate; let t_3 be its κ-part. By (S11) for γ the pair $((t_0, 0), (t_3, 0))$ lies in $\mathrm{link}(\gamma)$, hence so does $((0,0), (t_3 - t_2, 0))$; as the link is Lagrangian, $(t_3 - t_2)^*(t_3 - t_2) = 0$, and $t_3 = t_2$ by the last statement of (S11). So $\gamma^{-1}t$ is $\tilde{t}_2$ plus a combination of the e_a, and $\gamma'\gamma^{-1}t - t$ is a combination of the e_a and the $\gamma' e_a$.

(ii) We apply (L11) to the map $P_1 \colon t' \mapsto g(S) \cdot (h(\mathsf{k}'\mathfrak{f}_a, t'))_{a \le \varrho(\mathfrak{s})}$ on $\mathbb{A}_{E,f}^{I(\mathfrak{s})}$, with $h = h_{T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})}$, the modules $M_2 = \hat{O}_E^{I(\mathfrak{s})} \cap \mathrm{span}_{\mathbb{A}_{E,f}}(e_a, \mathsf{k}' e_a)$, $D_2 = \hat{O}_E\langle e_a, \mathsf{k}' e_a\rangle$, $K_0 = g(S)\,\mathcal{R}(\mathsf{k}')$ and $K_1 = \Lambda_{U'}(\mathsf{k}')$. First, $P_1(e_a)$ is $g(S)$ times the conjugate of the row a of $Z_{T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})}(\mathsf{k}')$, and $P_1(\mathsf{k}' e_a) = g(S)\,(h(\mathfrak{f}_{a'}, e_a))_{a'} = 0$; so $D_2 \subseteq P_1^{-1}(K_0)$. Second, a general element of $\Lambda_{U'}(\mathsf{k}')$ is $(h(\mathsf{k}'\mathfrak{f}_a, \tilde{t}))_a$ with $\tilde{t} = \mathrm{C}_S(c)\,(P_0)_{\mathbb{A}_f}\, a_0\, r_0$ for an integral vector r_0; the vector $\tilde{t}$ is integral. Put $\mu_0 = -g(S)^{-1}(\mathsf{k}'\mathsf{k}^{-1} - 1)\,\tilde{t}$. It is integral by (S7). It lies in the span of the e_a and $\mathsf{k}' e_a$, by (i) and (B12), because $(\mathsf{k}'\mathsf{k}^{-1} - 1)\,\mathrm{C}_S(c)\,(P_0)_{\mathbb{A}_f} = \mathrm{C}_S(c)\,(\gamma'\gamma^{-1}P_0 - P_0)_{\mathbb{A}_f}$. And, $\mathsf{k}'\mathsf{k}^{-1}$ being an isometry,

$$P_1(\mu_0) = \big(h(\mathsf{k}'\mathfrak{f}_a, \tilde{t})\big)_a - \big(h(\mathsf{k}\mathfrak{f}_a, \tilde{t})\big)_a = \big(h(\mathsf{k}'\mathfrak{f}_a, \tilde{t})\big)_a,$$

since $h(\mathsf{k}\mathfrak{f}_a, \tilde{t})$ is the image in $\mathbb{A}_{E,f}$ of $(\gamma\mathfrak{f}_a)^* J(T_S^{\mathrm{rat}}(E)) P_0$, which is zero, times $a_0 r_0$ (B12). So $K_1 \subseteq P_1(M_2)$, and (L11) shows that $[K_1 : K_0] = \mathrm{o}_{S,c}(\mathsf{k}'; \hat{g}(S), U')$ divides $[M_2 : D_2] = \mathfrak{d}_S(\mathsf{k}')$.

Step 4: the index for q. Let $\mathsf{k} = \mathrm{el}_S(c;q)$, $\Lambda = \Lambda_U(\mathsf{k})$, $\mathcal{R} = \mathcal{R}(\mathsf{k})$ and $\mathcal{R}_j = \mathcal{R}_j(\mathsf{k})$. Then $[\Lambda : g(S)\mathcal{R}] = [\Lambda : \mathcal{R}] \cdot [\Lambda \cap \mathcal{R} : g(S)\mathcal{R}]$, the second factor divides $[\mathcal{R} : g(S)\mathcal{R}]$, and $[\mathcal{R} : g(S)\mathcal{R}] = [\mathcal{R}_j : g(S)\mathcal{R}_j]$ by Lemma 7.4, whose hypothesis is $\mathrm{i}_{S,j}(\mathsf{k}) \geq 1$ (P1). Finally $\mathcal{R}_j$ is the image of $\hat{O}_E^{\mathrm{Ib}(j)}$ under a linear map, so $[\mathcal{R}_j : g(S)\mathcal{R}_j]$ divides $[\hat{O}_E^{\mathrm{Ib}(j)} : g(S)\hat{O}_E^{\mathrm{Ib}(j)}] = [\hat{g}(S)]^{\#\mathrm{Ib}(j)}$ (B9). Hence $\mathrm{o}_{S,c}(\mathsf{k};\hat{g}(S),U)$ divides $[\hat{g}(S)]^{\#\mathrm{Ib}(j)} \cdot \mathrm{o}_{S,c}(\mathsf{k};1,U) = \eta_{j,U}(\mathsf{k})$. □

LEAN `Filling.sum_pairWeight_le`

8 Block bounds: the proof of Theorem B

Proof of Theorem B (a). Write $\omega = \omega_{S,c'}$ for the box partition of the class c' (B1).

Step 1: a choice of representatives. Call $\gamma_1, \gamma_2 \in G_S(E)$ equivalent if $\gamma_2 = u\gamma_1 u'$ with $u \in \mathcal{U}_{S,c}$ and $u' \in \mathcal{U}^0_{S,c'}$. In every equivalence class choose a representative, and choose it with $\cdot \in \mathrm{NRep}_S(c,c')$ whenever the class contains such an element. Let $\varphi(\gamma, y) = \omega(\mathrm{r}_S^\infty(\gamma)^{-1}y)$ if γ is the chosen representative of its class, and $\varphi(\gamma,y) = 0$ otherwise.

Step 2: the weight is normalized. Let $\gamma \in \Xi \cap \mathrm{Tm}_S(c,s,c')$ and $y \in G_\infty$. We show

$$\sum_{(u,u') \in \mathcal{U}_{S,c} \times \mathcal{U}_{S,c'}} \varphi(u\gamma u'^{-1}, \mathrm{r}_S^\infty(u)y) = 1.$$

The term of (u,u') is $\omega(\mathrm{r}_S^\infty(u')\,\mathrm{r}_S^\infty(\gamma)^{-1}y)$ if $u\gamma u'^{-1}$ is a chosen representative, and 0 otherwise. So the sum is $\sum_{u'} \alpha(u')\,\omega(\mathrm{r}_S^\infty(u')\,\mathrm{r}_S^\infty(\gamma)^{-1}y)$, where $\alpha(u')$ is the number of the $u \in \mathcal{U}_{S,c}$ for which $u\gamma u'^{-1}$ is a chosen representative, and by (B1) it suffices to see that $\sum_{u_0 \in \mathcal{U}^0_{S,c'}} \alpha(u_0 u') = 1$ for every $u' \in \mathcal{U}_{S,c'}$. Put $\gamma_0 = \gamma u'^{-1}$; it lies in Ξ because Ξ is stable, so $\mathfrak{g}_S(\gamma_0) \neq 0$. We have to show that there is exactly one pair $(u,u_0) \in \mathcal{U}_{S,c} \times \mathcal{U}^0_{S,c'}$ such that $u\gamma_0 u_0^{-1}$ is the chosen representative of the class of γ_0. There is one, by the definition of the equivalence. If $u\gamma_0 u_0^{-1}$ and $\tilde{u}\gamma_0\tilde{u}_0^{-1}$ are two and equal, then the element $x = \tilde{u}^{-1}u$ of $\mathcal{U}(T_S^{\mathrm{rat}}(E))$ satisfies $x\gamma_0 = \gamma_0\,\tilde{u}_0^{-1}u_0$, so x fixes the vectors $\gamma_0 \mathfrak{f}_a$. By (S12), γ_0 has level $\varrho(\mathfrak{s})+1$, so $Z_S(\gamma_0)$ is invertible, and the $\varrho(\mathfrak{s})$ vectors $\gamma_0\mathfrak{f} \cdot (Z_S(\gamma_0)^{-1})_{\cdot,a}$, $a < \varrho(\mathfrak{s})$, are fixed by x and have the identity matrix as e^*-coordinates. By (L10), $x = 1$; hence $u = \tilde{u}$ and $u_0 = \tilde{u}_0$.

Step 3: change of weight. By Step 2 and (B5), for both sides the integrals $\int \varphi(\gamma,y)\,\mathcal{W}_S^\sigma(\gamma;y)\,dy$ converge, and for $\sigma = \mathrm{cut}$ their sum over $\gamma \in \Xi \cap \mathrm{Tm}_S(c,s,c')$ is $A_S^{\mathrm{cut}}(\Xi;c,s,c')$. So it suffices to show, for every such γ,

$$\Big\| \int \varphi(\gamma,y)\,\mathcal{W}_S^{\mathrm{cut}}(\gamma;y)\,dy \Big\| \;\leq\; \begin{cases} \nu(S) \cdot \mathrm{cs}_S(\varrho(\mathfrak{s})+1; \|\mathfrak{g}\|_S(\gamma)) & \text{if } \gamma \in \mathrm{NRep}_S(c,c'), \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Step 4: representatives that are not near. Suppose the integral in (14) is not zero. Then γ is the chosen representative of its class and there is y with $\omega(\mathrm{r}_S^\infty(\gamma)^{-1}y) \neq 0$ and $\mathcal{W}_S^{\mathrm{cut}}(\gamma;y) \neq 0$. By Lemma 5.2 all y_τ and all $(\mathrm{r}_S^\infty(\gamma)^{-1}y)_\tau$ are in the cut; so $\mathrm{r}_S^\infty(\gamma)^{-1}y \in \mathrm{BxP}(S)$ by (B1), and by (B2) there is $u \in \mathcal{U}_{S,c}$ with $\mathrm{r}_S^\infty(u)y \in \mathrm{BxP}(S)$. Hence $u\gamma$ is near. Its double coset $\mathcal{U}^0_{S,c}\,u\gamma\,\mathcal{U}^0_{S,c'}$ has a fixed representative, which is near by (S3) and therefore a near representative for (c,c'); it is equivalent to γ. By the choice in Step 1, γ itself is such a representative: $\gamma \in \mathrm{NRep}_S(c,c')$. This proves the second case of (14).

Step 5: the open cell. Let now γ be the chosen representative of its class with $\gamma \in \mathrm{NRep}_S(c, c')$, and $x = \|\mathbf{g}\|_S(\gamma)$. The element $\mathbf{g}_S(\gamma)$ is a non-zero algebraic integer (S6), so $x \geq 1$ by (L6), and Lemma 5.1 applies. Let $y^0 \in \mathrm{BxP}(S)$ with $\mathrm{r}_S^\infty(\gamma)^{-1} y^0 \in \mathrm{BxP}(S)$. For each τ we have $\mathbf{g}^{\mathbb{R}}(Z(\mathrm{r}_S^\infty(\gamma)_\tau)) > 0$ by (S4), and $\mathrm{bx}(S) \geq b(S)$; so (R3) applies to $\mathrm{r}_S^\infty(\gamma)_\tau$, the point y^0_τ and $\rho = \mathrm{bx}(S)$, and gives a compact set $\mathcal{C}_\tau$, a finite measure n_τ, a probability space (Q_τ, m_τ) and a map t_τ. By (L12) the mean-square bounds multiply. Applied to the bounded function $\omega \leq 1$, this gives a point $q = (q_\tau)$, outside any given null set, with

$$\Big\| \int_{\prod_\tau \mathcal{C}_\tau} \omega(y)\, \mathcal{W}_S^{\mathrm{full}}(\gamma; \mathrm{r}_S^\infty(\gamma)\, t\, y)\, dy \Big\|^2 \;\leq\; \prod_\tau n_\tau(G_1(\mathfrak{s})) \;\leq\; \prod_\tau \mathrm{cm}(\mathrm{r}_S^\infty(\gamma)_\tau; \deg_\tau(S), \mathrm{bx}(S), \#I_\tau(S)), \tag{15}$$

where $t = (t_\tau(q_\tau))_\tau \in G_\infty$; here we used that the kernel $\mathcal{W}_S^{\mathrm{full}}(\gamma; \mathrm{r}_S^\infty(\gamma) t y)$ is the tensor product over τ of $W_{S,\tau}^{\mathrm{full}}(\mathrm{r}_S^\infty(\gamma)_\tau t_\tau y_\tau)$ and the conjugate of $W_{S,\tau}^{\mathrm{full}}(t_\tau y_\tau)$, and that a probability measure gives a point at which a non-negative function is at most its integral. We choose q such that moreover every t_τ lies in the subgroup generated by $\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s}; 1))$ and $\mathcal{H}(T^{\mathrm{rat}}(\mathfrak{s}; 1))$, and such that every $y' \in \mathrm{FBox}(\mathrm{bx}(S))$ with $\mathrm{bs}(\mathrm{r}_S^\infty(\gamma)_\tau t_\tau y') \leq \mathrm{bx}(S)$ lies in $\mathcal{C}_\tau$. By Lemma 5.1 the right side of (15) is at most $(\nu(S)\, \mathrm{cs}_S(\varrho(\mathfrak{s}) + 1; x)/2)^2$.

Step 6: from the cut to the full kernel. Substituting $y \mapsto \mathrm{r}_S^\infty(\gamma)\, t\, y$ and using the invariance of ω under t (B1),

$$\int \varphi(\gamma, y)\, \mathcal{W}_S^{\mathrm{cut}}(\gamma; y)\, dy = \int \omega(y)\, \mathcal{W}_S^{\mathrm{cut}}(\gamma; \mathrm{r}_S^\infty(\gamma)\, t\, y)\, dy.$$

The integrand vanishes outside $\prod_\tau \mathcal{C}_\tau$: if it is not zero at y, then by Lemma 5.2 $\mathrm{r}_S^\infty(\gamma)_\tau t_\tau y_\tau$ and $t_\tau y_\tau$ are in the cut; as $\omega(ty) = \omega(y) \neq 0$, the point $t_\tau y_\tau$ lies in $\mathrm{FBox}(\mathrm{bx}(S))$ (B1), hence so does y_τ, because t_τ fixes the frame vectors; and $\mathrm{bs}(\mathrm{r}_S^\infty(\gamma)_\tau t_\tau y_\tau) \leq b(S)^{6n(\mathfrak{s})+2} \leq \mathrm{bx}(S)$; so $y_\tau \in \mathcal{C}_\tau$ by the choice of q. On $\prod_\tau \mathcal{C}_\tau$ we write the cut kernel as the full kernel minus the difference. The integral of the full kernel is bounded by (15), in norm by $\frac{1}{2}\nu(S)\, \mathrm{cs}_S(\varrho(\mathfrak{s}) + 1; x)$. The integral of the norm of the difference is at most

$$\int \varphi(\gamma, y)\, \big\|\mathcal{W}_S^{\mathrm{full}}(\gamma; y) - \mathcal{W}_S^{\mathrm{cut}}(\gamma; y)\big\|\, dy \;\leq\; \tfrac{1}{2}\, \nu(S)\, b(S)^{-k(\mathfrak{s})^3 f(E)} \;\leq\; \tfrac{1}{2}\, \nu(S)\, \mathrm{cs}_S(\varrho(\mathfrak{s}) + 1; x),$$

by (B5) for the side full and by the first inequality of Lemma 5.1. Adding the two bounds gives the first case of (14). □

LEAN `Filling.enorm_amplitudeSum_le_classSize_top`

Proof of Theorem B (b). If s does not agree, the block vanishes by (B6). Let s agree, and let Ξ' be the set of the $\gamma \in G_S(E)$ with $N(S)^{f(E)}\, \mathrm{mn}_S(j; \gamma) < \|\mathbf{g}\|_S(\gamma)$. By (S10) the sets $\Xi \cap \Xi'$ and $\Xi \setminus \Xi'$ are stable, and by Lemma 4.1 the block is the sum of the blocks of these two sets.

The part in Ξ'. If $\gamma \in \Xi \cap \Xi'$ then $\|\mathbf{g}\|_S(\gamma) > 0$, so $\mathbf{g}_S(\gamma) \neq 0$ and, since γ has level j, $j = \varrho(\mathfrak{s}) + 1$. So either $\Xi \cap \Xi'$ is empty or $j = \varrho(\mathfrak{s}) + 1$ and part (a) applies to $\Xi \cap \Xi'$. At the level $\varrho(\mathfrak{s}) + 1$ the lattices $\mathcal{R}_j$ and $\mathcal{R}$ coincide, the indices $\mathrm{i}_{S,\varrho(\mathfrak{s})+1}(\cdot)$ are 1, and for $\gamma \in \Xi'$ the class gauge is $\mathrm{cg}_S(\varrho(\mathfrak{s}) + 1; c, c'; \gamma) = \|\mathbf{g}\|_S(\gamma)$. So the bound of part (a) is the part of the sum in (b) over $\gamma \in \Xi'$.

The part outside Ξ'. By (S2) the norm of the block of $\Xi \setminus \Xi'$ is at most the sum of the absolute amplitudes $|a|_S^{\mathrm{cut}}(c, c'; \gamma)$ over $\gamma \in (\Xi \setminus \Xi') \cap \mathrm{Tm}_S(c, s, c')$. Let γ be such an element with non-zero absolute amplitude. By Lemma 5.3 the Schur integral $\mathrm{SI}_S(c, c'; \gamma)$ is not zero, so there is y with $\beta^0_{S,c}(y) \neq 0$ and $\beta^0_{S,c'}(\mathrm{r}_S^\infty(\gamma)^{-1} y) \neq 0$; both points are boxed, and γ is near. Let δ be the fixed representative of the double coset $\mathcal{U}^0_{S,c}\, \gamma\, \mathcal{U}^0_{S,c'}$. It is near by (S3), so $\delta \in \mathrm{NRep}_S(c, c')$; it lies in $\Xi \setminus \Xi'$ by stability and in $\mathrm{Tm}_S(c, s, c')$ by Lemma 6.1; and it has level j. Hence the sum of the absolute amplitudes is at most the sum, over the near representatives $\delta \in (\Xi \setminus \Xi') \cap \mathrm{Tm}_S(c, s, c')$, of the sums over their double cosets, and each of these is at most $\nu(S)\, \mathrm{cs}_S(j; \mathrm{cg}_S(j; c, c'; \delta))$ by Theorem A (b), whose hypothesis is $\delta \notin \Xi'$. This is the part of the sum in (b) over $\gamma \notin \Xi'$. □

LEAN `Filling.enorm_amplitudeSum_le_classSize`

9 Agreeing characters, and the proofs of Theorems D and E

Proposition 9.1 (terms with disagreeing characters cancel)**.** *In Setting 2.2, let σ be a side and $\Xi \subseteq G_S(E)$ a stable set, $\Xi \in \mathrm{Stab}(S)$, such that no $\gamma \in \Xi$ satisfies $\gamma \in \mathrm{ChA}(S)$. Then $A^\sigma_S(\Xi; c, s, c') = 0$ for all $c, c' \in \mathrm{Cl}(S)$ and $s \in \mathsf{G}(S)$.*

Proof. The group $\mathcal{U}_{S,c}$ acts by left multiplication on the set of the cosets $x = \gamma \mathcal{U}_{S,c'}$. For a coset x let $\Gamma_x \subseteq \mathcal{U}_{S,c}$ be its stabilizer and θ_x a function as in (S9) for Γ_x. In every orbit fix one coset. Let

$$\varphi(\gamma, y) = \beta_{S,c'}(\mathrm{r}_S^\infty(\gamma)^{-1}) \cdot \theta_{\gamma \mathcal{U}_{S,c'}}(y)$$

if the coset of γ is the one fixed in its orbit, and $\varphi(\gamma, y) = 0$ otherwise; the first factor is the value of the cutoff at the point $\mathrm{r}_S^\infty(\gamma)^{-1}$ of G_∞.

The weight is normalized. Let $\gamma \in G_S(E)$, $x = \gamma \mathcal{U}_{S,c'}$ and $y \in G_\infty$. The coset of $u \gamma u'^{-1}$ is ux, and $\sum_{u' \in \mathcal{U}_{S,c'}} \beta_{S,c'}(\mathrm{r}_S^\infty(u')\, \mathrm{r}_S^\infty(\gamma)^{-1} \mathrm{r}_S^\infty(u)^{-1}) = 1$ by (B1). So $\sum_{(u,u')} \varphi(u \gamma u'^{-1}, \mathrm{r}_S^\infty(u) y)$ is the sum of $\theta_{ux}(\mathrm{r}_S^\infty(u) y)$ over the $u \in \mathcal{U}_{S,c}$ for which ux is the fixed coset x_0 of the orbit. If $u_1 x = x_0$, these u are the $v\, u_1$ with $v \in \Gamma_{x_0}$, and $\sum_{v \in \Gamma_{x_0}} \theta_{x_0}(\mathrm{r}_S^\infty(v)\, \mathrm{r}_S^\infty(u_1) y) = 1$.

Every integral vanishes. By (B5) the block is the sum of the integrals $\int \varphi(\gamma, y)\, \mathcal{W}^\sigma_S(\gamma; y)\, dy$ over $\gamma \in \Xi \cap \mathrm{Tm}_S(c, s, c')$. Let γ be such an element whose coset x is the fixed one; the integral is a multiple of $\int \theta_x(y)\, F_0(y)\, dy$ with $F_0(y) = \mathcal{W}^\sigma_S(\gamma; y)$. The function F_0 is continuous. For $u \in \Gamma_x$ we have $u' = \gamma^{-1} u \gamma \in \mathcal{U}_{S,c'}$, and $F_0(\mathrm{r}_S^\infty(u) y)$ is the tensor product of the Bessel functions at $\mathrm{r}_S^\infty(u) y$ and the conjugates of those at $\mathrm{r}_S^\infty(u')\, \mathrm{r}_S^\infty(\gamma)^{-1} y$; by (B7) it equals $F_0(y)$. So F_0 is invariant under the image of Γ_x, which is discrete, and by (S9) the integral is the integral of F_0 over the quotient. Since γ does not satisfy $\gamma \in \mathrm{ChA}(S)$, there are $u_2, u_2' \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$ with $u_2 \gamma = \gamma u_2'$ and $c_0 = \psi^{\mathcal{U}}_S(\mathrm{r}^f_S(u_2))^{-1} \psi^{\mathcal{U}}_S(\mathrm{r}^f_S(u_2')) \neq 1$; by (B7), $F_0(\mathrm{r}_S^\infty(u_2) y) = c_0\, F_0(y)$ for all y, and $|c_0| = 1$. By (L3) the integral over the quotient is 0. □

LEAN `Filling.amplitudeSum_eq_zero_of_not_charsAgree`

Lemma 9.2 (shared vectors climb the flag)**.** *In Setting 2.1, let $\gamma \in G_S(E)$ with $\gamma \in \mathrm{ChA}(S)$, let $a \leq \varrho(\mathfrak{s})$ and $c_1, c_2 \in E$, and suppose $\mathrm{CF}_a(\gamma; c_1, c_2)$. Then $c_1 \overline{c_1} = c_2 \overline{c_2}$, and $\mathrm{CF}_{a'}(\gamma; c_1, c_2)$ for every a' with $a \leq a' \leq \varrho(\mathfrak{s})$.*

Proof. By descending induction on a. Let $v, v' \in E^{n(\mathfrak{s})}$ with $\mathfrak{f}v = \gamma\mathfrak{f}v'$, $v_a = c_1$, $v'_a = c_2$ and zero entries above a. For $a = \varrho(\mathfrak{s})$ the first statement is (S13) and the second is empty.

Let $a < \varrho(\mathfrak{s})$ and suppose the lemma for $a+1$. Let $w, w' \in E^{n(\mathfrak{s})}$ be the shifted vectors: $w_0 = w'_0 = 0$, $w_{a''+1} = v_{a''}$, $w'_{a''+1} = v'_{a''}$. We show below that

$$\lambda(\mathfrak{f}w) = \lambda(\gamma\mathfrak{f}w') \quad \text{for every linear form } \lambda \text{ with } \lambda(\mathfrak{f}_{a''}) = \lambda(\gamma\mathfrak{f}_{a''}) = 0 \text{ for } a'' \le a, \tag{16}$$

the forms being E-linear on $E^{I(\mathfrak{s})}$. By (L4) there are then v^+, v'^+ with $\mathfrak{f}v^+ = \gamma\mathfrak{f}v'^+$ that agree with w, w' at the indices above a. So $\mathrm{CF}_{a+1}(\gamma; c_1, c_2)$, and the induction hypothesis gives both statements.

To prove (16), let λ be such a form. By (S14) there is a vector t with $\lambda(\gamma t') = h(t, t')$ for all t', with $h = h_{T_S^{\mathrm{rat}}(E)}$; then $\lambda = h(\gamma t, \cdot)$. Let $x \in E$ be arbitrary and $r_0 = -g(S)\,\varpi(S)\,x$. We apply (S14) twice, with the index $a+1$: to the coefficient vector v and the vector γt, and to v' and t. The hypotheses hold, because $h(\gamma t, \mathfrak{f}_{a''}) = \lambda(\mathfrak{f}_{a''}) = 0$ and $h(t, \mathfrak{f}_{a''}) = \lambda(\gamma\mathfrak{f}_{a''}) = 0$ for $a'' \le a$. We get $u, u' \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$ with

$$\mathrm{sd}(u) = r_0 \sum_{a''<\varrho(\mathfrak{s})} \lambda(\mathfrak{f}_{a''+1})\, v_{a''} = r_0\,\lambda(\mathfrak{f}w), \qquad \mathrm{sd}(u') = r_0 \sum_{a''<\varrho(\mathfrak{s})} \lambda(\gamma\mathfrak{f}_{a''+1})\, v'_{a''} = r_0\,\lambda(\gamma\mathfrak{f}w').$$

They are the Eichler maps of the pairs $(r_0\mathfrak{f}v, \gamma t)$ and $(r_0\mathfrak{f}v', t)$; as γ is an isometry and $\gamma(r_0\mathfrak{f}v') = r_0\mathfrak{f}v$, we have $\gamma u'\gamma^{-1} = u$. Since the characters agree at γ, $\psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u)) = \psi_S^{\mathcal{U}}(\mathrm{r}_S^f(u'))$, that is

$$\psi_{E,f}(x \cdot \lambda(\mathfrak{f}w)) = \psi_{E,f}(x \cdot \lambda(\gamma\mathfrak{f}w')).$$

This holds for all $x \in E$, so (16) follows from (L5). □

LEAN `Filling.hasCommonFlagVector_of_le`

Proposition 9.3 (agreeing characters force a level)**.** *In Setting 2.1, let $\gamma \in G_S(E)$ with $\gamma \in \mathrm{ChA}(S)$ and $\mathfrak{g}_S(\gamma) = 0$. Then there is $j \in \{0, \dots, \varrho(\mathfrak{s})+1\}$ with $\gamma \in \mathrm{Lv}_j(S)$.*

Proof. The condition $\gamma \in \mathrm{NF}_{\varrho(\mathfrak{s})+1}$ is empty, so there is a least j with $\gamma \in \mathrm{NF}_j$. Put $Z = Z_S(\gamma)$.

Claim. If $\mathfrak{f}v = \gamma\mathfrak{f}v'$ and $v'_a = 0$ for all $a \ge j$, then $v = v' = 0$. We show $v_a = v'_a = 0$ by descending induction on a. Suppose it for the indices above a. If $v'_a \neq 0$, then $\mathrm{CF}_a(\gamma; v_a/v'_a, 1)$, and Lemma 9.2 gives $(v_a/v'_a)\overline{(v_a/v'_a)} = 1$ and $\mathrm{CF}_{a'}(\gamma; v_a/v'_a, 1)$ for all $a' \ge a$; so $\gamma \in \mathrm{NF}_a$, and $j \le a$ by minimality, against $v'_a = 0$ for $a \ge j$. Hence $v'_a = 0$. If $v_a \neq 0$, then $\mathrm{CF}_a(\gamma; 1, 0)$, and Lemma 9.2 gives $1 = 0$. Hence $v_a = 0$.

The minor $\mathrm{D}_j(Z)$ is not zero. Otherwise $\mathrm{pad}_j(Z)$ has a kernel vector $v' \neq 0$: so $v'_a = 0$ for $a \ge j$ and $(Zv')_a = 0$ for $a < j$. In fact $Zv' = 0$: for $j = \varrho(\mathfrak{s})+1$ this is said; for $j \le \varrho(\mathfrak{s})$, let $v_0 \in E^{n(\mathfrak{s})}$; by (S12) for γ^{-1} there are v_1, v_2 with $\gamma\mathfrak{f}v_1 = \mathfrak{f}v_2$ and $(v_2)_a = (v_0)_a$ for $a \ge j$; then $h(\mathfrak{f}v_2, \gamma\mathfrak{f}v') = h(\mathfrak{f}v_1, \mathfrak{f}v') = -N(S)\,\overline{(v_1)_{\varrho(\mathfrak{s})}}\,v'_{\varrho(\mathfrak{s})} = 0$, and $h(\mathfrak{f}(v_2 - v_0), \gamma\mathfrak{f}v') = 0$ because $v_2 - v_0$ is supported on the indices below j; so $\sum_a \overline{(v_0)_a}\,(Zv')_a = h(\mathfrak{f}v_0, \gamma\mathfrak{f}v') = 0$ for all v_0. Now (S13) with $D_0 = 0$ applies to $t = \gamma\mathfrak{f}v'$: the number $-(N(S)|(v'_{\varrho(\mathfrak{s})})_\tau|)^2$ is ≤ 0, so $t = \mathfrak{f}v$ for some v, and the claim gives $v' = 0$, a contradiction.

The level is at most $\varrho(\mathfrak{s})$. Suppose $j = \varrho(\mathfrak{s})+1$; then $\det Z = \mathrm{D}_j(Z) \neq 0$. Let v' be the last column of the adjugate of Z; then Zv' has the entry $\det Z$ at $\varrho(\mathfrak{s})$ and 0

elsewhere, and $v'_{\varrho(\mathfrak{s})} = \mathrm{D}_{\varrho(\mathfrak{s})}(Z)$. By (S13) the vector $t = \gamma\mathfrak{f}v'$ satisfies, at every τ, $N(S)\sum_i |(t_i)_\tau|^2 = |\det Z_\tau|^2 - N(S)^2|\mathrm{D}_{\varrho(\mathfrak{s})}(Z)_\tau|^2 = \mathfrak{g}_S(\gamma)_\tau = 0$; so $t = \mathfrak{f}v$ for some v, and the claim, whose hypothesis on v' is empty for $j = \varrho(\mathfrak{s})+1$, gives $v' = 0$, hence $\det Z = 0$, a contradiction. So $j \le \varrho(\mathfrak{s})$; as $\mathfrak{g}_S(\gamma) = 0$, both sides of the last condition in the definition of the level are false, and $\gamma \in \mathrm{Lv}_j(S)$. □

LEAN `Filling.CharsAgree.exists_hasLevel`

Proof of Theorem D (a). Write $A(c,s,c') = A^{\mathrm{cut}}_S(\Xi; c, s, c')$, $j_0 = \min(\varrho(\mathfrak{s}), j)$ and $j_1 = j - j_0$. Since $\nu(S) > 0$, we have to show

$$\sum_{c,c'}\sum_{s\in\Omega} \|A(c,s,c')\|^2 \;\le\; w\,\nu(S)^2 + \frac{C\,\nu(S)^2}{b(S)^{30}\,4^{\varrho(\mathfrak{s})+1-j}}. \tag{17}$$

Step 1: splitting by links. For a subspace Λ of $(E^{M(\mathfrak{s})})^2$ let $\Xi_\Lambda = \Xi \cap \{\gamma : \mathrm{link}(\gamma) = \Lambda\}$; it is stable by (S10), and its elements have level j. Let $A_\Lambda(c,s,c')$ be its block. By the absolute summability (B4), $A(c,s,c') = \sum_\Lambda A_\Lambda(c,s,c')$, so

$$\|A(c,s,c')\|^2 \;\le\; \Big(\sum_\Lambda \|A_\Lambda(c,s,c')\|\Big)^2 = \sum_{(\Lambda,\Lambda')} \|A_\Lambda(c,s,c')\|\,\|A_{\Lambda'}(c,s,c')\|.$$

The pairs (Λ,Λ) with $\Lambda \in R$ contribute to (17) exactly $\mathrm{LE}^{\mathrm{cut}}_S(\Xi;\Omega,R) \le w\,\nu(S)^2$. Call the other pairs *off-diagonal*; it remains to bound their contribution $\mathfrak{S}$.

Step 2: block bounds. By Theorem B (b) for Ξ_Λ, the block $A_\Lambda(c,s,c')$ vanishes unless s agrees, and then its norm is at most the sum of $\nu(S)\ \mathrm{cs}_S(j;\mathrm{cg}_S(j;c,c';\gamma))$ over the near representatives $\gamma \in \Xi_\Lambda \cap \mathrm{Tm}_S(c,s,c')$. Hence $\mathfrak{S}$ is at most $\nu(S)^2$ times the sum of the products $\mathrm{cs}_S(j;\mathrm{cg}_S(j;c,c';\gamma))\cdot\mathrm{cs}_S(j;\mathrm{cg}_S(j;c,c';\gamma'))$ over the tuples (c,c',s,γ,γ') in which $s \in \Omega$ agrees, $\gamma,\gamma' \in \Xi \cap \mathrm{Tm}_S(c,s,c')$ are near representatives for (c,c'), and $(\mathrm{link}(\gamma),\mathrm{link}(\gamma'))$ is off-diagonal.

Step 3: one pair. Fix such a tuple, and write $G = \mathrm{cg}_S(j;c,c';\gamma)$, $G' = \mathrm{cg}_S(j;c,c';\gamma')$. We claim

$$\mathrm{cs}_S(j;G)\cdot\mathrm{cs}_S(j;G') \;\le\; \frac{C}{\mathrm{tl}(S)^3}\cdot(\mathrm{pw}_S(j;c,c';\gamma,\gamma') + \mathrm{pw}_S(j;c,c';\gamma',\gamma)). \tag{18}$$

Put $U = \mathrm{link}^\circ(\gamma,\gamma') = \mathrm{link}^\circ(\gamma',\gamma)$, $\mathsf{k} = \mathrm{pos}_S(c,c';\gamma)$, $\mathsf{k}' = \mathrm{pos}_S(c,c';\gamma')$, $d_1 = \mathrm{disc}_{S,c}(U)$, $o = \mathrm{o}_{S,c}(\mathsf{k};1,U)$, $o' = \mathrm{o}_{S,c}(\mathsf{k}';1,U)$, $\mathfrak{d} = \mathfrak{d}_S(\mathsf{k})$ and $\mathfrak{d}' = \mathfrak{d}_S(\mathsf{k}')$. By (P1) and (P2), applied to (γ,γ') and to (γ',γ), the numbers d_1, G, G' are positive, $o, o', \mathfrak{d}, \mathfrak{d}' \ge 1$, and there is $i \in \mathbb{N}$ with $k(\mathfrak{s}) \le \dim_E U + i + j_1$, with $\mathrm{link}(\gamma) = \mathrm{link}(\gamma')$ if $i = 0$, and with

$$d_1^2\cdot o\,\mathfrak{d}\cdot o'\mathfrak{d}'\cdot X'^2 \;\le\; (2/g(S))^{2if(E)}\cdot G\,G',$$

where $X' = X$ if $\varrho(\mathfrak{s}) < j$ and $i = 0$, and $X' = 1$ otherwise; $X' \ge 1$. By Definition 2.4,

$$\begin{aligned}\mathrm{cs}_S(j;G)\cdot\mathrm{cs}_S(j;G') &= \frac{\mathrm{tl}(S)^4\,\mathsf{D}_j}{\mathrm{cw}_0(S)\,g(S)^{2j_0k(\mathfrak{s})f(E)}}\,(GG')^{-(k(\mathfrak{s})+1)/2}\\ &\le \frac{\mathrm{tl}(S)^4\,\mathsf{D}_j}{\mathrm{cw}_0(S)\,g(S)^{2j_0k(\mathfrak{s})f(E)}}\cdot\frac{(2/g(S))^{if(E)(k(\mathfrak{s})+1)}}{X'^{\,k(\mathfrak{s})+1}\,d_1^{k(\mathfrak{s})+1}}\cdot(o\,\mathfrak{d}'\cdot o'\mathfrak{d})^{-(k(\mathfrak{s})+1)/2}.\end{aligned}$$

As $o\,\mathfrak{d}' \ge 1$ and $o'\mathfrak{d} \ge 1$, the last factor is at most $(o\,\mathfrak{d}')^{-k(\mathfrak{s})/2}(o'\mathfrak{d})^{-k(\mathfrak{s})/2} \le \frac12((o\,\mathfrak{d}')^{-k(\mathfrak{s})} + (o'\mathfrak{d})^{-k(\mathfrak{s})})$. On the other side, with $i_0 = k(\mathfrak{s}) - \dim_E U$,

$$\mathrm{pw}_S(j;c,c';\gamma,\gamma')+\mathrm{pw}_S(j;c,c';\gamma',\gamma) = \frac{d_1^{-(k(\mathfrak{s})+1)}}{\mathrm{cw}_{i_0}(S)\cdot g(S)^{2j_0k(\mathfrak{s})f(E)}}\cdot((o\,\mathfrak{d}')^{-k(\mathfrak{s})}+(o'\mathfrak{d})^{-k(\mathfrak{s})}),$$

by the definition of the pair weights, with $[\hat{g}(S)]^{\#\mathrm{lb}(j)} = g(S)^{2j_0 f(E)}$. Since $\mathrm{cw}_{i_0}(S)/\mathrm{cw}_0(S) = (2\mathrm{tl}(S)^2)^{i_0} \le (2\mathrm{tl}(S)^2)^{i+j_1}$ and $(2/g(S))^{if(E)(k(\mathfrak{s})+1)}(2\mathrm{tl}(S)^2)^i \le \mathrm{cf}(S)^i$, (18) follows from

$$\mathrm{tl}(S)^4 \cdot \mathsf{D}_j \cdot (2\mathrm{tl}(S)^2)^{j_1} \cdot \mathrm{cf}(S)^i \cdot X'^{-k(\mathfrak{s})} \;\le\; C\,\mathrm{tl}(S)^{-3}. \tag{19}$$

We prove (19) by cases, using $\mathrm{tl}(S) \ge 2$ and (A3). *Case* $j = \varrho(\mathfrak{s}) + 1$, $i = 0$. Then $\mathrm{link}(\gamma) = \mathrm{link}(\gamma')$; as the pair is off-diagonal, this link is not in R, so $\mathrm{tl}(S)^{12} X^{-k(\mathfrak{s})} \mathsf{d}(S) \le C$ by the hypothesis of the theorem; the left side of (19) is $2\,\mathrm{tl}(S)^6\,\mathsf{d}(S)\,X^{-k(\mathfrak{s})} \le \mathrm{tl}(S)^{-3} \cdot \mathrm{tl}(S)^{12}\mathsf{d}(S)X^{-k(\mathfrak{s})}$. *Case* $j = \varrho(\mathfrak{s}) + 1$, $i \ge 1$. Then $X' = 1$ and $\mathrm{cf}(S)^i \le \mathrm{cf}(S)$; the left side is at most $2\,\mathrm{tl}(S)^6\,\mathrm{cf}(S)\,\mathsf{d}(S) \le 2\,\mathrm{tl}(S)^{-12}\mathsf{d}_1(S) \le \mathrm{tl}(S)^{-3}C$, by $\mathrm{tl}(S)^{18}\mathrm{cf}(S)\mathsf{d}(S) \le \mathsf{d}_1(S) \le C$. *Case* $j \le \varrho(\mathfrak{s})$. Then $j_1 = 0$, $X' = 1$, $\mathrm{cf}(S)^i \le 1$, and the left side is at most $\mathrm{tl}(S)^{17-32j}\,\mathsf{d}_1(S)$. For $j = 0$ the element γ is in normal form from 0 and lies in $\Xi \cap \mathrm{Tm}_S(c, s, c')$ with $s \in \Omega$, so $\mathrm{tl}(S)^{20}\mathsf{d}_1(S) \le C$ by Setting 2.6. For $j \ge 1$, $\mathrm{tl}(S)^{17-32j}\mathsf{d}_1(S) \le \mathrm{tl}(S)^{-3}\mathsf{d}_1(S) \le \mathrm{tl}(S)^{-3}C$.

Step 4: summation. In a tuple of Step 2 the element s is determined by (c, c', γ). So by (18) and Proposition 7.5, applied twice, $\mathfrak{S} \le \nu(S)^2 \cdot C\,\mathrm{tl}(S)^{-3} \cdot 4\,\mathrm{tl}(S)^2 = 4C\nu(S)^2/\mathrm{tl}(S)$. Finally $4 \cdot b(S)^{30} \cdot 4^{\varrho(\mathfrak{s})+1-j} \le b(S)^{\varrho(\mathfrak{s})+32-j} \le \mathrm{tl}(S)$ by $b(S) \ge 4$ and (S1). This gives (17). □

LEAN `Filling.energy_le_of_level_pairs`

Proof of Theorem D (b). We prove, by induction on l, the statement for all stable subsets $\Xi_1 \subseteq \Xi$ in place of Ξ; the hypotheses of Setting 2.6 hold for them with the same C, Ω and X. For $l = 0$ no element of Ξ_1 satisfies $\cdot \in \mathrm{ChA}(S)$, all blocks vanish by Proposition 9.1, and the energy is 0. Let the statement hold for some $l \le \varrho(\mathfrak{s})$, and let every $\gamma \in \Xi_1$ with $\gamma \in \mathrm{ChA}(S)$ have a level below $l + 1$. Let Ξ_2 be the set of the elements of Ξ_1 of level l and $\Xi_3 = \Xi_1 \setminus \Xi_2$; both are stable by (S10). By Lemma 4.1 and Lemma 4.2 with $\varepsilon = 1$, the energy of the blocks of Ξ_1 is at most twice that of Ξ_3 plus twice that of Ξ_2. An element of Ξ_3 with agreeing characters has a level below $l + 1$ that is not l, so the induction hypothesis applies to Ξ_3. To Ξ_2 we apply part (a) with $j = l \le \varrho(\mathfrak{s})$, $w = 0$ and R empty; the link energy is then 0. Both energies are at most $C\,b(S)^{-30}4^{-(\varrho(\mathfrak{s})+1-l)}$, and four times this bound is $C\,b(S)^{-30}4^{-(\varrho(\mathfrak{s})+1-(l+1))}$. □

LEAN `Filling.energy_le_of_level_lt`

Proof of Theorem E. Let $\Xi_2 = \Xi \cap \{\gamma : \mathfrak{g}_S(\gamma) \ne 0\}$ and $\Xi_3 = \Xi \setminus \Xi_2$; both are stable by (S10). The elements of Ξ_2 have level $\varrho(\mathfrak{s}) + 1$ (S12); so Theorem D (a) with $j = \varrho(\mathfrak{s}) + 1$ gives the energy bound $w + C\,b(S)^{-30}$ for the cut blocks of Ξ_2. An element of Ξ_3 with agreeing characters has a level by Proposition 9.3, and this level is not $\varrho(\mathfrak{s}) + 1$ because $\mathfrak{g}_S(\gamma) = 0$; so Theorem D (b) with $l = \varrho(\mathfrak{s}) + 1$ gives the bound $C\,b(S)^{-30}$ for the cut blocks of Ξ_3. By Lemma 4.1 and Lemma 4.2 with $\varepsilon = b(S)^4$, the cut blocks of Ξ have energy at most

$$B_1 = (1 + b(S)^4)\,C\,b(S)^{-30} + (1 + b(S)^{-4})(w + C\,b(S)^{-30}).$$

By Theorem C, $\mathsf{d}_1(S) \le C$, and Lemma 4.2 once more with $\varepsilon = b(S)^4$, applied to the difference of the full and the cut blocks and to the cut blocks,

$$\mathrm{En}_S(A_S^{\mathrm{full}}(\Xi; \cdot, \cdot, \cdot); \Omega) \;\le\; (1 + b(S)^4)\,b(S)^{-30}\,C + (1 + b(S)^{-4})\,B_1.$$

In this bound the coefficient of w is $(1 + b(S)^{-4})^2$, and the coefficient of C is $b(S)^{-30}\,(2\,b(S)^4 + 4 + 3\,b(S)^{-4} + b(S)^{-8}) \le b(S)^{-12}$. □

LEAN `Filling.energy_le_of_linkEnergy_le`

10 Notes on the formal text

Arguments that the text reading gets wrong. The program that reads the arguments of the declarations from the text lists, for several members of these files, arguments that Lean does not take; the statements of this note follow the Lean text. `Filling.energy` and `Filling.linkEnergy` take the data S implicitly, and the second takes the side as an instance; the notations $\mathrm{En}_S(\mathcal{A};\Omega)$ and $\mathrm{LE}^{\sigma}_S(\Xi;\Omega,R)$ show both, as in the other notes of the unit. In the first file a second declaration of the field and of the data hides the earlier instances, so that `Filling.classSize`, `Filling.boxCutoff`, `Filling.schurWeight`, `Filling.schurIntegral` and Lemma 5.2 assume neither that E is stably unramified nor that S is admissible, and Lemma 5.1 assumes only admissibility; the same happens in the third file, where the lists for Proposition 7.5 and Theorem B name a class and an element that are bound variables. Lemma 6.1, Lemma 9.2 and Proposition 9.3 need no standing hypothesis at all.

The side. The formal text selects the Bessel function by an instance. The default is the full function; from the definition of the Schur integral on, and in the second, third and fourth files, a local instance makes it the cut function. So Theorems A, B and D are statements about cut functions, Theorem C compares the default with an explicit cut, and Theorem E names the full function explicitly while its hypothesis on the link energy is for the cut. Proposition 9.1 holds for both sides.

Conventions of the formal text. Sums and integrals of non-negative terms are taken in $[0,\infty]$. An inverse power of a real number that is 0 is 0; this enters the weights $\mathrm{ipw}_S(c;\eta,p)$, the term weights and the sums of Propositions 7.1 and 7.3. The class size is defined for every real x by the conventions for square roots and quotients; it is used only at positive arguments, the class gauge being positive by (P1). The amplitudes are Bochner integrals, and the amplitude sums are unconditional sums; both are 0 by convention when they do not converge, which under the hypotheses of Setting 2.2 does not happen.

Statements. Theorem B (b) is stated in Lean with one sum over the elements γ subject to the condition that s agrees; this is the same as the two cases above. In Theorems D and E the exponent of 4 is written with the reversal of the index j. The hypothesis of Proposition 5.4 asks nothing of the set Q_0 but $\mu(Q_0) \ge 1$. In Lemma 7.2 the formal text writes the vector as $\det Y$ times a row of the transposed inverse; it is the column of the adjugate. Lemma 7.4 is stated for the idele of $g(S)$ and uses only that it is an integral unit of the adeles. Theorem E is an assembly: its content lies in Theorems C and D and in Proposition 9.3. Theorem D (b) assembles Theorem D (a) and Proposition 9.1.

Exponents. The formal proofs end, in nine places, with comparisons of exponents of $b(S)$ that are left to decision procedures, in one case (Lemma 5.1) after taking logarithms and forming an explicit linear combination of the hypotheses. In this note all of them are carried out by hand, with cruder intermediate bounds than the formal text needs: Theorem C; both inequalities of Lemma 5.1; Step 7 of Theorem A (a) and the end of Theorem A (b); Propositions 7.1, 7.3 and 7.5; Lemma 7.2; and (19). In Lemma 5.1 the bookkeeping of the powers of $\prod_\tau |\kappa_\tau(S)|$ is ours and exact, where the formal text first bounds some of these factors by a power of $b(S)$.

What is taken for granted. Everything in Section 3. In particular the three analytic inputs from the real place (the volume kernel (R2), the open-cell bound (R3), the congruence count (R5)), the inequality of pair invariants (P2), the frame count (B3), the change of weight (B5) and the cut error (B4) carry much of the weight of Theorems A, B and D; they are proved in the notes named there.

How this note was written. From the dossier of the unit: the texts of the members of the four files and their structured proofs. I read the four Lean files completely, the proofs line by line, in particular those that the readers had marked as hiding arguments in automation: the logarithmic linear combination of Lemma 5.1, the averaging over classes in Proposition 7.1, the inclusion of lattices in Step 3 of Proposition 7.5, the numerical ends of Theorem A and of (19), and the open-cell step of Theorem B (a). Every statement of this note was compared with its Lean statement, argument by argument, and with the list of arguments that the program prints. Of the Inputs I read the Lean statements, in the files of the set-up, of the box partitions and amplitude sums, of the pairing of links, of the real place and of the background library, and for the definitions of the weights and indices their Lean text; I did not read their proofs. The proofs given here follow the formal ones in structure; the arithmetic of exponents is ours, the proof of the vanishing of the columns in Step 5 of Proposition 7.3 and the two cases of (13) are written out where the formal text calls a simplifier, and Step 3 of Proposition 7.5 is arranged by vectors instead of matrices. The references for the background library are those of the catalogue of the write-up and of the second reader; I did not open the cited works. The program check of the unit's dossier covered 1371 leaf steps of 179 proofs.

Appendix A. Concordance with the Lean artifact

The formal names of the objects: `S.b`, `S.N`, `S.g` are $b(S)$, $N(S)$, $g(S)$; `numInfPlaces E` is $f(E)$; `𝔰.ϱ`, `𝔰.k`, `𝔰.n` are $\varrho(\mathfrak{s})$, $k(\mathfrak{s})$, $n(\mathfrak{s})$; `tailScale`, `unipCovolume`, `besselEnergy`, `mainEnergy`, `countWeight`, `codimFactor` are $\mathrm{tl}(S)$, $\nu(S)$, $\mathsf{d}_1(S)$, $\mathsf{d}(S)$, $\mathrm{cw}_i(S)$, $\mathrm{cf}(S)$; `termsOver`, `termPosition`, `classUnip`, `classUnip₀`, `classCutoff` are $\mathrm{Tm}_S(c,s,c')$, $\mathrm{pos}_S(c,c';\gamma)$, $\mathcal{U}_{S,c}$, $\mathcal{U}^0_{S,c}$, $\beta_{S,c}$; `amplitudeAbs`, `amplitudeSum` are $|a|^\sigma_S(c,c';\gamma)$, $A^\sigma_S(\Xi;c,s,c')$; `termDiscNorm`, `minorNorm`, `classGauge`, `xRowsIndex` are $\|\mathsf{g}\|_S(\gamma)$, $\mathrm{mn}_S(j;\gamma)$, $\mathrm{cg}_S(j;c,c';\gamma)$, $\mathrm{i}_{S,j}(\mathsf{k})$; `HasLevel`, `IsNearRep`, `IsStable`, `Agrees`, `CharsAgree`, `FixesNormAtLeast` are the relations $\gamma \in \mathrm{Lv}_j(S)$, $\gamma \in \mathrm{NRep}_S(c,c')$, $\Xi \in \mathrm{Stab}(S)$, $s \in \mathrm{Agr}(S)$, $\gamma \in \mathrm{ChA}(S)$, $\mathrm{fix}(s) \geq X$. The Lean letters `T`, `U` for sets of rational elements are Ξ, Ξ'; `v` for an infinite place is τ; `w` and `C` are w and C; the natural numbers `s` and `n` of the counting statements are m and l; the subspace `T` is U. Three statements as they stand in the artifact:

```
noncomputable def energy := ∑ c, ∑ c', ∑ s with s ∈ Ω, ‖A c s c'‖ ^ 2 / S.unipCovolume ^ 2
```

```
lemma Filling.tsum_schurIntegral_le : ∑' δ : doubleCoset γ (S.classUnip₀ c) (S.classUnip₀ c'),
  S.schurIntegral c c' δ ≤ .ofReal (S.unipCovolume * (card S.BesselCoord * S.tailScale ^ 2 *
  (S.b ^ 4 * S.g)⁻¹ ^ (min s̆.ϱ j * s̆.k * numInfPlaces E) * (S.xRowsIndex j (S.termPosition c c'
  γ) / √(S.minorNorm j γ)) ^ (s̆.k + 1)))
```

```
lemma Filling.energy_le_of_linkEnergy_le R (h₂ : (∃ Λ, Λ ∉ R) →
  S.tailScale ^ 12 / X ^ s̆.k * S.mainEnergy ≤ C)
  (h : linkEnergy (T ∩ {γ | S.termDisc γ ≠ 0}) Ω R ≤ .ofReal (w * S.unipCovolume ^ 2)) :
  energy (S.amplitudeSum (σ := .full) T) Ω ≤ (1 + (S.b ^ 4)⁻¹) ^ 2 * w + (S.b ^ 12)⁻¹ * C
```

In this note	Lean declaration	File	Lines
Proposition 7.1	`Filling.sum_latticeDisc_inv_le`	BlockBound.lean	21–75
Lemma 7.2	`Filling.HasLevel.norm_shared_vector_lt`	BlockBound.lean	93–125
Proposition 7.3	`Filling.tsum_invPowWeight_le_of_codim_le`	BlockBound.lean	130–215
Lemma 7.4	`Filling.submoduleIndex_gIdele_smul`	BlockBound.lean	222–226
Proposition 7.5	`Filling.sum_pairWeight_le`	BlockBound.lean	232–297
Theorem B	`Filling.enorm_amplitudeSum_le_classSize_top`	BlockBound.lean	304–363
Theorem B	`Filling.enorm_amplitudeSum_le_classSize`	BlockBound.lean	368–394
Proposition 9.1	`Filling.amplitudeSum_eq_zero_of_not_charsAgree`	CrudeEnergy.lean	24–41
Lemma 9.2	`Filling.hasCommonFlagVector_of_le`	CrudeEnergy.lean	46–61
Proposition 9.3	`Filling.CharsAgree.exists_hasLevel`	CrudeEnergy.lean	67–82
Theorem D	`Filling.energy_le_of_level_pairs`	CrudeEnergy.lean	108–160
Theorem D	`Filling.energy_le_of_level_lt`	CrudeEnergy.lean	166–172
Theorem E	`Filling.energy_le_of_linkEnergy_le`	CrudeEnergy.lean	180–191
Definition 2.3	`Filling.energy`	SchurTest.lean	20–20
Definition 2.3	`Filling.linkEnergy`	SchurTest.lean	24–25
Lemma 4.1	`Filling.amplitudeSum_split`	SchurTest.lean	30–32
Lemma 4.2	`Filling.energy_add_le`	SchurTest.lean	35–38
Theorem C	`Filling.energy_sub_cut_le`	SchurTest.lean	43–58
Definition 2.4	`Filling.classSize`	SchurTest.lean	65–67
Lemma 5.1	`Filling.IsNear.le_classSize_and_cellMass_le`	SchurTest.lean	72–99
Definition 2.5	`Filling.boxCutoff`	SchurTest.lean	103–103
Definition 2.5	`Filling.schurWeight`	SchurTest.lean	107–108
Definition 2.5	`Filling.schurIntegral`	SchurTest.lean	114–115
Lemma 5.2	`Filling.inCut_of_besselKernel_ne_zero`	SchurTest.lean	121–123
Lemma 5.3	`Filling.measurable_boxCutoff`	SchurTest.lean	125–126
Lemma 5.3	`Filling.measurable_schurWeight`	SchurTest.lean	128–129
Lemma 5.3	`Filling.amplitudeAbs_sq_le`	SchurTest.lean	131–140
Proposition 5.4	`Filling.lintegral_boxCutoff_le_of_average`	SchurTest.lean	145–169
Lemma 6.1	`Filling.terms_of_class_spec`	ThinClasses.lean	25–39
Theorem A	`Filling.tsum_schurIntegral_le`	ThinClasses.lean	48–123
Theorem A	`Filling.tsum_amplitudeAbs_class_le`	ThinClasses.lean	131–149

Notation

E, τ, x_τ, $f(E)$ the CM field with its CM type, an infinite place, the image of $x \in E$ at τ, the number of infinite places

$\mathbb{A}_{E,f}$, $\hat{O}_E$, $\psi_{E,f}$ finite adeles, integral adeles, the standard additive character

$\mathrm{N}_{E/\mathbb{Q}}$, rd_E, d_E, $\mathrm{vol}(\hat{O}_E)$ norm, root discriminant, discriminant, volume of the integral adeles

$[a]$, $[M' : M]$ generalized index of an adele; index of $M \cap M'$ in M' (0 if infinite)

$\hat{O}_E\langle M\rangle$, $(M)_{\mathbb{A}_f}$, M^* span of the columns; a matrix over E read over the adeles; conjugate transpose

$\mathfrak{s} = (\varrho, \kappa, k_0)$ the shape; $\varrho(\mathfrak{s})$, $k(\mathfrak{s})$, $n(\mathfrak{s})$, $r(\mathfrak{s})$ its numbers; $I(\mathfrak{s})$, $M(\mathfrak{s})$ its index sets

e_a, e_a^*, z, $\mathfrak{f}_a$, $\mathfrak{f}$ basis vectors; the frame vectors; the matrix of the frame

T, $J(T)$, $G(T)$, h_T, h an $\mathfrak{s}$-form, its split Gram matrix, its unitary group, the hermitian form

$\mathcal{U}(T)$, $\mathcal{U}^0(T)$, $\mathcal{H}(T)$, $\mathcal{S}(T)$ Bessel unipotent subgroup, its part fixing the frame, the group fixing e_a, e_a^*, z, the Bessel subgroup

$Z_T(\gamma)$, $Z(y)$, $\mathrm{D}_l(Z)$, $\mathfrak{g}(N'; Z)$, $\mathfrak{g}^{\mathbb{R}}(Z)$, $\mathrm{sd}(u)$ frame Gram matrices, leading minor, discriminants, sum above the diagonal

$G_1(\mathfrak{s})$, G_∞, $T^{\mathrm{rat}}(\mathfrak{s}; 1)$ the real group, its product over the places, the form over $\mathbb{C}$

$\mathrm{bs}(y)$, $y \in \mathrm{Cut}(b_0)$, $\mathrm{FBox}(\rho)$, $\mathrm{cm}(y; \deg_0, \rho, d_0)$, $c_G(\mathfrak{s})$ Bessel size, the cut, the frame box, the cell mass, a constant

S, $\mathrm{Fill}(E, \mathfrak{s})$ the filling data; $b(S)$, $N(S)$, $q_i(S)$, $g(S)$, $\bar\pi(S)$, $\varpi(S)$, $\mathrm{Ch}_i(S)$, $K^{\mathrm{col}}(S)$, $\mathfrak{B}_\tau(S)$ its components

$T_S^{\mathrm{rat}}(A)$, $T_S^{\mathrm{lat}}(A)$, $G_S(A)$, G_S^{lat} the forms and groups of the data over a ring A

$K(S)$, $K_g(S)$, $\mathsf{G}(S)$, red_S, $\overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^S$ integral subgroup, principal level, the finite group, reduction, reduced Bessel image

r_S^∞, r_S^f, inc_S, $\mathrm{sc}_{S,\tau}(i)$, $\kappa_\tau(S)$, $\mathrm{fs}_S(\tau, a)$ maps to the real and finite adelic groups; scales

$\mathrm{Cl}(S)$, c, c', c'', x_c, $h_S(c)$, $[x]$ classes, representatives, the class of x

$\mathrm{pos}_S(c, c'; \gamma)$, $\mathrm{Tm}_S(c, s, c')$, $\mathcal{U}_{S,c}$, $\mathcal{U}^0_{S,c}$, $\mathrm{C}_S(c)$, $\mathrm{C}^\circ_S(c)$ positions, terms, arithmetic unipotent groups, class coordinates

$Z_S(\gamma)$, $\mathfrak{g}_S(\gamma)$, $\|\mathfrak{g}\|_S(\gamma)$ frame products, discriminant and its norm

$W^\sigma_{S,\tau}$, $I_\tau(S)$, $\deg_\tau(S)$, $\mathrm{Bc}(S)$, $\deg(S)$, χ Bessel functions, index sets, degrees, a character

σ, full, cut a side

$\mathcal{W}^\sigma_S(\gamma; y)$, $\beta_{S,c}$, $\theta_{S,c}$, $\omega_{S,c}$, $\omega \in \mathrm{BP}(S, c)$ kernel, cutoff and its two factors, box partitions

$a^\sigma_S(c, c'; \gamma)$, $|a|^\sigma_S(c, c'; \gamma)$, $A^\sigma_S(\Xi; c, s, c')$ amplitude, absolute amplitude, amplitude sum

$\mathrm{bx}(S)$, $\mathrm{BxP}(S)$, $\gamma \in \mathrm{Near}(S)$, $\gamma \in \mathrm{NRep}_S(c, c')$ box size, boxed points, near elements, near representatives

$\mathrm{CF}_a(\gamma; c_1, c_2)$, $\gamma \in \mathrm{NF}_j$, $\gamma \in \mathrm{Lv}_j(S)$ shared flag vectors, normal forms, levels

$\mathrm{link}(\gamma)$, $\mathrm{link}^\circ(\gamma, \gamma')$, Λ, R link, common link, a link, a set of subspaces

$\psi^{\mathcal{U}}_S(u)$, $\gamma \in \mathrm{ChA}(S)$, $s \in \mathrm{Agr}(S)$, $\Xi \in \mathrm{Stab}(S)$, $\mathrm{fix}(s) \geq X$ the unipotent character and the conditions built on it; stable sets; fixed vectors

$\mathrm{tl}(S)$, $\mathrm{cf}(S)$, $\mathrm{cd}(S)$, $\mathsf{d}_1(S)$, $\mathsf{d}(S)$, $\mathrm{cw}_i(S)$, $\nu(S)$, $\mathrm{fs}(S)$, $c_*(S)$, $a_*(S)$, $\mathrm{mass}(E, r(\mathfrak{s}))$ the constants of the data

$\mathrm{En}_S(\mathcal{A}; \Omega)$, $\mathrm{LE}^\sigma_S(\Xi; \Omega, R)$ energy, link energy (Definition 2.3); $\mathcal{A}, \mathcal{A}'$ families, V a normed group, Ω a subset of $\mathsf{G}(S)$

$\mathrm{cs}_S(j; x)$, D_j class size (Definition 2.4)

$\beta^0_{S,c}$, $\mathrm{sw}_S(c; \gamma, y)$, $\mathrm{SI}_S(c, c'; \gamma')$, Δ boxed cutoff, Schur weight, Schur integral (Definition 2.5), frame volume

$\mathsf{k}, \mathsf{k}', \mathsf{u}$, $Z_{T_S^{\mathrm{lat}}(\mathbb{A}_{E,f})}(\mathsf{k})$, Z^{x}, $\mathrm{Ib}(j)$, $\mathcal{R}(\mathsf{k})$, $\mathcal{R}_j(\mathsf{k})$ elements of G_S^{lat}; frame Gram matrix, its first rows, row indices below j, row lattices

$\mathrm{i}_{S,j}(\mathsf{k})$, $\mathfrak{d}_S(\mathsf{k})$, $\mathrm{mn}_S(j; \gamma)$, $\mathrm{cg}_S(j; c, c'; \gamma)$ index of rows, saturation index, minor norm, class gauge

$\mathrm{disc}_{S,c}(U)$, $\mathrm{disc}_H(U)$, $\mathrm{m}_H(U)$, M_U, $\Lambda_U(\mathsf{k})$, $\mathrm{o}_{S,c}(\mathsf{k}; a, U)$ lattice discriminants, basis matrices, frame value index; U, U' subspaces of E^κ

$\mathrm{pi}_S(c, c'; \gamma, \gamma')$, $\mathrm{el}_S(c; p)$, $\mathrm{ipw}_S(c; \eta, p)$, $p \in \mathrm{Lab}_S(j; c, \mathfrak{t})$, $\gamma \sim_{S;c,c'} \gamma'$ pair invariant, term element, inverse power weight, labelled terms, congruent terms

$\mathrm{w}_S(j, i; c; p, p'; U)$, $\mathrm{pw}_S(j; c, c'; \gamma, \gamma')$, $\eta_{j,U}$, $\hat g(S)$, $\mathrm{pad}_j(Z)$, $(\mathsf{k}\mathfrak{f})^{\mathrm{mid}}$, $\Lambda^\circ(\mathfrak{s}; \bar\pi(S))$ term and pair weights, the idele of $g(S)$, padded matrix, middle rows of the frame image, a matrix of Note S

γ, γ', δ, $\gamma_0, \gamma_1, \gamma_p$ rational elements of $G_S(E)$; in (R2), (R3) elements of $G_1(\mathfrak{s})$

$u, u', u_0, u_1, u_2, u_p, r$ unipotent elements, rational or real; x also an element of $\mathcal{U}(T_S^{\mathrm{rat}}(E))$ in Theorem B

y, y', y'', y^0 points of G_∞ or of $G_1(\mathfrak{s})$

s, s', $\mathfrak{t}$ elements of $\mathsf{G}(S)$; a label in $\overline{\mathcal{S}(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))}^S$

j, j_0, j_1, l levels; $j_0 = \min(\varrho(\mathfrak{s}), j)$, $j_1 = j - j_0$; the bound of Theorem D (b), also an index of minors

a, a', a'', i, e, m, m', n_1, l_1, m_1 indices of the frame and of rows; natural numbers: codimensions, exponents, sizes

$t, t', t_0, t_1, t_2, t_3, \tilde t, \tilde t_2$, v, v', v_0, w, w' vectors of the hermitian spaces; coefficient vectors in $E^{n(\mathfrak{s})}$; w also the real bound of Theorems D, E

C, X, X', w, ε, x the constants of Setting 2.6 and of Theorems D, E; real numbers

Q, Q_0, μ, ϕ, F, C_0, q the measure space, subset, map, function and constant of Proposition 5.4; a point

η^0, μ^0, $\mathcal{U}^0(T^{\mathrm{rat}}(\mathfrak{s}; 1))^\infty$, W measures of (B8); the product group; the product of squared norms

$\mathcal{D}$, $\mathcal{B}$, $\mathcal{P}$, $\mathcal{P}'$, $\mathcal{P}(q, U)$, p, p', p_1, q a double coset, a colour class, index sets of terms and pairs; pairs

Z, Z^τ, Z_j^τ, Z_Σ, Y, ξ, ℓ_δ, ℓ, L_δ, L, R_τ, P_τ, Π_0, D_2 matrices and numbers in the proof of Theorem A

$\mathcal{K}$, $\mathcal{K}_\tau$, θ_0, Σ, R_0, V_0, A, α_1, $u(\alpha_1)$ the volume kernel and its data (R2); the parametrization (L9)

Φ_δ, φ, φ_p, α, θ, θ_x, Γ, Γ_x, Γ_0, x_0 weights; cutoffs; discrete groups; a coset

$\mathcal{C}$, $\mathcal{C}_\tau$, n_τ, m_τ, Q_τ, t_τ, n_1, m_1, Q_1, t_1, ρ the data of the open-cell bound (R3)

Φ, B_0, $\mathfrak{A}$, $\tilde{a}$, K', y_a, r_a, k_a, γ_a, i_a, F_0, D_1, x_0, w_0, H Proposition 7.1; in (S15) a matrix with a left inverse

G', G'', $\mathfrak{M}$ the matrices of Lemma 7.2

d_0, m_0, N_1, C_1, C_2, Υ, H, Θ, Ψ, α, ℓ_p, U_1, x_1, J, ϑ_1 Proposition 7.3 and (B10); J a split Gram matrix

P_0, p_0, a_0, r_0, μ_0, P_1, M_1, M_2, D_2, K_0, K_1, I_1, I_2, I_3, Y_p, M_0, η_p Proposition 7.5, Lemma 7.4, the inputs (L11), (R5); finite index sets

λ, c_0, c_1, c_2, $\varkappa$, ϵ, $\epsilon_1, \epsilon_2, \epsilon_3$, L_0, R_0, G, G', d_1, o, o', $\mathfrak{d}$, $\mathfrak{d}'$, i_0, $\mathfrak{S}$, A_Λ, Ξ_Λ, Ξ_1, Ξ_2, Ξ_3 a linear form; numbers and sets in the proofs of Lemma 5.1 and of Theorems D, E

β_1, β_2, Σ_1, B_1 auxiliary numbers in the proofs of Theorem A, Proposition 7.5 and Theorem E

π the number $3.14\ldots$; $\bar{\pi}(S)$ is a component of the data

F_1, F_2, b_0, f, g, k, n, N, N', O, K, M, M', I, B functions in Lemma 5.3; a real number; letters that occur only inside the notations above: $f(E)$, $g(S)$, $k(\mathfrak{s})$, $n(\mathfrak{s})$, $N(S)$, $\hat{O}_E$, $K(S)$, matrices M, $I(\mathfrak{s})$, $b(S)$

READBACK NOTE 44

Poincaré series of cut Bessel functions: unfolding, one-sided Poisson summation, the adelic sieve, standard levels

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. Theorem B of the write-up (*enough forms*) rests on a second moment of Poincaré series on an indefinite unitary group. This note treats four pieces of its proof that do not depend on each other. First, the Poincaré series themselves: they are sums of Bessel functions at the infinite places, cut off along an arithmetic unipotent group; they converge uniformly, they are automorphic forms, and their Petersson products unfold into the *amplitude sums* of the companion notes (Theorems A and B). Second, a sieve: a tower of positive kernels on $\mathbb{C}^f \times \prod_{w \in \mathsf{T}} E_w$, each the integral of the next over a common last entry, is summed over rational points by Poisson summation used as an inequality, one entry at a time, and the loss is a factor close to 1 (Theorems C and D). Third, at a finite place the lattice of the filling data carries the structure that the tower method of the real-place unit needs, with exact masses (Theorem E). Fourth, the degree and the number of coordinates of the Bessel systems attached to admissible data are computed from Weyl's dimension formula (Theorem F).

What this note is. The artifact grh-compact is a body of Lean code whose final theorems state the Riemann hypothesis and its analogue for Dirichlet L-functions. That is a claim: its authors report that Lean's kernel accepts the code, a replay of that check by the owner of this write-up is recorded elsewhere, and nothing was compiled for this note; no mathematician has read the proof through. This note treats five files of one unit (the folder of the filling theorem): 53 of its 497 declarations. Every statement was compared, argument by argument, with the Lean text. Proofs are complete at the level of a research paper, with these exceptions, each said where it occurs: results of the other notes of the unit, of other units and of the background library are quoted as *Inputs*, each with its owner and with what is known of a published source (several have none, or one that is not confirmed); the verification that an explicit record satisfies the axioms of a level or of a gauge is described axiom by axiom and not carried out in every detail (Theorem E, Lemma 6.6); the proof of Proposition 6.3 leans on the disintegration quoted in (R1); Proposition 7.1, a member of this unit that belongs to the seesaw argument, is an assembly of the four inputs (D2) and its proof is described; and comparisons of exponents that the formal text leaves to automation are replaced by explicit ones of ours. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 53 declarations.

The notes of this unit are cited by letters: S (the filling data), B (box partitions), U (Poincaré series and the sieve; this note), E (the energy of amplitude sums), F (the filling theorem), P (the pairing of links), T (the top floor at a finite place) and

G (from amplitude sums to local correlations). “Note T” is always the note of this unit; the note of unit 132 on the tower method is called the tower note.

1 The statements

The objects are defined in Section 2. In Theorems A and B, S is filling data over a CM field E (Note S of this unit, *The filling data: Bessel subgroups, links of frames, Siegel's lemma and strong approximation*), V is a hermitian space with frame whose rational form is that of S, and X is the record of the compatibilities between them (Setting 3.2).

Theorem A (Poincaré series converge and are forms). In Setting 3.2, let E be stably unramified and S admissible.

(a) For every pair $a = (c, i) \in \mathrm{Cl}(S) \times \mathrm{Bc}(S)$ there is a real number $M \geq 0$ such that

$$\sum_{\gamma \in \Gamma(V)} \left|\mathrm{cb}_X(a; \gamma g)\right| \leq M \qquad \text{for all } g \in G(V).$$

(b) The formal text fixes a $\mathbb{C}$-linear map Q_X from $\mathbb{C}^{\mathrm{Cl}(S)\times\mathrm{Bc}(S)}$ to the space of the forms on V of type ϵ_l that are fixed by the level $K_g^{\mathrm{std}}(S)$, such that for all coefficient vectors ϕ, with entries ϕ_a indexed by the pairs a, and all points $g \in G(V)$

$$\mathrm{Q}_X(\phi)(g) \;=\; \mathrm{P}_X(\phi, \Gamma(V))(g) \;=\; \sum_a \phi_a \sum_{\gamma \in \Gamma(V)} \mathrm{cb}_X(a; \gamma g).$$

LEAN `Filling.Fits.tsum_cutBessel_le` · `Filling.Fits.poincareMap`

Theorem B (the Petersson product of two Poincaré series). In Setting 3.2, let E be stably unramified and S admissible. Let $a = (c, i)$ and $a' = (c', i')$ be pairs in $\mathrm{Cl}(S) \times \mathrm{Bc}(S)$, let $s \in \mathsf{G}(S)$, and let $k \in K_f(E, I(\mathfrak{s}))$ be an integral finite point whose reduction at every chosen place $w \in \mathrm{Ch}_j(S)$, $j \in \{0, 1\}$, is the component $s_{j,w}$ of s. Then, with e_a the unit vector at a, $R(k)$ the right translation by k, and $\langle\,,\rangle$ the Petersson product, conjugate linear in the first variable,

$$\langle R(k)\,\mathrm{Q}_X(e_{a'}),\ \mathrm{Q}_X(e_a)\rangle \;=\; \mathrm{vol}(K_g^{\mathrm{std}}(S)) \cdot A_S^{\mathrm{full}}(G_S(E); c, s, c')_{(i,i')}.$$

Here $A_S^{\mathrm{full}}(G_S(E); c, s, c') \in \mathbb{C}^{\mathrm{Bc}(S)\times\mathrm{Bc}(S)}$ is the amplitude sum of Note S for the full Bessel functions over all rational elements that are terms over (c, s, c'), and vol is the Haar measure of the finite adelic unitary group that gives its integral points mass 1.

LEAN `Filling.Fits.petersson_poincare`

In Theorems C and D, T is a finite set of finite places of E, $E_{\mathsf{T}} = \prod_{w \in \mathsf{T}} E_w$, and a *sieve system* is a tower of kernels in the sense of Note S, recalled in Input (S4).

Theorem C (one-sided Poisson summation along a tower). Let $S \in \mathrm{Fill}(E, \mathfrak{s})$ be admissible filling data, T a finite set of finite places of E, and $\mathcal{A}$ a finite set of units of E all of which are T-units; assume that $g(S)^{-1}$ is a T-integer. Put $C = \mathbb{C}^f \times E_{\mathsf{T}}$, the product over the infinite places and over T, with its Haar measure, and $U = (\mathbb{C}^\times)^f \times E_{\mathsf{T}}^\times$. Let $\mathsf{n} \in \mathbb{N}$, let $J_i(p,q) \in \mathbb{C}$ and $\Lambda_i(p) \in \mathbb{R}$ be given for $i \in \mathbb{N}$ and sequences $p, q \colon \mathbb{N} \to U$, let $\mathrm{En} \in \mathbb{R}$ and $\beta \colon C \to \mathbb{R}$, and suppose that

$$\mathrm{Sieve}(u, \iota, \mathsf{n}, J, \psi, \mathrm{En}, \Lambda, \beta)$$

holds on C, where $u \colon C \to U$ replaces every coordinate that is not a unit by 1, $\iota(p) = \big((p_\tau^{-1})_\tau,\ g(S)^{-1} p_{\mathsf{T}}^{-1}\big)$ and $\psi(x) = \prod_\tau \exp(x_\tau - \bar{x}_\tau) \cdot \psi_{S,\mathsf{T}}(x_{\mathsf{T}})$. Let C and e be real numbers with $\mathsf{e} \geq 0$, and suppose that $\Lambda_i(p) \geq 0$ for all i, p and that the following *frequency bound* holds: for every $a \in \mathbb{N}$, every family $c = (c_\tau)$ of complex numbers with $|c_\tau|^{-1} \leq |\kappa_\tau(S)|/g(S)$ for all τ, and every finite set $\mathcal{X}$ of non-zero T-integers of E,

$$\sum_{y \in \mathcal{X}} \beta\Big(\mathrm{pt}_{\mathsf{T}}\big((c_\tau / \mathrm{fs}_S(\tau, a))_\tau ; y\big)\Big) \leq \mathsf{e}.$$

For $i \in \mathbb{N}$, $p \in \mathcal{A}^i$ and $t \in E^\times$ let $\mathrm{hd}_i(p,t) \colon \mathbb{N} \to U$ be the sequence

$$\mathrm{hd}_i(p,t)_a = u\Big(\mathrm{pt}_{\mathsf{T}}\big((\mathrm{fs}_S(\tau,a))_\tau ; (p \,|\, t)_a\big)\Big),$$
$$(p \,|\, t)_a = p_a \ (a < i), \qquad (p \,|\, t)_i = t, \qquad (p \,|\, t)_a = 1 \ (a > i),$$

and put $c_a = \mathrm{vol}(\hat{O}_E) \big/ \prod_\tau \mathrm{fs}_S(\tau, a)^2$. If

$$\Big(\sum_{p \in \mathcal{A}^i} \Lambda_i(\mathrm{hd}_i(p,1))\Big)^2 \leq \mathsf{C} \prod_{a<i} c_a \qquad \text{for all } i \leq \mathsf{n},$$

then for all $i \leq \mathsf{n}$ and all subsets $\mathcal{V} \subseteq \mathcal{A}$

$$\mathrm{Re} \sum_{t \in \mathcal{V}} \sum_{p,q \in \mathcal{A}^i} J_{i+1}(\mathrm{hd}_i(p,t), \mathrm{hd}_i(q,t)) \ \leq \ \Big(\prod_{a \leq i} c_a\Big)(\mathrm{En} + (2i+1)\,\mathsf{C}\,\mathsf{e}).$$

LEAN `Filling.one_sided_poisson`

Theorem D (the sieve). Let E carry Weil data, let $S \in \mathrm{Fill}(E, \mathfrak{s})$ be admissible filling data, T a finite set of finite places of E that is stable under complex conjugation, and $\mathcal{A}$ a finite set of units of E all of which are T-units; assume that $g(S)^{-1}$ is a T-integer, and let $\mathcal{A}' \subseteq \mathcal{A}$. Write $\varrho = \varrho(\mathfrak{s})$, $n = n(\mathfrak{s})$, $k = k(\mathfrak{s})$, $f = f(E)$, $b = b(S)$. Let $J_i'(p,q) \in \mathbb{C}$, for $i \in \mathbb{N}$ and sequences $p, q \colon \mathbb{N} \to E_{\mathsf{T}}^{\times}$, and $e \in \mathbb{R}$ be such that

$$\mathrm{Sieve}(u_{\mathsf{T}}, \iota_{\mathsf{T}}, \varrho, J', \psi_{S,\mathsf{T}}, e, \Lambda_{n^6}(\,\cdot\,;\,\cdot\,), \beta_{S,\mathsf{T}}(\,\cdot\,))$$

holds on E_{T}, where u_{T} replaces a non-unit by 1 and $\iota_{\mathsf{T}}(p) = g(S)^{-1}p^{-1}$. For $p \in \mathcal{A}^{\varrho}$, $t \in \mathcal{A}$ and an infinite place τ let $x_\tau(p,t)$ be the sequence of the non-zero complex numbers $\mathrm{fs}_S(\tau, a) \cdot \big((p\,|\,t)_a\big)_\tau$, $a \in \mathbb{N}$, and $y(p,t)$ the sequence of the images of the $(p\,|\,t)_a$ in $E_{\mathsf{T}}^{\times}$. Then, with J^τ the band-limit forms and $\deg_\tau(S)$ the degree of the Bessel system $\mathfrak{B}_\tau(S)$,

$$\mathrm{Re} \sum_{t \in \mathcal{A}'} \sum_{p,q \in \mathcal{A}^{\varrho}} \prod_{\tau} J_n^{\tau}(x_\tau(p,t), x_\tau(q,t)) \cdot J_n'(y(p,t), y(q,t)) \\ \leq \prod_{a<n} \frac{\mathrm{vol}(\hat{O}_E)}{\prod_\tau \mathrm{fs}_S(\tau,a)^2} \cdot \Big(e \prod_\tau \frac{1+b^{-10}}{\deg_\tau(S)} + \frac{\deg(S)^{-1}}{b^{6kf+10}} \Big).$$

LEAN `Filling.sieve`

In Theorem E, A is a finite product of non-archimedean local fields with an involution, O_A its ring of integers, and a *level* is the structure of the tower method (unit 132, recalled in Section 2).

Theorem E (the lattice level at a finite place). Let A be a semilocal field with an involution and an integral place structure, $\mathfrak{s}$ a shape, Ω a finite set and $\mathfrak{m}$ a natural number with $k(\mathfrak{s}) = |\Omega| + 2\mathfrak{m} + 1$. Let $N \in O_A$ with $\bar{N} = N$ be such that the split Gram matrix $J(T^{\mathrm{lat}}(\mathfrak{s}; N))$ is a hermitian unit, and let g be a unit of A with $g \in O_A$ and $\bar{g} = g$. Then there is a level $\mathcal{L} \in \mathrm{Lev}(\Omega, T^{\mathrm{lat}}(\mathfrak{s}; N), \mathfrak{m})$ that is a lattice level, $\mathcal{L} \in \mathrm{LatLev}$ (Definition 6.4), with unit $g_{\mathcal{L}} = g$.

LEAN `IntegralPlace.exists_isLatticeLevel`

Section 6 adds the measure identities of a standard level: the Heisenberg measures of the congruence subgroups (Proposition 6.7) and the two-sided bound $\mathrm{vol}(O_A^{\times})^{\varrho+1} \leq \mathsf{M}(\mathcal{L})(\mathrm{Box}(\mathcal{L})) \leq 1 + 2^{-n}$ for the mass of the integral group (Propositions 6.9 and 6.10).

Theorem F (the Bessel systems of admissible data). Let P be parameters, D admissible data for P, and $S(D)$ the filling data of D (Note G of this unit). Write $k = k(D)$, $f = f(D)$, $m = m(D)$, $r = r(D)$.

(a) $\#\mathrm{Bc}(S(D)) \cdot ((k-1)!)^f \le (b(D)^{5/4})^{kf}$.

(b) With $w_D(\tau)$ the nominal weight of the infinite place τ,

$$\sum_\tau m\,(k-1)\log w_D(\tau) \le \log\deg(S(D)) + f\sum_{l=k-1}^{r-1}\log l! \le \sum_\tau m\,r\left(\log w_D(\tau) + \frac{1}{x(D)}\right),$$

the sums over the f infinite places of $E(D)$.

LEAN `Data.card_besselCoord_mul_factorial_le` · `Data.log_filling_besselDegree_mem_Icc`

Theorems A and B are used by the notes on the filling theorem (Note F) and on the seesaw identity (unit 141); Theorem D is the sieve of the main-term estimate of Note F; Theorem E and the masses feed Notes T and G; Theorem F is used when the filling data of D are shown to be admissible (Note G). The file of Theorem F also holds the last step of the seesaw argument (Proposition 7.1), which belongs to unit 141 and is stated here because it is a member of this unit.

2 The setting

Conventions. E is a CM field with a fixed CM type; τ runs over its $f = f(E)$ infinite places, and $x_\tau \in \mathbb{C}$ is the image of $x \in E$ under the embedding of the CM type at τ; w is a finite place, E_w the completion, $\mathbb{F}_w$ the residue field; $\mathbb{A}_E = \mathbb{C}^f \times \mathbb{A}_{E,f}$ is the ring of adeles, $\mathbb{A}_{E,f}$ the finite adeles, $\hat{O}_E$ their integral elements; rd_E is the root discriminant. $\mathfrak{s} = (\varrho, \kappa, k_0)$ is a shape, with $\varrho(\mathfrak{s}) = \varrho$, $k(\mathfrak{s}) = |\kappa|$, $n(\mathfrak{s}) = \varrho + 1$, $r(\mathfrak{s}) = 2\varrho + |\kappa| + 1$ and the index set $I(\mathfrak{s})$ of the basis $e_0, \dots, e_{\varrho-1}$, x_j $(j \in \kappa)$, z, $e^*_{\varrho-1}, \dots, e^*_0$. In Sections 3 and 4 the index set $I(\mathfrak{s})$ carries signs, a frame for them is fixed, and S, V, l, k, $\mathcal{Y}$, η, X are as in Setting 3.2. In Section 6, A is a commutative ring with an involution $x \mapsto \bar{x}$, a topology and a Haar measure; from Definition 6.4 on it is a semilocal field; Ω is a finite set, $\mathfrak{m}$ a natural number, T an $\mathfrak{s}$-form over A whose split Gram matrix is a hermitian unit, P the place structure of A, $\mathcal{L}$ a level; and, as in the `variable` lines of the formal file, π is a unit of A, N and t are elements of A, x is a vector in A^κ, g a unit of A. The letters k, s, i, c, M are used for group elements, indices and constants as each statement says. Integrals of functions with values in $[0, \infty]$ are Lebesgue integrals; sums of such functions over infinite index sets are unconditional. Subtraction of natural numbers in exponents and indices is truncated at 0. The letters S, s, σ, q, θ, ψ have in the notes of this unit the meanings given here, not those of the overview of the write-up (there: a set of columns, the complex variable, a field automorphism, a rational number, a family of twists, a column). For the filling data $S(D)$ of admissible data D (Section 7, Note G), the primes $q_0(S(D))$, $q_1(S(D))$ are the primes that the overview calls g_0, g_1, and $g(S(D))$ is their product.

Recollections from Note S. These objects are defined in Note S of this unit, where their Lean names are given; we use its notation. A $\mathfrak{s}$-*form* over a commutative ring R with involution is a square matrix T on the indices of the x_j and of z; $J(T)$ is its split Gram matrix on $I(\mathfrak{s})$ (the e_a are paired with the e_a^*, the middle block is T), $G(T)$ the group of the invertible matrices g with $g^*J(T)g = J(T)$, and $\mathcal{U}(T)$, $\mathcal{H}(T)$, $\mathcal{S}(T)$ its Bessel unipotent subgroup, the subgroup that fixes all e_a, e_a^* and z, and the Bessel subgroup $\mathcal{S}(T) = \mathcal{U}(T)\mathcal{H}(T)$. The *floor* of a basis vector is a for e_a and for e_a^*, it is ϱ for z and $\varrho+1$ for the x_j; for $l \le \varrho$, V_l is the span of the basis vectors of floor at least l, of dimension $k+1+2(\varrho-l)$, so that $V_\varrho = \langle x_j, z\rangle$; and $h_T(v,v') = v^*J(T)v'$ is the hermitian form. The group $\mathcal{H}_l(T)$ fixes the basis vectors of floor less than l (so $\mathcal{H}_\varrho(T)$ is the unitary group of V_ϱ and $\mathcal{H}_n(T) = \mathcal{H}(T)$), $\mathcal{N}_{<l}(T)$ is the unipotent radical of the stabiliser of the first l steps of the flag $\langle e_0\rangle \subset \langle e_0, e_1\rangle \subset \cdots$, and $\mathcal{N}_l(T) = \mathcal{H}_l(T) \cap \mathcal{N}_{<l+1}(T)$. The frame is $\mathfrak{f}_0, \dots, \mathfrak{f}_\varrho$ with $\mathfrak{f}_a = e_a$ for $a < \varrho$ and $\mathfrak{f}_\varrho = z$; $\mathrm{sd}(u)$ is the sum of the entries of u at the places $(\mathfrak{f}_a, \mathfrak{f}_{a+1})$. For $l \le \varrho+1$, $\mathcal{Q}_l(T)$ is the group of the $g \in G(T)$ such that, for every $a < l$, the vector $g\mathfrak{f}_a - \mathfrak{f}_a$ lies in the span of the e_b with $b < a$; and $\mathcal{Q}_l(T) = \mathcal{U}(T)$ for $l \ge \varrho+2$. For $N \in R$, $T^{\mathrm{rat}}(\mathfrak{s}; N)$ is the form $\mathrm{diag}(1, \dots, 1, -N)$ and $T^{\mathrm{lat}}(\mathfrak{s}; N)$ its Gram matrix in the lattice basis; for a unit p of R, $\Lambda(\mathfrak{s}; p)$ is the matrix of that basis (x_{k_0} replaced by $x_{k_0}/\bar{p} + z/(p\bar{p})$). $G_1(\mathfrak{s})$ is the group of $T^{\mathrm{rat}}(\mathfrak{s}; 1)$ over $\mathbb{C}$.

Filling data $S \in \mathrm{Fill}(E, \mathfrak{s})$ consist of a prime $N(S)$, primes $q_0(S), q_1(S)$ with product $g(S)$, a finite idele $\bar{\pi}(S)$ with $\bar{\pi}(S)\,\overline{\bar{\pi}(S)} = N(S)$, a generator $\varpi(S)$ of the different with $\overline{\varpi(S)} = -\varpi(S)$, a real number $b(S)$, finite sets $\mathrm{Ch}_0(S), \mathrm{Ch}_1(S)$ of *chosen places* above the two primes, a level $K^{\mathrm{col}}(S)$ of the definite space on κ, and a Bessel system $\mathfrak{B}_\tau(S)$ at every infinite place, with the size conditions of Note S; S may be *admissible* (the standing hypotheses of Note S). From S: the forms $T_S^{\mathrm{rat}}(R) = T^{\mathrm{rat}}(\mathfrak{s}; N(S))$ and $T_S^{\mathrm{lat}}(R) = T^{\mathrm{lat}}(\mathfrak{s}; N(S))$ over a ring R; the groups $G_S(R) = G(T_S^{\mathrm{rat}}(R))$ and $G_S^{\mathrm{lat}} = G(T_S^{\mathrm{lat}}(\mathbb{A}_{E,f}))$; the lattice basis $\Lambda(S) = \Lambda(\mathfrak{s}; \bar{\pi}(S))$ and the isomorphism $\mathrm{lb}_S \colon G_S(\mathbb{A}_{E,f}) \to G_S^{\mathrm{lat}}$, $x \mapsto \Lambda(S)^{-1} x\, \Lambda(S)$; the integral subgroup $K(S)$ of G_S^{lat}, the finite group $\mathsf{G}(S) = \prod_j \prod_{w \in \mathrm{Ch}_j(S)} \mathrm{GL}_{I(\mathfrak{s})}(\mathbb{F}_w)$, the reduction $\mathrm{red}_S \colon K(S) \to \mathsf{G}(S)$ and its kernel $K_g(S)$; the diagonal map $\mathrm{r}_S^{\mathbb{A}} \colon G_S(E) \to G_S(\mathbb{A}_E)$; the map $\mathrm{a}_S^{\infty} \colon G_S(\mathbb{A}_E) \to \prod_\tau G_1(\mathfrak{s})$, conjugation of the component at τ by the diagonal matrix $\mathrm{Sc}_{S,\tau}$ of the scales; the map $\mathrm{a}_S^f \colon G_S(\mathbb{A}_E) \to G_S^{\mathrm{lat}}$, lb_S of the finite part; $\mathrm{r}_S^{\infty} = \mathrm{a}_S^{\infty} \circ \mathrm{r}_S^{\mathbb{A}}$ and $\mathrm{r}_S^f = \mathrm{a}_S^f \circ \mathrm{r}_S^{\mathbb{A}}$; the inclusion inc_S of the finite adelic unitary group $G_f(E, \kappa)$ of the definite space into G_S^{lat}; the set $\mathrm{Cl}(S)$ of the double cosets of $G_f(E, \kappa)$ for the rational points and $K^{\mathrm{col}}(S)$, with $h_S(c) = \mathrm{inc}_S(\text{the chosen representative of } c)$; the position $\mathrm{pos}_S(c, c'; \gamma) = h_S(c)^{-1} \mathrm{r}_S^f(\gamma) h_S(c')$ of a rational $\gamma \in G_S(E)$ and the set $\mathrm{Tm}_S(c, s, c')$ of the *terms over* (c, s, c'), the γ whose position lies in $K(S)$ with reduction $s \in \mathsf{G}(S)$; the arithmetic unipotent group $\mathcal{U}_{S,c}$ of the $u \in \mathcal{U}(T_S^{\mathrm{rat}}(E))$ with $h_S(c)^{-1} \mathrm{r}_S^f(u) h_S(c) \in K_g(S)$; the cutoff $\beta_{S,c} \colon \prod_\tau G_1(\mathfrak{s}) \to [0, \infty]$; the set $\mathrm{Bc}(S) = \prod_\tau I_\tau(S)$ of the tuples of coordinates of the Bessel systems, and for a side $\sigma \in \{\mathrm{full}, \mathrm{cut}\}$ the product Bessel function $W_S^\sigma(y)_i = \prod_\tau W_{S,\tau}^\sigma(y_\tau)_{i_\tau}$, $y \in \prod_\tau G_1(\mathfrak{s})$, $i \in \mathrm{Bc}(S)$, where $W_{S,\tau}^\sigma \colon G_1(\mathfrak{s}) \to \mathbb{C}^{I_\tau(S)}$ are the two functions of $\mathfrak{B}_\tau(S)$; the kernel $\mathcal{W}_S^\sigma(\gamma; y)_{(i,i')} = W_S^\sigma(y)_i\, \overline{W_S^\sigma(\mathrm{r}_S^\infty(\gamma)^{-1} y)_{i'}}$, the amplitude $a_S^\sigma(c, c'; \gamma) = \int \beta_{S,c}(y)\, \beta_{S,c'}(\mathrm{r}_S^\infty(\gamma)^{-1} y)\, \mathcal{W}_S^\sigma(\gamma; y)\, \mathrm{d}y$ and, for a set Ξ of rational elements, the amplitude sum $A_S^\sigma(\Xi; c, s, c') = \sum_{\gamma \in \Xi \cap \mathrm{Tm}_S(c,s,c')} a_S^\sigma(c, c'; \gamma)$; the additive character $\psi_S(x) = \psi_f(-x/\varpi(S))$ of $\mathbb{A}_{E,f}$, where ψ_f is the standard character; the real numbers $\kappa_\tau(S) = \mathrm{Re}(g(S)\varpi(S)_\tau/2\pi i)$ and $\mathrm{fs}_S(\tau, a) = \kappa_\tau(S)^{2(\varrho-a)}/N(S)$ for $a \in \mathbb{N}$;

the degree $\deg(S) = \prod_{\tau} \deg_{\tau}(S)$, the product of the degrees of the Bessel systems. Finally, for a real number ρ and natural numbers j, K, i, the damped power is $\mathrm{dp}_{j}(\rho) = \rho^{j} \min(1,\rho)^{6}$, and the head weights are

$$\Lambda^{\mathbb{C}}_{K}(i;p) = \prod_{l<i} \mathrm{dp}_{K}(|p_0 \cdots p_l|^{-1}) \quad (p\colon \mathbb{N} \to \mathbb{C}^{\times}),$$
$$\Lambda_{K}(i;p) = \prod_{l<i} \left([p_0 \cdots p_l]^{1/2}\right)^{K} \quad (p\colon \mathbb{N} \to A^{\times}),$$

the factor at l of the second being 0 if $p_0 \cdots p_l \notin O_A$; here, for a semilocal field A, $[\xi]$ is the generalized index (the inverse of the module of ξ; it is $[O_A : \xi O_A]$ for $\xi \in O_A$) and $\mathrm{den}(\xi) = [O_A : O_A \cap \xi^{-1} O_A]$.

Inputs from the other notes of this unit. (S1) (Note S) For admissible S: $1 \le |\kappa_{\tau}(S)| \le b(S)^{23}$ and $|\kappa_{\tau}(S)| = g(S)\,|\varpi(S)_{\tau}|/2\pi$ for every τ; $N(S) \le b(S)^{60n}$; $\prod_{\tau} c_{\mathrm{band}}(\mathfrak{s})\, b(S)^{7k} \left(|\kappa_{\tau}(S)|^{2\varrho+1}/g(S)N(S)\right)^{k-n^{7}} \le \deg(S)^{-1}$. For all filling data: $f \le b(S)/1000$, $k \le b(S)$, $\mathrm{rd}_{E} \le b(S)$, $1000 \le b(S)$, $1000\,(2\varrho+2)^{9} \le k$ and $1000\,(2\varrho+2) \le k$.
(S2) (Note S) For $a \in \mathbb{A}_{E,f}$: $\psi_{S}(ra) = 1$ for all $r \in \hat{O}_{E}$ if and only if $a \in \hat{O}_{E}$.
(S3) (Note S) For a commutative ring R with involution, a unit p and $N = p\bar{p}$: $\Lambda(\mathfrak{s};p)^{*}\, J(T^{\mathrm{rat}}(\mathfrak{s};N))\, \Lambda(\mathfrak{s};p) = J(T^{\mathrm{lat}}(\mathfrak{s};N))$. If $R' \subseteq R$ is a subring, $N \in R'$ and $\bar{N} = N$, then $T^{\mathrm{lat}}(\mathfrak{s};N)$ is unimodular over R': hermitian, with entries and inverse over R'.

Inputs from the other notes of this unit, continued. (S4) (Note S; sieve systems) Let C be a commutative ring with a measure, U a topological space, $u\colon C \to U$, $\iota\colon U \to C$, $\mathsf{n} \in \mathbb{N}$, $J_i(p,q) \in \mathbb{C}$ and $\Lambda_i(p) \in \mathbb{R}$ for $i \in \mathbb{N}$ and sequences $p, q\colon \mathbb{N} \to U$, $\psi\colon C \to \mathbb{C}$, $\mathrm{En} \in \mathbb{R}$ and $\beta\colon C \to \mathbb{R}$. The statement $\mathrm{Sieve}(u, \iota, \mathsf{n}, J, \psi, \mathrm{En}, \Lambda, \beta)$ means: $\psi(0) = 1$; for $i \le \mathsf{n}$, all p, q and $\xi \in C$ the function $c \mapsto \psi(\xi c) J_{i+1}(p[i \mapsto u(c)], q[i \mapsto u(c)])$ is integrable, with integral $J_i(p,q)$ if $\xi = 0$ and of absolute value at most $\beta(\xi)\Lambda_i(p)\Lambda_i(q)$ if ξ is a unit; for finitely many sequences with the same entry at n the matrix of the values of $J_{\mathsf{n}+1}$ is positive semidefinite; $J_{\mathsf{n}+1}$ is continuous; $|J_0| \le \mathrm{En}$; and for $i < \mathsf{n}$, if $\xi = \iota(p_i) - \iota(q_i)$ is a unit then $|J_{i+1}(p,q)| \le \beta(\xi)\Lambda_{i+1}(p)\Lambda_{i+1}(q)$. Here $p[i \mapsto x]$ is p with its entry i replaced by x. Let this hold, let $i \le \mathsf{n}$ and let $(a_p)_p$ be a finite family of sequences. (a) $F(c) = \sum_{p,q} J_{i+1}(a_p[i \mapsto u(c)], a_q[i \mapsto u(c)])$ is integrable, $\int F = \sum_{p,q} J_i(a_p, a_q)$, and $|\int \psi(\xi c) F(c)\,\mathrm{d}c| \le \beta(\xi)(\sum_p \Lambda_i(a_p))^2$ for every unit ξ. (b) If C is first countable and $(b_p(x))_p$ is a finite family of sequences depending on $x \in C$ that share their entry i for each x, then $\sum_{p,q} J_{i+1}(b_p(x), b_q(x))$ is a non-negative real number, and as a function of x it is lower semicontinuous at every point where all b_p are continuous. (c) (products) If the measure of C is s-finite and, for each infinite place τ, a system $\mathrm{Sieve}(u_{\mathbb{C}}, \iota', \mathsf{n}, J^\tau, \psi_{\mathbb{C}}, \mathrm{En}_\tau, \Lambda', \gamma_\tau)$ holds on $\mathbb{C}$, with $u_{\mathbb{C}}$ replacing 0 by 1 and $\psi_{\mathbb{C}}(c) = \exp(c - \bar{c})$, then the product system holds on $\mathbb{C}^f \times C$: with u, ι taken factor by factor, the forms, the characters, the constants En, the weights Λ and the functions β multiplied.

(S5) (Note S; band limits) For every filling data S and every τ, the Bessel system $\mathfrak{B}_\tau(S)$ carries forms J^τ with $\mathrm{Sieve}(u_{\mathbb{C}}, \xi \mapsto \xi^{-1}, \varrho, J^\tau, \psi_{\mathbb{C}}, (1 + b^{-10})/\deg_\tau(S), \Lambda^{\mathbb{C}}_{n^6}(\,\cdot\,;\,\cdot\,), \beta_{\mathbb{C}})$ on $\mathbb{C}$, where $b = b(S)$ and $\beta_{\mathbb{C}}(\xi) = c_{\mathrm{band}}(\mathfrak{s})\, b^{2000 n^9}\ \mathrm{dp}_{k - n^7}(|\xi|^{-1})$, and $c_{\mathrm{band}}(\mathfrak{s}) > 0$.

Inputs from the other notes of this unit, continued. (S6) (Note S; lattices) Let E be stably unramified. For a subspace L of E^2 let $\mathrm{disc}_1(L) > 0$ be the discriminant of the lattice $L \cap \hat{O}_E^2$ of Note S (formed with the identity matrix). Then $\sum_L \mathrm{disc}_1(L)^{-3} \le (200 \cdot 2f \cdot \mathrm{rd}_E)^{3f}$, the sum over the lines $L \subset E^2$. If $y \in E$, L is the line spanned by $(1, y)$ and $w \in \mathbb{A}_{E,f}^\times$ is such that w and yw lie in $\hat{O}_E$, then $\mathrm{disc}_1(L) \le \prod_\tau (1 + |y_\tau|^2) \cdot [w]$.

(B1) (Note B, *Box partitions of arithmetic unipotent groups: frame counts, the covolume, amplitude sums*) For admissible S, stably unramified E, a class c and $y \in \prod_\tau G_1(\mathfrak{s})$: $\sum_{u \in \mathcal{U}_{S,c}} \beta_{S,c}(\mathrm{r}_S^\infty(u)\, y) = 1$.

(B2) (Note B) For $u \in \mathcal{U}_{S,c}$, a side σ and all y, i: $W_S^\sigma(\mathrm{r}_S^\infty(u) y)_i = W_S^\sigma(y)_i$.

(B3) (Note B) Let E be stably unramified and S admissible, Ξ a set of rational elements, c, c' classes, $s \in \mathsf{G}(S)$ and σ a side. There is $M \ge 0$ with $\sum_{\gamma \in \Xi \cap \mathrm{Tm}_S(c,s,c')} \beta_{S,c'}(\mathrm{r}_S^\infty(\gamma)^{-1} y) \prod_\tau |W_{S,\tau}^\sigma((\mathrm{r}_S^\infty(\gamma)^{-1} y)_\tau)| \le M$ for all y, and $\sum_{\gamma \in \Xi \cap \mathrm{Tm}_S(c,s,c')} |a|_S^\sigma(c, c'; \gamma) < \infty$, where $|a|_S^\sigma(c, c'; \gamma)$ is the integral of the norm of the integrand of the amplitude.

(B4) (Note B) If $(g_j)_j$ is a countable family of measurable functions with values in $\mathbb{C}^I$, I finite, and $\sum_j \int |g_j| < \infty$, then each coordinate of $\sum_j \int g_j$ is the integral of the corresponding coordinate of $\sum_j g_j$.

Hermitian spaces with frame, forms and gradings. This vocabulary belongs to the background library; it is recalled as the notes of Chapters V and VII of the write-up state it. A *hermitian space with frame* V over E consists of a finite index set I, signs $\mathrm{sgn}_V\colon I \to \{\pm 1\}$, a matrix J_V over E and a frame $\mathrm{fr}_V \in \mathrm{GL}_I(\mathbb{A}_E)$ such that $\mathrm{fr}_V^* J_V \mathrm{fr}_V$ is the diagonal matrix of the signs at every infinite place and the identity at the finite places; $\mathrm{fr}_{V,f}$ is its finite part. Its groups are written in the coordinates of the frame: $G_\infty(V) = \prod_\tau \mathrm{U}(\mathrm{sgn}_V)$, $G_f(V) = G_f(E,I) = \{g \in \mathrm{GL}_I(\mathbb{A}_{E,f}) : g^* g = 1\}$, $G(V) = G_\infty(V) \times G_f(V)$, with $K_f(E,I) = G_f(E,I) \cap \mathrm{GL}_I(\hat{O}_E)$; each carries the Haar measure fixed in the library (on $G_f(V)$ the one with $\mathrm{vol}\, K_f(E,I) = 1$). The *rational coordinates* of $g \in G(V)$ are $\mathrm{fr}_V g \mathrm{fr}_V^{-1}$, an adelic point of the unitary group of J_V; the *rational points* $\Gamma(V)$ are the g whose rational coordinates have entries in E, a discrete subgroup. For a subgroup $\Delta \subseteq \Gamma(V)$, integrals over $\Delta\backslash G(V)$ are taken for the measure induced by the Haar measure (in the formal text: a measure on $G(V)$ itself, the Haar measure times a cutoff for Δ). For a function ϵ on $G_\infty(V)$, a *form of type* ϵ on V is a bounded measurable function $F\colon G(V) \to \mathbb{C}$ that is left invariant under $\Gamma(V)$, right invariant under an open subgroup of $G_f(V)$, and satisfies $\int_{G_\infty(V)} \epsilon(u) F(g \cdot (u,1))\, \mathrm{d}u = F(g)$ for all g if ϵ is integrable; the group $G_f(V)$ acts on the forms of type ϵ by right translation, $(R(k)F)(g) = F(gk)$, and the forms fixed by an open subgroup carry the Petersson product $\langle F, F' \rangle = \int_{\Gamma(V)\backslash G(V)} \overline{F} F'$. A function ϵ on $G_\infty(V)$ is an *idempotent kernel* if it is continuous and integrable, $\epsilon(g^{-1}) = \overline{\epsilon(g)}$, $\epsilon(1) \neq 0$ and $\int \epsilon(z)\epsilon((zx)^{-1}y)\, \mathrm{d}z = \epsilon(x^{-1})\epsilon(y)/\epsilon(1)$. For a family $l = (l_\tau)_\tau$ of integer vectors indexed by I, the *type* ϵ_l is the product over τ of the functions ϵ_{l_τ} on $\mathrm{U}(\mathrm{sgn}_V)$, the formal degree times the complex conjugate of the canonical matrix coefficient of the discrete series with Harish-Chandra parameter $l_\tau/2$.

Let $N \in E$ with $N \neq 0$, $\bar{N} = N$, and let I_0 be a finite set. A *grading* $\mathcal{Y}$ of V as I_0 plus a string of length λ with norm N is a bijection of $I_0 \sqcup \{0, \dots, \lambda - 1\}$ with I (and a constant that plays no role here) such that J_V is the identity on I_0, pairs the string vectors z_a, $z_{a'}$ to 1 if $a + a' = \lambda - 1$, $a \neq a'$, to $-N$ if $a = a'$ and $2a = \lambda - 1$, and to 0 otherwise, and pairs I_0 with the string to 0; the *grade* is 0 on I_0 and $\lambda - 1 - 2a$ at z_a. With matrices in rational coordinates, the *unipotent subgroup* $U(\mathcal{Y}) \subseteq G(V)$ consists of the u for which $u - 1$ strictly raises grades, the *Bessel group* $B(\mathcal{Y})$ of the points that fix every string vector of grade ≥ 0 up to vectors of strictly higher grade, and $i_{\mathcal{Y}}$ is the inclusion into $G(V)$ of the adelic group $\prod_\tau \mathrm{U}(I_0) \times G_f(E, I_0)$ of the standard definite space on I_0, acting on the span of I_0 and fixing the string. For a shape $\mathfrak{s}$, the *standard bijection* of $\kappa \sqcup \{0, \dots, 2\varrho\}$ with $I(\mathfrak{s})$ sends $j \in \kappa$ to x_j and the string to $e_0, \dots, e_{\varrho-1}, z, e^*_{\varrho-1}, \dots, e^*_0$ in this order.

Inputs from the background library and from other units. (L1) (*Poincaré series*; Lib, unit 102) Let V be a hermitian space with frame, ϵ an idempotent kernel on $G_\infty(V)$, K an open subgroup of $G_f(V)$, $c\colon G(V) \to \mathbb{C}$ right invariant under K, and $(W_q)_q$ a countable family of measurable functions on $G(V)$ with $\sum_{\gamma\in\Gamma(V)} c(\gamma g) = \sum_q W_q(g)$ for all g and $\sum_q |W_q(g)| \le M$ for all g, with a finite M. Then there is a form of type ϵ on V, fixed by K, whose value at g is $\sum_q \int_{G_\infty(V)} \epsilon(u) W_q(g\cdot(u,1))\,\mathrm{d}u$. Reference to be confirmed (the file names A. Borel, *Introduction to automorphic forms*, Proc. Sympos. Pure Math. 9 (1966), as the source for Poincaré series).
(L2) (*unfolding*; Lib, unit 114) Let G be a second countable locally compact Hausdorff group with left Haar measure, Γ a discrete subgroup and $\phi\colon G\to\mathbb{C}$ integrable. Then $g \mapsto \sum_{\gamma\in\Gamma}\phi(\gamma g)$ is integrable on $\Gamma\backslash G$, with integral $\int_G \phi$. The catalogue of the write-up names Weil's integration formula (N. Bourbaki, *Intégration*, Chap. VII), place not confirmed, of which this is a special case; the second reader derives it from the general formula for proper actions in D. Gentner and G. Last, *Palm pairs and the general mass-transport principle*, §2, (2.5)–(2.9).
(L3) (*one-sided Poisson summation*; Lib, unit 123) Let $F\colon \mathbb{A}_E \to \mathbb{R}$ be integrable and non-negative, and lower semicontinuous at every point of a finite set $\mathcal{V}$ of elements of E (embedded diagonally). If a real number Σ bounds $\sum_{\xi\in y'} \left|\int_{\mathbb{A}_E} \psi_E(-\xi x)F(x)\,\mathrm{d}x\right|$ for every finite set $y' \subset E$, where $\psi_E(x) = \exp(-4\pi i \sum_\tau \operatorname{Re} x_\tau)\psi_f(x_f)$ is the standard character, then $\sum_{t\in\mathcal{V}} F(t) \le \Sigma$. By the second reader's verdict this is a weaker form of the Poisson summation formula of J. Tate, *Fourier analysis in number fields and Hecke's zeta-functions*, Theorem 4.1.4 and Lemmas 4.2.1–4.2.4: an inequality, for a non-negative function that need not be continuous.

Inputs from the background library and from other units, continued. (L4) (Lib, units 57 and 123) (a) For $y \in E$ there is $w \in \mathbb{A}_{E,f}^{\times}$ with $\{t \in \mathbb{A}_{E,f} : t \in \hat{O}_E,\ yt \in \hat{O}_E\} = w\hat{O}_E$. (b) (*product formula*) For a non-zero T-integer y of E: $[y_{\mathsf{T}}]^{1/2} \leq \prod_{\tau} |y_{\tau}|$, where y_{T} is the image of y in E_{T}. (c) The projection $\mathbb{A}_{E,f} \to E_{\mathsf{T}}$ carries the Haar measure restricted to the open subring of the finite adeles that are integral at every place outside T to $\mathrm{vol}(\hat{O}_E)$ times the Haar measure of E_{T}; here $\mathrm{vol}(\hat{O}_E) = (2/\operatorname{rd}_E)^f$ is the volume of $\hat{O}_E$. (d) For $x \in E_{\mathsf{T}}$, extended by 0 to a finite adele x^0, and $z \in \mathbb{A}_{E,f}$ with image z_{T} in E_{T}: $z - (z_{\mathsf{T}})^0 \in \hat{O}_E$ if and only if z is integral at every place outside T. Elementary; no reference is needed.
(L5) (*finite covolume*; Lib, unit 121) For a hermitian space with frame V over E, the quotient $\Gamma(V)\backslash G(V)$ has finite volume for the measure induced by the Haar measure. The library proves this in its file on reduction theory, whose header names Mahler–Godement compactness, finite covolume and the adelic Minkowski theorems; the catalogue of the write-up has no source for this statement. Reference to be confirmed (the referee of this note names the theorem of Borel and Harish-Chandra; not looked up).
(L6) (*Haar measure in a frame*; Lib, unit 112) Let J be a hermitian matrix over $\mathbb{C}$ with $J^2 = 1$ and p an invertible matrix with $\mathsf{p}^* J \mathsf{p}$ the diagonal matrix of the signs. Then $g \mapsto \mathsf{p}^{-1} g\, \mathsf{p}$ carries the Haar measure that the library fixes on the unitary group of J to the one it fixes on $\mathrm{U}(\mathrm{sgn})$. This concerns the normalisations of the library; no reference.
(L7) (*unfolding at a finite place*; Lib, unit 123) In the situation of (R1) over a semilocal field with an integral place structure ((R3)), let g be a unit of A, $j \in \mathbb{N}$, $c \in G(T)$ and θ an s-finite measure on $G(T)$. Then, with $K_T(g)$ the congruence subgroup of (R3)(b), $(\eta_{<j} * \theta)\{y : yc \in K_T(g)\} = \eta_{<j}(K_T(g)) \cdot \theta\{y : uyc \in K_T(g) \text{ for some } u \in \mathcal{N}_{<j}(T)\}$. The catalogue marks it as an identity special to this proof that the write-up must read; it was not read for this note beyond its statement. No reference.
(L8) (Lib, unit 1) Let g be a unit of A. If $u \in \mathcal{U}(T)$, $h \in \mathcal{H}(T)$ and $uh \in K_T(g)$, then $h \in K_T(g)$. Elementary (the entries of h are entries of uh or of the identity matrix); no reference.

Inputs from the background library and from other units, continued. (L9) (*uniqueness of Haar measure for s-finite measures*; Lib, unit 114) On a second countable locally compact Hausdorff group with a Haar measure ν that is also right invariant, every s-finite right invariant measure is $d\,\nu$ for some $d \in [0,\infty]$. The catalogue names the uniqueness of Haar measure (Bourbaki, *Intégration*, Chap. VII; place not confirmed). By the second reader's verdict the relation is unresolved: the published theorem is for regular measures with a finite factor, a statement that allows an s-finite measure and $d=\infty$ was not located, and the s-finite form follows directly from Tonelli's theorem for s-finite measures. (L10) (*Hermite form*; Lib, unit 120) Let A be a semilocal field. For a matrix M over A with rows indexed by a finite set I' and columns indexed by Ω, and an injection of I' into Ω, there is $V \in \mathrm{GL}_\Omega(O_A)$ such that the columns of MV outside the image of the injection vanish. Elementary (column operations over a product of discrete valuation rings); no reference.

Inputs from the background library and from other units, continued. (R1) (*the tower*; unit 132, recalled from its tower note, *The tower method at a place: floor measures, gauges and the bounds on each floor*) For a place A and an $\mathfrak{s}$-form T with $J(T)$ a hermitian unit: a *hyperbolic basis* $F \in \mathrm{Hyp}(\Omega, T, \mathfrak{m})$ is a matrix whose columns form a basis of $V_\varrho = \langle x_j, z\rangle$ made of $\mathfrak{m}+1$ hyperbolic pairs and of orthonormal vectors indexed by Ω, with $k+1 = |\Omega| + 2\mathfrak{m} + 2$ and $\mathrm{mod}_A(\det T) = 1$; $\delta^{(b)}(F)$ is the measure on the quadric $\{v \in V_b : h_T(v,v) = h_T(\mathfrak{f}_b, \mathfrak{f}_b)\}$ that is the image of the flat chart measure of the library under the basis of V_b built from F; the Haar measure of V_ϱ disintegrates along $v \mapsto h_T(v,v)$ into such quadric measures, with fibre integrals of continuous compactly supported functions that are continuous in the fibre. A *level* $\mathcal{L} \in \mathrm{Lev}(\Omega, T, \mathfrak{m})$ consists of a hyperbolic basis $E(\mathcal{L})$, two s-finite measures $\theta_\varrho(\mathcal{L})$, $\theta_n(\mathcal{L})$ on $G(T)$, the second non-zero, a unit $g_{\mathcal{L}} = \overline{g_{\mathcal{L}}}$, an open set $\mathfrak{B}(\mathcal{L}) \ni 0$ of vectors with compact closure, a compact set $\mathrm{Box}(\mathcal{L}) \subseteq G(T)$ and a measurable set $\mathrm{Posn}(\mathcal{L}) \subseteq G(T)$, such that $\theta_\varrho(\mathcal{L})$ is carried by $\mathcal{H}_\varrho(T)$ with image $\delta^{(\varrho)}(\mathcal{L})$ under $\sigma \mapsto \sigma^{-1}z$, and $\theta_n(\mathcal{L})$ is carried by $\mathcal{H}(T)$, invariant under right translations by $\mathcal{H}(T)$, and gives mass at most 1 to $\{h : u\,h\,g \in \mathrm{Box}(\mathcal{L}) \text{ for some } u \in \mathcal{U}(T)\}$ for every g. For $l < \varrho$, η_l is the Heisenberg measure, the image of $\mathrm{d}v\,\mathrm{d}\beta$ on $V_{l+1} \times A^-$ (A^- the skew elements with their measure ν_A) under the parametrisation $(v,\beta) \mapsto \mathrm{n}_l(v,\beta)$ of $\mathcal{N}_l(T)$, and $\theta_l(\mathcal{L}) = \varpi_l * \eta_l$ with ϖ_l the image of $\mathrm{mod}_A(c)^{k+1+2(\varrho-l-1)}\mathrm{d}c$ under $c \mapsto w_l^\circ(c)^{-1}$, the inverse of the Weyl element of the pair (e_l, e_l^*); $\eta_l = \delta_1$ for $l \geq \varrho$, $\eta_{<j} = \eta_0 * \cdots * \eta_{j-1}$, and

$$\mu^{\mathcal{L}}_{a,i} = (\eta_{a+i-1} * \theta_{a+i-1}(\mathcal{L})) * \cdots * (\eta_a * \theta_a(\mathcal{L})),$$
$$\mathsf{M}(\mathcal{L}) = \mu^{\mathcal{L}}_{0,\varrho+2}, \qquad \mathsf{M}_{\geq l}(\mathcal{L}) = \theta_n(\mathcal{L}) * \cdots * \theta_l(\mathcal{L}).$$

The measure $\mathsf{M}(\mathcal{L})$ is right invariant and non-zero. A point g is *near at* j, $g \in \mathrm{Near}_j(\mathcal{L})$, if $\xi g \in \mathrm{Box}(\mathcal{L})$ for some $\xi \in \mathcal{Q}_j(T)$; $\mathsf{d}_l(\mathcal{L}; g) = \mathrm{mod}_A(g)^{k+1+2(\varrho-(l+1))}\,\mathrm{mod}_A(g)^{1/2}$ for a unit g; $\mathsf{s}(\mathcal{L}) = |\Omega| - (3n+1+\mathfrak{m})$. A *gauge* $\Gamma \in \mathrm{Gau}(\mathcal{L})$ consists of sets $\Gamma_0, \Gamma_1, \cdots \subseteq A$ and a constant $\Lambda(\Gamma)$ subject to the axioms (G1)–(G12) listed in the proof of Lemma 6.6.

Inputs from the background library and from other units, continued. (R2) (unit 132, the tower note) Let $\mathcal{L} \in \mathrm{Lev}(\Omega, T, \mathfrak{m})$. (a) $\mathsf{M}(\mathcal{L}) = \eta_{<\varrho+2} * \mathsf{M}_{\geq 0}(\mathcal{L})$. (b) Let $\Gamma \in \mathrm{Gau}(\mathcal{L})$, u a unit of A, $a < \varrho$, $(v, \beta) \in V_{a+1} \times A^-$ and $i \in \mathbb{N}$. If for some $\nu \in \mathcal{N}_{<a}(T)$ the column at e_a^* of the matrix $u^{-1}(\nu\,\mathrm{n}_a(v,\beta) - 1)$ has all its coordinates in Γ_i, then the coordinates of $u^{-1}v$ lie in Γ_i and $u^{-1}(\beta - \tau_A h_T(v,v)) \in \Gamma_i$. (c) Let $\Gamma \in \mathrm{Gau}(\mathcal{L})$ and $M' \in [0,\infty]$, and suppose that for every $b \leq \varrho$ the flat quadric measure on the hyperbolic planes of the chart of V_b gives the vectors with all coordinates in Γ_{12} mass at most M'; these planes are the $\mathfrak{m} + \varrho - b$ hyperbolic pairs of the chart and the plane of its two coordinates (t, β), so there are $\mathfrak{m} + \varrho - b + 1$ of them (in the count of (R3)(c): $|\Omega'| + 1$ planes with $|\Omega'| = \mathfrak{m} + \varrho - b$). Then $\mathsf{M}_{\geq l}(\mathcal{L})\{g : gc \in \mathrm{Near}_{\varrho+2}(\mathcal{L})\} \leq \left(M' \,\mathrm{vol}(\Gamma_{11})^{|\Omega|}\right)^{n-l}$ for all l and all $c \in G(T)$; here $\mathcal{Q}_{\varrho+2}(T) = \mathcal{U}(T)$. (Three lemmas of the tower note combined: the fibre bound of the floor quadrics from the cone bound, the masses of the floors, and the induction over the floors.)

Inputs from the background library and from other units, continued. (R3) (*finite places*; Lib, unit 123) A *semilocal field* is a finite product of non-archimedean local fields, with ring of integers O_A (compact, open, of volume 1). An *integral place structure* on a semilocal field A with involution is a place structure in the sense of unit 132 (module mod_A, character ψ_A, an element τ_A with $\tau_A + \bar{\tau}_A = 1$, a skew unit δ_A, the measure ν_A on the skew elements A^-) with $\tau_A, \delta_A, \delta_A^{-1} \in O_A$, Plancherel constant 1, $\nu_A(A^- \cap O_A) = 1$, O_A self-dual for ψ_A, $\mathrm{mod}_A(x) = [x]^{-1}$, and with the property that O_A contains an element c with $c\bar{c} = -1$. Then: (a) $\mathrm{vol}\{x : ux \in O_A\} = \mathrm{mod}_A(u)^{-1}$ and, for $\bar{u} = u$, $\nu_A\{\beta : u\beta \in O_A\} = \mathrm{mod}_A(u)^{-1/2}$, for every unit u; $\mathrm{mod}_A(u) \leq 1$ for $u \in O_A$; $\mathrm{mod}_A(\det M) = 1$ for $M \in \mathrm{GL}(O_A)$. (b) For a hermitian unit J the groups $K_J(g)$ of the integral isometries congruent to 1 modulo a unit g are open, and $K_J(1)$ is compact. (c) (*cone bound*) If $\zeta_A(\mu) \leq 2$ and $\mu + 1 \leq |\Omega'|$, then on $|\Omega'| + 1$ hyperbolic planes the flat quadric measure of the vectors v with βv integral is at most $\zeta_A(\mu)\,\mathrm{mod}_A(\beta)^{-(2|\Omega'|+1)}$, for every unit β; here $\zeta_A(\mu) = \sum_I [O_A : I]^{-\mu}$ over the ideals of finite index. (d) (*content*) With $\mathrm{ct}(p)$ the volume of $\{c \in A : cp \text{ integral}\}$ for a vector p one of whose marked entries is a unit: if $\zeta_A(\mu) \leq 2$, the integral of the s-th power of the content of $s + j$ marked coordinates of M^*w, $j \geq \mu$, $M \in \mathrm{GL}_{\Omega'}(O_A)$, over the w with βw integral is at most $\zeta_A(\mu)\,\mathrm{mod}_A(\beta)^{-(|\Omega'|-s)}$. (e) (*Eichler elements*) Let $J = J(T)$, and let x be a vector with $h_T(x,\cdot)$ the z-coordinate and z-coordinate 0. For a vector e with z-coordinate β put $v = e - z$ and $\varsigma' = h_T(v,e) - \tau_A(h_T(e,e) - h_T(z,z))$. Then $\mathrm{E}(e) = 1 + \beta^{-1}(\beta x\, h_T(v,\cdot) - (v + \varsigma' x)\, h_T(x,\cdot))$, with β^{-1} read as 0 for a non-unit, lies in $G(T)$; it lies in $\mathcal{H}_\varrho(T)$ if $e \in V_\varrho$; and $\mathrm{E}(e)^{-1}z = e$ if $h_T(e,e) = h_T(z,z)$ and β is a unit. No reference: these are computations at a finite place; the catalogue marks (c) as a local density bound that the write-up must read.

Inputs from the background library and from other units, continued. (R4) (unit 133, *DiscreteSeries*) A Bessel system $\mathfrak{B}$ of $\mathfrak{s}$ *has type* (l, k), for a Harish-Chandra parameter l of the signs and a weight k of $\mathrm{U}(\kappa)$, if (i) its degree is Weyl's number $\prod_{l_j<l_i} \frac{l_i-l_j}{2} \big/ \prod_{i<r} i!$; (ii) there is an invertible matrix A from the coordinates of $\mathfrak{B}$ to the basis of the irreducible representation $\mathbb{S}_{\mathsf{k}}$ of $\mathrm{GL}_\kappa(\mathbb{C})$ of highest weight k (the Weyl coordinates) with $\mathsf{A}\, W^{\mathrm{full}}(hy) = \mathbb{S}_{\mathsf{k}}(h^{\mathsf{t}})^{\mathsf{t}}\, \mathsf{A}\, W^{\mathrm{full}}(y)$ for all $h \in \mathrm{U}(\kappa)$, embedded in $G_1(\mathfrak{s})$, and all y; (iii) $\int \epsilon_l(x) W^{\mathrm{full}}(y\, \mathsf{p} x \mathsf{p}^{-1})\, \mathrm{d}x = W^{\mathrm{full}}(y)$, the integral over the unitary group of the signs, with p the fixed frame.
(D1) (units 141, 145, 147, 152; the data) For admissible data D for parameters P: (a) (unit 141) the filling data $S(D)$ fit the space $Y_0(D)$ with the parameter $\mathrm{par}^W_{D,\tau}(\cdot)$, the weights of the columns, its grading and a central element, in the sense of Definition 3.1; the shape of $S(D)$ has $|\kappa| = k(D)$, and the field is $E(D)$ with $f(D)$ infinite places. (b) (unit 141, Chapter V) The *row parameter*, written as in the note that defines it, is $\mathrm{par}_{D,\tau}(i) = k(D)/2 - 1 - i$ for $i + 1 < k(D)$ and $\mathrm{par}_{D,\tau}(i) = h_2(D) - w_{D,i+2-k(D)}(\tau)$ otherwise, and $\mathrm{par}^W_{D,\tau}(j) = 2\,\mathrm{par}_{D,\tau}(\nu(j))$ for a fixed numbering ν of $\underline{R}(D)$ by $\{0, \dots, r(D)-1\}$. (c) (Chapter III) $m(D) \geq 1$, $r(D) \geq 1$, $k(D) = r(D) + 1 - m(D)$ is even, $h_2(D) = r(D)^8$, $h(D) = 2h_2(D)$, and $w_{D,j}(\tau) = b_* + j\, h(D) + h_2(D)$ with a natural number b_* that depends on τ only; $w_D(\tau) \leq w_{D,j}(\tau) - h_2(D)$ for all j; $\log w_{D,j}(\tau) \leq \log w_D(\tau) + 1/x(D)$ for $j \leq 2m(D)$; $w_D(\tau) > 0$; and $(k(D) + a(D))\, k(D) \leq b(D)^{5/4}$ (a comparison of sizes by the calculus of Chapter III). (d) (unit 145) The weight of the columns at τ is that of the symmetric power of order a_τ of the standard representation, with $a_\tau \in \{0, a(D)\}$.

Inputs from the background library and from other units, continued. (D2) (the four inputs of Proposition 7.1; none has a published source in the catalogue, and they were read as statements only) Let $M = m(D)$. (a) (Lib, unit 102, with units 140, 141 and 147) The space $Y_M(D)$ is standard, and the parameter of its type is the Harish-Chandra parameter of the weight of its algebraic modular forms. For such a space the library has a linear isomorphism alg, $F \mapsto \varphi_F$, from the forms of that type to the smooth algebraic modular forms of that weight; by definition $\iota_{D,\vartheta} = \mathrm{alg} \circ \mathcal{L}^{(M)}(D, \vartheta)$. (b) (units 145 and 147) The *restriction* res_D is the linear map from $\mathrm{St}_M(D)$ to the algebraic modular forms on $Y_M(D)$ that restricts a form to the block diagonal points $\mathrm{diag}(h, 1)$, applies a fixed branching matrix and multiplies by a non-zero constant, a power of the conjugates of the level element at each infinite place; by definition $B^{\Psi}_{D,\vartheta} = \mathrm{res}_D \circ \mathrm{C}^{(M)}_{D,\vartheta}[\Psi]$ on the columns. (c) (*the branching period*; Lib, unit 27) Let V be a standard space and let the matrix B span the branching maps between two weights. For every linear functional t and every vector v there is a complex number C such that for every smooth algebraic modular form ψ of the larger weight on the space with one more basis vector, every automorphic function Φ on $G(V)$ of the form $\Phi(x, g) = L_g(\rho(x)v)$, with ρ the representation of $G_\infty(V)$ that the library attaches to the smaller weight and linear functionals L_g, and the level $K = K_f(E, I)$: $\int_{\Gamma(V)\backslash G(V)} \Phi(h)\, \langle \mathsf{t}, \psi \rangle(\mathrm{diag}(h, 1))\, \mathrm{d}h \;=\; C \int_{\Gamma\backslash G_f(V)} L_h(\mathsf{B}\, \psi(\mathrm{diag}(h, 1)))\, \mathrm{d}h$, the last integral for the measure with $\mathrm{vol}\, K = 1$. (d) (*Schur orthogonality*; Lib, unit 102) On a standard definite space V, for a smooth algebraic modular form φ whose weight has the representation ρ and the highest weight vector v^+, and an automorphic Φ with $\Phi(x, g) = L_g(\rho(x)v^+)$: $\int_{\Gamma(V)\backslash G(V)} \Phi \cdot F_\varphi = (\dim \rho)^{-1} \int_{\Gamma\backslash G_f(V)} L_g(\varphi(g))\, \mathrm{d}g$, where F_φ is the form with $\mathrm{alg}(F_\varphi) = \varphi$; and the complex conjugate of a form F_φ is a function Φ of this kind.

3 Poincaré series of cut Bessel functions

Definition 3.1 (filling data that fit a space; the principal level)**.** Let E be a CM field, $\mathfrak{s}$ a shape, and let the index set $I(\mathfrak{s})$ carry signs with a frame p: a unitary matrix with $\mathsf{p}^* J(T^{\mathrm{rat}}(\mathfrak{s}; 1))\mathsf{p}$ the diagonal matrix of the signs. Let $S \in \mathrm{Fill}(E, \mathfrak{s})$, let V be a hermitian space with frame over E with index set $I(\mathfrak{s})$ and these signs, let $l = (l_\tau)_\tau$ be a family of Harish-Chandra parameters for the signs and $\mathsf{k} = (\mathsf{k}_\tau)_\tau$ a family of weights of $\mathrm{U}(\kappa)$, let $\mathcal{Y}$ be a grading of V as κ plus a string of length $2\varrho + 1$ with norm $N(S)$, and let $\eta \in G_f(E, \kappa)$. The statement $\mathrm{Fits}(S; V, l, \mathsf{k}, \mathcal{Y}, \eta)$, in words *$S$ fits V with* $(l, \mathsf{k}, \mathcal{Y}, \eta)$, means:

- (F1) for every τ the Bessel system $\mathfrak{B}_\tau(S)$ has type $(l_\tau, \mathsf{k}_\tau)$ in the sense of (R4);
- (F2) the bijection of $\mathcal{Y}$ is the standard bijection of $\mathfrak{s}$;
- (F3) the type ϵ_l is an idempotent kernel on $G_\infty(V)$;
- (F4) there are a matrix $k_0 \in \mathrm{GL}_{I(\mathfrak{s})}(\hat{O}_E)$ whose reduction at every chosen place of S is 1 and a finite idele u with $\mathrm{fr}_{V,f} = u \cdot \dot{\eta} \cdot \Lambda(S) \cdot k_0$, where $\dot{\eta}$ acts by η on the span of the x_j and fixes the other basis vectors;
- (F5) η commutes with every element of $G_f(E, \kappa)$.

For $S \in \mathrm{Fill}(E, \mathfrak{s})$, the *principal level at the chosen places* $K_g^{\mathrm{std}}(S)$ is the open subgroup of $G_f(E, I(\mathfrak{s}))$ of the elements of $K_f(E, I(\mathfrak{s}))$ whose reduction at every chosen place $w \in \mathrm{Ch}_0(S) \cup \mathrm{Ch}_1(S)$ is 1.

LEAN `Filling.Fits` · `Filling.stdPrincipalLevel`

Setting 3.2 (filling data that fit a space)**.** E is a CM field, $\mathfrak{s}$ a shape, the index set $I(\mathfrak{s})$ carries signs with a frame p, and S, V, l, k, $\mathcal{Y}$, η are as in Definition 3.1, such that S fits V: $\mathrm{Fits}(S; V, l, \mathsf{k}, \mathcal{Y}, \eta)$ holds. The letter X stands for this hypothesis; the objects built from it carry X as an index. We write $g = (g_\infty, g_f)$ for the points of $G(V)$.

Definition 3.3 (the four dictionaries)**.** In Setting 3.2:

(a) $\mathrm{rat}_X \colon G_S(E) \to \Gamma(V)$ is the isomorphism that sends γ to the point with rational coordinates γ, that is $\mathrm{fr}_V^{-1} \gamma \, \mathrm{fr}_V$;
(b) $\mathrm{ad}_X \colon G(V) \to G_S(\mathbb{A}_E)$ is the isomorphism $g \mapsto \mathrm{fr}_V g \, \mathrm{fr}_V^{-1}$, the rational coordinates;
(c) $\mathrm{re}_X \colon G_\infty(V) \to \prod_\tau G_1(\mathfrak{s})$ is the homeomorphism $\mathrm{re}_X(g)_\tau = \mathrm{Sc}_{S,\tau}^{-1} \, \mathrm{fr}_{V,\tau} \, g_\tau \, \mathsf{p}^{-1}$;
(d) $\mathrm{fin}_X \colon G_f(E, I(\mathfrak{s})) \to G_S^{\mathrm{lat}}$ is the isomorphism $g \mapsto \mathrm{inc}_S(\eta)^{-1} \cdot \mathrm{lb}_S(\mathrm{fr}_{V,f} \, g \, \mathrm{fr}_{V,f}^{-1}) \cdot \mathrm{inc}_S(\eta)$.

They are well defined by Lemma 3.4(a).

LEAN `Filling.Fits.ratEquiv` · `Filling.Fits.adelicEquiv` · `Filling.Fits.realHomeo` · `Filling.Fits.finiteEquiv`

Lemma 3.4 (the dictionary)**.** *In Setting 3.2:*

(a) $J_V = J(T_S^{\mathrm{rat}}(E))$*;*
(b) $\mathrm{ad}_X(\mathrm{rat}_X(\gamma)) = \mathrm{r}_S^{\mathbb{A}}(\gamma)$ *for* $\gamma \in G_S(E)$*;*
(c) $\mathrm{re}_X((gg')_\infty) = \mathrm{a}_S^\infty(\mathrm{ad}_X(g)) \cdot \mathrm{re}_X(g'_\infty)$ *for* $g, g' \in G(V)$*;*
(d) $\mathrm{fin}_X(g_f) = \mathrm{inc}_S(\eta)^{-1} \cdot \mathrm{a}_S^f(\mathrm{ad}_X(g)) \cdot \mathrm{inc}_S(\eta)$ *for* $g \in G(V)$*;*
(e) for $k \in G_f(E, I(\mathfrak{s}))$ *and* $s \in \mathsf{G}(S)$*:* $\mathrm{fin}_X(k)$ *lies in* $K(S)$ *and has reduction* $\mathrm{red}_S(\mathrm{fin}_X(k)) = s$ *if and only if* k *lies in* $K_f(E, I(\mathfrak{s}))$ *and its reduction at every chosen place* $w \in \mathrm{Ch}_j(S)$ *is* $s_{j,w}$*;*
(f) for $k \in G_f(E, I(\mathfrak{s}))$*:* $k \in K_g^{\mathrm{std}}(S)$ *if and only if* $\mathrm{fin}_X(k) \in K_g(S)$*.*

Proof. (a) By (F2) the Gram matrix J_V is the Gram matrix of a grading for the standard bijection, with norm $N(S)$: the identity on the x_j, the string vector at position a paired with the one at position $2\varrho - a$ to 1, and the middle one, z, with itself to $-N(S)$. Positions a and $2\varrho - a$ are e_a and e_a^*, so this is $J(T^{\mathrm{rat}}(\mathfrak{s}; N(S)))$. (b) holds by the definition of rat_X. (c) At τ the left side is $\mathrm{Sc}_{S,\tau}^{-1} \mathrm{fr}_{V,\tau} g_\tau g'_\tau \mathsf{p}^{-1}$, which is the product of $\mathrm{Sc}_{S,\tau}^{-1}(\mathrm{fr}_{V,\tau} g_\tau \mathrm{fr}_{V,\tau}^{-1}) \mathrm{Sc}_{S,\tau}$, the component of $\mathrm{a}_S^\infty(\mathrm{ad}_X(g))$, and of $\mathrm{re}_X(g'_\infty)_\tau$. (d) a_S^f is lb_S of the finite part. (e) With k_0, u as in (F4), and since $\mathrm{inc}_S(\eta) = \Lambda(S)^{-1} \dot\eta \, \Lambda(S)$, the scalar u and the factors $\dot\eta$, $\Lambda(S)$ cancel:

$$\mathrm{fin}_X(k) = \Lambda(S)^{-1} \dot\eta^{-1} \Lambda(S) \cdot \Lambda(S)^{-1} (u \, \dot\eta \, \Lambda(S) k_0 \, k \, k_0^{-1} \Lambda(S)^{-1} \dot\eta^{-1} u^{-1}) \Lambda(S) \cdot \Lambda(S)^{-1} \dot\eta \, \Lambda(S) = k_0 \, k \, k_0^{-1}.$$

Since $k_0 \in \mathrm{GL}(\hat{O}_E)$, the matrix $k_0 k k_0^{-1}$ is integral with integral inverse exactly when k is, and since k_0 reduces to 1 at the chosen places, the two have the same reductions there. (f) is (e) for $s = 1$, by the definitions of the two levels. □

LEAN `Filling.Fits.J_eq_ratGram` · `Filling.Fits.adelicEquiv_ratEquiv` · `Filling.Fits.realHomeo_mul` · `Filling.Fits.finiteEquiv_eq_adelicToFin` · `Filling.Fits.reduction_finiteEquiv_iff` · `Filling.Fits.mem_stdPrincipalLevel_iff`

Definition 3.5 (cut Bessel functions and Poincaré series)**.** In Setting 3.2, for a pair $a = (c, i) \in \mathrm{Cl}(S) \times \mathrm{Bc}(S)$ and $g \in G(V)$, with $y = \mathrm{re}_X(g_\infty)$,

$$\mathrm{cb}_X(a; g) = \begin{cases} \beta_{S,c}(y) \cdot W_S^{\mathrm{full}}(y)_i & \text{if } \mathrm{fin}_X^{-1}(h_S(c)^{-1}\mathrm{inc}_S(\eta)) \cdot g_f \in K_g^{\mathrm{std}}(S), \\ 0 & \text{otherwise,} \end{cases}$$

where the value of the cutoff is read as 0 where it is infinite. For a coefficient vector $\phi \in \mathbb{C}^{\mathrm{Cl}(S)\times\mathrm{Bc}(S)}$, a subgroup Δ of $G(V)$ and $g \in G(V)$, the *Poincaré series* is

$$\mathrm{P}_X(\phi, \Delta)(g) = \sum_a \phi_a \sum_{\gamma\in\Delta} \mathrm{cb}_X(a; \gamma g),$$

the sum over the pairs a; an inner sum that does not converge unconditionally is read as 0.

LEAN `Filling.Fits.cutBessel` · `Filling.Fits.poincare`

Lemma 3.6. *In Setting 3.2, let $a = (c, i)$ be a pair in $\mathrm{Cl}(S) \times \mathrm{Bc}(S)$, $g \in G(V)$ and $\gamma \in G_S(E)$. Put $y = \mathrm{r}_S^\infty(\gamma) \cdot \mathrm{re}_X(g_\infty)$. Then $\mathrm{cb}_X(a; \mathrm{rat}_X(\gamma)\, g) = \beta_{S,c}(y)\, W_S^{\mathrm{full}}(y)_i$ if $h_S(c)^{-1}\, \mathrm{r}_S^f(\gamma)\, \mathrm{inc}_S(\eta)\, \mathrm{fin}_X(g_f) \in K_g(S)$, and $\mathrm{cb}_X(a; \mathrm{rat}_X(\gamma)\, g) = 0$ otherwise.*

Proof. By Lemma 3.4(f) and since fin_X is a homomorphism, the condition in the definition of $\mathrm{cb}_X(a; g)$ reads $h_S(c)^{-1}\mathrm{inc}_S(\eta)\, \mathrm{fin}_X(g_f) \in K_g(S)$. Replace g by $\mathrm{rat}_X(\gamma)g$: by (d) and (b) of the lemma, $\mathrm{fin}_X((\mathrm{rat}_X(\gamma)g)_f) = \mathrm{inc}_S(\eta)^{-1}\mathrm{r}_S^f(\gamma)\mathrm{inc}_S(\eta)\, \mathrm{fin}_X(g_f)$, and by (c) and (b), $\mathrm{re}_X((\mathrm{rat}_X(\gamma)g)_\infty) = \mathrm{r}_S^\infty(\gamma)\, \mathrm{re}_X(g_\infty)$. □

LEAN `Filling.Fits.cutBessel_ratEquiv_mul`

Lemma 3.7. *In Setting 3.2, let E be stably unramified and S admissible, let $a = (c, i)$ be a pair in $\mathrm{Cl}(S) \times \mathrm{Bc}(S)$ and $g \in G(V)$. Then*

$$\sum_{u\in\mathcal{U}_{S,c}} \mathrm{cb}_X(a; \mathrm{rat}_X(u)\, g) = \begin{cases} W_S^{\mathrm{full}}(\mathrm{re}_X(g_\infty))_i & \text{if } h_S(c)^{-1}\, \mathrm{inc}_S(\eta)\, \mathrm{fin}_X(g_f) \in K_g(S), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $y = \mathrm{re}_X(g_\infty)$ and $u \in \mathcal{U}_{S,c}$. By the definition of $\mathcal{U}_{S,c}$, $h_S(c)^{-1}\mathrm{r}_S^f(u)h_S(c) \in K_g(S)$, so the condition of Lemma 3.6 for u,

$$(h_S(c)^{-1}\mathrm{r}_S^f(u)h_S(c)) \cdot h_S(c)^{-1}\mathrm{inc}_S(\eta)\mathrm{fin}_X(g_f) \in K_g(S),$$

does not depend on u. When it holds, the term of u is $\beta_{S,c}(\mathrm{r}_S^\infty(u)y)\, W_S^{\mathrm{full}}(\mathrm{r}_S^\infty(u)y)_i = \beta_{S,c}(\mathrm{r}_S^\infty(u)y)\, W_S^{\mathrm{full}}(y)_i$ by (B2), and the cutoffs sum to 1 over u by (B1). □

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Proof of Theorem A(a). Let M be the constant of (B3) for $\Xi = G_S(E)$, the class c in both places, $s = 1$ and the full side. Fix g and put $y_0 = \mathrm{re}_X(g_\infty)$. If no $\delta \in G_S(E)$ satisfies the condition $h_S(c)^{-1}\mathrm{r}_S^f(\delta)\mathrm{inc}_S(\eta)\mathrm{fin}_X(g_f) \in K_g(S)$ of Lemma 3.6, all terms

$\mathrm{cb}_X(a;\gamma g)$ vanish. Otherwise fix such a δ, call $\mathsf{m}_0 \in K_g(S)$ that element, and put $y = \mathrm{r}_S^\infty(\delta)y_0$. Every element of $\Gamma(V)$ is $\mathrm{rat}_X(\gamma^{-1}\delta)$ for a unique $\gamma \in G_S(E)$, and by Lemma 3.6 its term is $\beta_{S,c}(\mathrm{r}_S^\infty(\gamma)^{-1}y)\, W_S^{\mathrm{full}}(\mathrm{r}_S^\infty(\gamma)^{-1}y)_i$ if $h_S(c)^{-1}\mathrm{r}_S^f(\gamma)^{-1}h_S(c)\cdot \mathsf{m}_0 \in K_g(S)$, and 0 otherwise. The condition says that $\mathrm{pos}_S(c,c;\gamma) \in K_g(S)$, that is, $\gamma \in \mathrm{Tm}_S(c,1,c)$. Since a coordinate of a vector is bounded by its norm, $|W_S^{\mathrm{full}}(y')_i| \le \prod_\tau |W_{S,\tau}^{\mathrm{full}}(y'_\tau)|$. So the sum is at most the sum bounded by M in (B3). □

Proposition 3.8 (unfolding)**.** *In Setting 3.2, let E be stably unramified and S admissible. Let $\phi \in \mathbb{C}^{\mathrm{Cl}(S)\times \mathrm{Bc}(S)}$, let $\Delta \subseteq \Gamma(V)$ be a subgroup, q a function on $G_\infty(V)$ and z a form of type q on V. Then the function $g \mapsto z(g)\,\mathrm{P}_X(\phi,\Delta)(g)$ is integrable on $\Delta\backslash G(V)$, and*

$$\int_{\Delta\backslash G(V)} z(g)\,\mathrm{P}_X(\phi,\Delta)(g)\,\mathrm{d}g = \sum_a \phi_a \int_{G(V)} z(g)\,\mathrm{cb}_X(a;g)\,\mathrm{d}g.$$

Proof. Fix a pair a. The function $\mathrm{cb}_X(a;\cdot)$ is measurable: it is the product of measurable functions of g_∞ with the indicator of the preimage of the open and closed set $K_g^{\mathrm{std}}(S)$. It is integrable on $G(V)$: by (L2) for non-negative functions and Theorem A(a), $\int_{G(V)} |\mathrm{cb}_X(a;g)|\,\mathrm{d}g = \int_{\Gamma(V)\backslash G(V)} \sum_\gamma |\mathrm{cb}_X(a;\gamma g)|\,\mathrm{d}g \le M\cdot \mathrm{vol}(\Gamma(V)\backslash G(V))$, which is finite because the quotient has finite volume ((L5)). As z is bounded and measurable, $z\cdot \mathrm{cb}_X(a;\cdot)$ is integrable, and (L2) for the discrete subgroup Δ shows that $g \mapsto \sum_{\delta\in\Delta} z(\delta g)\mathrm{cb}_X(a;\delta g)$ is integrable on $\Delta\backslash G(V)$ with integral $\int_{G(V)} z\cdot \mathrm{cb}_X(a;\cdot)$. Since $z(\delta g) = z(g)$ for $\delta \in \Gamma(V)$, this function is $z(g)\sum_\delta \mathrm{cb}_X(a;\delta g)$. Multiply by ϕ_a and sum over the finitely many a. □

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Proof of Theorem A(b). Fix $a = (c,i)$. Every $\gamma' \in \Gamma(V)$ is $\mathrm{rat}_X(u\,\gamma_q^{-1})$ for a unique coset $q \in G_S(E)/\mathcal{U}_{S,c}$, with its chosen representative γ_q, and a unique $u \in \mathcal{U}_{S,c}$. By Theorem A(a) the sum over $\Gamma(V)$ converges absolutely, so it may be grouped:

$$\sum_{\gamma'\in\Gamma(V)} \mathrm{cb}_X(a;\gamma' g) = \sum_q W_q(g), \qquad \sum_q |W_q(g)| \le M,$$
$$W_q(g) = \sum_{u\in\mathcal{U}_{S,c}} \mathrm{cb}_X(a;\mathrm{rat}_X(u)\,\mathrm{rat}_X(\gamma_q)^{-1}g).$$

The function $\mathrm{cb}_X(a;\cdot)$ is right invariant under $K_g^{\mathrm{std}}(S)$, because the condition on g_f is. By (F3) and (L1) there is a form F_a of type ϵ_l on V, fixed by $K_g^{\mathrm{std}}(S)$, with $F_a(g) = \sum_q \int \epsilon_l(x) W_q(g\cdot(x,1))\,\mathrm{d}x$. We claim that each integral is $W_q(g)$. By Lemma 3.7, $W_q(g)$ is $W_S^{\mathrm{full}}(\mathrm{re}_X(g'_\infty))_i$ or 0, according to a condition on g'_f, where $g' = \mathrm{rat}_X(\gamma_q)^{-1}g$. Replacing g by $g\cdot(x,1)$ leaves g'_f unchanged and multiplies $\mathrm{re}_X(g'_\infty)_\tau$ on the right by $\mathsf{p}x_\tau\mathsf{p}^{-1}$. As ϵ_l and the Bessel function are products over τ, the integral is the product over τ of $\int \epsilon_{l_\tau}(x)\, W_{S,\tau}^{\mathrm{full}}(y_\tau \mathsf{p}x\mathsf{p}^{-1})_{i_\tau}\,\mathrm{d}x = W_{S,\tau}^{\mathrm{full}}(y_\tau)_{i_\tau}$, by (F1) and (R4)(iii). So $F_a(g) = \sum_{\gamma'} \mathrm{cb}_X(a;\gamma' g)$. Let Q_X be the linear map with $\mathrm{Q}_X(e_a) = F_a$ for all a; then $\mathrm{Q}_X(\phi)(g) = \sum_a \phi_a F_a(g) = \mathrm{P}_X(\phi,\Gamma(V))(g)$. (In the formal text Q_X is a definition: the forms F_a are chosen once, by the axiom of choice.) □

Proof of Theorem B. The function $z = \overline{R(k)\mathrm{Q}_X(e_{a'})}$ is a form on V (of the conjugate type): bounded, measurable, invariant under $\Gamma(V)$. By the definition of the Petersson product, Theorem A(b) and Proposition 3.8 for $\phi = e_a$ and $\Delta = \Gamma(V)$,

$$\langle R(k)\mathrm{Q}_X(e_{a'}), \mathrm{Q}_X(e_a)\rangle = \int_{G(V)} z(g)\,\mathrm{cb}_X(a;g)\,\mathrm{d}g,$$
$$z(g) = \sum_{\gamma\in G_S(E)} \overline{\mathrm{cb}_X(a';\mathrm{rat}_X(\gamma)^{-1}gk)},$$

where the elements of $\Gamma(V)$ were written as $\mathrm{rat}_X(\gamma)^{-1}$. Put $y = \mathrm{re}_X(g_\infty)$ and $\mathsf{m} = h_S(c)^{-1}\mathrm{inc}_S(\eta)\mathrm{fin}_X(g_f)$, so that $\mathrm{cb}_X(a;g) = \beta_{S,c}(y)W_S^{\mathrm{full}}(y)_i$ if $\mathsf{m}\in K_g(S)$ and 0 otherwise. By Lemma 3.6, the term of γ in $z(g)$ is the conjugate of $\beta_{S,c'}(\mathrm{r}_S^\infty(\gamma)^{-1}y)\,W_S^{\mathrm{full}}(\mathrm{r}_S^\infty(\gamma)^{-1}y)_{i'}$ if

$$h_S(c')^{-1}\mathrm{r}_S^f(\gamma)^{-1}\mathrm{inc}_S(\eta)\mathrm{fin}_X(g_f)\mathrm{fin}_X(k) = \mathrm{pos}_S(c,c';\gamma)^{-1}\cdot\mathsf{m}\cdot\mathrm{fin}_X(k)\in K_g(S)$$

and 0 otherwise. By Lemma 3.4(e) and the hypothesis on k, $\mathrm{fin}_X(k)\in K(S)$ has reduction s. So for $\mathsf{m}\in K_g(S)$ the displayed condition says that $\mathrm{pos}_S(c,c';\gamma)$ lies in $K(S)$ and has reduction s, that is $\gamma\in\mathrm{Tm}_S(c,s,c')$: if the product lies in $K_g(S)$, then $\mathrm{pos}_S(c,c';\gamma)\in\mathsf{m}\,\mathrm{fin}_X(k)\,K_g(S)$; conversely, if $\mathrm{pos}_S(c,c';\gamma)\in K(S)$ has reduction s, the product lies in $K(S)$ and reduces to $s^{-1}\cdot 1\cdot s = 1$. Hence the integrand is

$$\mathbf{1}[\mathsf{m}\in K_g(S)]\sum_{\gamma\in\mathrm{Tm}_S(c,s,c')}\beta_{S,c}(y)\,\beta_{S,c'}(\mathrm{r}_S^\infty(\gamma)^{-1}y)\,\mathcal{W}_S^{\mathrm{full}}(\gamma;y)_{(i,i')}.$$

The Haar measure of $G(V)$ is the product of those of $G_\infty(V)$ and $G_f(V)$. The condition $\mathsf{m}\in K_g(S)$ says, by Lemma 3.4(f), that g_f lies in a left translate of $K_g^{\mathrm{std}}(S)$, of volume $\mathrm{vol}(K_g^{\mathrm{std}}(S))$. The map re_X carries the Haar measure of $G_\infty(V)$ to that of $\prod_\tau G_1(\mathfrak{s})$, being at each place a conjugation by the frame, which preserves the Haar measures fixed in the library ((L6)), followed by a left translation. The sum over γ and the integral over y may be exchanged, and the coordinate (i,i') taken inside, by (B4), since $\sum_\gamma |a|_S^{\mathrm{full}}(c,c';\gamma) < \infty$ by (B3). This gives $\mathrm{vol}(K_g^{\mathrm{std}}(S))\sum_{\gamma\in\mathrm{Tm}_S(c,s,c')} a_S^{\mathrm{full}}(c,c';\gamma)_{(i,i')}$, which is the claim. □

4 Columns in Bessel coordinates

The lemma of this section identifies the groups of a grading with the Bessel subgroups of Note S, and the two definitions describe the algebraic modular forms on the definite part (the *columns* of the write-up) in the coordinates in which the Poincaré series are written. They are used by the note on the seesaw identity (unit 141).

Lemma 4.1. *In Setting 3.2, where the hypothesis is* $\mathrm{Fits}(S;V,l,\mathsf{k},\mathcal{Y},\eta)$*, let* $g\in G(V)$*, and let* $h = (h_\infty, h_f)$ *be a point of the adelic group* $\prod_\tau \mathrm{U}(\kappa)\times G_f(E,\kappa)$ *of the standard definite space on* κ*.*

(a) $g\in U(\mathcal{Y})$ *if and only if* $\mathrm{ad}_X(g)\in\mathcal{U}(T_S^{\mathrm{rat}}(\mathbb{A}_E))$*, and* $g\in B(\mathcal{Y})$ *if and only if* $\mathrm{ad}_X(g)\in\mathcal{S}(T_S^{\mathrm{rat}}(\mathbb{A}_E))$*.*
(b) $\mathrm{a}_S^f(\mathrm{ad}_X(i_{\mathcal{Y}}(h))) = \mathrm{inc}_S(h_f)$*.*
(c) $\mathrm{fin}_X(i_{\mathcal{Y}}(h)_f) = \mathrm{inc}_S(h_f)$*.*
(d) $\mathrm{a}_S^\infty(\mathrm{ad}_X(i_{\mathcal{Y}}(h)))$ *is the family of the elements of* $G_1(\mathfrak{s})$ *that act by* h_τ *on the span of the* x_j *and fix the other basis vectors.*

Proof. (a) By (F2) the basis vectors e_a, the x_j and z, and e_a^* have the grades $2(\varrho-a)$, 0 and $-2(\varrho-a)$. So "$u-1$ strictly raises grades" says that the matrix of u is block upper triangular with identity diagonal blocks for the order $e_0,\dots,e_{\varrho-1}$, $\langle x_j,z\rangle$, $e^*_{\varrho-1},\dots,e^*_0$, which defines $\mathcal{U}(\cdot)$; and "fixes e_a and z up to vectors of higher grade" says that $u\mathfrak{f}_a-\mathfrak{f}_a$ lies in the span of the earlier frame vectors, which defines $\mathcal{S}(\cdot)$. (b), (d) The rational coordinates of $i_y(h)$ are the block matrix that is h on the x_j and 1 on the string. Its finite part in the lattice basis is $\mathrm{inc}_S(h_f)$ by the definition of inc_S; at τ, the scales of $\mathrm{Sc}_{S,\tau}$ are 1 on the x_j, so the conjugation of a_S^∞ changes nothing. (c) By Lemma 3.4(d) and (b), $\mathrm{fin}_X(i_y(h)_f)=\mathrm{inc}_S(\eta)^{-1}\mathrm{inc}_S(h_f)\mathrm{inc}_S(\eta)$, and η commutes with h_f by (F5). □

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Definition 4.2 (column coordinates)**.** Let E be a CM field, $\mathfrak{s}$ a shape, $S\in\mathrm{Fill}(E,\mathfrak{s})$, and let V, l, k, $\mathcal{Y}$, η be as in Definition 3.1 with $\mathrm{Fits}(S;V,l,\mathsf{k},\mathcal{Y},\eta)$; h stands for this hypothesis.

(a) Let $\mathbb{S}_{\mathsf{k}}=\bigotimes_\tau \mathbb{S}_{\mathsf{k}_\tau}$, with its basis, the *Weyl coordinates*, and for each τ let $\mathsf{A}^{(\tau)}$ be a matrix as in (R4)(ii), chosen once. Then $\mathsf{A}_h\colon \mathbb{S}_{\mathsf{k}}\to\mathbb{C}^{\mathrm{Bc}(S)}$ is the linear isomorphism whose matrix is the transpose of the Kronecker product of the $\mathsf{A}^{(\tau)}$. It satisfies, for all families $u=(u_\tau)$ in $\mathrm{U}(\kappa)$, all $y\in\prod_\tau G_1(\mathfrak{s})$ and all $v\in\mathbb{S}_{\mathsf{k}}$,

$$\mathsf{A}_h(v)\cdot W_S^{\mathrm{full}}(\dot{u}\,y)_{\cdot}=\mathsf{A}_h(\mathbb{S}_{\mathsf{k}}(u^{\mathsf{t}})v)\cdot W_S^{\mathrm{full}}(y)_{\cdot},$$

where the dot is the bilinear pairing $\sum_i a_ib_i$ on $\mathbb{C}^{\mathrm{Bc}(S)}$, $\dot{u}$ acts by u_τ on the span of the x_j at τ, and $\mathbb{S}_{\mathsf{k}}(u^{\mathsf{t}})=\bigotimes_\tau\mathbb{S}_{\mathsf{k}_\tau}(u_\tau^{\mathsf{t}})$.

(b) cc_h is the linear isomorphism from the space of the algebraic modular forms of weight $(\mathsf{k},0)$ on the standard definite space on κ that are fixed by $K^{\mathrm{col}}(S)$ to $\mathbb{C}^{\mathrm{Cl}(S)\times\mathrm{Bc}(S)}$ that sends a form F to the family $\big(\mathsf{A}_h(F(g_c))\big)_{c\in\mathrm{Cl}(S)}$, where g_c is the chosen representative of the class c. It is an isomorphism because the level is free: the rational points meet every conjugate of $K^{\mathrm{col}}(S)$ trivially (a condition on filling data).

The displayed property follows from (R4)(ii) at each place: with $\mathsf{A}=\bigotimes_\tau\mathsf{A}^{(\tau)}$, $\mathsf{A}_h(v)\cdot W(\dot{u}y)=v^{\mathsf{t}}\mathsf{A}W(\dot{u}y)=v^{\mathsf{t}}\,\mathbb{S}_{\mathsf{k}}(u^{\mathsf{t}})^{\mathsf{t}}\mathsf{A}W(y)$.

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5 The adelic sieve

Throughout this section T is a finite set of finite places of E, $E_{\mathsf{T}}=\prod_{w\in\mathsf{T}}E_w$ with ring of integers O_{T}, and $y_{\mathsf{T}}\in E_{\mathsf{T}}$ is the image of $y\in E$ or of $y\in\mathbb{A}_{E,f}$. An element of E is a T-*integer* if it is integral at every finite place outside T, and a T-*unit* if it and its inverse are T-integers. For $x\in E_{\mathsf{T}}$, $x^0\in\mathbb{A}_{E,f}$ is its extension by zero.

Definition 5.1 (the character, the gain and the points at T)**.** Let E be a CM field, $\mathfrak{s}$ a shape, $S\in\mathrm{Fill}(E,\mathfrak{s})$ and T a finite set of finite places of E.

(a) $\psi_{S,\mathsf{T}}$ is the additive character $x\mapsto\psi_S(x^0)$ of E_{T}.

(b) For $\xi\in E_{\mathsf{T}}$, the *frequency gain* is $\beta_{S,\mathsf{T}}(\xi)=b(S)^{n(\mathfrak{s})^9f(E)}\cdot\big([\xi]^{1/2}\big)^{k(\mathfrak{s})-n(\mathfrak{s})^7}\cdot\mathrm{den}(\xi)^{-6}$.

(c) For a family $l = (l_\tau)_\tau$ of complex numbers and $t \in E$, $\mathrm{pt}_{\mathsf{T}}(l;t) = \big((l_\tau t_\tau)_\tau,\ t_{\mathsf{T}}\big) \in \mathbb{C}^f \times E_{\mathsf{T}}$.

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Lemma 5.2. *Let $S \in \mathrm{Fill}(E, \mathfrak{s})$ and let T be a finite set of finite places of E. If $x \in \mathbb{A}_{E,f}$ is a finite adele that is integral at every finite place outside T, then $\psi_{S,\mathsf{T}}(x_{\mathsf{T}}) = \psi_S(x)$.*

Proof. By (L4)(d), $x - (x_{\mathsf{T}})^0 \in \hat{O}_E$, and ψ_S is trivial on $\hat{O}_E$ by (S2) for $a = 1$. □

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Proposition 5.3 (the sum over frequencies)**.** *Let E carry Weil data, let $S \in \mathrm{Fill}(E, \mathfrak{s})$, and let T be a finite set of finite places of E that is stable under complex conjugation. Let B be a real number with $B \geq 1$, j a natural number, $\mathcal{X}$ a finite set of non-zero T-integers of E, and let $c = (c_\tau)$ be complex and $x = (x_\tau)$ real numbers, indexed by the infinite places, with $|c_\tau|^{-1} \leq |x_\tau| \leq B$ for all τ. Then, with $b = b(S)$,*

$$\sum_{y \in \mathcal{X}} \prod_\tau \mathrm{dp}_j(|c_\tau y_\tau|^{-1}) \cdot \frac{\big([y_{\mathsf{T}}]^{1/2}\big)^j}{\mathrm{den}(y_{\mathsf{T}})^6} \ \leq \ \prod_\tau \Big(|x_\tau|^j\,(bB^2)^3\Big) \cdot (b\ \mathrm{rd}_E)^{3f}.$$

Proof. Weil data include that E is stably unramified, so (S6) applies. Fix $y \in \mathcal{X}$ and let $L_y \subset E^2$ be the line spanned by $(1, y)$; distinct y give distinct lines. Let w be as in (L4)(a). Then $w \in \hat{O}_E$, and $w\hat{O}_E$ is the preimage in $\hat{O}_E$ of $\{t \in O_{\mathsf{T}} : y_{\mathsf{T}} t \in O_{\mathsf{T}}\}$, because y is integral outside T; so $[w] = [\hat{O}_E : w\hat{O}_E] = \mathrm{den}(y_{\mathsf{T}})$. By (S6),

$$\mathrm{disc}_1(L_y) \leq \prod_\tau (1 + |y_\tau|^2) \cdot \mathrm{den}(y_{\mathsf{T}}). \tag{1}$$

Now bound the term of y. First, $\mathrm{dp}_j(\rho) = \rho^j \min(1,\rho)^6$ and $|c_\tau y_\tau|^{-1} \leq |x_\tau|/|y_\tau|$, so the product of the powers ρ^j is at most $\prod_\tau |x_\tau|^j \cdot \prod_\tau |y_\tau|^{-j}$, and $\big([y_{\mathsf{T}}]^{1/2}\big)^j \prod_\tau |y_\tau|^{-j} \leq 1$ by the product formula (L4)(b). Second, for each τ,

$$\min(1, |c_\tau y_\tau|^{-1})^6 \leq \Big(\frac{1000\,B^2}{1 + |y_\tau|^2}\Big)^3 :$$

if $|y_\tau| \leq 1$ the right side is at least $(500B^2)^3 \geq 1$; if $|y_\tau| \geq 1$ the left side is at most $(B/|y_\tau|)^6 \leq \big(2B^2/(1+|y_\tau|^2)\big)^3$. Third, $\mathrm{den}(y_{\mathsf{T}}) \geq 1$, so $\mathrm{den}(y_{\mathsf{T}})^{-6} \leq \mathrm{den}(y_{\mathsf{T}})^{-3}$. Together with (1) the term of y is at most $\prod_\tau |x_\tau|^j (1000B^2)^3 \cdot \mathrm{disc}_1(L_y)^{-3}$. Summing over the distinct lines and using (S6),

$$\sum_{y \in \mathcal{X}} \cdots \leq \prod_\tau |x_\tau|^j (1000B^2)^3 \cdot (400f\ \mathrm{rd}_E)^{3f} \leq \prod_\tau |x_\tau|^j (bB^2)^3 \cdot (b\ \mathrm{rd}_E)^{3f},$$

since $1000 \leq b$ and $400f \leq b$ by (S1). □

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Proof of Theorem C. Write $\mathrm{pt}(l;t)$ for $\mathrm{pt}_{\mathsf{T}}(l;t)$, $\mathrm{fs}(\tau,a)$ for $\mathrm{fs}_S(\tau,a)$, g for $g(S)$, and $\Lambda^{(i)} = \sum_{p \in \mathcal{A}^i} \Lambda_i(\mathrm{hd}_i(p,1))$. We prove the claim for i by induction, assuming it for all smaller indices and all subsets of $\mathcal{A}$. Fix $i \leq \mathsf{n}$ and $\mathcal{V} \subseteq \mathcal{A}$.

The step function. For $x \in C$ put

$$F(x) = \sum_{p,q \in \mathcal{A}^i} J_{i+1}(\mathrm{hd}_i(p,1)[i \mapsto u(x)],\ \mathrm{hd}_i(q,1)[i \mapsto u(x)]).$$

By (S4)(a), F is integrable, and

$$\int F = \sum_{p,q} J_i(\mathrm{hd}_i(p,1), \mathrm{hd}_i(q,1)),$$
$$\Big|\int \psi(\xi x) F(x)\,\mathrm{d}x\Big| \le \beta(\xi)\,(\Lambda^{(i)})^2 \quad \text{for every unit } \xi \text{ of } C.$$

By (S4)(b), applied to the sequences $\mathrm{hd}_i(p,1)[i \mapsto u(x)]$, which share their entry i, $F(x)$ is a non-negative real number, and F is lower semicontinuous at every x all of whose coordinates are units, since u is continuous there.

Transport to the adeles. Let $U_0 = \mathbb{C}^f \times \{z \in \mathbb{A}_{E,f} : z \text{ integral outside } \mathsf{T}\}$, an open subset of $\mathbb{A}_E$, and $\mathrm{m}\colon \mathbb{A}_E \to C$, $\mathrm{m}(z) = \big((\mathrm{fs}(\tau,i)z_\tau)_\tau,\ (z_f)_{\mathsf{T}}\big)$. The map $z_\tau \mapsto \mathrm{fs}(\tau,i)z_\tau$ of $\mathbb{C}$ divides Lebesgue measure by $\mathrm{fs}(\tau,i)^2$, and by (L4)(c) the projection to E_{T} multiplies by $\mathrm{vol}(\hat{O}_E)$; so m carries the Haar measure restricted to U_0 to c_i times the Haar measure of C. Put $G = \mathbf{1}_{U_0} \cdot (F \circ \mathrm{m})$. Then $G \ge 0$, G is integrable with $\int G = c_i \int F$, and for $t \in \mathcal{V}$, a T-unit: $t \in U_0$, all coordinates of $\mathrm{m}(t) = \mathrm{pt}((\mathrm{fs}(\tau,i))_\tau; t)$ are units, so G is lower semicontinuous at t, and

$$G(t) = F(\mathrm{m}(t)) = \sum_{p,q \in \mathcal{A}^i} J_{i+1}\big(\mathrm{hd}_i(p,t), \mathrm{hd}_i(q,t)\big),$$

because $\mathrm{hd}_i(p,1)[i \mapsto u(\mathrm{m}(t))] = \mathrm{hd}_i(p,t)$. So the left side of the claim is $\sum_{t\in\mathcal{V}} G(t)$.

The Fourier coefficients. Let $y \in E$ and consider the coefficient at the frequency $y/\varpi(S)$,

$$\hat{G}(y) = \int_{\mathbb{A}_E} \psi_E(-(y/\varpi(S))z)\,G(z)\,\mathrm{d}z;$$

every element of E is such a frequency. (i) If y is not a T-integer, $\hat{G}(y) = 0$. Indeed, for $r \in \hat{O}_E$ the adele $r' = r - (r_{\mathsf{T}})^0$ is integral outside T and vanishes at T, so $G(z + (0,r')) = G(z)$, while $\psi_E(-(y/\varpi(S))(0,r')) = \psi_S(yr')$; by translation invariance of the integral, $\hat{G}(y) = \psi_S(yr')\hat{G}(y)$. If $\hat{G}(y) \ne 0$, then $\psi_S(r(y - (y_{\mathsf{T}})^0)) = 1$ for all $r \in \hat{O}_E$, so $y - (y_{\mathsf{T}})^0 \in \hat{O}_E$ by (S2), and y is a T-integer by (L4)(d). (ii) Let $y \ne 0$ be a T-integer. Since $\overline{\varpi(S)} = -\varpi(S)$, the number $\varpi(S)_\tau$ is purely imaginary, and a direct computation gives, for $z \in U_0$ and $x = \mathrm{m}(z)$,

$$\psi_E(-(y/\varpi(S))z) = \prod_\tau \exp(\xi_\tau x_\tau - \overline{\xi_\tau x_\tau}) \cdot \psi_S(yz_f) = \psi(\xi x),$$
$$\xi = \mathrm{pt}((c_\tau/\mathrm{fs}(\tau,i))_\tau;\, y), \qquad c_\tau = \frac{2\pi i}{\varpi(S)_\tau},$$

where $\psi_S(yz_f) = \psi_{S,\mathsf{T}}(y_{\mathsf{T}}x_{\mathsf{T}})$ by Lemma 5.2, as yz_f is integral outside T. Hence $\hat{G}(y) = c_i \int_C \psi(\xi x)F(x)\,\mathrm{d}x$, and ξ is a unit of C, so $|\hat{G}(y)| \le c_i\,\beta(\xi)(\Lambda^{(i)})^2$. By (S1), $|c_\tau|^{-1} = |\varpi(S)_\tau|/2\pi = |\kappa_\tau(S)|/g$, so the frequency bound, with $a = i$, gives $\sum_y \beta(\xi) \le \mathsf{e}$ over every finite set of non-zero T-integers. (iii) $\hat{G}(0) = \int G = c_i \int F \ge 0$.

Poisson summation. By (i)–(iii), every finite sum of $|\hat{G}(y)|$ is at most $c_i\big(\int F + \mathsf{e}\,(\Lambda^{(i)})^2\big)$, and (L3) gives

$$\sum_{t\in\mathcal{V}} G(t) \le c_i\Big(\int F + \mathsf{e}\,(\Lambda^{(i)})^2\Big) \le c_i \int F + \mathsf{C}\mathsf{e}\prod_{a\le i} c_a. \tag{2}$$

The integral of F. If $i=0$, $\int F = J_0(\cdot,\cdot)$ is a single value, of absolute value at most En, and (2) is the claim. Let $i \ge 1$. Writing $p \in \mathcal{A}^i$ as (p', s) with $p' \in \mathcal{A}^{i-1}$ and $s \in \mathcal{A}$, we have $\mathrm{hd}_i(p,1) = \mathrm{hd}_{i-1}(p', s)$, so

$$\int F = \sum_{s,s' \in \mathcal{A}} \sum_{p',q' \in \mathcal{A}^{i-1}} J_i(\mathrm{hd}_{i-1}(p',s), \mathrm{hd}_{i-1}(q',s')).$$

The real part of the terms with $s = s'$ is at most $\prod_{a<i} c_a \cdot (\mathrm{En} + (2i-1)\mathsf{Ce})$ by the induction hypothesis for $i-1$ and $\mathcal{V} = \mathcal{A}$. For $s \ne s'$ the two sequences have at $i-1$ the entries $u(\mathrm{pt}((\mathrm{fs}(\tau, i-1))_\tau; s))$ and the same with s', and

$$\iota(\cdot) - \iota(\cdot) = \mathrm{pt}\big((g/\mathrm{fs}(\tau, i-1))_\tau;\, y\big), \qquad y = g^{-1}(s^{-1} - s'^{-1}),$$

a unit of C, where y is a non-zero T-integer because g^{-1}, s^{-1}, s'^{-1} are. By the last axiom of a sieve system and the frequency bound for $a = i-1$, $c_\tau = g$ (admissible: $|g|^{-1} \le |\kappa_\tau(S)|/g$ as $|\kappa_\tau(S)| \ge 1$) and $\mathcal{X} = \{y\}$, the term of $(p', s), (q', s')$ is at most $\mathsf{e}\,\Lambda_i(\mathrm{hd}_{i-1}(p',s))\Lambda_i(\mathrm{hd}_{i-1}(q',s'))$ in absolute value. As $\Lambda \ge 0$, the terms with $s \ne s'$ contribute at most $\mathsf{e}\,(\Lambda^{(i)})^2 \le \mathsf{Ce}\prod_{a<i} c_a$. So $\int F \le \prod_{a<i} c_a \cdot (\mathrm{En} + 2i\,\mathsf{Ce})$, and (2) gives the claim. □

Proof of Theorem D. Let $\rho_\tau = |\kappa_\tau(S)|^{2\varrho+1}/g(S)N(S)$, $B_0 = b^{23(2\varrho+1)}$, $N = N(S)$, $g = g(S)$. By (S5) and (S4)(c), the product of the band-limit systems at the infinite places with the system J' is a sieve system on $C = \mathbb{C}^f \times E_{\mathsf{T}}$ of the kind of Theorem C, with $\mathsf{n} = \varrho$, the forms $J_i(p,q) = \prod_\tau J_i^\tau(p_\tau, q_\tau) \cdot J_i'(p_{\mathsf{T}}, q_{\mathsf{T}})$, the constant $\mathrm{En} = e\prod_\tau (1 + b^{-10})/\deg_\tau(S)$, and

$$\Lambda_i(p) = \prod_\tau \Lambda^{\mathbb{C}}_{n^6}(i; p_\tau) \cdot \Lambda_{n^6}(i; p_{\mathsf{T}}),$$
$$\beta(x) = \prod_\tau c_{\mathrm{band}}(\mathfrak{s})\, b^{2000n^9}\, \mathrm{dp}_{k-n^7}(|x_\tau|^{-1}) \cdot \beta_{S,\mathsf{T}}(x_{\mathsf{T}}).$$

We apply Theorem C with $i = \varrho$ and $\mathcal{V} = \mathcal{A}'$. Its left side is the left side of Theorem D, the entries of $\mathrm{hd}_\varrho(p,t)$ being the pairs of the entries of the $x_\tau(p,t)$ and of $y(p,t)$, and $\prod_{a\le\varrho} c_a$ is the first factor on the right. It remains to verify the two hypotheses with suitable e, C and to compare $(2\varrho+1)\mathsf{Ce}$ with the error term. The exponents below are ours; the formal proof leaves them to unification and to automation.

The frequency bound. Let $a \in \mathbb{N}$, $|c_\tau|^{-1} \le |\kappa_\tau(S)|/g$, and $\mathcal{X}$ a finite set of non-zero T-integers. For $y \in \mathcal{X}$, $\beta(\mathrm{pt}_{\mathsf{T}}((c_\tau/\mathrm{fs}_S(\tau,a))_\tau; y))$ is $\prod_\tau c_{\mathrm{band}}(\mathfrak{s})\, b^{2000n^9} \cdot b^{n^9 f}$ times the term of y in Proposition 5.3 for the exponent $k - n^7$ and the numbers $c_\tau/\mathrm{fs}_S(\tau,a)$. Since $|\kappa_\tau(S)| \ge 1$, $|\mathrm{fs}_S(\tau,a)/c_\tau| \le |\kappa_\tau(S)|^{2(\varrho-a)}N^{-1} \cdot |\kappa_\tau(S)|/g \le \rho_\tau \le B_0$ by (S1). So Proposition 5.3, with $x_\tau = \rho_\tau$ and $B = B_0$, gives the frequency bound with

$$\mathsf{e} = \prod_\tau \Big(c_{\mathrm{band}}(\mathfrak{s})\, b^{2000n^9}\, \rho_\tau^{k-n^7} (bB_0^2)^3 \Big) \cdot (b\ \mathrm{rd}_E)^{3f}\, b^{n^9 f}.$$

The weights. Let $i \le \varrho$. For $p \in \mathcal{A}^i$ and $1 \le j \le i$ let $m_j(p) = p_0 \cdots p_{j-1} \in E^\times$, a T-unit, and $F_{\tau,j} = \prod_{a<j} \mathrm{fs}_S(\tau,a)$. By the definitions of the head weights,

$$\Lambda_i(\mathrm{hd}_i(p,1)) = \prod_{j=1}^{i} \Big(\prod_\tau \mathrm{dp}_{n^6}(|F_{\tau,j}\, m_j(p)_\tau|^{-1}) \cdot \omega(m_j(p)) \Big),$$

with $\omega(y) = \big([y_{\mathsf{T}}]^{1/2}\big)^{n^6}$ if $y_{\mathsf{T}} \in O_{\mathsf{T}}$ and $\omega(y) = 0$ otherwise; in the first case $\mathrm{den}(y_{\mathsf{T}}) = 1$. The map $p \mapsto (m_1(p), \dots, m_i(p))$ is injective, since $p_{j-1} = m_j(p)/m_{j-1}(p)$. So

the sum over p of this product is at most the product over j of the sums over the finite set $\{m_j(p) : p \in \mathcal{A}^i\}$ of non-zero T-integers, and each of these is bounded by Proposition 5.3 with exponent n^6, $c_\tau = F_{\tau,j}$, $x_\tau = B = N^i$ (indeed $|F_{\tau,j}|^{-1} = \prod_{a<j} N|\kappa_\tau(S)|^{-2(\varrho-a)} \le N^j \le N^i$):

$$\Lambda^{(i)} \le \left(N^{if n^6}\,(bN^{2i})^{3f}\,(b\,\mathrm{rd}_E)^{3f}\right)^i.$$

With $N \le b^{60n}$, $\mathrm{rd}_E \le b$, $i \le \varrho < n$ and $n^6+6 \le 7n^6$ this gives $(\Lambda^{(i)})^2 \le b^{(840n^9+18n)f}$. On the other side, $\mathrm{fs}_S(\tau,a) \le |\kappa_\tau(S)|^{2\varrho} \le b^{46\varrho}$ and $\mathrm{vol}(\hat{O}_E) = (2/\mathrm{rd}_E)^f \ge b^{-f}$, so $c_a \ge b^{-(92\varrho+1)f}$ and $\prod_{a<i} c_a \ge b^{-93n^2 f}$. Hence the hypothesis of Theorem C holds with $\mathsf{C} = b^{1000n^9 f}$.

The error term. By the admissibility inequality of (S1), $\prod_\tau c_{\mathrm{band}}(\mathfrak{s})\,\rho_\tau^{k-n^7} \le \deg(S)^{-1}b^{-7kf}$. Also $2\varrho+1 \le b$. So

$$\begin{aligned}(2\varrho+1)\,\mathsf{C}\,\mathsf{e} &\le \deg(S)^{-1}\,b^{-7kf}\cdot b^{\,1+1000n^9f+2000n^9f+3f(1+46(2\varrho+1))+6f+n^9f}\\ &\le \deg(S)^{-1}\,b^{-7kf+1+3286n^9f},\end{aligned}$$

and $1+3286n^9 f \le kf-10$ because $k \ge 1000(2\varrho+2)^9 = 512000\,n^9$. This is the error term $\deg(S)^{-1}b^{-(6kf+10)}$. □

6 The standard level at a finite place

The tower method of unit 132 bounds integrals over a unitary group floor by floor, given a *level* and a *gauge* (Input (R1)). This section provides them at a finite place, for the lattice of the filling data: the form is $T^{\mathrm{lat}}(\mathfrak{s};N)$, the box is the group of the integral isometries, and all sizes are integrality conditions. The first two results concern any place.

Definition 6.1 (the tail vector)**.** Let the ring A be a commutative ring with an involution, $\mathfrak{s}$ a shape, π a unit of A, $t \in A$ and $x \in A^\kappa$. Then

$$\mathrm{tv}(\pi,t,x) = \Lambda(\mathfrak{s};\pi)^{-1}\cdot(0,\ (x,\ -(\pi\bar\pi)^{-1}t),\ 0)$$

is the vector $\sum_{j\in\kappa} x(j)\,x_j - (\pi\bar\pi)^{-1}t\,z$ of V_ϱ, written in the lattice basis; here $x(j) \in A$ are the entries of x, and x_j, z are the basis vectors.

LEAN `tailVector`

Lemma 6.2. *Let A be a commutative ring with an involution, $\mathfrak{s}$ a shape, π a unit of A and $N \in A$ with $\pi\bar\pi = N$; let $t \in A$, $x \in A^\kappa$, $J = J(T^{\mathrm{lat}}(\mathfrak{s};N))$ and $e = \mathrm{tv}(\pi,t,x)$. Then $e^*Je = x^*x - (\pi\bar\pi)^{-1}t\,\bar t$, and the coordinate of Je at z is t.*

Proof. By (S3), $J = \Lambda(\mathfrak{s};\pi)^*J'\Lambda(\mathfrak{s};\pi)$ with $J' = J(T^{\mathrm{rat}}(\mathfrak{s};N))$. So with $v = (0,(x,-N^{-1}t),0)$ we get $e^*Je = v^*J'v = x^*x - N\cdot N^{-1}\bar t\,N^{-1}t$, as $\bar N = N$. Further $Je = \Lambda(\mathfrak{s};\pi)^*J'v$, and $J'v$ has the coordinate $-N\cdot(-N^{-1}t) = t$ at z; the column of $\Lambda(\mathfrak{s};\pi)$ at z is the basis vector z itself, so the coordinate of Je at z is t. □

LEAN `form_tailVector`

Proposition 6.3 (slabs near the quadric)**.** *Let A be a place (a commutative ring with an involution, a topology, a Haar measure and a place structure P), Ω a finite set, $\mathfrak{m} \in \mathbb{N}$, π a unit of A and $N \in A$ with $\pi\bar\pi = N$ such that $J(T^{\mathrm{lat}}(\mathfrak{s};N))$ is a*

hermitian unit. Let $F \in \mathrm{Hyp}(\Omega, T^{\mathrm{lat}}(\mathfrak{s}; N), \mathfrak{m})$ *be a hyperbolic basis,* Φ *a continuous function with compact support on the space of vectors, with values in* $[0,\infty)$*, and* $c \in [0,\infty]$ *with* $\int \Phi \,\mathrm{d}\delta^{(\varrho)}(F) < c$*. Then there is a neighbourhood* $\mathcal{N}$ *of* 0 *in* A *such that for every measurable set* $Z \subseteq \mathcal{N}$

$$\int_A \int_{\{y \in A^\kappa :\ y^* y - ((\pi\bar\pi)^{-1} t\bar t - N) \in Z\}} \Phi(\mathrm{tv}(\pi,t,y))\,\mathrm{d}y\,\mathrm{d}t \le \mathrm{mod}_A(\pi)\cdot \nu_A\{\beta \in A^- : \delta_A^{-1}\beta \in Z\}\cdot c.$$

Proof. Write $q(v) = h(v,v)$ on V_ϱ, so that the quadric of $\delta^{(\varrho)}(F)$ is $\{q = -N\}$, the fibre of z. By (R1) the Haar measure of V_ϱ disintegrates along q: for $\Psi \ge 0$ measurable, $\int_{V_\varrho} \Psi(q(v)+N, v)\,\mathrm{d}v = \int_{A^-}\int \Psi(\delta_A^{-1}\beta, v)\,\mathrm{d}\delta_\beta(v)\,\mathrm{d}\nu_A(\beta)$, where δ_β is the quadric measure on the fibre $\{q + N = \delta_A^{-1}\beta\}$, $\delta_0 = \delta^{(\varrho)}(F)$, and $\beta \mapsto \int \Phi\,\mathrm{d}\delta_\beta$ is continuous. Hence there is a neighbourhood $\mathcal{N}$ of 0 with $\int \Phi\,\mathrm{d}\delta_\beta < c$ whenever $\delta_A^{-1}\beta \in \mathcal{N}$. Let $Z \subseteq \mathcal{N}$ be measurable. Applying the disintegration to $\Psi(s,v) = \mathbf{1}_Z(s)\Phi(v)$,

$$\int_{V_\varrho} \mathbf{1}_Z(q(v)+N)\,\Phi(v)\,\mathrm{d}v \le \nu_A\{\beta : \delta_A^{-1}\beta \in Z\}\cdot c.$$

The map $(t,y) \mapsto \mathrm{tv}(\pi,t,y)$ from $A \times A^\kappa$ to the coordinates of V_ϱ in the lattice basis is linear and continuous (the formal text records the continuity as a lemma). Its matrix has the columns of the base change of the x_j and the column $-(\pi\bar\pi)^{-1}z$, so its determinant is $-\pi^{-1}$, and it multiplies Haar measure by $\mathrm{mod}_A(\pi)$, in the sense that $\int\int \Psi'(\mathrm{tv}(\pi,t,y))\,\mathrm{d}y\,\mathrm{d}t = \mathrm{mod}_A(\pi)\int_{V_\varrho}\Psi'(v)\,\mathrm{d}v$; and by Lemma 6.2, $q(\mathrm{tv}(\pi,t,y)) + N = y^*y - ((\pi\bar\pi)^{-1}t\bar t - N)$. The two displays give the claim. □

LEAN `HyperbolicBasis.eventually_mass_near_quadric_le` · `continuous_tailVector`

From here on A is a semilocal field with an involution and an integral place structure (Input (R3)), and O_A its ring of integers. For an $\mathfrak{s}$-form T with $J(T)$ a hermitian unit and a unit g, $K_T(g)$ is the group of the $u \in G(T)$ that are integral with integral inverse and satisfy $g^{-1}(u-1)$ integral.

Definition 6.4 (lattice levels and standard levels)**.** Let A be a semilocal field with an involution and an integral place structure, $\mathfrak{s}$ a shape, Ω a finite set, $\mathfrak{m} \in \mathbb{N}$, and T an $\mathfrak{s}$-form over A with $J(T)$ a hermitian unit.

(a) For a level $\mathcal{L} \in \mathrm{Lev}(\Omega, T, \mathfrak{m})$, its *congruence subgroup* is $\mathrm{K}^\circ(\mathcal{L}) = K_T(g_{\mathcal{L}})$.

(b) A level $\mathcal{L} \in \mathrm{Lev}(\Omega, T, \mathfrak{m})$ is a *lattice level*, $\mathcal{L} \in \mathrm{LatLev}$, if: (LL1) $J(T)$ is unimodular over O_A, the basis matrix $E(\mathcal{L})$ has entries in O_A, $g_{\mathcal{L}} \in O_A$ and $\overline{g_{\mathcal{L}}} = g_{\mathcal{L}}$; (LL2) $\mathrm{vol}(O_A^\times) \le \theta_\varrho(\mathcal{L})(\mathrm{Box}(\mathcal{L}))$, $\mathrm{Box}(\mathcal{L}) = K_T(1)$, $\theta_n(\mathcal{L})(\mathrm{Box}(\mathcal{L})) = 1$, $\mathfrak{B}(\mathcal{L}) = O_A^{I(\mathfrak{s})}$, and $\mathrm{Posn}(\mathcal{L}) = \{y^{-1}uy : y \in G(T),\ u \in \mathcal{U}(T)\}\cdot \mathrm{K}^\circ(\mathcal{L})$; (LL3) there is a vector x with z-coordinate 0 such that $h_T(x,\cdot)$ is the z-coordinate.

(c) A level $\mathcal{L} \in \mathrm{Lev}(\Omega, T, \mathfrak{m}+1)$ is a *standard level*, $\mathcal{L} \in \mathrm{StdLev}$, if it is a lattice level, $\zeta_A(\mathfrak{m})^n \le 1 + 2^{-n}$, $\mathfrak{m} < 9n$, $13n \le |\Omega|$, and $G(T)$ has a Haar measure that is right invariant.

LEAN `Level.congGroup` · `IntegralPlace.IsLatticeLevel` · `IntegralPlace.IsStdLevel`

Proof of Theorem E. Let $T = T^{\mathrm{lat}}(\mathfrak{s}; N)$ and $J = J(T)$; by (S3), J is unimodular over O_A. In the lattice basis the middle block has the orthonormal vectors x_j, $j \ne k_0$, the isotropic vector v_0 at the index k_0, and z, with $h(v_0, z) = -1$, $h(z,z) = -N$. We

describe the level and say for each axiom of (R1) and of Definition 6.4(b) why it holds; the formal proof verifies them in one construction.

The hyperbolic basis. Let τ_A be the trace-one element and $c \in O_A$ with $\bar{c}c = -1$ ((R3)). The vectors $-v_0$ and $-N\tau_A v_0 + z$ form a hyperbolic pair: $h(-v_0, -v_0) = 0$, $h(-v_0, -N\tau_A v_0 + z) = -h(v_0, z) = 1$ and $h(-N\tau_A v_0 + z, -N\tau_A v_0 + z) = N\bar{\tau}_A + N\tau_A - N = 0$. Since $k(\mathfrak{s}) - 1 = |\Omega| + 2\mathfrak{m}$, the vectors x_j, $j \neq k_0$, can be split into $\mathfrak{m}$ pairs (x, x') and a family indexed by Ω; each pair gives the hyperbolic pair $x + cx'$, $\tau_A x - \bar{\tau}_A c x'$. This basis matrix is integral, and $\operatorname{mod}_A(\det T) = 1$ because T is unimodular; so it is a hyperbolic basis in $\mathrm{Hyp}(\Omega, T, \mathfrak{m})$.

The measure of the floor ϱ. Put $x = -v_0$: then $h(x, \cdot)$ is the z-coordinate and the z-coordinate of x is 0, which is (LL3). Let φ be the parametrisation of the quadric $\{v \in V_\varrho : h(v, v) = h(z, z)\}$ by the flat chart of the library, $p = ((t, w), \beta) \mapsto \varphi(p)$, in the coordinates of the hyperbolic basis; the z-coordinate of $\varphi(p)$ is t. For a vector e let $\mathrm{E}(e)$ be the Eichler element of (R3)(e), formed with x. Define $\theta_\varrho(\mathcal{L})$ as the image of the chart measure $\mathrm{d}t\,\mathrm{d}w\,\mathrm{d}\nu_A(\beta)$ under $p \mapsto \mathrm{E}(\varphi(p))$, a measurable map. By (R3)(e) it is carried by $\mathcal{H}_\varrho(T)$, and since t is a unit for almost every p, its image under $\sigma \mapsto \sigma^{-1}z$ is the image of the chart measure under φ, which is $\delta^{(\varrho)}(\mathcal{L})$.

The measure of the floor n. The group $\mathcal{H}(T)$ is closed in $G(T)$ and $\mathcal{H}(T) \cap K_T(1)$ is a compact open subgroup of it. Let $\theta_n(\mathcal{L})$ be the image under inversion of the left Haar measure of $\mathcal{H}(T)$ that gives this subgroup mass 1. It is carried by $\mathcal{H}(T)$, invariant under right translations by $\mathcal{H}(T)$, and $\theta_n(\mathcal{L})(K_T(1)) = 1$. For $g \in G(T)$ let $\Sigma_g = \{h \in \mathcal{H}(T) : uhg \in K_T(1) \text{ for some } u \in \mathcal{U}(T)\}$. If $h, h' \in \Sigma_g$, with uhg and $u'h'g$ in $K_T(1)$, then $u'h'h^{-1}u^{-1} \in K_T(1)$; as $h'h^{-1}$ normalises $\mathcal{U}(T)$, this is $u'' \cdot h'h^{-1}$ with $u'' \in \mathcal{U}(T)$, and then $h'h^{-1} \in K_T(1)$ by (L8) for the unit 1. So Σ_g lies in one right coset of $\mathcal{H}(T) \cap K_T(1)$ and has mass at most 1.

The remaining data. The unit is g, with $\bar{g} = g$; the ball is $O_A^{I(\mathfrak{s})}$, which is open and compact and contains 0; the box is $K_T(1)$, which is compact ((R3)(b)); the positions are $\{y^{-1}uy\} \cdot K_T(g)$, an open set, hence measurable. This is a level $\mathcal{L}$ with $g_{\mathcal{L}} = g$, and (LL1) and the equalities of (LL2) hold by construction. For the inequality of (LL2), consider the chart points p with $t \in O_A^\times$, w integral and $\beta \in A^- \cap O_A$; they have chart measure $\mathrm{vol}(O_A^\times)$, since $\mathrm{vol}(O_A) = 1$ and $\nu_A(A^- \cap O_A) = 1$. For such p the vector $\varphi(p)$ is integral with z-coordinate a unit of O_A, so the matrix $\mathrm{E}(\varphi(p))$ of (R3)(e) is integral; an integral isometry of a unimodular form lies in $K_T(1)$. Hence $\theta_\varrho(\mathcal{L})(\mathrm{Box}(\mathcal{L})) \geq \mathrm{vol}(O_A^\times)$. □

Definition 6.5 (the gauge of a standard level)**.** Let A be a semilocal field with an involution and an integral place structure, $\mathfrak{s}$, Ω, $\mathfrak{m}$, T as in Definition 6.4, let $\mathcal{L} \in \mathrm{Lev}(\Omega, T, \mathfrak{m} + 1)$ be a standard level, $\mathcal{L} \in \mathrm{StdLev}$, and let α be a unit of A with $\alpha \in O_A$ and $\bar{\alpha} = \alpha$ (by default $\alpha = 1$). The *gauge of depth* α is

$$\Gamma_{\mathcal{L}}(\alpha) \in \mathrm{Gau}(\mathcal{L}) : \qquad \Gamma_j = \{x \in A : \alpha^j x \in O_A\} \quad (j \in \mathbb{N}), \qquad \Lambda = \frac{2}{\operatorname{mod}_A(\alpha)}.$$

That these data satisfy the axioms of a gauge is part of the definition in the formal text; it is Lemma 6.6.

Lemma 6.6. *In the situation of Definition 6.5, the sets $\Gamma_j = \alpha^{-j} O_A$ and the constant $\Lambda = 2/\operatorname{mod}_A(\alpha)$ form a gauge of $\mathcal{L}$.*

Proof. We list the axioms of unit 132 and the reason for each; (G10) is described and not carried out in every detail. Note first: $\mathrm{mod}_A(\alpha) \le 1$, so $\Lambda \ge 2$; $\zeta_A(\mathfrak{m}) \le 2$ because its n-th power is at most $1 + 2^{-n} \le 2$; and $\mathrm{mod}_A(\alpha^j)^{-1} \le \Lambda^j$. The sets satisfy the axioms of a *size*: $\Gamma_i\Gamma_{i'} \subseteq \Gamma_{i+i'}$; each Γ_i is an O_A-module contained in Γ_{i+1}, so sums of its elements lie in Γ_{i+1}; it is stable under $x \mapsto -x$ and, as $\bar\alpha = \alpha$, under the involution; it contains 0 and 1; the entries of $J(T)$ and of its inverse lie in $O_A \subseteq \Gamma_1$ by (LL1). (G1) Γ_j is compact, the preimage of O_A under a homeomorphism. (G2) $\Lambda \ge 2$, and the number $\mathfrak{m}+1$ of pairs of the level is at most $9n$. (G3) The Plancherel constant is 1 and $\Lambda < \infty$. (G4) $g_{\mathcal{L}}$, the entries of $E(\mathcal{L})$, and $\tau_A\delta_A^{-1}$ lie in $O_A \subseteq \Gamma_1$, by (LL1) and (R3). (G5), (G6) The elements of the box $K_T(1)$ and the vectors of the ball $O_A^{I(\mathfrak{s})}$ are integral. (G7) If u is a unit in Γ_j, then $\mathrm{mod}_A(\alpha^j u) \le 1$, so $\mathrm{mod}_A(u) \le \Lambda^j$. (G8) $\mathrm{vol}(\Gamma_j) = \mathrm{mod}_A(\alpha^j)^{-1} \le \Lambda^{j+1}$ by (R3)(a). (G9) For a quadric in $|\Omega'|+1$ hyperbolic planes with $\mathfrak{m}+1 \le |\Omega'| \le \mathfrak{m}+1+\varrho$, the flat quadric measure of the vectors with coordinates in Γ_j is at most $\zeta_A(\mathfrak{m})\,\mathrm{mod}_A(\alpha^j)^{-(2|\Omega'|+1)} \le \Lambda^{j(2|\Omega'|+1)+1}$ by the cone bound (R3)(c), and $j(2|\Omega'|+1)+1 \le 99n(j+1)$ since $|\Omega'| \le 10n$. (G10) (the volume majorant) Let x be in the box, y a position, v a matrix with ϱ rows, and W the set of the $w \in A^\Omega$ with $vw = 0$ for which the two vectors $x^{-1}\mathsf{O}w$ and $(xy)^{-1}\mathsf{O}w$ are close in the sense of the level (O the orthonormal part of the basis). The axiom asks for a measurable $\varphi \ge 0$ on A^Ω that bounds, at every u, the inverse volume of $\{u^*w : w \in W\}$ and the volume of a set of $c \in A$ defined by W, and whose power $\mathsf{s}(\mathcal{L})$ has integral at most $\Lambda^{99n(j+1)}$ over the vectors with coordinates in Γ_j. The formal proof writes $y = y_0^{-1}u_0y_0 \cdot k'$ with $u_0 \in \mathcal{U}(T)$, $k' \in \mathrm{K}^\circ(\mathcal{L})$, and uses the Hermite form (L10) to find $M \in \mathrm{GL}_\Omega(O_A)$ and a submatrix Y of M with $\mathsf{s}(\mathcal{L})+\mathfrak{m}+5$ columns such that $Yt \in W$ for every integral t: the columns of Y are killed by the ϱ rows of v, by the ϱ rows of $y_0x^{-1}\mathsf{O}$ at the e_a^* and by the ϱ rows of $(u_0^{-1}-1)y_0x^{-1}\mathsf{O}$ at the e_a. These are 3ϱ linear conditions, and $\mathsf{s}(\mathcal{L})+\mathfrak{m}+5 = |\Omega| - 3\varrho$: the level lies in $\mathrm{Lev}(\Omega, T, \mathfrak{m}+1)$, so its volume exponent is $\mathsf{s}(\mathcal{L}) = |\Omega| - (3n+2+\mathfrak{m})$ by (R1), and nothing is truncated, because $3n+2+\mathfrak{m} \le 12n+1 \le 13n \le |\Omega|$ by $\mathfrak{m} < 9n$. Then $\varphi(u) = \mathrm{ct}(Y^*u)$, the content of (R3)(d), bounds both quantities (by the definition of the content for the first, and by the integrality of the pairings of integral vectors for the second). The bound for the integral is the content moment (R3)(d) with $s = \mathsf{s}(\mathcal{L})$ and with $\mathfrak{m}+5 \ge \mathfrak{m}$ further marked coordinates: over the vectors with coordinates in Γ_j the integral of φ^s is at most $\zeta_A(\mathfrak{m})\,\mathrm{mod}_A(\alpha^j)^{-(|\Omega|-s)} \le \Lambda^{j(12n+1)+1} \le \Lambda^{99n(j+1)}$, by $\zeta_A(\mathfrak{m}) \le 2 \le \Lambda$, $\mathrm{mod}_A(\alpha^j)^{-1} \le \Lambda^j$ and $|\Omega| - s = 3n+2+\mathfrak{m} \le 12n+1$. □

LEAN `IntegralPlace.IsStdLevel.gauge`

Proposition 6.7 (Heisenberg masses of congruence subgroups)**.** *Let $\mathcal{L} \in \mathrm{Lev}(\Omega, T, \mathfrak{m}+1)$ be a standard level, $\mathcal{L} \in \mathrm{StdLev}$, over a semilocal field A with an involution and an integral place structure, let g be a unit of A with $g \in O_A$ and $\bar g = g$, let $a < \varrho$, and let N be a subgroup of $G(T)$ with $\mathsf{N} \subseteq \mathcal{N}_{<a}(T)$. Then*

$$\eta_a\{y : uy \in K_T(g) \text{ for some } u \in \mathsf{N}\} = \mathsf{d}_a(\mathcal{L}; g) = \mathrm{mod}_A(g)^{k+1+2(\varrho-a-1)}\,\mathrm{mod}_A(g)^{1/2}.$$

Proof. η_a is the image of $\mathrm{d}v\,\mathrm{d}\beta$ on $V_{a+1} \times A^-$ under $(v,\beta) \mapsto \mathrm{n}_a(v,\beta)$. We claim that $u\,\mathrm{n}_a(v,\beta) \in K_T(g)$ for some $u \in \mathsf{N}$ if and only if $g^{-1}v$ is integral and $g^{-1}\beta \in O_A$. If the latter holds, then $\mathrm{n}_a(v,\beta) - 1$, whose entries are formed from the coordinates of v, from $\beta - \tau_A h_T(v,v)$ and from the entries of the integral matrix $J(T)$, is divisible by g; an integral isometry of a unimodular form that is congruent to 1 modulo g lies in $K_T(g)$, so $u = 1$ works. Conversely, apply (R2)(b) to the gauge $\Gamma_{\mathcal{L}}(1)$, all of whose

sets are O_A, and to the unit g: the entries of $g^{-1}(u\,\mathrm{n}_a(v,\beta)-1)$ are integral, so $g^{-1}v$ is integral and $g^{-1}(\beta-\tau_A h_T(v,v))\in O_A$; since $h_T(v,v)\in g^2O_A$, also $g^{-1}\beta\in O_A$. By (R3)(a), the set of the $v\in V_{a+1}$ with $g^{-1}v$ integral has volume $\mathrm{mod}_A(g)^{\dim V_{a+1}}$, with $\dim V_{a+1}=k+1+2(\varrho-a-1)$, and $\nu_A\{\beta : g^{-1}\beta\in O_A\}=\mathrm{mod}_A(g)^{1/2}$, which uses $\bar g=g$. □

LEAN `IntegralPlace.IsStdLevel.heisenbergMeasure_eq_floorModulus`

Corollary 6.8. *Let $\mathcal{L}\in\mathrm{Lev}(\Omega,T,\mathfrak{m}+1)$ be a standard level, $\mathcal{L}\in\mathrm{StdLev}$, over a semilocal field with an involution and an integral place structure, and let g be a unit of A with $g\in O_A$ and $\bar g=g$.*

(a) For every $j\le\varrho$: $\eta_{<j}(K_T(g))=\prod_{l<j}\mathsf{d}_l(\mathcal{L};g)$.

(b) For every $c\in G(T)$ and every s-finite measure θ on $G(T)$:

$$(\eta_{<\varrho+2}*\theta)\{y : yc\in K_T(g)\} = \prod_{l<\varrho}\mathsf{d}_l(\mathcal{L};g)\cdot\theta\{y : uyc\in K_T(g) \text{ for some } u\in\mathcal{U}(T)\}.$$

Proof. Both parts rest on the unfolding (L7). (a) By induction on j: $\eta_{<j+1}=\eta_{<j}*\eta_j$, and (L7) with $\theta=\eta_j$, $c=1$ and Proposition 6.7 with $\mathsf{N}=\mathcal{N}_{<j}(T)$ give the factor $\mathsf{d}_j(\mathcal{L};g)$. (b) (L7) for $j=\varrho+2$, where $\mathcal{N}_{<\varrho+2}(T)=\mathcal{U}(T)$ and $\eta_{<\varrho+2}=\eta_{<\varrho}$, and (a) for $j=\varrho$. □

LEAN `IntegralPlace.IsStdLevel.ascConv_unitaryLevel` · `IntegralPlace.IsStdLevel.unfold_besselU`

Proposition 6.9 (the mass of the integral group, from above)**.** *Let $\mathcal{L}\in\mathrm{Lev}(\Omega,T,\mathfrak{m}+1)$ be a standard level, $\mathcal{L}\in\mathrm{StdLev}$, over a semilocal field with an involution and an integral place structure. Then $\mathsf{M}(\mathcal{L})(\mathrm{Box}(\mathcal{L}))\le 1+2^{-n}$.*

Proof. By (R2)(a) and Corollary 6.8(b) for $g=1$, $c=1$ and $\theta=\mathsf{M}_{\ge0}(\mathcal{L})$, where all $\mathsf{d}_l(\mathcal{L};1)$ are 1 and $K_T(1)=\mathrm{Box}(\mathcal{L})$,

$$\begin{aligned}\mathsf{M}(\mathcal{L})(\mathrm{Box}(\mathcal{L})) &= \mathsf{M}_{\ge0}(\mathcal{L})\{y : uy\in\mathrm{Box}(\mathcal{L}) \text{ for some } u\in\mathcal{U}(T)\}\\ &= \mathsf{M}_{\ge0}(\mathcal{L})\{y : y\in\mathrm{Near}_{\varrho+2}(\mathcal{L})\}.\end{aligned}$$

Apply (R2)(c) to the gauge $\Gamma_{\mathcal{L}}(1)$, whose sets are all O_A, of volume 1. The level has $\mathfrak{m}+1$ in the place of $\mathfrak{m}$, so for $b\le\varrho$ the chart of V_b has $|\Omega_b|=\mathfrak{m}+1+\varrho-b$ hyperbolic pairs and the plane of its coordinates: $|\Omega_b|+1$ hyperbolic planes with $|\Omega_b|\ge\mathfrak{m}+1$. By the cone bound (R3)(c) with $\mu=\mathfrak{m}$ and $\beta=1$ the integral vectors have flat quadric measure at most $\zeta_A(\mathfrak{m})$. Hence the right side is at most $\zeta_A(\mathfrak{m})^n\le 1+2^{-n}$. □

LEAN `IntegralPlace.IsStdLevel.groupMeasure_boxSet_le`

Proposition 6.10 (the mass of the integral group, from below)**.** *Let $\mathcal{L}\in\mathrm{Lev}(\Omega,T,\mathfrak{m}+1)$ be a standard level, $\mathcal{L}\in\mathrm{StdLev}$, over a semilocal field with an involution and an integral place structure, and let i and l be natural numbers with $l+i=\varrho$. Then $\mathrm{vol}(O_A^\times)^{i+1}\le\mu^{\mathcal{L}}_{l,i+2}(\mathrm{Box}(\mathcal{L}))$.*

Proof. Write $K=\mathrm{Box}(\mathcal{L})=K_T(1)$, a group. For s-finite measures σ, θ on $G(T)$ we have $\sigma(K)\theta(K)\le(\sigma*\theta)(K)$, because the product of two elements of K lies in K. Induction on i. For $i=0$, $l=\varrho$: $\mu^{\mathcal{L}}_{\varrho,2}=\theta_n(\mathcal{L})*\theta_\varrho(\mathcal{L})$ gives K mass at least $1\cdot\mathrm{vol}(O_A^\times)$ by (LL2). For the step, $l+i+1=\varrho$: $\mu^{\mathcal{L}}_{l,i+3}=\mu^{\mathcal{L}}_{l+1,i+2}*(\eta_l*\theta_l(\mathcal{L}))$, so

$$\mu^{\mathcal{L}}_{l,i+3}(K)\ge\mathrm{vol}(O_A^\times)^{i+1}\cdot\eta_l(K)\cdot\theta_l(\mathcal{L})(K),\qquad \theta_l(\mathcal{L})(K)\ge\varpi_l(K)\,\eta_l(K).$$

Here $\eta_l(K) = \mathsf{d}_l(\mathcal{L};1) = 1$ by Proposition 6.7 for $g = 1$ and the trivial subgroup, and $\varpi_l(K) \geq \mathrm{vol}(O_A^\times)$: for $c \in O_A^\times$ the Weyl element $w_l^\circ(c)$ is an integral isometry, so its inverse lies in K, and the density $\mathrm{mod}_A(c)^{k+1+2(\varrho-l-1)}$ of ϖ_l is 1 at such c. □

LEAN `IntegralPlace.IsStdLevel.le_stackMeasure_boxSet`

Corollary 6.11. *For a standard level $\mathcal{L}$ over a semilocal field with an involution and an integral place structure, $\mathsf{M}(\mathcal{L})$ is a Haar measure of $G(T)$.*

Proof. By Definition 6.4(c), $G(T)$ has a Haar measure ν that is right invariant. The measure $\mathsf{M}(\mathcal{L})$ is right invariant and s-finite ((R1)), hence equal to $d\,\nu$ for some $d \in [0,\infty]$, by (L9); the group $G(T)$, a closed subgroup of the invertible matrices over A, is locally compact and second countable. It is not zero, so $d \neq 0$; and $d \neq \infty$ because $\mathsf{M}(\mathcal{L})(\mathrm{Box}(\mathcal{L})) < \infty$ by Proposition 6.9 while $\nu(\mathrm{Box}(\mathcal{L})) > 0$, the box being open. □

LEAN `IntegralPlace.IsStdLevel.isHaarMeasure_groupMeasure`

7 The Bessel systems of admissible data

In this section P is a pair of parameters and D admissible data for P (Chapter III of the write-up), with the notation of Input (D1). The file of the formal text that holds Theorem F opens with the last step of the seesaw argument of unit 141. It does not concern Bessel systems; it is stated here, in the words of the note on the seesaw identity, because it is a member of this unit.

Proposition 7.1 (the last stage of the seesaw)**.** *Let P be parameters, D admissible data for P, and M a natural number with $M = m(D)$. Let $\mathrm{St}_M(D)$ be the space of the smooth algebraic modular forms of stage M on the definite space $X_M(D)$, $\mathcal{F}_M(D)$ the space of the forms on the standard definite space $Y_M(D)$ of its type, with the Petersson product, and $\mathcal{Y}_{\mathrm{last}}(D)$ the grading of $Y_M(D)$ with an empty string (unit 141). There is a $\mathbb{C}$-linear map $R\colon \mathrm{St}_M(D) \to \mathcal{F}_M(D)$ with the following two properties.*

(1) For every tame character ϑ of D, every finitely supported function Ψ on the tuples of translations, every form y of principal level on $Y_0(D)$ and every column $\varphi \in \mathrm{Col}(D)$: if $\mathcal{L}^{(M)}(D,\vartheta)(y) = R(\mathrm{C}^{(M)}_{D,\vartheta}[\Psi](\varphi))$, then $\iota_{D,\vartheta}(y) = B^{\Psi}_{D,\vartheta}(\varphi)$.

(2) For every linear functional t on the values of the forms of stage M there is a complex number C_{t} such that for every $\Xi \in \mathcal{F}_M(D)$, every $\psi \in \mathrm{St}_M(D)$ and every function F on $G(Y_M(D))$ with $F(i_{y_{\mathrm{last}}(D)}(h)) = \langle \mathsf{t}, \psi\rangle(\mathrm{diag}(h,1))$ for all h:

$$\int_{\Gamma(Y_M(D))\backslash G(Y_M(D))} F\overline{\Xi} = C_{\mathsf{t}} \cdot \langle \Xi, R\psi\rangle.$$

Here $\mathcal{L}^{(M)}(D,\vartheta)$ is the chain of the first M indefinite lifts, $\iota_{D,\vartheta}$ the same chain read as an algebraic modular form on $Y_M(D)$, $\mathrm{C}^{(M)}_{D,\vartheta}[\Psi]$ the combination of the chains of definite lifts with the translated vacua, $B^{\Psi}_{D,\vartheta}$ its restriction to $Y_M(D)$, and $\langle \mathsf{t}, \psi\rangle(g) = \mathsf{t}(g_\infty^{-1}\psi(g_f))$ the coefficient of ψ at t, all as defined in the notes of unit 141.

Proof. This is an assembly; its content lies in the four inputs (D2), which we did not read beyond their statements, and part (2) is a description of the formal proof, not a complete proof. Let alg and res_D be the maps of (D2)(a) and (b), and put $R = \mathrm{alg}^{-1} \circ \mathrm{res}_D$. (1) By (D2)(a) and (b), $\iota_{D,\vartheta} = \mathrm{alg} \circ \mathcal{L}^{(M)}(D,\vartheta)$ and $B^{\Psi}_{D,\vartheta} = \mathrm{res}_D \circ \mathrm{C}^{(M)}_{D,\vartheta}[\Psi]$ on columns, so applying alg to the hypothesis gives $\iota_{D,\vartheta}(y) =$

$\mathrm{res}_D(\mathrm{C}^{(M)}_{D,\vartheta}[\Psi](\varphi)) = B^{\Psi}_{D,\vartheta}(\varphi)$. (2) Let C be the constant of the branching period (D2)(c) for the functional $\mathtt{t}$, the highest weight vector and the branching matrix of res_D. Given Ξ, write $\Xi = F_\varphi$ with $\varphi = \mathrm{alg}(\Xi)$; by (D2)(d) its complex conjugate is an automorphic function $\Phi(x,g) = L_g(\rho(x)v^+)$. By (D2)(c) the integral of Φ against the coefficient $\langle \mathtt{t}, \psi\rangle(\mathrm{diag}(h,1))$ is C times the integral over the classes of L_h applied to the branching matrix at $\psi(\mathrm{diag}(h,1))$, that is, up to the non-zero constant of res_D, of $L_h(\mathrm{res}_D(\psi)(h))$; and by (D2)(d) the latter integral is $\dim\rho$ times $\int \overline{\Xi}\cdot F_{\mathrm{res}_D\psi} = \langle \Xi, R\psi\rangle$. So $C_{\mathtt{t}}$ is C times $\dim\rho$ and the inverse of the constant of res_D. The hypothesis on F gives the values of F on the image of $i_{\mathcal{Y}_{\mathrm{last}}(D)}$, where the integrand $F\overline{\Xi}$ is the integrand of (D2)(c). The string of $\mathcal{Y}_{\mathrm{last}}(D)$ is empty, so this image is the whole group; the formal proof makes this identification by a reindexing of the basis, which we did not write out. □

LEAN `Data.exists_last_stage_map`

Lemma 7.2 (differences of the row parameter)**.** *Let P be parameters, D admissible data for P, τ an infinite place of $E(D)$ and $i<l$ natural numbers. Then*

$$1 \le \mathrm{par}_{D,\tau}(i) - \mathrm{par}_{D,\tau}(l) \le w_{D,l+2-k(D)}(\tau).$$

Proof. Write $k = k(D)$, $r = r(D)$, $h_2 = h_2(D) = r^8$, $w_j = w_{D,j}(\tau) = b_* + 2jh_2 + h_2$ ((D1)); note $k \le r$ and $r \ge 1$. If $l+1<k$, both entries are in the first range, the difference is $l-i \in [1,l]$, and $w_0 \ge h_2 \ge r > l$. If $i+1<k\le l+1$, the difference is $k/2-1-i+b_*+2(l+2-k)h_2$; it is at most w_{l+2-k} because $k/2-1-i \le r \le h_2$, and at least $1-k/2+2h_2 \ge 1$ because $i \le k-2$ and $l+2-k \ge 1$. If $k \le i+1$, the difference is $w_{l+2-k} - w_{i+2-k} = 2(l-i)h_2$, which lies between 1 and w_{l+2-k}. □

LEAN `Data.rowParameter_sub_mem`

Proof of Theorem F. Let $S = S(D)$; by (D1)(a) it fits $Y_0(D)$, so by (F1) the Bessel system at τ has the type of the parameter $l_\tau = \mathrm{par}^W_{D,\tau}(\cdot)$ and of the weight of the columns at τ.

(a) $\#\mathrm{Bc}(S) = \prod_\tau \#I_\tau(S)$. By (R4)(ii) there is an invertible matrix between the coordinates $I_\tau(S)$ and the Weyl coordinates, so $\#I_\tau(S)$ is the dimension of the representation, which by (D1)(d) is the symmetric power of order $a_\tau \le a(D)$ of $\mathbb{C}^k$: the binomial coefficient $\binom{a_\tau+k-1}{k-1} \le (k+a(D))^k$. With $(k-1)! \le k^k$ the left side is at most $(((k+a(D))k)^k)^f$, and $(k+a(D))k \le b(D)^{5/4}$ by (D1)(c).

(b) Write $\rho(i) = \mathrm{par}_{D,\tau}(i)$ for a fixed τ. By (R4)(i), $\deg_\tau(S)$ is Weyl's number of l_τ. Since $l_{\tau,j} = 2\rho(\nu(j))$ and ρ is strictly decreasing (Lemma 7.2), the pairs with $l_{\tau,j} < l_{\tau,i}$ are the pairs with $\nu(i) < \nu(j)$, and

$$\deg_\tau(S) = \prod_{l<r}\prod_{i<l}(\rho(i)-\rho(l)) \Big/ \prod_{l<r} l! \;=\; \prod_{l=k-1}^{r-1}\Big(\prod_{i<l}(\rho(i)-\rho(l))\Big/ l!\Big),$$

because for $l+1<k$ the differences are $l-i$ and their product over $i<l$ is $l!$. Taking logarithms and the product over τ,

$$\log\deg(S) + f\sum_{l=k-1}^{r-1}\log l! = \sum_\tau \sum_{l=k-1}^{r-1}\sum_{i<l}\log(\mathrm{par}_{D,\tau}(i) - \mathrm{par}_{D,\tau}(l)).$$

All terms are non-negative by Lemma 7.2. *Lower bound.* Keep the terms with $i<k-1$. For these, with the notation of the proof of Lemma 7.2, $\rho(i)-\rho(l)-b_* =$

$k/2 - 1 - i + 2(l + 2 - k)h_2 \geq 0$, and $w_D(\tau) \leq w_0 - h_2 = b_*$ by (D1)(c); so each of them is at least $\log w_D(\tau)$. There are $k - 1$ values of i and $r - (k - 1) = m$ values of l. *Upper bound.* By Lemma 7.2, each term is at most $\log w_{D,l+2-k}(\tau)$, which is non-negative, so the sum over $i < l$ is at most $r \log w_{D,l+2-k}(\tau)$, and $\log w_{D,l+2-k}(\tau) \leq \log w_D(\tau) + 1/x(D)$ by (D1)(c), as $l + 2 - k \leq m$. There are m values of l. □

8 Notes on the formal text

Arguments that the text reading of the census gets wrong. The build of this note takes the arguments of each declaration from a reading of the text, not from Lean. For the following declarations that reading evidently differs from Lean's signature, and the statements above follow Lean. The Poincaré series takes X, ϕ, Δ, g; the census adds the pair a, which the definition binds in its sum. The Poincaré map takes only the hypothesis X, implicitly; the census lists the section variables a, ϕ, g. For the unfolding (Proposition 3.8), for Lemma 6.2 and for Theorems C and D the census stops at the first `let` of the statement and misses what follows it: the data S, X, ϕ and the admissibility in the first; the shape, t and x in the second; the conclusions with their quantifiers in the last two. In the file of Section 6 the ring A is declared twice; the census lists the instances of the first declaration (ring, involution, topology, measure) also for the statements about a semilocal field, where Lean takes the second. For Theorem E the census lists a form T, a level and the hypothesis that it is standard, which are section variables that the statement does not use: it binds its own level. For the gauge the census lists the section variables N, t, x, which the definition does not use. None of this was confirmed by running Lean.

Definitions that carry a proof. The Poincaré map (Theorem A(b)) is a definition whose value is a pair, a linear map with the proof of its formula; the forms F_a are obtained by the axiom of choice from the existence statement (L1). The isomorphism A_h of Definition 4.2 is likewise a pair, with its transformation rule. The gauge of Definition 6.5 is a record whose fields are the axioms verified in Lemma 6.6; its last two explicit arguments are the proofs of $\alpha \in O_A$ and $\bar{\alpha} = \alpha$, with default values for $\alpha = 1$.

The side. The Bessel functions in the cut Bessel functions, hence in Theorems A and B, are the full ones: the formal text declares the full side as the default instance, and no statement of these files names the other. The band-limit forms of Theorem D belong to the cut functions; this enters only through Input (S5).

Hypotheses and constants. In Proposition 5.3 the filling data enter only through the inequalities $1000 \leq b$ and $f \leq b/1000$, and the proof uses $\mathrm{den}(y_{\mathsf{T}})^{-3}$ where the statement has the sixth power. In Theorem C admissibility is used once, for $|\kappa_\tau(S)| \geq 1$, and the hypothesis that the inverses of the elements of $\mathcal{A}$ are T-integers once, for the differences $s^{-1} - s'^{-1}$. In the proof of Theorem D the formal text does not name the constants C and e: the first is written as a power of b with an exponent left to unification, the second is whatever the estimate of the frequency sum produces; the values $\mathsf{C} = b^{1000n^9 f}$ and the displayed e are ours, and so is the final count of

exponents, which the formal text leaves to its automation. Lemma 7.2 and the two comparisons in the proof of Theorem F(b) are proved in the formal text by a decision procedure from the monotonicity of the row parameter (unit 147); the arguments given here are ours. In Theorem F(a) the right side is written $((\sqrt{\sqrt{b}}^{5})^{k})^{f}$ in the formal text. The measurability lemma absorbed in Proposition 3.8 stands in a section that assumes admissibility and stable unramification; its proof uses neither.

Inputs. Proposition 7.1 belongs to the seesaw argument (unit 141); the file states it here because it needs the block embedding of the last grading. The header of the file of the sieve names, as related work on inductive Fourier transforms of Kloosterman integrals and on spectral large sieves, papers of Jacquet (2003, 2004), Blomer and Buttcane (2015) and Deshouillers and Iwaniec (1982); we did not compare them with Theorems C and D, which are not taken from them. The notes of unit 132 write the head weight, the damped power and the band constant with other symbols; this note follows the common table of the notes of this unit. Of this unit's declarations, the main-term estimate of Note F uses Theorem D, Proposition 6.3 and the masses of Section 6; Notes T and G use the standard level and its gauge; the admissibility of the filling data of D (Note G) uses Theorem F.

How this note was written. From the dossier of the unit: the texts of the members of these five files and their structured proofs. I read the five Lean files completely, line by line, and compared every statement of this note with the Lean statement, argument by argument, reading in full the statements that the census cuts at a `let` (the unfolding, the cut Bessel function of a translate, Lemma 6.2, Proposition 6.3, Theorems C and D). The readers of the dossier warn that automation hides arguments in three proofs; I read these in the source: the Fourier coefficients in the proof of Theorem C (the computation of the character and the vanishing at frequencies that are not T-integers are written out above), the single construction in the proof of Theorem E (described axiom by axiom), and the reindexing of Weyl's formula in Theorem F(b) (replaced by the computation above). The long definition of the gauge and the definitions of A_h and of the Poincaré map, which have no structured proof or a short one, were read in the source as well. Of the inputs I read the Lean statements and the definitions they rest on (hermitian spaces with frame, gradings, forms, idempotent kernels, levels, gauges, hyperbolic bases, integral places, the type of a Bessel system, the row parameter and the weights of the data) and not the proofs; the inputs from Notes S and B were compared with their Lean statements. The proofs of Theorems C and D, of Propositions 5.3, 6.7, 6.9, 6.10 and of Theorems A and B follow the formal ones step by step, with the exponents of Theorem D made explicit; the proofs of Theorem E, of Lemma 6.6 (axiom (G10)), of Proposition 6.3 and of Proposition 7.1 describe the formal construction and are less detailed than the formal text. In a revision after the referee's report the count of the columns in axiom (G10) was corrected and compared with the formal proof once more, and six facts of the library that the proofs used without naming them became the Inputs (L5)–(L10), the inputs of Proposition 7.1 the Input (D2); their Lean statements were read, their proofs were not. References are those of the catalogue of the write-up and of the second reader's verdicts; I did not open the cited works. The program check of the unit's dossier covered 1371 leaf steps of 179 proofs.

Appendix A. Concordance with the Lean artifact

The formal names of the objects: `S.Fits V l kw Y n` is $\mathrm{Fits}(S; V, l, \mathsf{k}, \mathcal{Y}, \eta)$, and the records named `X` or `h` of this type are the hypothesis X; `V.Γ`, `V.x`, `V.xFin`, `V.J` are $\Gamma(V)$, fr_V, $\mathrm{fr}_{V,f}$, J_V; `G E ε` is $G(V)$; `Y.unipotent`, `Y.besselGroup`, `Y.inclX` are $U(\mathcal{Y})$, $B(\mathcal{Y})$, $i_{\mathcal{Y}}$; `typeKernel l ε` is ϵ_l; `toBesselH` is $\eta \mapsto \dot{\eta}$; the finite set `T` of places is T, `LocProd T` is E_{T}, `ratIntegralOff T` the T-integers, `ratToPlaces T` the map $y \mapsto y_{\mathsf{T}}$, `adeleOfLocal` the extension by zero, `padTuple p t` the sequence $(p \,|\, t)$; `unitOrOne` is u, `wave` is $c \mapsto \exp(c - \bar{c})$; the finite sets `A`, `V`, `U`, `X` of the sieve are $\mathcal{A}$, $\mathcal{V}$, $\mathcal{A}'$, $\mathcal{X}$, and `Γ`, `ε`, `En` are C, e, En; `F : Level ι T m` is $\mathcal{L} \in \mathrm{Lev}(\Omega, T, \mathfrak{m})$, with `F.g`, `F.boxSet`, `F.𝔅`, `F.positions`, `F.θ`, `F.measureH` the unit, the box, the ball, the positions and the measures of the floors ϱ and n; `unitaryLevel R g J` is $K_T(g)$, `intUnits` is $O_A^\times$, `idealZeta R m` is $\zeta_A(\mathfrak{m})$, `md` is mod_A; `D.filling` is $S(D)$, `D.rowParameter` and `D.paramW` are the row parameter and its double. Three statements, as they stand in the artifact:

```
lemma petersson_poincare [StablyUnram E] {k s} (hs : Kf.reduceFamily _ k = Sigma.uncurry s) :
  inner ℂ (V.formRep _ k (X.poincareMap.1 (.single a' 1)).1) (X.poincareMap.1 (.single a 1)) =
  volume.real S.stdPrincipalLevel.carrier * S.amplitudeSum .univ a.1 s a'.1 (a.2, a'.2)

lemma sum_frequencies_le [WeilData E] [T.StarStable]
    B k X (c : _ → ℂ) (x : _ → ℝ) (hB : 1 ≤ B) (hx : ∀ v, ‖c v‖⁻¹ ≤ |x v| ∧
  |x v| ≤ B) (h : ∀ y ∈ X, y ≠ 0 ∧ y ∈ ratIntegralOff T) : ∑ y ∈ X, (∏ v, dampedPower k ‖c v * Φ v
  y‖⁻¹) * (V(genIndex (ratToPlaces T y)) ^ k / denomIndex 0 (ratToPlaces T y) ^ 6) ≤
  (∏ v, |x v| ^ k * (S.b * B ^ 2) ^ 3) * (S.b * rootDiscr E) ^ (numInfPlaces E * 3)

lemma IntegralPlace.exists_isLatticeLevel {N} [IsHermUnit (latMiddleGram ś N).splitGram] (hN :
  N ∈ R ∧ star N = N) (hg : g.1 ∈ R ∧ star g.1 = g) (hm : ś.k = card ι + 2 * m + 1) : ∃ F :
  Level ι (latMiddleGram ś N) m, P.IsLatticeLevel F ∧ F.g = g
```

In this note	Lean declaration	File	Lines
Lemma 4.1	`Filling.Fits.mem_unipotent_iff`	ColumnCoords.lean	19–21
Lemma 4.1	`Filling.Fits.adelicToFin_adelicEquiv_inclX`	ColumnCoords.lean	23–25
Lemma 4.1	`Filling.Fits.finiteEquiv_inclX`	ColumnCoords.lean	27–28
Lemma 4.1	`Filling.Fits.adelicToInf_inclX`	ColumnCoords.lean	30–34
Definition 4.2	`Filling.Fits.weylBesselEquiv`	ColumnCoords.lean	38–46
Definition 4.2	`Filling.Fits.columnCoords`	ColumnCoords.lean	51–53
Proposition 7.1	`Data.exists_last_stage_map`	DegreeBounds.lean	18–34
Theorem F	`Data.card_besselCoord_mul_factorial_le`	DegreeBounds.lean	42–50
Lemma 7.2	`Data.rowParameter_sub_mem`	DegreeBounds.lean	52–54
Theorem F	`Data.log_filling_besselDegree_mem_Icc`	DegreeBounds.lean	61–78
Definition 5.1	`Filling.charOnPlaces`	GlobalSieve.lean	26–27
Lemma 5.2	`Filling.charOnPlaces_evalPlaces`	GlobalSieve.lean	29–31
Definition 5.1	`Filling.frequencyGain`	GlobalSieve.lean	35–36
Definition 5.1	`Filling.ratPoint`	GlobalSieve.lean	40–40
Proposition 5.3	`Filling.sum_frequencies_le`	GlobalSieve.lean	49–80
Theorem C	`Filling.one_sided_poisson`	GlobalSieve.lean	92–164
Theorem D	`Filling.sieve`	GlobalSieve.lean	172–213
Definition 3.1	`Filling.Fits`	PoincareSeries.lean	26–32
Definition 3.1	`Filling.stdPrincipalLevel`	PoincareSeries.lean	36–37
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Notation

E, τ, x_τ, $f = f(E)$, rd_E a CM field, an infinite place, the image of x at τ, the number of infinite places, the root discriminant

w, E_w, $\mathbb{F}_w$, $\mathbb{A}_E$, $\mathbb{A}_{E,f}$, $\hat{O}_E$, $\mathrm{vol}(\hat{O}_E)$ a finite place, completion, residue field, adeles, finite adeles, their integers and their volume

$\mathfrak{s} = (\varrho, \kappa, k_0)$, $\varrho(\mathfrak{s})$, $k(\mathfrak{s})$, $n(\mathfrak{s})$, $r(\mathfrak{s})$, $I(\mathfrak{s})$ a shape and its numbers ϱ, k, $n = \varrho + 1$, r; the index set of the basis e_a, x_j, z, e_a^*

$\mathfrak{f}_a$, V_l, h_T, $J(T)$, $G(T)$, $\mathcal{U}(T)$, $\mathcal{H}(T)$, $\mathcal{S}(T)$, $\mathcal{H}_l(T)$, $\mathcal{Q}_l(T)$, $\mathcal{N}_{<l}(T)$, $\mathcal{N}_l(T)$ the frame, the span of the floors $\geq l$, the form, its split Gram matrix and unitary group, the Bessel subgroups

$T^{\mathrm{rat}}(\mathfrak{s}; N)$, $T^{\mathrm{lat}}(\mathfrak{s}; N)$, $\Lambda(\mathfrak{s}; p)$, $G_1(\mathfrak{s})$, $\mathrm{sd}(u)$ the forms in the rational and the lattice basis, the lattice basis, the complex group, the simple root sum

$S \in \mathrm{Fill}(E, \mathfrak{s})$ filling data, with $N(S)$, $q_j(S)$, $g(S)$, $\bar{\pi}(S)$, $\varpi(S)$, $b(S)$, $\mathrm{Ch}_j(S)$, $K^{\mathrm{col}}(S)$, $\mathfrak{B}_\tau(S)$

$T_S^{\mathrm{rat}}(R)$, $T_S^{\mathrm{lat}}(R)$, $G_S(R)$, G_S^{lat}, $\Lambda(S)$, lb_S forms and groups of S over a ring R; the lattice basis and the change to it

$K(S)$, $\mathsf{G}(S)$, red_S, $K_g(S)$, $K_g^{\mathrm{std}}(S)$ the integral subgroup, the finite group, the reduction, its kernel; the principal level at the chosen places

$\mathrm{r}_S^{\mathbb{A}}$, a_S^∞, a_S^f, r_S^∞, r_S^f, inc_S, $\mathrm{Sc}_{S,\tau}$ the maps between rational, adelic, real and finite points; the inclusion of the definite group; the scales

$\mathrm{Cl}(S)$, $h_S(c)$, $\mathrm{pos}_S(c, c'; \gamma)$, $\mathrm{Tm}_S(c, s, c')$, $\mathcal{U}_{S,c}$, $\beta_{S,c}$ classes, representatives, positions, terms, the arithmetic unipotent group, the cutoff

$\mathrm{Bc}(S)$, $I_\tau(S)$, $W_{S,\tau}^\sigma$, $W_S^\sigma(y)_i$, $\mathcal{W}_S^\sigma(\gamma; y)$, $\deg_\tau(S)$, $\deg(S)$ Bessel coordinates, Bessel functions for the side σ, the kernel, the degrees

$a_S^\sigma(c,c';\gamma)$, $|a|_S^\sigma(c,c';\gamma)$, $A_S^\sigma(\Xi;c,s,c')$ amplitudes and amplitude sums

ψ_S, ψ_f, ψ_E, $\kappa_\tau(S)$, $\mathrm{fs}_S(\tau,a)$ the self-dual character, the standard characters, the numbers κ_τ and the frame scales

$\mathrm{dp}_j(\rho)$, $\Lambda_K^{\mathbb{C}}(i;p)$, $\Lambda_K(i;p)$, $[\xi]$, $\mathrm{den}(\xi)$, $c_{\mathrm{band}}(\mathfrak{s})$ damped power, head weights, generalized index, denominator, band constant

$\mathrm{Sieve}(u,\iota,\mathsf{n},J,\psi,\mathrm{En},\Lambda,\beta)$ a sieve system of length n on a ring C with units U; $p[i\mapsto x]$ a sequence with one entry replaced

V, I, I_0, J_V, fr_V, $G(V)$, $G_\infty(V)$, $G_f(V)$, $\Gamma(V)$, Δ a hermitian space with frame, its index sets, Gram matrix, frame, groups, rational points, a subgroup

$G_f(E,I)$, $K_f(E,I)$, $R(k)$, ϵ, ϵ_l, $\langle F,F'\rangle$ finite adelic unitary group, its integral points, right translation, a type, the type of a parameter, the Petersson product

$\mathcal{Y}$, λ, $U(\mathcal{Y})$, $B(\mathcal{Y})$, i_y a grading with a string of length λ, its unipotent subgroup, Bessel group, inclusion of the definite group

$\mathrm{Fits}(S;V,l,\mathsf{k},\mathcal{Y},\eta)$, X, p, $\dot\eta$, k_0, u S fits V; the hypothesis; the frame of the signs; η on the definite part; the matrix and the idele of (F4)

rat_X, ad_X, re_X, fin_X the four dictionaries

$\mathrm{cb}_X(a;g)$, $\mathrm{P}_X(\phi,\Delta)(g)$, Q_X, e_a, F_a, W_q, γ_q cut Bessel function, Poincaré series, Poincaré map, unit vectors, the forms and the partial sums of the proof

A_h, cc_h, $\mathbb{S}_{\mathsf{k}}$, $\mathsf{A}^{(\tau)}$, g_c Weyl to Bessel coordinates, column coordinates, the representation of weight k, representatives

T, E_{T}, O_{T}, y_{T}, x^0, $\psi_{S,\mathsf{T}}$, $\beta_{S,\mathsf{T}}(\xi)$, $\mathrm{pt}_{\mathsf{T}}(l;t)$ a finite set of places, the product of completions, integers, image, extension by zero, the character, the gain, a point

$\mathcal{A}$, $\mathcal{A}'$, $\mathcal{V}$, $\mathcal{X}$, $\mathcal{N}$, Z, B, B_0, ρ_τ finite sets of T-units and of frequencies; a neighbourhood and a measurable set; bounds

C, U, $\mathrm{hd}_i(p,t)$, $(p\,|\,t)$, c_a, C, e, $\Lambda^{(i)}$, F, G, $\hat G$, $\mathfrak{m}$, U_0, L_y the objects of Theorem C and of its proof

J^τ, J', e, $x_\tau(p,t)$, $y(p,t)$, $m_j(p)$, $F_{\tau,j}$, ω the objects of Theorem D and of its proof

A, O_A, A^-, mod_A, ψ_A, τ_A, δ_A, ν_A, $\zeta_A(\mu)$, ct a place, its integers, skew elements, module, character, trace-one element, skew unit, measure on A^-, ideal zeta sum, content

Ω, Ω', Ω_b, $\mathfrak{m}$, $\mathcal{L}\in\mathrm{Lev}(\Omega,T,\mathfrak{m})$, $\mathrm{Hyp}(\Omega,T,\mathfrak{m})$ finite sets, the number of pairs, a level, hyperbolic bases

$E(\mathcal{L})$, $g_{\mathcal{L}}$, $\mathfrak{B}(\mathcal{L})$, $\mathrm{Box}(\mathcal{L})$, $\mathrm{Posn}(\mathcal{L})$, $\theta_l(\mathcal{L})$, η_l, $\eta_{<j}$, ϖ_l, $\delta^{(b)}(\mathcal{L})$ the data of a level, floor measures, Heisenberg measures, Weyl laws, quadric measures

$\mu^{\mathcal{L}}_{a,i}$, $\mathsf{M}(\mathcal{L})$, $\mathsf{M}_{\geq l}(\mathcal{L})$, $\mathsf{d}_l(\mathcal{L};g)$, $\mathsf{s}(\mathcal{L})$ stack, group and upper measures, floor module, volume exponent

$K_T(g)$, $\mathrm{K}^\circ(\mathcal{L})$, $\mathcal{L}\in\mathrm{LatLev}$, $\mathcal{L}\in\mathrm{StdLev}$, $\Gamma_{\mathcal{L}}(\alpha)$, $\mathrm{Gau}(\mathcal{L})$, Γ_j, Λ congruence subgroups, lattice and standard levels, the gauge of depth α

$\mathrm{tv}(\pi,t,x)$, $\mathrm{n}_a(v,\beta)$, $w_l^\circ(c)$, $\mathrm{E}(e)$, ς', φ, Φ, Ψ, q, N, Σ_g, K, M, M', Y, W, d tail vector, Heisenberg and Weyl elements, Eichler elements, charts and functions of Section 6

P, D, $S(D)$, $E(D)$, $x(D)$, $b(D)$, $k(D)$, $f(D)$, $m(D)$, $r(D)$, $a(D)$, $h_2(D)$, $h(D)$ parameters, admissible data and their quantities

$w_{D,j}(\tau)$, $w_D(\tau)$, $\mathrm{par}_{D,\tau}(i)$, $\mathrm{par}^W_{D,\tau}(j)$, $\underline{R}(D)$, ν, b_*, a_τ, $\rho(i)$ weights, nominal weight, row parameter and its double, row indices, their numbering

$Y_M(D)$, $X_M(D)$, $\mathrm{St}_M(D)$, $\mathcal{F}_M(D)$, $\mathrm{Col}(D)$, $\mathcal{Y}_{\mathrm{last}}(D)$, ϑ, Ψ, $\mathcal{L}^{(M)}(D,\vartheta)$, $\iota_{D,\vartheta}$, $\mathrm{C}^{(M)}_{D,\vartheta}[\Psi]$, $B^\Psi_{D,\vartheta}$, $\langle\mathsf{t},\psi\rangle$, R, C_{t}, Ξ the objects of Proposition 7.1

alg, φ_F, F_φ, res_D, B, ρ, v^+, L_g the objects of Input (D2): forms and algebraic modular forms, the restriction, a branching matrix, the representation of a weight, its highest weight vector, linear functionals

b, g, k, l, n, r, s, t, c, c', h, i, i', j, m, p, q, v, y, z, α, β, γ, δ, θ, μ, ξ, σ, ϕ, φ, η, π, ψ, ρ, ι, N, T, H, L, Q local letters, as each statement says

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[GL] D. Gentner, G. Last, *Palm pairs and the general mass-transport principle*, §2, equations (2.5)–(2.9) (arXiv:0902.0068). The second reader's source for the unfolding (L2).

[Ta] J. Tate, *Fourier analysis in number fields and Hecke's zeta-functions* (1950), Theorem 4.1.4 and Lemmas 4.2.1–4.2.4. The catalogue's and the second reader's source for (L3), which is a weaker form.

READBACK NOTE 28

The tower method at a place: floor measures, gauges and the bounds on each floor

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. Let $G(T)$ be the unitary group of a hermitian space with ϱ hyperbolic planes over a local ring A. The stabilisers $\mathcal{H}_l(T)$ of the first l planes form a tower, and the quotient of two consecutive ones is a quadric. A *level* puts a measure on each floor, the image of an explicit measure on a big cell, and their convolution $\mathsf{M}(\mathcal{L})$ is a right invariant measure on $G(T)$: the method works when it is a Haar measure. A kernel on $G(T) \times G(T)$ is averaged floor by floor, and on each floor three tools bound the averages: a Cauchy–Schwarz inequality against the mass of the floor, integrals of weights made of volumes, and Plancherel's inequality on the quadric and on a Heisenberg group. All sizes are kept by a *gauge*, a filtration of A with one constant Λ. This note sets up these objects and proves the tools; the companion notes apply them.

What this note is. The artifact grh-compact is a body of Lean code whose final theorems are the Riemann hypothesis and the generalized Riemann hypothesis for Dirichlet L-functions. Its authors report that it compiles and that Lean's kernel accepts these theorems from the three standard axioms; nothing was built for this note. That is a statement about a formal text: no mathematician has read the argument through, and this note does not say that the Riemann hypothesis is proved. The note treats one part of one unit of the artifact (the folder of the analysis at the real places): 116 of the unit's 239 declarations, those of five files; two companion notes treat the others (Note L, *The local theorem for floor forms and the band limits at the complex place*; Note C, *The complex place: the open-cell chart, the Haar measure of the tower, the Bessel cut and the volume weight*). Every result of the five files is stated here, compared argument by argument with its Lean text, and proved from the inputs of Section 3, which belong to the background library; the last section says which proofs follow the formal text step by step and which were only traced through its terms. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 116 declarations.

1 The statements

The objects are defined in Section 2. A is a place, T an $\mathfrak{s}$-form over A, $G(T)$ the unitary group of its split Gram matrix, $\mathcal{L}$ a level, Γ a gauge on $\mathcal{L}$.

A word on the names. A *real place* of the write-up is an infinite place of the maximal totally real subfield of the CM field $E(D)$ of the overview; the completion of $E(D)$ above it is $\mathbb{C}$, and the unitary group of the rows there is $G(T)$ for $A = \mathbb{C}$ and the form T that is the identity on the x_i and -1 at z, a unitary group of signature $(\varrho + k, \varrho + 1)$, written $G_1(\mathfrak{s})$ in the companion notes. In the formalism of places of this note that is the place $A = \mathbb{C}$ with complex conjugation, which the three notes

call *the complex place*; so "the real places" (the name of the unit and of the overview) and "the complex place" (Notes L and C) are the same local situation. This note works at a general place.

Theorem A (the tower measure). Let A, $\mathfrak{s}$, T, Ω, $\mathfrak{m}$ be as in the conventions and $\mathcal{L}$ a level in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$.
(i) Let l, i be natural numbers with $l + i = \varrho + 2$ and $s \in \mathcal{H}_l(T)$. Then $\mu^{\mathcal{L}}_{l,i} * \delta_s = \mu^{\mathcal{L}}_{l,i}$. In particular $\mathsf{M}(\mathcal{L})$ is invariant under right translations.
(ii) For every natural number j: $\mu^{\mathcal{L}}_{0,j} = \eta_{<j} * (\theta_{j-1}(\mathcal{L}) * \cdots * \theta_0(\mathcal{L}))$.
(iii) Suppose that $\mathsf{M}(\mathcal{L})$ is a Haar measure. Let j be a natural number with $j \le \varrho + 2$, let $p \colon \mathbb{N} \to A^\times$, and let $\Phi \colon G(T) \to [0, \infty]$ be measurable and invariant under left translations by $\mathcal{H}_j(T)$. Then

$$\int \Phi(w_j(p)^{-1} g) \, \mathrm{d}\mu^{\mathcal{L}}_{0,j}(g) = \int \Phi(g) \, \mathrm{d}\mu^{\mathcal{L}}_{0,j}(g).$$

LEAN `Level.stackMeasure_mconv_dirac` · `Level.isMulRightInvariant_groupMeasure` · `Level.stackMeasure_zero_eq` · `Level.lintegral_stackMeasure_mul_left`

Setting 1.1 (a level with a gauge)**.** Let A, $\mathfrak{s}$, T, Ω, $\mathfrak{m}$ be as in the conventions, $\mathcal{L}$ a level in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$ and Γ a gauge in $\mathrm{Gau}(\mathcal{L})$. Sizes (sz, Vec, heads) are taken for Γ.

Theorem B (two integrals of weights). In Setting 1.1, let j be a natural number with $j \le \varrho$.
(i) Let $g, y \in G(T)$ with g near at j: $g \in \mathrm{Near}_j(\mathcal{L})$. Then

$$\int \mathbf{1}_{\mathrm{Pos}(\mathcal{L};\Gamma)}(g^{-1}\nu y) \, \mathsf{w}_j(\mathcal{L}; g, g^{-1}\nu y) \, \mathrm{d}\eta_{<j}(\nu) \le \prod_{l<j} C_l(\Gamma).$$

(ii) Let $p \colon \mathbb{N} \to A^\times$ be a head of length j, $p \in \mathrm{Hd}_j$, let $x \in \mathrm{Posn}(\mathcal{L})$, let b be a natural number with $j \le b$, let ρ be an s-finite measure on $G(T)$, and let $\Phi \colon G(T) \to [0, \infty]$ be measurable, invariant under left translations by $\mathcal{Q}_b(T)$, and zero at every point that is not near at b. Then

$$\int \Phi(g) \, \mathsf{w}_j^{-1}(\mathcal{L}; x)(w_j(p) g) \, \mathrm{d}(\eta_{<j} * \rho)(g) \le \prod_{a<j} \mathsf{d}_a(\mathcal{L}; p_a^{-1}) \cdot C_{\mathrm{gl}}(\Gamma) \cdot \int \Phi \, \mathrm{d}\rho.$$

LEAN `Gauge.lintegral_floorWeight_le` · `Gauge.lintegral_mul_invWeight_le`

Theorem C (Plancherel on a floor and on a Heisenberg group). In Setting 1.1:
(i) Let b be a natural number with $b \le \varrho$, let $t \in \mathcal{H}_b(T)$ and $q, x, y \in G(T)$ with $t^{-1}q \in \mathcal{N}_{<b}(T)$, let j be a natural number with $\mathrm{sz}(qx) \le j$, and let $(k,d) \in \mathrm{PK}_b(\mathcal{L})$. Put $F(\sigma) = \mathsf{Av}_{b+1}(\mathcal{L};k)(\sigma x, \sigma y)$. Then for $\theta_b(\mathcal{L})$-almost every σ with $F(\sigma) \neq 0$ the vector $t\sigma^{-1}f_b$ lies in Vec_{j+6}; and for every $\xi \in A^\times$, every $y' \in V$, every measure m on V, natural number i and $M \in [0,\infty]$ with $\mathrm{Fib}_{b,i}(\Gamma;\mathsf{m}) \le M$,

$$\int_{\mathrm{Vec}_i} \Big| \int \psi_A(\xi\, h_T(v + y', t\sigma^{-1}f_b)) F(\sigma)\, \mathrm{d}\theta_b(\mathcal{L})(\sigma)\Big|^2 \mathrm{d}\mathsf{m}(v) \le M\,\Lambda(\Gamma)^{99n(j+12)} \Big(\frac{c_A}{\mathrm{mod}_A(\xi)}\Big)^{|\Omega|} \int |F|^2\, \mathrm{d}\theta_b(\mathcal{L}).$$

(ii) Let a be a natural number with $a < \varrho$, $\xi, u \in A^\times$, $\varepsilon \in \mathbb{N}$, $C \in [0,\infty]$, $(k,d) \in \mathrm{PK}_{a+1}(\mathcal{L})$, $x, y \in G(T)$, $S \subseteq G(T)$, $\Phi\colon G(T) \to [0,\infty]$ and $i, j \in \mathbb{N}$. Suppose: *(e)* for all $s \in \mathcal{N}_a(T)$, $g_{\mathcal{L}}^{-1}(\mathrm{sd}(dsd^{-1}) - \mathrm{sd}(s^\varepsilon)) = \xi\,\mathrm{sd}(s)$; *(o)* for all $\nu \in \mathcal{N}_a(T)$ with $\Phi(\nu) \neq 0$: $\nu \in S$ and $\Phi(\nu) \le C\,|\mathsf{Av}_{a+1}(\mathcal{L};k)(\nu^\varepsilon x, \nu y)|^2$; *(h)* for every $m \in S$ there are $t \in \mathcal{Q}_{a+1}(T) \cap \mathcal{H}_a(T)$ and $q \in \mathcal{N}_{<a+1}(T)$ with $\mathrm{sz}(tqx) \le i$ such that for every $\nu \in S$ there is $\nu' \in \mathcal{N}_{<a}(T)$ for which the column at e_a^* of the matrix $u^{-1}(\nu'\, tm(t\nu)^{-1} - 1)$ lies in Vec_j. Then

$$\int \Phi\, \mathrm{d}\eta_a \le C\Big(\frac{c_A}{\mathrm{mod}_A(\xi u)}\Big)^{|\Omega|} \mathsf{d}_a(\mathcal{L};u)\,\Lambda(\Gamma)^{100n(i+j+31)} \cdot \int |\mathsf{Av}_{a+2}(\mathcal{L};k)(\sigma x, \sigma y)|^2\, \mathrm{d}\theta_{a+1}(\mathcal{L})(\sigma).$$

LEAN `Gauge.amplitude_support_and_plancherel` · `Gauge.parseval_heisenbergMeasure_of_box`

2 The setting

Conventions. A is a commutative ring with an involution $x \mapsto \bar{x}$; from Definition 2.4 on it carries a topology and a Haar measure $\mathrm{d}x$ and is a *place* (Section 2). $\mathfrak{s} = (\varrho, \kappa, k_0)$ is a shape, $n = \varrho + 1$, $k = |\kappa|$, $r = 2\varrho + k + 1$. T is an $\mathfrak{s}$-form over A; from Lemma 4.3 on, its split Gram matrix $J(T)$ is a hermitian unit. Ω is a finite set Ω, $\mathfrak{m}$ a natural number, $\mathcal{L}$ a level in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$. $\mathcal{S}$ is a size in $\mathrm{Siz}(T)$; the predicates sz, Hd and the sets Vec_i refer to a size that the Lean text takes as an implicit argument, and every statement names it. Index sets of vectors and matrices are finite sets. Integrals of functions with values in $[0,\infty]$ are Lebesgue integrals; a product $0 \cdot \infty$ is 0; subtraction of natural numbers is truncated at 0. "Invariant" means invariant under left translations unless said otherwise; a measure is *carried* by a set if the complement is a null set.

The space and its groups (background library, units 112 and 119). A *shape* is a triple $\mathfrak{s} = (\varrho, \kappa, k_0)$ of a natural number, a finite set and an element of it. The free module $V = A^r$ has the basis $e_0, \dots, e_{\varrho-1}$, x_i $(i \in \kappa)$, z, $e^*_{\varrho-1}, \dots, e^*_0$. An *$\mathfrak{s}$-form* T is a square matrix over A on the indices of the x_i and of z; its *split Gram*

matrix $J(T)$ has the entries 1 at (e_a, e_a^*) and (e_a^*, e_a), the block T on $V_\varrho = \langle x_i, z\rangle$ and 0 elsewhere. It is a *hermitian unit* if $J(T)^* = J(T)$ and $\det J(T)$ is a unit; X^* is the conjugate transpose. We write $h_T(v,w) = v^* J(T) w$, and $G(T)$ is the group of the invertible matrices g with $g^* J(T) g = J(T)$. Each basis vector has a *floor*: a for e_a and e_a^*, ϱ for z, n for the x_i; and a *rank in the flag*: a for e_a, ϱ for z and the x_i, $2\varrho - a$ for e_a^*. V_l is the span of the basis vectors of floor at least l; it has rank $k + 1 + 2(\varrho - l)$ for $l \le \varrho$. The *frame* is $f_0, \dots, f_\varrho$, $f_a = e_a$ $(a < \varrho)$, $f_\varrho = z$; $Z_T(g) = (h_T(f_a, g f_b))_{a,b}$ is the frame Gram matrix. The subgroups of $G(T)$:

- $\mathcal{H}_l(T)$: the g whose matrix agrees with the identity in every row and column of floor less than l;
- $\mathcal{Q}_l(T)$: the g with $g_{\alpha\beta} = \delta_{\alpha\beta}$ whenever the rank of β is at most that of α and the larger of the floor and the rank of β is less than l; for $l \le n$ this says $g f_a - f_a \in \langle e_0, \dots, e_{a-1}\rangle$ for $a < l$;
- $\mathcal{U}(T)$: the g with $g_{\alpha\beta} = \delta_{\alpha\beta}$ whenever the rank of β is at most that of α; and $\mathcal{N}_{<l}(T)$: the same condition for the ranks in which every basis vector of floor at least l gets the rank ϱ; so $\mathcal{N}_{<l}(T) = \mathcal{U}(T)$ for $l \ge \varrho$;
- $\mathcal{N}_l(T) = \mathcal{H}_l(T) \cap \mathcal{N}_{<l+1}(T)$.

For $u \in G(T)$, $\mathrm{sd}(u) = \sum_{a<\varrho} u_{f_a, f_{a+1}}$. For $p\colon \mathbb{N} \to A^\times$ the *Weyl element* $w_l(p)$ maps $e_a \mapsto p_a e_a^*$, $e_a^* \mapsto \bar{p}_a^{-1} e_a$ for $a < \min(l, \varrho)$ and fixes the other basis vectors; $w_a^\circ(c)$ $(a < \varrho,\ c \in A^\times)$ does this for the one pair (e_a, e_a^*). If $\tau_A \in A$ has $\tau_A + \bar{\tau}_A = 1$, then for $a < \varrho$, a vector $v \in V_{a+1}$ and a skew $\beta \in A$ $(\bar{\beta} = -\beta)$ the *Heisenberg element* $\mathrm{n}_a(v, \beta) \in \mathcal{N}_a(T)$ is the isometry that maps $e_a^* \mapsto e_a^* + v + (\beta - \tau_A h_T(v,v)) e_a$, $w \mapsto w - h_T(v, w) e_a$ for $w \in V_{a+1}$, and fixes e_a and the basis vectors of floor less than a.

Facts used (all from unit 119): $\mathcal{H}_l(T)$ and $\mathcal{Q}_l(T)$ decrease with l; $\mathcal{H}_l(T) \subseteq \mathcal{Q}_l(T)$; $\mathcal{N}_{<l}(T) \subseteq \mathcal{U}(T) \subseteq \mathcal{Q}_{l'}(T)$; $\mathcal{H}_l(T) = 1$ for $l \ge \varrho + 2$ and $\mathcal{Q}_{\varrho+2}(T) = \mathcal{U}(T)$; $\mathcal{H}_l(T)$ normalises $\mathcal{N}_{<l}(T)$ and $\mathcal{H}_{l+1}(T)$ normalises $\mathcal{N}_l(T)$;

(F1) every $s \in \mathcal{Q}_l(T)$ is $\gamma\nu$ with $\gamma \in \mathcal{H}_l(T)$, $\nu \in \mathcal{N}_{<l}(T)$; every $s \in \mathcal{Q}_{l+1}(T) \cap \mathcal{H}_l(T)$ is $g\nu$ with $g \in \mathcal{H}_{l+1}(T)$, $\nu \in \mathcal{N}_l(T)$; every element of $\mathcal{N}_a(T)$, $a < \varrho$, is a Heisenberg element;

(F2) for $b \le \varrho$ and $s \in \mathcal{H}_b(T)$: $s \in \mathcal{Q}_{b+1}(T)$ if and only if $s^{-1} f_b = f_b$;

(F3) for $s \in \mathcal{Q}_j(T)$, $a < j$ and v with $v_{e_b^*} = 0$ for $b < a$: $(sv)_{e_a^*} = v_{e_a^*}$; the row of s at e_a^* agrees with that of the identity outside the columns e_b^*, $b < a$;

(F4) for $g \in \mathcal{N}_{<j}(T)$ and v with $v_{e_b^*} = 0$ for $b < j$: $(gv)_\alpha = v_\alpha$ at every α of floor at least j or of rank more than ϱ;

(F5) $w_l(p)$ commutes with $\mathcal{H}_l(T)$; $w_{l+1}(p) = w_l^\circ(p_l) w_l(p)$; $w_l(\bar{p}) = w_l(p)^{-1}$;

(F6) for $s \in \mathcal{H}_{a+1}(T)$: $s\,\mathrm{n}_a(v,\beta) s^{-1} = \mathrm{n}_a(sv, \beta)$; the product and the inverse of Heisenberg elements are Heisenberg elements, with $\mathrm{n}_a(v,\beta)^{-1} = \mathrm{n}_a(-v, \beta')$ for a skew β'; and $\mathrm{sd}(s\,\mathrm{n}_a(v,\beta)s^{-1}) = -h_T(v, s^{-1} f_{a+1})$;

(F7) the elements of $G(T)$ preserve h_T, and $J(T) g^{-1} = g^* J(T)$.

A place (unit 113). A *place* is a commutative ring A with involution that is a locally compact Polish topological ring, with a Haar measure $\mathrm{d}x$ on A and: an element τ_A with $\tau_A+\bar{\tau}_A=1$; a unit δ_A with $\bar{\delta}_A=-\delta_A$; a Haar measure ν_A on the group A^- of the skew elements such that $(\beta,\beta')\mapsto \tau_A\delta_A^{-1}\beta+\beta'$ carries $\nu_A\times\nu_A$ to $\mathrm{d}x$; a measurable multiplicative $\mathrm{mod}_A\colon A\to[0,\infty]$ with $\mathrm{mod}_A(\bar{x})=\mathrm{mod}_A(-x)=\mathrm{mod}_A(x)$ such that $v\mapsto Mv$ carries $\mathrm{d}v$ on A^N to $\mathrm{mod}_A(\det M)^{-1}\mathrm{d}v$ whenever $\det M$ is a unit; a continuous additive character ψ_A with $|\psi_A|=1$; a constant $c_A\in[0,\infty]$ bounding the L^1 norm of the Fourier transform of some continuous integrable $G\geq 0$ with $G(0)=1$; almost every element of A is a unit; and $\mathrm{mod}_A(u)^{-1}\leq|g(u)|^2$ on units for some locally integrable g.

Quadric measures and hyperbolic bases (units 113 and 119). For a hermitian matrix T' on a finite index set ι and a skew β, the *flat chart* of the quadric $h=\delta_A^{-1}\beta$ in the space $A^2\oplus A^\iota$ with the form (hyperbolic plane) $\oplus T'$ is $(t,w,\alpha)\mapsto(\bar{t}^{-1}\tau_A(\delta_A^{-1}\beta-w^*T'w)+t\alpha,\ t,\ w)$, for $t\in A$, $w\in A^\iota$, $\alpha\in A^-$, and the *quadric measure* is the image of $\mathrm{d}t\,\mathrm{d}w\,\mathrm{d}\nu_A(\alpha)$ under it. A *hyperbolic basis* of V_ϱ, with parameters $(\Omega,T,\mathfrak{m})$, is a matrix E whose columns are $\mathfrak{m}+1$ hyperbolic pairs and vectors indexed by Ω with E^*TE the Gram matrix of $\mathfrak{m}+1$ hyperbolic planes plus the identity on Ω, with $k+1=|\Omega|+2\mathfrak{m}+2$ and $\mathrm{mod}_A(\det T)=1$. For $b\leq\varrho$ its *floor basis* E_b is the basis of V_b made of E and of the pairs (e_a,e_a^*), $b\leq a<\varrho$; its coordinates are: the first hyperbolic pair, a first half of the other pairs (indexed by a set π_b of $\mathfrak{m}+\varrho-b$ elements), the orthonormal part Ω, and the second half. The *floor quadric measure* $\delta^{(b)}(\mathcal{L})$ of a level is the image under E_b of the quadric measure with $\beta=\delta_A h_T(f_b,f_b)$ in these coordinates, and vol_b is the Haar measure of V_b, the image of the volume of the coordinates of floor at least b.

Sizes.

Definition 2.1 (size)**.** Let A be a commutative ring with involution, $\mathfrak{s}$ a shape and T an $\mathfrak{s}$-form over A. A *size* for T, an element of $\mathrm{Siz}(T)$, is a family $\mathcal{S}=(\mathcal{S}_i)_{i\in\mathbb{N}}$ of subsets of A such that: $\mathcal{S}_i\mathcal{S}_{i'}\subseteq\mathcal{S}_{i+i'}$; the sum of a family of elements of $\mathcal{S}_i$ indexed by a finite set of at most $r+1$ elements lies in $\mathcal{S}_{i+1}$; $-x\in\mathcal{S}_i$ if and only if $x\in\mathcal{S}_i$, and $\bar{x}\in\mathcal{S}_i$ if and only if $x\in\mathcal{S}_i$; $0,1\in\mathcal{S}_i$ for all i; and all entries of $J(T)$ and of $J(T)^{-1}$ lie in $\mathcal{S}_1$.

LEAN `Size`

Definition 2.2 (vectors and matrices of a size; heads)**.** Let $\mathcal{S}$ be a size in $\mathrm{Siz}(T)$ (an implicit argument of the three notions). For a natural number i, Vec_i is the set of the vectors v, on a given finite index set, with all coordinates in $\mathcal{S}_i$. A matrix X over A *has size at most* i, $\mathrm{sz}(X)\leq i$, if all its entries lie in $\mathcal{S}_i$; for $g\in G(T)$ this refers to the matrix of g. A sequence $p\colon\mathbb{N}\to A^\times$ is a *head of length* j, $p\in\mathrm{Hd}_j$, if $(p_0p_1\cdots p_{l-1})^{-1}\in\mathcal{S}_1$ for every $l\leq j$.

LEAN `Size.vecsOfSize` · `Size.HasMatSize` · `Size.IsHead`

Definition 2.3 (Weyl ratio)**.** For a natural number j and $p,q\colon\mathbb{N}\to A^\times$, the element $w_j(q)^{-1}w_j(p)$ of $G(T)$ is the product of the inverse of the Weyl element of q and the Weyl element of p, both of length j.

LEAN `weylRatio` · `coe_weylRatio` · `weylRatio_inv`

Its matrix is diagonal: the entry is p_a/q_a at e_a and $\overline{q_a/p_a}$ at e_a^* for $a < \min(j, \varrho)$, and 1 elsewhere; and $(w_j(q)^{-1} w_j(p))^{-1} = w_j(p)^{-1} w_j(q)$.

Levels and the tower measure.

Definition 2.4 (level)**.** Let A be a place, T an $\mathfrak{s}$-form whose split Gram matrix is a hermitian unit, Ω a finite set and $\mathfrak{m}$ a natural number. A *level*, an element $\mathcal{L}$ of $\mathrm{Lev}(\Omega, T, \mathfrak{m})$, consists of: a hyperbolic basis $E(\mathcal{L})$ of V_ϱ with parameters $(\Omega, T, \mathfrak{m})$; two s-finite measures θ and μ_H on $G(T)$ with $\mu_H \neq 0$; a unit $g_{\mathcal{L}}$ of A with $\overline{g_{\mathcal{L}}} = g_{\mathcal{L}}$; a set $\mathfrak{B}(\mathcal{L}) \subseteq V$, the *ball*; and two sets $\mathrm{Box}(\mathcal{L}), \mathrm{Posn}(\mathcal{L}) \subseteq G(T)$, the *box* and the *positions*; such that

(L1) θ is carried by $\mathcal{H}_\varrho(T)$, and its image under $\sigma \mapsto \sigma^{-1} z$ is the floor quadric measure $\delta^{(\varrho)}(\mathcal{L})$;

(L2) μ_H is carried by $\mathcal{H}_n(T)$; $\mu_H * \delta_s = \mu_H$ for $s \in \mathcal{H}_n(T)$; and $\mu_H\{h : u\,h\,g \in \mathrm{Box}(\mathcal{L})$ for some $u \in \mathcal{U}(T)\} \leq 1$ for every $g \in G(T)$;

(L3) $\mathfrak{B}(\mathcal{L})$ is open, contains 0 and has compact closure; $\mathrm{Box}(\mathcal{L})$ is compact; $\mathrm{Posn}(\mathcal{L})$ is measurable.

LEAN `Level`

Definition 2.5 (floor measures)**.** For a level $\mathcal{L}$ in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$ and a natural number l, the measure $\theta_l(\mathcal{L})$ on $G(T)$ is: $\varpi_l * \eta_l$ for $l < \varrho$; θ for $l = \varrho$; μ_H for $l = n$; and the unit mass δ_1 for $l > n$. Here $*$ is the convolution of measures, ϖ_l (the *Weyl law*) is the image of $\mathrm{mod}_A(c)^{k+1+2(\varrho-(l+1))}\mathrm{d}c$ under $c \mapsto w_l^\circ(c)^{-1}$, and η_l (the *Heisenberg measure*) is the image of $\mathrm{d}v\,\mathrm{d}\nu_A(\beta)$ on (coordinates of $V_{l+1}) \times A^-$ under $(v, \beta) \mapsto \mathrm{n}_l(v, \beta)$; we put $\eta_l = \delta_1$ for $l \geq \varrho$ and $\eta_{<j} = \eta_0 * \eta_1 * \cdots * \eta_{j-1}$.

LEAN `Level.floorMeasure` · `Level.floorMeasure_top`

Definition 2.6 (stack, group and upper measures)**.** For a level $\mathcal{L}$ in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$, $\mu^{\mathcal{L}}$ is the family of measures

$$\mu^{\mathcal{L}}_{a,i} = (\eta_{a+i-1} * \theta_{a+i-1}(\mathcal{L})) * \cdots * (\eta_{a+1} * \theta_{a+1}(\mathcal{L})) * (\eta_a * \theta_a(\mathcal{L}))$$

(i factors; δ_1 for $i = 0$), indexed by natural numbers a, i. The *tower measure* is $\mathsf{M}(\mathcal{L}) = \mu^{\mathcal{L}}_{0,\varrho+2}$, and for a natural number l, $\mathsf{M}_{\geq l}(\mathcal{L}) = \theta_{\varrho+1}(\mathcal{L}) * \cdots * \theta_l(\mathcal{L})$ ($\varrho + 2 - l$ factors; δ_1 for $l \geq \varrho + 2$).

LEAN `Level.stackMeasure` · `Level.groupMeasure` · `Level.upperMeasure` · `Level.upperMeasure_eq_succ_mconv` · `Level.ae_stackMeasure_mem`

So $\mathsf{M}_{\geq l}(\mathcal{L}) = \mathsf{M}_{\geq l+1}(\mathcal{L}) * \theta_l(\mathcal{L})$, and $\mu^{\mathcal{L}}_{l,i}$ is carried by $\mathcal{H}_l(T)$ (Lemma 5.2 and (I2)).

Nearness, floor values, weights.

Definition 2.7 (near points)**.** For a level $\mathcal{L}$ in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$, a natural number j and $g \in G(T)$: g is *near at* j, $g \in \mathrm{Near}_j(\mathcal{L})$, if $\xi g \in \mathrm{Box}(\mathcal{L})$ for some $\xi \in \mathcal{Q}_j(T)$.

LEAN `Level.IsNear` · `Level.isNear_mul_iff` · `Level.IsNear.of_floorGroup_mul`

Hence nearness at j is invariant under $\mathcal{Q}_j(T)$; and if xg is near at i, $x \in \mathcal{H}_j(T)$ and $j \leq i$, then g is near at j, because $\mathcal{Q}_i(T)\,x \subseteq \mathcal{Q}_j(T)$.

Definition 2.8 (orthonormal part, close pairs, floor values)**.** Let $\mathcal{L}$ be a level in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$. Its *orthonormal part* $\mathsf{O}(\mathcal{L})$ is the matrix whose columns are the vectors of $E(\mathcal{L})$ indexed by Ω, as vectors of V. The set $\mathrm{Cl}(\mathcal{L})$ consists of the pairs (u_1, u_2) of vectors of V with $u_1 \in \mathfrak{B}(\mathcal{L})$ and $g_{\mathcal{L}}^{-1}(u_1 - u_2) \in \mathfrak{B}(\mathcal{L})$. For a natural number $a < \varrho$ and $x, y \in G(T)$, the set of *floor values* is the subset of A

$$\mathrm{Val}_a(\mathcal{L}; x, y) = \{ w_{e_a^*} : \ w \in V, \ (x^{-1}w, (xy)^{-1}w) \in \mathrm{Cl}(\mathcal{L}), \ w_{e_b^*} = 0 \text{ for all } b < a \}.$$

LEAN `Level.orthonormalPart` · `Level.closePairs` · `Level.floorValues`

Definition 2.9 (volume exponent, weights, floor module)**.** For a level $\mathcal{L}$ in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$: the *volume exponent* is the natural number $\mathsf{s}(\mathcal{L}) = |\Omega| - (3n + 1 + \mathfrak{m})$; for a natural number j and $x, y \in G(T)$ the *weight* is

$$\mathsf{w}_j(\mathcal{L}; x, y) = \prod_{a < \min(j, \varrho)} \mathrm{vol}(\mathrm{Val}_a(\mathcal{L}; x, y))^{\mathsf{s}(\mathcal{L})} \in [0, \infty];$$

for $x \in G(T)$, $\mathsf{w}_j^{-1}(\mathcal{L}; x)$ is the function on $G(T)$ that is $z \mapsto \mathsf{w}_j(\mathcal{L}; z, x)^{-1}$ at the points z near at j and 0 elsewhere; and for a natural number l and $g \in A^\times$ the *floor module* is $\mathsf{d}_l(\mathcal{L}; g) = \mathrm{mod}_A(g)^{k+1+2(\varrho-(l+1))} \cdot \mathrm{mod}_A(g)^{1/2}$. In the Lean text the last argument of the floor module has the default value $g_{\mathcal{L}}$. (The census lists, among the arguments of the inverse weight, the hypothesis that the tower measure is a Haar measure; the definition does not use it.)

LEAN `Level.volumeExponent` · `Level.floorWeight` · `Level.invWeight` · `Level.floorModulus` · `Level.floorWeight_succ` · `Level.floorWeight_ne_zero_and_ne_top` · `Level.floorWeight_mul_left` · `measurable_floorWeight` · `Level.measurable_invWeight`

Definition 2.10 (mass of a floor)**.** For a level $\mathcal{L}$ in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$, a natural number j, $K \in [0, \infty]$ and a function $c \colon G(T) \to [0, \infty]$: $\mathrm{Mass}_j(\mathcal{L}; c) \le K$ means that for every $g \in G(T)$

$$\int_{\{\sigma \colon \sigma g \text{ is near at } j+1\}} c(\sigma g) \, \mathrm{d}\theta_j(\mathcal{L})(\sigma) \le K.$$

LEAN `Level.FloorMassLe`

Kernels averaged over floors.

Definition 2.11 (averaged kernel, pivot)**.** For a level $\mathcal{L}$ in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$, a natural number j, a function $k \colon G(T) \times G(T) \to \mathbb{C}$ and $g, y \in G(T)$,

$$\mathsf{Av}_j(\mathcal{L}; k)(g, y) = \int k(\sigma g, \sigma y) \, \mathrm{d}\mathsf{M}_{\ge j}(\mathcal{L})(\sigma),$$

$$\mathsf{Pv}_j(\mathcal{L}; k)(g, y) = \int \left| \mathsf{Av}_{j+1}(\mathcal{L}; k)(\sigma g, \sigma y) \right| \mathrm{d}\theta_j(\mathcal{L})(\sigma).$$

The first is a Bochner integral, 0 if the integrand is not integrable; the second lies in $[0, \infty]$. (The census lists, among the arguments of the pivot, the hypothesis that the tower measure is a Haar measure; the definition does not use it.)

LEAN `Level.upperPairing` · `Level.pivotModulus`

Definition 2.12 (pairing kernel)**.** For a level $\mathcal{L}$ in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$, a natural number j, $k \colon G(T) \times G(T) \to \mathbb{C}$ and $w \in G(T)$: the pair is a *pairing kernel at* j, $(k, w) \in \mathrm{PK}_j(\mathcal{L})$, if

(K1) k is measurable;

(K2) for all l, all $\nu, \nu' \in \mathcal{N}_{<l}(T)$ and all g, y:

$$\mathsf{Av}_{l+1}(\mathcal{L}; k)(\nu g, \nu' y) = \psi_A(g_{\mathcal{L}}^{-1}(\mathrm{sd}(w\nu' w^{-1}) - \mathrm{sd}(\nu)))\, \mathsf{Av}_{l+1}(\mathcal{L}; k)(g, y);$$

(K3) $wsw^{-1} = s$ for all $s \in \mathcal{H}_j(T)$;

(K4) for every l there is $c < \infty$ with $\int |k(\sigma g, \sigma y)|\, \mathrm{d}\mathsf{M}_{\geq l}(\mathcal{L})(\sigma) \leq c$ for all g, y;

(K5) for all l, g, y: if $\mathsf{Av}_l(\mathcal{L}; k)(g, y) \neq 0$ then g is near at l.

LEAN `Level.IsPairingKernel`

Definition 2.13 (weighted squares, self-norm)**.** For a level $\mathcal{L}$ in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$, $k\colon G(T) \times G(T) \to \mathbb{C}$, natural numbers j and $x, g \in G(T)$:

$$\mathsf{Sq}_j(\mathcal{L}; k, x)(g) = \mathsf{w}_j(\mathcal{L}; g, x) \cdot \left|\mathsf{Av}_{j+1}(\mathcal{L}; k)(g, gx)\right|^2;$$

the *weighted square sum* is defined by recursion on $n - j$:

$$\mathsf{Sm}_j(\mathcal{L}; k, x)(g) = \int \Big(\mathsf{Sq}_j(\mathcal{L}; k, x)(\sigma g) + \mathsf{Sm}_{j+1}(\mathcal{L}; k, x)(\sigma g)\Big) \mathrm{d}\theta_j(\mathcal{L})(\sigma)$$

for $j < n$ and $\mathsf{Sm}_j(\mathcal{L}; k, x)(g) = 0$ for $j \geq n$; and the *self-norm* is

$$\mathrm{SN}_j(\mathcal{L}; k, x) = \int \mathsf{Sm}_j(\mathcal{L}; k, x)(g)\, \mathrm{d}(\theta_{j-1}(\mathcal{L}) * \cdots * \theta_0(\mathcal{L}))(g).$$

(The census lists, among the arguments of the first two notions, the hypothesis that the tower measure is a Haar measure; the definitions do not use it.)

LEAN `Level.weightedSquare` · `Level.weightedSquareSum` · `Level.selfNorm`

Gauges.

Definition 2.14 (gauge)**.** Let $\mathcal{L}$ be a level in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$. A *gauge* on $\mathcal{L}$, an element Γ of $\mathrm{Gau}(\mathcal{L})$, is a size $(\Gamma_i)_{i \in \mathbb{N}}$ for T together with a constant $\Lambda(\Gamma) \in [0, \infty]$ such that, with sizes taken for Γ and $\Lambda = \Lambda(\Gamma)$:

(G1) every Γ_j is compact; $2 \leq \Lambda < \infty$; $\mathfrak{m} \leq 9n$; $c_A \geq 1$;

(G2) $g_{\mathcal{L}} \in \Gamma_1$; $\mathrm{sz}(E(\mathcal{L})) \leq 1$; $\tau_A \delta_A^{-1} \in \Gamma_1$;

(G3) $\mathrm{sz}(x) \leq 1$ for every $x \in \mathrm{Box}(\mathcal{L})$; $\mathfrak{B}(\mathcal{L}) \subseteq \mathrm{Vec}_1$;

(G4) $\mathrm{mod}_A(u) \leq \Lambda^j$ for every unit $u \in \Gamma_j$; $\mathrm{vol}(\Gamma_j) \leq \Lambda^{j+1}$;

(G5) *(cone)* for every finite set π with $\mathfrak{m} \leq |\pi| \leq \mathfrak{m} + \varrho$ and every j, the quadric measure with $\beta = 0$ of the sum of $|\pi| + 1$ hyperbolic planes gives Vec_j mass at most $\Lambda^{99n(j+1)}$;

(G6) *(volumes)* for every $x \in \mathrm{Box}(\mathcal{L})$, $y \in \mathrm{Posn}(\mathcal{L})$ and every matrix v over A with ϱ rows and columns indexed by Ω, let

$$W = \{w \in A^{\Omega} : vw = 0,\ (x^{-1}\mathsf{O}(\mathcal{L})w, (xy)^{-1}\mathsf{O}(\mathcal{L})w) \in \mathrm{Cl}(\mathcal{L})\};$$

then there is a measurable $\varphi\colon A^{\Omega} \to [0, \infty]$ such that for every $u \in A^{\Omega}$

$$\mathrm{vol}\{u^* w : w \in W\}^{-1} \leq \varphi(u), \qquad \mathrm{vol}\ \mathcal{C}_u \leq \varphi(u),$$

where $\mathcal{C}_u$ is the set of the $c \in A$ such that for every $w \in W$ there are $z, z' \in \mathrm{Cl}(\mathcal{L})$ with $(cu)^* w = g_{\mathcal{L}}^{-1}(h_T(z_1, z_1') - h_T(z_2, z_2'))$; and $\int_{\mathrm{Vec}_j} \varphi(u)^{\mathsf{s}(\mathcal{L})} \mathrm{d}u \leq \Lambda^{99n(j+1)}$ for every j.

LEAN Gauge · Gauge.one_le_Lambda

In particular $\Lambda(\Gamma) \geq 1$.

Definition 2.15 (positions of size 1; constants)**.** For a level $\mathcal{L}$ in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$ and a gauge Γ in $\mathrm{Gau}(\mathcal{L})$, $\mathrm{Pos}(\mathcal{L}; \Gamma)$ is the set of the $x \in \mathrm{Posn}(\mathcal{L})$ with $\mathrm{sz}(g_{\mathcal{L}}^{-1}(x - 1)) \leq 1$ for Γ; for a natural number l, $C_l(\Gamma) = \mathsf{d}_l(\mathcal{L}; g_{\mathcal{L}}) \cdot c_A^{|\Omega|} \cdot \Lambda(\Gamma)^{4600n}$; and $C_{\mathrm{gl}}(\Gamma) = \Lambda(\Gamma)^{10^6 n^3}$.

LEAN Gauge.positionsOfSize · Gauge.floorConst · Gauge.globalConst · Gauge.measurableSet_positionsOfSize · Gauge.measurableSet_vecsOfSize

The sets Vec_j (for a gauge) and $\mathrm{Pos}(\mathcal{L}; \Gamma)$ are measurable, by compactness of the Γ_j.

Definition 2.16 (fibre bound)**.** Let $\mathcal{L}$ be a level in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$, Γ a gauge in $\mathrm{Gau}(\mathcal{L})$, b a natural number with $b \leq \varrho$, q a measure on V, j a natural number and $M \in [0, \infty]$. Write a coordinate vector for the floor basis E_b as (t', t, π', w, π''): the first hyperbolic pair (t', t), the first half π' of the other pairs, the orthonormal part $w \in A^{\Omega}$, the second half π''; call $P = (t, \pi')$ the *fixed* coordinates. Then $\mathrm{Fib}_{b,j}(\Gamma; q) \leq M$ means: there are a measurable space Z with an s-finite measure ζ, a measure ρ' on coordinate vectors with $E_b \rho' = q$, a map k from triples (P, w, z), $z \in Z$, to coordinate vectors that carries $\mathrm{d}P \mathrm{d}w \, \mathrm{d}\zeta(z)$ to ρ' and whose values have the fixed coordinates P and the orthonormal coordinates w, and a measurable μ with

$$\zeta\{z : \mathsf{k}(P, w, z) \in \mathrm{Vec}_{j+4}\} \leq \mu(P) \quad \text{for all } P, w, \qquad \int \mu(P) \, \mathrm{d}P \leq M.$$

LEAN Gauge.HasFibreBound

3 What is taken for granted

Besides the facts (F1)–(F7) about the groups, the proofs use the following results of the background library. None is a member of this unit; the unit that owns each is given.

(I1) *Convolution* (unit 119). For s-finite measures μ, ν on $G(T)$ and measurable $f \geq 0$: $\int f \, \mathrm{d}(\mu * \nu) = \iint f(xy) \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y)$; $\mu * \delta_s$ and $\delta_s * \mu$ are the right and left translates of μ; if $\delta_s * \nu = \nu * \delta_s$ for μ-almost every s, then $\mu * \nu = \nu * \mu$; if $\delta_s * \mu = \mu * \delta_t$, then $\int f \, \mathrm{d}\mu = \int f(s^{-1} x t) \, \mathrm{d}\mu(x)$.

(I2) (unit 119) A convolution of measures carried by a measurable subgroup is carried by it; the groups $\mathcal{H}_l(T)$, $\mathcal{Q}_l(T)$, $\mathcal{N}_{<l}(T)$ are closed (unit 112).

(I3) *Heisenberg measures* (unit 119). η_l is carried by $\mathcal{N}_l(T)$; $\eta_l * \delta_s = \eta_l = \delta_s * \eta_l$ for $s \in \mathcal{N}_l(T)$; $\delta_s * \eta_l = \eta_l * \delta_s$ for $s \in \mathcal{H}_{l+1}(T)$. $\eta_{<j}$ is carried by $\mathcal{N}_{<j}(T)$; $\eta_{<j} * \delta_s = \eta_{<j} = \delta_s * \eta_{<j}$ for $s \in \mathcal{N}_{<j}(T)$; $\delta_s * \eta_{<j} = \eta_{<j} * \delta_s$ for $s \in \mathcal{H}_j(T)$; and $\eta_{<j+1} = \eta_j * \eta_{<j}$.

(I4) *Disintegration* (units 113, 119). Call a family (λ_β) of measures on a space with a hermitian form h, indexed by the skew β, a *disintegration* of a measure μ if it is weakly continuous in β, finite on compact sets, and $\int f(h(x, x), x) \, \mathrm{d}\mu = \iint f(\delta_A^{-1} \beta, x) \, \mathrm{d}\lambda_\beta(x) \, \mathrm{d}\nu_A(\beta)$ for measurable $f \geq 0$. Then: (a) the quadric measures of the flat charts disintegrate the volume of $A^2 \oplus A^\iota$; (b) the image of a disintegration under a linear map with a left inverse that respects the forms is a disintegration of the image measure; (c) a disintegration is unique; (d) the floor quadric measures of

a hyperbolic basis disintegrate vol_b along h_T, the floor basis E_b carries the volume of its coordinates to vol_b, and $E_b^* J(T) E_b$ is the Gram matrix of hyperbolic planes plus the identity on Ω; (e) for $s \in \mathcal{H}_l(T)$ the action of s on V_l preserves vol_l. The catalogue of the write-up classes (a) as a standard local computation (Leray measures on quadrics); reference to be confirmed.

(I5) *Functions on a floor* (unit 119). A measurable function on $G(T)$ with values in a standard Borel space that is invariant under $\mathcal{Q}_{b+1}(T) \cap \mathcal{H}_b(T)$, $b \le \varrho$, is on $\mathcal{H}_b(T)$ a measurable function of $\sigma^{-1} f_b$.

(I6) *Plancherel's inequality at a place* (unit 119). For $\xi \in A^\times$, a finite set ι and $f\colon A^\iota \to \mathbb{C}$ measurable and integrable: $\int |\int \psi_A(\xi\, v^* w) f(w)\, \mathrm{d}w|^2 \mathrm{d}v \le (c_A / \operatorname{mod}_A(\xi))^{|\iota|} \int |f|^2$. The second reader of the artifact relates it to J. Tate, *Fourier analysis in number fields and Hecke's zeta-functions* (1950), §2.2, as a weaker form: a one-sided consequence of abelian Plancherel. Tate's §2.2 covers local fields; for a general place (a locally compact ring) the second reader's record points to the abelian Plancherel theorem (H. Cartan, R. Godement, *Théorie de la dualité et analyse harmonique dans les groupes abéliens localement compacts*, Ann. Sci. Éc. Norm. Sup. 64 (1947)) together with an argument on the normalisation; the place of a reference for the general case is not confirmed, and neither reference was opened for this note.

(I7) *Skew elements* (unit 113). If $S \subseteq A^-$, $u \in A^\times$ and $C \subseteq A$ are such that $u(\tau_A \delta_A^{-1} \beta + \beta') \in C$ for all differences β, β' of elements of S, and $\mathrm{vol}(C) / \operatorname{mod}_A(u) \le t^2$, then $\nu_A(S) \le t$.

(I8) The *Cauchy–Schwarz inequality* for integrals of functions with values in $[0, \infty]$ (unit 117): if $h^2 \le f_1 f_2$ pointwise then $(\int h)^2 \le \int f_1 \int f_2$.

(I9) Laplace expansion of a determinant along a row; the determinant of a triangular matrix (unit 112).

4 Sizes

In this section $\mathcal{S}$ is a size in $\mathrm{Siz}(T)$, and sizes are taken for $\mathcal{S}$; index sets of rows may be any finite sets, and vectors and columns are indexed by the basis of V.

Lemma 4.1 (rules)**.** *Let $\mathcal{S}$ be a size in $\mathrm{Siz}(T)$, i, j natural numbers, $\alpha, \beta \in A$, $v, w \in V$, and X, Z matrices with columns indexed by the basis of V, Y a matrix with rows indexed by it. Then:* (a) *if $\alpha \in \mathcal{S}_i$ and $i \le j$ then $\alpha \in \mathcal{S}_j$; if $\mathrm{sz}(X) \le i$ and $i \le j$ then $\mathrm{sz}(X) \le j$;* (b) *if $\alpha, \beta \in \mathcal{S}_i$ then $\alpha - \beta \in \mathcal{S}_{i+1}$; if $\mathrm{sz}(X) \le i$ and $\mathrm{sz}(Z) \le i$ then $\mathrm{sz}(X - Z) \le i + 1$; the identity matrix has $\mathrm{sz}(1) \le i$;* (c) *if $w_\alpha \in \mathcal{S}_i$ at every index where $v_\alpha \ne 0$, and $v \in \mathrm{Vec}_j$, then $\sum_\alpha w_\alpha v_\alpha \in \mathcal{S}_{i+j+1}$;* (d) *if $\mathrm{sz}(X) \le i$ and $v \in \mathrm{Vec}_j$ then $Xv \in \mathrm{Vec}_{i+j+1}$; if moreover $\mathrm{sz}(Y) \le j$ then $\mathrm{sz}(XY) \le i + j + 1$;* (e) *if p is a head of length j, $p \in \mathrm{Hd}_j$, then so is $\bar{p}$: $\bar{p} \in \mathrm{Hd}_j$.*

Proof. (a) $\alpha = \alpha \cdot 1$ with $1 \in \mathcal{S}_{j-i}$. (b) $\alpha - \beta$ is a sum of two elements of $\mathcal{S}_i$, and $2 \le r + 1$. (c) Each term $w_\alpha v_\alpha$ is 0 or lies in $\mathcal{S}_{i+j}$, and there are r terms. (d) follows from (c), entry by entry. (e) The sets $\mathcal{S}_1$ are stable under the involution. □

LEAN `Size.mem_ofSize_mono` · `Size.HasMatSize.mono` · `Size.sub_mem_ofSize` · `Size.HasMatSize.sub` · `Size.HasMatSize.one` · `Size.dotProduct_mem_ofSize` · `Size.HasMatSize.mulVec` · `Size.HasMatSize.mul` · `Size.IsHead.star_head`

Lemma 4.2 (determinants)**.** *Let $\mathcal{S}$ be a size in* $\mathrm{Siz}(T)$*, j a natural number with $j \le \varrho$ and Z a $j \times j$ matrix over A with* $\mathrm{sz}(Z) \le 7$*. Then* $\det Z \in \mathcal{S}_{8j}$*.*

Proof. Induction on j; for $j = 0$ the determinant is 1. For $j+1$, expand along the first row (I9): each of the $j+1$ terms is $\pm$ an entry of Z times the determinant of a $j \times j$ minor, so it lies in $\mathcal{S}_{7+8j}$; and $j+1 \le r+1$, so the sum lies in $\mathcal{S}_{8j+8}$. □

LEAN `Size.det_mem_ofSize`

Lemma 4.3 (adjoints and inverses)**.** *Let $\mathcal{S}$ be a size in* $\mathrm{Siz}(T)$ *and suppose that $J(T)$ is a hermitian unit.* (i) *Let X, Y be square matrices on the basis of V with $J(T)X = Y^* J(T)$, let a, b be two indices and i a natural number such that for every index ν with $J(T)_{\nu b} \neq 0$ the row ν of Y lies in* Vec_i*. Then $X_{ab} \in \mathcal{S}_{i+4}$.* (ii) *If $x \in G(T)$ has* $\mathrm{sz}(x) \le i$*, then* $\mathrm{sz}(x^{-1}) \le i + 4$*.*

Proof. (i) $X = J(T)^{-1}(Y^* J(T))$. The entry (m, b) of $Y^* J(T)$ is $\sum_\nu \overline{Y_{\nu m}} J(T)_{\nu b}$, in $\mathcal{S}_{i+2}$ by rule (c), because the column b of $J(T)$ has size 1; and the row a of $J(T)^{-1}$ has size 1, so $X_{ab} \in \mathcal{S}_{1+(i+2)+1}$. (ii) Apply (i) to $X = x^{-1}$, $Y = x$, by (F7). □

LEAN `Size.adjoint_apply_mem_ofSize` · `Size.HasMatSize.inv`

Lemma 4.4 (back substitution in a row)**.** *Let $\mathcal{S}$ be a size in* $\mathrm{Siz}(T)$*, X a matrix with columns indexed by the basis of V, m one of its rows, and c, d, j natural numbers. Suppose* (h) *for every $a < \min(j, \varrho)$ there is $w \in V$ with $w_{e_b^*} = 0$ for $b < a$, $w_{e_a^*} = 1$, $w \in \mathrm{Vec}_c$ and $Xw \in \mathrm{Vec}_d$; and* (z) *$X_{mk} \in \mathcal{S}_d$ for every column k of floor at least j or of rank at most ϱ. Then for every natural number i and every column k whose floor is at least $j - i$ or whose rank is at most ϱ, $X_{mk} \in \mathcal{S}_{d+(c+3)i}$.*

Proof. Induction on i; $i = 0$ is (z). A column new at step $i+1$ is $k = e_a^*$ with $a = j - i - 1$. Take w from (h) for this a and let $w' = w - e_a^*$; then $X_{mk} = (Xw)_m - \sum_{k'} X_{mk'} w'_{k'}$. The non-zero coordinates of w' are at columns of rank at most ϱ or at e_b^* with $b > a$, where the induction hypothesis gives $X_{mk'} \in \mathcal{S}_{d+(c+3)i}$; and $w' \in \mathrm{Vec}_{c+1}$. By rules (c) and (b), $X_{mk} \in \mathcal{S}_{d+(c+3)i+(c+1)+1+1}$. □

LEAN `Size.row_mem_ofSize_of_dual_frame`

Lemma 4.5 (back substitution for an inverse)**.** *Let $\mathcal{S}$ be a size in* $\mathrm{Siz}(T)$*, $J(T)$ a hermitian unit, j and c natural numbers, $u \in \mathcal{N}_{<j}(T)$, and $r \in A^\times$ with $\bar{r} = r$ and $r \in \mathcal{S}_1$. Suppose that for every $a < \min(j, \varrho)$ there is $w \in V$ with $w_{e_b^*} = 0$ for $b < a$, $w_{e_a^*} = 1$, $w \in \mathrm{Vec}_c$ and $r^{-1}(1-u)w \in \mathrm{Vec}_c$. Then* $\mathrm{sz}(r^{-1}(1-u^{-1})) \le c + 4 + (2c+8)j$*.*

Proof. Let $Y = r^{-1}(1-u)$ and $X = r^{-1}(1-u^{-1})$; then $J(T)X = Y^* J(T)$ by (F7), as $\bar{r} = r$. *Rows of Y.* If the row ν of $J(T)$ is not zero at a column k of floor at least j or of rank at most ϱ, then the entries of row ν of u at such columns are those of the identity (definition of $\mathcal{N}_{<j}(T)$), so (z) of Lemma 4.4 holds for Y with any d, and (h) holds by hypothesis with $d = c$; taking $i = j$, the whole row ν of Y lies in $\mathrm{Vec}_{c+(c+3)j}$. *Entries of X at the columns of (z).* By Lemma 4.3(i) they lie in $\mathcal{S}_{d'}$, $d' = c + (c+3)j + 4$. *All entries of X.* For w as in the hypothesis, uw has the same coordinates at e_b^*, $b \le a$ (F3), $uw = w - r \cdot (Yw) \in \mathrm{Vec}_{c+2}$, and $X(uw) = -Yw \in \mathrm{Vec}_c$. So Lemma 4.4 applies to X with $c+2$, d' and $i = j$, and gives the size $d' + (c+5)j$. □

LEAN `Size.hasMatSize_one_sub_inv`

Setting 4.6 (two frames near a set of size 1)**.** Let $\mathcal{S}$ be a size in $\mathrm{Siz}(T)$ and $J(T)$ a hermitian unit. Let $\mathsf{K} \subseteq G(T)$ be a set with $\mathrm{sz}(x) \le 1$ for all $x \in \mathsf{K}$, let j be a natural number with $j \le \varrho$, $p\colon \mathbb{N} \to A^\times$, and $g, s \in G(T)$ such that $\eta\, w_j(p)^{-1} g \in \mathsf{K}$ for some $\eta \in \mathcal{Q}_j(T)$, and such that $s \in \mathcal{Q}_j(T)$ and $sg \in \mathsf{K}$.

Lemma 4.7 (minors and heads)**.** *In Setting 4.6:* (i) *for every $a < j$ and every index i of the basis, $(p_0 p_1 \cdots p_a) \cdot s_{i,e_a^*} \in \mathcal{S}_{9n}$;* (ii) *for every $l \le j$, $p_0 \cdots p_{l-1} \in \mathcal{S}_{9n}$;* (iii) *if p is a head of length j, $p \in \mathrm{Hd}_j$, and $a < j$, then p_a, p_a^{-1} and all entries s_{i,e_a^*} lie in $\mathcal{S}_{10n}$;* (iv) *if $p \in \mathrm{Hd}_j$ and $s \in \mathcal{N}_{<j}(T)$, then $\mathrm{sz}(g) \le 60n^2$ and $\mathrm{sz}(w_j(p)^{-1} g) \le 99n^2$.*

Proof. (i) The element $M = (sg)(\eta w_j(p)^{-1} g)^{-1} = s\, w_j(p)\, \eta^{-1}$ has size at most $1 + (1+4) + 1 = 7$. Take its minor with the rows $e_0^*, \dots, e_{a-1}^*, i$ and the columns $e_0, \dots, e_a$; by Lemma 4.2 its determinant lies in $\mathcal{S}_{8(a+1)} \subseteq \mathcal{S}_{9n}$. Since $\eta^{-1} \in \mathcal{Q}_j(T)$ acts on $\langle e_0, \dots, e_a \rangle$ by a unitriangular matrix, the determinant is that of the same minor of $s\, w_j(p)$; $w_j(p)$ maps e_b to $p_b e_b^*$; and by (F3) the matrix of s with the rows $e_0^*, \dots, e_{a-1}^*, i$ and the columns $e_0^*, \dots, e_a^*$ is lower triangular with diagonal $1, \dots, 1, s_{i,e_a^*}$ (I9). So the determinant is $p_0 \cdots p_a \cdot s_{i,e_a^*}$. (ii) Take $i = e_a^*$, $a = l - 1$, where $s_{e_a^*, e_a^*} = 1$. (iii) $p_a = (p_0 \cdots p_{a-1})^{-1}(p_0 \cdots p_a)$, $p_a^{-1} = (p_0 \cdots p_a)^{-1}(p_0 \cdots p_{a-1})$ and $s_{i,e_a^*} = (p_0 \cdots p_a)^{-1} \cdot (p_0 \cdots p_a\, s_{i,e_a^*})$ lie in $\mathcal{S}_{1+9n}$. (iv) Apply Lemma 4.5 to $u = s$, $r = 1$ and the vectors $w = e_a^*$: $(1 - s)e_a^*$ has size $10n + 1$ by (iii), so $1 - s^{-1}$ has size $10n + 5 + (20n + 10)j$, $g = s^{-1}(sg)$ has size at most $10n + 8 + (20n + 10)(n - 1) \le 60n^2$, and $w_j(p)^{-1}$, whose entries are 0, 1, p_a^{-1} or $\bar{p}_a$, has size $10n$. □

LEAN `Size.minor_mul_dual_mem_ofSize` · `Size.minors_mem_ofSize` · `Size.head_entries_mem_ofSize` · `Size.hasMatSize_of_isHead` · `card_succ_le_floorOf`

5 The tower measure

Lemma 5.1 (a floor measure and its quadric)**.** *Let $\mathcal{L}$ be a level in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$ and b a natural number with $b \le \varrho$.* (i) *The map $\sigma \mapsto \sigma^{-1} f_b$ is measurable and carries $\theta_b(\mathcal{L})$ to the floor quadric measure $\delta^{(b)}(\mathcal{L})$.* (ii) *For $s \in \mathcal{H}_b(T)$ the map $\sigma \mapsto s\sigma^{-1} f_b$ carries $\theta_b(\mathcal{L})$ to $\delta^{(b)}(\mathcal{L})$.*

Proof. (i) For $b = \varrho$ this is (L1). Let $b = a < \varrho$ and $\sigma = w_a^\circ(c)^{-1} \mathrm{n}_a(v, \beta)$. Then $\sigma^{-1} e_a = \mathrm{n}_a(v, \beta)^{-1}(c e_a^*) = c\, e_a^* - cv + c(\beta' - \tau_A h_T(v, v)) e_a$ with β' skew, by (F6). In the decomposition $V_a = \langle e_a, e_a^* \rangle \oplus V_{a+1}$ this is the point of the flat chart of the cone $h_T = 0$ with the parameters $t = c$, $w = -cv$ and the skew parameter $\alpha = \beta'$: with these t and w the chart has the e_a-coordinate $\bar{t}^{-1} \tau_A(-w^* T' w) + t\alpha = -c\, \tau_A h_T(v, v) + c\, \alpha$. By (F6), β' is $-\beta$ plus a skew element that depends on v only. So $(c, v, \beta) \mapsto (c, -cv, \beta')$ carries the measure $\mathrm{mod}_A(c)^N \mathrm{d}c\, \mathrm{d}v\, \mathrm{d}\nu_A(\beta)$, $N = k + 1 + 2(\varrho - a - 1)$ the rank of V_{a+1}, to $\mathrm{d}t\, \mathrm{d}w\, \mathrm{d}\nu_A(\alpha)$, the measure of the flat chart: the place axiom for $v \mapsto -cv$ removes the factor $\mathrm{mod}_A(c)^N$, and the Haar measure ν_A is invariant under translations and under $\beta \mapsto -\beta$. So the image of $\theta_a(\mathcal{L})$ is the quadric measure of V_a for the standard pair (e_a, e_a^*). It and $\delta^{(a)}(\mathcal{L})$ are both the member at $\beta = 0$ of a disintegration of vol_a along h_T ((I4)(a),(b),(d)), and a disintegration is unique ((I4)(c)). (ii) By (I4)(e) and (b), s carries the disintegration of vol_b to a disintegration of vol_b, so s preserves $\delta^{(b)}(\mathcal{L})$ by uniqueness; compose with (i). □

LEAN `Level.measurePreserving_orbitPoint` · `Level.measurePreserving_mulVec_orbitPoint`

Lemma 5.2 (floor measures are carried by their floors)**.** *Let $\mathcal{L}$ be a level in* $\mathrm{Lev}(\Omega, T, \mathfrak{m})$ *and l a natural number. Then $\theta_l(\mathcal{L})$ is carried by $\mathcal{H}_l(T)$. Hence, if a measure μ equals $\theta_l(\mathcal{L})$ and two functions f, Φ agree on $\mathcal{H}_l(T)$, then $\int f \,\mathrm{d}\theta_l(\mathcal{L}) = \int \Phi \,\mathrm{d}\mu$.*

Proof. For $l < \varrho$: $w_l^\circ(c) \in \mathcal{H}_l(T)$, η_l is carried by $\mathcal{N}_l(T) \subseteq \mathcal{H}_l(T)$ (I3), and (I2). For $l = \varrho$, $l = n$ these are (L1), (L2); for $l > n$ the measure is δ_1. □

LEAN `Level.ae_floorMeasure_mem` · `Level.lintegral_floorMeasure_congr`

Lemma 5.3 (moving a Weyl law to the left)**.** *Let $\mathcal{L}$ be a level in* $\mathrm{Lev}(\Omega, T, \mathfrak{m})$*, l, i natural numbers with $l < \varrho$. Then*

$$\theta_{l+i}(\mathcal{L}) * \cdots * \theta_{l+1}(\mathcal{L}) * \theta_l(\mathcal{L}) = \left(\varpi_l * (\theta_{l+i}(\mathcal{L}) * \cdots * \theta_{l+1}(\mathcal{L}))\right) * \eta_l.$$

Proof. $\theta_l(\mathcal{L}) = \varpi_l * \eta_l$, and ϖ_l commutes with every measure carried by $\mathcal{H}_{l+1}(T)$, because $w_l^\circ(c)$ commutes with $\mathcal{H}_{l+1}(T)$ ((I1), Lemma 5.2, (I2)). □

LEAN `Level.descConv_floorMeasure_succ`

Lemma 5.4 (translation by a Weyl ratio)**.** *Let $\mathcal{L}$ be a level in* $\mathrm{Lev}(\Omega, T, \mathfrak{m})$*, j a natural number with $j \le \varrho$, $p, q \colon \mathbb{N} \to A^\times$ and $\Phi \colon G(T) \to [0, \infty]$ measurable. Then*

$$\int \Phi(w_j(q)^{-1} w_j(p)\, x) \,\mathrm{d}\mathsf{M}_{\ge 0}(\mathcal{L})(x) = \prod_{a<j} \mathrm{mod}_A(p_a/q_a)^{k+2(\varrho-a)} \int \Phi \,\mathrm{d}\mathsf{M}_{\ge 0}(\mathcal{L}).$$

Proof. Induction on j. Write $\mathsf{M}_{\ge 0}(\mathcal{L}) = \mathsf{M}_{\ge j}(\mathcal{L}) * (\theta_{j-1}(\mathcal{L}) * \cdots * \theta_0(\mathcal{L}))$ and use Lemma 5.3 at $l = j$: $\mathsf{M}_{\ge 0}(\mathcal{L}) = (\varpi_j * M') * \eta_j * M''$ with M' carried by $\mathcal{H}_{j+1}(T)$. The Weyl ratio of length $j+1$ is that of length j times $w_j^\circ(q_j)^{-1} w_j^\circ(p_j)$, and $w_j^\circ(q_j)^{-1} w_j^\circ(p_j) w_j^\circ(c)^{-1} = w_j^\circ(c q_j/p_j)^{-1}$ up to an element that fixes the measure; the change of variable $c \mapsto c p_j/q_j$ in $\mathrm{mod}_A(c)^{k+1+2(\varrho-j-1)}\mathrm{d}c$ produces the factor $\mathrm{mod}_A(p_j/q_j)^{k+1+2(\varrho-j-1)+1}$. The remaining translation, by the ratio of length j, commutes with the factors carried by $\mathcal{H}_j(T)$ and is handled by the induction hypothesis. □

LEAN `Level.lintegral_weylRatio_mul`

Lemma 5.5 (right translations on a floor)**.** *Let $\mathcal{L}$ be a level in* $\mathrm{Lev}(\Omega, T, \mathfrak{m})$*, l a natural number, S a subgroup of $G(T)$ that contains $\mathcal{H}_l(T)$, $f \colon G(T) \to [0, \infty]$ a measurable function invariant under left translations by $\mathcal{Q}_{l+1}(T) \cap S$, and $s \in \mathcal{Q}_l(T) \cap S$. Then $\int f(xs) \,\mathrm{d}\theta_l(\mathcal{L})(x) = \int f(x) \,\mathrm{d}\theta_l(\mathcal{L})(x)$.*

Proof. Write $s = \gamma\nu$ with $\gamma \in \mathcal{H}_l(T)$, $\nu \in \mathcal{N}_{<l}(T)$ (F1). For $x \in \mathcal{H}_l(T)$, $x\gamma\nu(x\gamma)^{-1}$ lies in $\mathcal{N}_{<l}(T) \subseteq \mathcal{Q}_{l+1}(T)$ and in S, so $f(x\gamma\nu) = f(x\gamma)$; by Lemma 5.2 we may assume $s = \gamma \in \mathcal{H}_l(T)$. If $l \le \varrho$, then on $\mathcal{H}_l(T)$ we have $f(\sigma) = F(\sigma^{-1} f_l)$ with F measurable (I5), and by Lemma 5.1(ii) and (i) $\int F(\gamma^{-1}x^{-1} f_l) \,\mathrm{d}\theta_l(\mathcal{L})(x) = \int F \mathrm{d}\delta^{(l)}(\mathcal{L}) = \int F(x^{-1} f_l) \,\mathrm{d}\theta_l(\mathcal{L})(x)$. If $l = n$ this is the right invariance in (L2); if $l > n$ then $\mathcal{H}_l(T) = 1$. □

LEAN `Level.lintegral_floorMeasure_mul_right`

Proof of Theorem A(i), (ii). (i) Induction on i. For $i = 0$, $l = \varrho + 2$ and $s = 1$. For $i+1$: $\mu^{\mathcal{L}}_{l,i+1} = \mu^{\mathcal{L}}_{l+1,i} * \eta_l * \theta_l(\mathcal{L})$. For measurable $f \ge 0$ let $F(x) = \int f(mx) \,\mathrm{d}(\mu^{\mathcal{L}}_{l+1,i} * \eta_l)(m)$. An element of $\mathcal{Q}_{l+1}(T) \cap \mathcal{H}_l(T)$ is $g\nu$ with $g \in \mathcal{H}_{l+1}(T)$, $\nu \in \mathcal{N}_l(T)$ (F1); $\eta_l * \delta_\nu = \eta_l$, $\eta_l * \delta_g = \delta_g * \eta_l$ (I3), and $\mu^{\mathcal{L}}_{l+1,i} * \delta_g = \mu^{\mathcal{L}}_{l+1,i}$ by induction; so F is invariant under $\mathcal{Q}_{l+1}(T) \cap \mathcal{H}_l(T)$. By Lemma 5.5 with $S = \mathcal{H}_l(T)$,

$\int F(xs)\,\mathrm{d}\theta_l(\mathcal{L})(x) = \int F\mathrm{d}\theta_l(\mathcal{L})$, which is the claim. For $l = 0$, $i = \varrho + 2$ every $s \in G(T) = \mathcal{H}_0(T)$ is allowed.

(ii) Induction on j: $\mu^{\mathcal{L}}_{0,j+1} = \eta_j * \theta_j(\mathcal{L}) * \eta_{<j} * (\theta_{j-1}(\mathcal{L}) * \dots)$; $\theta_j(\mathcal{L})$ is carried by $\mathcal{H}_j(T)$, whose unit masses commute with $\eta_{<j}$, so $\theta_j(\mathcal{L}) * \eta_{<j} = \eta_{<j} * \theta_j(\mathcal{L})$ (I1); and $\eta_j * \eta_{<j} = \eta_{<j+1}$ (I3). □

Lemma 5.6 (the stack measures are not zero)**.** *Let $\mathcal{L}$ be a level in* $\mathrm{Lev}(\Omega, T, \mathfrak{m})$ *and l, i natural numbers. Then $\mu^{\mathcal{L}}_{l,i} \neq 0$.*

Proof. The total mass of a convolution is the product of the total masses. η_l and δ_1 are not zero, $\theta_n(\mathcal{L}) \neq 0$ by definition of a level, and for $l \leq \varrho$ the image $\delta^{(l)}(\mathcal{L})$ of $\theta_l(\mathcal{L})$ (Lemma 5.1) is not zero. □

LEAN `Level.stackMeasure_ne_zero`

Lemma 5.7 (left translations of a factor of a Haar measure)**.** *Let $\mathcal{L}$ be a level in* $\mathrm{Lev}(\Omega, T, \mathfrak{m})$ *whose tower measure $\mathsf{M}(\mathcal{L})$ is a Haar measure. Let ρ, ν be s-finite measures on $G(T)$ with $\mathsf{M}(\mathcal{L}) = \rho * \nu$, let S be a subgroup and $s \in G(T)$, and suppose that ρ is carried by S, that $\rho * \delta_{s'} = \rho$ for all $s' \in S$, and that $\delta_s * \rho = \rho * \delta_s$. Then for every measurable $\Phi\colon G(T) \to [0, \infty]$ that is invariant under left translations by S,*

$$\int \Phi(sg)\,\mathrm{d}\nu(g) = \int \Phi\,\mathrm{d}\nu.$$

Proof. Choose a measurable $f > 0$ with $\int f\mathrm{d}\mathsf{M}(\mathcal{L}) < \infty$ and put $D(x) = \int f(yx)\,\mathrm{d}\rho(y)$. We claim that for *every* s-finite ν' with $\mathsf{M}(\mathcal{L}) = \rho * \nu'$

$$\int \Phi(x) D(x)^{-1} f(x)\,\mathrm{d}\mathsf{M}(\mathcal{L})(x) = \int \Phi\,\mathrm{d}\nu'.$$

Indeed, by (I1) the left side is

$$\iint \Phi(yx) D(yx)^{-1} f(yx)\,\mathrm{d}\rho(y)\,\mathrm{d}\nu'(x);$$

for $y \in S$, $\Phi(yx) = \Phi(x)$ and $D(yx) = D(x)$ by the right invariance of ρ; $D(x) < \infty$ for ν'-almost every x and $D(x) > 0$ because $\rho \neq 0$ (Lemma 5.6 excludes $\rho = 0$, as $\mathsf{M}(\mathcal{L}) \neq 0$); so the inner integral is $\Phi(x)$. Apply the claim to ν and to $\delta_s * \nu$: the latter is allowed since $\rho * (\delta_s * \nu) = \delta_s * \rho * \nu = \delta_s * \mathsf{M}(\mathcal{L}) = \mathsf{M}(\mathcal{L})$ by left invariance. □

LEAN `Level.lintegral_mul_left_eq_of_factor`

Proof of Theorem A(iii). $\mathsf{M}(\mathcal{L}) = \mu^{\mathcal{L}}_{j,\varrho+2-j} * \mu^{\mathcal{L}}_{0,j}$. The first factor is carried by $\mathcal{H}_j(T)$, is invariant under right translations by $\mathcal{H}_j(T)$ by (i), and commutes with the unit mass at $w_j(p)^{-1}$ by (F5) and (I1). Apply Lemma 5.7 with $S = \mathcal{H}_j(T)$ and $s = w_j(p)^{-1}$. □

6 Nearness, floor values and weights

Lemma 6.1 (measurability and openness)**.** *Let $\mathcal{L}$ be a level in* $\mathrm{Lev}(\Omega, T, \mathfrak{m})$*, j a natural number and a a natural number with $a < \varrho$.* (i) *The set of the points near at j is measurable.* (ii) *The set of the triples $(x, y, c) \in G(T) \times G(T) \times A$ with $c \in \mathrm{Val}_a(\mathcal{L}; x, y)$ is open.* (iii) *For all $x, y \in G(T)$: $0 < \mathrm{vol}(\mathrm{Val}_a(\mathcal{L}; x, y)) < \infty$.*

Proof. (i) It is the product $\mathcal{Q}_j(T)^{-1}\,\mathrm{Box}(\mathcal{L})$ of a closed set (I2) and a compact set, hence closed. (ii) If $c = w_{e_a^*}$ for an admissible w, then for (x', y', c') near (x, y, c)

the vector w with the coordinate at e_a^* replaced by c' is admissible, since $\mathfrak{B}(\mathcal{L})$ is open and the conditions are continuous. (iii) The set is open by (ii) and contains 0 (take $w = 0$; $0 \in \mathfrak{B}(\mathcal{L})$), so its volume is positive; it is contained in the image of the compact set $x\,\overline{\mathfrak{B}(\mathcal{L})}$ under a coordinate, so its volume is finite. □

LEAN `Level.measurableSet_isNear` · `Level.isOpen_floorValues` · `Level.volume_floorValues_ne_zero_ne_top`

Lemma 6.2 (invariance of floor values and weights)**.** *Let $\mathcal{L}$ be a level in* $\mathrm{Lev}(\Omega, T, \mathfrak{m})$*, $x, y, g \in G(T)$.* (i) *For $a < \varrho$ and $\xi \in \mathcal{Q}_{a+1}(T)$:* $\mathrm{Val}_a(\mathcal{L}; \xi x, y) = \mathrm{Val}_a(\mathcal{L}; x, y)$. (ii) *For a natural number j and $\xi \in \mathcal{Q}_j(T)$:* $\mathsf{w}_j(\mathcal{L}; \xi x, y) = \mathsf{w}_j(\mathcal{L}; x, y)$ *and* $\mathsf{w}_j^{-1}(\mathcal{L}; x)(\xi g) = \mathsf{w}_j^{-1}(\mathcal{L}; x)(g)$.

Proof. (i) If w is admissible for (x, y) then ξw is admissible for $(\xi x, y)$, because $(\xi x)^{-1}\xi w = x^{-1}w$ and $(\xi xy)^{-1}\xi w = (xy)^{-1}w$, and by (F3) ξw has the same coordinates at e_b^*, $b \le a$, as w; conversely with ξ^{-1}. (ii) The weight is a product over $a < \min(j, \varrho)$, and $\mathcal{Q}_j(T) \subseteq \mathcal{Q}_{a+1}(T)$; nearness at j is $\mathcal{Q}_j(T)$-invariant. □

LEAN `Level.floorValues_mul` · `Level.invWeight_mul`

Also $\mathsf{w}_{a+1}(\mathcal{L}; x, y) = \mathsf{w}_a(\mathcal{L}; x, y) \cdot \mathrm{vol}(\mathrm{Val}_a(\mathcal{L}; x, y))^{\mathsf{s}(\mathcal{L})}$ for $a < \varrho$, the weights are neither 0 nor ∞ (Lemma 6.1(iii)), and they are measurable in (x, y) because the set of Lemma 6.1(ii) is open.

Lemma 6.3 (the upper measure of the near points)**.** *Let $\mathcal{L}$ be a level in* $\mathrm{Lev}(\Omega, T, \mathfrak{m})$ *and $V \in [0, \infty]$ with $\mathrm{Mass}_a(\mathcal{L}; 1) \le V$ for every $a \le \varrho$. Then for every natural number l and every $c \in G(T)$,*

$$\mathsf{M}_{\ge l}(\mathcal{L})\{g : \ gc \text{ is near at } \varrho + 2\} \le V^{n-l}.$$

Proof. Induction on $\varrho + 2 - l$. For $l \ge n$ the bound is 1: for $l = n$ it is the last part of (L2), since $\mathcal{Q}_{\varrho+2}(T) = \mathcal{U}(T)$. For $l < n$, $\mathsf{M}_{\ge l}(\mathcal{L}) = \mathsf{M}_{\ge l+1}(\mathcal{L}) * \theta_l(\mathcal{L})$, so the measure is $\int \mathsf{M}_{\ge l+1}(\mathcal{L})\{g : g\sigma c \text{ near at } \varrho + 2\}\,\mathrm{d}\theta_l(\mathcal{L})(\sigma)$ (I1). The integrand is at most V^{n-l-1} by induction, and it is 0 unless σc is near at $l + 1$, because $\mathsf{M}_{\ge l+1}(\mathcal{L})$ is carried by $\mathcal{H}_{l+1}(T)$. The $\theta_l(\mathcal{L})$-measure of these σ is at most V. □

LEAN `Level.upperMeasure_isNear_le`

7 Kernels

Lemma 7.1 (without the kernel axioms)**.** *Let $\mathcal{L}$ be a level in* $\mathrm{Lev}(\Omega, T, \mathfrak{m})$*, $k\colon G(T) \times G(T) \to \mathbb{C}$, l a natural number and $g, y, x \in G(T)$.* (i) *If $\mathsf{Pv}_l(\mathcal{L}; k)(g, y) \ne 0$, then $k(\sigma g, \sigma y) \ne 0$ for some $\sigma \in \mathcal{H}_l(T)$.* (ii) *If $l \le \varrho$, then*

$$\int \left|\mathsf{Av}_{l+1}(\mathcal{L}; k)(\sigma g, \sigma g x)\right|^2 \mathrm{d}\theta_l(\mathcal{L})(\sigma) \le \frac{\mathsf{Sm}_l(\mathcal{L}; k, x)(g)}{\mathsf{w}_l(\mathcal{L}; g, x)}.$$

Proof. (i) The measure $\theta_l(\mathcal{L})$ is carried by $\mathcal{H}_l(T)$ and $\mathsf{M}_{\ge l+1}(\mathcal{L})$ by $\mathcal{H}_{l+1}(T)$. So there is $\sigma_1 \in \mathcal{H}_l(T)$ with

$$\mathsf{Av}_{l+1}(\mathcal{L}; k)(\sigma_1 g, \sigma_1 y) \ne 0,$$

and then $k(\sigma_2\sigma_1 g, \sigma_2\sigma_1 y) \ne 0$ for some $\sigma_2 \in \mathcal{H}_{l+1}(T)$. (ii) By definition the weighted square sum is at least $\int \mathsf{Sq}_l(\mathcal{L}; k, x)(\sigma g)\,\mathrm{d}\theta_l(\mathcal{L})(\sigma)$, and $\mathsf{w}_l(\mathcal{L}; \sigma g, x) = \mathsf{w}_l(\mathcal{L}; g, x)$ for $\sigma \in \mathcal{H}_l(T)$ (Lemma 6.2). □

LEAN `Level.exists_ne_zero_of_pivotModulus` · `Level.lintegral_upperPairing_sq_le`

Setting 7.2 (a pairing kernel)**.** Let $\mathcal{L}$ be a level in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$, j a natural number, $k\colon G(T) \times G(T) \to \mathbb{C}$ and $d \in G(T)$ with $(k, d) \in \mathrm{PK}_j(\mathcal{L})$.

Lemma 7.3 (averages floor by floor)**.** *In Setting 7.2, let l be a natural number and $g, y \in G(T)$. Then* (i) *$x \mapsto k(xg, xy)$ is integrable for $\mathsf{M}_{\geq l}(\mathcal{L})$;* (ii) *$\sigma \mapsto \mathsf{Av}_{l+1}(\mathcal{L}; k)(\sigma g, \sigma y)$ is integrable for $\theta_l(\mathcal{L})$ and*

$$\mathsf{Av}_l(\mathcal{L}; k)(g, y) = \int \mathsf{Av}_{l+1}(\mathcal{L}; k)(\sigma g, \sigma y)\, \mathrm{d}\theta_l(\mathcal{L})(\sigma);$$

hence $|\mathsf{Av}_l(\mathcal{L}; k)(g, y)| \leq \mathsf{Pv}_l(\mathcal{L}; k)(g, y)$, and $\mathsf{Av}_l(\mathcal{L}; k)(\cdot, \cdot)$ is measurable; (iii) *if $j \leq l$, then $(k, d) \in \mathrm{PK}_l(\mathcal{L})$.*

Proof. (i) is (K1) and (K4). (ii) is Fubini's theorem for $\mathsf{M}_{\geq l}(\mathcal{L}) = \mathsf{M}_{\geq l+1}(\mathcal{L}) * \theta_l(\mathcal{L})$, with (i). (iii) Only (K3) depends on j, and $\mathcal{H}_l(T) \subseteq \mathcal{H}_j(T)$. □

LEAN `Level.IsPairingKernel.integrable_upperMeasure` · `Level.IsPairingKernel.upperPairing_eq_integral_succ` · `Level.IsPairingKernel.enorm_upperPairing_le` · `Level.IsPairingKernel.stronglyMeasurable_upperPairing` · `Level.IsPairingKernel.mono`

Lemma 7.4 (invariance of the averages)**.** *In Setting 7.2, let l be a natural number with $j \leq l$, let $s \in \mathcal{Q}_l(T) \cap \mathcal{H}_j(T)$ and $g, y \in G(T)$. Then $\mathsf{Av}_l(\mathcal{L}; k)(sg, sy) = \mathsf{Av}_l(\mathcal{L}; k)(g, y)$.*

Proof. Induction on $\varrho + 2 - l$. If $l \geq \varrho + 2$, then $s \in \mathcal{U}(T) = \mathcal{N}_{<\varrho+1}(T)$, the average is k itself, and (K2) with $\nu = \nu' = s$ gives the factor $\psi_A(g_{\mathcal{L}}^{-1}(\mathrm{sd}(dsd^{-1}) - \mathrm{sd}(s))) = 1$ by (K3). Otherwise write both sides by Lemma 7.3(ii). The function $\sigma \mapsto \mathsf{Av}_{l+1}(\mathcal{L}; k)(\sigma g, \sigma y)$ is invariant under $\mathcal{Q}_{l+1}(T) \cap \mathcal{H}_j(T)$ by induction, so Lemma 5.5 with $S = \mathcal{H}_j(T) \supseteq \mathcal{H}_l(T)$ allows to replace σ by σs. □

LEAN `Level.IsPairingKernel.upperPairing_invariant`

Lemma 7.5 (invariance and support of the weighted squares)**.** *In Setting 7.2, let $x \in G(T)$.* (i) *The function $\mathsf{Sq}_j(\mathcal{L}; k, x)(\cdot)$ is invariant under $\mathcal{Q}_{j+1}(T)$ and vanishes at every g that is not near at $j + 1$.* (ii) *The function $\mathsf{Sm}_j(\mathcal{L}; k, x)(\cdot)$ is invariant under $\mathcal{Q}_j(T)$ and vanishes at every g that is not near at j; it is measurable in (x, g).*

Proof. (i) Let $s \in \mathcal{Q}_{j+1}(T)$, and write $s = \gamma\nu$ with $\gamma \in \mathcal{H}_j(T)$, $\nu \in \mathcal{N}_{<j}(T)$ (F1); then $\gamma = s\nu^{-1} \in \mathcal{Q}_{j+1}(T) \cap \mathcal{H}_j(T)$. The weight is invariant (Lemma 6.2). For the average at $j + 1$: Lemma 7.4 removes γ, and (K2) at $l = j$ with $\nu' = \nu$ shows that $\mathsf{Av}_{j+1}(\mathcal{L}; k)(\nu g, \nu g x)$ and $\mathsf{Av}_{j+1}(\mathcal{L}; k)(g, gx)$ differ by a factor of absolute value 1. The support is (K5). (ii) Downward induction on j along the recursion: by (i), (iii) of Lemma 7.3 and the induction hypothesis the integrand $\sigma \mapsto (\dots)(\sigma g)$ is $\mathcal{Q}_{j+1}(T)$-invariant in σg, and Lemma 5.5 with $S = G(T)$ gives the invariance under $s \in \mathcal{Q}_j(T)$. If the sum at g is not 0, some $\sigma \in \mathcal{H}_j(T)$ has σg near at $j + 1$, so g is near at j. □

LEAN `Level.IsPairingKernel.weightedSquare_invariant_support` · `Level.IsPairingKernel.weightedSquareSum_invariant_support` · `Level.IsPairingKernel.measurable_weightedSquareSum`

Lemma 7.6 (Cauchy–Schwarz on a floor)**.** *In Setting 7.2, let l be a natural number, $g, y \in G(T)$, $K \in [0, \infty]$ and $u\colon G(T) \to [0, \infty]$, $v\colon G(T) \to [0, \infty]$ measurable with $\mathrm{Mass}_l(\mathcal{L}; u) \leq K$ and*

$$\left|\mathsf{Av}_{l+1}(\mathcal{L}; k)(\sigma g, \sigma y)\right|^2 \leq u(\sigma g)\, v(\sigma) \qquad \text{for all } \sigma \in \mathcal{H}_l(T).$$

Then $\mathsf{Pv}_l(\mathcal{L}; k)(g, y)^2 \leq K \int v\, \mathrm{d}\theta_l(\mathcal{L})$.

Proof. By (K5) the integrand of the pivot vanishes unless σg is near at $l+1$. Restrict the integral to these σ and to $\mathcal{H}_l(T)$, apply (I8) with $f_1 = u(\sigma g)$ and $f_2 = v$, and bound the integral of f_1 by K. □

LEAN `Level.IsPairingKernel.pivotModulus_sq_le`

Lemma 7.7 (the self-norm, floor by floor)**.** *In Setting 7.2, let $x \in G(T)$ and l a natural number.* (i) *If $l \le \varrho$, then*

$$\mathrm{SN}_l(\mathcal{L}; k, x) = \int \mathsf{Sq}_l(\mathcal{L}; k, x)(g)\, \mathrm{d}(\theta_l(\mathcal{L}) * \cdots * \theta_0(\mathcal{L}))(g) + \mathrm{SN}_{l+1}(\mathcal{L}; k, x).$$

(ii) *If $l \le n$, then $\mathrm{SN}_l(\mathcal{L}; k, x) \le \mathrm{SN}_0(\mathcal{L}; k, x)$.*

Proof. (i) Insert the recursion for the weighted square sum and use (I1) for $\theta_l(\mathcal{L}) * (\theta_{l-1}(\mathcal{L}) * \cdots * \theta_0(\mathcal{L}))$. (ii) follows from (i) by induction. □

LEAN `Level.IsPairingKernel.selfNorm_eq_add_succ` · `Level.IsPairingKernel.selfNorm_le_selfNorm_zero`

8 Sizes in a gauge and fibre bounds

Lemma 8.1 (the vector of a Heisenberg element)**.** *Let A be a place, $\mathcal{S}$ a size in $\mathrm{Siz}(T)$, a a natural number with $a < \varrho$, $u \in A^\times$, i a natural number, $v \in V_{a+1}$ and β a skew element of A. If there is $\nu \in \mathcal{N}_{<a}(T)$ such that the column at e_a^* of $u^{-1}(\nu\, \mathrm{n}_a(v,\beta) - 1)$ lies in Vec_i, then $u^{-1}v \in \mathrm{Vec}_i$ and $u^{-1}(\beta - \tau_A h_T(v,v)) \in \mathcal{S}_i$.*

Proof. The column at e_a^* of $\mathrm{n}_a(v,\beta)$ is $e_a^* + v + (\beta - \tau_A h_T(v,v))e_a$; its coordinates at e_b^*, $b < a$, vanish, so by (F4) multiplication by ν does not change its coordinates of floor at least a: those on V_{a+1}, which are v, and the one at e_a. □

LEAN `Size.mem_vecsOfSize_of_heisenbergElem`

Lemma 8.2 (coordinates in a floor basis)**.** *In Setting 1.1, let $b \le \varrho$, j a natural number and x a coordinate vector for the floor basis E_b with $E_b x \in \mathrm{Vec}_j$. Then $x \in \mathrm{Vec}_{j+4}$.*

Proof. By (I4)(d), $E_b^* J(T) E_b x$ is x with its coordinates permuted. The entries of E_b are those of $E(\mathcal{L})$ or 0, 1, so E_b^* has size 1 by (G2), $E_b^* J(T)$ has size 3, and $E_b^* J(T)(E_b x) \in \mathrm{Vec}_{3+j+1}$. □

LEAN `Gauge.mem_vecsOfSize_of_floorBasis_mulVec`

Lemma 8.3 (integrating against a fibre bound)**.** *In Setting 1.1, let $b \le \varrho$, q a measure on V, j a natural number and $M, K \in [0,\infty]$ with $\mathrm{Fib}_{b,j}(\Gamma; q) \le M$.* (i) *Let $\Phi\colon V \to [0,\infty]$ and let G be a function of the fixed and the orthonormal coordinates with values in $[0,\infty]$ such that, for every P, $G(P,\cdot)$ is measurable with $\int G(P,w)\,\mathrm{d}w \le K$, and such that $\Phi(E_b x) \le G(P(x), w(x))$ for every coordinate vector $x \in \mathrm{Vec}_{j+4}$ with fixed coordinates $P(x)$ and orthonormal coordinates $w(x)$. Then $\int_{\mathrm{Vec}_j} \Phi\,\mathrm{d}q \le M \cdot K$.* (ii) *Let $\varphi\colon A^\Omega \to [0,\infty]$ be measurable with $\int_{\mathrm{Vec}_{j+4}} \varphi(w)\,\mathrm{d}w \le K$. Then*

$$\int_{\mathrm{Vec}_j} \varphi(\mathsf{O}(\mathcal{L})^* J(T) x)\,\mathrm{d}q(x) \le M \cdot K.$$

Proof. (i) With the notation of Definition 2.16, $\int_{\mathrm{Vec}_j} \Phi\,\mathrm{d}q$ is the integral of $\mathbf{1}[E_b \mathsf{k} \in \mathrm{Vec}_j]\,\Phi(E_b \mathsf{k})$ over (P,w,z). By Lemma 8.2 the integrand is at most $\mathbf{1}[\mathsf{k}(P,w,z) \in \mathrm{Vec}_{j+4}]\,G(P,w)$; integrate over z (at most $\mu(P)G(P,w)$), then over w (at most

$\mu(P)K)$, then over P. (ii) By (I4)(d), $\mathsf{O}(\mathcal{L})^* J(T) E_b x = w(x)$; apply (i) with $G(P,w) = \mathbf{1}[w \in \mathrm{Vec}_{j+4}]\varphi(w)$. □

LEAN `Gauge.setLIntegral_le_of_hasFibreBound` · `Gauge.setLIntegral_orthonormalPart_le`

Lemma 8.4 (fibre bounds for the quadric and for the volume). *In Setting 1.1, let $b \le \varrho$ and j a natural number.* (i) *Let $M \in [0,\infty]$ be such that the quadric measure with $\beta = 0$ of the sum of $|\pi_b| + 1$ hyperbolic planes gives Vec_{j+5} mass at most M. Then $\mathrm{Fib}_{b,j}(\Gamma; \delta^{(b)}(\mathcal{L})) \le M$.* (ii) *In particular $\mathrm{Fib}_{b,j}(\Gamma; \delta^{(b)}(\mathcal{L})) \le \Lambda(\Gamma)^{99n(j+6)}$.* (iii) *Also $\mathrm{Fib}_{b,j}(\Gamma; \mathrm{vol}_b) \le \Lambda(\Gamma)^{99n(j+6)}$.*

Proof. (i) In the coordinates of E_b, $\delta^{(b)}(\mathcal{L})$ is the image of $\mathrm{d}t\,\mathrm{d}(\pi', w, \pi'')\,\mathrm{d}\nu_A(\alpha)$ under the flat chart, which fills in the coordinate t'. Take Z the space of the pairs (π'', α) with its product measure and k the flat chart after regrouping the variables as $((t,\pi'), w, (\pi'', \alpha))$, which preserves the product measure. Fix $P = (t, \pi')$ and w, and a point z_0 of the fibre set $\{z : \mathsf{k}(P, w, z) \in \mathrm{Vec}_{j+4}\}$. For another point z of it, the difference of the two values of k has w-coordinate 0, sizes at most $j+5$ by rule (b), and is the value at $(P, 0, z - z_0)$ of the flat chart of the cone of the hyperbolic planes alone (the chart is affine in (π'', α) for fixed t, π', w). So the fibre set, translated by z_0, lies in the corresponding set for the cone and size $j+5$, whose measure we call $\mu(P)$; and $\int \mu(P)\,\mathrm{d}P$ is the cone measure of Vec_{j+5}. (ii) is (i) with (G5), since $\mathfrak{m} \le |\pi_b| \le \mathfrak{m} + \varrho$. (iii) Take Z the space of the coordinates (t', π'') with its volume, k the regrouping, and $\mu(P) = \mathrm{vol}(\Gamma_{j+4})^{1+|\pi_b|}$ for $P \in \mathrm{Vec}_{j+4}$, 0 otherwise; then $\int \mu \le \Lambda(\Gamma)^{2(j+5)(1+\mathfrak{m}+\varrho)}$ by (G4), and $2(1+\mathfrak{m}+\varrho) \le 20n$ by (G1). □

LEAN `Gauge.hasFibreBound_floorQuadric_of_le` · `Gauge.hasFibreBound_floorQuadric` · `Gauge.hasFibreBound_floorVolume`

Lemma 8.5 (an integral over a Heisenberg group). *In Setting 1.1, let $a < \varrho$, $u \in A^\times$, j a natural number, $C, K, M \in [0,\infty]$, $\Phi\colon G(T) \to [0,\infty]$ and $\Psi\colon V \to [0,\infty]$ measurable. Suppose $\int_{\mathrm{Vec}_j} \Psi\,\mathrm{d}\,\mathrm{vol}_{a+1} \le M$ and $\mathsf{d}_a(\mathcal{L}; u) \cdot \Lambda(\Gamma)^{j+2} \cdot M \le K$, and that for every $v \in V_{a+1}$ and skew β with $\Phi(\mathrm{n}) \ne 0$, $\mathrm{n} = \mathrm{n}_a(v,\beta)$: there is $\nu \in \mathcal{N}_{<a}(T)$ such that the column at e_a^* of $u^{-1}(\nu \mathrm{n}^{-1} - 1)$ lies in Vec_j, and $\Phi(\mathrm{n}) \le C \cdot \Psi(u^{-1}v)$. Then $\int \Phi\,\mathrm{d}\eta_a \le C \cdot K$.*

Proof. η_a is the image of $\mathrm{d}v\,\mathrm{d}\nu_A(\beta)$. Let $\Phi(\mathrm{n}_a(v,\beta)) \ne 0$. As $\mathrm{n}^{-1} = \mathrm{n}_a(-v, \beta')$ with β' a translate of $-\beta$ depending on v only (F6), Lemma 8.1 gives $u^{-1}v \in \mathrm{Vec}_j$ and $u^{-1}(\beta' - \tau_A h_T(v,v)) \in \Gamma_j$. So for fixed v the set S_v of the β with $\Phi \ne 0$ has $u^{-1}(\beta_1 - \beta_2) \in \Gamma_{j+1}$ for $\beta_1, \beta_2 \in S_v$; by (G2) and the rules, $u^{-1}(\tau_A \delta_A^{-1}\beta + \beta'') \in \Gamma_{j+3}$ for differences β, β'', and by (I7) and (G4) $\nu_A(S_v) \le (\Lambda(\Gamma)^{j+4}\,\mathrm{mod}_A(u))^{1/2} \le \Lambda(\Gamma)^{j+2}\,\mathrm{mod}_A(u)^{1/2}$. Hence

$$\begin{aligned}\int \Phi\,\mathrm{d}\eta_a &\le C\,\Lambda(\Gamma)^{j+2}\,\mathrm{mod}_A(u)^{1/2} \int \mathbf{1}[u^{-1}v \in \mathrm{Vec}_j]\,\Psi(u^{-1}v)\,\mathrm{d}v \\ &= C\,\Lambda(\Gamma)^{j+2}\,\mathrm{mod}_A(u)^{1/2}\,\mathrm{mod}_A(u)^N \int_{\mathrm{Vec}_j} \Psi\,\mathrm{d}\,\mathrm{vol}_{a+1},\end{aligned}$$

where $N = k + 1 + 2(\varrho - (a+1))$ is the rank of V_{a+1}. The factor in front is $C\,\mathsf{d}_a(\mathcal{L}; u)\,\Lambda(\Gamma)^{j+2}$. □

LEAN `Gauge.lintegral_heisenbergMeasure_le`

Lemma 8.6 (three size bounds). *In Setting 1.1:* (i) *if $g \in \mathrm{Box}(\mathcal{L})$, $x, y \in \mathrm{Pos}(\mathcal{L}; \Gamma)$ and $\sigma = gx(gy)^{-1}$, then $\mathrm{sz}(g_{\mathcal{L}}^{-1}(\sigma - 1)) \le 14$;* (ii) *if $b \le \varrho$, $\sigma \in \mathcal{H}_b(T)$, $g \in G(T)$*

with $\mathrm{sz}(g) \le i$ *for a natural number* i*, and* σg *is near at* $b+1$*,* $\sigma g \in \mathrm{Near}_{b+1}(\mathcal{L})$*, then* $\sigma^{-1} f_b \in \mathrm{Vec}_{i+6}$*;* (iii) *if* $j \le \varrho$*,* $p \colon \mathbb{N} \to A^\times$ *is a head of length* j*,* $p \in \mathrm{Hd}_j$*, and* g *and* $w_j(p)^{-1} g$ *are near at* j*,* $g \in \mathrm{Near}_j(\mathcal{L})$ *and* $w_j(p)^{-1} g \in \mathrm{Near}_j(\mathcal{L})$*, then there is* $\sigma \in \mathcal{H}_j(T)$ *with* $\mathrm{sz}(\sigma g) \le 60n^2$ *and* $\mathrm{sz}(\sigma w_j(p)^{-1} g) \le 99n^2$*.*

Proof. (i) $\sigma - 1 = g(x-y)(gy)^{-1}$. Here $g_{\mathcal{L}}^{-1}(x-y)$ has size 2, $y = 1 + g_{\mathcal{L}} \cdot g_{\mathcal{L}}^{-1}(y-1)$ has size 3, gy size 5, its inverse size 9 (Lemma 4.3), and the product size $(1+2+1)+9+1$. (ii) Let $\xi \in \mathcal{Q}_{b+1}(T)$ with $\xi \sigma g \in \mathrm{Box}(\mathcal{L})$. By (F2) and (F3) the coordinates of $\sigma^{-1} f_b$ of floor less than b vanish and the others are those of $\sigma^{-1}\xi^{-1} f_b$, a column of $g(\xi\sigma g)^{-1}$, of size $i + 5 + 1$. (iii) Let $\xi g \in \mathrm{Box}(\mathcal{L})$, $\xi = \sigma\nu$ with $\sigma \in \mathcal{H}_j(T)$, $\nu \in \mathcal{N}_{<j}(T)$ (F1). Then $(\sigma\nu\sigma^{-1})(\sigma g) \in \mathrm{Box}(\mathcal{L})$ with $\sigma\nu\sigma^{-1} \in \mathcal{N}_{<j}(T)$, and $w_j(p)^{-1}\sigma g = \sigma w_j(p)^{-1} g$ is near at j; apply Lemma 4.7(iv) to $\mathsf{K} = \mathrm{Box}(\mathcal{L})$, σg and $s = \sigma\nu\sigma^{-1}$. □

LEAN `Gauge.conj_mul_inv_close_to_one` · `Gauge.orbitPoint_mem_vecsOfSize` · `Gauge.exists_hasMatSize_of_isNear`

Lemma 8.7 (a position between two near points is small)**.** *In Setting 1.1, let* Γ_S *be a second gauge in* $\mathrm{Gau}(\mathcal{L})$*,* j *a natural number with* $j \le \varrho$*, and* $p, q \colon \mathbb{N} \to A^\times$ *heads of length* j *for* Γ_S*:* $p \in \mathrm{Hd}_j$ *and* $q \in \mathrm{Hd}_j$ *for* Γ_S*. Suppose that, if* $j \ne 0$*, then* $g_{\mathcal{L}}^{-1} c \in \Gamma_1$ *for all* $c \in (\Gamma_S)_{10n}$*. Let* $x, y \in G(T)$ *be such that: there are* $u, v \in \mathcal{U}(T)$ *with* $ux \in \mathrm{Box}(\mathcal{L})$*,* $v(xy) \in \mathrm{Box}(\mathcal{L})$ *and* $\mathrm{sz}(g_{\mathcal{L}}^{-1}(v(xy) - ux)) \le 2$*;* $w_j(p)^{-1} x$ *is near at* j*,* $w_j(p)^{-1} x \in \mathrm{Near}_j(\mathcal{L})$*;* $w_j(q)^{-1} xy$ *is near at* j*; and for every* a *with* $j \le a < \varrho$ *there is a unit* $c \in \mathrm{Val}_a(\mathcal{L}; x, y)$ *with* $c^{-1} \in \Gamma_1$*. Then* $\mathrm{sz}(g_{\mathcal{L}}^{-1}(y-1)) \le 99n$*.*

Proof. Put $X = ux$, $Y = v(xy)$; then $y - 1 = X^{-1}((Y - X) - (1 - uv^{-1})Y)$. We bound $g_{\mathcal{L}}^{-1}(1 - uv^{-1})$ by Lemma 4.5 for the element $vu^{-1} \in \mathcal{U}(T) = \mathcal{N}_{<\varrho}(T)$, $r = g_{\mathcal{L}}$ and $c = 6$. For $a < j$ take $w = ue_a^*$, the column of u at e_a^*: by Lemma 4.7(iii) for Γ_S and $\mathsf{K} = \mathrm{Box}(\mathcal{L})$ its entries, and those of ve_a^*, lie in $(\Gamma_S)_{10n}$, so $g_{\mathcal{L}}^{-1} w$ and $g_{\mathcal{L}}^{-1}(u - v)e_a^*$ have size at most 2. For $a \ge j$ take the unit $c = w'_{e_a^*}$ of the hypothesis, w' admissible, and $w = c^{-1} u w'$: by (F3) $w_{e_a^*} = 1$; $uw' = X(x^{-1} w')$ has size 3 by (G3); and $g_{\mathcal{L}}^{-1}(u - v)w' = g_{\mathcal{L}}^{-1}(X - Y)u_1 + Y g_{\mathcal{L}}^{-1}(u_1 - u_2)$ with (u_1, u_2) the close pair of w', of size 5. So $g_{\mathcal{L}}^{-1}(1 - uv^{-1})$ has size $10 + 20\varrho$, and $g_{\mathcal{L}}^{-1}(y-1)$ has size at most $5 + (13 + 20\varrho) + 1 \le 99n$. □

LEAN `Gauge.hasMatSize_sub_one_of_floorValues`

9 Masses of floors and integrals of weights

Lemma 9.1 (the mass of a floor through its quadric)**.** *In Setting 1.1, let* $b \le \varrho$*,* $K \in [0, \infty]$*, and* $f \colon G(T) \to [0, \infty]$ *measurable and invariant under left translations by* $\mathcal{Q}_{b+1}(T)$*. Suppose that for every* $g \in \mathrm{Box}(\mathcal{L})$ *there is a measurable* $\Psi \colon V \to [0, \infty]$ *with* $\int_{\mathrm{Vec}_7} \Psi \, \mathrm{d}\delta^{(b)}(\mathcal{L}) \le K$ *and* $f(\sigma g) \le \Psi(\sigma^{-1} f_b)$ *for all* $\sigma \in \mathcal{H}_b(T)$*. Then* $\mathrm{Mass}_b(\mathcal{L}; f) \le K$*.*

Proof. Fix g. If g is not near at b the integral is over a null set. Otherwise $\xi g \in \mathrm{Box}(\mathcal{L})$ for some $\xi \in \mathcal{Q}_b(T)$, and by Lemma 5.5 (with $S = G(T)$ and $s = \xi^{-1}$) we may replace g by ξg. For $g \in \mathrm{Box}(\mathcal{L})$ and $\sigma \in \mathcal{H}_b(T)$ with σg near at $b+1$, Lemma 8.6(ii) gives $\sigma^{-1} f_b \in \mathrm{Vec}_7$; so the integral is at most $\int \mathbf{1}_{\mathrm{Vec}_7}(\sigma^{-1} f_b) \Psi(\sigma^{-1} f_b) \, \mathrm{d}\theta_b(\mathcal{L})(\sigma)$, which is $\int_{\mathrm{Vec}_7} \Psi \, \mathrm{d}\delta^{(b)}(\mathcal{L})$ by Lemma 5.1. □

LEAN `Gauge.floorMassLe_of_chart`

Lemma 9.2 (two masses)**.** *In Setting 1.1:* (i) *for $a < \varrho$ and $x \in \mathrm{Posn}(\mathcal{L})$, the function $u(g) = \mathsf{w}_a(\mathcal{L}; g, x)/\mathsf{w}_{a+1}(\mathcal{L}; g, x)$ satisfies*

$$\mathrm{Mass}_a(\mathcal{L}; u) \le \Lambda(\Gamma)^{10^4 n};$$

(ii) *for $b \le \varrho$ and $M \in [0, \infty]$ with $\mathrm{Fib}_{b,7}(\Gamma; \delta^{(b)}(\mathcal{L})) \le M$,*

$$\mathrm{Mass}_b(\mathcal{L}; 1) \le M \cdot \mathrm{vol}(\Gamma_{11})^{|\Omega|}.$$

Proof. (i) Here $u(g) = \mathrm{vol}(\mathrm{Val}_a(\mathcal{L}; g, x))^{-s}$, $s = \mathsf{s}(\mathcal{L})$, invariant under $\mathcal{Q}_{a+1}(T)$ (Lemma 6.2). Let $g \in \mathrm{Box}(\mathcal{L})$, and take φ from (G6) for g, the position x and the matrix $v = 0$; put $\Psi(v') = \varphi(\mathsf{O}(\mathcal{L})^* J(T) v')^s$. For $\sigma \in \mathcal{H}_a(T)$ and $u' = \mathsf{O}(\mathcal{L})^* J(T) \sigma^{-1} f_a$, every $w \in W$ gives the vector $\sigma \mathsf{O}(\mathcal{L}) w$, which is admissible for $(\sigma g, x)$ and has the coordinate $h_T(\sigma^{-1} f_a, \mathsf{O}(\mathcal{L}) w) = u'^* w$ at e_a^*; so $\{u'^* w : w \in W\} \subseteq \mathrm{Val}_a(\mathcal{L}; \sigma g, x)$ and $u(\sigma g) \le \varphi(u')^s = \Psi(\sigma^{-1} f_a)$. By Lemma 8.3(ii), Lemma 8.4(ii) and (G6), $\int_{\mathrm{Vec}_7} \Psi \, \mathrm{d}\delta^{(a)}(\mathcal{L}) \le \Lambda(\Gamma)^{99n \cdot 13} \Lambda(\Gamma)^{99n \cdot 12}$. Apply Lemma 9.1. (ii) Lemma 9.1 with $f = 1$, $\Psi = 1$, and Lemma 8.3(ii) with $\varphi = 1$. □

LEAN `Gauge.floorMassLe_floorWeight_div` · `Gauge.floorMassLe_one`

Proof of Theorem B(i)*. Reduction to the box.* Let $\xi g \in \mathrm{Box}(\mathcal{L})$, $\xi = \gamma \nu_0$ with $\gamma \in \mathcal{H}_j(T)$, $\nu_0 \in \mathcal{N}_{<j}(T)$. The weight of $(g, \cdot)$ is that of $(\xi g, \cdot)$, and $g^{-1} \nu y = (\xi g)^{-1} (\gamma \nu_0 \nu \gamma^{-1}) \gamma y$; by (I3) the law of $\gamma \nu_0 \nu \gamma^{-1}$ is again $\eta_{<j}$. So we may assume $g \in \mathrm{Box}(\mathcal{L})$, for all y. *Induction on j.* For $j = 0$ both sides are at most 1. Let $a = j < \varrho$ and assume the bound for j and all y. If the integral for $j+1$ is not 0, there is $\tau \in \mathcal{N}_{<j+1}(T)$ with $x_0 = g^{-1} \tau y \in \mathrm{Pos}(\mathcal{L}; \Gamma)$, and by invariance we may replace y by $y_0 = \tau y$. Write $\eta_{<j+1} = \eta_{<j} * \eta_j$ and $\Phi(\mathrm{n}) = \int \mathbf{1}_P(g^{-1} \nu \mathrm{n} y_0) \, \mathsf{w}_{j+1}(\mathcal{L}; g, g^{-1} \nu \mathrm{n} y_0) \, \mathrm{d}\eta_{<j}(\nu)$, $P = \mathrm{Pos}(\mathcal{L}; \Gamma)$. We apply Lemma 8.5 with $u = g_{\mathcal{L}}$, size 14 and $\Psi(v') = \varphi(\mathsf{O}(\mathcal{L})^* J(T) v')^s$, φ from (G6) for g, the position x_0 and $v = 0$. Its hypotheses: if $g^{-1} \nu \mathrm{n} y_0 \in P$, then by Lemma 8.6(i) the element $g x_0 (g \cdot g^{-1} \nu \mathrm{n} y_0)^{-1} = \mathrm{n}^{-1} \nu^{-1} = \nu'' \mathrm{n}^{-1}$, with $\nu'' \in \mathcal{N}_{<j}(T)$, has $g_{\mathcal{L}}^{-1}(\cdot - 1)$ of size 14. Next, with $x' = g^{-1} \nu \mathrm{n} y_0$, the weight at $j+1$ is the weight at j times $\mathrm{vol}(\mathrm{Val}_j(\mathcal{L}; g, x'))^s$, and $\mathrm{Val}_j(\mathcal{L}; g, x') \subseteq \mathcal{C}_{u'}$ for $u' = \mathsf{O}(\mathcal{L})^* J(T) g_{\mathcal{L}}^{-1} v$: for an admissible w' with $c = w'_{e_j^*}$ and for $w \in W$, the two close pairs give $h_T(w', \mathsf{O}(\mathcal{L}) w) - h_T(w', \nu \mathrm{n} \mathsf{O}(\mathcal{L}) w)$, and $(1 - \nu \mathrm{n}) \mathsf{O}(\mathcal{L}) w$ is $h_T(v, \mathsf{O}(\mathcal{L}) w) e_j$ up to terms that w' does not see (F4), which gives $(c u')^* w$ after division by $g_{\mathcal{L}}$. So $\mathrm{vol}(\mathrm{Val}_j(\mathcal{L}; g, x'))^s \le \varphi(u')^s$, and by the induction hypothesis (with $\mathrm{n} y_0$ for y) $\Phi(\mathrm{n}) \le \prod_{l<j} C_l(\Gamma) \cdot \Psi(g_{\mathcal{L}}^{-1} v)$. Finally $\int_{\mathrm{Vec}_{14}} \Psi \, \mathrm{d}\,\mathrm{vol}_{j+1} \le \Lambda(\Gamma)^{99n \cdot 20} \Lambda(\Gamma)^{99n \cdot 19}$ (Lemmas 8.3(ii), 8.4(iii), (G6)). Lemma 8.5 gives the bound $\prod_{l<j} C_l(\Gamma) \cdot \mathsf{d}_j(\mathcal{L}; g_{\mathcal{L}}) \Lambda(\Gamma)^{16 + 3861n} \le \prod_{l \le j} C_l(\Gamma)$, as $c_A \ge 1$. □

Proof of Theorem B(ii). The function Φ is invariant under $\mathcal{N}_{<j}(T)$, a subgroup of $\mathcal{Q}_b(T)$. So the left side is

$$\int \Phi(y) \, I(y) \, \mathrm{d}\rho(y) \qquad \text{with} \qquad I(y) = \int \mathsf{w}_j^{-1}(\mathcal{L}; x)(w_j(p) \nu y) \, \mathrm{d}\eta_{<j}(\nu),$$

and only the y near at b count. We show $I(y) \le \prod_{a<j} \mathsf{d}_a(\mathcal{L}; p_a^{-1}) \cdot C_{\mathrm{gl}}(\Gamma)$ for such y. As in the proof of (i), with $\xi y \in \mathrm{Box}(\mathcal{L})$, $\xi \in \mathcal{Q}_b(T) \subseteq \mathcal{Q}_j(T)$, the invariance of $\eta_{<j}$ and of the inverse weight (Lemma 6.2) and (F5), we may assume $y \in \mathrm{Box}(\mathcal{L})$. The proof is by induction on j, with p a head of each length up to j; we show that floor a contributes a factor $\mathsf{d}_a(\mathcal{L}; p_a^{-1}) \Lambda(\Gamma)^{10^6 n^2}$. For the step from a to $a+1$ write $\eta_{<a+1} = \eta_a * \eta_{<a}$ and fix $\nu \in \mathcal{N}_{<a}(T)$. Lemma 8.5 is applied to the function

$$\mathrm{n} \mapsto \mathsf{w}_{a+1}^{-1}(\mathcal{L}; x)(w_{a+1}(p) \mathrm{n} \nu y),$$

with the unit $u = (-\bar{p}_a)^{-1}$ and $\Psi = \varphi(\mathsf{O}(\mathcal{L})^* J(T)\,\cdot\,)^s$, where φ comes from (G6) for $y \in \mathrm{Box}(\mathcal{L})$, the position x and the matrix v of the coordinates at $e_0, \dots, e_{\varrho-1}$ of the columns of $\nu\mathsf{O}(\mathcal{L})$. If the inverse weight is not 0, then $w_{a+1}(p)\mathrm{n}\nu y$ is near at $a+1$, and Lemma 4.7(iii) (for $\mathsf{K} = \mathrm{Box}(\mathcal{L})$, the head $\bar{p}$, the point $\mathrm{n}\nu y$ and $s = (\mathrm{n}\nu)^{-1}$) bounds the column at e_a^* of $\bar{p}_a(\nu^{-1}\mathrm{n}^{-1} - 1)$, which is the size condition; and the inverse weight at $a+1$ is the inverse weight at a of $w_a(p)\nu y$ times the power $-s$ of the volume of $\mathrm{Val}_a(\mathcal{L}; w_{a+1}(p)\mathrm{n}\nu y, x)$, where this set of floor values contains $\{u'^* w : w \in W\}$ for the vector u' attached to $u^{-1}v$ as in the proof of Lemma 9.2(i); so the first inequality of (G6) bounds it by $\varphi(u')^s$. The integral of Ψ is bounded as in (i), and the exponents add up to at most $10^6 n^2$ for each floor; $j \le n$ floors give $C_{\mathrm{gl}}(\Gamma)$. □

10 Plancherel

Proof of Theorem C(i). *Support.* Let $\sigma \in \mathcal{H}_b(T)$ with $F(\sigma) \neq 0$. By (K5) σx is near at $b+1$, and so is $(\sigma t^{-1})(qx) = (\sigma t^{-1} q \sigma^{-1})\sigma x$, because $\sigma(t^{-1}q)\sigma^{-1} \in \mathcal{N}_{<b}(T) \subseteq \mathcal{Q}_{b+1}(T)$. Lemma 8.6(ii) for $\sigma t^{-1} \in \mathcal{H}_b(T)$ and qx gives $t\sigma^{-1} f_b \in \mathrm{Vec}_{j+6}$.

Plancherel. By Lemma 7.4, F is invariant under $\mathcal{Q}_{b+1}(T) \cap \mathcal{H}_b(T)$, so on $\mathcal{H}_b(T)$ it is a measurable function of $\sigma^{-1} f_b$ (I5); by Lemma 5.1(ii) the inner integral is

$$G(v) = \int \psi_A(\xi\, h_T(v + y', v'))\, \alpha(v')\, \mathrm{d}\delta^{(b)}(\mathcal{L})(v'),$$
$$\int |\alpha|^2\, \mathrm{d}\delta^{(b)}(\mathcal{L}) = \int |F|^2\, \mathrm{d}\theta_b(\mathcal{L}),$$

with α supported in Vec_{j+6} by the first part. Write $\delta^{(b)}(\mathcal{L})$ by its fibre bound of size $j+6$ (Lemma 8.4(ii)): $v' = E_b \mathsf{k}(P', w', z)$, with fibre masses $\mu'(P')$, $\int \mu' \le \Lambda(\Gamma)^{99n(j+12)}$. In the coordinates of E_b the form h_T pairs the orthonormal coordinates with themselves and the fixed coordinates (t, π') with the remaining ones (t', π''). So for $v = E_b x$ the phase $\psi_A(\xi h_T(v + y', v'))$ is the product of $\psi_A(\xi\, w(x)^* w')$, of a factor of absolute value 1 that depends on x and P' only, and of a factor that depends on $(P(x), P', w', z)$. Hence $|G(E_b x)| \le \int |\widehat{\beta_{P(x),P'}}(w(x))|\, \mathrm{d}P'$, where $\beta_{P,P'}(w')$ is the integral over z of the last factor times α and the hat is the Fourier transform $\int \psi_A(\xi\, w^* w') \cdot \mathrm{d}w'$. We apply Lemma 8.3(i) to the measure m with $G(P, w) = (\int |\widehat{\beta_{P,P'}}(w)|\, \mathrm{d}P')^2$. By the Cauchy–Schwarz inequality in P' with the weight μ', by (I6) in w, and by the Cauchy–Schwarz inequality in z on the fibre, whose mass is at most $\mu'(P')$ because α lives on Vec_{j+6} (Lemma 8.2),

$$\int G(P, w)\, \mathrm{d}w \le \int \mu' \cdot \Big(\frac{c_A}{\mathrm{mod}_A(\xi)}\Big)^{|\Omega|} \iint \frac{|\beta_{P,P'}(w')|^2}{\mu'(P')}\, \mathrm{d}w'\, \mathrm{d}P'$$
$$\le \Lambda(\Gamma)^{99n(j+12)} \Big(\frac{c_A}{\mathrm{mod}_A(\xi)}\Big)^{|\Omega|} \int |\alpha|^2\, \mathrm{d}\delta^{(b)}(\mathcal{L}).$$

The integrability that (I6) needs comes from Lemma 7.3(ii). □

Proof of Theorem C(ii). We may assume $\int \Phi\, \mathrm{d}\eta_a \neq 0$; then $\Phi(m) \neq 0$ for some $m \in \mathcal{N}_a(T)$ (I3), and $m \in S$ by (o). Take t, q from (h) for this m and write $t = t_0 r$ with $t_0 \in \mathcal{H}_{a+1}(T)$, $r \in \mathcal{N}_a(T)$ (F1). By (I3) and (I1), η_a is invariant under $\nu \mapsto r^{-1}\nu r m$ and under conjugation by t_0, so

$$\int \Phi(\nu)\, \mathrm{d}\eta_a(\nu) = \int \Phi(t^{-1}\nu t m)\, \mathrm{d}\eta_a(\nu).$$

We apply Lemma 8.5 to $\nu \mapsto \Phi(t^{-1}\nu t m)$, with u and the size j. *The size condition.* If $\Phi(\nu_1) \neq 0$ for $\nu_1 = t^{-1}\nu t m$, then $\nu_1 \in S$ by (o), and (h) gives $\nu' \in \mathcal{N}_{<a}(T)$ with the column at e_a^* of $u^{-1}(\nu' t m (t\nu_1)^{-1} - 1) = u^{-1}(\nu'\nu^{-1} - 1)$ in Vec_j. *The pointwise bound.* By (o), $\Phi(\nu_1) \le C|\mathsf{Av}_{a+1}(\mathcal{L};k)(\nu_1^{\varepsilon}x, \nu_1 y)|^2$. By Lemma 7.3(ii) this average is the integral over $\sigma \in \mathcal{H}_{a+1}(T)$ of $\mathsf{Av}_{a+2}(\mathcal{L};k)(c^{\varepsilon}\sigma x, c\,\sigma y)$ with $c = \sigma\nu_1\sigma^{-1} \in \mathcal{N}_a(T)$, which by (K2) and (e) is $\psi_A(\xi\,\mathrm{sd}(c)) \cdot F_2(\sigma)$, $F_2(\sigma) = \mathsf{Av}_{a+2}(\mathcal{L};k)(\sigma x, \sigma y)$. By (F6), $\mathrm{sd}(\sigma\nu_1\sigma^{-1}) = -h_T(v_1, \sigma^{-1}f_{a+1})$, and the vector v_1 of ν_1 is t_0^{-1} applied to the sum of the vector v of ν and a vector that does not depend on ν. So, with $\xi' = -\xi\bar{u}$ and a fixed $y' \in V$,

$$\Phi(\nu_1) \le C\,\Psi(u^{-1}v),$$
$$\Psi(v') = \Big| \int \psi_A(\xi'\,h_T(v'+y', t_0\sigma^{-1}f_{a+1}))F_2(\sigma)\,\mathrm{d}\theta_{a+1}(\mathcal{L})(\sigma)\Big|^2.$$

The integral of Ψ. Theorem C(i) at $b = a+1$, with t_0, the element tq (for which $t_0^{-1}tq = rq \in \mathcal{N}_{<a+1}(T)$ and $\mathrm{sz}(tqx) \le i$), the unit ξ' and the measure vol_{a+1} with its fibre bound (Lemma 8.4(iii)), gives

$$\int_{\mathrm{Vec}_j} \Psi\,\mathrm{d}\,\mathrm{vol}_{a+1} \le \Lambda(\Gamma)^{99n(j+6)}\Lambda(\Gamma)^{99n(i+12)}\Big(\frac{c_A}{\mathrm{mod}_A(\xi u)}\Big)^{|\Omega|} \int |F_2|^2 \mathrm{d}\theta_{a+1}(\mathcal{L}).$$

Lemma 8.5 multiplies this by $C\,\mathsf{d}_a(\mathcal{L};u)\Lambda(\Gamma)^{j+2}$, and $99n(i+j+18)+j+2 \le 100n(i+j+31)$. □

11 Where the results are used

Note L uses almost everything: Theorem A in the descent between floors and in every change of variables; Theorems B and C and Lemmas 7.6, 9.2, 4.7, 8.6, 8.7 in the three bounds of the local theorem and in the integration over positions. Note C constructs the level and the gauges at the complex place, proves there that the tower measure is a Haar measure (the hypothesis of Theorem A(iii)), and verifies the axioms (G1)–(G6). No other unit of the artifact uses this one directly except through these two.

12 Notes on the formal text

Implicit arguments. The size is an implicit argument of sz, Vec and Hd; a gauge is a size with further data, and for a gauge these predicates are applied to it. The level is an explicit argument of all notions of Section 2 except the volume exponent, whose only argument it is.

Argument lists not confirmed by Lean. For 43 declarations of the five files the reader's dossier derived the list of arguments by Lean's rules for section variables, because the census reads a name that the declaration binds anew (n, i, a, j, x) as the section variable of that name, misses the section variables after a `let` in a statement, or drops the variable `m` where a hypothesis has that name. Nothing was compiled, so these lists are not confirmed by Lean; among them are the definitions of sizes, of vectors and matrices of a size, of heads, of floor values, of the mass of a floor, of the averaged kernel, of pairing kernels, of the self-norm, of gauges and of fibre bounds. The statements above follow the Lean text of each declaration. Three statements

continue after a `let` and were read to their end: Lemma 4.7(iii), Lemma 8.1 and Theorem C(i).

Differences from the natural statements. (1) Averaged kernels are Bochner integrals, 0 when the integrand is not integrable; (K4) makes them honest integrals for pairing kernels. (2) In (L2) the third condition is stated with $\mathcal{U}(T)$, which is $\mathcal{Q}_{\varrho+2}(T)$; it is the bound $\mathrm{Mass}_n(\mathcal{L};1)\le 1$. (3) The Weyl law is defined with the exponent $k+1+2(\varrho-(l+1))$ in truncated subtraction for all l, and used for $l<\varrho$. (4) A gauge allows $c_A=\infty$; the bounds are then empty. (5) In Lemma 4.4 and Lemma 4.5 the hypothesis runs over the $a<\varrho$ with $a<j$; j may exceed ϱ. (6) In Lemma 4.1(d) the Lean text takes the first matrix with columns indexed by the basis of V and any finite set of rows. (7) Lemma 5.7 takes two s-finite measures and does not ask that ρ be a probability. (8) In Theorem B(i) the indicator of the positions of size 1 and the weight are one function in the Lean text (an indicator of the weight). (9) In Lemma 5.4 the product runs over $a<j$ with exponents $k+2(\varrho-a)$, the rank of V_{a+1} plus one. (10) The count of the basis vectors of floor at least $a+1$, $k+1+2(\varrho-(a+1))$, is a lemma of the formal text; it is used as the rank of V_{a+1}. (11) One line of the file of the charts that the census gives to the invariance of the inverse weight belongs to the next declaration.

Small results. The declarations without a statement of their own are cited where they are used: the matrix and the inverse of a Weyl ratio after Definition 2.3; the two rules for floor measures after Definitions 2.5 and 2.6; nearness after Definition 2.7; the rules for weights after Definition 2.9 and Lemma 6.2; measurability after Definition 2.15; the bound $\Lambda\ge 1$ after Definition 2.14; and the count of (10) in Lemma 4.7.

How this note was written. From the structured proofs of the reader's dossier and from the Lean text. I read the five Lean files completely, statements and proofs, and the definitions of the background library that they use (shapes, split Gram matrices, the groups of the tower, Weyl and Heisenberg elements, places, flat charts, quadric measures, hyperbolic bases, convolutions). The proofs of Sections 4, 5, 6 and 7, of Lemmas 8.1, 8.2, 8.3, 8.5, 8.6, 8.7, 9.1, 9.2 and of Theorem C(ii) follow the formal proofs step by step; the exponents in them were recomputed. The proofs flagged by the reader as least certain were read line by line in the Lean text: the two integrals of Theorem B, the two parts of Theorem C, and Lemma 8.4(i). For these the text gives the argument as I could rebuild it from the terms: which lemma is applied to which objects, and with which sizes. Three places are less than a full proof and say so here: in Lemma 8.4(i) the comparison of a fibre of the quadric with the cone uses a library lemma on measures of sets whose differences lie in a given set, whose statement I took from its use; in Theorem B the inclusions of sets of floor values in the sets of (G6) are checked in the formal text by a computation with Heisenberg elements that I describe and did not redo entry by entry, and the exponent per floor in Theorem B(ii) was not recomputed; in Theorem C(i) the splitting of the phase in the coordinates of the floor basis is the formal text's computation. Lemma 5.4 and Lemma 5.1(i) are written from the formal proofs, with the change of variables described and not carried out. The program check of the unit's dossier covered 639 leaf steps of 108 proofs.

Appendix A. Concordance with the Lean artifact

The letters of this note and the names of the formal text: $\mathcal{L}$ is `F`; Ω is `ι`; $\mathfrak{m}$ is `m`; $\mathcal{S}$ is `S`; Γ_i and $\mathcal{S}_i$ are `ofSize i`; mod_A, ψ_A, c_A, τ_A, δ_A, ν_A are the fields `md`, `ψ`, `c`, `τ`, `skewUnit`, `skewMeasure` of the place; $\mathcal{H}_l(T)$, $\mathcal{Q}_l(T)$, $\mathcal{N}_{<l}(T)$, $\mathcal{N}_l(T)$, $\mathcal{U}(T)$ are `GT`, `unitriGroup`, `unipotentBelow`, `floorUnipotent`, `besselU`; n_a is `heisenbergElem`; η_l, $\eta_{<j}$, ϖ_l are `heisenbergMeasure`, `ascConv heisenbergMeasure`, `weylLaw`; $\delta^{(b)}(\mathcal{L})$ and vol_b are `floorQuadric` and `floorVolume`; E_b is `floorBasis`; sd is `simpleRootSum`; $\sigma^{-1} f_b$ is `orbitPoint`. Two statements as the artifact has them:

```
lemma stackMeasure_mconv_dirac (h : l + i = ś.ϱ + 2) (hs : s ∈ GT l) :
  F.stackMeasure l i ∗ₘ dirac s = F.stackMeasure l i

lemma pivotModulus_sq_le {υ v} (hK : F.FloorMassLe l K υ) (h :
  ∀ σ ∈ GT l, ‖F.upperPairing (l + 1) k (σ • g) (σ • y)‖ₑ ^ 2 ≤ υ (σ
  • g) • v σ := by exact fun _ _ ↦ (one_mul _).ge) (hu : Measurable υ := by fun_prop) (hv :
  Measurable v := by fun_prop) : F.pivotModulus l k g y ^ 2 ≤ K • ∫⁻ σ, v σ ∂F.floorMeasure l
```

In this note	Lean declaration	File	Lines
Lemma 8.1	`Size.mem_vecsOfSize_of_heisenbergElem`	FlatCharts.lean	21–26
Definition 2.7	`Level.isNear_mul_iff`	FlatCharts.lean	28–29
Definition 2.7	`Level.IsNear.of_floorGroup_mul`	FlatCharts.lean	31–33
Lemma 6.1	`Level.measurableSet_isNear`	FlatCharts.lean	35–37
Lemma 6.3	`Level.upperMeasure_isNear_le`	FlatCharts.lean	41–51
Lemma 6.1	`Level.isOpen_floorValues`	FlatCharts.lean	53–55
Lemma 6.2	`Level.floorValues_mul`	FlatCharts.lean	57–61
Lemma 6.1	`Level.volume_floorValues_ne_zero_ne_top`	FlatCharts.lean	63–66
Definition 2.9	`Level.floorWeight_succ`	FlatCharts.lean	68–70
Definition 2.9	`Level.floorWeight_ne_zero_and_ne_top`	FlatCharts.lean	72–73
Definition 2.9	`Level.floorWeight_mul_left`	FlatCharts.lean	76–77
Lemma 6.2	`Level.invWeight_mul`	FlatCharts.lean	80–83
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Notation

A, $\bar{x}$, $\mathrm{d}x$, A^{-} the place; involution; Haar measure; skew elements

mod_A, ψ_A, c_A, τ_A, δ_A, ν_A module, character, Plancherel constant, trace-one element, skew unit, measure on A^{-}

$\mathfrak{s} = (\varrho, \kappa, k_0)$, n, k, r the shape; $\varrho + 1$; $|\kappa|$; $2\varrho + k + 1$

V, V_l, e_a, e_a^*, x_i, z, f_a the space, its floors, basis, frame

T, $J(T)$, h_T, $G(T)$, $Z_T(g)$ form, split Gram matrix, hermitian form, unitary group, frame Gram matrix

$\mathcal{H}_l(T)$, $\mathcal{Q}_l(T)$, $\mathcal{N}_{<l}(T)$, $\mathcal{N}_l(T)$, $\mathcal{U}(T)$ the subgroups of Section 2

sd, $w_l(p)$, $w_a^\circ(c)$, $\mathrm{n}_a(v, \beta)$ superdiagonal sum; Weyl elements; Heisenberg element

Ω, $\mathfrak{m}$, E, E_b, π_b parameters of a hyperbolic basis; basis matrices; index set of the extra pairs

$\mathcal{L}$, $E(\mathcal{L})$, $g_{\mathcal{L}}$, θ, μ_H a level and its data

$\mathfrak{B}(\mathcal{L})$, $\mathrm{Box}(\mathcal{L})$, $\mathrm{Posn}(\mathcal{L})$ ball, box, positions

Lev, Siz, Gau levels, sizes, gauges

$\theta_l(\mathcal{L})$, η_l, $\eta_{<j}$, ϖ_l floor measure, Heisenberg measures, Weyl law

$\mu^{\mathcal{L}}_{a,i}$, $\mathsf{M}(\mathcal{L})$, $\mathsf{M}_{\geq l}(\mathcal{L})$ stack, tower and upper measures

$\delta^{(b)}(\mathcal{L})$, vol_b, vol, δ_s, $*$ quadric measure, volume of V_b, volume, unit mass, convolution

$\mathcal{S}$, $\mathcal{S}_i$, Γ, Γ_i, Λ, Γ_S a size, a gauge, its constant, a second gauge

sz, Vec_i, Hd_j size of a matrix, vectors of size i, heads

Near_j, O, Cl, Val_a near points, orthonormal part, close pairs, floor values

s, w_j, w_j^{-1}, d_l, Mass_j volume exponent, weight, inverse weight, floor module, mass

Av_j, Pv_j, PK_j averaged kernel, pivot, pairing kernels

Sq_j, Sm_j, SN_j weighted square, its sum, self-norm

Pos, C_l, C_{gl}, Fib positions of size 1, constants, fibre bound

W, φ, $\mathcal{C}_u$ the objects of axiom (G6)

P, w, t, t', π', π'', k, Z, ζ, μ, ρ' coordinates and data of a fibre bound

p, q, d sequences of units; also q a group element or a measure, d the element of a pairing kernel or a size

g, x, y, s, u, v, m, h group elements, vectors, units; u, v also functions

σ, τ, ν, γ, ξ, η, α, β variables of integration, group elements, units, skew elements

i, j, l, a, b, c, N natural numbers: sizes, floors, ranks; c also an element of A

k, K, C, M, V a kernel (also $k = |\kappa|$); constants

K, m the set of Setting 4.6; the measure on V of Theorem C(i)

$E(D)$, $G_{\mathbf{1}}(\mathfrak{s})$ the CM field of the admissible data D of the overview; the group at the complex place (only in Section 1)

d, λ, K, f, Θ letters that are local to the notes of this unit and differ from those of the overview; this note uses d (an element or a size), λ (measures), K (constants) and f (functions)

f, F, G, Φ, Ψ, D, X, Y, S, I auxiliary functions, matrices, sets

ε, ρ, λ, ι, θ, ψ, ϱ, κ, Λ, δ an exponent; measures; an index set; letters of names

$\mathbf{1}$, ∞, L, H, O, Q, U, B, e, n, o, r, z indicator; infinity; letters inside names

READBACK NOTE 29

The local theorem for floor forms and the band limits at the complex place

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. A unitary group $G(T)$ over a local ring A is cut into *floors*: a tower of subgroups, each with a measure, whose convolution is a Haar measure. A *cut Bessel function* W on $G(T)$ is paired with its translates by Weyl elements, and the pairing is averaged floor by floor; this gives *floor forms* $\mathcal{I}_j(p, q; W, \omega)$, one family for each floor j. The *local theorem* bounds them by one quantity, the self-norm product, in three situations: by a volume, by Plancherel's inequality against a character of the floor, and by Plancherel's inequality on a Heisenberg group when the two Weyl elements differ. A descent identity writes each form as an integral of the forms of the next floor. At the complex place the bounds are summed over shells of gauges and give the *band limits* that the second moment of Poincaré series (Theorem B of the write-up) needs.

What this note is. The artifact grh-compact is a body of Lean code whose final theorems are the Riemann hypothesis and the generalized Riemann hypothesis for Dirichlet L-functions. Its authors report that it compiles and that Lean's kernel accepts these theorems from the three standard axioms; nothing was built for this note. That is a statement about a formal text: no mathematician has read the argument through, and this note does not say that the Riemann hypothesis is proved. The note treats one part of one unit of the artifact (the folder of the analysis at the real places): 42 of the unit's 239 declarations, those of three files; two companion notes treat the others. Proved here in full, from stated inputs: Theorems A to D. Theorem E is proved up to three comparisons of explicit exponents (in Lemmas 8.4, 8.6 and in (S3)), which are named where they occur. Taken for granted, with precise statements: the results of the two companion notes (Inputs (T1)–(T14) and (C1)–(C4)), definitions of the unit on the second moment, and classical results of the background library (Section 3). Every statement of a result of this note was compared, argument by argument, with its Lean text. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 42 declarations.

1 The statements

The objects are defined in Section 2; here is what the letters stand for. A is a *place*: a locally compact ring with involution, with a module mod_A, an additive character ψ_A and a Plancherel constant c_A; the model cases are $\mathbb{C}$ and products of non-archimedean local fields. V is a hermitian space over A with ϱ hyperbolic planes and an anisotropic part, $G(T)$ its unitary group, and $\mathcal{H}_0(T) = G(T) \supset \mathcal{H}_1(T) \supset \ldots$ the tower of the stabilisers of the first hyperbolic planes. A *level* $\mathcal{L}$ gives a measure $\theta_l(\mathcal{L})$ on each floor $\mathcal{H}_l(T)$ and a measure $\mathsf{M}(\mathcal{L})$ on $G(T)$, their convolution; a *gauge* Γ on $\mathcal{L}$ measures the size of matrix entries by a filtration $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$ of A and has a constant $\Lambda(\Gamma) \geq 2$. A *head* is a sequence $p\colon \mathbb{N} \to A^\times$ whose partial products have

inverses of size 1; $w_j(p)$ is the Weyl element that exchanges the first j hyperbolic pairs with the entries of p.

Setting 1.1 (the local theorem)**.** Let A, $\mathfrak{s}$, T, Ω, $\mathfrak{m}$, I be as in the conventions (Section 2), let $\mathcal{L}$ be a level in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$ whose measure $\mathsf{M}(\mathcal{L})$ is a Haar measure of $G(T)$, and let Γ be a gauge in $\mathrm{Gau}(\mathcal{L})$. Let $W\colon G(T) \to \mathbb{C}^I$ be a cut Bessel function for $\mathcal{L}$: $W \in \mathrm{Cut}$. Let j be a natural number with $j \le \varrho$, let $p, q\colon \mathbb{N} \to A^\times$ be sequences of units such that p is a head of length j for Γ, that is $p \in \mathrm{Hd}_j$ for Γ, and let $x \in \mathrm{Posn}(\mathcal{L})$ be a position.

Theorem A (the local theorem). In Setting 1.1, the three quantities on the left are bounded by the self-norm product

$$\mathrm{SNP}_j(\Gamma; p, q; W, W', x) = \Big(\prod_{a<j} \mathsf{d}_a(\mathcal{L}; p_a^{-1})\Big)^2 C_{\mathrm{gl}}(\Gamma)^3 \ \mathrm{SN}_0(\mathcal{L}; \langle W(\cdot), W(1\cdot)\rangle, x) \cdot \mathrm{SN}_j(\mathcal{L}; \langle W(\cdot), W'(d\cdot)\rangle, x), \qquad d = w_j(q)^{-1} w_j(p).$$

(a) Volume. Let $W'\colon G(T) \to \mathbb{C}^I$ satisfy $W' \in \mathrm{Bes}(g_{\mathcal{L}}^{-1})$ and vanish at every y that is not near at j (if $W'(y) \neq 0$ then $y \in \mathrm{Near}_j(\mathcal{L})$). Let $\mathsf{V} \in [0, \infty]$ with $\mathsf{V} \ge 1$ and $\mathrm{Mass}_j(\mathcal{L}; 1) \le \mathsf{V}$, and let $\sigma_0 \in [0, \infty]$. Suppose that, if $\sigma_0 < 1$, then for every $y \in G(T)$ with $\langle W(y), W(yx)\rangle \neq 0$, $w_j(p)^{-1} y \in \mathrm{Near}_j(\mathcal{L})$ and $w_j(q)^{-1} yx \in \mathrm{Near}_j(\mathcal{L})$ there is a with $j \le a < \varrho$ and $\mathrm{vol}(\mathrm{Val}_a(\mathcal{L}; y, x)) \le \sigma_0$. Then

$$\mathcal{J}_j^{\mathrm{maj}}(p, q; W, W', x)^2 \ \le \ \sigma_0^{\mathfrak{s}(\mathcal{L})} \ \mathsf{V}^2 \ \mathrm{SNP}_j(\Gamma; p, q; W, W', x).$$

(b) A character on the floor. Let $\xi \in A^\times$ and $\eta \in A$ be such that $\psi_A(\eta z) = \psi_A(\xi z)$ for all $z \in \Gamma_{200n^2}$, and let $\omega_\eta(y) = \psi_A(\eta\, Z_T(y)_{jj})$ for $y \in G(T)$. Then

$$\mathcal{J}_j^{\mathrm{mod}}(p, q; W, x, \omega_\eta)^2 \ \le \ \Big(\frac{c_A}{\mathrm{mod}_A(\xi)}\Big)^{|\Omega|} \ \mathrm{SNP}_j(\Gamma; p, q; W, W, x).$$

(c) Unequal heads. Let i be a natural number with $j = i + 1$, and let $\xi \in A^\times$ with $\xi = g_{\mathcal{L}}^{-1}(p_i^{-1} - q_i^{-1})$. Then

$$\mathcal{J}_j^{\mathrm{mod}}(p, q; W, x, 1)^2 \ \le \ \Big(\frac{c_A}{\mathrm{mod}_A(\xi)}\Big)^{|\Omega|} \ \mathrm{SNP}_j(\Gamma; p, q; W, W, x).$$

LEAN `Gauge.formMajorant_sq_le` · `Gauge.formModulus_sq_le_of_char` · `Gauge.formModulus_sq_le_of_heads`

So a form is small when the pair of positions is spread out (a), when it is tested against a character of large frequency ξ in the diagonal entry of floor j (b), and when the last entries of the two heads differ by a large amount (c); in (b) and (c) the gain is the factor $\mathrm{mod}_A(\xi)^{-|\Omega|}$, where $|\Omega|$ is the number of anisotropic directions that the floor leaves free.

Setting 1.2 (a cut function and a gauge)**.** Let A, $\mathfrak{s}$, T, Ω, $\mathfrak{m}$, I be as in the conventions, let $\mathcal{L}$ be a level in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$, let Γ be a gauge in $\mathrm{Gau}(\mathcal{L})$, and let $W\colon G(T) \to \mathbb{C}^I$ be a cut Bessel function for $\mathcal{L}$: $W \in \mathrm{Cut}$.

Theorem B (descent). In Setting 1.2, let $p, q \colon \mathbb{N} \to A^{\times}$, let a be a natural number with $a < \varrho$, let $t \in \mathbb{C}$ and $\eta \in A$, and suppose that $\int \mathcal{J}_a^{\mathrm{maj}}(p, q; W, W, x)\, \mathrm{dM}(\mathcal{L})(x) < \infty$. For $c \in A$ let $u(c) = c$ if c is a unit and $u(c) = 1$ otherwise, and let $p[a \mapsto u(c)]$ be the sequence p with its entry a replaced by $u(c)$. Then the function

$$f(c) = \psi_A(\eta c) \cdot \Big(t \cdot \mathcal{J}_{a+1}(p[a \mapsto u(c)], q[a \mapsto u(c)]; W, 1) \Big)$$

is integrable on A, and

$$\int_A f(c)\, \mathrm{d}c \;=\; t \cdot \mathcal{J}_a(p, q; W, \omega_\eta), \qquad \omega_\eta(y) = \psi_A\big(\eta\, Z_T(y)_{aa}\big).$$

LEAN `Gauge.integral_floorForm_succ`

For $\eta = 0$ this says that the form of floor a is the integral of the forms of floor $a+1$ over a common entry at the place a; for $\eta \neq 0$ it identifies the Fourier transform in that entry, which Theorem A(b) bounds.

Theorem C (the self-norm, integrated over positions). Let A, $\mathfrak{s}$, T, Ω, $\mathfrak{m}$ be as in the conventions, let $\mathcal{L}$ be a level in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$ whose measure $\mathsf{M}(\mathcal{L})$ is a Haar measure, and Γ a gauge in $\mathrm{Gau}(\mathcal{L})$. Let j be a natural number with $j \le \varrho$, let $k \colon G(T) \times G(T) \to \mathbb{C}$ and $d \in G(T)$ form a pairing kernel, $(k, d) \in \mathrm{PK}_j(\mathcal{L})$, and let

$$c = \Big(\prod_{l < \varrho} C_l(\Gamma) \Big) \cdot \iint |k(g, y)|^2\, \mathrm{dM}_{\ge 0}(\mathcal{L})(y)\, \mathrm{dM}_{\ge 0}(\mathcal{L})(g).$$

Then

$$\int_{\mathrm{Pos}(\mathcal{L};\Gamma)} \mathrm{SN}_j(\mathcal{L}; k, x)\ \mathrm{dM}(\mathcal{L})(x) \;\le\; n \cdot c.$$

LEAN `Gauge.lintegral_selfNorm_le`

Theorem D (a bound on a set of positions). In Setting 1.2, suppose that $\mathsf{M}(\mathcal{L})$ is a Haar measure. Let Γ' be a second gauge in $\mathrm{Gau}(\mathcal{L})$, let $W' \colon G(T) \to \mathbb{C}^I$ satisfy $W' \in \mathrm{Bes}(g_{\mathcal{L}}^{-1})$, let $j \le \varrho$ and let $p, q \colon \mathbb{N} \to A^{\times}$ with q a head of length j for Γ: $q \in \mathrm{Hd}_j$ for Γ. Let $\Theta_0, e \in [0, \infty]$, let $S \subseteq G(T)$ and $f \colon G(T) \to [0, \infty]$, and suppose that

$$\mathrm{PN}(W, W) \cdot \mathrm{PN}(W, W') \le \Theta_0^2, \qquad S \subseteq \mathrm{Pos}(\mathcal{L}; \Gamma'),$$
$$f(x)^2 \le e^2 \cdot \mathrm{SNP}_j(\Gamma; p, q; W, W', x) \quad \text{for all } x \in S.$$

Then

$$\mathsf{h}_j(\mathcal{L}; p, q) \int_S f\, \mathrm{dM}(\mathcal{L}) \;\le\; \Big(e \prod_{l < \varrho} \mathsf{d}_l(\mathcal{L}; g_{\mathcal{L}}) \Big)\, c_A^{|\Omega|\varrho} \cdot \Lambda(\Gamma)^{1500000\, n^3}\ \Lambda(\Gamma')^{50000\, n^2}\ \Theta_0\ \Lambda(\Gamma).$$

LEAN `Gauge.lintegral_positions_le`

The last theorem is at the complex place. A word on the names: a *real place* of the write-up is an infinite place of the maximal totally real subfield of the CM field $E(D)$ of the overview; the completion of $E(D)$ above it is $\mathbb{C}$, and the unitary group of the rows there is the group $G_1(\mathfrak{s})$ of Section 2, a unitary group of signature $(\varrho+k, \varrho+1)$. In the formalism of places this is the place $A = \mathbb{C}$ with complex conjugation, which these notes call *the complex place*; so "the real places" (the name of the unit and of the overview) and "the complex place" (Notes L and C) are the same local situation. There $\psi_{\mathbb{C}}(c) = \exp(c - \bar{c})$, the level $\mathcal{L}_{\mathbb{C}}(\mathfrak{h})$ and its gauges are those of the companion note on the complex place (Inputs (C1)–(C3)), and a pair of a Bessel function and its cut is given by the unit on the second moment (Definition 2.9). The notion of band limits is recalled in Section 3: it is a system of forms $J_0, \dots, J_{\varrho+1}$ on sequences of non-zero complex numbers, each the integral of the next over a common entry, with decay in the frequency and off the diagonal.

Theorem E (band limits at the complex place). Let $\mathfrak{s}$ be a shape, I a non-empty finite set, $\Phi, \Phi_0 \colon G_1(\mathfrak{s}) \to \mathbb{C}^I$ continuous functions, rep a unitary representation of $\mathrm{U}(\kappa)$ on $\mathbb{C}^I$, ρ and b real numbers, s_0 and $\mathfrak{n}$ natural numbers, and let $\mathfrak{c}$ be cut Bessel data: $\mathrm{CB}(\Phi, \mathrm{rep}, \rho, s_0, b, \Phi_0, \mathfrak{n})$ (Definition 2.9; Φ is the cut, Φ_0 the full function). Let deg be a real number with $\deg > 0$ such that Φ_0 has Schur's bound with constant $\deg^{-1}|I|$ (Section 3), and suppose that $\deg \cdot \Theta(\mathfrak{s}) \le b^{4r^2}$. Then there is a family $J = (J_i)_{i \in \mathbb{N}}$ of functions of two sequences $p, q \colon \mathbb{N} \to \mathbb{C}^\times$ such that Φ has the band limits with respect to b, deg and J. One such family is

$$J_i(p,q) = c_{\mathrm{f}}(\mathfrak{c}) \cdot \mathcal{J}_i(p, q; \Phi, 1) \quad (i \le \varrho), \qquad J_i(p,q) = \langle \Xi(\Phi; p), \Xi(\Phi; q) \rangle \quad (i > \varrho),$$

with the floor forms of the level $\mathcal{L}_{\mathbb{C}}(\mathfrak{h})$ of the sizes $\mathfrak{h}$ that underlie $\mathfrak{c}$.

LEAN `hasBandLimits_of_isCutBessel`

2 The setting

Conventions. A is a commutative ring with an involution $x \mapsto \bar{x}$, a topology and a Haar measure $\mathrm{d}x$, and it is a *place* in the sense of Section 2. $\mathfrak{s} = (\varrho, \kappa, k_0)$ is a shape, with $n = \varrho + 1$, $k = |\kappa|$ and $r = 2\varrho + k + 1$. T is an $\mathfrak{s}$-form over A whose split Gram matrix $J(T)$ is a hermitian unit. Ω is a finite set Ω and $\mathfrak{m}$ a natural number; $\mathcal{L}$ denotes a level in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$. I is a finite set I, and $\mathbb{C}^I$ has the inner product $\langle v, w \rangle = \sum_{i \in I} \overline{v_i}\, w_i$ and the norm $|v|$. In statements at the complex place, $A = \mathbb{C}$ with complex conjugation, T is the form of $G_1(\mathfrak{s})$, and Φ denotes a continuous function $G_1(\mathfrak{s}) \to \mathbb{C}^I$. Integrals of functions with values in $[0, \infty]$ are Lebesgue integrals and need no integrability; a product $0 \cdot \infty$ is 0. Subtraction of natural numbers in exponents is truncated at 0.

The space, its groups, and a place. These objects belong to the background library (units 112, 113 and 119 of the artifact); the companion Note T (*The tower method at a place: floor measures, gauges and the bounds on each floor*) defines them in full. A *shape* is a triple $\mathfrak{s} = (\varrho, \kappa, k_0)$ of a natural number, a finite set and

an element $k_0 \in \kappa$. The free module $V = A^r$ has the basis $e_0, \dots, e_{\varrho-1}$, x_i ($i \in \kappa$), z, $e^*_{\varrho-1}, \dots, e^*_0$. An *$\mathfrak{s}$-form* T is a square matrix on the indices of the x_i and of z; its *split Gram matrix* $J(T)$ pairs e_a with e^*_a (entries 1), restricts to T on $V_\varrho = \langle x_i, z\rangle$, and is zero elsewhere. It is a *hermitian unit* if it is hermitian with invertible determinant. We write $h_T(v,w) = v^* J(T) w$, with v^* the conjugate transpose, and $G(T)$ for the group of the invertible matrices g with $g^* J(T) g = J(T)$. The *floor* of e_a and of e^*_a is a, that of z is ϱ, that of the x_i is n; V_l is the span of the basis vectors of floor at least l. The *frame* is $f_0, \dots, f_\varrho$ with $f_a = e_a$ for $a < \varrho$ and $f_\varrho = z$, and $Z_T(g) = \big(h_T(f_a, g f_b)\big)_{0 \le a,b \le \varrho}$ is the frame Gram matrix of $g \in G(T)$. The subgroups are:

- $\mathcal{H}_l(T)$, the elements that fix every basis vector of floor less than l; so $\mathcal{H}_0(T) = G(T)$, $\mathcal{H}_\varrho(T)$ is the unitary group of V_ϱ, $\mathcal{H}_n(T)$ that of $\langle x_i\rangle$, and $\mathcal{H}_l(T) = 1$ for $l \ge n+1$;
- $\mathcal{Q}_l(T)$, the elements g with $gf_a - f_a \in \langle e_0, \dots, e_{a-1}\rangle$ for every $a < l$, $a \le \varrho$ (and, for $l \ge n+1$, also $gx_i - x_i \in \langle e_0, \dots, e_{\varrho-1}\rangle$);
- $\mathcal{U}(T) = \mathcal{Q}_{n+1}(T)$, the unipotent radical of the stabiliser of the flag $\langle e_0\rangle \subset \langle e_0, e_1\rangle \subset \dots$, and $\mathcal{N}_{<l}(T)$, the unipotent radical of the stabiliser of the first l steps of this flag, so that $\mathcal{N}_{<l}(T) = \mathcal{U}(T)$ for $l \ge \varrho$;
- $\mathcal{N}_l(T) = \mathcal{H}_l(T) \cap \mathcal{N}_{<l+1}(T)$, a Heisenberg group.

They satisfy $\mathcal{H}_l(T) \subseteq \mathcal{Q}_l(T)$, $\mathcal{N}_{<l}(T) \subseteq \mathcal{U}(T) \subseteq \mathcal{Q}_{l'}(T)$ for all l, l', the families $\mathcal{H}_l(T)$ and $\mathcal{Q}_l(T)$ decrease with l, $\mathcal{H}_l(T)$ normalises $\mathcal{N}_{<l}(T)$, every element of $\mathcal{Q}_l(T)$ is a product $\gamma\nu$ with $\gamma \in \mathcal{H}_l(T)$ and $\nu \in \mathcal{N}_{<l}(T)$, and for $b \le \varrho$ the group $\mathcal{Q}_{b+1}(T) \cap \mathcal{H}_b(T)$ is the stabiliser of f_b in $\mathcal{H}_b(T)$. For $u \in G(T)$, $\mathrm{sd}(u)$ is the sum of the entries of u at the places (f_a, f_{a+1}), $a < \varrho$. For $p \colon \mathbb{N} \to A^\times$ the *Weyl element* $w_l(p) \in G(T)$ maps $e_a \mapsto p_a e^*_a$ and $e^*_a \mapsto \bar{p}_a^{-1} e_a$ for $a < \min(l, \varrho)$ and fixes the other basis vectors; it commutes with $\mathcal{H}_l(T)$, and $w_l(\bar{p}) = w_l(p)^{-1}$. The *floor Weyl element* $w^\circ_a(c)$, for $a < \varrho$ and $c \in A^\times$, does the same with the one pair (e_a, e^*_a), so that $w_{a+1}(p) = w^\circ_a(p_a) w_a(p)$ and $Z_T(w^\circ_a(c))_{aa} = c$.

A *place* is a commutative ring A with involution that is a locally compact Polish topological ring with a Haar measure, together with: an element τ_A with $\tau_A + \bar{\tau}_A = 1$; a unit δ_A with $\bar{\delta}_A = -\delta_A$; a Haar measure on the skew elements; a measurable multiplicative *module* $\mathrm{mod}_A \colon A \to [0,\infty]$, invariant under $x \mapsto \bar{x}$ and $x \mapsto -x$, such that multiplication by a matrix M with invertible determinant multiplies Haar measure on A^N by $\mathrm{mod}_A(\det M)^{-1}$; a continuous additive character $\psi_A \colon A \to \mathbb{C}$ of absolute value 1; a constant $c_A \in [0,\infty]$; and further axioms (almost every element is a unit; a compatibility of the measure on skew elements; a local bound for mod_A^{-1}; a Plancherel kernel) that enter this note only through the inputs. For $A = \mathbb{C}$: $\mathrm{mod}_A(x) = |x|^2$, $\psi_{\mathbb{C}}(c) = \exp(c - \bar{c})$, $c_{\mathbb{C}} = \pi^2$.

The tower (recalled from Note T). A *level* $\mathcal{L} \in \mathrm{Lev}(\Omega, T, \mathfrak{m})$ consists of: a *hyperbolic basis* of V_ϱ, that is a basis matrix $E(\mathcal{L})$ made of $\mathfrak{m}+1$ hyperbolic pairs and of orthonormal vectors indexed by Ω (so $k+1 = |\Omega| + 2\mathfrak{m} + 2$); two s-finite measures on $G(T)$, written $\theta_\varrho(\mathcal{L})$ and $\theta_n(\mathcal{L})$, the second non-zero; a unit $g_{\mathcal{L}} = \overline{g_{\mathcal{L}}}$ of A; an open set $\mathfrak{B}(\mathcal{L}) \subseteq V$ that contains 0 and has compact closure; a compact set $\mathrm{Box}(\mathcal{L}) \subseteq G(T)$, the *box*; and a measurable set $\mathrm{Posn}(\mathcal{L}) \subseteq G(T)$ of *positions*. It is required that $\theta_\varrho(\mathcal{L})$ is carried by $\mathcal{H}_\varrho(T)$ and that its image under $\sigma \mapsto \sigma^{-1} z$ is

the quadric measure $\delta^{(\varrho)}(\mathcal{L})$ of Note T on $\{v \in V_\varrho : h_T(v,v) = h_T(z,z)\}$; and that $\theta_n(\mathcal{L})$ is carried by $\mathcal{H}_n(T)$, is invariant under right translations by $\mathcal{H}_n(T)$, and gives mass at most 1 to $\{h : u\,h\,g \in \mathrm{Box}(\mathcal{L}) \text{ for some } u \in \mathcal{U}(T)\}$, for every g. For $l < \varrho$ the *floor measure* is $\theta_l(\mathcal{L}) = \varpi_l * \eta_l$, where $*$ is convolution of measures on $G(T)$, ϖ_l is the image of the measure $\mathrm{mod}_A(c)^{k+1+2(\varrho-l-1)}\mathrm{d}c$ on A under $c \mapsto w_l^\circ(c)^{-1}$, and η_l is the *Heisenberg measure*, a Haar measure of $\mathcal{N}_l(T)$ in the coordinates of Note T; and $\theta_l(\mathcal{L}) = \delta_1$, the unit mass at 1, for $l > n$. We put $\eta_l = \delta_1$ for $l \geq \varrho$ and $\eta_{<j} = \eta_0 * \cdots * \eta_{j-1}$. Then

$$\mu^{\mathcal{L}}_{a,i} = (\eta_{a+i-1} * \theta_{a+i-1}(\mathcal{L})) * \cdots * (\eta_a * \theta_a(\mathcal{L})),$$
$$\mathsf{M}(\mathcal{L}) = \mu^{\mathcal{L}}_{0,\varrho+2}, \qquad \mathsf{M}_{\geq l}(\mathcal{L}) = \theta_n(\mathcal{L}) * \cdots * \theta_l(\mathcal{L}),$$

with $\mathsf{M}_{\geq l}(\mathcal{L}) = \delta_1$ for $l > n$. The measure $\theta_l(\mathcal{L})$ is carried by $\mathcal{H}_l(T)$, and $\mathsf{M}(\mathcal{L})$ is right invariant.

A point $g \in G(T)$ is *near at* j, $g \in \mathrm{Near}_j(\mathcal{L})$, if $\xi g \in \mathrm{Box}(\mathcal{L})$ for some $\xi \in \mathcal{Q}_j(T)$. For $a < \varrho$ and $x, y \in G(T)$, the set $\mathrm{Val}_a(\mathcal{L}; x, y) \subseteq A$ consists of the coordinates at e_a^* of the vectors $w \in V$ whose coordinates at e_b^*, $b < a$, vanish and for which $u_1 = x^{-1}w$ and $u_2 = (xy)^{-1}w$ satisfy $u_1 \in \mathfrak{B}(\mathcal{L})$ and $g_{\mathcal{L}}^{-1}(u_1 - u_2) \in \mathfrak{B}(\mathcal{L})$. With the *volume exponent* $\mathsf{s}(\mathcal{L}) = |\Omega| - (3n + 1 + \mathfrak{m})$, the *weight* and the *floor module* are

$$\mathsf{w}_j(\mathcal{L}; x, y) = \prod_{a < \min(j,\varrho)} \mathrm{vol}\big(\mathrm{Val}_a(\mathcal{L}; x, y)\big)^{\mathsf{s}(\mathcal{L})},$$
$$\mathsf{d}_l(\mathcal{L}; g) = \mathrm{mod}_A(g)^{k+1+2(\varrho-(l+1))}\ \mathrm{mod}_A(g)^{1/2} \quad (g \in A^\times),$$

and $\mathsf{w}_j^{-1}(\mathcal{L}; x)$ is the function $z \mapsto \mathsf{w}_j(\mathcal{L}; z, x)^{-1}$ on the points near at j, extended by 0. For a function $c\colon G(T) \to [0, \infty]$ and $K \in [0, \infty]$, $\mathrm{Mass}_j(\mathcal{L}; c) \leq K$ means that $\int c(\sigma g)\,\mathrm{d}\theta_j(\mathcal{L})(\sigma) \leq K$ for every g, the integral taken over the σ with σg near at $j+1$.

Kernels averaged over floors (recalled from Note T). For a measurable $k\colon G(T) \times G(T) \to \mathbb{C}$ the *averaged kernel*, its *pivot*, the *weighted square* and the *weighted square sum* at a point $x \in G(T)$ are

$$\mathsf{Av}_j(\mathcal{L}; k)(g, y) = \int k(\sigma g, \sigma y)\,\mathrm{d}\mathsf{M}_{\geq j}(\mathcal{L})(\sigma),$$
$$\mathsf{Pv}_j(\mathcal{L}; k)(g, y) = \int \big|\mathsf{Av}_{j+1}(\mathcal{L}; k)(\sigma g, \sigma y)\big|\,\mathrm{d}\theta_j(\mathcal{L})(\sigma),$$
$$\mathsf{Sq}_j(\mathcal{L}; k, x)(g) = \mathsf{w}_j(\mathcal{L}; g, x) \cdot \big|\mathsf{Av}_{j+1}(\mathcal{L}; k)(g, gx)\big|^2,$$
$$\mathsf{Sm}_j(\mathcal{L}; k, x)(g) = \int \Big(\mathsf{Sq}_j(\mathcal{L}; k, x)(\sigma g) + \mathsf{Sm}_{j+1}(\mathcal{L}; k, x)(\sigma g)\Big)\mathrm{d}\theta_j(\mathcal{L})(\sigma)$$

for $j \leq \varrho$, with $\mathsf{Sm}_j(\mathcal{L}; k, x)(g) = 0$ for $j > \varrho$, and the *self-norm* is $\mathrm{SN}_j(\mathcal{L}; k, x) = \int \mathsf{Sm}_j(\mathcal{L}; k, x)(g)\,\mathrm{d}(\theta_{j-1}(\mathcal{L}) * \cdots * \theta_0(\mathcal{L}))(g)$. A pair of such a k and an element $w \in G(T)$ is a *pairing kernel at* j, $(k, w) \in \mathrm{PK}_j(\mathcal{L})$, if: k is measurable; for all l, all $\nu, \nu' \in \mathcal{N}_{<l}(T)$ and all g, y,

$$\mathsf{Av}_{l+1}(\mathcal{L}; k)(\nu g, \nu' y) = \psi_A\Big(g_{\mathcal{L}}^{-1}(\mathrm{sd}(w\nu' w^{-1}) - \mathrm{sd}(\nu))\Big)\,\mathsf{Av}_{l+1}(\mathcal{L}; k)(g, y);$$

w commutes with $\mathcal{H}_j(T)$; for every l the integrals $\int |k(\sigma g, \sigma y)|\,\mathrm{d}\mathsf{M}_{\geq l}(\mathcal{L})(\sigma)$ are bounded by a finite constant independent of g, y; and $\mathsf{Av}_l(\mathcal{L}; k)(g, y) \neq 0$ implies $g \in \mathrm{Near}_l(\mathcal{L})$.

Sizes and gauges (recalled from Note T). A *gauge* $\Gamma \in \mathrm{Gau}(\mathcal{L})$ consists of sets $\Gamma_0, \Gamma_1, \dots \subseteq A$, compact, containing 0 and 1, stable under $x \mapsto -x$ and $x \mapsto \bar{x}$, with $\Gamma_i \Gamma_{i'} \subseteq \Gamma_{i+i'}$ and with sums of at most $r+1$ elements of Γ_i in Γ_{i+1}, and of a constant $\Lambda(\Gamma) \in [2, \infty)$, subject to the axioms listed in Note T. Those used here by name: the entries of $J(T)$ and $J(T)^{-1}$ lie in Γ_1; $\mathfrak{m} \le 9n$ and $c_A \ge 1$; $g_{\mathcal{L}} \in \Gamma_1$; the entries of the elements of $\mathrm{Box}(\mathcal{L})$ lie in Γ_1; the coordinates of the vectors of $\mathfrak{B}(\mathcal{L})$ lie in Γ_1; $\mathrm{mod}_A(u) \le \Lambda(\Gamma)^i$ for units $u \in \Gamma_i$; and $\mathrm{vol}(\Gamma_i) \le \Lambda(\Gamma)^{i+1}$. A matrix X over A *has size at most i for* Γ, $\mathrm{sz}(X) \le i$, if all its entries lie in Γ_i; for $g \in G(T)$ this refers to the matrix of g; Vec_i is the set of the vectors with all coordinates in Γ_i. A sequence $p \colon \mathbb{N} \to A^\times$ is a *head of length j for* Γ, $p \in \mathrm{Hd}_j$, if $(p_0 \cdots p_{l-1})^{-1} \in \Gamma_1$ for all $l \le j$; then $\bar{p}$ is one too. The predicates sz, Hd and the sets Vec refer to a gauge that the Lean text takes as an implicit argument; every statement below names it. Further,

$$\mathrm{Pos}(\mathcal{L}; \Gamma) = \{x \in \mathrm{Posn}(\mathcal{L}) : \mathrm{sz}(g_{\mathcal{L}}^{-1}(x-1)) \le 1 \text{ for } \Gamma\},$$
$$C_l(\Gamma) = \mathsf{d}_l(\mathcal{L}; g_{\mathcal{L}})\, c_A^{|\Omega|}\, \Lambda(\Gamma)^{4600n}, \qquad C_{\mathrm{gl}}(\Gamma) = \Lambda(\Gamma)^{10^6 n^3}.$$

Bessel functions and the kernels of two functions.

Definition 2.1 (Bessel functions)**.** Let $W \colon G(T) \to \mathbb{C}^I$ and $\eta \in A^\times$. Then W is a *Bessel function of frequency η*, $W \in \mathrm{Bes}(\eta)$, if W is measurable and bounded and

$$W(ux) = \psi_A(\eta\, \mathrm{sd}(u))\, W(x) \qquad \text{for all } u \in \mathcal{U}(T),\ x \in G(T).$$

LEAN `Place.IsBessel`

Definition 2.2 (cut functions)**.** Let $\mathcal{L}$ be a level in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$ and $W \colon G(T) \to \mathbb{C}^I$. Then W is *cut for* $\mathcal{L}$, $W \in \mathrm{Cut}$, if $W \in \mathrm{Bes}(g_{\mathcal{L}}^{-1})$ and, for every x with $W(x) \ne 0$, there is $u \in \mathcal{U}(T)$ with $ux \in \mathrm{Box}(\mathcal{L})$. The level is an implicit argument of this predicate in the Lean text.

LEAN `Level.IsCut` · `Level.IsCut.isNear_of_ne_zero`

Since $\mathcal{U}(T) \subseteq \mathcal{Q}_j(T)$ for every j, a cut function vanishes at every point that is not near at j: if $W(x) \ne 0$ then $x \in \mathrm{Near}_j(\mathcal{L})$, for every j.

Definition 2.3 (the kernel of two functions)**.** For $W, W' \colon G(T) \to \mathbb{C}^I$ and $d, x, y \in G(T)$, $\langle W(x), W'(dy) \rangle$ is the inner product of $W(x)$ and $W'(dy)$ in $\mathbb{C}^I$. As a function of (x, y) it is written $\langle W(\cdot), W'(d\cdot) \rangle$.

LEAN `translatePairing`

Definition 2.4 (pairing norm)**.** For a level $\mathcal{L}$ in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$ (an implicit argument) and $W, W' \colon G(T) \to \mathbb{C}^I$,

$$\mathrm{PN}(W, W') = \iint \left|\langle W(x), W'(y) \rangle\right|^2 \mathrm{dM}_{\ge 0}(\mathcal{L})(y)\, \mathrm{dM}_{\ge 0}(\mathcal{L})(x) \ \in [0, \infty].$$

LEAN `Level.pairingNorm`

The floor forms.

Definition 2.5 (head density)**.** For a natural number j, sequences $p, q \colon \mathbb{N} \to A^\times$ and a level $\mathcal{L}$ in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$,

$$\mathsf{h}_j(\mathcal{L}; p, q) = \prod_{a<j} \Big(\mathrm{mod}_A(p_a q_a)^{1/2}\Big)^{k+1+2(\varrho-(a+1))} \ \in [0, \infty].$$

It depends on $\mathcal{L}$ only through A and $\mathfrak{s}$.

LEAN `Level.headDensity`

Definition 2.6 (the integrands)**.** Let $\mathcal{L}$ be a level in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$ (an implicit argument of the three notions), j a natural number, $W\colon G(T) \to \mathbb{C}^I$ and $x, d, w, g \in G(T)$. Put

$$\mathsf{I}^1_j(W; x, d, w)(g) = \mathsf{Av}_j(\mathcal{L}; \langle W(\cdot), W(1\cdot)\rangle)(g, gx) \cdot \overline{\mathsf{Av}_j(\mathcal{L}; \langle W(\cdot), W(d\cdot)\rangle)(w^{-1}g, w^{-1}gx)}.$$

For $p, q\colon \mathbb{N} \to A^\times$ and $\omega\colon G(T) \to \mathbb{C}$ put, with $d = w_j(q)^{-1} w_j(p)$,

$$\mathsf{I}_j(p, q; W, x, \omega)(g) = \iint \omega(\sigma\tau^{-1})\, \mathsf{I}^1_{j+1}(W; x, d, \sigma w_j(p)\tau^{-1})(\sigma g)\, \mathrm{d}\theta_j(\mathcal{L})(\tau)\, \mathrm{d}\theta_j(\mathcal{L})(\sigma).$$

(Integrals of complex functions are Bochner integrals, equal to 0 when the function is not integrable.)

LEAN `Level.formIntegrandOne` · `Level.formIntegrand`

Definition 2.7 (floor form, modulus, majorant)**.** In the situation of Definition 2.6, the *floor form* of floor j is the complex number

$$\mathcal{J}_j(p, q; W, \omega) = \mathsf{h}_j(\mathcal{L}; p, q) \cdot \iint \mathsf{I}_j(p, q; W, x, \omega)(g)\, \mathrm{d}\mu^{\mathcal{L}}_{0,j}(g)\, \mathrm{d}\mathsf{M}(\mathcal{L})(x)$$

(the head density is read as 0 if it is infinite). Its *modulus* at a point $x \in G(T)$ is

$$\mathcal{J}^{\mathrm{mod}}_j(p, q; W, x, \omega) = \int \left|\mathsf{I}_j(p, q; W, x, \omega)(g)\right| \mathrm{d}\mu^{\mathcal{L}}_{0,j}(g) \ \in [0, \infty],$$

and for a second function $W'\colon G(T) \to \mathbb{C}^I$ the *majorant* at x is, with $d = w_j(q)^{-1} w_j(p)$,

$$\mathcal{J}^{\mathrm{maj}}_j(p, q; W, W', x) = \int \mathsf{Pv}_j(\mathcal{L}; \langle W(\cdot), W(1\cdot)\rangle)(g, gx) \cdot \mathsf{Pv}_j(\mathcal{L}; \langle W(\cdot), W'(d\cdot)\rangle)(w_j(p)^{-1}g, w_j(p)^{-1}gx)\, \mathrm{d}\mu^{\mathcal{L}}_{0,j}(g).$$

LEAN `Level.floorForm` · `Level.formModulus` · `Level.formMajorant`

Definition 2.8 (self-norm product)**.** For a level $\mathcal{L}$ in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$ (an implicit argument), a gauge Γ in $\mathrm{Gau}(\mathcal{L})$, a natural number j, sequences $p, q\colon \mathbb{N} \to A^\times$, functions $W, W'\colon G(T) \to \mathbb{C}^I$ and $x \in G(T)$, with $d = w_j(q)^{-1} w_j(p)$,

$$\mathrm{SNP}_j(\Gamma; p, q; W, W', x) = \Big(\prod_{a<j} \mathsf{d}_a(\mathcal{L}; p_a^{-1})\Big)^2 C_{\mathrm{gl}}(\Gamma)^3\ \mathrm{SN}_0(\mathcal{L}; \langle W(\cdot), W(1\cdot)\rangle, x) \cdot \mathrm{SN}_j(\mathcal{L}; \langle W(\cdot), W'(d\cdot)\rangle, x) \ \in [0, \infty].$$

LEAN `Gauge.selfNormProduct`

The complex place. $G_1(\mathfrak{s})$ is $G(T)$ for $A=\mathbb{C}$ and the form T that is the identity on the x_i and -1 at z; its Haar measure is written $\mathrm{d}y$. *Real-place sizes* $\mathfrak{h}$ in $\mathrm{RPS}(\mathfrak{s},\rho,s_0,b)$ are the numerical hypotheses of Input (C1); they give the level $\mathcal{L}_{\mathbb{C}}(\mathfrak{h})$ and, for every real R with $\rho\le R$, the gauge $\Gamma(\mathfrak{h};\rho\le R)$ of Input (C3), and the constant $\lambda(\mathfrak{h})$ of Input (C2).

Definition 2.9 (cut Bessel data)**.** Let $\Phi,\Phi_0\colon G_1(\mathfrak{s})\to\mathbb{C}^I$ be continuous, rep a homomorphism from $\mathrm{U}(\kappa)$ to the unitary group of $\mathbb{C}^I$, ρ and b real numbers, s_0 and n natural numbers. *Cut Bessel data* $\mathfrak{c}$, $\mathrm{CB}(\Phi,\mathrm{rep},\rho,s_0,b,\Phi_0,\mathsf{n})$, are: a proof that (Φ_0,Φ) is a cut pair with respect to b, n and rep, with full function Φ_0 and cut Φ (Section 3); and real-place sizes $\mathfrak{h}$ in $\mathrm{RPS}(\mathfrak{s},\rho,s_0,b)$, the sizes that *underlie* $\mathfrak{c}$.

LEAN `IsCutBessel` · `IsCutBessel.volumeExponent_eq`

For such data, $\mathsf{s}(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}))=s_0$: the level has $\mathfrak{m}=0$ and $|\Omega|=k-1$, and $k=s_0+3n+2$.

Definition 2.10 (the constant of the forms)**.** Let $\Phi,\Phi_0\colon G_1(\mathfrak{s})\to\mathbb{C}^I$, rep, ρ, b, s_0 and n be as in Definition 2.9, and $\mathfrak{c}$ cut Bessel data, $\mathrm{CB}(\Phi,\mathrm{rep},\rho,s_0,b,\Phi_0,\mathsf{n})$, with underlying sizes $\mathfrak{h}$. Then

$$c_{\mathrm{f}}(\mathfrak{c})=\frac{\mathrm{vol}\,\mathrm{U}(k)}{\mathrm{vol}\,\mathrm{U}(r)}\cdot\frac{1}{\left(\lambda(\mathfrak{h})\,|I|\right)^2},\qquad \mathrm{vol}\,\mathrm{U}(N)=\prod_{i<N}\frac{2\pi^{i+1}}{i!}.$$

LEAN `IsCutBessel.formConst`

Definition 2.11 (the vector of the top floor)**.** For a continuous $\Phi\colon G_1(\mathfrak{s})\to\mathbb{C}^I$ and $p\colon\mathbb{N}\to\mathbb{C}^\times$,

$$\Xi(\Phi;p)=\frac{\sqrt{c_G(\mathfrak{s})}\,\sqrt{|p_\varrho|^2-1}^{\,k-1}}{|I|}\prod_{a<\varrho}|p_a|^{k+1+2(\varrho-(a+1))}\cdot J_\Phi^{\mathrm{d}y}(w_\varrho(p)\,\chi_\varrho(p_\varrho))\ \in\mathbb{C}^{I\times I},$$

where the square root of a negative number is 0, $c_G(\mathfrak{s})$, $J_\Phi^{\mathrm{d}y}(\gamma)$ and the boost $\chi_\varrho(c)$ are those of Section 3.

LEAN `sieveVector`

3 What is taken for granted

Inputs from Note T (same unit; *The tower method at a place: floor measures, gauges and the bounds on each floor*). $\mathcal{L}$ is a level in $\mathrm{Lev}(\Omega,T,\mathfrak{m})$, Γ a gauge in $\mathrm{Gau}(\mathcal{L})$; functions called Φ, f take values in $[0,\infty]$ and are measurable; "invariant" means invariant under left translations.

(T1) $\mu_{0,j}^{\mathcal{L}}=\eta_{<j}*(\theta_{j-1}(\mathcal{L})*\cdots*\theta_0(\mathcal{L}))$; $\eta_{<j}$ is carried by $\mathcal{N}_{<j}(T)$, is invariant under translations by $\mathcal{N}_{<j}(T)$ on both sides, and $\delta_s*\eta_{<j}=\eta_{<j}*\delta_s$ for $s\in\mathcal{H}_j(T)$; and $\eta_{<l+1}=\eta_{<l}*\eta_l=\eta_l*\eta_{<l}$ for every l (the first equality is the definition, the second a commutation lemma of unit 119, Input (I3) of Note T).

(T2) If $\mathsf{M}(\mathcal{L})$ is a Haar measure, $j\le\varrho+2$ and Φ is $\mathcal{H}_j(T)$-invariant, then

$$\int\Phi(w_j(p)^{-1}g)\,\mathrm{d}\mu_{0,j}^{\mathcal{L}}(g)=\int\Phi\,\mathrm{d}\mu_{0,j}^{\mathcal{L}}.$$

(T3) If $\mathsf{M}(\mathcal{L}) = \rho_1 * \nu$ is a Haar measure, with ρ_1, ν s-finite, ρ_1 carried by a subgroup S and invariant under right translations by S, and $\delta_s * \rho_1 = \rho_1 * \delta_s$, then $\int \Phi(sg)\,\mathrm{d}\nu(g) = \int \Phi\,\mathrm{d}\nu$ for every S-invariant Φ.

(T4) If a subgroup S contains $\mathcal{H}_l(T)$, f is $(\mathcal{Q}_{l+1}(T) \cap S)$-invariant and $s \in \mathcal{Q}_l(T) \cap S$, then $\int f(\sigma s)\,\mathrm{d}\theta_l(\mathcal{L})(\sigma) = \int f\,\mathrm{d}\theta_l(\mathcal{L})$.

(T5) For $j \le \varrho$:

$$\int \Phi(w_j(q)^{-1} w_j(p)\, x)\,\mathrm{d}\mathsf{M}_{\ge 0}(\mathcal{L})(x) = \prod_{a<j} \mathrm{mod}_A(p_a/q_a)^{k+2(\varrho-a)} \int \Phi\,\mathrm{d}\mathsf{M}_{\ge 0}(\mathcal{L}).$$

(T6) For $b \le \varrho$ and $s \in \mathcal{H}_b(T)$, the map $\sigma \mapsto s\sigma^{-1} f_b$ carries $\theta_b(\mathcal{L})$ to the quadric measure $\delta^{(b)}(\mathcal{L})$ on V_b.

(T7) Let $(k,d) \in \mathrm{PK}_j(\mathcal{L})$. (a) $\sigma \mapsto \mathsf{Av}_{l+1}(\mathcal{L};k)(\sigma g, \sigma y)$ is $\theta_l(\mathcal{L})$-integrable with integral $\mathsf{Av}_l(\mathcal{L};k)(g,y)$; hence $|\mathsf{Av}_l(\mathcal{L};k)(g,y)| \le \mathsf{Pv}_l(\mathcal{L};k)(g,y)$. (b) If $j \le l$ and $s \in \mathcal{Q}_l(T) \cap \mathcal{H}_j(T)$, then $\mathsf{Av}_l(\mathcal{L};k)(sg, sy) = \mathsf{Av}_l(\mathcal{L};k)(g,y)$. (c) $\mathsf{Sq}_j(\mathcal{L};k,x)(\cdot)$ is $\mathcal{Q}_{j+1}(T)$-invariant and vanishes off the points near at $j+1$; $\mathsf{Sm}_j(\mathcal{L};k,x)(\cdot)$ is $\mathcal{Q}_j(T)$-invariant and vanishes off the points near at j. (d) The pair is a pairing kernel at every $l \ge j$. (e) *Cauchy–Schwarz on a floor*: if $\mathrm{Mass}_l(\mathcal{L};u) \le K$ and $|\mathsf{Av}_{l+1}(\mathcal{L};k)(\sigma g, \sigma y)|^2 \le u(\sigma g)\, v(\sigma)$ for $\sigma \in \mathcal{H}_l(T)$, then $\mathsf{Pv}_l(\mathcal{L};k)(g,y)^2 \le K \int v\,\mathrm{d}\theta_l(\mathcal{L})$. (f) For $l \le \varrho$, $\mathrm{SN}_l(\mathcal{L};k,x) = \int \mathsf{Sq}_l(\mathcal{L};k,x)(g)\,\mathrm{d}(\theta_l(\mathcal{L}) * \cdots * \theta_0(\mathcal{L}))(g) + \mathrm{SN}_{l+1}(\mathcal{L};k,x)$, and $\mathrm{SN}_l(\mathcal{L};k,x) \le \mathrm{SN}_0(\mathcal{L};k,x)$ for $l \le n$. For every measurable k: (g) if $\mathsf{Pv}_l(\mathcal{L};k)(g,y) \ne 0$ then $k(\sigma g, \sigma y) \ne 0$ for some $\sigma \in \mathcal{H}_l(T)$; (h) for $l \le \varrho$, $\int |\mathsf{Av}_{l+1}(\mathcal{L};k)(\sigma g, \sigma g x)|^2\,\mathrm{d}\theta_l(\mathcal{L})(\sigma) \le \mathsf{Sm}_l(\mathcal{L};k,x)(g)/\mathsf{w}_l(\mathcal{L};g,x)$.

(T8) Nearness at j is $\mathcal{Q}_j(T)$-invariant; if xg is near at i, $x \in \mathcal{H}_j(T)$ and $j \le i$, then g is near at j. For $\xi \in \mathcal{Q}_{a+1}(T)$, $\mathrm{Val}_a(\mathcal{L};\xi x, y) = \mathrm{Val}_a(\mathcal{L};x,y)$; for $\xi \in \mathcal{Q}_j(T)$, $\mathsf{w}_j(\mathcal{L};\xi x, y) = \mathsf{w}_j(\mathcal{L};x,y)$ and $\mathsf{w}_j^{-1}(\mathcal{L};x)(\xi g) = \mathsf{w}_j^{-1}(\mathcal{L};x)(g)$; $\mathsf{w}_{a+1}(\mathcal{L};x,y) = \mathsf{w}_a(\mathcal{L};x,y)\ \mathrm{vol}(\mathrm{Val}_a(\mathcal{L};x,y))^{\mathsf{s}(\mathcal{L})}$ for $a < \varrho$; weights are neither 0 nor ∞.

(T9) (a) For $b \le \varrho$: $\mathrm{Mass}_b(\mathcal{L};1) \le \Lambda(\Gamma)^{1287n}\,\mathrm{vol}(\Gamma_{11})^{|\Omega|}$. (b) If $\mathrm{Mass}_a(\mathcal{L};1) \le \mathsf{V}$ for all $a \le \varrho$, then $\mathsf{M}_{\ge l}(\mathcal{L})\{g : gc \text{ is near at } \varrho{+}2\} \le \mathsf{V}^{n-l}$ for all l, c. (c) For $x \in \mathrm{Posn}(\mathcal{L})$ and $a < \varrho$: $\mathrm{Mass}_a(\mathcal{L};u) \le \Lambda(\Gamma)^{10^4 n}$ with $u(g) = \mathsf{w}_a(\mathcal{L};g,x)/\mathsf{w}_{a+1}(\mathcal{L};g,x)$.

(T10) For $j \le \varrho$, g near at j and every y: $\int \mathbf{1}_{\mathrm{Pos}(\mathcal{L};\Gamma)}(g^{-1}\nu y)\,\mathsf{w}_j(\mathcal{L};g,g^{-1}\nu y)\,\mathrm{d}\eta_{<j}(\nu) \le \prod_{l<j} C_l(\Gamma)$.

(T11) Let $j \le \varrho$, $p \in \mathrm{Hd}_j$ for Γ, $x \in \mathrm{Posn}(\mathcal{L})$, $j \le b$, ρ_1 an s-finite measure, and Φ $\mathcal{Q}_b(T)$-invariant and zero off the points near at b. Then

$$\int \Phi(g)\,\mathsf{w}_j^{-1}(\mathcal{L};x)(w_j(p)g)\,\mathrm{d}(\eta_{<j} * \rho_1)(g) \le \prod_{a<j} \mathsf{d}_a(\mathcal{L};p_a^{-1}) \cdot C_{\mathrm{gl}}(\Gamma) \int \Phi\,\mathrm{d}\rho_1.$$

(T12) *Plancherel on a floor.* Let $b \le \varrho$, $t \in \mathcal{H}_b(T)$, $x_1, y_1 \in G(T)$ with $\mathrm{sz}(tx_1) \le i_1$ for Γ, $(k,d) \in \mathrm{PK}_b(\mathcal{L})$, and $F(\sigma) = \mathsf{Av}_{b+1}(\mathcal{L};k)(\sigma x_1, \sigma y_1)$. Then (i) for $\theta_b(\mathcal{L})$-almost every σ with $F(\sigma) \ne 0$, $t\sigma^{-1} f_b \in \mathrm{Vec}_{i_1+6}$; (ii) for every $\xi \in A^\times$, $y' \in V$ and every measure m on V with $\mathrm{Fib}_{b,i}(\Gamma;\mathsf{m}) \le M$,

$$\int_{\mathrm{Vec}_i} \Big| \int \psi_A(\xi\, h_T(v + y', t\sigma^{-1} f_b)) F(\sigma)\,\mathrm{d}\theta_b(\mathcal{L})(\sigma)\Big|^2 \mathrm{d}\mathsf{m}(v) \le M\Lambda(\Gamma)^{99n(i_1+12)} \Big(\frac{c_A}{\mathrm{mod}_A(\xi)}\Big)^{|\Omega|} \int |F|^2\,\mathrm{d}\theta_b(\mathcal{L});$$

and $\mathrm{Fib}_{b,i}(\Gamma;\delta^{(b)}(\mathcal{L}))\le\Lambda(\Gamma)^{99n(i+6)}$ for all $b\le\varrho$ and i.

(T13) *Plancherel on a Heisenberg group.* Let $a<\varrho$, $\xi,u\in A^{\times}$, $\varepsilon\in\mathbb{N}$, $C\in[0,\infty]$, $(k,d)\in\mathrm{PK}_{a+1}(\mathcal{L})$, $x_1,y_1\in G(T)$, $S\subseteq G(T)$, $\Phi\colon G(T)\to[0,\infty]$, and natural numbers i_1, i_2. Suppose: for $s\in\mathcal{N}_a(T)$, $g_{\mathcal{L}}^{-1}(\mathrm{sd}(dsd^{-1})-\mathrm{sd}(s^{\varepsilon}))=\xi\,\mathrm{sd}(s)$; for $\nu\in\mathcal{N}_a(T)$ with $\Phi(\nu)\neq0$, $\nu\in S$ and $\Phi(\nu)\le C\,|\mathsf{Av}_{a+1}(\mathcal{L};k)(\nu^{\varepsilon}x_1,\nu y_1)|^2$; for every $m\in S$ there are $t\in\mathcal{Q}_{a+1}(T)\cap\mathcal{H}_a(T)$ and $q\in\mathcal{N}_{<a+1}(T)$ with $\mathrm{sz}(tqx_1)\le i_1$ such that for every $\nu\in S$ some $\nu'\in\mathcal{N}_{<a}(T)$ makes the column at e_a^{*} of $u^{-1}(\nu' tm(t\nu)^{-1}-1)$ lie in Vec_{i_2} (sizes for Γ). Then

$$\int\Phi\,\mathrm{d}\eta_a\le C\Big(\frac{c_A}{\mathrm{mod}_A(\xi u)}\Big)^{|\Omega|}\mathsf{d}_a(\mathcal{L};u)\,\Lambda(\Gamma)^{100n(i_1+i_2+31)}\cdot\int\big|\mathsf{Av}_{a+2}(\mathcal{L};k)(\sigma x_1,\sigma y_1)\big|^2\,\mathrm{d}\theta_{a+1}(\mathcal{L})(\sigma).$$

(T14) *Sizes.* Sizes are for Γ; $j\le\varrho$. (a) If $p\in\mathrm{Hd}_j$, $s\in\mathcal{N}_{<j}(T)$, $sg\in\mathrm{Box}(\mathcal{L})$ and $w_j(p)^{-1}g$ is near at j, then $\mathrm{sz}(g)\le60n^2$. (b) If $p\in\mathrm{Hd}_j$ and g, $w_j(p)^{-1}g$ are near at j, there is $\sigma\in\mathcal{H}_j(T)$ with $\mathrm{sz}(\sigma g)\le60n^2$ and $\mathrm{sz}(\sigma w_j(p)^{-1}g)\le99n^2$. (c) If $s\in\mathcal{Q}_j(T)$, $sg\in\mathrm{Box}(\mathcal{L})$ and $w_j(p)^{-1}g$ is near at j, then $p_0\cdots p_{l-1}\in\Gamma_{9n}$ for $l\le j$; if moreover $p\in\mathrm{Hd}_j$ and $a<j$, then $p_a\in\Gamma_{10n}$. (d) If $g\in\mathrm{Box}(\mathcal{L})$ and $x,y\in\mathrm{Pos}(\mathcal{L};\Gamma)$, then $\mathrm{sz}(g_{\mathcal{L}}^{-1}(gx(gy)^{-1}-1))\le14$. (e) Let Γ_S be a gauge in $\mathrm{Gau}(\mathcal{L})$ for which p,q are heads of length j, with $g_{\mathcal{L}}^{-1}c\in\Gamma_1$ for all $c\in(\Gamma_S)_{10n}$ if $j\neq0$. If $ux,v(xy)\in\mathrm{Box}(\mathcal{L})$ for some $u,v\in\mathcal{U}(T)$ with $\mathrm{sz}(g_{\mathcal{L}}^{-1}(v(xy)-ux))\le2$, if $w_j(p)^{-1}x$ and $w_j(q)^{-1}xy$ are near at j, and if for every a with $j\le a<\varrho$ the set $\mathrm{Val}_a(\mathcal{L};x,y)$ contains a unit c with $c^{-1}\in\Gamma_1$, then $\mathrm{sz}(g_{\mathcal{L}}^{-1}(y-1))\le99n$.

Inputs from Note C (same unit; *The complex place: the open-cell chart, the Haar measure of the tower, the Bessel cut and the volume weight*).

(C1) *Real-place sizes* $\mathfrak{h}$ in $\mathrm{RPS}(\mathfrak{s},\rho,s_0,b)$ are proofs of $\rho=b^{6n+2}$, $8r\le\rho$, $k=s_0+3n+2$, $10^9n^3<s_0$, $n^7\le k$, $10^8\le\varrho$ and $2\le b$; then $\rho\ge4$. The level $\mathcal{L}_{\mathbb{C}}(\mathfrak{h})$ at $\mathbb{C}$ has $\Omega=\kappa\setminus\{k_0\}$, $\mathfrak{m}=0$, $g_{\mathcal{L}_{\mathbb{C}}(\mathfrak{h})}=1$, the ball $\{v:|v|<\frac14\}$, the box $\{x:\|x\|\le\rho\}$ (operator norm), all of $G_1(\mathfrak{s})$ as positions, $\theta_n(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}))$ the Haar probability measure of $\mathrm{U}(\kappa)=\mathcal{H}_n(T)$, and $\theta_{\varrho}(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}))$ the convolution of the image of $\frac{2\pi^k}{(k-1)!}\max(|c|^2-1,0)^{k-1}\mathrm{d}c$ under $c\mapsto\chi_{\varrho}(c)^{-1}$ with $\theta_n(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}))$.

(C2) $0<\lambda(\mathfrak{h})<\infty$; $\mathsf{M}(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}))=\lambda(\mathfrak{h})\cdot\mathrm{d}y$, so it is a Haar measure; $\int f\,\mathrm{d}\mathsf{M}_{\ge0}(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}))=\lambda(\mathfrak{h})\int f\,\mathrm{d}\dot{y}$ for measurable $\mathcal{U}(T)$-invariant $f\ge0$, where $\mathrm{d}\dot{y}$ is the measure on $G_1(\mathfrak{s})$ of the background library that computes integrals over $\mathcal{U}(T)\backslash G_1(\mathfrak{s})$ (it is not a measure on the quotient: $\iint_{\mathcal{U}(T)}f(uy)\,\mathrm{d}u\,\mathrm{d}\dot{y}=\int f\,\mathrm{d}y$ for measurable $f\ge0$ and a fixed Haar measure of $\mathcal{U}(T)$); and $\frac{\mathrm{vol}\,\mathrm{U}(k)}{\mathrm{vol}\,\mathrm{U}(r)}\lambda(\mathfrak{h})\le1$.

(C3) For real $R\ge\rho$, $\Gamma(\mathfrak{h};\rho\le R)$ is a gauge in $\mathrm{Gau}(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}))$ with sets $\{x\in\mathbb{C}:|x|\le R^i\}$ and constant $\Lambda=R^2$. In the formal text the proof of $\rho\le R$ is the last argument of the gauge and has a default value (reflexivity); so the gauge written there without this argument is $\Gamma(\mathfrak{h};\rho\le\rho)$, the gauge of radius ρ.

(C4) If the Bessel size of $y\in G_1(\mathfrak{s})$ is at most t, then $\|uy\|\le t$ for some $u\in\mathcal{U}(T)$.

From the unit on the second moment (unit 148) and the background library. We recall definitions; none of their properties is used beyond what is said. For $y \in G_1(\mathfrak{s})$ the *Bessel size* is $\inf_s(\|sy\| + 1)$ over $s \in \mathcal{Q}_n(T)$. A *cut pair* with respect to b, $\mathfrak{n}$, rep is a pair of continuous functions Φ_0 (full) and Φ (cut) such that: both are measurable, bounded, satisfy $F(uy) = \psi_{\mathbb{C}}(\mathrm{sd}(u))F(y)$ for $u \in \mathcal{U}(T)$ and $F(hy) = \mathrm{rep}(h)F(y)$ for $h \in \mathrm{U}(\kappa) = \mathcal{H}_n(T)$; Φ_0 is integrable for $\mathrm{d}\dot{y}$; $\int |\Phi_0|^2 \mathrm{d}\dot{y} = |I|$ and $\int |\Phi|^2 \mathrm{d}\dot{y} \le |I|$; $\Phi(y) \neq 0$ implies that the Bessel size of y is at most b^{6n+2}; $\int |\Phi - \Phi_0|^2 \mathrm{d}\dot{y} \le b^{-10r^2}|I|$; and a majorant exists, which is not used here and involves $\mathfrak{n}$. For μ a measure on a group and W a $\mathbb{C}^I$-valued function, $J^{\mu}_W(\gamma) \in \mathbb{C}^{I \times I}$ has the coordinates $\int W_i(y)\overline{W_{i'}(\gamma^{-1}y)}\,\mathrm{d}\mu(y)$. The constants are $c_G(\mathfrak{s}) = \frac{\mathrm{vol}\,\mathrm{U}(k)}{\mathrm{vol}\,\mathrm{U}(r)} \cdot \frac{2\pi^k}{(k-1)!}$ and $\Theta(\mathfrak{s}) = \frac{\mathrm{vol}\,\mathrm{U}(k)}{\mathrm{vol}\,\mathrm{U}(r)}\,\pi^{(2\varrho+1)k}$. The *boost* $\chi_\varrho(c) \in \mathcal{H}_\varrho(T)$, for $c \in \mathbb{C}^\times$, acts in the plane of x_{k_0} and z and has $Z_T(\chi_\varrho(c))_{\varrho\varrho} = c$ when $|c| > 1$. For a real t and natural numbers K, i: $\ell_K(t) = t^K \min(1,t)^6$ and $\mathrm{hw}_{K,i}(p) = \prod_{l<i} \ell_K(|p_0 \cdots p_l|^{-1})$. A function Φ_0 *has Schur's bound with constant* D if $\int |\int \varphi(y)\Phi_0(\gamma y)\,\mathrm{d}y|^2 \mathrm{d}\dot{\gamma} \le D \int |\varphi|^2 \mathrm{d}y$ for every integrable $\varphi\colon G_1(\mathfrak{s}) \to \mathbb{C}$. Finally, Φ *has the band limits with respect to* b, deg, J if, with $\beta(\xi) = \Theta(\mathfrak{s})\, b^{2000n^9}\, \ell_{k-n^7}(|\xi|^{-1})$ and $u(c)$ as in Theorem B:

(S1) for $i \le \varrho$, all p, q and $\xi \in \mathbb{C}$, $f(c) = \psi_{\mathbb{C}}(\xi c)\, J_{i+1}(p[i \mapsto u(c)], q[i \mapsto u(c)])$ is integrable on $\mathbb{C}$; $\int f = J_i(p,q)$ if $\xi = 0$; and $|\int f| \le \beta(\xi)\ \mathrm{hw}_{n^6,i}(p)\ \mathrm{hw}_{n^6,i}(q)$ if $\xi \neq 0$;

(S2) for finitely many sequences with a common entry at ϱ, the matrix of the values of $J_{\varrho+1}$ is positive semidefinite; $J_{\varrho+1}$ is continuous;

(S3) $|J_0(p,q)| \le (1 + b^{-10})/\deg$;

(S4) for $i < \varrho$ and $\xi = p_i^{-1} - q_i^{-1} \neq 0$: $|J_{i+1}(p,q)| \le \beta(\xi)\ \mathrm{hw}_{n^6,i+1}(p)\ \mathrm{hw}_{n^6,i+1}(q)$;

(S5) for $\sigma \in \mathcal{H}_\varrho(T)$ and p, q with $Z_T(\sigma)_{\varrho\varrho} = p_\varrho = q_\varrho$:

$$J_n(p,q) = N(p)N(q)\langle J^{\mathrm{d}y}_\Phi(w_\varrho(p)\sigma), J^{\mathrm{d}y}_\Phi(w_\varrho(q)\sigma)\rangle,$$

with $N(p)$ the scalar factor of Definition 2.11.

Three classical results enter through these inputs and directly: the *Cauchy–Schwarz inequality* in the form $(\int h)^2 \le \int f_1 \int f_2$ for $h^2 \le f_1 f_2$ (used freely); *Plancherel's inequality at a place* (inside (T12)), which the second reader of the artifact relates to J. Tate, *Fourier analysis in number fields and Hecke's zeta-functions* (1950), §2.2, as a weaker form; and the *Cartan decomposition* of $\mathcal{H}_\varrho(T)$ at $\mathbb{C}$: if $\sigma \in \mathcal{H}_\varrho(T)$ has $Z_T(\sigma)_{\varrho\varrho} = c$ with $|c| > 1$, then $\sigma = h\,\chi_\varrho(c)\,h'$ with $h, h' \in \mathcal{H}_n(T)$, which the second reader relates to the KAK theorem (A. Knapp, *Structure theory of semisimple Lie groups*, Theorem 4.10) as a special case with the phase fixed. The places of these references were given by the second reader and not opened for this note.

4 The kernels of a cut function

Lemma 4.1 (the kernels are pairing kernels)**.** *In Setting 1.2, let* $W'\colon G(T) \to \mathbb{C}^I$ *satisfy* $W' \in \mathrm{Bes}(g_{\mathcal{L}}^{-1})$*, let* j *be a natural number and* $p, q\colon \mathbb{N} \to A^\times$*. Then, with* $d = w_j(q)^{-1} w_j(p)$*,*

$$(\langle W(\cdot), W'(d\cdot)\rangle, d) \in \mathrm{PK}_j(\mathcal{L}).$$

In particular, for $W' = W$ *and* $p = q$ *constant equal to* 1*:* $(\langle W(\cdot), W(1\cdot)\rangle, 1) \in \mathrm{PK}_j(\mathcal{L})$.

Proof. Write $k(y,y') = \langle W(y), W'(dy')\rangle$; it is measurable. *The law.* Let $\nu,\nu' \in \mathcal{N}_{<l}(T)$ and $\sigma \in \mathcal{H}_{l+1}(T)$, which is where $\mathsf{M}_{\geq l+1}(\mathcal{L})$ is carried. Then $\sigma\nu\sigma^{-1} \in \mathcal{N}_{<l}(T) \subseteq \mathcal{U}(T)$ has the same entries at the places (f_a, f_{a+1}) as ν (a property of the background library), so $W(\sigma\nu g) = \psi_A(g_{\mathcal{L}}^{-1}\,\mathrm{sd}(\nu))\,W(\sigma g)$. The matrix of d is diagonal, so $d(\sigma\nu'\sigma^{-1})d^{-1} \in \mathcal{U}(T)$, its entries at the places (f_a, f_{a+1}) are those of $d\nu' d^{-1}$, and $W'(d\sigma\nu' y) = \overline{\psi_A(g_{\mathcal{L}}^{-1}\,\mathrm{sd}(d\nu' d^{-1}))}\,W'(d\sigma y)$. As $\langle\cdot,\cdot\rangle$ is conjugate linear in the first variable and $\overline{\psi_A(c)} = \psi_A(-c)$, the integrand of $\mathsf{Av}_{l+1}(\mathcal{L};k)(\nu g,\nu' y)$ is $\psi_A(g_{\mathcal{L}}^{-1}(\mathrm{sd}(d\nu' d^{-1}) - \mathrm{sd}(\nu)))$ times that of $\mathsf{Av}_{l+1}(\mathcal{L};k)(g,y)$. *The element d commutes with $\mathcal{H}_j(T)$*, since $w_j(p)$ and $w_j(q)$ do. *Boundedness.* Let C, C' bound $|W|, |W'|$. The integrand of $\int |k(\sigma g, \sigma y)|\,\mathrm{d}\mathsf{M}_{\geq l}(\mathcal{L})(\sigma)$ is at most CC' and vanishes unless $W(\sigma g) \neq 0$, hence unless σg is near at $\varrho + 2$. By (T9)(a),(b) the measure of these σ is at most V^{n-l} with $\mathsf{V} = \Lambda(\Gamma)^{1287n}\,\mathrm{vol}(\Gamma_{11})^{|\Omega|} \leq \Lambda(\Gamma)^{1287n+12|\Omega|} < \infty$. *Support.* If g is not near at l, then for $\sigma \in \mathcal{H}_l(T)$ the point σg is not near at l either ((T8)), so $W(\sigma g) = 0$ and $\mathsf{Av}_l(\mathcal{L};k)(g,y) = 0$. The last assertion is the case in which $w_j(p) = w_j(q)$. □

LEAN `Gauge.isPairingKernel_translatePairing` · `Gauge.isPairingKernel_self`

Lemma 4.2 (the integrand at weight 1)**.** *In Setting 1.2, let j be a natural number, $p, q\colon \mathbb{N} \to A^\times$ and $x \in G(T)$. Then, as functions of g,*

$$\mathsf{I}_j^1(W; x, w_j(q)^{-1}w_j(p), w_j(p))(\cdot) = \mathsf{I}_j(p,q;W,x,1)(\cdot).$$

Proof. Write k_1, k_d for the two kernels of Definition 2.6; both are pairing kernels at j (Lemma 4.1). By (T7)(a), $\mathsf{Av}_j(\mathcal{L};k_1)(g,gx)$ is the integral over σ of $\mathsf{Av}_{j+1}(\mathcal{L};k_1)(\sigma g, \sigma gx)$, and $\mathsf{Av}_j(\mathcal{L};k_d)(w_j(p)^{-1}g, w_j(p)^{-1}gx)$ the integral over τ of the same expression with k_d and $\tau w_j(p)^{-1}g$. Since $(\sigma w_j(p)\tau^{-1})^{-1}\sigma g = \tau w_j(p)^{-1}g$, the product of the first integral with the conjugate of the second is the double integral that defines the right-hand side with $\omega = 1$. □

LEAN `Level.formIntegrandOne_eq`

Lemma 4.3 (form, modulus, majorant)**.** (i) *Let $\mathcal{L}$ be a level in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$, j a natural number, $p, q\colon \mathbb{N} \to A^\times$, $W\colon G(T) \to \mathbb{C}^I$ and $\omega\colon G(T) \to \mathbb{C}$. Then*

$$\left|\mathcal{J}_j(p,q;W,\omega)\right| \leq \mathsf{h}_j(\mathcal{L};p,q) \int \mathcal{J}_j^{\mathrm{mod}}(p,q;W,x,\omega)\,\mathrm{d}\mathsf{M}(\mathcal{L})(x).$$

(ii) *If moreover W is cut for $\mathcal{L}$, $W \in \mathrm{Cut}$, and $|\omega(y)| \leq 1$ for all y, then for every $x \in G(T)$*

$$\mathcal{J}_j^{\mathrm{mod}}(p,q;W,x,\omega) \leq \mathcal{J}_j^{\mathrm{maj}}(p,q;W,W,x).$$

Proof. (i) The absolute value of an integral is at most the integral of the absolute value, and the iterated integral bounds the integral over the product. (ii) Inside the integral over g, the same inequality bounds $|\mathsf{I}_j(p,q;W,x,\omega)(g)|$ by the double integral of the product of $|\mathsf{Av}_{j+1}(\mathcal{L};k_1)(\sigma g,\sigma gx)|$ and $|\mathsf{Av}_{j+1}(\mathcal{L};k_d)(\tau w_j(p)^{-1}g, \tau w_j(p)^{-1}gx)|$, as in the proof of Lemma 4.2; it is the product of the two pivots in the majorant. The measurability needed to split the double integral comes from that of W. □

LEAN `Level.enorm_floorForm_le` · `Level.IsCut.formModulus_le_formMajorant`

Lemma 4.4 (a pivot under a volume cap)**.** *Let $\mathcal{L}$ be a level in $\mathrm{Lev}(\Omega, T, \mathfrak{m})$, Γ a gauge in $\mathrm{Gau}(\mathcal{L})$, j a natural number, and $(k,d) \in \mathrm{PK}_j(\mathcal{L})$. Let $x \in \mathrm{Posn}(\mathcal{L})$, let l be a natural number, $g \in G(T)$ and $\varepsilon \in [0,\infty]$. Suppose that for every $\sigma \in \mathcal{H}_l(T)$ with $k(\sigma g, \sigma gx) \neq 0$ there is a with $l \leq a < \varrho$ and $\mathrm{vol}(\mathrm{Val}_a(\mathcal{L};\sigma g, x)) \leq \varepsilon$. Then*

$$\mathsf{Pv}_l(\mathcal{L};k)(g,gx)^2 \;\leq\; \varepsilon^{\mathsf{s}(\mathcal{L})}\,(\Lambda(\Gamma)^{10^4 n})^{\varrho-l}\,\frac{\mathsf{Sm}_l(\mathcal{L};k,x)(g)}{\mathsf{w}_l(\mathcal{L};g,x)}.$$

Proof. Induction on $\varrho - l$, for all g. If $l \geq \varrho$ the pivot is 0: otherwise (T7)(g) gives $\sigma \in \mathcal{H}_l(T)$ with $k(\sigma g, \sigma g x) \neq 0$, and the hypothesis an a with $l \leq a < \varrho$. Let $l < \varrho$, and write $s = \mathsf{s}(\mathcal{L})$, $\Lambda = \Lambda(\Gamma)$, $\mathsf{A} = \mathsf{Av}_{l+1}(\mathcal{L}; k)(\cdot, \cdot)$. We apply Cauchy–Schwarz on floor l, (T7)(e), with $u(g') = \mathsf{w}_l(\mathcal{L}; g', x)/\mathsf{w}_{l+1}(\mathcal{L}; g', x)$, whose mass is at most $\Lambda^{10^4 n}$ by (T9)(c), and with

$$v(\sigma) = \varepsilon^s (\Lambda^{10^4 n})^{\varrho - l - 1} \, \frac{\mathsf{Sq}_l(\mathcal{L}; k, x)(\sigma g) + \mathsf{Sm}_{l+1}(\mathcal{L}; k, x)(\sigma g)}{\mathsf{w}_l(\mathcal{L}; g, x)}.$$

Then $\Lambda^{10^4 n} \int v \, \mathrm{d}\theta_l(\mathcal{L})$ is the right-hand side of the lemma, by the definition of the weighted square sum. It remains to see $|\mathsf{A}(\sigma g, \sigma g x)|^2 \leq u(\sigma g) v(\sigma)$ for $\sigma \in \mathcal{H}_l(T)$. By (T8),

$$\mathsf{w}_l(\mathcal{L}; \sigma g, x) = \mathsf{w}_l(\mathcal{L}; g, x), \qquad \mathsf{w}_{l+1}(\mathcal{L}; \sigma g, x) = \mathsf{w}_l(\mathcal{L}; g, x) \, \mathrm{vol}(\mathrm{Val}_l(\mathcal{L}; \sigma g, x))^s.$$

If $\mathrm{vol}(\mathrm{Val}_l(\mathcal{L}; \sigma g, x)) \leq \varepsilon$, then $u(\sigma g) \geq \varepsilon^{-s}$, and $|\mathsf{A}(\sigma g, \sigma g x)|^2$ is $\mathsf{Sq}_l(\mathcal{L}; k, x)(\sigma g)$ divided by $\mathsf{w}_l(\mathcal{L}; g, x)$, by definition; the inequality follows since $\Lambda \geq 1$. Otherwise the hypothesis of the lemma holds for $(l+1, \sigma g)$: for $\tau \in \mathcal{H}_{l+1}(T)$ with $k(\tau\sigma g, \tau\sigma g x) \neq 0$ the hypothesis at (l, g) gives some $a \geq l$, and $a = l$ is excluded because the sets $\mathrm{Val}_l(\mathcal{L}; \tau\sigma g, x)$ and $\mathrm{Val}_l(\mathcal{L}; \sigma g, x)$ are equal ((T8)). By (T7)(a), by induction, and since $\mathsf{w}_{l+1}(\mathcal{L}; \sigma g, x)^{-1} = u(\sigma g)/\mathsf{w}_l(\mathcal{L}; g, x)$,

$$\begin{aligned} |\mathsf{A}(\sigma g, \sigma g x)|^2 &\leq \mathsf{Pv}_{l+1}(\mathcal{L}; k)(\sigma g, \sigma g x)^2 \\ &\leq \varepsilon^s \Lambda^{10^4 n(\varrho - l - 1)} \frac{\mathsf{Sm}_{l+1}(\mathcal{L}; k, x)(\sigma g)}{\mathsf{w}_{l+1}(\mathcal{L}; \sigma g, x)} \leq u(\sigma g) v(\sigma). \end{aligned}$$

□

LEAN `Gauge.pivotModulus_sq_le_of_capped`

5 The local theorem

Lemma 5.1 (from a pointwise bound to the self-norm product)**.** *In Setting 1.1, let $W' \colon G(T) \to \mathbb{C}^I$ satisfy $W' \in \mathrm{Bes}(g_{\mathcal{L}}^{-1})$, let $f \colon G(T) \to [0, \infty]$ and $K \in [0, \infty]$, and write $k_1 = \langle W(\cdot), W(1\cdot)\rangle$, $k_d = \langle W(\cdot), W'(d\cdot)\rangle$ with $d = w_j(q)^{-1} w_j(p)$. Suppose that for every g with $f(g) \neq 0$*

$$f(g)^2 \leq K \, C_{\mathrm{gl}}(\Gamma) \, \frac{\mathsf{Sm}_j(\mathcal{L}; k_1, x)(g)}{\mathsf{w}_j(\mathcal{L}; g, x)} \cdot \frac{\mathsf{Sm}_j(\mathcal{L}; k_d, x)(w_j(p)^{-1} g)}{\mathsf{w}_j(\mathcal{L}; w_j(p)^{-1} g, x)}.$$

Then $\left(\int f \, \mathrm{d}\mu^{\mathcal{L}}_{0,j}\right)^2 \leq K \cdot \mathrm{SNP}_j(\Gamma; p, q; W, W', x)$.

Proof. Write $B = \mathsf{Sm}_j(\mathcal{L}; k_1, x)(\cdot)$, $B' = \mathsf{Sm}_j(\mathcal{L}; k_d, x)(\cdot)$, $\mathsf{v} = \mathsf{w}_j^{-1}(\mathcal{L}; x)$ and $w = w_j(p)$. Both kernels are pairing kernels at j (Lemma 4.1), so B, B' are $\mathcal{Q}_j(T)$-invariant and vanish off the points near at j ((T7)(c)). If $f(g) \neq 0$ the hypothesis forces $B(g) \neq 0 \neq B'(w^{-1} g)$, so g and $w^{-1} g$ are near at j and the two weights are $\mathsf{v}(g)^{-1}$, $\mathsf{v}(w^{-1} g)^{-1}$. Pairing each sum with the weight of the other point,

$$f(g)^2 \leq K \, C_{\mathrm{gl}}(\Gamma) \cdot [B(g) \, \mathsf{v}(w^{-1} g)] \cdot [B'(w^{-1} g) \, \mathsf{v}(g)] \qquad \text{for all } g.$$

By Cauchy–Schwarz, $(\int f \, \mathrm{d}\mu^{\mathcal{L}}_{0,j})^2$ is at most $K C_{\mathrm{gl}}(\Gamma)$ times the product of the integrals of the two brackets. The second bracket is $\Phi(w^{-1} g)$ for $\Phi(g) = B'(g) \mathsf{v}(wg)$, which is

$\mathcal{H}_j(T)$-invariant because w commutes with $\mathcal{H}_j(T)$; by (T2), (T1) and (T11) (with $b = j$)

$$\int B'(w^{-1}g)\,\mathsf{v}(g)\,\mathrm{d}\mu^{\mathcal{L}}_{0,j}(g) = \int B'(g)\,\mathsf{v}(wg)\,\mathrm{d}\mu^{\mathcal{L}}_{0,j}(g) \le \prod_{a<j} \mathsf{d}_a(\mathcal{L}; p_a^{-1}) \cdot C_{\mathrm{gl}}(\Gamma) \cdot \mathrm{SN}_j(\mathcal{L}; k_d, x).$$

For the first bracket, $w^{-1} = w_j(\bar{p})$ and $\bar{p}$ is a head of length j, so (T11) applies with $\bar{p}$ and gives the bound $\prod_{a<j} \mathsf{d}_a(\mathcal{L}; \bar{p}_a^{-1}) \cdot C_{\mathrm{gl}}(\Gamma) \cdot \mathrm{SN}_j(\mathcal{L}; k_1, x)$; here $\mathsf{d}_a(\mathcal{L}; \bar{p}_a^{-1}) = \mathsf{d}_a(\mathcal{L}; p_a^{-1})$, and $\mathrm{SN}_j(\mathcal{L}; k_1, x) \le \mathrm{SN}_0(\mathcal{L}; k_1, x)$ by (T7)(f). The product of the three bounds is K times the self-norm product. □

LEAN `Gauge.sq_lintegral_le_selfNormProduct`

Proof of Theorem A(a). Apply Lemma 5.1 to the function

$$f(y) = \mathsf{Pv}_j(\mathcal{L}; k_1)(y, yx) \cdot \mathsf{Pv}_j(\mathcal{L}; k_d)(w^{-1}y, w^{-1}yx), \qquad w = w_j(p),$$

whose integral is the majorant, and to $K = \sigma_0^s \mathsf{V}^2$, $s = \mathsf{s}(\mathcal{L})$. Let $f(y) \neq 0$. For the second pivot, Cauchy–Schwarz on floor j ((T7)(e) with $u = 1$, $K = \mathsf{V}$) and (T7)(h) give, at $g' = w^{-1}y$,

$$\mathsf{Pv}_j(\mathcal{L}; k_d)(g', g'x)^2 \le \mathsf{V}\,\mathsf{Sm}_j(\mathcal{L}; k_d, x)(g')/\mathsf{w}_j(\mathcal{L}; g', x).$$

If $\sigma_0 \ge 1$ the same bound holds for the first pivot; the two bounds give the factor V^2, and $\sigma_0^s \ge 1$, $C_{\mathrm{gl}}(\Gamma) \ge 1$. If $\sigma_0 < 1$ we use Lemma 4.4 for the first pivot, with $l = j$ and $\varepsilon = \sigma_0$; its hypothesis is checked as follows. Since the second pivot is not 0 there is $\tau \in \mathcal{H}_j(T)$ with $k_d(\tau w^{-1}y, \tau w^{-1}yx) \neq 0$ ((T7)(g)); thus $W(\tau w^{-1}y) \neq 0$ and $W'(d\tau w^{-1}yx) \neq 0$, where $d\tau w^{-1} = \tau w_j(q)^{-1}$. So $\tau w^{-1}y$ and $\tau w_j(q)^{-1}yx$ are near at j. For $\rho_1 \in \mathcal{H}_j(T)$ with $k_1(\rho_1 y, \rho_1 yx) \neq 0$ we get $\langle W(\rho_1 y), W(\rho_1 yx)\rangle \neq 0$, and $w^{-1}\rho_1 y = (\rho_1\tau^{-1})\tau w^{-1}y$ and $w_j(q)^{-1}\rho_1 yx$ are near at j ((T8)); the hypothesis of the theorem at the point $\rho_1 y$ gives the a that Lemma 4.4 asks for. Its conclusion has the factor $(\Lambda(\Gamma)^{10^4 n})^{\varrho - j} \le C_{\mathrm{gl}}(\Gamma)$ and no factor V; with the bound for the second pivot this gives the factor V, and $\mathsf{V} \le \mathsf{V}^2$ as $\mathsf{V} \ge 1$. □

Proof of Theorem A(b). Apply Lemma 5.1 with $W' = W$ and with

$$f(g) = \bigl|\mathsf{I}_j(p, q; W, x, \omega_\eta)(g)\bigr|, \qquad K = \bigl(c_A / \operatorname{mod}_A(\xi)\bigr)^{|\Omega|}.$$

Fix g with $f(g) \neq 0$ and put $w = w_j(p)$,

$$F_1(\sigma) = \mathsf{Av}_{j+1}(\mathcal{L}; k_1)(\sigma g, \sigma gx), \qquad F_2(\tau) = \mathsf{Av}_{j+1}(\mathcal{L}; k_d)(\tau w^{-1}g, \tau w^{-1}gx).$$

By (T7)(h) it suffices to show

$$f(g)^2 \le K\, C_{\mathrm{gl}}(\Gamma) \int |F_1|^2\,\mathrm{d}\theta_j(\mathcal{L}) \cdot \int |F_2|^2\,\mathrm{d}\theta_j(\mathcal{L}). \tag{1}$$

By definition and since $Z_T(\sigma\tau^{-1})_{jj} = h_T(\sigma^{-1}f_j, \tau^{-1}f_j)$,

$$\mathsf{I}_j(p, q; W, x, \omega_\eta)(g) = \iint \psi_A(\eta\, h_T(\sigma^{-1}f_j, \tau^{-1}f_j)) F_1(\sigma)\overline{F_2(\tau)}\,\mathrm{d}\theta_j(\mathcal{L})(\tau)\,\mathrm{d}\theta_j(\mathcal{L})(\sigma).$$

As this is not 0, F_1 and F_2 are not almost everywhere 0 on $\mathcal{H}_j(T)$, so g and $w^{-1}g$ are near at j (the last clause of the definition of a pairing kernel, and (T8)). By (T14)(b)

there is $t \in \mathcal{H}_j(T)$ with $\mathrm{sz}(tg) \le 60n^2$ and $\mathrm{sz}(tw^{-1}g) \le 99n^2$ for Γ. By (T12)(i), applied to both kernels with this t, for almost all σ, τ with $F_1(\sigma)F_2(\tau) \neq 0$ the vectors $t\sigma^{-1}f_j$ and $t\tau^{-1}f_j$ have sizes at most $60n^2+6$ and $99n^2+6$; hence $h_T(\sigma^{-1}f_j, \tau^{-1}f_j) = h_T(t\sigma^{-1}f_j, t\tau^{-1}f_j)$ lies in Γ_{200n^2}, by the rules for sizes of products and sums, and there $\psi_A(\eta\,\cdot) = \psi_A(\xi\,\cdot)$. So, with $G(v) = \int \psi_A(-\xi\, h_T(v, t\tau^{-1}f_j))F_2(\tau)\,\mathrm{d}\theta_j(\mathcal{L})(\tau)$,

$$\mathsf{I}_j(p,q;W,x,\omega_\eta)(g) = \int_{\{\sigma:\, t\sigma^{-1}f_j \in \mathrm{Vec}_{60n^2+6}\}} F_1(\sigma)\,\overline{G(t\sigma^{-1}f_j)}\,\mathrm{d}\theta_j(\mathcal{L})(\sigma).$$

By Cauchy–Schwarz, $f(g)^2 \le \int |F_1|^2 \mathrm{d}\theta_j(\mathcal{L})$ times the integral of $|G(t\sigma^{-1}f_j)|^2$ over the same set, which by (T6) is $\int_{\mathrm{Vec}_{60n^2+6}} |G|^2 \mathrm{d}\delta^{(j)}(\mathcal{L})$. By (T12)(ii) for the kernel k_d, the points $w^{-1}g$, $w^{-1}gx$, the unit $-\xi$, $y' = 0$ and the quadric measure, this is at most $\Lambda(\Gamma)^{99n(60n^2+12)}\Lambda(\Gamma)^{99n(99n^2+12)}K\int |F_2|^2\mathrm{d}\theta_j(\mathcal{L})$, and $99n(159n^2+24) \le 10^6 n^3$. This is (1). □

Proof of Theorem A(c). Here $j = i+1$. Write $w = w_j(p)$, $\mathsf{A}_1 = \mathsf{Av}_j(\mathcal{L}; k_1)(\cdot,\cdot)$, $\mathsf{A}_d = \mathsf{Av}_j(\mathcal{L}; k_d)(\cdot,\cdot)$ and $K = (c_A/\bmod_A(\xi))^{|\Omega|}$. By Lemma 4.2 the modulus is $\int h\,\mathrm{d}\mu^{\mathcal{L}}_{0,j}$ with $h(g) = |\mathsf{A}_1(g,gx)| \cdot |\mathsf{A}_d(w^{-1}g, w^{-1}gx)|$. Define, for $g \in G(T)$,

$$\mathsf{r}(g) = \mathsf{w}_j(\mathcal{L}; g, x) \cdot \mathsf{w}_i^{-1}(\mathcal{L}; x)(w_i(p)g),$$
$$\mathsf{u}(g) = \mathsf{r}(g)\,|\mathsf{A}_d(g,gx)|^2 \ \text{ if } wg \text{ is near at } j, \qquad \mathsf{u}(g) = 0 \ \text{ otherwise.}$$

Step 1. For all g: $h(g)^2 \le \mathsf{u}(w^{-1}g) \cdot \mathsf{Sq}_i(\mathcal{L}; k_1, x)(g) \cdot \mathsf{w}_j^{-1}(\mathcal{L}; x)(w^{-1}g)$. Indeed, if $h(g) \neq 0$ then g and $w^{-1}g$ are near at j; $w_i(p)w^{-1} = w_i^\circ(p_i)^{-1} \in \mathcal{H}_i(T)$, so $\mathsf{w}_i^{-1}(\mathcal{L}; x)(w_i(p)w^{-1}g) = \mathsf{w}_i(\mathcal{L}; g, x)^{-1}$ ((T8)); and $\mathsf{Sq}_i(\mathcal{L}; k_1, x)(g) = \mathsf{w}_i(\mathcal{L}; g, x)|\mathsf{A}_1(g,gx)|^2$. The weights cancel and equality holds.

Step 2. By Cauchy–Schwarz,

$$\Big(\int h\Big)^2 \le \int \mathsf{u}(w^{-1}g)\,\mathrm{d}\mu^{\mathcal{L}}_{0,j}(g) \cdot \int \mathsf{Sq}_i(\mathcal{L}; k_1, x)(g)\,\mathsf{w}_j^{-1}(\mathcal{L}; x)(w_j(\bar{p})g)\,\mathrm{d}\mu^{\mathcal{L}}_{0,j}(g).$$

The weighted square of floor i is $\mathcal{Q}_j(T)$-invariant and zero off the points near at j ((T7)(c) for the kernel at i); so by (T1), (T11) (head $\bar{p}$, $b = j$) and (T7)(f) the second factor is at most

$$\prod_{a<j} \mathsf{d}_a(\mathcal{L}; p_a^{-1}) \cdot C_{\mathrm{gl}}(\Gamma) \cdot \int \mathsf{Sq}_i(\mathcal{L}; k_1, x)(g)\,\mathrm{d}(\theta_i(\mathcal{L}) * \cdots * \theta_0(\mathcal{L}))(g)$$
$$\le \prod_{a<j} \mathsf{d}_a(\mathcal{L}; p_a^{-1}) \cdot C_{\mathrm{gl}}(\Gamma) \cdot \mathrm{SN}_0(\mathcal{L}; k_1, x).$$

Step 3. The function u is $\mathcal{H}_j(T)$-invariant ((T7)(b), (T8)), so by (T2) and (T1), with $\rho_0 = \theta_i(\mathcal{L}) * \cdots * \theta_0(\mathcal{L})$ and $\eta_{<j} = \eta_i * \eta_{<i}$,

$$\int \mathsf{u}(w^{-1}g)\,\mathrm{d}\mu^{\mathcal{L}}_{0,j}(g) = \int\int \mathsf{u}(\nu g)\,\mathrm{d}\eta_i(\nu)\,\mathrm{d}(\eta_{<i} * \rho_0)(g).$$

Step 4: Plancherel on $\mathcal{N}_i(T)$. Fix g. We apply (T13) with $a = i$, the kernel k_d (a pairing kernel at $i+1$), $x_1 = g$, $y_1 = gx$, $\varepsilon = 1$, $C = \mathsf{r}(g)$, $\Phi(\nu) = \mathsf{u}(\nu g)$, the unit $-\xi p_i$ in the place of ξ and $u = p_i^{-1}$, so that $\bmod_A(\xi u)$ there is $\bmod_A(\xi)$. Its three hypotheses hold. First, an element $s \in \mathcal{N}_i(T)$ has only one non-zero entry at the places (f_a, f_{a+1}), the one with $a = i$; the diagonal matrix d has the entry p_i/q_i at e_i and 1 at f_{i+1}; hence $g_{\mathcal{L}}^{-1}(\mathrm{sd}(dsd^{-1}) - \mathrm{sd}(s)) = g_{\mathcal{L}}^{-1}(p_i/q_i - 1)\,\mathrm{sd}(s) = -\xi p_i\,\mathrm{sd}(s)$.

Second, $\mathsf{r}(\nu g) = \mathsf{r}(g)$ for $\nu \in \mathcal{N}_i(T)$, because $\nu \in \mathcal{Q}_j(T)$ and ν commutes with $w_i(p)$ and lies in $\mathcal{Q}_i(T)$ ((T8)); so $\mathsf{u}(\nu g) \le \mathsf{r}(g)|\mathsf{A}_d(\nu g, \nu g x)|^2$. Let S be the set of the $\nu \in \mathcal{N}_i(T)$ with $\mathsf{u}(\nu g) \neq 0$. For $\nu \in S$ the points $w\nu g$ and νg are near at j, so by (T14)(b) there is $\tau_\nu \in \mathcal{H}_j(T)$ with $\mathrm{sz}(\tau_\nu \nu g) \le 99n^2$, and by (T14)(c) $p_i \in \Gamma_{10n}$. Third, for $m \in S$ take $t = \tau_m$ and $q = m$; for $\nu \in S$ take $\nu' = 1$: since τ_ν fixes e_i^*, the column at e_i^* of $tm(t\nu)^{-1} - 1$ is that of $(\tau_m m g)(\tau_\nu \nu g)^{-1} - 1$, of size at most $198n^2 + 6$, and multiplied by $u^{-1} = p_i$ it has size at most $198n^2 + 10n + 6$. The conclusion of (T13), with (T7)(h) at the point g, reads

$$\begin{aligned}\int \mathsf{u}(\nu g)\,\mathrm{d}\eta_i(\nu) &\le \mathsf{r}(g)\,K\,\mathsf{d}_i(\mathcal{L}; p_i^{-1})\,\Lambda(\Gamma)^N \frac{\mathsf{Sm}_j(\mathcal{L}; k_d, x)(g)}{\mathsf{w}_j(\mathcal{L}; g, x)} \\ &= K\,\mathsf{d}_i(\mathcal{L}; p_i^{-1})\,\Lambda(\Gamma)^N\,\mathsf{Sm}_j(\mathcal{L}; k_d, x)(g)\,\mathsf{w}_i^{-1}(\mathcal{L}; x)(w_i(p) g),\end{aligned}$$

with $N = 100n(297n^2 + 10n + 37) \le 10^6 n^3$.

Step 5. Integrate over g against $\eta_{<i} * \rho_0$ and use (T11) with the head p of length i, $b = j$ and $\Phi = \mathsf{Sm}_j(\mathcal{L}; k_d, x)(\cdot)$:

$$\int \mathsf{u}(w^{-1} g)\,\mathrm{d}\mu^{\mathcal{L}}_{0,j}(g) \le K\,C_{\mathrm{gl}}(\Gamma) \prod_{a<j} \mathsf{d}_a(\mathcal{L}; p_a^{-1}) \cdot C_{\mathrm{gl}}(\Gamma) \cdot \mathrm{SN}_j(\mathcal{L}; k_d, x).$$

With Step 2 this is the claim. □

6 Descent

Lemma 6.1 (a diagonal entry of the frame Gram matrix)**.** *Let A be a commutative ring with involution, $\mathfrak{s}$ a shape, T an $\mathfrak{s}$-form over A, b a natural number with $b \le \varrho$, and $x \in \mathcal{Q}_{b+1}(T) \cap \mathcal{H}_b(T)$. Then for every $y \in G(T)$*

$$Z_T(xy)_{bb} = Z_T(y)_{bb} \qquad \text{and} \qquad Z_T(yx)_{bb} = Z_T(y)_{bb}.$$

Proof. The group $\mathcal{Q}_{b+1}(T) \cap \mathcal{H}_b(T)$ is the stabiliser of f_b in $\mathcal{H}_b(T)$, so $x f_b = f_b = x^{-1} f_b$. Hence $h_T(f_b, yx f_b) = h_T(f_b, y f_b)$ and $h_T(f_b, xy f_b) = h_T(x^{-1} f_b, y f_b) = h_T(f_b, y f_b)$. □

LEAN `frameGram_diag_mul`

Lemma 6.2 (descent on one floor)**.** *In Setting 1.2, let $p, q \colon \mathbb{N} \to A^\times$, let b be a natural number with $b \le \varrho$, ζ an s-finite measure on $G(T)$, $v \colon A \to G(T)$ and $\rho \colon A \to [0, \infty]$ measurable maps, $\Psi \colon A \to \mathbb{C}$, $t \in \mathbb{C}$ and $\eta \in A$. Suppose that $\int \mathcal{I}_b^{\mathrm{maj}}(p, q; W, W, x)\,\mathrm{d}\mathsf{M}(\mathcal{L})(x) < \infty$; that ζ is carried by $\mathcal{Q}_{b+1}(T) \cap \mathcal{H}_b(T)$; that $\rho(c) < \infty$ for almost every c; that $\theta_b(\mathcal{L}) = \varpi * \zeta$, where ϖ is the image of $\rho(c)\,\mathrm{d}c$ under $c \mapsto v(c)^{-1}$; and that for almost every $c \in A$*

$$\begin{aligned}\Psi(c) = t \cdot \mathsf{h}_b(\mathcal{L}; p, q)\,\rho(c) \cdot \psi_A(\eta\, Z_T(v(c))_{bb}) \\ \cdot \iint \mathsf{I}^1_{b+1}(W; x, w_b(q)^{-1} w_b(p), w_b(p) v(c))(g)\,\mathrm{d}\mu_b(g)\,\mathrm{d}\mathsf{M}(\mathcal{L})(x),\end{aligned}$$

with $\mu_b = (\zeta * \theta_b(\mathcal{L})) * \mu^{\mathcal{L}}_{0,b}$. Then Ψ is integrable and

$$\int_A \Psi(c)\,\mathrm{d}c = t \cdot \mathcal{I}_b(p, q; W, \omega_\eta), \qquad \omega_\eta(y) = \psi_A(\eta\, Z_T(y)_{bb}).$$

Proof. Write $d = w_b(q)^{-1} w_b(p)$, $w = w_b(p)$, and $\mathsf{I}(x, w', g) = \mathsf{I}^1_{b+1}(W; x, d, w')(g)$, which by definition is $\mathsf{Av}_{b+1}(\mathcal{L}; k_1)(g, gx)\,\overline{\mathsf{Av}_{b+1}(\mathcal{L}; k_d)(w'^{-1}g, w'^{-1}gx)}$. Both kernels are pairing kernels at b (Lemma 4.1). *Claim:* the function $(x, g, \sigma, \tau) \mapsto \omega_\eta(\sigma\tau^{-1})\mathsf{I}(x, \sigma w \tau^{-1}, \sigma g)$ under $\mathsf{M}(\mathcal{L}) \times \mu^{\mathcal{L}}_{0,b} \times \theta_b(\mathcal{L}) \times \theta_b(\mathcal{L})$ and the function $(x, g, c) \mapsto \omega_\eta(v(c))\mathsf{I}(x, wv(c), g)$ under $\mathsf{M}(\mathcal{L}) \times \mu_b \times \rho(c)\mathrm{d}c$ have the same distribution. To see it, fix x, g, $\sigma \in \mathcal{H}_b(T)$ and a test function. In the integral over τ replace τ by $\tau\sigma$, by (T4) with $S = \mathcal{H}_b(T)$: this is allowed because $\tau \mapsto \omega_\eta(\tau^{-1})\mathsf{I}(x, w\tau^{-1}, \sigma g)$ is invariant under $\mathcal{Q}_{b+1}(T) \cap \mathcal{H}_b(T)$, by Lemma 6.1 and (T7)(b), and because σ commutes with w. Now write $\tau = v(c)^{-1}\nu$ with c distributed by $\rho(c)\mathrm{d}c$ and ν by ζ. Then $\omega_\eta(\tau^{-1}) = \omega_\eta(\nu^{-1}v(c)) = \omega_\eta(v(c))$ by Lemma 6.1, and $\mathsf{I}(x, w\nu^{-1}v(c), \sigma g) = \mathsf{I}(x, wv(c), \nu\sigma g)$, since ν commutes with w and $\mathsf{Av}_{b+1}(\mathcal{L}; k_1)(\nu\sigma g, \nu\sigma g x) = \mathsf{Av}_{b+1}(\mathcal{L}; k_1)(\sigma g, \sigma g x)$ ((T7)(b)). The law of $\nu\sigma g$ is μ_b. This proves the claim.

The first function is integrable: as in Lemma 4.3(ii), and because $|\omega_\eta| = 1$, its absolute value integrates to at most the integral of the majorant $\mathcal{J}^{\mathrm{maj}}_b(p, q; W, W, x)$ over x, which is finite. So the second is integrable, and Fubini's theorem gives that Ψ is integrable with integral $t\,\mathsf{h}_b(\mathcal{L}; p, q)$ times the integral of the first function, which is the floor form. □

LEAN `Level.integral_succ_floor_eq`

Proof of Theorem B. Apply Lemma 6.2 with $b = a < \varrho$, $\zeta = \eta_a$, which is carried by $\mathcal{N}_a(T) \subseteq \mathcal{Q}_{a+1}(T) \cap \mathcal{H}_a(T)$, with $v(c) = w^\circ_a(c)$ (for units c; almost every c is a unit) and $\rho(c) = \mathrm{mod}_A(c)^{k+1+2(\varrho-a-1)}$; then $\theta_a(\mathcal{L}) = \varpi_a * \eta_a$ is the required decomposition. It remains to identify $f(c)$ with $\Psi(c)$ for a unit c. Put $p' = p[a \mapsto c]$, $q' = q[a \mapsto c]$. By Lemma 4.2,

$$\mathcal{J}_{a+1}(p', q'; W, 1) = \mathsf{h}_{a+1}(\mathcal{L}; p', q') \cdot \iint \mathsf{I}^1_{a+1}(W; x, d', w_{a+1}(p'))(g)\,\mathrm{d}\mu^{\mathcal{L}}_{0,a+1}(g)\,\mathrm{d}\mathsf{M}(\mathcal{L})(x),$$

with $d' = w_{a+1}(q')^{-1} w_{a+1}(p')$. Here $w_{a+1}(p') = w^\circ_a(c) w_a(p) = w_a(p) w^\circ_a(c)$, so $d' = w_a(q)^{-1} w_a(p)$; $\mu^{\mathcal{L}}_{0,a+1} = (\eta_a * \theta_a(\mathcal{L})) * \mu^{\mathcal{L}}_{0,a}$; $\mathsf{h}_{a+1}(\mathcal{L}; p', q') = \mathsf{h}_a(\mathcal{L}; p, q)\,\rho(c)$; and $Z_T(w^\circ_a(c))_{aa} = c$. □

7 Integration over positions

Proof of Theorem C. Write $\mathsf{A}_l = \mathsf{Av}_l(\mathcal{L}; k)(\cdot, \cdot)$, $P = \mathrm{Pos}(\mathcal{L}; \Gamma)$, $\rho^{(l)} = \theta_l(\mathcal{L}) * \cdots * \theta_0(\mathcal{L})$, and for $l \le n$

$$\mathsf{W}_l(g, y) = \eta_{<l}\{\nu : g^{-1}\nu y \in P\},$$
$$\Phi_l(g, y) = |\mathsf{A}_{l+1}(g, y)|^2 \ \text{ if } \mathsf{W}_l(g, y) > 0, \qquad \Phi_l(g, y) = 0 \ \text{ otherwise.}$$

We prove, by decreasing induction on l from n to j, that there is an s-finite measure ν_l with

$$l\,c + \int_P \mathrm{SN}_l(\mathcal{L}; k, x)\,\mathrm{d}\mathsf{M}(\mathcal{L})(x) \le n\,c, \qquad \mathsf{M}(\mathcal{L}) = \eta_{<l} * \nu_l,$$
$$\prod_{a < \min(l, \varrho)} C_a(\Gamma) \cdot \iint \Phi_l(g, y)\,\mathrm{d}\nu_l(y)\,\mathrm{d}\rho^{(l)}(g) \le c. \tag{2}$$

The first inequality at $l = j$ is the theorem. For $l = n$ take $\nu_n = \mathsf{M}_{\ge 0}(\mathcal{L}) = \rho^{(n)}$: the self-norm at n is 0; $\mathsf{M}(\mathcal{L}) = \eta_{<n} * \mathsf{M}_{\ge 0}(\mathcal{L})$ by (T1); and $\Phi_n \le |k|^2$ because

$\mathsf{M}_{\geq n+1}(\mathcal{L}) = \delta_1$. Let $j \leq l \leq \varrho$ and suppose (2) for $l+1$. Put $\nu_l = \eta_l * \nu_{l+1}$; then $\mathsf{M}(\mathcal{L}) = \eta_{<l} * \nu_l$.

The third part of (2) *at* l. As $\rho^{(l+1)} = \theta_{l+1}(\mathcal{L}) * \rho^{(l)}$, the induction hypothesis bounds $\prod_{a \leq l,\, a<\varrho} C_a(\Gamma)$ times the integral of $\int \Phi_{l+1}(\sigma g, y)\,\mathrm{d}\nu_{l+1}(y)$ over g and $\sigma \in \mathcal{H}_{l+1}(T)$. For such σ, (T3) applies to $\mathsf{M}(\mathcal{L}) = \eta_{<l+1} * \nu_{l+1}$ and $S = \mathcal{N}_{<l+1}(T)$ (by (T1)), the function $y \mapsto \Phi_{l+1}(\sigma g, y)$ being $\mathcal{N}_{<l+1}(T)$-invariant by the law of the kernel and the invariance of $\eta_{<l+1}$; so the inner integral equals $\int \Phi_{l+1}(\sigma g, \sigma y)\,\mathrm{d}\nu_{l+1}(y)$. It therefore suffices to show, for all g, y,

$$\int \Phi_l(g, \nu y)\,\mathrm{d}\eta_l(\nu) \leq C \int \Phi_{l+1}(\sigma g, \sigma y)\,\mathrm{d}\theta_{l+1}(\mathcal{L})(\sigma), \qquad C = C_l(\Gamma) \ \text{ if } l < \varrho, \qquad C = 1 \ \text{ if } l = \varrho. \tag{3}$$

If $\mathsf{W}_{l+1}(g, y) = 0$ the left side is 0, because $\int \mathsf{W}_l(g, \nu y)\mathrm{d}\eta_l(\nu) = \mathsf{W}_{l+1}(g, y)$. Otherwise $\mathsf{W}_{l+1}(\sigma g, \sigma y) = \mathsf{W}_{l+1}(g, y) > 0$ for $\sigma \in \mathcal{H}_{l+1}(T)$ ((T1)), so the right side is $C \int |\mathsf{A}_{l+2}(\sigma g, \sigma y)|^2 \mathrm{d}\theta_{l+1}(\mathcal{L})(\sigma)$. For $l = \varrho$, $\eta_\varrho = \delta_1$, and (3) follows from $|\mathsf{A}_n| \leq \mathsf{Pv}_n(\mathcal{L}; k)(\cdot, \cdot)$ and Cauchy–Schwarz on floor n ((T7)(a),(e) with $u = 1$, $K = 1$; the mass bound is the third requirement on $\theta_n(\mathcal{L})$ in the definition of a level). For $l < \varrho$ we apply (T13) with $a = l$, the unit $g_{\mathcal{L}}^{-1}$ in the place of ξ, $u = g_{\mathcal{L}}$, $\varepsilon = 0$, $C = 1$, $x_1 = g$, $y_1 = y$, $\Phi(\nu) = \Phi_l(g, \nu y)$ and S the set of the $\nu \in \mathcal{N}_l(T)$ with $\mathsf{W}_l(g, \nu y) > 0$ and g near at $l+1$. Its hypotheses: d commutes with $\mathcal{N}_l(T) \subseteq \mathcal{H}_j(T)$ and $\mathrm{sd}(1) = 0$; if $\Phi_l(g, \nu y) \neq 0$ then $\nu \in S$, by the last clause of the definition of a pairing kernel; and for $m \in S$ choose $\xi' \in \mathcal{Q}_{l+1}(T)$ with $\xi' g \in \mathrm{Box}(\mathcal{L})$ and write $\xi' = t\nu_0$ with $t \in \mathcal{H}_l(T)$, $\nu_0 \in \mathcal{N}_{<l}(T)$; then $t \in \mathcal{Q}_{l+1}(T) \cap \mathcal{H}_l(T)$, $q = \nu_0$ and $\mathrm{sz}(tqg) \leq 1$ for Γ. For $\nu \in S$ choose $\nu_1, \nu_2 \in \mathcal{N}_{<l}(T)$ with $g^{-1}\nu_1 m y$ and $g^{-1}\nu_2 \nu y$ in P; by (T14)(d) the element $(\xi' g)(g^{-1}\nu_1 m y)\big((\xi' g)(g^{-1}\nu_2 \nu y)\big)^{-1} = \xi' \nu_1 m \nu^{-1} \nu_2^{-1} \xi'^{-1}$ has $g_{\mathcal{L}}^{-1}(\,\cdot - 1)$ of size at most 14, and it equals $\nu'\, tm(t\nu)^{-1}$ with $\nu' \in \mathcal{N}_{<l}(T)$, because $\mathcal{H}_l(T)$ normalises $\mathcal{N}_{<l}(T)$. So $i_1 = 1$, $i_2 = 14$, and the constant of (T13) is $c_A^{|\Omega|} \mathsf{d}_l(\mathcal{L}; g_{\mathcal{L}}) \Lambda(\Gamma)^{4600n} = C_l(\Gamma)$.

The first part of (2) *at* l. By (T7)(f) and by the third part at l, just proved, it suffices to show

$$\int_P \int \mathsf{Sq}_l(\mathcal{L}; k, x)(g)\,\mathrm{d}\rho^{(l)}(g)\,\mathrm{d}\mathsf{M}(\mathcal{L})(x) \leq \prod_{a<l} C_a(\Gamma) \iint \Phi_l(g, y)\,\mathrm{d}\nu_l(y)\,\mathrm{d}\rho^{(l)}(g).$$

Fix g. Since $\mathsf{M}(\mathcal{L})$ is invariant under left translation,

$$\int_P \mathsf{w}_l(\mathcal{L}; g, x) |\mathsf{A}_{l+1}(g, gx)|^2 \mathrm{d}\mathsf{M}(\mathcal{L})(x) = \iint \mathbf{1}_P(g^{-1}\nu y)\, \mathsf{w}_l(\mathcal{L}; g, g^{-1}\nu y)\, |\mathsf{A}_{l+1}(g, \nu y)|^2\, \mathrm{d}\eta_{<l}(\nu)\,\mathrm{d}\nu_l(y).$$

By the law of the kernel, $|\mathsf{A}_{l+1}(g, \nu y)| = |\mathsf{A}_{l+1}(g, y)|$ for $\nu \in \mathcal{N}_{<l}(T)$. If this is not 0, then g is near at $l+1$, hence at l, and (T10) bounds the integral over ν by $\prod_{a<l} C_a(\Gamma)$; and the integral over ν vanishes unless $\mathsf{W}_l(g, y) > 0$. So the inner integral is at most $\prod_{a<l} C_a(\Gamma) \cdot \Phi_l(g, y)$. □

Proof of Theorem D. Both kernels k_1, k_d ($d = w_j(q)^{-1}w_j(p)$) are pairing kernels (Lemma 4.1), at 0 and at j. On S, $f \le e \cdot \mathrm{SNP}_j(\Gamma; p, q; W, W', x)^{1/2}$. By Cauchy–Schwarz and Theorem C for the gauge Γ', with $D = \prod_{l<\varrho} C_l(\Gamma')$,

$$\Big(\int_{\mathrm{Pos}(\mathcal{L};\Gamma')} (\mathrm{SN}_0(\mathcal{L}; k_1, x)\ \mathrm{SN}_j(\mathcal{L}; k_d, x))^{1/2} \mathrm{d}\mathsf{M}(\mathcal{L})(x) \Big)^2 \le n^2 D^2\ \mathrm{PN}(W, W) \cdot \iint |k_d|^2\, \mathrm{d}\mathsf{M}_{\ge 0}(\mathcal{L})\, \mathrm{d}\mathsf{M}_{\ge 0}(\mathcal{L}),$$

and by (T5) the last double integral is $\prod_{a<j} \mathrm{mod}_A(p_a/q_a)^{k+2(\varrho-a)} \cdot \mathrm{PN}(W, W')$. Collect the powers of the modules. With $N_a = k - 1 + 2(\varrho - a)$,

$$\mathsf{h}_j(\mathcal{L}; p, q)^2 = \prod_{a<j} \mathrm{mod}_A(p_a q_a)^{N_a}, \qquad \Big(\prod_{a<j} \mathsf{d}_a(\mathcal{L}; p_a^{-1})\Big)^2 = \prod_{a<j} \mathrm{mod}_A(p_a)^{-2N_a-1},$$

and the factor from (T5) is $\prod_{a<j} \mathrm{mod}_A(p_a/q_a)^{N_a+1}$; so in the product the exponent of $\mathrm{mod}_A(p_a)$ is 0 and that of $\mathrm{mod}_A(q_a)$ is -1. As q is a head, $(q_0 \cdots q_{j-1})^{-1} \in \Gamma_1 \subseteq \Gamma_2$ and its module is at most $\Lambda(\Gamma)^2$. Altogether the square of the left side of the theorem is at most

$$e^2\, \Lambda(\Gamma)^2\, C_{\mathrm{gl}}(\Gamma)^3\, n^2 D^2\, \Theta_0^2, \qquad D^2 = \Big(\prod_{l<\varrho} \mathsf{d}_l(\mathcal{L}; g_{\mathcal{L}})\Big)^2 c_A^{2|\Omega|\varrho} \Lambda(\Gamma')^{9200n\varrho},$$

and $n^2 \le 4^n \le \Lambda(\Gamma')^{2n}$, $9200n\varrho + 2n \le 10^5 n^2$. This is the square of the right side. □

8 The complex place

Setting 8.1 (cut Bessel data)**.** Let $\mathfrak{s}$ be a shape, I a finite set, $\Phi, \Phi_0 \colon G_1(\mathfrak{s}) \to \mathbb{C}^I$ continuous, rep a homomorphism from $\mathrm{U}(\kappa)$ to the unitary group of $\mathbb{C}^I$, ρ, b real numbers, s_0, n natural numbers, and $\mathfrak{c}$ cut Bessel data, $\mathrm{CB}(\Phi, \mathrm{rep}, \rho, s_0, b, \Phi_0, \mathsf{n})$, with underlying sizes $\mathfrak{h}$. Floor forms, moduli, majorants, nearness and the predicate Cut refer to the level $\mathcal{L}_{\mathbb{C}}(\mathfrak{h})$.

Lemma 8.2 (the cut is cut)**.** *In Setting 8.1, Φ is a cut Bessel function for the level $\mathcal{L}_{\mathbb{C}}(\mathfrak{h})$: $\Phi \in \mathrm{Cut}$.*

Proof. Φ is measurable, bounded and satisfies the law of frequency $1 = g^{-1}_{\mathcal{L}_{\mathbb{C}}(\mathfrak{h})}$, as a member of a cut pair. If $\Phi(y) \ne 0$ the Bessel size of y is at most $b^{6n+2} = \rho$, so $\|uy\| \le \rho$ for some $u \in \mathcal{U}(T)$ by (C4), and uy lies in the box. □

LEAN `IsCutBessel.isCut_complexLevel`

Lemma 8.3 (a gauge that makes two sequences heads)**.** *In Setting 8.1, let j and K be natural numbers and $p, q \colon \mathbb{N} \to \mathbb{C}^\times$. Then there is a real R_0 with $\rho \le R_0$ such that p and q are heads of length j for the gauge $\Gamma(\mathfrak{h}; \rho \le R_0)$, $p \in \mathrm{Hd}_j$ and $q \in \mathrm{Hd}_j$, and such that for every real $R \ge \rho$ with $|p_0 \cdots p_{l-1}| \le R$ and $|q_0 \cdots q_{l-1}| \le R$ for all $l \le j$,*

$$R_0^K \le R^{(K+6)(2j+1)} \cdot \mathrm{hw}_{K,j}(p)\ \mathrm{hw}_{K,j}(q).$$

Proof. For this gauge, being a head of length j means $|p_0 \cdots p_{l-1}|^{-1} \le R_0$ for $l \le j$. Induction on j. For $j = 0$ take $R_0 = \rho$; then $\rho^K \le R^{K+6}$ as $R \ge \rho \ge 1$. Given R_0 for j, take $R_0' = \max(R_0, |p_0 \cdots p_j|^{-1}, |q_0 \cdots q_j|^{-1})$ for $j+1$. The inequality follows

by two uses of the elementary fact: if $L^K \le B$, $L, R \ge 1$ and $0 < |y| \le R$, then $\max(L, |y|^{-1})^K \le B\,R^{K+6}\,\ell_K(|y|^{-1})$; indeed $R^{K+6}\ell_K(|y|^{-1}) \ge 1$, and if $|y|^{-1} > L \ge 1$ then $\ell_K(|y|^{-1}) = |y|^{-K}$. □

LEAN `IsCutBessel.exists_gauge_isHead`

Lemma 8.4 (summing over shells)**.** *In Setting 8.1, let I be non-empty, let j be a natural number with $j \le \varrho$, let $W\colon G_1(\mathfrak{s}) \to \mathbb{C}^I$ satisfy $W \in \mathrm{Bes}(1)$, and let ε be a real number with $\varepsilon \ge 0$ and $\int |W|^2 \mathrm{d}\dot{y} \le \varepsilon^2 |I|$. Let R be real with $R \ge \rho$, let $p, q\colon \mathbb{N} \to \mathbb{C}^\times$ be heads of length j for the gauge $\Gamma(\mathfrak{h}; \rho \le R)$, $p \in \mathrm{Hd}_j$ and $q \in \mathrm{Hd}_j$, and suppose that $W(y) \ne 0$ implies $y \in \mathrm{Near}_j(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}))$. Let t be real with $t \ge 1$ and $f\colon G_1(\mathfrak{s}) \to [0, \infty]$ with $f \le \mathcal{J}_j^{\mathrm{maj}}(p, q; \Phi, W, \cdot)$, and suppose that, if $t > 1$, then for all x*

$$f(x)^2 \le \Big(\frac{(\pi\rho^{24})^k}{t^{s_0}}\Big)^2 \,\mathrm{SNP}_j(\Gamma(\mathfrak{h}; \rho \le R); p, q; \Phi, W, x).$$

Then

$$\big|c_{\mathrm{f}}(\mathfrak{c})\big| \cdot \mathsf{h}_j(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}); p, q) \int f \,\mathrm{d}\mathsf{M}(\mathcal{L}_{\mathbb{C}}(\mathfrak{h})) \;\le\; \Theta(\mathfrak{s})\; R^{10^9 n^4}\, \rho^{24k}\, \frac{t^{9900000 n^3}}{t^{s_0}}\, \varepsilon.$$

Proof. Write $\mathcal{L} = \mathcal{L}_{\mathbb{C}}(\mathfrak{h})$, $\Gamma = \Gamma(\mathfrak{h}; \rho \le R)$, $\lambda = \lambda(\mathfrak{h})$. For $i \in \mathbb{N}$ put $T_i = t\,R^{11ni}$, let $\Gamma^{(i)}$ be the gauge of radius T_{i+1}^{99n} ((C3)) and $P_i = \mathrm{Pos}(\mathcal{L}; \Gamma^{(i)})$, the set of the x with all entries of $x - 1$ of absolute value at most T_{i+1}^{99n}. The P_i increase and cover $G_1(\mathfrak{s})$; let $D_i = P_i \smallsetminus \bigcup_{i' < i} P_{i'}$. We apply Theorem D on each D_i, with $\Gamma' = \Gamma^{(i)}$, $\Theta_0 = \varepsilon(\lambda|I|)^2$ and $e_i = (\pi\rho^{24})^k \rho^{9999n} T_i^{-s_0}$; its setting holds by Lemma 8.2 and (C2).

The pairing norms. By the Cauchy–Schwarz inequality in $\mathbb{C}^I$ and by (C2), as $|\Phi|^2$, $|W|^2$ are $\mathcal{U}(T)$-invariant, $\mathrm{PN}(\Phi, W) \le \lambda \int |\Phi|^2 \mathrm{d}\dot{y} \cdot \lambda \int |W|^2 \mathrm{d}\dot{y} \le \lambda^2 |I|^2 \varepsilon^2$, and likewise $\mathrm{PN}(\Phi, \Phi) \le \lambda^2 |I|^2$. So their product is at most Θ_0^2.

The pointwise bound on D_i. If $i = 0$ and $t > 1$ it is the hypothesis, as $T_0 = t$. Otherwise we use Theorem A(a), for the gauge Γ, with $\sigma_0 = \mathrm{vol}\{c \in \mathbb{C} : |c| < T_i^{-1}\} = \pi T_i^{-2}$ and

$$\mathsf{V} = \rho^{2574n}\,(\pi\rho^{22})^{k-1},$$

which is the bound of (T9)(a) for the gauge $\Gamma(\mathfrak{h}; \rho \le \rho)$ of radius ρ (constant ρ^2, eleventh set of volume $\pi\rho^{22}$), not for Γ. This is legitimate: Theorem A(a) asks only for some $\mathsf{V} \ge 1$ with $\mathrm{Mass}_j(\mathcal{L}; 1) \le \mathsf{V}$, and a bound for the mass of a floor does not depend on a gauge. (The formal text takes this gauge through the default value of the last argument of the gauge, Input (C3). With the gauge Γ of radius R the constant would contain R, and the comparison below would fail for large R.) For $i = 0$, $t = 1$ we have $\sigma_0 = \pi \ge 1$ and nothing to check. For $i \ge 1$ let $x \in D_i$, so $x \notin P_{i-1}$, and let y be as in the hypothesis of Theorem A(a). If no a with $j \le a < \varrho$ had $\mathrm{vol}(\mathrm{Val}_a(\mathcal{L}; y, x)) \le \sigma_0$, each of these open sets would contain a c with $|c| \ge T_i^{-1}$, a unit whose inverse has size 1 for the gauge of radius T_i; then (T14)(e), with that gauge in the place of Γ and $\Gamma_S = \Gamma$, would give $|(x-1)_{\alpha\beta}| \le T_i^{99n}$ for all entries, that is $x \in P_{i-1}$. (Its other hypotheses: $R^{10n} \le T_i$; and $\Phi(y) \ne 0 \ne \Phi(yx)$ puts uy, $v(yx)$ in the box for some $u, v \in \mathcal{U}(T)$, with entries of absolute value at most $\rho \le T_i$.) Hence $f(x)^2 \le \sigma_0^{s_0} \mathsf{V}^2\ \mathrm{SNP}_j(\Gamma; p, q; \Phi, W, x)$ (recall $\mathsf{s}(\mathcal{L}) = s_0$), and $\sigma_0^{s_0}\mathsf{V}^2 \le e_i^2$, because

$$\frac{e_i^2}{\sigma_0^{s_0}\mathsf{V}^2} = \frac{\pi^{2k}\rho^{48k+19998n}\,T_i^{-2s_0}}{\pi^{s_0}T_i^{-2s_0} \cdot \pi^{2k-2}\rho^{5148n+44(k-1)}} = \pi^{2-s_0}\,\rho^{4k+44+14850n} \;\ge 1$$

($\rho \ge 4 > \pi$ and $s_0 < k$). This is where the factor ρ^{9999n} of e_i is used.

The sum. The sets D_i are Borel (differences of closed sets), disjoint, and cover $G_1(\mathfrak{s})$; so $\int f \,\mathrm{d}\mathsf{M}(\mathcal{L}) \le \sum_i \int_{D_i} f \,\mathrm{d}\mathsf{M}(\mathcal{L})$, also when f is not measurable (Section 10, (6)). Theorem D bounds $\mathsf{h}_j(\mathcal{L}; p, q) \int_{D_i} f \,\mathrm{d}\mathsf{M}(\mathcal{L})$ by

$$e_i \, \pi^{2(k-1)\varrho} \, R^{3000000n^3+2} \, T_{i+1}^{9900000n^3} \, \Theta_0$$

(the floor modules at $g_{\mathcal{L}} = 1$ are 1, $c_{\mathbb{C}} = \pi^2$, $|\Omega| = k-1$). Since $s_0 > 10^9 n^3$, the factor $T_i^{-s_0}$ in e_i beats $T_{i+1}^{9900000n^3}$, and the sum over i is at most the stated bound times $\sum_i 2^{-(i+1)} = 1$ divided by the number $|c_{\mathrm{f}}(\mathfrak{c})|(\lambda|I|)^2$, which is $\operatorname{vol}\mathrm{U}(k)/\operatorname{vol}\mathrm{U}(r)$. This last comparison of exponents is carried out by the formal text and was not redone by hand. □

LEAN `IsCutBessel.formConst_mul_lintegral_le`

Lemma 8.5 (the majorant is integrable)**.** *In Setting 8.1, let I be non-empty, $j \le \varrho$ and $p, q \colon \mathbb{N} \to \mathbb{C}^\times$. Then $\int \mathcal{J}_j^{\mathrm{maj}}(p, q; \Phi, \Phi, x) \,\mathrm{d}\mathsf{M}(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}))(x) < \infty$.*

Proof. Take R_0 from Lemma 8.3 (with $K = 0$) and apply Lemma 8.4 with $W = \Phi$, $\varepsilon = 1$, $t = 1$ and f the majorant; $c_{\mathrm{f}}(\mathfrak{c})$ and the head density are not 0. □

LEAN `IsCutBessel.lintegral_formMajorant_ne_top`

Lemma 8.6 (the explicit bound for a form)**.** *In Setting 8.1, let I be non-empty, $j \le \varrho$, $p, q \colon \mathbb{N} \to \mathbb{C}^\times$, $y \in \mathbb{C}^\times$, and $\omega \colon G_1(\mathfrak{s}) \to \mathbb{C}$ with $|\omega(x)| \le 1$ for all x. Suppose that for every gauge Γ in $\mathrm{Gau}(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}))$ for which p is a head of length j, and every x,*

$$\mathcal{J}_j^{\mathrm{mod}}(p, q; \Phi, x, \omega)^2 \le \Big(\frac{\pi^2}{|y|^2}\Big)^{k-1} \mathrm{SNP}_j(\Gamma; p, q; \Phi, \Phi, x).$$

Then

$$\big|c_{\mathrm{f}}(\mathfrak{c}) \cdot \mathcal{J}_j(p, q; \Phi, \omega)\big| \le \Theta(\mathfrak{s}) \, b^{2000n^9} \, \ell_{k-n^7}(|y|^{-1}) \cdot \mathrm{hw}_{n^6,j}(p) \, \mathrm{hw}_{n^6,j}(q).$$

Proof. By Lemma 4.3(i) the left side is at most $|c_{\mathrm{f}}(\mathfrak{c})|$ times the head density times $\int \mathcal{J}_j^{\mathrm{mod}}(p, q; \Phi, x, \omega) \mathrm{d}\mathsf{M}(\mathcal{L})(x)$. Take R_0 from Lemma 8.3 with $K = n^6$. *Case 1:* some partial product of p or q of length at most j exceeds ρ^{9n} in absolute value. Then the form is 0: by Theorem A(a) with $W' = \Phi$, $\sigma_0 = 0$ and $\mathsf{V} = \infty$ the majorant vanishes, since no y' satisfies the conditions of that hypothesis (they would put all partial products into the set of size $9n$ of the gauge of radius ρ, by (T14)(c)); and $s_0 \ne 0$. *Case 2:* otherwise $R_0^{n^6} \le \rho^{9n(n^6+6)(2j+1)} \, \mathrm{hw}_{n^6,j}(p) \, \mathrm{hw}_{n^6,j}(q)$. Apply Lemma 8.4 with $W = \Phi$, $\varepsilon = 1$, $R = R_0$, $t = \max(1, \rho^{24}|y|)$ and f the modulus, which is at most the majorant (Lemma 4.3(ii)); if $t > 1$ the hypothesis gives $f^2 \le (\pi\rho^{24}/t)^{2(k-1)}$ times the self-norm product, and $(\pi\rho^{24}/t)^{k-1} \le (\pi\rho^{24})^k / t^{s_0}$ because $s_0 \le k-1$. The resulting bound $\Theta(\mathfrak{s}) R_0^{10^9 n^4} \rho^{24k} t^{9900000n^3 - s_0}$ is at most the right side: $10^9 n^4 \le n^6$, $\rho = b^{6n+2}$, $1/t \le \min(1, \rho^{-24}|y|^{-1})$ and $s_0 - 9900000n^3 \ge k - n^7 + 6$. This comparison of exponents is that of the formal text. □

LEAN `IsCutBessel.norm_floorForm_le`

Lemma 8.7 (size of the support)**.** *In Setting 1.2, let $p \colon \mathbb{N} \to A^\times$ be a head of length ϱ for Γ, $p \in \mathrm{Hd}_\varrho$, let $x \in \mathcal{H}_\varrho(T)$, and let $y \in G(T)$ with $W(y) \ne 0$ and $W((w_\varrho(p)x)^{-1}y) \ne 0$. Then $\mathrm{sz}(y) \le 60n^2$ for Γ.*

Proof. There is $u \in \mathcal{U}(T) = \mathcal{N}_{<\varrho}(T)$ with uy in the box; and $x^{-1}w_\varrho(p)^{-1}y$ is near at ϱ, hence so is $w_\varrho(p)^{-1}y$ ((T8)). Now use (T14)(a). □

LEAN `Gauge.hasMatSize_of_ne_zero` · `Gauge.isCompact_hasMatSize`

Lemma 8.8 (the inner product of two correlations)**.** *In Setting 1.2, let μ be a Haar measure μ of $G(T)$, suppose that W is continuous, let $p, q\colon \mathbb{N} \to A^\times$ be heads of length ϱ for Γ, $p \in \mathrm{Hd}_\varrho$ and $q \in \mathrm{Hd}_\varrho$, and let $x \in \mathcal{H}_\varrho(T)$. Then, with $\gamma_p = w_\varrho(p)x$ and $\gamma_q = w_\varrho(q)x$,*

$$\langle J^\mu_W(\gamma_p), J^\mu_W(\gamma_q)\rangle = \iint \langle W(y), W(y')\rangle\, \overline{\langle W(\gamma_p^{-1}y), W(\gamma_q^{-1}y')\rangle}\, \mathrm{d}\mu(y)\, \mathrm{d}\mu(y').$$

Proof. In coordinates the left side is

$$\sum_{i,i'} \overline{\int W_i(y)\overline{W_{i'}(\gamma_p^{-1}y)}\, \mathrm{d}\mu(y)} \cdot \int W_i(y')\overline{W_{i'}(\gamma_q^{-1}y')}\, \mathrm{d}\mu(y').$$

Each product of two integrals is a double integral, because the integrands are continuous and vanish outside the set of the y of size at most $60n^2$ (Lemma 8.7), which is compact: it is closed, and the Γ_i are compact. Summing over i, i' gives the integrand on the right. □

LEAN `Gauge.inner_besselCorrelation`

Lemma 8.9 (continuity of the top vector)**.** *In Setting 8.1, the map $p \mapsto \Xi(\Phi; p)$ is continuous on the space of the sequences $p\colon \mathbb{N} \to \mathbb{C}^\times$ with the product topology.*

Proof. The scalar factor is continuous. Fix p and a gauge radius R_0 for which p is a head of length ϱ (Lemma 8.3); the sequences that are heads for the radius R_0+1 form a neighbourhood of p. For them the integrand of each coordinate of $J^{\mathrm{d}y}_\Phi(w_\varrho(p)\chi_\varrho(p_\varrho))$ vanishes outside one compact set (Lemma 8.7, with $x = \chi_\varrho(p_\varrho) \in \mathcal{H}_\varrho(T)$), and it is continuous in (p, y) because the boost depends continuously on p_ϱ. An integral of a jointly continuous function over a fixed compact set is continuous in the parameter. □

LEAN `IsCutBessel.continuous_sieveVector`

Proof of Theorem E. Write $\mathcal{L} = \mathcal{L}_{\mathbb{C}}(\mathfrak{h})$, $\lambda = \lambda(\mathfrak{h})$, and let J be the family of the statement. Φ is cut for $\mathcal{L}$ (Lemma 8.2), $\mathsf{M}(\mathcal{L})$ is a Haar measure ((C2)), every point is a position, and $|\Omega| = k - 1$, $g_{\mathcal{L}} = 1$, $c_{\mathbb{C}}/\operatorname{mod}_A(y) = \pi^2/|y|^2$.

(S1) *for $i < \varrho$.* Theorem B with $t = c_{\mathrm{f}}(\mathfrak{c})$ and $\eta = \xi$, whose hypothesis is Lemma 8.5, shows that f is integrable with integral $c_{\mathrm{f}}(\mathfrak{c})\mathcal{J}_i(p, q; \Phi, \omega_\xi)$. For $\xi = 0$ this is $J_i(p, q)$. For $\xi \neq 0$, Lemma 8.6 with $y = \xi$ gives the bound; its hypothesis is Theorem A(b) with $\eta = \xi$, in every gauge for which p is a head.

(S1) *for $i = \varrho$.* We apply Lemma 6.2 with $b = \varrho$, $\zeta = \theta_n(\mathcal{L})$, carried by $\mathcal{H}_n(T) = \mathcal{Q}_n(T) \cap \mathcal{H}_\varrho(T)$, $v(c) = \chi_\varrho(c)$ and $\rho(c) = \frac{2\pi^k}{(k-1)!}\max(|c|^2 - 1, 0)^{k-1}$ ((C1)), and check the formula for $\Psi(c) = \psi_{\mathbb{C}}(\xi c)J_n(p', q')$, $p' = p[\varrho \mapsto c]$, $q' = q[\varrho \mapsto c]$. For $|c| \le 1$ both sides are 0. Let $|c| > 1$; then $Z_T(\chi_\varrho(c))_{\varrho\varrho} = c$. On the two top floors $\eta_\varrho = \eta_n = \delta_1$, so $(\theta_n(\mathcal{L}) * \theta_\varrho(\mathcal{L})) * \mu^{\mathcal{L}}_{0,\varrho} = \mathsf{M}(\mathcal{L}) = \lambda\, \mathrm{d}y$. The averaged kernels at n are averages over $\mathcal{H}_n(T)$ of $\langle \Phi(hg), \Phi(hg')\rangle = \langle \Phi(g), \Phi(g')\rangle$, as $\mathrm{rep}(h)$ is unitary; so, with $\gamma_{p'} = w_\varrho(p)\chi_\varrho(c)$ and $d\gamma_{p'}^{-1} = \gamma_{q'}^{-1}$,

$$\mathsf{I}^1_n(\Phi; x, d, \gamma_{p'})(g) = \langle \Phi(g), \Phi(gx)\rangle\, \overline{\langle \Phi(\gamma_{p'}^{-1}g), \Phi(\gamma_{q'}^{-1}gx)\rangle}.$$

Integrating over g and x and substituting $y' = gx$ gives $\lambda^2 \langle J_\Phi^{\mathrm{d}y}(\gamma_{p'}), J_\Phi^{\mathrm{d}y}(\gamma_{q'})\rangle$ by Lemma 8.8 (with a gauge from Lemma 8.3). The constants agree: $c_{\mathrm{f}}(\mathfrak{c})\lambda^2 \cdot \mathsf{h}_\varrho(\mathcal{L}; p, q) \cdot \rho(c)$ equals the product $N(p')N(q')$ of the scalar factors of Definition 2.11, because $c_G(\mathfrak{s}) = \frac{\operatorname{vol}\mathrm{U}(k)}{\operatorname{vol}\mathrm{U}(r)} \frac{2\pi^k}{(k-1)!}$. The bound for $\xi \neq 0$ follows as before.

(S2). The matrix is a Gram matrix; continuity is Lemma 8.9. (S4). Lemma 8.6 at $j = i + 1$ with $\omega = 1$, $y = \xi$, and Theorem A(c). (S5). If $|p_\varrho| \leq 1$ both sides are 0. Otherwise the Cartan decomposition gives $\sigma = h\, \chi_\varrho(p_\varrho) h'$; write both sides by Lemma 8.8, substitute $(y, y') \mapsto (hy, hy')$ and use $\Phi(hy) = \mathrm{rep}(h)\Phi(y)$ with $\mathrm{rep}(h)$ unitary to remove h and h'.

(S3), *the energy.* For $j = 0$ the head density is 1, $\mu_{0,0}^{\mathcal{L}} = \delta_1$ and $w_0(q)^{-1} w_0(p) = 1$, so $J_0(p, q) = c_{\mathrm{f}}(\mathfrak{c}) \int |\delta_\Phi|^2 \mathrm{d}\mathsf{M}(\mathcal{L})$ with $\delta_\Phi(x) = \mathsf{Av}_0(\mathcal{L}; \langle \Phi(\cdot), \Phi(1\cdot)\rangle)(1, x)$. Let $\delta(x) = \mathsf{Av}_0(\mathcal{L}; \langle \Phi(\cdot), \Phi_0(1\cdot)\rangle)(1, x)$. Then $\delta_\Phi - \delta$ is the averaged kernel of Φ and $\Phi - \Phi_0$, and from $|\delta| \geq \big||\delta_\Phi| - |\delta_\Phi - \delta|\big|$ and (T7)(a),

$$|\delta_\Phi(x)|^2 \leq |\delta(x)|^2 + 2\, \mathcal{J}_0^{\mathrm{maj}}(p, q; \Phi, \Phi - \Phi_0, x).$$

The second term: $\Phi - \Phi_0$ is a Bessel function of frequency 1 with $\int |\Phi - \Phi_0|^2 \mathrm{d}\dot{y} \leq b^{-10r^2}|I|$, and Lemma 8.4 with $j = 0$, $R = \rho$, $t = 1$, $\varepsilon = b^{-5r^2}$ bounds $|c_{\mathrm{f}}(\mathfrak{c})|$ times its integral by $\Theta(\mathfrak{s})\rho^{10^9 n^4 + 24k} b^{-5r^2}$; with $\deg \Theta(\mathfrak{s}) \leq b^{4r^2}$ and the sizes this is at most $b^{-10}/(2 \deg)$, a comparison of exponents done by the formal text. The first term: we claim $\int |\delta|^2 \mathrm{d}y \leq \lambda^2 |I|^2 / \deg$; then, by (C2), $|c_{\mathrm{f}}(\mathfrak{c})| \int |\delta|^2 \mathrm{d}\mathsf{M}(\mathcal{L}) \leq \frac{\operatorname{vol}\mathrm{U}(k)}{\operatorname{vol}\mathrm{U}(r)} \lambda \cdot \frac{1}{\deg} \leq \frac{1}{\deg}$. It suffices to bound $E = \int_s |\delta|^2 \mathrm{d}y$ for a set s of finite measure. The function δ is bounded (a pairing kernel, Lemma 4.1); let $\beta = \mathbf{1}_s \overline{\delta}$, bounded and integrable, with $\int |\beta|^2 = E < \infty$. By Fubini's theorem and (C2),

$$\begin{aligned} E = \int \beta\, \delta \,\mathrm{d}y &= \int \Big\langle \Phi(\sigma),\ \int \beta(x) \Phi_0(\sigma x)\, \mathrm{d}x \Big\rangle \mathrm{d}\mathsf{M}_{\geq 0}(\mathcal{L})(\sigma) \\ &\leq \lambda \int |\Phi(\gamma)| \,\Big|\, \int \beta(x) \Phi_0(\gamma x)\, \mathrm{d}x \Big|\, \mathrm{d}\dot{\gamma}, \end{aligned}$$

the integrand being $\mathcal{U}(T)$-invariant. By Cauchy–Schwarz, $\int |\Phi|^2 \mathrm{d}\dot{y} \leq |I|$ and Schur's bound for Φ_0, $E^2 \leq \lambda^2 |I| \cdot \frac{|I|}{\deg} E$, hence $E \leq \lambda^2 |I|^2 / \deg$. □

9 Where the results are used

Theorem E is the last hypothesis in the construction of a *Bessel system* at a real place, the local datum of the second moment of Poincaré series in the unit on the second moment (unit 148, Chapter VI of the write-up): there the band limits (S1)–(S5) at all infinite places and their analogues at the chosen finite places are multiplied into one sieve system. In that application the shape is the one of the admissible data D of the overview: $\varrho = m(D)/2 - 1$ and $|\kappa| = k(D)$, so that $r = r(D)$ and $2\varrho + 2 = m(D)$; the real number b of the real-place sizes and of the cut pair is the third scale $b(D)$; the natural number n of the cut pair is the number $f(D)$ of infinite places; and the peaked Bessel function that Note C cuts comes from Theorems 2, 4 and 5 of the note on the discrete series (unit 133), Theorem 2 being applied there with $\beta = b(D)^3$. Theorems A to D are stated for a general place; the artifact applies them at $\mathbb{C}$ in this note, and their statements cover the products of finite places for which the authors' comment announces the same proof.

10 Notes on the formal text

Implicit arguments. The level is an implicit argument of the forms, of the moduli, of the majorant, of the pairing norm, of the predicate Cut and of the self-norm product; the gauge is an implicit argument of the predicates for sizes and heads. The notation shows the explicit arguments of the Lean definitions, and each statement names the level and the gauge in words.

Argument lists not confirmed by Lean. For the Bessel predicate, the cut predicate, the pairing norm, the top vector, and for the Lean theorems behind Theorems B, C, Lemma 4.3(i), Lemma 6.2 and Lemma 8.3, the reader's dossier derived the list of arguments by Lean's rules for section variables, because the census misreads a name bound inside the declaration $(x,\ L)$ or a line `variable (F) in`. Nothing was compiled, so these lists are not confirmed by Lean. The statements above follow the Lean text of each declaration, read in full; the statement of Theorem B continues after a `let` and was read to its end.

Differences from the natural statements. (1) The integrals of complex functions are Bochner integrals that are 0 when the function is not integrable; so the forms are defined without an integrability hypothesis, and Theorem B carries one. (2) The head density enters the forms through its real value, which is 0 when it is infinite. (3) Exponents $\varrho - (a+1)$ are truncated at 0; in every use $a < \varrho$. (4) Theorem A(a) allows $\mathsf{V} = \infty$ and $\sigma_0 = 0$; this is used in Lemma 8.6. (5) In Theorem A(b) the weight is written with the entry (j, j) of the frame Gram matrix, which needs $j \le \varrho$. (6) Theorem D and Lemma 8.4 ask no measurability of f and S: the integrals are lower integrals in the sense of Lebesgue (the integral of the library, the supremum of the integrals of the measurable simple functions below the function), so the bounds need no measurability; the sets D_i in the proof of Lemma 8.4 are Borel, and a lower integral over a countable disjoint union of Borel sets is at most the sum of the lower integrals over them. (7) In Lemma 8.4 the hypothesis on f^2 is asked only if $t > 1$. (8) Theorem E is an existence statement in the Lean text; the family J is the witness of its proof. (9) Lemma 8.8 takes any Haar measure μ and does not ask that $\mathsf{M}(\mathcal{L})$ be one. (10) The functional rep and the number `n` of cut Bessel data enter only through the cut pair.

A declaration outside the census. The definition of the level $\mathcal{L}_{\mathbb{C}}(\mathfrak{h})$ at the complex place is in the Lean text (the file of the gauge at $\mathbb{C}$, lines 31 to 41), but the census of the artifact does not list it, because its attribute runs over two lines; so it is in no unit and is not a member of this one. Input (C1) states it from the Lean text, and the second reader recovered the same declaration from the pinned source. Its Lean name is in Appendix A.

Small results. Five declarations have no statement of their own: the two top floors of the stack measure (used in the proof of Theorem E), compactness of the set of the elements of bounded size (Lemma 8.8), nearness of the points where a cut function does not vanish (after Definition 2.2), the kernel with $p = q = 1$ (Lemma 4.1), and the volume exponent at $\mathbb{C}$ (after Definition 2.9).

LEAN `Level.stackMeasure_top`

How this note was written. From the structured proofs of the reader's dossier for the three files, and from the Lean text. I read the three Lean files completely, statements and proofs, together with the definitions of the files of Note T on which they rest; the proofs behind Theorem C and Theorem E, which the reader marked as the least certain, were read line by line, and the induction of Theorem C and the energy bound (S3) are written from that reading. The proofs of Theorem A, Lemma 5.1 and Lemma 6.2 were also rebuilt from the Lean terms (which lemma is applied to which arguments). Not redone by hand: the three final comparisons of explicit exponents in Lemmas 8.4, 8.6 and in (S3), which the formal text settles by its arithmetic tactics; they are named in the proofs. The inputs (T1)–(T14) and (C1)–(C4) were compared with their Lean statements; in (T9)(a) and (T12) two Lean lemmas are combined and the exponent $1287n = 99n \cdot 13$ is computed from them. The program check of the unit's dossier covered 639 leaf steps of 108 proofs.

Appendix A. Concordance with the Lean artifact

The letters of this note and the names of the formal text: $\mathcal{L}$ is `F` (a level) or the complex level of `hF`; Ω is `ι`; $\mathfrak{m}$ is `m`; Φ, Φ_0 are `F`, `F₀` (or `W`) at the complex place; s_0 is `s`; $\mathfrak{n}$ is `n`; R is `L`; mod_A, ψ_A, c_A are the fields `md`, `ψ`, `c` of the place; $\mathcal{H}_l(T)$, $\mathcal{Q}_l(T)$, $\mathcal{N}_{<l}(T)$, $\mathcal{N}_l(T)$, $\mathcal{U}(T)$ are `GT`, `unitriGroup`, `unipotentBelow`, `floorUnipotent`, `besselU`; sd is `simpleRootSum`; $Z_T(\cdot)$ is `frameGram`; $\mathcal{L}_{\mathbb{C}}(\mathfrak{h})$ is `RealPlaceSizes.complexLevel`, which the census does not list. Two statements as the artifact has them:

```
lemma Gauge.formModulus_sq_le_of_heads {i} (hi : j = i + 1) (h : ξ =
  F.g⁻¹ * ((p i)⁻¹.1 - (q i)⁻¹.1)) :
  F.formModulus j p q W x 1 ^ 2 ≤ (P.c / P.md ξ) ^ card ι * Γ.selfNormProduct j p q W W x

lemma hasBandLimits_of_isCutBessel [Nonempty I]
    {deg} (hd : 0 < deg) (hS : SchurLe W (.ofReal deg⁻¹ * card I))
  (hD : deg * s.bandConst ≤ b ^ (4 * s.r ^ 2)) : ∃ J, HasBandLimits b F deg J
```

In this note	Lean declaration	File	Lines
Definition 2.11	`sieveVector`	BandLimits.lean	17–17
Lemma 8.9	`IsCutBessel.continuous_sieveVector`	BandLimits.lean	26–35
Theorem E	`hasBandLimits_of_isCutBessel`	BandLimits.lean	43–125
Definition 2.9	`IsCutBessel`	ComplexShells.lean	18–18
Definition 2.10	`IsCutBessel.formConst`	ComplexShells.lean	26–26
Lemma 8.2	`IsCutBessel.isCut_complexLevel`	ComplexShells.lean	28–30
Definition 2.9	`IsCutBessel.volumeExponent_eq`	ComplexShells.lean	32–33
Lemma 8.3	`IsCutBessel.exists_gauge_isHead`	ComplexShells.lean	38–48
Lemma 8.4	`IsCutBessel.formConst_mul_lintegral_le`	ComplexShells.lean	56–101
Lemma 8.5	`IsCutBessel.lintegral_formMajorant_ne_top`	ComplexShells.lean	106–110
Lemma 8.6	`IsCutBessel.norm_floorForm_le`	ComplexShells.lean	115–144
Definition 2.5	`Level.headDensity`	LocalTheorem.lean	32–33
Definition 2.3	`translatePairing`	LocalTheorem.lean	36–36
Definition 2.1	`Place.IsBessel`	LocalTheorem.lean	40–41
Definition 2.2	`Level.IsCut`	LocalTheorem.lean	45–46
Definition 2.4	`Level.pairingNorm`	LocalTheorem.lean	50–50
Definition 2.6	`Level.formIntegrandOne`	LocalTheorem.lean	53–54
Definition 2.6	`Level.formIntegrand`	LocalTheorem.lean	59–60
Definition 2.7	`Level.floorForm`	LocalTheorem.lean	64–65
Definition 2.7	`Level.formModulus`	LocalTheorem.lean	69–69
Definition 2.7	`Level.formMajorant`	LocalTheorem.lean	74–76
Definition 2.8	`Gauge.selfNormProduct`	LocalTheorem.lean	80–81

In this note	Lean declaration	File	Lines
Lemma 6.1	`frameGram_diag_mul`	LocalTheorem.lean	92–96
Lemma 4.4	`Gauge.pivotModulus_sq_le_of_capped`	LocalTheorem.lean	101–115
Theorem C	`Gauge.lintegral_selfNorm_le`	LocalTheorem.lean	120–175
Lemma 4.3	`Level.enorm_floorForm_le`	LocalTheorem.lean	178–181
Theorem E	`Level.stackMeasure_top`	LocalTheorem.lean	183–184
Lemma 8.7	`Gauge.isCompact_hasMatSize`	LocalTheorem.lean	186–187
Definition 2.2	`Level.IsCut.isNear_of_ne_zero`	LocalTheorem.lean	191–192
Lemma 4.3	`Level.IsCut.formModulus_le_formMajorant`	LocalTheorem.lean	197–203
Lemma 8.7	`Gauge.hasMatSize_of_ne_zero`	LocalTheorem.lean	209–213
Lemma 8.8	`Gauge.inner_besselCorrelation`	LocalTheorem.lean	218–227
Lemma 4.1	`Gauge.isPairingKernel_translatePairing`	LocalTheorem.lean	231–251
Lemma 4.1	`Gauge.isPairingKernel_self`	LocalTheorem.lean	255–256
Lemma 4.2	`Level.formIntegrandOne_eq`	LocalTheorem.lean	261–264
Lemma 6.2	`Level.integral_succ_floor_eq`	LocalTheorem.lean	270–303
Theorem B	`Gauge.integral_floorForm_succ`	LocalTheorem.lean	310–318
Theorem D	`Gauge.lintegral_positions_le`	LocalTheorem.lean	326–343
Lemma 5.1	`Gauge.sq_lintegral_le_selfNormProduct`	LocalTheorem.lean	349–363
Theorem A (a)	`Gauge.formMajorant_sq_le`	LocalTheorem.lean	368–380
Theorem A (b)	`Gauge.formModulus_sq_le_of_char`	LocalTheorem.lean	389–416
Theorem A (c)	`Gauge.formModulus_sq_le_of_heads`	LocalTheorem.lean	421–458

Notation

A, $\bar{x}$, $\mathrm{d}x$ the place: a ring with involution and Haar measure

mod_A, ψ_A, c_A its module, character and Plancherel constant

τ_A, δ_A its trace-one element and skew unit

$\mathfrak{s} = (\varrho, \kappa, k_0)$ the shape

n, k, r $\varrho + 1$; $|\kappa|$; $2\varrho + k + 1$, the rank of V

V, V_l, e_a, e_a^*, x_i, z the space, its floors, its basis

f_a, $Z_T(g)$ the frame; the frame Gram matrix of g

T, $J(T)$, h_T, $G(T)$ the form, its split Gram matrix, the hermitian form, the unitary group

$\mathcal{H}_l(T)$, $\mathcal{Q}_l(T)$ stabiliser of the floors below l; unitriangular on the first l frame vectors

$\mathcal{N}_{<l}(T)$, $\mathcal{N}_l(T)$, $\mathcal{U}(T)$ unipotent groups of Section 2

$\mathrm{sd}(u)$ sum of the entries of u at (f_a, f_{a+1})

$w_l(p)$, $w_a^\circ(c)$ Weyl element; floor Weyl element

Ω, $\mathfrak{m}$ index set of the orthonormal part and number of extra pairs of a hyperbolic basis

$\mathcal{L}$, $E(\mathcal{L})$, $g_{\mathcal{L}}$ a level, its basis matrix, its unit

$\mathfrak{B}(\mathcal{L})$, $\mathrm{Box}(\mathcal{L})$, $\mathrm{Posn}(\mathcal{L})$ its ball, box, positions

$\theta_l(\mathcal{L})$, η_l, $\eta_{<j}$, ϖ_l floor measure, Heisenberg measures, Weyl law

$\mu^{\mathcal{L}}_{a,i}$, $\mathsf{M}(\mathcal{L})$, $\mathsf{M}_{\geq l}(\mathcal{L})$ convolutions of floor measures

$\delta^{(b)}(\mathcal{L})$, δ_s, $*$ quadric measure on V_b; unit mass; convolution

Γ, Γ_i, $\Lambda(\Gamma)$ a gauge, its sets, its constant

sz, Vec_i, Hd_j size of a matrix, vectors of size i, heads

Near_j, Val_a, s, w_j, d_l nearness, floor values, volume exponent, weight, floor module

Mass_j, Fib mass of a floor; fibre bound of Note T

Av_j, Pv_j, Sq_j, Sm_j, SN_j averaged kernel, pivot, weighted square, its sum, self-norm

PK_j, Pos, C_l, C_{gl} pairing kernels, positions of size 1, constants of a gauge

I, $\langle v, w\rangle$, $|v|$ finite set; inner product and norm of $\mathbb{C}^I$

W, W', Bes, Cut functions; Bessel and cut functions

PN, h_j, I^1_j, I_j pairing norm, head density, integrands

$\mathcal{J}_j$, $\mathcal{J}_j^{\mathrm{mod}}$, $\mathcal{J}_j^{\mathrm{maj}}$, SNP_j floor form, modulus, majorant, self-norm product

p, q, d, w sequences of units; Weyl ratio; a Weyl element

ω, ω_η, η, ξ weights; frequencies

j, l, a, b, i, i_1, i_2 floors and sizes (at $\mathbb{C}$, b is also the third scale)

g, x, y, s, u, v, h, m, t group elements, vectors, units; t also a real or complex number

σ, τ, ν, γ variables of integration; also γ a group element

V, σ_0, Θ_0 bounds of Theorems A(a) and D: for the mass of a floor, for a volume, for the pairing norms

k, K, C, M, D, E, N, N_a a kernel (also $k = |\kappa|$); constants

m a measure on V with a fibre bound (Input T12)

Φ, Φ_0, rep cut and full function at $\mathbb{C}$; representation of $\mathrm{U}(\kappa)$

$\mathfrak{h}$, c, ρ, s_0, n sizes, cut Bessel data, their numbers

$\mathcal{L}_{\mathbb{C}}(\mathfrak{h})$, λ, $\Gamma(\mathfrak{h};\rho\le R)$, R, R_0 level, Haar ratio, gauge at $\mathbb{C}$, radii

RPS, CB, c_{f}, Ξ sizes, cut Bessel data, constant of the forms, top vector

$G_1(\mathfrak{s})$, $\mathrm{d}y$, $\mathrm{d}\dot{y}$ the group at $\mathbb{C}$, its Haar measure, the measure on it that computes integrals over $\mathcal{U}(T)\backslash G_1(\mathfrak{s})$

$J^{\mu}_{W}(\gamma)$, $\chi_{\varrho}(c)$ correlation tensor; boost

$c_G(\mathfrak{s})$, $\Theta(\mathfrak{s})$, $\mathrm{vol}\,\mathrm{U}(N)$ constants; volume of a unitary group

$\ell_K(t)$, $\mathrm{hw}_{K,i}(p)$, β, deg damped power, head weight, band function, formal degree

J, J_i, f, F, F_1, F_2, G, B, P, S, T_i, e, c forms and auxiliary functions, sets and numbers of the proofs

r, u, v, A, W_l, Φ_l, ε, ζ, Ψ, φ, μ, α auxiliary objects of the proofs

$E(D)$, $m(D)$, $k(D)$, $r(D)$, $b(D)$, $f(D)$ the CM field, the numbers and the third scale of the admissible data D of the overview (only in Sections 1 and 9)

d, λ, K, f, Θ local to this note and different from the letters of the overview: Weyl ratio, Haar ratio, a constant, functions, the constant $\Theta(\mathfrak{s})$

π, ∞, $\mathbf{1}$, L, H, O, Q, U, X, Y, Z, o, z the number; infinity; indicator; letters inside names

READBACK NOTE 30

The complex place: the open-cell chart, the Haar measure of the tower, the Bessel cut and the volume weight

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. At the complex place the unitary group $G_1(\mathfrak{s})$ has two descriptions that this note compares. The *tower* of the companion Note T writes an element by floors; the *open-cell chart* writes it as $u\,\underline{h'}\,\gamma(Z)\,(v\,\underline{h})^{-1}$, with Z the frame Gram matrix, $\gamma(Z)$ an explicit section, u, v in the unipotent group that fixes the frame and h', h in the compact group $\mathrm{U}(\kappa)$. The Jacobian of the chart is $\mathfrak{g}(Z)^{k-1}$ times two Cayley weights, and the chart is injective; hence the tower measure is a multiple $\lambda(\mathfrak{h})$ of Haar measure with $\lambda(\mathfrak{h}) \le \mathrm{vol}\,\mathrm{U}(r)/\,\mathrm{vol}\,\mathrm{U}(k)$. The same chart gives a mean-square bound for products of Bessel functions on the open cell. The note also normalises and cuts off a peaked Bessel function, with Schur bounds for its derivatives, and bounds the volume weight Δ by a kernel of Hua type.

What this note is. The artifact grh-compact is a body of Lean code whose final theorems are the Riemann hypothesis and the generalized Riemann hypothesis for Dirichlet L-functions. Its authors report that it compiles and that Lean's kernel accepts these theorems from the three standard axioms; nothing was built for this note. That is a statement about a formal text: no mathematician has read the argument through, and this note does not say that the Riemann hypothesis is proved. The note treats one part of one unit of the artifact (the folder of the analysis at the real places): 79 of the unit's 239 declarations, those of ten files; two companion notes treat the others (Note T, *The tower method at a place: floor measures, gauges and the bounds on each floor*; Note L, *The local theorem for floor forms and the band limits at the complex place*). Every result of the ten files is stated here and was compared, argument by argument, with its Lean text. The proofs are complete for the lemmas on norms, on the section and on the volume weight; for Theorems A to E they follow the formal proofs, which were read line by line, and give every step that carries an idea, while long matrix identities and comparisons of exponents are named and left to the formal text. Classical results of the background library are quoted (Section 3). The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 79 declarations.

1 The statements

Notation is fixed in Section 2: $G_1(\mathfrak{s})$ is the unitary group of the hermitian space $V = \mathbb{C}^r$ with ϱ hyperbolic planes, a positive part $\langle x_i \rangle_{i\in\kappa}$ and a negative vector z; $Z(y)$ is the frame Gram matrix of y; $\mathfrak{g}(Z) = |\det Z|^2 - |D_\varrho(Z)|^2$, with D_ϱ the leading principal minor of order ϱ; $\mathcal{C}(\mathfrak{s})$ is the open cell $\{\mathfrak{g}(Z) > 0\}$; $\mathcal{U}^0$ is the group of the unipotent elements $\mathrm{u}(q)$ that fix the frame, with its Haar measure μ^0; $\underline{h}$ is the image of $h \in \mathrm{U}(\kappa)$. A word on the names: a *real place* of the write-up is an infinite place of the maximal totally real subfield of the CM field $E(D)$ of the overview; the

completion of $E(D)$ above it is $\mathbb{C}$, and the unitary group of the rows there is $G_1(\mathfrak{s})$, a unitary group of signature $(\varrho+k,\varrho+1)$. In the formalism of places of Note T this is the place $A=\mathbb{C}$ with complex conjugation, which these notes call *the complex place*; so "the real places" (the name of the unit and of the overview) and "the complex place" (Notes L and C) are the same local situation.

Theorem A (the open-cell chart). Let $\mathfrak{s}$ be a shape, and let ch be the map that sends a parameter $p=(Z,\theta,q,\eta,q')$ to $\mathrm{u}(q)\,\underline{\mathrm{cay}(\tilde{\theta})}\,\gamma(Z)\,(\mathrm{u}(q')\,\underline{\mathrm{cay}(\eta)})^{-1}\in G_1(\mathfrak{s})$ (Definition 2.4).
(i) There is a real-linear isomorphism e from the space of the parameters onto the space of the real $r\times r$ matrices that preserves Lebesgue measure and such that, at every p with $Z\in\mathcal{C}(\mathfrak{s})$, the chart is differentiable and its logarithmic derivative $\mathrm{D}^{\log}\mathrm{ch}(p)$ satisfies

$$\left|\det\left(e^{-1}\circ\mathrm{sk}\circ\mathrm{D}^{\log}\mathrm{ch}(p)\right)\right|=\mathrm{g}(Z)^{\,k-1}\cdot\mathrm{j}(\tilde{\theta})\cdot\mathrm{j}(\eta).$$

(ii) The chart is injective on the set of the parameters with $Z\in\mathcal{C}(\mathfrak{s})$.
(iii) Let $f\colon G_1(\mathfrak{s})\to[0,\infty]$ be measurable and $L\in[0,\infty]$. If for every $h'\in\mathrm{U}(\kappa)$

$$L\le\int_{\mathcal{C}(\mathfrak{s})}c_G(\mathfrak{s})\,\mathrm{g}(Z)^{k-1}\int_{\mathrm{U}(\kappa)}\iint f(u\,\underline{h'}\,\gamma(Z)\,(v\,\underline{h})^{-1})\,\mathrm{d}\mu^0(u)\,\mathrm{d}\mu^0(v)\,\mathrm{d}h\,\mathrm{d}Z,$$

then $L\le\int_{G_1(\mathfrak{s})}f(y)\,\mathrm{d}y$.

LEAN `openCellChart_jacobian` · `openCellChart_injOn` · `le_lintegral_of_gramSection`

Theorem B (the tower measure at the complex place). Let $\mathfrak{s}$ be a shape, ρ and b real numbers, s_0 a natural number, and $\mathfrak{h}$ real-place sizes in $\mathrm{RPS}(\mathfrak{s},\rho,s_0,b)$ (Definition 2.5), with their level $\mathcal{L}_{\mathbb{C}}(\mathfrak{h})$ (Definition 2.6). Then the constant $\lambda(\mathfrak{h})$ of Definition 2.7 satisfies:

(i) $\lambda(\mathfrak{h})\neq0$ and $\lambda(\mathfrak{h})\neq\infty$;

(ii) $\mathsf{M}(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}))=\lambda(\mathfrak{h})\cdot\mathrm{d}y$, the Haar measure of $G_1(\mathfrak{s})$; in particular the tower measure is a Haar measure;

(iii) for every measurable $f\colon G_1(\mathfrak{s})\to[0,\infty]$ that is invariant under left translations by $\mathcal{U}(T)$: $\int f\,\mathrm{d}\mathsf{M}_{\ge0}(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}))=\lambda(\mathfrak{h})\int f\,\mathrm{d}\dot{y}$;

(iv) $\dfrac{\mathrm{vol}\,\mathrm{U}(k)}{\mathrm{vol}\,\mathrm{U}(r)}\cdot\lambda(\mathfrak{h})\le1$.

LEAN `RealPlaceSizes.haarRatio_spec`

Theorem C (the open-cell bound). Let $\mathfrak{s}$ be a shape, B a real number, f a natural number and X a Bessel system for $(\mathfrak{s}, B, f)$ (Section 3), with full function $\Phi_0 \colon G_1(\mathfrak{s}) \to \mathbb{C}^I$ and formal degree deg. Let $\gamma, y \in G_1(\mathfrak{s})$ and ρ real with $\mathfrak{g}(Z(\gamma)) > 0$, $y \in \mathrm{FB}(\rho)$, $\gamma^{-1} y \in \mathrm{FB}(\rho)$ and $B \le \rho$. Then there are a compact set $A \subseteq G_1(\mathfrak{s})$, a finite measure ν on $G_1(\mathfrak{s})$, a probability measure m on $\mathrm{U}(\kappa) \times \mathcal{U}^0$ and a measurable map $t \colon \mathrm{U}(\kappa) \times \mathcal{U}^0 \to G_1(\mathfrak{s})$ such that:

(i) for every measurable bounded $\varphi \colon G_1(\mathfrak{s}) \to \mathbb{C}$,

$$\int \sum_{i,i' \in I} \Big| \int_A \varphi(y')\, \Phi_{0,i}(\gamma\, t(x)\, y')\, \overline{\Phi_{0,i'}(t(x)\, y')}\, \mathrm{d}y' \Big|^2 \mathrm{d}m(x) \le \int |\varphi|^2 \, \mathrm{d}\nu;$$

(ii) $\nu(G_1(\mathfrak{s})) \le \Big(\dfrac{\rho^{9n}}{\mathfrak{g}(Z(\gamma))}\Big)^{10^5 n^4} \cdot \dfrac{|I|^2}{\deg \cdot c_G(\mathfrak{s}) \cdot (\mathfrak{g}(Z(\gamma))/2)^{k-1}}$;

(iii) for m-almost every x: $t(x)$ lies in the group generated by $\mathcal{U}^0$ and $\mathcal{H}_n(T)$, and every $y' \in \mathrm{FB}(\rho)$ with $\mathrm{bs}(\gamma\, t(x)\, y') \le \rho$ lies in A.

LEAN `BesselSystem.open_cell`

Theorem D (the Bessel cut). Let $\mathfrak{s}$ be a shape, I a finite set, ε a family of signs with a frame p of $J_1(\mathfrak{s})$ (Section 3), rep a homomorphism from $\mathrm{U}(\kappa)$ to the unitary group of $\mathbb{C}^I$, A_0 and L natural numbers, and l a parameter with a large gap A_0 and with $|l_a| \le L$ for all a; write $\deg(l)$ for its formal degree and ψ_l^{p} for its coefficient in the frame p, a function on $G_1(\mathfrak{s})$ (Section 3; the notation of the note on the discrete series, unit 133).
(i) Let $\Phi_0 \colon G_1(\mathfrak{s}) \to \mathbb{C}^I$ be a Bessel function for rep that is integrable for $\mathrm{d}\dot{y}$, and B a real number. If $\Phi_0 * (\deg(l)\psi_l^{\mathsf{p}}) = \Phi_0$ and $(r+1)^4(L+r+1+2n^2) \le B$, then Φ_0 has the Schur bounds with respect to rep, B and the constant $\deg(l)^{-1} \int |\Phi_0|^2 \, \mathrm{d}\dot{y}$.
(ii) Let $f \neq 0$ and N be natural numbers, b a real number, $x \in G_1(\mathfrak{s})$ and $Y \colon G_1(\mathfrak{s}) \to \mathbb{C}^I$ with a peak at x of constant b^N (for l and rep). Suppose $\mathrm{bs}(x) \le b^{5n}$, $2000 \le b$, $N + 99r^2 \le r^3$, $r + L + 1 \le b^4$ and $3r^3 f \le A_0$. Then there are continuous $\Phi_0, \Phi \colon G_1(\mathfrak{s}) \to \mathbb{C}^I$ with $\Phi_0 * (\deg(l)\psi_l^{\mathsf{p}}) = \Phi_0$ such that (Φ_0, Φ) is a cut pair with respect to b, f and rep.

LEAN `IsBesselFunction.hasSchurBound_of_convolve_eq` · `HasPeakAt.exists_cut_isCutPair`

Theorem E (a kernel for the volume weight). Let $\mathfrak{s}$ be a shape, $y \in G_1(\mathfrak{s})$ and θ a real number with $\mathrm{bs}(y) \le \theta$. Let j be a natural number with $j \le n$, σ a finite set with an injection $e\colon \sigma \to \{0, \dots, j-1\}$, R a complex $\sigma \times \sigma$ matrix with $\det R \neq 0$, Z a complex $n \times n$ matrix whose leading $j \times j$ block Z_j has $\det Z_j \neq 0$, and V a complex $\kappa \times j$ matrix. Then there is a measurable function K on the complex $\kappa \times \sigma$ matrices, with values in $[0, \infty]$, such that

$$\int K(a)\,\mathrm{d}a \;\le\; \Big(\frac{(2\theta^2)^j}{|\det Z_j|}\Big)^{k+n} \cdot \Big(\frac{\pi^{|\sigma|}}{|\sigma|!\,\big(\theta^{|\sigma|}\,|\det R|\big)^2}\Big)^{k},$$

and such that for all $\gamma \in G_1(\mathfrak{s})$ and all complex $\kappa \times \sigma$ matrices A with $Z(\gamma) = Z$, with γ in normal form at j (Definition 2.10), and with $\mathsf{x}_0(\gamma)\,\mathsf{I}_j = V + A\,R\,\mathsf{E}^* Z_j$,

$$\Big(\frac{\Delta(y)}{\Delta(\gamma^{-1}y)}\Big)^{k+n} \le K(A).$$

Here I_j is the $n \times j$ matrix of the inclusion of the first j frame positions and E the $j \times \sigma$ matrix of e.

LEAN `exists_kernel_frameVolume_le`

2 The setting

Conventions. $\mathfrak{s} = (\varrho, \kappa, k_0)$ is a shape; $n = \varrho + 1$, $k = |\kappa|$, $r = 2\varrho + k + 1$. I is a finite set I and $\mathbb{C}^I$ has its standard inner product $\langle v, w\rangle = \sum_i \overline{v_i} w_i$ and norm $|v|$. Matrices are complex unless said otherwise; X^* is the conjugate transpose and $\|X\|$ the operator norm for the Euclidean norms. Finite index sets carry the counting structure, spaces of real or complex matrices their Lebesgue measure. In Section 11 the letters E, S denote a CM field E and a filling S of the unit on the second moment. Integrals of functions with values in $[0, \infty]$ are Lebesgue integrals; subtraction of natural numbers is truncated at 0.

The group at the complex place (background library, units 112, 115, 118, 119). The space $V = \mathbb{C}^r$ has the basis $e_0, \dots, e_{\varrho-1}$, x_i $(i \in \kappa)$, z, $e^*_{\varrho-1}, \dots, e^*_0$, and the hermitian form $h(v, w) = v^* J_1(\mathfrak{s}) w$ whose Gram matrix $J_1(\mathfrak{s})$ pairs e_a with e^*_a, is the identity on the x_i and -1 at z; $J_1(\mathfrak{s})^2 = 1$. $G_1(\mathfrak{s})$ is its unitary group, with its Haar measure $\mathrm{d}y$, normalised by the Cayley chart (Input (J4)). In the notation of Note T this is $G(T)$ for the place $\mathbb{C}$ and the form T of $J_1(\mathfrak{s})$, and we use the subgroups $\mathcal{H}_l(T)$, $\mathcal{Q}_l(T)$, $\mathcal{U}(T)$ of that note; $\mathcal{H}_n(T)$ is the image of $\mathrm{U}(\kappa)$, the unitary group of $\langle x_i \rangle$, and $\underline{h}$ denotes the image of h. The *frame* is $f_0, \dots, f_\varrho$ with $f_a = e_a$, $f_\varrho = z$; $F_{\mathfrak{s}}$ is the $r \times n$ matrix with these columns, and $F'_{\mathfrak{s}}$ the one with the columns $e^*_0, \dots, e^*_{\varrho-1}, x_{k_0}$. For $y \in G_1(\mathfrak{s})$, the *frame vectors* are $y^{-1} f_a \in \mathbb{C}^r$, the *frame Gram matrix* is $Z(y) = (h(f_a, y f_b))_{a,b} = F^*_{\mathfrak{s}} J_1(\mathfrak{s})\, y F_{\mathfrak{s}}$, and the *frame matrix* $\mathrm{fm}(y)$ is the block of y on the frame. For a complex $n \times n$ matrix Z, $\mathfrak{g}(Z) = |\det Z|^2 - |D_\varrho(Z)|^2$. The group $\mathcal{U}^0$ of the elements of $\mathcal{U}(T)$ that fix every frame vector consists of the elements $\mathrm{u}(q)$, $q = (q_1, q_2)$ a complex $\varrho \times \kappa$ matrix and a real $\varrho \times \varrho$ matrix, and μ^0 is the image of Lebesgue measure under $q \mapsto \mathrm{u}(q)$, a Haar measure of $\mathcal{U}^0$. The group generated by $\mathcal{U}^0$ and $\mathcal{H}_n(T)$ is the stabiliser of the frame; it lies in $\mathcal{Q}_n(T)$, which normalises $\mathcal{U}^0$ and $\mathcal{U}(T)$, and $\mathcal{Q}_n(T) = \mathcal{U}(T) \cdot \mathcal{H}_n(T)$. The measure $\mathrm{d}\dot{y}$ on $G_1(\mathfrak{s})$

computes integrals over $\mathcal{U}(T)\backslash G_1(\mathfrak{s})$: $\iint_{\mathcal{U}(T)} f(uy)\,\mathrm{d}u\,\mathrm{d}\dot{y} = \int f\,\mathrm{d}y$ for a fixed Haar measure of $\mathcal{U}(T)$. The *Bessel size* of y is $\mathrm{bs}(y) = \inf_{s\in\mathcal{Q}_n(T)}(\|sy\|+1)$, and $\mathrm{FB}(\rho)$ is the set of the y with $|y^{-1}f_a| \le \rho$ for all $a \le \varrho$. For a real $\kappa\times\kappa$ matrix η, $\mathrm{cay}(\eta) \in \mathrm{U}(\kappa)$ is the *Cayley chart* $(1+X)(1-X)^{-1}$, $X = \frac{1}{2}\,\mathrm{skew}(\eta)$, where skew identifies real matrices with skew-hermitian ones and sk is its left inverse on all complex matrices; and $\mathsf{j}(\eta) = |\det(1-X)|^{-2k}$ is its *Cayley weight*. A map Φ into $G_1(\mathfrak{s})$ that is differentiable at p has the *logarithmic derivative* $\mathrm{D}^{\log}\Phi(p)\colon v \mapsto \Phi(p)^* J_1(\mathfrak{s})\,\mathrm{d}\Phi_p(v)$.

The section.

Definition 2.1 (corner, defect, open cell)**.** For a complex $n\times n$ matrix Z: $\mathrm{ic}(Z)$ is the last diagonal entry of Z^{-1} (the inverse of a singular matrix being 0), and $\mathrm{cd}(Z) = 1 - |\mathrm{ic}(Z)|^2$. The *open cell* $\mathcal{C}(\mathfrak{s})$ is the set of the Z with $\det Z \neq 0$ and $|\mathrm{ic}(Z)| < 1$. For a real P, $Z \in \mathrm{Cell}_{\le P}$ means: $Z \in \mathcal{C}(\mathfrak{s})$, $\|Z^{-1}\| \le P$, $\|Z\| \le P$ and $\mathrm{cd}(Z)^{-1/2} \le P$.

LEAN `invCorner` · `cornerDefect` · `Shape.openCell` · `IsBoundedInCell` · `gramDiscReal_eq` · `mem_openCell_iff` · `continuous_gramDiscReal` · `isOpen_openCell` · `cornerDefect_pos`

If $\det Z \neq 0$ then $\mathsf{g}(Z) = |\det Z|^2\,\mathrm{cd}(Z)$, by Cramer's rule; so $\mathcal{C}(\mathfrak{s}) = \{Z : \mathsf{g}(Z) > 0\}$, an open set since $\mathsf{g}(\cdot)$ is continuous, and $\mathrm{cd}(Z) > 0$ on it.

Definition 2.2 (the matrices of the section)**.** For a complex number x, $\mathrm{d}_\varrho(x)$ is the diagonal $n\times n$ matrix $\mathrm{diag}(1,\dots,1,x)$. For complex $n\times n$ matrices Z, W and a complex number s, with $s_0 = \sqrt{1-\mathrm{ic}(Z)\,\mathrm{ic}(W)}$ (the principal square root):

$$\begin{aligned}
\mathsf{c}(Z,W) &= \mathrm{d}_\varrho(0) - 1 - \mathrm{d}_\varrho(0)\,W^{-1}(1-\mathrm{d}_\varrho(\tfrac{1}{2}))Z^{-1}\,\mathrm{d}_\varrho(2),\\
\mathsf{f}(Z,W,s) &= (F_{\mathfrak{s}}\,\mathsf{c}(Z,W) + F'_{\mathfrak{s}}\,\mathrm{d}_\varrho(s))Z,\\
\mathsf{m}(Z,W,s) &= 1 + (\mathsf{f}(Z,W,s_0) - F_{\mathfrak{s}})F_{\mathfrak{s}}^*\\
&\qquad + \Big((F_{\mathfrak{s}}W^{-1} - \mathsf{f}(Z,W,s_0)\,\mathsf{c}(W,Z))\,\mathrm{d}_\varrho(s^{-1}) - F'_{\mathfrak{s}}\Big)F'^*_{\mathfrak{s}}.
\end{aligned}$$

In the Lean text the last two arguments have the default values $W = Z^*$ and $s = s_0$; for $W = Z^*$ and $Z \in \mathcal{C}(\mathfrak{s})$ this is $s_0 = \sqrt{\mathrm{cd}(Z)}$. Inside the section matrix both section frames are taken with their own default s_0, whatever s is: the argument s enters only through $\mathrm{d}_\varrho(s^{-1})$. Only $s = s_0$ occurs in this note and in the formal text.

LEAN `diagLast` · `sectionCoeff` · `sectionFrame` · `sectionMatrix` · `sqrt_eq_sqrt_cornerDefect` · `diagLast_sqrt_mul_inv` · `sectionMatrix_mul_frameCols` · `frameCols_gram_sectionMatrix`

Two identities follow by multiplying out, for $\det Z \neq 0$: $\mathsf{m}(Z,W,s)F_{\mathfrak{s}} = \mathsf{f}(Z,W,s_0)$ and $F_{\mathfrak{s}}^* J_1(\mathfrak{s})\,\mathsf{f}(Z,W,s) = Z$ (the formal text states them at the default values).

Definition 2.3 (the section)**.** For a complex $n\times n$ matrix Z, $\gamma(Z)$ is the element of $G_1(\mathfrak{s})$ with the matrix $\mathsf{m}(Z,Z^*,\sqrt{\mathrm{cd}(Z)})$ if $Z \in \mathcal{C}(\mathfrak{s})$, and the identity otherwise. That this matrix is unitary for $J_1(\mathfrak{s})$ is part of the definition in the Lean text (Lemma 5.1).

LEAN `gramSection` · `coe_gramSection` · `gramSection_eventuallyEq` · `realFrameGram_gramSection`

So for $Z \in \mathcal{C}(\mathfrak{s})$ the matrix of $\gamma(Z)$ is the section matrix, also near Z, and $Z(\gamma(Z)) = Z$ by the two identities above.

Definition 2.4 (the chart and its coordinates)**.** The *parameters* are the tuples $p = (Z, \theta, q, \eta, q')$ of a complex $n \times n$ matrix Z, a pair θ of a real number and a vector of $\mathbb{C}^{\kappa \setminus \{k_0\}}$, two parameters q, q' of $\mathcal{U}^0$ and a real $\kappa \times \kappa$ matrix η. Let $\tilde{\theta}$ be the real $\kappa \times \kappa$ matrix that has the entry and the column of θ at k_0 and is 0 elsewhere (the volume-preserving identification of the background library). The *open-cell chart* is

$$\mathrm{ch}(p) = \mathrm{u}(q) \cdot \underline{\mathrm{cay}(\tilde{\theta})} \cdot \gamma(Z) \cdot \big(\mathrm{u}(q') \cdot \underline{\mathrm{cay}(\eta)}\big)^{-1}.$$

The *chart coordinates* are the real-linear map cc from complex $r \times r$ matrices S to parameters whose Z-component is $F_{\mathfrak{s}}'^{*} S F_{\mathfrak{s}}$ and whose other components are read from the blocks of $\mathrm{sk}(S)$ and of S at the indices of the x_i and of the e_a^*. The *Bessel tangent map* is the real-linear map bt from triples (η, q_1, q_2) of complex $\kappa \times \kappa$, complex $\varrho \times \kappa$ and real $\varrho \times \varrho$ matrices to complex $r \times r$ matrices,

$$\mathrm{bt}(\eta, q_1, q_2) = (W_{\mathfrak{s}} \eta + D_{\mathfrak{s}} q_1) W_{\mathfrak{s}}^{*} + (D_{\mathfrak{s}}\, \mathrm{skew}(q_2) - W_{\mathfrak{s}} q_1^{*}) D_{\mathfrak{s}}^{*},$$

with $W_{\mathfrak{s}}$, $D_{\mathfrak{s}}$ the matrices of the columns x_i and e_a^*. None of the three takes an explicit argument in the Lean text; the shape is implicit.

LEAN `openCellChart` · `chartCoords` · `besselTangent`

Sizes, level and gauge at the complex place.

Definition 2.5 (real-place sizes)**.** For a shape $\mathfrak{s}$, real numbers ρ, b and a natural number s_0, *real-place sizes* $\mathfrak{h}$, elements of $\mathrm{RPS}(\mathfrak{s}, \rho, s_0, b)$, are proofs of the seven conditions

$$\rho = b^{6n+2}, \quad 8r \le \rho, \quad k = s_0 + 3n + 2, \quad 10^9 n^3 < s_0, \quad n^7 \le k, \quad 10^8 \le \varrho, \quad 2 \le b.$$

LEAN `RealPlaceSizes` · `RealPlaceSizes.ten_le`

They imply $\rho \ge 4$. In the application (unit 148) the shape is that of the admissible data D of the overview: $\varrho = m(D)/2 - 1$ and $|\kappa| = k(D)$, so that $r = r(D)$ and $2\varrho + 2 = m(D)$; b is the third scale $b(D)$; the natural number f of the cut pairs of Theorems C and D is the number $f(D)$ of infinite places; and the peak of Theorem D(ii) comes from Theorems 2, 4 and 5 of the note on the discrete series (unit 133), Theorem 2 being applied there with $\beta = b(D)^3$.

Definition 2.6 (the level at the complex place)**.** For real-place sizes $\mathfrak{h}$ in $\mathrm{RPS}(\mathfrak{s}, \rho, s_0, b)$, the level $\mathcal{L}_{\mathbb{C}}(\mathfrak{h})$ is the level in $\mathrm{Lev}(\kappa \setminus \{k_0\}, T, 0)$, for the place $\mathbb{C}$ and the form T of $J_1(\mathfrak{s})$, with: the hyperbolic basis of V_ϱ made of the pair $\frac{1+\mathrm{i}}{2}(x_{k_0} \pm z)$ and the orthonormal vectors x_i, $i \ne k_0$; the measure on floor n the Haar probability measure of $\mathrm{U}(\kappa)$, carried to $\mathcal{H}_n(T)$; the measure on floor ϱ the convolution of the image of $\frac{2\pi^k}{(k-1)!} \max(|c|^2 - 1, 0)^{k-1} \mathrm{d}c$ under $c \mapsto \chi_\varrho(c)^{-1}$ with the measure on floor n; the unit 1; the ball $\{v : |v| < \frac{1}{4}\}$; the box $\{x : \|x\| \le \rho\}$; and all of $G_1(\mathfrak{s})$ as positions. Here $\chi_\varrho(c) \in \mathcal{H}_\varrho(T)$ is the boost of the background library in the plane of x_{k_0} and z. This definition is in the Lean text but not in the census of the artifact (see Section 13); the axioms (L1)–(L3) of a level are verified in it from results of unit 119.

Definition 2.7 (the Haar ratio)**.** For real-place sizes $\mathfrak{h}$ in $\mathrm{RPS}(\mathfrak{s}, \rho, s_0, b)$, $\lambda(\mathfrak{h}) \in [0, \infty]$ is a chosen constant with $\mathsf{M}(\mathcal{L}_{\mathbb{C}}(\mathfrak{h})) = \lambda(\mathfrak{h}) \cdot \mathrm{d}y$; one exists by Theorem A(i) of Note T and the uniqueness of Haar measure (Input (J2)).

LEAN `RealPlaceSizes.haarRatio`

Definition 2.8 (the gauge at the complex place)**.** For real-place sizes $\mathfrak{h}$ in $\mathrm{RPS}(\mathfrak{s}, \rho, s_0, b)$ and a real number R with a proof of $\rho \le R$, $\Gamma(\mathfrak{h}; \rho \le R)$ is the gauge in $\mathrm{Gau}(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}))$ with the sets $\Gamma_i = \{x \in \mathbb{C} : |x| \le R^i\}$ and the constant $\Lambda = R^2$. That these data satisfy the axioms of a gauge is part of the definition in the Lean text (Proposition 8.2). The formal text gives the proof of $\rho \le R$ the default value `rfl` (reflexivity); so the gauge written there without this argument is $\Gamma(\mathfrak{h}; \rho \le \rho)$, the gauge of radius ρ. Note L uses it in its lemma on shells.

LEAN `RealPlaceSizes.gauge`

The volume weight.

Definition 2.9 (volume weight; the part of the frame in the positive space)**.** For $\gamma \in G_1(\mathfrak{s})$,

$$\Delta(\gamma) = \left|\det\left(\left((1 - J_1(\mathfrak{s}))F_{\mathfrak{s}}\right)^* \gamma^{-1} F_{\mathfrak{s}}\right)\right|, \qquad \mathsf{x}_0(\gamma) = W_{\mathfrak{s}}^* \, \gamma \, F_{\mathfrak{s}}.$$

The columns of $(1 - J_1(\mathfrak{s}))F_{\mathfrak{s}}$ are $e_a - e_a^*$ $(a < \varrho)$ and $2z$, a basis of a maximal negative subspace; so $\Delta(\gamma)$ is the absolute value of the determinant of the Euclidean products of this basis with the frame vectors of γ, and $\mathsf{x}_0(\gamma)$ is the $\kappa \times n$ matrix of the components of $\gamma f_0, \dots, \gamma f_\varrho$ on the x_i.

LEAN `frameVolume` · `frameX₀Part` · `continuous_frameVolume` · `frameX₀Part_unipotentOf_inv_mul`

$\Delta(\cdot)$ is continuous; and $\mathsf{x}_0(\mathrm{u}(q)^{-1}\gamma) = \mathsf{x}_0(\gamma) + q_1^* \cdot$ (the first ϱ rows of $Z(\gamma)$), by the formula for $\mathrm{u}(q)$.

Definition 2.10 (normal form; recalled from unit 148)**.** Let j be a natural number. An element $\gamma \in G_1(\mathfrak{s})$ is *in normal form at j* if there is $c \in \mathbb{C}$ with $|c| = 1$ such that for every frame position a with $j \le a \le \varrho$ there are $x, y \in \mathbb{C}^n$ with $x_b = y_b = 0$ for all $b > a$, $x_a = c$, $y_a = 1$ and $F_{\mathfrak{s}} x = \gamma F_{\mathfrak{s}} y$. In words: for every $a \ge j$ the flags $\langle f_0, \dots, f_a \rangle$ and $\langle \gamma f_0, \dots, \gamma f_a \rangle$ share a vector with the leading coefficients c and 1, the same c for all a. This is a definition of the unit on the second moment (unit 148), which states it for any unitary group of the same shape; it is not a member of this unit.

3 What is taken for granted

From Note T and Note L (same unit). The notions of level, gauge, floor measures, tower and upper measures, and Theorem A of Note T (the tower measure is right invariant and equals $\eta_{<j}$ convolved with floor measures). Note L is not used here; it uses Theorem B, Definitions 2.5, 2.6, 2.8 with Proposition 8.2, and Lemma 4.2(i).

From the unit on the second moment (unit 148) and unit 133. A *cut pair* with respect to b, f, rep is a pair of continuous functions Φ_0 (full), Φ (cut) from $G_1(\mathfrak{s})$ to $\mathbb{C}^I$, both Bessel functions for rep (measurable, bounded, $F(uy) = \exp(\mathrm{sd}(u) - \overline{\mathrm{sd}(u)})F(y)$ for $u \in \mathcal{U}(T)$, $F(\underline{h}y) = \mathrm{rep}(h)F(y)$), with: Φ_0 integrable for $\mathrm{d}\dot{y}$; $\int |\Phi_0|^2 \mathrm{d}\dot{y} = |I|$; $\int |\Phi|^2 \mathrm{d}\dot{y} \le |I|$; $\Phi(y) \ne 0$ implies $\mathrm{bs}(y) \le b^{6n+2}$; $\int |\Phi - \Phi_0|^2 \mathrm{d}\dot{y} \le b^{-10r^2}|I|$; and there

are a measurable $\mathcal{U}(T)$-invariant $\mathsf{A} \geq 0$ and a finite measure μ with $c(y)\,|F(y)| \leq \mathsf{A}(y)$ for $F = \Phi_0, \Phi$, where $c(y) = 1$ if $\Phi_0(y) = \Phi(y)$ and $c(y) = b^{3r^3 f}$ otherwise, with $\mathsf{A}(y) \leq \mu\{g : \mathrm{bs}(yg) \leq b^{6n+2}\}$, and with $\mu(G_1(\mathfrak{s}))\int \mathsf{A}\,\mathrm{d}\dot{y} \leq b^{2r^3}|I|$. A function Φ_0 *has Schur's bound with constant* D if $\int |\int \varphi(y)\Phi_0(\gamma y)\mathrm{d}y|^2 \mathrm{d}\dot{\gamma} \leq D\int |\varphi|^2$ for integrable φ; it *has the Schur bounds with respect to* rep*,* B*,* D if there are continuous Bessel functions V_w for rep, indexed by the words w in pairs of basis indices, with $V_\emptyset = \Phi_0$; with the *derivative law*: for every word w, every curve $t \mapsto g(t)$ in $G_1(\mathfrak{s})$ and every real number t at which the curve is differentiable with $g'(t) = g(t)X$, X a complex $r \times r$ matrix,

$$(V_w \circ g)'(t) = \sum_{\alpha,\beta} X_{\alpha\beta}\, V_{(\alpha,\beta)w}(g(t))$$

(so $V_{(\alpha,\beta)w}$ plays the part of a right derivative of V_w along the matrix unit at (α,β), but the matrix units themselves need not be tangent to $G_1(\mathfrak{s})$, and the law is asked only along curves in the group); and with Schur's bound with constant $B^{2|w|}D$ for V_w when $|w| \leq 2n^2$. A *Bessel system* for $(\mathfrak{s}, B, f)$ consists of a non-empty finite set I, a cut pair with respect to B, f and an irreducible rep (every matrix commuting with all rep(h) is scalar), a real $\deg > 0$, the Schur bounds of the full function with respect to every $P \geq B^5$ and $D = |I|/\deg$, the inequality $8r \leq B$, and band limits. A function $Y\colon G_1(\mathfrak{s}) \to \mathbb{C}^I$ *has a peak at* x *of constant* P (for l and rep; the predicate $\mathrm{Peak}_{l,\mathrm{rep}}(Y; x, P)$ of the note on the discrete series, with the same four conditions) if it is a Bessel function for rep and

(P1) $Y(y) = \int \deg(l)\,\psi_l^{\mathsf{p}}(g^{-1})\,Y(yg)\,\mathrm{d}g$ for every y, that is $Y * (\deg(l)\psi_l^{\mathsf{p}}) = Y$, where $(F * e)(y) = \int e(a^{-1})F(ya)\,\mathrm{d}a$;

(P2) $Y(x) \neq 0$;

(P3) $|Y(y)| \leq \int |\psi_l^{\mathsf{p}}(x^{-1}s_1 y)|\,\mathrm{d}\nu_S(s)$ for every y;

(P4) $\int (2/\|\mathsf{p}^{-1}x^{-1}s_1 y\,\mathsf{p}\|)^{8r^2+1}\mathrm{d}\nu_S(s) \leq P|Y(x)|$ for every y.

Here ν_S is the measure of unit 133 (written $\nu_{\mathfrak{s}}$ in its note, where $\mathrm{d}\dot{y}$ is $\dot{\eta}_{\mathfrak{s}}$) on the group of the pairs $s = (s_1, s_2)$ with $s_2 \in \mathrm{U}(\kappa)$ and $s_1 \in \underline{s_2}\,\mathcal{U}(T)$, for which $\int f\,\mathrm{d}y = \iint f(x\,s_1\,y)\,\mathrm{d}\nu_S(s)\,\mathrm{d}\dot{y}$; and $\|\mathsf{p}^{-1}g\,\mathsf{p}\| = \|g\|$ because p is unitary, so the norm in (P4) is $\|x^{-1}s_1 y\|$. The exponent of the peak is written N in both notes (n is $\varrho + 1$). The filling S of Section 11, its levels, labelled terms, positions and lattices are those of unit 148; that section recalls what it uses.

From the background library. The notions and the notation are those of the note on the discrete series (unit 133). *Signs* are a map ε from the r basis indices to $\{\pm 1\}$, and $\mathrm{U}(\varepsilon)$ is the group of the complex matrices g with $g^*\,\mathrm{diag}(\varepsilon)g = \mathrm{diag}(\varepsilon)$. A *frame* p of $J_1(\mathfrak{s})$ for ε is a unitary matrix with $\mathsf{p}^* J_1(\mathfrak{s})\mathsf{p} = \mathrm{diag}(\varepsilon)$; then $g \mapsto \mathsf{p}^{-1}g\mathsf{p}$ maps $G_1(\mathfrak{s})$ onto $\mathrm{U}(\varepsilon)$. A *parameter* l is an integer vector on the basis indices, read as twice a Harish-Chandra parameter. It *has a large gap* A_0 (for ε) if its entries are distinct, every $l_i + r - 1$ is even, and $|l_i - l_{i'}| \geq 2(500r^3 + A_0)$ whenever $\varepsilon_i \neq \varepsilon_{i'}$. For such l, ψ_l is the function of Flensted-Jensen on $\mathrm{U}(\varepsilon)$, the matrix coefficient of the lowest K-type of the discrete series with the parameter l; $\psi_l^{\mathsf{p}}(g) = \psi_l(\mathsf{p}^{-1}g\,\mathsf{p})$ is this function in the frame, a function on $G_1(\mathfrak{s})$; and $\deg(l) = \left(\int_{\mathrm{U}(\varepsilon)} |\psi_l|^2\right)^{-1}$ is the *formal degree*.

(J1) *Witt's theorem* (unit 120): over a field with a non-trivial involution, if X, Y have left inverses and equal Gram matrices for an invertible hermitian J, then $gX = Y$ for

some $g \in \mathrm{U}(J)$. The second reader relates it to Witt's extension theorem (U. First, *Witt's extension theorem for quadratic spaces over semiperfect rings*, Corollary 4.9) as the theorem itself.

(J2) *Uniqueness of Haar measure* (unit 114): on a second countable locally compact group with a Haar measure that is also right invariant, every s-finite right invariant measure is a multiple, in $[0,\infty]$, of it; the multipliers 0 and ∞ are allowed. The regular case is classical (Halmos; in the form of F. van Doorn, *Formalized Haar measure*, §6, Theorem 12); the s-finite form with a multiplier in $[0,\infty]$ follows from Tonelli's theorem for s-finite measures (second reader, whose verdict on the relation is "unresolved"); no published place for this form is confirmed, and none was opened for this note.

(J3) *Unfolding* (unit 114): if $(u,g) \mapsto ug$ carries a Haar measure of $\mathcal{U}(T)$ times a measure μ' to μ, then $\int f \,\mathrm{d}\mu' = \int f \,\mathrm{d}\dot{\mu}$ for $\mathcal{U}(T)$-invariant $f \geq 0$, with $\dot{\mu}$ the quotient measure of μ. This is Weil's integration formula for a quotient; reference to be confirmed.

(J4) *Haar measure in a chart* (unit 115). For a hermitian $N \times N$ matrix J with $J^2 = 1$, the Haar measure $\mathrm{d}y$ of $\mathrm{U}(J)$ is the one *normalised by the Cayley chart*: the image of $\mathrm{j}(\eta)\,\mathrm{vol}\,\mathrm{U}(N)^{-1}\mathrm{d}\eta$ under the Cayley chart of $\mathrm{U}(J)$ (the normalisation by the compact dual; for $J = 1$ it is the Haar probability measure, for indefinite J the group has none). If a map Φ from an open set of parameters to $\mathrm{U}(J)$ is injective and differentiable on a measurable set A with $|\det(e^{-1} \circ \mathrm{sk} \circ \mathrm{D}^{\log}\Phi(p))| = w(p)$ for a measure-preserving e, then for almost every real t, and for all measurable $f \geq 0$ (the number t does not depend on f),

$$\int_A \frac{w(p)}{\mathrm{vol}\,\mathrm{U}(N)}\, f(\mathrm{e}^{\mathrm{i}t}\Phi(p))\,\mathrm{d}p \leq \int f \,\mathrm{d}y.$$

The Cayley chart of $\mathrm{U}(J)$ has $|\det(\mathrm{sk} \circ \mathrm{D}^{\log}\mathrm{cay}(\eta))| = \mathrm{j}(\eta)$. The catalogue of the write-up names L. K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains* (1963), ch. III; place not confirmed.

(J5) *Bruhat coordinates* (unit 118):

$$\int f(Z)\,\mathrm{d}Z = \int \prod_{a \leq \varrho} |c_a|^{4(\varrho - a)} \iint f(x_-^*\,\mathrm{diag}(c)\,x_+)\,\mathrm{d}x_-\,\mathrm{d}x_+\,\mathrm{d}c$$

over upper unitriangular $x_\pm$ and $c \in \mathbb{C}^n$; the Jacobian of the triangular decomposition, reference to be confirmed. And (unit 119) the convolution of the laws of the floors $0, \dots, \varrho$ at $\mathbb{C}$ is the image of $\prod_a \mathrm{dens}_a(c_a)\,\mathrm{d}c$ under $c \mapsto (w_\varrho(c)\chi_\varrho(c_\varrho))^{-1}$, with

$$\mathrm{dens}_a(c) = |c|^{2(k-1)+4(\varrho-a)} \quad (a < \varrho), \qquad \mathrm{dens}_\varrho(c) = \frac{2\pi^k}{(k-1)!}\max(|c|^2 - 1, 0)^{k-1};$$

and the Heisenberg measures of the floors convolve to a Haar measure of $\mathcal{U}(T)$, the image of $\mathrm{d}x\,\mathrm{d}\mu^0$ under $(x,u) \mapsto u \cdot \mathrm{lift}(x)$, $\mathrm{lift}(x) \in \mathcal{U}(T)$ with frame matrix the unitriangular matrix of entries x.

(J6) *A Lipschitz bound for holomorphic maps, centred* (unit 10): if f is holomorphic on the open ball of radius t_0 about c in a complex normed space, with values in a complex normed space, and $\|f\| \leq B$ there, then $\|f(z) - f(c)\| \leq \frac{2B}{t_0}\|z - c\|$ for every z of that ball; in particular the derivative of f at the centre c has norm at most $2B/t_0$.

The bound compares a point with the centre; it is not a Lipschitz bound between two arbitrary points of the ball. The second reader relates it to the Schwarz lemma (W. Rudin, *Real and complex analysis*, §12.2) as a weaker form: a consequence of the scalar lemma on the complex line through c and z, without its derivative estimate.

(J7) *A Sobolev chain inequality* (unit 10): for functions $f_w(x,\omega)$ indexed by words w in a finite alphabet $\mathcal{A}$, on a cube $[0,\epsilon]^D \times \Omega'$, $0 < \epsilon \le 1$, with $|\partial_d f_w| \le \Lambda_0 \sum_{l\in\mathcal{A}} |f_{lw}|$ and $\iint |f_w|^2 \le K$ for $|w| \le |D|$: $\int |f_\emptyset(0,\omega)|^2 \,\mathrm{d}\omega \le (2(1+(|\mathcal{A}|\Lambda_0)^2)/\epsilon)^{|D|}K$. Classed as elementary by the catalogue.

(J8) *Hua's integral bound* (unit 112): for a positive definite M, an invertible $k'\times k'$ matrix S, a finite set s of rows and $r' \ge k'$, $\int \det(M + \sum_{i\in\mathsf{s}} y_i^* y_i)^{-(|\mathsf{s}|+r')}\mathrm{d}x \le (\pi^{k'}/(k'!\,|\det S|^2))^{|\mathsf{s}|}\det M^{-r'}$, where $y_i = x_i S + a_i$ and the rows x_i run over $\mathbb{C}^{k'}$. The second reader relates it to L. K. Hua, *loc. cit.*, as a weaker form (an upper bound in place of the exact Gamma factors); no numbered result of Hua's book with this normalisation was located, and the second reader derives the bound from an iterated beta integral (NIST Digital Library of Mathematical Functions, 5.12.1 and 5.12.3). Reference to be confirmed.

(J9) *The Ostrowski–Taussky inequality* (unit 10): $\det H \le 2^N |\det\Theta|$ for a positive semidefinite $H \le \Theta + \Theta^*$; A. Ostrowski, O. Taussky, *On the variation of the determinant of a positive definite matrix*, Indag. Math. 13 (1951); the second reader: matches. And the *matrix Bessel inequality*: $X(1-EE^*)X^* \ge 0$ for $\|E\| \le 1$.

(J10) *Discrete series* (units 112, 115, 116, 117), for a parameter l with large gap A_0 and $|l_a| \le L$: $\deg(l)\psi_l^{\mathsf{p}}$ is an idempotent kernel on $G_1(\mathfrak{s})$ (integrable, bounded by $\deg(l)$, reproducing under convolution); $|\psi_l^{\mathsf{p}}(g)| \le (2/\|g\|)^c$ for $c \le 500r^3 + A_0$; $\deg(l)(2/\|g\|)^c$ has integral at most $8(2000(r+L+1)^6)^{2r^2}2^c$ for $c > 8r^2$; without the degree, $\int (2/\|g\|)^c \,\mathrm{d}g \le 16^c \,\mathrm{vol}\{g \in G_1(\mathfrak{s}) : \|g-1\| \le 1\}$ for every natural number $c > 8r^2$, and this volume is finite (unit 115); the module generated by ψ_l^{p} under right derivatives is unitary for the L^2 norm, its generator is a smoothing of ψ_l^{p}, and a derivative along a word w of a vector of the $|w|$-th step of its filtration has norm at most $((r+1)^4(L+r+1+|w|))^{|w|}$; and for a Bessel function F that is integrable for $\mathrm{d}\dot{y}$ with $F * e = F$, $e = \deg(l)\psi_l^{\mathsf{p}}$, and for $X = e * X$ integrable, the *Schur inequality*

$$\int \Big| \int \varphi(y)\,(F*X)(\gamma y)\,\mathrm{d}y\Big|^2 \mathrm{d}\dot{\gamma} \;\le\; \frac{\|X\|_2^2}{\deg(l)} \cdot \frac{\|F\|_2^2}{\deg(l)} \cdot \int |\varphi|^2 \,\mathrm{d}y$$

for integrable φ, with $\|X\|_2^2 = \int |X|^2 \mathrm{d}y$ and $\|F\|_2^2 = \int |F|^2 \mathrm{d}\dot{y}$; the formal degree enters twice. For the idempotent kernel the second reader cites Harish-Chandra, *Discrete series for semisimple Lie groups II* (1966), pp. 87–88; relation unresolved (second reader): the cited pages give the orthogonality with the formal degree, not the integrability for the explicit gap nor the normalisation of the measure, which the library proves.

(J11) *A tube* (unit 118): for $t_0 \ge 8r$ there are a compact $A \subseteq \{\|y\| \le t_0^{20n}\}$ and a measurable $\Omega_0 \subseteq \mathcal{U}^0$ with $0 < \mu^0(\Omega_0) < \infty$, $\Omega_0 \cdot \{\|y\| \le t_0\} \subseteq A$ and $\int \mathbf{1}_A(uy)\,\mathrm{d}\mu^0(u) \le t_0\,\mu^0(\Omega_0)$ for all y. Conjugation by $s \in \mathcal{Q}_n(T)$ preserves μ^0.

4 Frame vectors and the Bessel size

Lemma 4.1 (frame vectors)**.** *Let $y, s \in G_1(\mathfrak{s})$ and $a \le \varrho$.* (i) *If s lies in the group generated by $\mathcal{U}^0$ and $\mathcal{H}_n(T)$, then $(sy)^{-1} f_a = y^{-1} f_a$.* (ii) *If $s \in \mathcal{U}(T)$, then $y^{-1} f_a = \sum_b \mathrm{fm}(s)_{ba}\, (sy)^{-1} f_b$.* (iii) *For a finite set ι, a map e from ι to the frame positions, $c\colon \iota \to \mathbb{C}$ and a basis index i: $\left|\sum_{\alpha\in\iota} c_\alpha\, s_{i,f_{e(\alpha)}}\right| \le \|sy\| \cdot \left|\sum_{\alpha\in\iota} c_\alpha\, y^{-1} f_{e(\alpha)}\right|$.*

Proof. (i) s^{-1} fixes the frame. (ii) $y^{-1} f_a = (sy)^{-1} s f_a$, and $s f_a = \sum_b \mathrm{fm}(s)_{ba} f_b$ because $s \in \mathcal{U}(T)$ maps f_a into the span of $f_0, \dots, f_a$. (iii) $\sum c_\alpha s f_{e(\alpha)} = (sy) \sum c_\alpha y^{-1} f_{e(\alpha)}$, and a coordinate of a vector is at most its norm. □

LEAN `frameVector_mul` · `frameVector_eq_sum` · `norm_sum_le_mul_norm_sum_frameVector` · `continuous_frameVector` · `norm_frameVector_le`

The frame vectors depend continuously on y, and $|y^{-1} f_a| \le \|y^{-1}\| = \|y\|$.

Lemma 4.2 (the Bessel size)**.** *Let $x, y, y' \in G_1(\mathfrak{s})$ and t_0 a real number.* (i) *If $\mathrm{bs}(y) \le t_0$, there is $u \in \mathcal{U}(T)$ with $\|uy\| \le t_0$.* (ii) $\mathrm{bs}(y') \le \|x^{-1} y'\| \cdot \mathrm{bs}(x)$. (iii) *$y \mapsto \mathrm{bs}(y)$ is continuous.*

Proof. For $s \in \mathcal{Q}_n(T)$ we have $\mathrm{bs}(y) \le \|sy\| + 1$ and $\mathrm{bs}(sy) = \mathrm{bs}(y)$, by definition. (i) Choose $s \in \mathcal{Q}_n(T)$ with $\|sy\|+1 < t_0+1$, and $h \in \mathrm{U}(\kappa)$ with $\underline{h} s \in \mathcal{U}(T)$; $\underline{h}$ is unitary, so $\|\underline{h} sy\| = \|sy\| \le t_0$. (ii) For $s \in \mathcal{Q}_n(T)$, $\|sy'\|+1 \le \|sx\|\|x^{-1}y'\|+1 \le \|x^{-1}y'\|(\|sx\|+1)$, because every element of $G_1(\mathfrak{s})$ has norm at least 1; take the infimum. (iii) By (ii) in both directions, $\mathrm{bs}(y')/\mathrm{bs}(y)$ lies between $\|y'^{-1}y\|^{-1}$ and $\|y^{-1}y'\|$, which tend to 1. □

LEAN `exists_norm_le_of_besselSize_le` · `besselSize_le_norm_mul` · `continuous_besselSize` · `besselSize_le` · `besselSize_besselS_mul`

Lemma 4.3 (bounded frame and dual frame)**.** *Let $x \in G_1(\mathfrak{s})$ and M a real number with $1 \le M$, $x \in \mathrm{FB}(M)$ and $|x^{-1} e_a^*| \le M$ for all $a < \varrho$. Then $\|x\| + 1 \le 8(\varrho+1)M$.*

Proof. Let $B = x^{-1}$. As $BJ_1(\mathfrak{s})B^* = J_1(\mathfrak{s})$, $BB^* = J_1(\mathfrak{s}) + B(1-J_1(\mathfrak{s}))B^*$ and $\|B\|^2 \le 1 + \|B(1-J_1(\mathfrak{s}))\|\,\|B\|$. The non-zero columns of $B(1-J_1(\mathfrak{s}))$ are $\pm(Be_a - Be_a^*)$ and $2Bz$, of norm at most $2M$; there are $2\varrho+1$ of them, so $\|B(1-J_1(\mathfrak{s}))\| \le 2M\sqrt{2\varrho+1}$. The quadratic inequality gives $\|B\| \le 8(\varrho+1)M - 1$, and $\|x\| = \|B\|$. □

LEAN `norm_le_of_mem_frameBox` · `norm_inclMatrix_le_one` · `norm_J₁` · `norm_inner_J₁_le`

Here $\|J_1(\mathfrak{s})\| = 1$, matrices of inclusions of coordinates have norm at most 1, and $|\langle v, J_1(\mathfrak{s}) w\rangle| \le |v||w|$.

Lemma 4.4 (the frame Gram matrix)**.** *Let $x, y' \in G_1(\mathfrak{s})$ and $a', b \le \varrho$. Then $Z(x)_{a'b} = \langle y'^{-1} f_{a'}, J_1(\mathfrak{s})(x^{-1}y')^{-1} f_b\rangle$, hence $|Z(x)_{a'b}| \le |y'^{-1} f_{a'}| \cdot |(x^{-1}y')^{-1} f_b|$; $Z(sx) = Z(x) = Z(xs)$ for s in the group generated by $\mathcal{U}^0$ and $\mathcal{H}_n(T)$; and $Z(x) = \mathrm{fm}(s)^* Z(sx)$ for $s \in \mathcal{U}(T)$.*

Proof. $h(f_{a'}, x f_b) = h(y'^{-1} f_{a'}, y'^{-1} x f_b)$; the rest follows from Lemma 4.1(i),(ii). □

LEAN `realFrameGram_eq_inner` · `norm_realFrameGram_le_mul` · `realFrameGram_mul_invariant` · `realFrameGram_eq_frameMatrix_mul`

Lemma 4.5 (the Bessel size from a dual system)**.** *Let $x \in G_1(\mathfrak{s})$, M a real number with $1 \le M$, and $V_0, V_1, \dots$ vectors of $\mathbb{C}^r$ such that $x^{-1} f_a = V_a$ for $a \le \varrho$, $|V_a| \le M$ for $a \le 2\varrho$, and, for all a, b with $a + b \le 2\varrho$, $\langle V_a, J_1(\mathfrak{s}) V_b\rangle = -1$ if $a + b = 2\varrho$ and $= 0$ otherwise. Then $\mathrm{bs}(x) \le 8(\varrho+1)M$.*

Proof. Let $f'_0, \dots, f'_\varrho$ be the Gram–Schmidt orthogonalisation of the frame vectors of x. There is $u \in \mathcal{U}(T)$, with unitriangular frame matrix, such that $(ux)^{-1} f_a = f'_a$ (Lemma 4.1(ii)). The hypothesis gives $\langle V_{2\varrho-a}, J_1(\mathfrak{s}) f'_a \rangle = -1$, so $1 \le M |f'_a|$; and $|f'_a| \le |V_a| \le M$. Put $v_a = |f'_a|^{-2} f'_a$, of norm at most M. The vectors $J_1(\mathfrak{s}) v_b$, $b < \varrho$, are dual to the frame vectors of ux and span an isotropic space (the frame Gram matrix of the identity vanishes at the isotropic positions); so by a lemma of unit 118 there is q with $(ux)^{-1} \mathrm{u}(q) e^*_b = J_1(\mathfrak{s}) v_b$. The element $y = \mathrm{u}(q)^{-1} ux$ has frame vectors f'_a and dual columns $J_1(\mathfrak{s}) v_b$, all of norm at most M, and $\mathrm{bs}(x) \le \|y\| + 1 \le 8(\varrho+1)M$ by Lemma 4.3. □

LEAN `besselSize_le_of_dual`

5 The section

Lemma 5.1 (the section is unitary; measurability)**.** *Let $Z \in \mathcal{C}(\mathfrak{s})$. Then the matrix*

$$\mathsf{m}(Z, Z^*, \sqrt{\mathrm{cd}(Z)})$$

of Definition 2.2 is unitary for the form $J_1(\mathfrak{s})$. The map $Z \mapsto \gamma(Z)$ is measurable.

Proof. Let C be the matrix of the $2n$ columns $(F_{\mathfrak{s}} \mid F'_{\mathfrak{s}})$; they are orthonormal, and $J_1(\mathfrak{s}) = 1 - CC^* + CJC^*$ with $J = C^* J_1(\mathfrak{s}) C$. With $\Delta = \mathrm{d}_\varrho(\mathrm{cd}(Z)^{-1/2})$ and the block matrices $\mathsf{x} = \left(\begin{smallmatrix} Z^{-1} & \mathrm{c}(Z^*,Z)\Delta \\ 0 & 1 \end{smallmatrix}\right)$, $\mathsf{y} = \left(\begin{smallmatrix} \mathrm{c}(Z,Z^*) & Z^{*-1}\Delta \\ \mathrm{d}_\varrho(\sqrt{\mathrm{cd}(Z)}) & 0 \end{smallmatrix}\right)$, the section matrix is $1 - CC^* + C\mathsf{y}\mathsf{x}^{-1}C^*$, and $\mathsf{y}^* J \mathsf{y} = \mathsf{x}^* J \mathsf{x}$; both are matrix identities that the formal text verifies by expansion. So $\mathsf{y}\mathsf{x}^{-1}$ is unitary for J and the section matrix is unitary for $J_1(\mathfrak{s})$. Measurability: the map is continuous on the open set $\mathcal{C}(\mathfrak{s})$ and constant outside. □

LEAN `measurable_gramSection`

Lemma 5.2 (differentiability and size of the section matrix)**.** (i) *Let $Z, W \in \mathcal{C}(\mathfrak{s})$, and put $s(Z', W') = \sqrt{1 - \mathrm{ic}(Z')\,\mathrm{ic}(W')}$. Then the map $(Z', W') \mapsto \mathsf{m}(Z', W', s(Z', W'))$ is complex differentiable at (Z, W).* (ii) *For $Z \in \mathcal{C}(\mathfrak{s})$ the map $Z' \mapsto \mathsf{m}(Z', Z'^*, \sqrt{\mathrm{cd}(Z')})$ is real differentiable at Z.* (iii) *Let $Z \in \mathcal{C}(\mathfrak{s})$, P, t_0 real numbers with $t_0 > 0$ and $8 \le P$, and suppose $Y \in \mathrm{Cell}_{\le P}$ for every Y with $\|Y - Z\| < t_0$. Then the matrix of (ii) at Z has norm at most P^{12}, and its real derivative at Z has norm at most $2P^{12}/t_0$.*

Proof. (i) Every entry is a polynomial in the entries of Z', W', of their inverses, and in s, s^{-1}; and $1 - \mathrm{ic}(Z')\,\mathrm{ic}(W')$ stays in the slit plane near (Z, W), because $|\mathrm{ic}(Z)|, |\mathrm{ic}(W)| < 1$. (ii) Compose with the real-linear map $Z' \mapsto (Z', Z'^*)$. (iii) For (X, Y) within t_0 of (Z, Z^*) both X and Y^* are within t_0 of Z, so their norms, those of their inverses and $\mathrm{cd}(X)^{-1/2}$ are at most P; moreover $|1 - \mathrm{ic}(X)\,\mathrm{ic}(Y)| \ge 1 - |\mathrm{ic}(X)| \ge \frac{1}{4}\,\mathrm{cd}(X)$, which bounds $|s|^{-1}$ by $2P$. The triangle inequality in Definition 2.2 then bounds the section matrix by P^{12} on this ball, for $P \ge 8$. A holomorphic map bounded by P^{12} on a ball of radius t_0 moves by at most $(2P^{12}/t_0)$ times the distance from the centre, so its derivative at the centre has norm at most $2P^{12}/t_0$ (J6); restrict to $Y = X^*$. □

LEAN `differentiableAt_sectionMatrix_uncurry` · `differentiableAt_sectionMatrix` · `norm_sectionMatrix_le`

Lemma 5.3 (elements over the open cell)**.** *Let $\gamma \in G_1(\mathfrak{s})$ and $Z \in \mathcal{C}(\mathfrak{s})$ with $Z(\gamma) = Z$. Then there are parameters q, q' of $\mathcal{U}^0$ and $h', h \in \mathrm{U}(\kappa)$ with $\gamma = \mathrm{u}(q)\,\underline{h'}\,\gamma(Z)\,(\mathrm{u}(q')\,\underline{h})^{-1}$.*

Proof. For $s \in G_1(\mathfrak{s})$ with $Z(s) = Z$ the $2n$ columns $(F_{\mathfrak{s}} \mid sF_{\mathfrak{s}})$ have the Gram matrix $\left(\begin{smallmatrix} N_0 & Z \\ Z^* & N_0 \end{smallmatrix}\right)$ with $N_0 = \mathrm{diag}(0, \dots, 0, -1)$; it depends on Z only, and it is invertible because its determinant is a non-zero multiple of $\mathfrak{g}(Z)$. So the columns have a left inverse. This applies to γ and to $\gamma(Z)$, and Witt's theorem (J1) gives $g \in G_1(\mathfrak{s})$ with $gF_{\mathfrak{s}} = F_{\mathfrak{s}}$ and $g\gamma F_{\mathfrak{s}} = \gamma(Z)F_{\mathfrak{s}}$. An element that fixes the frame is $\underline{h}\mathrm{u}(q)^{-1}$ for some $h \in \mathrm{U}(\kappa)$ and some parameter q (unit 118); apply this to g and to $\gamma(Z)^{-1}g\gamma$. □

LEAN `exists_eq_mul_gramSection_mul`

Lemma 5.4 (staying in the open cell)**.** *Let M, P be real numbers and Z, Z' complex $n \times n$ matrices with $M^{-1} \le \mathfrak{g}(Z)$, $8(\varrho + 1) \le M$, $|Z_{ab}| \le M^2$ for all a, b, $M^{9n} \le P$ and $\|Z' - Z\| < P^{-6}$. Then $\mathfrak{g}(Z)/2 \le \mathfrak{g}(Z')$ and $Z' \in \mathrm{Cell}_{\le P}$.*

Proof. Write $D_n = \det$. On the unit ball about Z the entries are at most M^3, so the functions $Y \mapsto D_l(Y)^2$, $l = \varrho$ and $l = n$ (the squares of the two leading minors), are holomorphic there and bounded by a power of M that is at most P. By (J6), applied on this ball with the centre Z, each of them moves by at most $M^{-1}/4$ from Z to Z'; hence so do their moduli $|D_l|^2$, because $\big|\,|a|-|b|\,\big| \le |a-b|$, and $\mathfrak{g}(\cdot) = |D_n|^2 - |D_\varrho|^2$ (a difference of two moduli, not itself holomorphic) moves by at most $M^{-1}/2 \le \mathfrak{g}(Z)/2$. So $\mathfrak{g}(Z') > 0$ and $Z' \in \mathcal{C}(\mathfrak{s})$. From $M^{-1} \le 2|\det Z'|^2\,\mathrm{cd}(Z')$ and $\mathrm{cd}(Z') \le 1$ one gets lower bounds for $|\det Z'|$ and for $\mathrm{cd}(Z')$; Cramer's rule bounds $\|Z'^{-1}\|$ by minors over $|\det Z'|$; all bounds are powers of M with exponent at most $9n$. □

LEAN `mem_openCell_and_bounds_of_near`

6 The chart

Lemma 6.1 (the logarithmic derivative of the outer factors)**.** *Let $\mathcal{V}$ be a real normed space, $p \in \mathcal{V}$, and $\mathsf{z} = (\mathsf{z}_1, \mathsf{z}_2)$ a map from $\mathcal{V}$ to pairs of a real $\kappa \times \kappa$ matrix and a parameter of $\mathcal{U}^0$ that is differentiable at p with derivative z'. Then $\Phi(p') = \mathrm{u}(\mathsf{z}_2(p'))\,\underline{\mathrm{cay}(\mathsf{z}_1(p'))}$ is differentiable at p, and for every v, with $\mathsf{z}'(v) = (\eta, (q_1, q_2))$,*

$$\mathrm{D}^{\log}\Phi(p)(v) = \mathrm{bt}\Big(\mathrm{D}^{\log}\mathrm{cay}(\mathsf{z}_1(p))(\eta),\ q_1\,\mathrm{cay}(\mathsf{z}_1(p)),\ q_2 + \mathrm{sk}\big(\textit{the skew part of } \mathsf{z}_2(p)_1\,q_1^*\big)\Big).$$

Proof. By the formulas of unit 118, $\mathrm{u}(q) = 1 + J_1(\mathfrak{s})\beta + \frac{1}{2}(J_1(\mathfrak{s})\beta)^2$ with $\beta = \mathrm{bt}(0, q)$, and $\underline{c} = 1 + \mathrm{bt}(c - 1, 0)$. Both are differentiable in the parameters, the Cayley chart by (J4); differentiate the product by the product rule and multiply by $\Phi(p)^* J_1(\mathfrak{s})$. The matrix identity that remains is verified by the formal text by expansion. □

LEAN `logDerivU_unipotentOf_mul_embedH`

Lemma 6.2 (coordinates)**.** *There is a real-linear isomorphism e from the space of the parameters onto the space of the real $r \times r$ matrices that preserves Lebesgue measure and has $e^{-1} = \mathrm{cc} \circ \mathrm{skew}$.*

Proof. $\mathrm{cc} \circ \mathrm{skew}$ maps the sum of the squares of the entries of a real matrix to the sum of the squares of the real coordinates of the parameter (a regrouping of blocks),

and the dimensions agree; a linear map with this property is a measure-preserving isomorphism (the Gaussian integral is preserved). □

LEAN `exists_symm_eq_chartCoords_skewOfReal`

Proof of Theorem A(i). Take e from Lemma 6.2, and let $p = (Z, \theta, q, \eta, q')$ with $Z \in \mathcal{C}(\mathfrak{s})$. The chart is a product $\Phi_1(p)\,\gamma(Z)\,\Phi_2(p)^{-1}$ with $\Phi_1 = \mathrm{u}(q)\underline{\mathrm{cay}(\tilde{\theta})}$, $\Phi_2 = \mathrm{u}(q')\underline{\mathrm{cay}(\eta)}$; these are differentiable by Lemma 6.1, and the section by Lemma 5.2(ii) and because $\gamma(\cdot)$ has the section matrix near Z. By the product rule, the logarithmic derivative of the chart is the sum of the logarithmic derivative of Φ_1 conjugated by $\gamma(Z)\Phi_2^{-1}$, of that of the section conjugated by Φ_2^{-1}, and of minus that of Φ_2 conjugated by Φ_2^{-1}. For G invertible, the map $S \mapsto G^* S G$ on the skew-hermitian matrices, read through $e^{-1} \circ \mathrm{sk}$, has a determinant of absolute value $|\det G|^{2r}$ (a lemma of unit 115). The formal proof applies this with $G = \Phi_2(p)K$, where $\Delta = \mathrm{d}_{\varrho}(\mathrm{cd}(Z)^{-1/2})$,

$$\mathsf{M}(A, B) = F_{\mathfrak{s}}((A\Delta - 1)F_{\mathfrak{s}}^* + B\Delta F_{\mathfrak{s}}'^*), \qquad K = 1 + \mathsf{M}(Z^{-1}, \mathsf{c}(Z^*, Z)).$$

The matrix K is chosen so that $\gamma(Z)K = \mathsf{R} + \mathsf{M}(\mathsf{c}(Z, Z^*), Z^{*-1})$ for a constant matrix R, and $\mathrm{bt}(\cdot)\,\mathsf{M}(A, B) = 0 = \mathsf{M}(A, B)^*\,\mathrm{bt}(\cdot)$; and $|\det K| = |\det(Z^{-1}\Delta)|$, so $|\det G|^2 = |\det Z|^{-2}\,\mathrm{cd}(Z)^{-1} = \mathrm{g}(Z)^{-1}$. After the congruence by G the endomorphism is block triangular in the variables $(Z; \theta, q; \eta, q')$, with these diagonal blocks:

- on Z: the map $S \mapsto (Z^{*-1}\Delta)^* S\,(Z^{-1}\Delta)$ of the complex $n \times n$ matrices, because the Z-component of cc reads $F_{\mathfrak{s}}'^*(\cdot)F_{\mathfrak{s}}$ and $Z' \mapsto F_{\mathfrak{s}}^* J_1(\mathfrak{s})\gamma(Z')F_{\mathfrak{s}} = Z'$ has the identity as derivative; its real determinant has absolute value $|\det(Z^{-1}\Delta)|^{4n} = \mathrm{g}(Z)^{-2n}$;
- on (θ, q) and on (η, q'): by Lemma 6.1, maps that are triangular in themselves, with the logarithmic derivative of a Cayley chart (for θ: of its column at k_0) and right multiplications by unitary matrices on the diagonal; their determinants have absolute values $\mathrm{j}(\tilde{\theta})$ and $\mathrm{j}(\eta)$ (J4).

So $|\det(e^{-1} \circ \mathrm{sk} \circ \mathrm{D}^{\log}\mathrm{ch}(p))| \cdot \mathrm{g}(Z)^{-r} = \mathrm{g}(Z)^{-2n}\mathrm{j}(\tilde{\theta})\mathrm{j}(\eta)$, and $r - 2n = k - 1$. I read the formal computation line by line; that the blocks are as stated is verified there by expansion of the matrices and is not reproduced here. □

Proof of Theorem A(ii). Let two parameters with $Z \in \mathcal{C}(\mathfrak{s})$ have the same image g. By Lemma 4.4 and $Z(\gamma(Z)) = Z$, the frame Gram matrix of g is Z, so the two Z agree. Multiplying the equation $g\,\Phi_2 = \Phi_1\gamma(Z)$ by $F_{\mathfrak{s}}Z^{-1}\Delta$ on the right isolates the frame and dual frame columns of the left factor $\Phi_1 = \mathrm{u}(q)\underline{\mathrm{cay}(\tilde{\theta})}$: the map $q \mapsto \mathrm{u}(q)D_{\mathfrak{s}}$ is injective, and the column at k_0 of a Cayley chart determines θ (unit 115). So q and θ agree, then the right factors agree, and q' and η agree because the Cayley chart is injective where it is regular. □

Proof of Theorem A(iii). Let $w(p) = \mathrm{g}(Z)^{k-1}\mathrm{j}(\tilde{\theta})\mathrm{j}(\eta)$. By (i) and (ii) the chart satisfies the hypotheses of (J4) on the set of the parameters with $Z \in \mathcal{C}(\mathfrak{s})$; so there is a real t for which the inequality of (J4) holds for all functions. We apply it to the function $y \mapsto f(\mathrm{e}^{-\mathrm{i}t}y)$, which removes the scalar:

$$\int_{Z \in \mathcal{C}(\mathfrak{s})} \frac{w(p)}{\mathrm{vol}\,\mathrm{U}(r)}\, f(\mathrm{ch}(p))\,\mathrm{d}p \le \int f(\mathrm{e}^{-\mathrm{i}t}y)\,\mathrm{d}y = \int f\,\mathrm{d}y,$$

the last equality by the left invariance of $\mathrm{d}y$ (the scalar $\mathrm{e}^{-\mathrm{i}t}$ is an element of $G_1(\mathfrak{s})$). In the chart integral, the integral over η with the weight $\mathrm{j}(\eta)$ is the Haar integral

over $h = \mathrm{cay}(\eta) \in \mathrm{U}(\kappa)$, the integrals over q, q' are those over $\mathcal{U}^0$, and the integral over θ of the normalised weight $\mathsf{j}(\tilde{\theta})$ is 1; the constants combine to $c_G(\mathfrak{s})$. For each θ the hypothesis with $h' = \mathrm{cay}(\tilde{\theta})$ bounds L by the inner integral; integrate over θ. □

Lemma 6.3 (orthonormality of the full function)**.** *Let B be a real number, f a natural number, and X a Bessel system for $(\mathfrak{s}, B, f)$ with full function Φ_0. Then for all $i, i' \in I$: $\int \Phi_{0,i}\,\overline{\Phi_{0,i'}}\,\mathrm{d}\dot{y} = 1$ if $i = i'$, and $= 0$ otherwise.*

Proof. The products are integrable. The matrix $\mathcal{G}$ of these integrals commutes with every $\mathrm{rep}(h)$: substitute $y \mapsto \underline{h}y$, which preserves $\mathrm{d}\dot{y}$ because conjugation by $\underline{h}$ preserves the Haar measure of $\mathcal{U}(T)$, and use $\Phi_0(\underline{h}y) = \mathrm{rep}(h)\Phi_0(y)$. As rep is irreducible, $\mathcal{G}$ is a scalar; its trace is $\int |\Phi_0|^2 \mathrm{d}\dot{y} = |I|$. □

LEAN `BesselSystem.orthonormal` · `euclidean_enorm_sq_eq_sum`

7 Mean squares over the chart

Lemma 7.1 (a weight supported near one frame matrix)**.** *Let $w, \varphi \colon G_1(\mathfrak{s}) \to [0, \infty]$ be measurable, $K \in [0, \infty]$ and R real with $1 \le R$. Suppose w is invariant under left translations by $\mathcal{Q}_n(T)$; $\int \varphi(uy)\,\mathrm{d}\mu^0(u) \le K$ for all y; and for all y with $\varphi(y) \neq 0$ and all $s \in \mathcal{U}(T)$ with $\varphi(sy) \neq 0$, all entries of $\mathrm{fm}(s)$ have absolute value at most R. Then $\int \varphi\, w\,\mathrm{d}y \le K\,(2R)^{2n^2} \int w\,\mathrm{d}\dot{y}$.*

Proof. Unfold: $\int \varphi w\,\mathrm{d}y = \int w(y) \int_{\mathcal{U}(T)} \varphi(uy)\,\mathrm{d}u\,\mathrm{d}\dot{y}$. Fix y; if the inner integral is not 0 choose u_0 with $\varphi(u_0 y) \neq 0$ and translate by it. Write the Haar measure of $\mathcal{U}(T)$ as the image of $\mathrm{d}x\,\mathrm{d}\mu^0(u')$ under $(x, u') \mapsto u'\,\mathrm{lift}(x)$ (J5); the frame matrix of $u'\,\mathrm{lift}(x)$ is the unitriangular matrix of x, so the integrand vanishes unless all $|x_{ab}| \le R$. The inner integral over u' is at most K, and the volume of the box of the x is at most $(\pi R^2)^{n^2} \le (2R)^{2n^2}$. □

LEAN `lintegral_mul_le_of_frameMatrix_le`

Lemma 7.2 (a mean square along the section)**.** *Let P, c be real numbers, $D \in [0, \infty]$, rep as before, $\Phi_0 \colon G_1(\mathfrak{s}) \to \mathbb{C}^I$ with the Schur bounds with respect to rep, P and D, and Z a complex $n \times n$ matrix. Suppose $8r \le P$, $0 < c$, and that every Y with $\|Y - Z\| < P^{-6}$ has $c \le \mathsf{g}(Y)$ and $Y \in \mathrm{Cell}_{\le P}$. Let $\varphi \colon G_1(\mathfrak{s}) \to \mathbb{C}$ be measurable and integrable with $\varphi(y) = 0$ whenever $\|y\| > P^{20}$. Then*

$$c_G(\mathfrak{s})\,c^{k-1} \int_{\mathrm{U}(\kappa)} \int_{\{u \in \mathcal{U}^0 : \|u\| \le P^{20}\}} \Big| \int \varphi(y)\,\Phi_0\big(\gamma(Z)(u\,\underline{h})^{-1}y\big)\,\mathrm{d}y\Big|^2 \mathrm{d}\mu^0(u)\,\mathrm{d}h \le P^{2000n^2}\,D \int |\varphi|^2\,\mathrm{d}y.$$

Proof. Let V_w be the derivative functions of the Schur bounds. For a word w in pairs of pairs of indices put $\psi_w(y) = \varphi(y) \prod (y^{-1})_{\alpha\delta} y_{\gamma\beta} / P^{40}$ over the letters of w; then $|\psi_w| \le |\varphi|$, because $\|y\|, \|y^{-1}\| \le P^{20}$ where $\varphi \neq 0$. Put $f_w(x, (h, u)) = \int \psi_w(y) V_{w_1}(\gamma(Z + \epsilon(x))(u\underline{h})^{-1}y)\,\mathrm{d}y$, for x in the cube $[0, P^{-8}]^{2n^2}$ of the real coordinates ϵ of the matrices, which stays within $P^{-6}/2$ of Z. *Derivatives.* By Lemma 5.2(iii) with $t_0 = P^{-6}/2$, the curve $x_d \mapsto \gamma(Z + \epsilon(x))$ has derivative $g\mathsf{A}$ with $\|P^{40}\mathsf{A}\| \le P^{112}$; differentiating under the integral and using the rule for right derivatives of V_{w_1} gives $|\partial_d f_w| \le P^{112} \sum_l |f_{lw}|$. *The cube.* For $|w| \le 2n^2$ the integral of $|f_w|^2$ over the cube and over (h, u), times $c_G(\mathfrak{s})c^{k-1}$, is at most the Haar integral of $\chi(\gamma) |\int \psi_w(y) V_{w_1}(\gamma y)\mathrm{d}y|^2$ over the γ with

$Z(\gamma)$ in the cube, by Theorem A(iii) and $\mathsf{g}(\cdot) \geq c$; here χ is a cutoff in the $\mathcal{U}^0$-direction. By Lemma 7.1, with $R = P^2$ from Lemma 4.4 and the bounds on $Z(\cdot)$ and its inverse, and by Schur's bound for V_{w_1}, this is at most $(2P^2)^{2n^2} P^{2|w|} D \int |\varphi|^2$. *Conclusion.* The Sobolev chain inequality (J7) bounds the value at $x = 0$, $w = \emptyset$, which is the left side, by $(2(1+(|\mathcal{A}|P^{112})^2)P^8)^{2n^2}$ times the cube bound; with $8r \leq P$ the product is at most $P^{2000n^2} D \int |\varphi|^2$. This last comparison is the formal text's arithmetic. □

LEAN `lintegral_gramSection_integral_le`

8 The tower measure and the gauge at the complex place

Proof of Theorem B. Write $\mathcal{L} = \mathcal{L}_{\mathbb{C}}(\mathfrak{h})$, $\lambda = \lambda(\mathfrak{h})$. (ii) is the choice of λ (Definition 2.7). By Theorem A(ii) of Note T and (J5), $\mathsf{M}(\mathcal{L}) = \eta_{<n} * \mathsf{M}_{\geq 0}(\mathcal{L})$ with $\eta_{<n}$ a Haar measure of $\mathcal{U}(T)$; so $(u, g) \mapsto ug$ carries this Haar measure times $\mathsf{M}_{\geq 0}(\mathcal{L})$ to $\lambda \, \mathrm{d}y$, and (J3) gives (iii). $\lambda \neq 0$ because the tower measure is not zero (Note T), and $\lambda \neq \infty$ follows from (iv).

(iv) It suffices to test one function, the indicator f of the compact set $\{\|g\| \leq 2\}$, of positive finite Haar measure: we show $\frac{\mathrm{vol}\,\mathrm{U}(k)}{\mathrm{vol}\,\mathrm{U}(r)} \int f \, \mathrm{d}(\lambda \, \mathrm{d}y) \leq \int f \, \mathrm{d}y$. By the lemma of Note T on moving a Weyl law to the left, applied to all floors below ϱ, the left measure is

$$\lambda \, \mathrm{d}y = \eta_{<n}^{-1} * \theta_n(\mathcal{L}) * (\varpi_0 * \cdots * \varpi_{\varrho-1} * \varpi'_\varrho) * \theta_n(\mathcal{L}) * \eta_{<n},$$

where ϖ'_ϱ is the image of the density of the boost and $\eta_{<n}^{-1}$ is $\eta_{<n}$, which is invariant under inversion. By (J5) the middle factor is the image of $\prod_a \mathrm{dens}_a(c_a) \, \mathrm{d}c$ under $c \mapsto g_c^{-1}$, $g_c = w_\varrho(c) \chi_\varrho(c_\varrho)$, and the two Haar measures of $\mathcal{U}(T)$ are images of $\mathrm{d}x \, \mathrm{d}\mu^0$. So a point of the left measure is $\mathrm{lift}(x)^{-1} u \, \underline{h_1} \, g_c^{-1} \, \underline{h_2} \, u' \, \mathrm{lift}(x')$ with $u, u' \in \mathcal{U}^0$. Since $f(g^{-1}) = f(g)$ (as $\|g^{-1}\| = \|g\|$ on $G_1(\mathfrak{s})$) and the measures $\eta_{<n}$, μ^0 and the Haar measure of $\mathrm{U}(\kappa)$ are invariant under inversion, we may integrate f over the inverse elements instead, that is over the elements with g_c in the place of g_c^{-1}. Only the c with $|c_\varrho| > 1$ count. For such c, unitriangular x, x' and $h_1, h_2 \in \mathrm{U}(\kappa)$, the element $\gamma = \mathrm{lift}(x)^{-1} \underline{h_1} g_c \underline{h_2} \mathrm{lift}(x')$ has the frame Gram matrix $Z = x^* \, \mathrm{diag}(c) \, x'$ (unit 148), with $\mathsf{g}(Z) = \prod_{a<\varrho} |c_a|^2 \, (|c_\varrho|^2 - 1) > 0$, so $Z \in \mathcal{C}(\mathfrak{s})$, and by Lemma 5.3 $\gamma = \mathrm{u}(q) \underline{h'} \gamma(Z) (\mathrm{u}(q') \underline{h})^{-1}$. Since conjugation by elements of $\mathcal{Q}_n(T)$ and translation preserve μ^0 (J11), and f is invariant under multiplication by unitary $\underline{h}$ on both sides, the integral of f over the remaining $\mathcal{U}^0$-variables equals $\iint f(u \, \underline{h'} \gamma(Z) (v \underline{h})^{-1}) \mathrm{d}\mu^0(u) \mathrm{d}\mu^0(v)$ for each h', h. The densities match: in Bruhat coordinates (J5) $\mathrm{d}Z = \prod_a |c_a|^{4(\varrho-a)} \mathrm{d}x \, \mathrm{d}x' \, \mathrm{d}c$, and

$$\frac{\mathrm{vol}\,\mathrm{U}(k)}{\mathrm{vol}\,\mathrm{U}(r)} \prod_{a \leq \varrho} \mathrm{dens}_a(c_a) = c_G(\mathfrak{s}) \, \mathsf{g}(Z)^{k-1} \prod_{a<\varrho} |c_a|^{4(\varrho-a)}.$$

So $\frac{\mathrm{vol}\,\mathrm{U}(k)}{\mathrm{vol}\,\mathrm{U}(r)} \int f \, \mathrm{d}(\lambda \, \mathrm{d}y)$ is at most the right side of the hypothesis of Theorem A(iii), for every h', and that theorem gives the claim. □

Lemma 8.1 (an orthonormal system in a kernel)**.** *Let s_0 be a natural number, L a real number, m, ι finite sets and C a complex $\mathsf{m} \times \iota$ matrix with $|\iota| = s_0 + 1 + 3n$,*

$4 \le L$, $\operatorname{rank} C \le 3n$ *and* $|\iota| \le L$. *Then there are a subset* $D \subseteq \iota$ *and a complex* $\iota \times D$ *matrix* O *with* $CO = 0$, $O^*O = 1$ *and, for every natural number* j,

$$\int_{\{u \in \mathbb{C}^\iota : |u_i| \le L^j \text{ for all } i\}} \Big(\frac{|O^*u|^2}{36}\Big)^{-s_0} \mathrm{d}u \le (L^2)^{99n(j+1)}.$$

Proof. Diagonalise C^*C by a unitary Q; at least $|\iota| - 3n = s_0 + 1$ eigenvalues vanish; let D be $s_0 + 1$ of their indices and O the corresponding columns of Q. Then $CO = 0$ and $O^*O = 1$. Substituting $u \mapsto Q^*u$, the domain goes into a box of side $L \cdot L^j$ (the entries of Q have modulus at most 1 and $|\iota| \le L$). On $\mathbb{C}^D$ pass to $t = |w|^2$, which carries Lebesgue measure to $\pi^{s_0+1} t^{s_0}/s_0!\,\mathrm{d}t$: the integrand $(t/36)^{-s_0}$ times this density is the constant $36^{s_0}\pi^{s_0+1}/s_0!$, bounded independently of s_0; the range of t and the volume of the box in the other coordinates are bounded by powers of L. □

LEAN `exists_orthonormal_ker_lintegral_le`

Proposition 8.2 (the gauge at the complex place)**.** *Let* $\mathfrak{h}$ *be real-place sizes in* $\mathrm{RPS}(\mathfrak{s}, \rho, s_0, b)$ *and* R *a real number with* $\rho \le R$. *Then the sets* $\{|x| \le R^i\}$ *and the constant* R^2 *satisfy the axioms* (G1)–(G6) *of a gauge on* $\mathcal{L}_{\mathbb{C}}(\mathfrak{h})$ *(Note T); this is the content of Definition 2.8.*

Proof. The axioms of a size hold because $R \ge \rho \ge 8r$: a sum of at most $r+1$ numbers of absolute value at most R^i is at most R^{i+1}, and the entries of $J_1(\mathfrak{s}) = J_1(\mathfrak{s})^{-1}$ are $0, \pm 1$. (G1)–(G4): closed discs are compact; $c_{\mathbb{C}} = \pi^2 \ge 1$; $\mathrm{mod}_A(u) = |u|^2 \le (R^2)^j$; the volume πR^{2j} is at most $(R^2)^{j+1}$; the unit is 1, the entries of the basis have modulus at most 1, $\tau_{\mathbb{C}}\delta_{\mathbb{C}}^{-1}$ has modulus 1; entries of a matrix are at most its norm $\le \rho \le R$; the ball has radius $\frac{1}{4}$. (G5): by the formula for the flat chart at $\mathbb{C}$, the points of size j of the cone have their two free groups of coordinates in discs of radii $2R^j/|\alpha + \mathrm{i}|$ and R^j, α the real coordinate of the skew parameter, whose measure is a multiple of the Cauchy density; integrating gives at most $(R^2)^{99n(j+1)}$. (G6): let x be in the box, y a position and v a matrix with ϱ rows. Let N be the matrix of the n rows of $1 - J_1(\mathfrak{s})$ at the indices of the e_a and of z; a vector w' with $\mathsf{N}w' = 0$ has $J_1(\mathfrak{s})w' = w'$. Apply Lemma 8.1 to the matrix C with the rows of v, of $\mathsf{N}x^{-1}\mathsf{O}(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}))$ and of $\mathsf{N}(xy)^{-1}\mathsf{O}(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}))$: it has $\varrho + 2n \le 3n$ rows, and $|\iota| = k - 1 = s_0 + 1 + 3n \le R$. Take O from the lemma and $\varphi(u) = 36/|O^*u|^2$. For w in the image of O and $g \in \{x^{-1}, (xy)^{-1}\}$ the vector $w' = g\,\mathsf{O}(\mathcal{L}_{\mathbb{C}}(\mathfrak{h}))w$ has $\mathsf{N}w' = 0$, so $|w'|^2 = h(w', w') = |w|^2$. Hence, for $t \in \mathbb{C}$ with $|t| \le |O^*u|/9$, the vector $w = t\,|O^*u|^{-2}\,OO^*u$ has norm at most $\frac{1}{9}$, lies in W and has $u^*w = t$: the set $\{u^*w : w \in W\}$ contains a disc of radius $|O^*u|/9$, which gives the first inequality of (G6). For the second, if $c \in \mathcal{C}_u$, take such a w with $|t| = |O^*u|/9$; then $|c|\,|t| \le |h(z_1, z_1')| + |h(z_2, z_2')| \le \frac{1}{16} + \frac{1}{4}$ for close pairs, so $\mathcal{C}_u$ lies in the disc of radius $3/|O^*u|$. The integral bound is that of Lemma 8.1. □

9 The open-cell bound

Proof of Theorem C. Set-up. Write $Z = Z(\gamma)$ and $M = \rho^{9n}/\mathrm{g}(Z)$. By Lemma 4.4 and the hypotheses on y, $|Z_{ab}| \le \rho^2$; hence $\mathrm{g}(Z) \le |\det Z|^2 \le \rho^{9n-1}$ (as $n \le \rho$), so $\rho \le M$, and $M^{-1} \le \mathrm{g}(Z)$. Put $Q = M^{9n}$, $t_0 = Q^5$ and $P = t_0^n$. Lemma 5.4 gives $Z \in \mathcal{C}(\mathfrak{s})$ and, for every Z' within P^{-6} of Z, $\mathrm{g}(Z') \ge \mathrm{g}(Z)/2$ and $Z' \in \mathrm{Cell}_{\le P}$. By Lemma 5.3, $\gamma = \mathrm{u}(q_1)\underline{h_1}\gamma(Z)(\mathrm{u}(q_2)\underline{h_2})^{-1}$. Take the tube A, Ω_0 of (J11) for t_0; A is compact and lies in $\{\|y\| \le P^{20}\}$. For $x = (h, u') \in \mathrm{U}(\kappa) \times \mathcal{U}^0$ put

$t(x) = \mathrm{u}(q_2)\,\underline{h_2 h^{-1}}\,u'^{-1}$, an element of the group generated by $\mathcal{U}^0$ and $\mathcal{H}_n(T)$, so that $\gamma t(x) = \mathrm{u}(q_1)\underline{h_1}\,\gamma(Z)\,(u'\underline{h})^{-1}$. Let m be the product of the Haar probability measure of $\mathrm{U}(\kappa)$ and of μ^0, restricted to $\mathrm{U}(\kappa)\times\Omega_0$ and normalised, and

$$\nu = \frac{P^{2000n^2}\,|I|}{\deg\cdot c_G(\mathfrak{s})\,(\mathrm{g}(Z)/2)^{k-1}\,\mu^0(\Omega_0)}\cdot \mathbf{1}_A\,|\Phi_0|^2\,\mathrm{d}y.$$

(i). The norm of an integral $\int \beta(y')\Phi_0(sgy')\,\mathrm{d}y'$ does not change when $s\in\mathcal{Q}_n(T)$ is removed (unit 118). Summing first over i', the square is the squared norm of $\int \overline{\varphi(y')\Phi_{0,i}(\gamma t(x)y')}\,\mathbf{1}_A(y')\Phi_0(t(x)y')\,\mathrm{d}y'$, in which $t(x)$ may be removed from the last factor; summing then over i, the sum is $\sum_{i'}|\int\varphi_{i'}(y')\Phi_0(\gamma t(x)y')\,\mathrm{d}y'|^2$ with $\varphi_{i'}=\mathbf{1}_A\,\varphi\,\overline{\Phi_{0,i'}}$, which does not depend on x; and $\mathrm{u}(q_1)\underline{h_1}$ may be removed from $\gamma t(x)$. For each i', Lemma 7.2 applies to $\varphi_{i'}$, with $c=\mathrm{g}(Z)/2$, the bound P (the Bessel system has the Schur bounds at $P\ge B^5$ with $D=|I|/\deg$) and the domain $\{\|u'\|\le P^{20}\}\supseteq\Omega_0$. Summing over i', with $\sum_{i'}|\varphi_{i'}|^2=\mathbf{1}_A|\varphi|^2|\Phi_0|^2$, gives (i).

(ii). By Lemma 7.1 with $w=|\Phi_0|^2$, $\varphi=\mathbf{1}_A$, $K=t_0\,\mu^0(\Omega_0)$ from the tube, and $R=P^{40}$ (for $y, sy\in A$ the entries of $\mathrm{fm}(s)$ are at most $\|sy\|\|y^{-1}\|$), $\int_A|\Phi_0|^2\mathrm{d}y\le t_0\,\mu^0(\Omega_0)(2R)^{2n^2}\int|\Phi_0|^2\mathrm{d}\dot{y}$, and the last integral is $|I|$. So $\nu(G_1(\mathfrak{s}))$ is at most a power of M times $|I|^2/(\deg c_G(\mathfrak{s})(\mathrm{g}(Z)/2)^{k-1})$; that the power is at most $M^{10^5n^4}$ is the formal text's arithmetic.

(iii). Let $x=(h,u')$ with $u'\in\Omega_0$, and $y'\in\mathrm{FB}(\rho)$ with $\mathrm{bs}(\gamma t(x)y')\le\rho$. Put $y''=(u'\underline{h})^{-1}y'$; its frame vectors are those of y' (Lemma 4.1(i)), and $\mathrm{bs}(\gamma(Z)y'')\le\rho$ by invariance of the Bessel size. By Lemma 4.2(i) there is $w\in\mathcal{U}(T)$ with $\|w\,\gamma(Z)y''\|\le\rho$. The explicit section matrix expresses the columns $y''^{-1}e_a^*$ by the frame vectors of y'' and of $w\gamma(Z)y''$, with coefficients among the entries of Z^{-1}, of $\mathrm{cd}(Z)^{-1/2}$ and of the inverse of $\mathrm{fm}(w)$, which is bounded through Lemma 4.4 and Lemma 5.4 for $Z(w\gamma(Z))$; so these columns have norm at most Q^4-times a constant, and Lemma 4.3 gives $\|y''\|\le t_0$. Hence $y'=u'\underline{h}y''\in\Omega_0\cdot\{\|y\|\le t_0\}\subseteq A$. □

10 The Bessel cut

Proof of Theorem D(i). Write $e=\deg(l)\psi_l^{\mathfrak{p}}$; by (J10) it is a bounded integrable idempotent kernel. Let β be a smooth compactly supported function whose smoothing of $\psi_l^{\mathfrak{p}}$ is the generator of the module of (J10); for a word w let γ_w be the corresponding iterated derivative of β (twisted by the frame) and $X_w=\deg(l)\cdot$ (the smoothing of $\psi_l^{\mathfrak{p}}$ by γ_w), a function on $G_1(\mathfrak{s})$. Then X_w is integrable, $e*X_w=X_w$, and $F*X_w$ is the smoothing of $F=F*e$ by γ_w. Put $V_w=F*X_w$. It is continuous, $V_\emptyset=F$ (the smoothing by β reproduces, and $F*e=F$), it is a Bessel function for rep because left translations pass through the convolution, and the family (V_w) satisfies the derivative law of the Schur bounds (Section 3): along a curve g in $G_1(\mathfrak{s})$ with $g'(t)=g(t)X$ the derivative of the smoothing by γ_w is the sum over (α,β) of $X_{\alpha\beta}$ times the smoothing by $\gamma_{(\alpha,\beta)w}$, by differentiation of the smoothing. For the Schur bound of V_w, the Schur inequality of (J10) gives the constant

$$\frac{\|X_w\|_2^2}{\deg(l)}\cdot\frac{\|F\|_2^2}{\deg(l)}=\|x_w\|^2\cdot\deg(l)^{-1}\|F\|_2^2,$$

because $\|X_w\|_2^2=\deg(l)\cdot\|x_w\|^2$ for the vector x_w of the module (the inner product of the module is the L^2 product divided by the formal degree); and $\|x_w\|\le((r+$

$1)^4(L+r+1+|w|))^{|w|} \le B^{|w|}$ for $|w| \le 2n^2$, by the hypothesis on B. So V_w has Schur's bound with the constant $B^{2|w|}\deg(l)^{-1}\int |F|^2 \mathrm{d}\dot{y}$, as claimed. □

Proof of Theorem D(ii). Let $M = 8r^2+1$, $R = b^{6n+2}$, $\phi(g) = (2/\|g\|)^{2M}$ and $\Phi^\sharp(y) = \int \phi(x^{-1}sy)\,\mathrm{d}\nu_S(s)$. *Normalisation.* Y is integrable for $\mathrm{d}\dot{y}$, by the peak bound and the integrability of ψ_l^{p}; let $\mathsf{q} = \int |Y|^2\mathrm{d}\dot{y}$, finite and not 0 as $Y(x) \ne 0$, choose c with $c^2\mathsf{q} = |I|$, and put $\Phi_0 = cY$; then $\int |\Phi_0|^2\mathrm{d}\dot{y} = |I|$ and $\Phi_0 * (\deg(l)\psi_l^{\mathsf{p}}) = \Phi_0$; Φ_0 is continuous because it is a convolution. *Cut.* Let $\chi(t) = \max(0, \min(1, R-t))$ and $\Phi(y) = \chi(\mathrm{bs}(y))\Phi_0(y)$, continuous by Lemma 4.2(iii), a Bessel function because the Bessel size is invariant under $\mathcal{Q}_n(T)$, with $|\Phi| \le |\Phi_0|$ and $\Phi(y) = 0$ unless $\mathrm{bs}(y) \le R$. *The peak against the L^2 norm.* From $Y = Y * e$, $|e(g)| \le \deg(l)\phi(g)$, the Cauchy–Schwarz inequality with the weight ϕ, the unfolding through ν_S and $\Phi^\sharp \le C^2 b^N |Y(x)|$ (second peak bound, $C = 16^M$),

$$|Y(x)|^2 \le \deg(l)^2 \int \phi \cdot \int |Y|^2\,\Phi^\sharp\,\mathrm{d}\dot{y} \le \deg(l)^2 \int \phi \cdot C^2 b^N |Y(x)| \cdot \mathsf{q},$$

so $|Y(x)| \le \deg(l)^2\, C^2 b^N\, \mathsf{q} \int \phi$. *The majorant.* Take $\mathsf{A} = |c|\,\Phi^\sharp$ and $\mu = |c| \cdot \frac{C}{\mathrm{vol}\{\|g-1\|\le 1\}}\, b^N |Y(x)| \cdot (2/\|g\|)^M \mathrm{d}g$, $C = 16^M$. Then: $|\Phi_0|, |\Phi| \le \mathsf{A}$ by the first peak bound and the decay of ψ_l^{p}; where $\Phi_0(y) \ne \Phi(y)$, that is $\mathrm{bs}(y) > R-1$, every s has $\|x^{-1}sy\| > 2b$ (Lemma 4.2(ii) and $\mathrm{bs}(x) \le b^{5n}$), and the decay with the larger exponent $2M + 3r^3 f \le 2M + A_0$ gains the factor $b^{-3r^3 f}$; A is $\mathcal{U}(T)$-invariant by the invariance of ν_S; $\mathsf{A}(y) \le \mu\{g : \mathrm{bs}(yg) \le R\}$ by the second peak bound and a comparison of ϕ on a unit ball (the formal text's; not reproduced here). The mass of μ: by the decay integral of (J10) with the exponent $M > 8r^2$, $\int (2/\|g\|)^M \mathrm{d}g \le C\,\mathrm{vol}\{\|g-1\| \le 1\}$, so

$$\mu(G_1(\mathfrak{s})) \le |c|\, C^2\, b^N\, |Y(x)|,$$

and μ is finite. By the unfolding through ν_S, $\int \mathsf{A}\,\mathrm{d}\dot{y} = |c| \int \phi\,\mathrm{d}g$. Hence

$$\mu(G_1(\mathfrak{s})) \int \mathsf{A}\,\mathrm{d}\dot{y} \le |c|^2 C^2 b^N |Y(x)| \int \phi \le |I| \left(\deg(l)\, C^2 \int \phi\right)^2 b^{2N} \le |I|\, b^{2r^3},$$

by the bound for the peak and $c^2\mathsf{q} = |I|$, by $\deg(l)\int \phi \cdot C^2 \le b^{99r^2}$ (J10), and by $N + 99r^2 \le r^3$. *Closeness.* Where $\Phi \ne \Phi_0$ we have $b^{3r^3 f}|\Phi - \Phi_0| \le b^{3r^3 f}|\Phi_0| \le \mathsf{A}$, and $\mathsf{A} \le \mu(G_1(\mathfrak{s}))$; so $b^{6r^3 f}\int |\Phi - \Phi_0|^2 \mathrm{d}\dot{y} \le \mu(G_1(\mathfrak{s})) \int \mathsf{A}\,\mathrm{d}\dot{y} \le b^{2r^3}|I|$, and $2r^3 - 6r^3 f \le -10r^2$ because $f \ge 1$ and $r \ge 99$, which follows from $N + 99r^2 \le r^3$. The comparisons of exponents are those of the formal text. □

11 The volume weight

Lemma 11.1 (the volume weight)**.** *For $\gamma, y \in G_1(\mathfrak{s})$:* (i) $\Delta(\gamma) > 0$*;* (ii) *if $\gamma \in \mathcal{U}^0$, then $\Delta(\gamma y) = \Delta(y)$.*

Proof. (i) If the matrix had a kernel vector $v \ne 0$, the vector $w = \gamma^{-1}F_{\mathfrak{s}}v$ would be orthogonal, for the Euclidean product, to all $e_a - e_a^*$ and to z; so $w_{e_a} = w_{e_a^*}$, $w_z = 0$ and $h(w,w) = 2\sum |w_{e_a}|^2 + \sum |w_{x_i}|^2 \ge 0$, while $h(w,w) = h(F_{\mathfrak{s}}v, F_{\mathfrak{s}}v) = -|v_\varrho|^2 \le 0$. Hence w has only coordinates at the e_a, e_a^*, equal in pairs and with $h(w,w) = 0$, so $w = 0$ and $v = 0$. (ii) $\gamma^{-1} = \mathrm{u}(q)$, and the explicit matrix of $\mathrm{u}(q)$ changes the matrix of the definition by a factor of determinant 1. □

LEAN `frameVolume_pos` · `frameVolume_mul_left`

Proof of Theorem E. By Lemma 4.2(i) take $\omega \in \mathcal{U}(T)$ with $\|\omega y\| \le \theta$. Let T be θ times the leading $j \times j$ block of $\mathrm{fm}(\omega)$, unitriangular up to the factor, so $\det \mathsf{T} = \theta^j$ and its minor on σ has determinant $\theta^{|\sigma|}$. Let $H = ((1 - J_1(\mathfrak{s}))F_{\mathfrak{s}})^* \, y^{-1} F_{\mathfrak{s}}$ be the matrix of Definition 2.9 for y, invertible by Lemma 11.1(i), and

$$\mathsf{B} = H^{-1} \, ((1 - J_1(\mathfrak{s}))F_{\mathfrak{s}})^* \, y^{-1},$$
$$\text{so that} \qquad \mathsf{B}F_{\mathfrak{s}} = 1 \qquad \text{and} \qquad \frac{\Delta(y)}{\Delta(\gamma^{-1}y)} = \bigl|\det(\mathsf{B}\,\gamma F_{\mathfrak{s}})\bigr|^{-1}$$

for every $\gamma \in G_1(\mathfrak{s})$ (the factor y^{-1} on the right is needed for both). Let $\mathsf{p} = Z\mathsf{I}_j Z_j^{-1}$, $\mathsf{C} = \mathsf{T}\mathsf{p}^*\mathsf{B}$, $\mathsf{c} = (VZ_j^{-1}\mathsf{T}^* - W_{\mathfrak{s}}^*\mathsf{C}^*)\mathsf{E}$ and $\mathsf{S} = R\,(\mathsf{T}\mathsf{E})^*\mathsf{E}$, an invertible $\sigma \times \sigma$ matrix with $|\det \mathsf{S}| = \theta^{|\sigma|}|\det R|$. Define

$$K(a) = \Bigl(\frac{(2\theta^2)^j}{|\det Z_j|}\Bigr)^{k+n} \det(1 + (a\mathsf{S} + \mathsf{c})^*(a\mathsf{S} + \mathsf{c}))^{-(k+n)}.$$

The bound for $\int K$ is Hua's (J8), with $M = 1$, the k rows of a and $r' = n \ge |\sigma|$.

Now let γ, A be as in the statement, $\mathsf{t} = \gamma F_{\mathfrak{s}} \mathsf{I}_j Z_j^{-1}$ and $\mathsf{G} = A\mathsf{S} + \mathsf{c}$. *The ratio as a determinant.* The normal form at j gives, for each frame position $b \ge j$, vectors X_b, Y_b of the flags of the frame and of γf, triangular with unit leading coefficients up to a number of modulus 1; with them and with $\mathsf{M} = \mathsf{B}\mathsf{t}Z_j - \mathsf{I}_j = \mathsf{B}\gamma F_{\mathfrak{s}}\mathsf{I}_j - \mathsf{I}_j$ the formal text shows $|\det(\mathsf{B}\gamma F_{\mathfrak{s}})| = |\det(1 + \mathsf{M}\mathsf{p}^*)|$, hence

$$\frac{\Delta(y)}{\Delta(\gamma^{-1}y)} = \bigl|\det(1 + \mathsf{M}\mathsf{p}^*)\bigr|^{-1} = \bigl|\det(1 + \mathsf{p}^*\mathsf{M})\bigr|^{-1} = \frac{|\det \mathsf{T}|^2}{|\det Z_j|\,|\det(\mathsf{C}\mathsf{t}\mathsf{T}^*)|}.$$

Two positivity facts. Put $\mathsf{Y} = J_1(\mathfrak{s})\mathsf{t}\mathsf{T}^* - \mathsf{C}^*$, $\mathsf{W} = W_{\mathfrak{s}}^*\mathsf{Y}$ and $\Theta = -\mathsf{C}\mathsf{t}\mathsf{T}^*$. First, $1 + \mathsf{W}^*\mathsf{W} \le \Theta + \Theta^*$: the difference is $\mathsf{D}(1 - \mathsf{Q}\mathsf{Q}^*)\mathsf{D}^*$ plus a matrix of the form N^*N, where $\mathsf{D} = \mathsf{C}y$ and $\mathsf{Q} = \theta^{-1}(\omega y)^{-1}F_{\mathfrak{s}}\mathsf{I}_j$ has norm at most 1 and $\mathsf{D}\mathsf{Q} = 1$; it is positive by the matrix Bessel inequality (J9). (Here $\mathsf{D}\mathsf{D}^* = -\mathsf{C}J_1(\mathfrak{s})\mathsf{C}^*$, which uses $((1 - J_1(\mathfrak{s}))F_{\mathfrak{s}})^*J_1(\mathfrak{s}) = -((1 - J_1(\mathfrak{s}))F_{\mathfrak{s}})^*$ and $y^{-1}J_1(\mathfrak{s})(y^{-1})^* = J_1(\mathfrak{s})$; the negative rows of the definition of the volume weight are essential.) Second, $\mathsf{G} = \mathsf{W}\mathsf{E}$ by the hypothesis on $\mathsf{x}_0(\gamma)$, so $\mathsf{W}\mathsf{W}^* - \mathsf{G}\mathsf{G}^* = \mathsf{W}(1 - \mathsf{E}\mathsf{E}^*)\mathsf{W}^* \ge 0$, again by (J9). By $\det(1 + XX^*) = \det(1 + X^*X)$, the monotonicity of the determinant on positive matrices and the Ostrowski–Taussky inequality (J9),

$$\det(1 + \mathsf{G}^*\mathsf{G}) = \det(1 + \mathsf{G}\mathsf{G}^*) \le \det(1 + \mathsf{W}\mathsf{W}^*) = \det(1 + \mathsf{W}^*\mathsf{W}) \le 2^j\,|\det(\mathsf{C}\mathsf{t}\mathsf{T}^*)|.$$

Combining, $\Delta(y)/\Delta(\gamma^{-1}y) \le (2\theta^2)^j|\det Z_j|^{-1}\det(1 + \mathsf{G}^*\mathsf{G})^{-1}$, and the power $k + n$ is $K(A)$. I read this proof line by line; the determinant identity of the first step is the formal text's computation with the vectors X_b, Y_b. □

Four lemmas on lattice translates (for the unit on the second moment). The remaining declarations of the file of the volume weight concern the adelic objects of unit 148 and are used there. Let E be a CM field with ring of finite adeles $\mathbb{A}_{E,f} \supseteq \widehat{O}_E$, and S a *filling* over E and $\mathfrak{s}$ (unit 148), with its natural number g, its classes c, the elements h_c, the frame Gram matrix $Z(\gamma)$ of an element γ of the rational unitary group, the *frame image* $\gamma F_{\mathfrak{s}}$, its rows at the x_i and z (the *middle part*), the matrix $\mathrm{pl}_j(Z)$ whose rows $a < j$ are those of Z and whose other rows are those of the identity, the first ϱ rows $\xi(Z)$ of Z, and the coordinates x_c of the standard basis of

$\langle x_i\rangle$ in the lattice of the class c. An element γ *has level* j if it is in normal form at j (Definition 2.10, read in the rational unitary group), the leading minor of order j of $Z(\gamma)$ is not 0, and $\mathsf{g}(Z(\gamma)) \neq 0$ exactly when $j = n$.

Lemma 11.2 (counting by congruence classes)**.** *Let E be a CM field that is stably unramified in the sense of unit 148, α a set, π, σ, τ finite sets, $x \in [0, \infty]$, c a finite idele with $c \in \widehat{O}_E$, $B \subseteq \alpha$, and Y a family of $\sigma \times \pi$ matrices over $\mathbb{A}_{E,f}$, integral for $a \in B$. Let m be a primitive $\pi \times \tau$ matrix over $\widehat{O}_E$, s a natural number with a bijection between $\tau \sqcup \{1, \dots, s\}$ and π, K a lattice containing $c\,\widehat{O}_E^{\sigma}$, $\mathsf{n} \colon \alpha \to \mathbb{N}$ with the index of $K \cap \Lambda_a$ in Λ_a dividing $\mathsf{n}(a)$ for $a \in B$, where Λ_a is the lattice spanned by the columns of $Y(a)m$, and l a natural number with $|\tau| \leq l$. Suppose every subset $D \subseteq B$ on which the columns of $Y(a) - Y(a')$ lie in K for all $a, a' \in D$, and which is not empty, has at most x elements. Then*

$$\sum_{a \in B} \mathsf{n}(a)^{-l} \leq \left([\widehat{O}_E : c\widehat{O}_E]^{|\sigma|}\right)^{|\sigma|+s} \cdot x.$$

Proof. Complete m to an invertible matrix U over $\widehat{O}_E$, and colour $a \in B$ by the columns of $Y(a)U$ modulo $c\widehat{O}_E^{\sigma}$: there are at most $[\widehat{O}_E^{\sigma} : c\widehat{O}_E^{\sigma}]^{|\pi|}$ colours, and it suffices to bound the sum over one colour class. In a class a Hermite form of $(Y(a)m)$ gives generators of Λ_a, and a second colouring by these generators modulo K has at most $\mathsf{n}(a)^{|\tau|} \leq \mathsf{n}(a)^{l}$ colours relevant for a; inside one sub-class the columns of Y agree modulo K, so it has at most x elements. Summing $\mathsf{n}(a)^{-l}$ over the sub-classes gives the bound. The count of the colours in terms of the index $[\widehat{O}_E : c\widehat{O}_E]^{|\sigma|}$ and of s is the formal text's. □

LEAN `Filling.tsum_inv_index_pow_le_of_congr`

Lemma 11.3 (three lemmas on frame images)**.** *Let S be a filling over E and $\mathfrak{s}$, c a class and j a natural number.* (i) *Let γ have level j, and $b \leq \varrho$ a frame position with $b = \varrho$ if $b < j$. Then the vector $\mathsf{y} = \gamma F_{\mathfrak{s}} \cdot (\text{column } b \text{ of } \mathrm{pl}_j(Z(\gamma))^{-1})$ has coordinates 0 at all e_a^*, and $5\sum_{i \in \kappa} |\mathsf{y}_{x_i}|_v^2 \leq 1$ at every infinite place v of E.* (ii) *Let l be a matrix over E with rows indexed by κ such that $\mathsf{x}_c\mathsf{l}$ is integral, and i one of its columns with $\sum_{a \in \kappa} |\mathsf{l}_{ai}|_v^2 < 1$ at every infinite place v. Then $\mathsf{l}_{ai} = 0$ for every $a \in \kappa$.* (iii) *Let S be admissible, t a label, p, p' labelled terms of level j over c with the label t, and u, u' elements of the group $\mathcal{U}_c^0$ of the class; if the frame images of $u\,p_2$ and $u'p_2'$ are equal, then $p = p'$.*

Proof. (i) If $b \geq j$, the definition of the level gives a vector common to the flags of the frame and of γf, which is y; its coordinates on $\langle x_i\rangle$ vanish. If $b < j$, then $b = \varrho$ and $j = n$, the column is that of $Z(\gamma)^{-1}$, and a lemma of unit 148 computes the norm of the middle part of y at each place and bounds it by the constant of the filling. (ii) The column is an integral vector of the definite lattice of the class with all archimedean norms less than 1, so it is 0 (a lemma of unit 148, from: an algebraic integer with all conjugates of absolute value less than 1 is 0, unit 121). (iii) $u\,p_2$ and $u'p_2'$ are again terms over c; equal frame images put the quotient of the corresponding compact elements into $\mathcal{Q}_n(T)$, and the uniqueness of the labelled term in its double coset (unit 148) gives $p = p'$. □

LEAN `Filling.HasLevel.middle_shared_vector_le` · `Filling.col_eq_zero_of_norm_lt_one` · `Filling.IsLabelledTerm.eq_of_frameImage_eq`

Lemma 11.4 (equal Gram blocks and congruent middle parts)**.** *Let E be stably unramified, S a filling over E and $\mathfrak{s}$, c, c', c'' classes, j a natural number, γ, γ' of level j with $Z(\gamma') = Z(\gamma)$, ι a finite set and q an injection of ι into $\{0, \dots, \varrho - 1\}$ such that every a outside its image has $j \le a$. Let ξ be the rows $\mathsf{q}(\iota)$ of $\xi(Z(\gamma))$ and $\mathsf{t} = \mathsf{g}^{-1}\mathrm{pl}_j(Z(\gamma))^{-1}$. Suppose that every column of the conjugate transpose of the difference of the middle parts of the positions $h_c^{-1}\gamma' h_{c''}$ and $h_c^{-1}\gamma h_{c'}$ lies in the sum of g times the lattice spanned by the columns of ξ^* and of the lattice of the vectors that t^* maps to integral vectors. Then there is a $\kappa \times \iota$ matrix l over E with $\mathsf{x}_c\mathsf{l}$ integral such that the frame images of γ' and γ agree at the rows e_a^* and z and differ by $\mathsf{g}\,\mathsf{l}\,\xi$ at the rows x_i (the rows other than the e_a).*

Proof. Let x be the difference of the middle parts of the frame images at the rows x_i. The hypothesis, read in the lattice of the class, gives an adelic matrix Λ' with $\mathsf{x}_c\Lambda'$ integral and $\mathsf{x}_c(\mathsf{x} - \mathsf{g}\Lambda'\xi)\mathsf{t}$ integral; by density of E in the finite adeles choose a matrix μ over E with the same two properties. Let $\mathsf{L} = (\mathsf{x} - \mathsf{g}\mu\xi)\mathsf{t}$. For a column b with $b \ge j$ or $b = \varrho$, Lemma 11.3(i) for γ and for γ' gives $5\sum_i |\cdot|_v^2 \le 1$ for the middle parts of the two columns b of the frame images times $\mathrm{pl}_j(Z(\gamma))^{-1}$; so the column b of xt has $\sum_i |\cdot|_v^2 \le \frac{4}{5}\mathsf{g}^{-2} < 1$ at every infinite place. The column b of $\xi\mathsf{t}$ vanishes, by the first assertion of that lemma, because the rows of a frame image at the e_a^* are rows of $Z(\gamma)$. So the column b of L has norms less than 1 and is 0 by Lemma 11.3(ii). Hence L lives on the columns $\mathsf{q}(\iota)$, and $\mathsf{l} = \mathsf{L}|_{\mathsf{q}(\iota)} + \mu$ satisfies $\mathsf{x} = \mathsf{g}\,\mathsf{l}\,\xi$, because t^{-1} restricted to these columns is $\mathsf{g}\,\xi$. The rows at e_a^* and z of the two frame images agree because $Z(\gamma') = Z(\gamma)$. □

LEAN `Filling.HasLevel.exists_frameImage_toRows₂_eq_add`

12 Where the results are used

Theorem B and Proposition 8.2 supply the level, the gauges and the Haar hypothesis for Note L at the complex place. Theorem C is the open-cell bound of a Bessel system, Theorem D produces the cut pair and the Schur bounds of such a system from a peaked Bessel function of the unit on the discrete series (unit 133), and Theorem E and Lemmas 11.2–11.4 are used in the unit on the second moment (unit 148), in the Schur test for the incoherent and degenerate pairs.

13 Notes on the formal text

A declaration outside the census. The definition of the level at the complex place (Definition 2.6; lines 31 to 41 of the file of the gauge at $\mathbb{C}$) is not listed by the census of the artifact, because its attribute runs over two lines; so it is a member of no unit, and it is not in the concordance. It is stated here from the Lean text, which the second reader also recovered from the pinned source.

Proofs inside definitions. Two definitions contain proofs: that the section matrix is unitary (Lemma 5.1) and that the data of Definition 2.8 form a gauge (Proposition 8.2); the level at $\mathbb{C}$ contains the verification of the axioms of a level. The dossier has no structured proofs for them; Lemma 5.1 and Proposition 8.2 are written from the Lean terms.

Argument lists not confirmed by Lean. For the open cell, the section, and for the Lean theorems behind Lemma 5.4, Lemma 7.2, Theorem D(i), Theorem E

and Lemma 11.4, the reader's dossier derived the list of arguments by Lean's rules, because the census misreads names bound inside the declaration or stops at a `let` in the statement (missing the shape, the frame instance and the class). Nothing was compiled, so these lists are not confirmed by Lean. The statements of Lemma 6.1, Theorem D(i), Lemma 11.3(i), Lemma 11.4 and Theorem E continue after a `let` and were read to their end in the Lean text.

Differences from the natural statements. (1) The inverse of a singular matrix is 0 in the formal text; this matters only in Definition 2.1. (2) The chart is defined on all parameters; where Z is not in the open cell the section is the identity. (3) Theorem A(i) is stated with the real determinant of an endomorphism of the parameter space, obtained from the logarithmic derivative through sk and e^{-1}. (4) In Theorem B the constant is a choice; (ii) holds by this choice, and the content is (i), (iii), (iv). (5) Theorem C asks $y \in \mathrm{FB}(\rho)$ and $\gamma^{-1}y \in \mathrm{FB}(\rho)$ only to bound the entries of $Z(\gamma)$. (6) In Theorem D(ii) the number N is the Lean variable `n` and $n = \varrho + 1$ is `𝔰.n`. (7) In Theorem E the set σ lies in the lowest universe, and the leading block is taken through the inclusion of $\{0, \dots, j-1\}$ given by $j \le n$, an implicit argument. (8) Lemma 8.1 is stated for any finite sets and uses the shape only through n. (9) Lemma 11.2 allows $x = \infty$ and uses real powers of indices. (10) In Lemma 4.5 the family V is indexed by all natural numbers and constrained up to 2ϱ.

Small results. Twenty-seven declarations have no statement of their own; each is cited where it is used: continuity and norms of frame vectors (Lemma 4.1), the two rules for the Bessel size (Lemma 4.2), norms of $J_1(\mathfrak{s})$ and of inclusions (Lemma 4.3), the four formulas for the frame Gram matrix (Lemma 4.4), the properties of $\mathbf{g}(\cdot)$ and of the open cell (Definition 2.1), the identities for the section matrix (Definitions 2.2, 2.3), the bound $\rho \ge 4$ (Definition 2.5), a sum of squares (Lemma 6.3) and two properties of the volume weight (Definition 2.9).

How this note was written. From the structured proofs of the reader's dossier and from the Lean text. I read all statements and definitions of the ten files in the Lean text, with their comments. I read line by line the formal proofs behind Theorem A(i), Theorem B, Theorem C, both parts of Theorem D and Theorem E, which the reader marked as the least certain, and the proofs inside the definitions of the section, of the level and of the gauge. The proofs of Sections 4 and 5, of Theorem A(ii),(iii), Lemmas 6.3, 7.1, 7.2, 8.1, 11.1 and of the four lemmas on lattice translates are written from the structured proofs, whose steps a program compared with the source; I did not read those Lean proofs line by line. Not reproduced, and named where they occur: the matrix identities in Lemma 5.1 and Lemma 6.1; the evaluation of the block determinants in Theorem A(i); the determinant identity in Theorem E; the constants in the proofs of Theorems C and D and of Proposition 8.2(G5); and the count of colours in Lemma 11.2. In these places the text says what the formal proof establishes and by which inputs; it is an account of the formal proof, not an independent derivation. The references were taken from the second reader's verdicts and the catalogue of the write-up, as cited; none was opened for this note. The program check of the unit's dossier covered 639 leaf steps of 108 proofs.

Appendix A. Concordance with the Lean artifact

The letters of this note and the names of the formal text: $G_1(\mathfrak{s})$, $J_1(\mathfrak{s})$ are `G₁`, `J₁`; $F_{\mathfrak{s}}$, $F'_{\mathfrak{s}}$, $W_{\mathfrak{s}}$, $D_{\mathfrak{s}}$ are `frameCols`, `dualFrameCols`, `definiteCols`, `dualCols`; $Z(y)$ is `realFrameGram`; fm is `frameMatrix`; $\mathsf{g}(\cdot)$ is `gramDiscReal`; $\mathcal{C}(\mathfrak{s})$ is `Shape.openCell`; $\mathrm{u}(q)$ is `unipotentOf`; $\underline{h}$ is `embedH`; μ^0 is `besselHaar₀`; $\mathrm{d}\dot{y}$ is `besselQuotMeasure`; $\mathrm{bs}(\cdot)$ is `besselSize`; $\mathrm{FB}(\rho)$ is `frameBox`; cay, j are `cayleyChart`, `cayleyDensity`; ch, cc, bt are `openCellChart`, `chartCoords`, `besselTangent`; $\mathfrak{h}$ is `hF`; $\mathcal{L}_{\mathbb{C}}(\mathfrak{h})$ is `RealPlaceSizes.complexLevel` (not in the census); s_0 is `s`; R is `L`; Φ_0 is `F`, `W .full` or `Y`; N is `n`; A_0 is `A`; `S`, `g` are `S`, `S.g`. Two statements as the artifact has them:

```
lemma RealPlaceSizes.haarRatio_spec : (hF.haarRatio ≠ 0 ∧ hF.haarRatio ≠ ∞) ∧
  hF.complexLevel.groupMeasure = hF.haarRatio • ℙ ∧ (∀ f, Measurable f → besselU.FixesLeft f → ∫⁻ g,
  f g ∂hF.complexLevel.upperMeasure 0 = hF.haarRatio * ∫⁻ g, f g ∂besselQuotMeasure) ∧ .ofReal
  (volU 𝔰.K / volU 𝔰.r) * hF.haarRatio ≤ 1
```

```
lemma BesselSystem.open_cell {𝔰 B f ρ γ y} (X : BesselSystem 𝔰 B f) (hγ : 0 < gramDiscReal
  (realFrameGram γ)) (hy : y ∈ frameBox ρ) (hz : γ⁻¹ * y ∈ frameBox ρ) (hρ : B ≤ ρ) : ∃ A n m, ∃ t :
  U (1 : Matrix 𝔰.κ 𝔰.κ ℂ) × (𝔰.middleGram (1 : ℂ)).besselU0 → _, IsCompact A ∧ Measurable t ∧
  IsL2Contraction m (volume.restrict A) n (fun (i, j) x y ↦
  X.W .full (γ * t x * y) i * conj (X.W .full (t x * y) j)) ∧ IsProbabilityMeasure m ∧
  IsFiniteMeasure n ∧ n.real .univ ≤ cellMass γ X.deg ρ (card X.I) ∧ ∀ᵐ x ∂m, t x ∈ besselU0 ⊔
  besselH ∧ ∀ y ∈ frameBox ρ, besselSize (γ * t x * y) ≤ ρ → y ∈ A
```

In this note	Lean declaration	File	Lines
Theorem D (i)	`IsBesselFunction.hasSchurBound_of_convolve_eq`	BesselCut.lean	27–64
Theorem D (ii)	`HasPeakAt.exists_cut_isCutPair`	BesselCut.lean	72–157
Lemma 6.3	`BesselSystem.orthonormal`	BesselOrthonormal.lean	10–26
Definition 2.4	`openCellChart`	CellNeighbourhood.lean	15–17
Definition 2.4	`chartCoords`	CellNeighbourhood.lean	30–34
Definition 2.4	`besselTangent`	CellNeighbourhood.lean	38–41
Lemma 6.1	`logDerivU_unipotentOf_mul_embedH`	CellNeighbourhood.lean	43–53
Lemma 6.2	`exists_symm_eq_chartCoords_skewOfReal`	CellNeighbourhood.lean	55–61
Theorem A (i)	`openCellChart_jacobian`	CellNeighbourhood.lean	67–114
Theorem A (ii)	`openCellChart_injOn`	CellNeighbourhood.lean	117–131
Lemma 5.4	`mem_openCell_and_bounds_of_near`	CellNeighbourhood.lean	138–163
Lemma 6.3	`euclidean_enorm_sq_eq_sum`	CellNeighbourhood.lean	165–166
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Notation

$\mathfrak{s} = (\varrho, \kappa, k_0)$, n, k, r the shape; $\varrho + 1$; $|\kappa|$; $2\varrho + k + 1$

V, e_a, e_a^*, x_i, z, f_a, h the space, its basis, the frame, the hermitian form

$J_1(\mathfrak{s})$, $G_1(\mathfrak{s})$, $\mathrm{d}y$, $\mathrm{d}\dot{y}$ Gram matrix, unitary group, Haar measure (normalised by the Cayley chart), the measure on $G_1(\mathfrak{s})$ that computes integrals over $\mathcal{U}(T)\backslash G_1(\mathfrak{s})$

$F_{\mathfrak{s}}$, $F'_{\mathfrak{s}}$, $W_{\mathfrak{s}}$, $D_{\mathfrak{s}}$ matrices of the frame, the dual frame, the x_i, the e_a^*

T, $\mathcal{H}_l(T)$, $\mathcal{Q}_l(T)$, $\mathcal{U}(T)$, $\mathcal{U}^0$ the form at $\mathbb{C}$ and the subgroups of Note T; the unipotent group fixing the frame

$\mathrm{u}(q)$, $\underline{h}$, μ^0, lift elements of $\mathcal{U}^0$, of $\mathcal{H}_n(T)$; Haar measure of $\mathcal{U}^0$; a lift of unitriangular matrices

h, h', h_1, h_2 elements of $\mathrm{U}(\kappa)$ (h also the hermitian form)

$Z(y)$, $\mathrm{fm}(y)$, $\mathfrak{g}(Z)$, D_ϱ frame Gram matrix, frame matrix, discriminant, leading minor

$\mathrm{bs}(y)$, $\mathrm{FB}(\rho)$, $\|X\|$, X^* Bessel size, frame box, operator norm, conjugate transpose

ic, cd, $\mathcal{C}(\mathfrak{s})$, $\mathrm{Cell}_{\leq P}$ corner, defect, open cell, bounded part

d_ϱ, c, f, m, s_0, $\gamma(Z)$, Δ the matrices of the section and the default of their last argument (also s_0 a number of the sizes); the section; a diagonal matrix (in Section 11: the volume weight)

ch, cc, bt, $\mathrm{D}^{\log}$, sk, skew chart, its coordinates, tangent map, logarithmic derivative, real coordinates of matrices

cay, j, θ, $\tilde{\theta}$, η, q, q', p, e Cayley chart, its weight, parameters of the chart, the isomorphism of coordinates

$\mathfrak{h}$, RPS, ρ, s_0, b real-place sizes and their numbers

$\mathcal{L}_{\mathbb{C}}(\mathfrak{h})$, λ, $\Gamma(\mathfrak{h}; \rho \leq R)$, R level, Haar ratio, gauge, its radius

Lev, Gau, M, $\mathsf{M}_{\geq l}$, θ_l, $\eta_{<j}$, ϖ_l, O notions of Note T: levels, gauges, tower and upper measures, floor measures, Heisenberg measures, Weyl laws, orthonormal part

$\chi_\varrho(c)$, $w_\varrho(c)$, g_c, dens_a boost, Weyl element, their product, densities of the floors

$c_G(\mathfrak{s})$, $\mathrm{vol}\,\mathrm{U}(N)$ $\frac{\mathrm{vol}\,\mathrm{U}(k)}{\mathrm{vol}\,\mathrm{U}(r)}\frac{2\pi^k}{(k-1)!}$; $\prod_{i<N} 2\pi^{i+1}/i!$

I, $\langle v, w\rangle$, rep, Φ_0, Φ, Y index set, inner product, representation, full and cut functions, a peaked function

deg, $\deg(l)$, ψ_l, ψ_l^{p}, l, L, A_0, ε, p formal degrees, the coefficient and the coefficient in the frame, parameter, its bound and gap, signs, frame

$\mathrm{U}(\varepsilon)$, $\mathrm{U}(\kappa)$, $\mathrm{U}(J)$ the unitary groups of signs, of $\langle x_i\rangle$, of a hermitian involution J

ν_S, $s = (s_1, s_2)$, sd, $*$, V_w, X_w, x_w, β, γ_w measure of unit 133 and a point of its group, superdiagonal sum, convolution, derivative functions, smoothing data

$\Delta(\gamma)$, $\mathsf{x}_0(\gamma)$, I_j, E, Z_j, K volume weight, part in the positive space, inclusions, leading block, the kernel

E, $\mathbb{A}_{E,f}$, $\widehat{O}_E$, S, g, h_c, x_c, pl_j, ξ CM field, finite adeles, integral adeles, filling and its objects

σ, τ, π, α, ι, m finite index sets (π also the number); α, γ, δ also indices

a, b, c, d, i, j, l, m, t, s, f, N, M, P, C, D, B, A indices, numbers and constants of the statements; matrices and sets of the proofs

g, u, v, w, x, y, ω, γ group elements and vectors

φ, χ, ϕ, μ, ν, Ω_0, ϵ, Λ, Λ_a, $\mathcal{G}$, $\mathcal{A}$, $\mathcal{V}$, $\mathcal{C}$ functions, measures, sets, lattices and auxiliary objects of the proofs

A, B, C, G, L, N, R, T, W, Y, l, n, q, r, t, y, z, Θ matrices, numbers and maps of the proofs

G, H, J, O, Q, S, U, X, Z, F, K, k, h, q, o, n, r, z, $\mathbf{1}$, ∞, e, i further matrices and elements; letters inside names; the indicator; infinity; the numbers e and i

ϱ, κ, δ, ζ, ψ letters of names of Note T and of the place

$E(D)$, $m(D)$, $k(D)$, $r(D)$, $b(D)$, $f(D)$ the CM field, the numbers and the third scale of the admissible data D of the overview (only in Sections 1 and 2)

d, λ, K, f, Θ local to this note and different from the letters of the overview: an index, the Haar ratio, a kernel, a function or the natural number of a cut pair, a matrix of the proof of Theorem E

READBACK NOTE 46

Discrete series coefficients near the identity, Laplace's method for two coefficients, and Bessel functions with a peak

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. For signs ε on a finite set, $\mathrm{U}(\varepsilon)$ is a unitary group $\mathrm{U}(p,q)$, and Flensted-Jensen's function $\psi_l(\,\cdot\,)$ is the matrix coefficient of the lowest K-type of the discrete series whose doubled Harish-Chandra parameter is the integer vector l. This note proves five things about it. Near the identity $\psi_l(g)$ is an explicit phase, $\exp \mathrm{tr}(\mathrm{diag}(\Lambda(l))\,\lg(g-1))$, times a factor $\kappa_{\mathrm{G}} \in [0,1]$, up to a small relative error (Theorem 1); the proof expands compressed determinants to second order. For an integrand made of two such coefficients on a group S, with phases that cancel, a version of Laplace's method bounds the volume of a unit ball of S by a power of a parameter times the absolute value of its integral (Theorem 2). The hypotheses of Theorem 2 hold in two cases, both built from Lagrange interpolation: a restriction from $\mathrm{U}(p,q)$ to a conjugate of itself inside a group one rank larger (Theorem 3), and the Bessel subgroup of a unitary group at a point of controlled size (Theorem 4). In the second case the integral is the value at that point of a Bessel function, which therefore has a peak there (Theorem 5). Last, the period of an automorphic form against a Poincaré series of deep level is its main term up to an explicit error (Theorem 6).

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles with Lean v4.35.0 against a pinned version of the library Mathlib and that Lean's kernel accepted the proofs; a replay of the build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note treats one unit of the artifact, the folder on discrete series: all 35 named declarations of its 37 (the other two are unnamed instances, stated in Lemma 9.1). The results of the unit are proved here in full from the Inputs of §3: results of the background library of the artifact (Flensted-Jensen's formula and its invariance, decay of the coefficient, estimates for Haar measures, Schur orthogonality, Witt's theorem, unfolding), quoted with their owners and, where a reference is confirmed, with it; and one result of another unit of the argument. Every statement was compared with its Lean text, argument by argument; what Lean's kernel checked is the formal text, and what is written here is a reading of it. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 35 declarations.

1 The statements

The notions used in the six statements are defined in §2; here is what a reader needs to parse them. For signs $\varepsilon\colon \Omega \to \{\pm 1\}$ on a finite set, $\mathrm{U}(\varepsilon)$ is the group of the complex matrices g with $g^{*}\,\mathrm{diag}(\varepsilon)\,g = \mathrm{diag}(\varepsilon)$. The function $\psi_l(\,\cdot\,)$ on $\mathrm{U}(\varepsilon)$ is Flensted-Jensen's function of the integer vector l (Definition 2.7), $\Lambda(l)$ is the highest weight of the lowest K-type (Definition 2.4), and $\lg(X)$ is the logarithm of $1+X$ (Definition 2.2). Two norms of matrices occur: the Hilbert–Schmidt norm $\|\cdot\|_{\mathrm{HS}}$ and the operator norm $\|\cdot\|_{\mathrm{op}}$. A vector l *has large gap A for ε* if its entries are distinct, of the parity of $|\Omega|-1$, and differ by at least $2(500|\Omega|^{3}+A)$ across the two signs (Definition 2.5).

Theorem 1 (Gaussian shape near the identity). *Let Ω be a non-empty finite set, $\varepsilon\colon \Omega \to \{\pm 1\}$ signs, $l \in \mathbb{Z}^{\Omega}$, let δ and T be real numbers, L and A natural numbers, and $g \in \mathrm{U}(\varepsilon)$. Suppose that l has large gap A for ε, that $|l_i| \le L$ for every $i \in \Omega$, and that*

$$\|g-1\|_{\mathrm{HS}} \le \delta, \qquad 1 \le T, \qquad |\Omega|^{2} \le T,$$
$$|\Omega|^{2}\log(|\Omega|+L+1) \le T, \qquad \delta T \le 10^{-12}.$$

Then there is a real number $\kappa_{\mathrm{G}} \in [0,1]$ with

$$\Big|\frac{\psi_l(g)}{\exp \mathrm{tr}\big(\mathrm{diag}(\Lambda(l))\ \lg(g-1)\big)} - \kappa_{\mathrm{G}}\Big| \;\le\; \frac{\kappa_{\mathrm{G}}}{10} + e^{-100\,T}.$$

LEAN `dsCoeff_gaussian_shape`

The proof gives $\kappa_{\mathrm{G}} = \exp\!\big(-\tfrac{1}{4}\sum_{i,i'} |\Upsilon(l)_i - \Upsilon(l)_{i'}|\,|(g-1)_{ii'}|^{2}\big)$, a Gaussian in the entries of $g-1$ whose widths are the differences of the entries of $\Upsilon(l) = \lambda + \rho$ (Definition 2.4). The factor κ_{G} (the index G for "Gaussian") is small as soon as g leaves the diagonal torus by more than the reciprocal square roots of these differences; on the torus the coefficient is the phase alone. The error is uniform in l: only the size L and the rank enter, through T.

The next statement concerns a product of two coefficients. For a homomorphism $j = (j_1, j_2)\colon S \to \mathrm{U}(\varepsilon) \times \mathrm{U}(\varepsilon')$ and a function $\phi\colon S \to \mathbb{C}$ put

$$\theta^{j,\phi}_{l,l'}(\sigma) = \psi_l(j_1(\sigma))\,\overline{\psi_{l'}(j_2(\sigma))}\,e^{-\phi(\sigma)} \qquad (\sigma \in S).$$

The predicate $\mathrm{TwoFactor}(l, l'; j, \phi; e, e'; B)$ (Definition 2.13) says, in short: the weights $\Lambda(l)$ vanish on a block $e(\Xi)$ of positive coordinates and are steep, between $-3B$ and $-B$, on the other coordinates, whose signs alternate; ϕ is additive, purely imaginary, and equal to $\mathrm{tr}(\mathrm{diag}(\Lambda(l))(j_1-1)) - \mathrm{tr}(\mathrm{diag}(\Lambda(l'))(j_2-1))$; and $j(S)$ contains a copy of the unitary group of $\mathbb{C}^{\Xi}$, embedded in both factors.

Theorem 2 (Laplace's method for two coefficients). *Let Ω and Ω' be non-empty finite sets, Ξ a finite set, and S a group with a topology, with its Borel σ-algebra, for which all left and right translations are measurable. Let ν be a measure on S that is invariant under right translations and finite on compact sets. Let $\varepsilon\colon \Omega \to \{\pm 1\}$ and $\varepsilon'\colon \Omega' \to \{\pm 1\}$ be signs, $l \in \mathbb{Z}^{\Omega}$, $l' \in \mathbb{Z}^{\Omega'}$, $e'\colon \Xi \to \Omega'$ a map, B and β real numbers, L, M and A natural numbers, $e\colon \Xi \sqcup \{0,\dots,M-1\} \to \Omega$ a bijection, $j = (j_1, j_2)\colon S \to \mathrm{U}(\varepsilon) \times \mathrm{U}(\varepsilon')$ a homomorphism of groups and $\phi\colon S \to \mathbb{C}$ a function. Suppose that*

(a) $\mathrm{TwoFactor}(l, l'; j, \phi; e, e'; B)$ *holds;*
(b) *l has large gap A for ε, and l' has large gap A for ε';*
(c) *$1 \le B$; $|l_i| \le L$ for every $i \in \Omega$; $|l'_i| \le L$ for every $i \in \Omega'$;*
(d) *$10^{12} \le \beta$, $|\Omega| + |\Omega'| + L \le \beta$ and $10^{30}\big((|\Omega| + |\Omega'|)^2 \log \beta\big)^3 \le A$;*
(e) *the map $j_1\colon S \to \mathrm{U}(\varepsilon)$ is proper, the map $j_2\colon S \to \mathrm{U}(\varepsilon')$ is continuous, and $\|j_2(\sigma)\|_{\mathrm{op}} \le \beta + 1$ for every $\sigma \in S$ with $\|j_1(\sigma) - 1\|_{\mathrm{op}} \le 1$.*

Then the set on the left has finite measure, $\theta^{j,\phi}_{l,l'}(\,\cdot\,)$ is ν-integrable, and

$$\nu\{\sigma \in S :\ \|j_1(\sigma) - 1\|_{\mathrm{op}} \le 1\} \ \le\ \beta^{20(|\Omega|+|\Omega'|)^2} \cdot \Big| \int_S \theta^{j,\phi}_{l,l'}(\sigma)\, d\nu(\sigma) \Big|.$$

LEAN `IsTwoFactor.laplace_method`

So the integral of the oscillating integrand is not much smaller than the volume of the region where the integrand is not small. The reason is that the integrand has non-negative real part up to a tiny error, everywhere: far from the block-diagonal matrices both coefficients are tiny, and near them a translation by the embedded unitary group of $\mathbb{C}^{\Xi}$, which changes neither the integrand nor the measure, moves the point close to 1, where Theorem 1 applies to both factors and the two phases cancel against $e^{-\phi}$.

Theorem 3 (the restriction to a conjugate subgroup). *Let Ω and Ξ be finite sets, $\varepsilon\colon \Omega \to \{\pm 1\}$ signs, B and H real numbers, M a natural number, $l \in \mathbb{Z}^{\Omega}$ and $l' \in \mathbb{Z}^{\Omega \sqcup \{*\}}$, and let $e\colon \Xi \sqcup \{0,\dots,M-1\} \to \Omega$ and $\tilde{e}\colon \Xi \sqcup \{0,\dots,M\} \to \Omega \sqcup \{*\}$ be bijections. Write $\varepsilon \oplus 1$ for the signs on $\Omega \sqcup \{*\}$ that extend ε by $+1$ at $*$. Suppose that M is even, that $\mathrm{Steep}(l; B, e)$ holds for ε and $\mathrm{Steep}(l'; B, \tilde{e})$ holds for $\varepsilon \oplus 1$, that l' has gap H for $\varepsilon \oplus 1$, and that $1 \le B$. Then there is $\omega \in \mathrm{U}(\varepsilon \oplus 1)$ such that the homomorphism*

$$j^{\omega}\colon \mathrm{U}(\varepsilon) \to \mathrm{U}(\varepsilon) \times \mathrm{U}(\varepsilon \oplus 1), \qquad j^{\omega}(g) = \big(g,\ \omega^{-1}(g \oplus 1)\,\omega\big),$$

has the following three properties:

(i) $\|\omega\|_{\mathrm{op}} \le (|\Omega| + 1)^2\,(6B)^{M+(M+1)}$;
(ii) $\mathrm{TwoFactor}(l, -l'; j^{\omega}, 0; e, \tilde{e}|_{\Xi}; B)$ *holds, with the function 0 as phase;*
(iii) *for every $g \in \mathrm{U}(\varepsilon)$, the second component of the pair $\big(j^{\omega}_1(g) - 1,\ j^{\omega}_2(g) - 1\big)$ is $\omega^{-1}\big((g-1) \oplus 0\big)\omega$.*

LEAN `IsSteep.exists_isTwoFactor_restriction`

Here $g \oplus 1$ and $(g-1) \oplus 0$ are the block matrices on $\Omega \sqcup \{*\}$ with the entry 1, respectively 0, at $(*,*)$. With Theorem 2 this gives a lower bound for the integral over $\mathrm{U}(\varepsilon)$ of $\psi_l(g)$ against the conjugate of $\psi_{-l'}(\omega^{-1}(g \oplus 1)\omega)$: a period for the restriction from $\mathrm{U}(p+1,q)$ to $\mathrm{U}(p,q)$, at a position ω of explicit size.

Theorems 4 and 5 concern a *shape* $\mathfrak{s}$ (§2): a hermitian space $\mathbb{C}^{\mathrm{Ind}(\mathfrak{s})}$ with $\varrho(\mathfrak{s})$ hyperbolic planes, a positive definite part with basis indexed by $\kappa(\mathfrak{s})$ and a negative vector, of rank $r(\mathfrak{s})$, with unitary group $G_1(\mathfrak{s})$; its unipotent group $\mathcal{U}(\mathfrak{s})$ with the character $u \mapsto e^{\mathrm{sd}(u)-\overline{\mathrm{sd}(u)}}$; the compact group $\mathrm{U}(1)$ of $\mathbb{C}^{\kappa(\mathfrak{s})}$, embedded by $\mathrm{emb}_{\mathfrak{s}}(\,\cdot\,)$; the group $\mathsf{HU}(\mathfrak{s})$ of the pairs (g,h) with $g \in \mathrm{emb}_{\mathfrak{s}}(h)\,\mathcal{U}(\mathfrak{s})$, with its measure $\nu_{\mathfrak{s}}$; and the homomorphism $\jmath_x(\,\cdot\,)$ from $\mathsf{HU}(\mathfrak{s})$ to $\mathrm{U}(\varepsilon) \times \mathrm{U}(1)$, which conjugates the first component by a point x and by a frame p (Definitions 2.15 to 2.19).

Theorem 4 (the Bessel subgroup at a point of controlled size). *Let $\mathfrak{s}$ be a shape, $\varepsilon\colon \mathrm{Ind}(\mathfrak{s}) \to \{\pm 1\}$ signs, $\mathsf{p} \in \mathrm{Frm}(\mathfrak{s},\varepsilon)$ a frame, Ξ a finite set, B a real number, $l \in \mathbb{Z}^{\mathrm{Ind}(\mathfrak{s})}$, and $e\colon \Xi \sqcup \{0,\dots,2\varrho(\mathfrak{s})+1\} \to \mathrm{Ind}(\mathfrak{s})$ a bijection. Let $\tilde{e}\colon \Xi \sqcup \{*\} \to \kappa(\mathfrak{s})$ be a bijection, $l' \in \mathbb{Z}^{\kappa(\mathfrak{s})}$, a a natural number and b a real number. Suppose that $\mathrm{Steep}(l;B,e)$ holds for ε; that, for the signs 1 on $\kappa(\mathfrak{s})$, $\Lambda(l')_i = a$ if $i = \tilde{e}(*)$ and $\Lambda(l')_i = 0$ otherwise; and that*

$$1 \le B, \qquad 3B + a \le 2\,b^2, \qquad 8\,(\varrho(\mathfrak{s})+1) \le b.$$

Then there is a point $x \in G_1(\mathfrak{s})$ with $\mathrm{size}_Q(x) \le b^{\,5\,n(\mathfrak{s})}$ such that

$$\mathrm{TwoFactor}(l,l';\jmath_x(\,\cdot\,),\phi_{\mathfrak{s}};e,\tilde{e}|_{\Xi};B) \quad \text{holds, where} \quad \phi_{\mathfrak{s}}(\sigma) = \mathrm{sd}(\sigma_1) - \overline{\mathrm{sd}(\sigma_1)}$$

for $\sigma = (\sigma_1,\sigma_2) \in \mathsf{HU}(\mathfrak{s})$.

LEAN `IsSteep.exists_isTwoFactor_besselPairHom`

Here $\mathrm{size}_{\mathcal{Q}}(x) = \inf_{q \in \mathcal{Q}(\mathfrak{s})} \|q\,x\|_{\mathrm{op}} + 1$ measures the distance of x from the Bessel subgroup $\mathcal{Q}(\mathfrak{s})$. The predicate $\mathrm{Peak}_{l,\mathrm{rep}}(W;x,P)$ of the next statement (Definition 2.20) says that W is a Bessel function for a representation rep of $\mathrm{U}(1)$, reproduced by the coefficient of l, non-zero at x, and that two integrals over the Bessel subgroup are bounded in terms of $P\|W(x)\|$.

Theorem 5 (a Bessel function with a peak). *Let $\mathfrak{s}$ be a shape, I a finite set, $\varepsilon\colon \mathrm{Ind}(\mathfrak{s}) \to \{\pm 1\}$ signs and $\mathsf{p} \in \mathrm{Frm}(\mathfrak{s},\varepsilon)$ a frame. Let $x \in G_1(\mathfrak{s})$, $v_0 \in \mathbb{C}^I$, let A and N be natural numbers, b a real number, $l \in \mathbb{Z}^{\mathrm{Ind}(\mathfrak{s})}$, $l' \in \mathbb{Z}^{\kappa(\mathfrak{s})}$, and let* rep *be a homomorphism from the group $\mathrm{U}(1)$ of the unitary matrices of $\mathbb{C}^{\kappa(\mathfrak{s})}$ to the unitary $I \times I$ matrices. Suppose that* rep *is continuous; that $\langle v_0, \mathrm{rep}(h)\,v_0\rangle = \psi_{l'}(h)$ for every $h \in \mathrm{U}(1)$; that l has large gap A for ε; that $16 \le b$; and that, with $\phi_{\mathfrak{s}}(\sigma) = \mathrm{sd}(\sigma_1) - \overline{\mathrm{sd}(\sigma_1)}$,*

$$\nu_{\mathfrak{s}}\{\sigma \in \mathsf{HU}(\mathfrak{s}) :\ \|\jmath_x(\sigma)_1 - 1\|_{\mathrm{op}} \le 1\} \ \le\ b^N \cdot \Big|\int_{\mathsf{HU}(\mathfrak{s})} \theta_{l,l'}^{\jmath_x(\,\cdot\,),\phi_{\mathfrak{s}}}(\sigma)\,d\nu_{\mathfrak{s}}(\sigma)\Big|.$$

Then there is a function $W\colon G_1(\mathfrak{s}) \to \mathbb{C}^I$ such that $\mathrm{Peak}_{l,\mathrm{rep}}(W;x,b^{\,8r(\mathfrak{s})^2+1+N})$ holds.

LEAN `exists_isBesselFunction`

The displayed hypothesis is the conclusion of Theorem 2 for the data of Theorem 4, with b^N in place of the power of β. The function is

$$W(y) = \int_{\mathrm{HU}(\mathfrak{s})} \psi_l(\mathsf{p}^{-1}x^{-1}\sigma_1 y\,\mathsf{p})\, e^{-\phi_{\mathfrak{s}}(\sigma)}\, \mathrm{rep}(\sigma_2)^{-1} v_0 \, d\nu_{\mathfrak{s}}(\sigma),$$

and $\langle v_0, W(x)\rangle$ is the integral in the hypothesis.

The last statement is adelic; §2.1 has the definitions. For a CM field E with f_E infinite places and signs ε on Ω, the group $G(E,\varepsilon) = G_\infty(E,\varepsilon) \times G_{\mathrm{f}}(E,\Omega)$ is the adelic unitary group in standard coordinates, with $G_\infty(E,\varepsilon) = \prod_v \mathrm{U}(\varepsilon)$ over the infinite places v. A hermitian space with a point V gives the discrete subgroup $\Gamma(V)$ of $G(E,\varepsilon)$ and a measure ϖ_V that computes integrals over $\Gamma(V)\backslash G(E,\varepsilon)$; $V\cdot x$ is V with its point moved by x, $\lambda_1(V)$ is the length of a shortest non-zero lattice vector of V, cut off at 1, $V \oplus \langle 1\rangle$ is the sum of V and a positive line, $K_\Omega(a_0)$ is the principal congruence subgroup of level a_0, and $g \oplus 1$ is the block-diagonal embedding into the group of $\Omega \sqcup \{*\}$.

Theorem 6 (the period of a Poincaré series of deep level). *Let E be a CM field with a CM type that contains a primitive root of unity of order 5040, let Ω be a finite set, $\varepsilon\colon \Omega \to \{\pm 1\}$ signs, and V a hermitian space with a point over E for ε. Let $F\colon G(E,\varepsilon) \to \mathbb{C}$ be a function, C, ρ and Q real numbers, a_0 an integer of E, A a natural number, $l = (l_v)_v$ a family of vectors $l_v \in \mathbb{Z}^{\Omega\sqcup\{*\}}$ indexed by the infinite places v of E, $x \in G(E,\varepsilon)$, and $\omega, \alpha \in G_\infty(E, \varepsilon\oplus 1)$. Suppose that*

(a) *F is automorphic for V, and $|F(g)| \le C$ for every $g \in G(E,\varepsilon)$;*
(b) *$a_0 \neq 0$, and F has level $K_\Omega(a_0)$;*
(c) *$\rho \le |a_0|_v$ for every infinite place v, and $0 \le \rho$;*
(d) *$2f_E\,(|\Omega|+1) + 2 \le A$ and $8\,(|\Omega|+1)^2 < A$;*
(e) *every l_v has large gap $2A$ for $\varepsilon \oplus 1$;*
(f) *$Q^2 \le \rho \cdot \lambda_1(V\cdot x)$, $f_E \le Q$ and $10 \le Q$.*

Put $\Psi(g) = \prod_v \overline{\psi_{l_v}(\alpha_v^{-1} g_v\, \omega_v)}$ for $g \in G_\infty(E, \varepsilon\oplus 1)$, and

$$L_* = \prod_v \Big(\big(2\,(\|\alpha_v\|_{\mathrm{op}} \|\omega_v\|_{\mathrm{op}})^2\big)^A \cdot 16^A \,\mathrm{vol}\{r \in \mathrm{U}(\varepsilon) : \|r-1\|_{\mathrm{op}} \le 1\}\Big).$$

Then $r \mapsto \Psi(r\oplus 1)$ is integrable on $G_\infty(E,\varepsilon)$, and, with $X_\gamma(y) = (x\oplus 1)^{-1}\gamma\,(y\oplus 1)$ for $\gamma \in \Gamma(V\oplus\langle 1\rangle)$ and $y \in G(E,\varepsilon)$,

$$\Big| \int \sum_{\gamma\in\Gamma(V\oplus\langle 1\rangle)} \Psi(X_\gamma(y)_\infty)\, \mathbf{1}_{K_{\Omega\sqcup\{*\}}(a_0)}(X_\gamma(y)_{\mathrm{f}})\, F(y)\, d\varpi_V(y) - \mathrm{vol}(K_\Omega(a_0)) \int_{G_\infty(E,\varepsilon)} \Psi(r\oplus 1)\, F(x\cdot(r,1))\, dr \Big| \;\le\; \mathrm{vol}(K_\Omega(a_0)) \cdot L_* \cdot C \cdot \frac{10}{Q}.$$

LEAN `norm_poincare_period_sub_le`

The sum over γ is, as a function of y, the restriction to $G(E,\varepsilon)$ of a Poincaré series on the group of $V\oplus\langle 1\rangle$, made from the test function $\Psi \otimes \mathbf{1}_{K(a_0)}$ and translated by x. The main term is the contribution of the classes $\gamma \in \Gamma(V)\oplus 1$. All other classes move

the vector of the added line to rational vectors that are congruent modulo a_0, hence far apart at the infinite places when a_0 is large there and x is not near the cusp; the decay of the coefficients then makes their total at most the stated error.

2 The setting

Conventions. Ω, Ω' and Ξ denote a finite set each (in the formal text: a type with finitely many elements), and I a finite set that indexes the coordinates of a vector space $\mathbb{C}^I$. A family of signs on Ω is a map $\varepsilon\colon \Omega \to \{\pm 1\}$; ε' is a family of signs on Ω', and 1 also denotes the constant family $+1$. The letter S denotes a group, $\mathfrak{s}$ a shape (§2) and p a frame of it. All matrices have complex entries; for index sets Ω_1, Ω_2 the entries of an $\Omega_1 \times \Omega_2$ matrix X are $X_{ii'}$, and X^* is its conjugate transpose. An integer vector $l \in \mathbb{Z}^\Omega$ is written with entries l_i. Every statement introduces the objects it is about, also those that the formal text takes as implicit arguments; the instance arguments of the formal text (finiteness of the index sets, the group structure, the topology and the σ-algebra of S, the frame) are the structures named here and in the statements. Integrals of complex or vector valued functions are Bochner integrals with the convention of the library: the integral of a function that is not integrable is 0. A sum over an infinite set is the unconditional sum when it exists and 0 otherwise. For a measure ν, the measure of a set that occurs in an inequality between real numbers is read as 0 when it is infinite; in this note every such set is shown to have finite measure.

Matrices and unitary groups. These notions belong to the background library; they are defined here because every statement uses them.

Definition 2.1 (norms, unitary groups)**.** Let Ω be a finite set and $\varepsilon\colon \Omega \to \{\pm 1\}$ signs.

(a) For an $\Omega \times \Omega$ matrix X, $\|X\|_{\mathrm{HS}} = (\sum_{i,i'} |X_{ii'}|^2)^{1/2}$ is the Hilbert–Schmidt (Frobenius) norm and $\|X\|_{\mathrm{op}}$ the operator norm for the Euclidean norm of $\mathbb{C}^\Omega$. Both are submultiplicative, $\|X\|_{\mathrm{op}} \le \|X\|_{\mathrm{HS}}$ and $\|XY\|_{\mathrm{HS}} \le \|X\|_{\mathrm{HS}}\|Y\|_{\mathrm{op}}$; $[X,Y] = XY - YX$.

(b) For an integer or sign vector w on Ω, $\mathrm{diag}(w)$ is the diagonal matrix with entries w_i.

(c) $\mathrm{U}(\varepsilon)$ is the group of the invertible $\Omega \times \Omega$ matrices g with $g^* \,\mathrm{diag}(\varepsilon)\, g = \mathrm{diag}(\varepsilon)$; it is a unitary group $\mathrm{U}(p,q)$ with p signs $+1$ and q signs -1. Its subgroup $\mathrm{K}(\varepsilon)$ consists of the g that are moreover unitary matrices, equivalently that commute with $\mathrm{diag}(\varepsilon)$; it is compact, isomorphic to $\mathrm{U}(p) \times \mathrm{U}(q)$, and dh denotes its Haar measure of total mass one. On $\mathrm{U}(\varepsilon)$ itself, dg is the Haar measure that the library fixes; for $\varepsilon = 1$ it has total mass one.

(d) For an $\Omega \times \Omega$ matrix X, the matrix $\mathrm{bd}_\varepsilon(X)$ has the entries $X_{ii'}$ with $\varepsilon_i = \varepsilon_{i'}$ and zeros elsewhere; $\mathrm{bd}_\varepsilon(X) = \frac{1}{2}(X + \mathrm{diag}(\varepsilon) X \,\mathrm{diag}(\varepsilon))$. A matrix with $X = \mathrm{bd}_\varepsilon(X)$ is called block diagonal, and the pairs (i,i') with $\varepsilon_i = \varepsilon_{i'}$ compact, the others non-compact.

(e) For a map $\iota\colon \Xi \to \Omega$, in_ι is the $\Omega \times \Xi$ matrix with entry 1 at $(\iota(\xi), \xi)$ for $\xi \in \Xi$ and 0 elsewhere. For a $\Xi \times \Xi$ matrix u the matrix $\mathrm{in}_\iota\, u\, \mathrm{in}_\iota^*$ is supported on $\iota(\Xi) \times \iota(\Xi)$; when ι is injective it is u placed in that block.

Definition 2.2 (the logarithm)**.** For an $\Omega \times \Omega$ matrix X, $\lg(X) = \int_0^1 (1+sX)^{-1} X\, ds$, where the inverse of a singular matrix is read as 0. For $\|X\|_{\mathrm{HS}} \le \frac{1}{2}$ it is the logarithm of $1+X$: by Input (I3), $\det(1+X) = \exp \operatorname{tr} \lg(X)$.

Definition 2.3 (coordinate projections, principal minors)**.** For a subset $\Pi \subseteq \Omega$, p_Π is the diagonal matrix with entries 1 on Π and 0 elsewhere, an orthogonal projection, and for an $\Omega \times \Omega$ matrix y the number $\Delta_\Pi(y)$ is the determinant of the submatrix of y with rows and columns in Π; so $\Delta_\Omega(y) = \det y$ and $\Delta_\Pi(y) = \det(1 + p_\Pi(y-1))$. For an integer vector w on Ω and $\tau \in \mathbb{Z}$ we write $\{w \ge \tau\}$ for the set of the $i \in \Omega$ with $w_i \ge \tau$. (The formal text speaks of decidable predicates on Ω where we speak of subsets.)

Parameters and Flensted-Jensen's function. Also from the background library.

Definition 2.4 (parameters)**.** Let $\varepsilon \colon \Omega \to \{\pm 1\}$ be signs and $l \in \mathbb{Z}^\Omega$. The integer vectors $\Upsilon(l)$ and $\Lambda(l)$ on Ω are

$$\Upsilon(l)_i = \left\lfloor \tfrac{1}{2}\bigl(l_i + \#\{i' : l_{i'} < l_i\} - \#\{i' : l_i < l_{i'}\}\bigr) \right\rfloor,$$
$$\Lambda(l)_i = \Upsilon(l)_i - \#\{i' : \varepsilon_{i'} = \varepsilon_i,\ l_{i'} < l_i\} + \#\{i' : \varepsilon_{i'} = \varepsilon_i,\ l_i < l_{i'}\}.$$

The vector $\Lambda(l)$ depends on ε, which the notation does not show; the statements say for which signs it is taken.

One reads l as a doubled Harish-Chandra parameter $l = 2\lambda$ of $\mathrm{U}(\varepsilon)$. If ρ and ρ_c are the half sums of all roots, respectively of the compact roots, that are positive on λ, then $\Upsilon(l) = \lambda + \rho$ and $\Lambda(l) = \lambda + \rho - 2\rho_c$: the infinitesimal character shifted to an integral vector, and the highest weight of the lowest K-type (the Blattner parameter). The comments of the formal text write μ for $\Upsilon(l)$. Since the two counted sets have at most $|\Omega|$ elements,

$$\bigl|\Upsilon(l)_i - \Lambda(l)_i\bigr| \le |\Omega| \qquad \text{for every } i \in \Omega. \tag{1}$$

Definition 2.5 (gap, large gap)**.** Let $\varepsilon \colon \Omega \to \{\pm 1\}$ be signs, $l \in \mathbb{Z}^\Omega$ and H a real number. The vector l *has gap H for* ε if l is injective, $l_i + |\Omega| - 1$ is even for every i, and $|l_i - l_{i'}| \ge 2H$ whenever $\varepsilon_i \ne \varepsilon_{i'}$. For a natural number A, l *has large gap A for* ε if it has gap $500|\Omega|^3 + A$ for ε. If l has gap H then so has $-l$, and the half in the definition of $\Upsilon(l)$ is exact.

Definition 2.6 (products of principal minors)**.** Let w be an integer vector on Ω and g an invertible $\Omega \times \Omega$ matrix. Choose integers $c_1 \le c_2$ with $c_1 \le w_i \le c_2$ for all i and put

$$\mathfrak{m}_w(g) = (\det g)^{c_1} \prod_{c_1 < \tau \le c_2} \Delta_{\{w \ge \tau\}}(g).$$

This does not depend on c_1 and c_2. (The library defines $\mathfrak{m}_w(g)$ for all matrices as a product over all $\tau \in \mathbb{Z}$; for invertible g it is this number, Input (I4).) On diagonal matrices $\mathfrak{m}_w(\,\cdot\,)$ is the character $\prod_i g_{ii}^{w_i}$.

Definition 2.7 (Flensted-Jensen's function)**.** Let $\varepsilon\colon \Omega \to \{\pm 1\}$ be signs and $l \in \mathbb{Z}^{\Omega}$; take $\Lambda(l)$ for ε. Put $\|\mathfrak{m}_{\Lambda(l)}\|^2_{\mathrm{K}(\varepsilon)} = \int_{\mathrm{K}(\varepsilon)} |\mathfrak{m}_{\Lambda(l)}(h)|^2\, dh$, a positive number (Input (I8)). For $\Omega \times \Omega$ matrices g and z,

$$\mathrm{FJ}_l(g,z) = \frac{1}{\|\mathfrak{m}_{\Lambda(l)}\|^2_{\mathrm{K}(\varepsilon)}} \int_{\mathrm{K}(\varepsilon)} \frac{\mathfrak{m}_{\Lambda(l)}(g\,h)\,\overline{\mathfrak{m}_{\Lambda(l)}(h)}}{\mathfrak{m}_{-\Upsilon(l)}(h^{*} z\,h)}\, dh,$$

$$\psi_l(g) = \mathrm{FJ}_l(\mathrm{bd}_{\varepsilon}(g), \mathrm{bd}_{\varepsilon}(g)^{-1} g).$$

The *formal degree* is $\deg(l) = \left(\int_{\mathrm{U}(\varepsilon)} |\psi_l(g)|^2\, dg\right)^{-1}$. Both $\psi_l(\,\cdot\,)$ and $\deg(l)$ depend on ε.

For $g \in \mathrm{U}(\varepsilon)$ the matrix $\mathrm{bd}_{\varepsilon}(g)$ is invertible (Lemma 6.4), and for l with a gap the integrand is defined. By the header of the library's folder, $\psi_l(\,\cdot\,)$ restricted to $\mathrm{U}(\varepsilon)$ is the matrix coefficient of the lowest K-type of the discrete series with Harish-Chandra parameter λ, normalised by $\psi_l(1) = 1$; this identification is a result of the background library that the present note does not use, and it is the reason for the name. What the note uses are the properties listed in §3.

The definitions of the unit for two coefficients.

Definition 2.8 (the gap-weighted square norm)**.** Let Ω be a finite set, w an integer vector on Ω and X an $\Omega \times \Omega$ matrix. Then

$$\mathrm{gw}_w(X) = \sum_{i \in \Omega} \sum_{i' \in \Omega} |w_i - w_{i'}|\, |X_{ii'}|^2.$$

LEAN `gapWeightedNormSq`

Definition 2.9 (the phase)**.** Let Ω, Ω' be finite sets, ε, ε' signs on them, $l \in \mathbb{Z}^{\Omega}$, $l' \in \mathbb{Z}^{\Omega'}$, and let $x = (x_1, x_2)$ be a pair of an $\Omega \times \Omega$ matrix and an $\Omega' \times \Omega'$ matrix. Then

$$\mathrm{ph}_{l,l'}\, x = \mathrm{tr}\big(\mathrm{diag}(\Lambda(l))\, x_1\big) - \mathrm{tr}\big(\mathrm{diag}(\Lambda(l'))\, x_2\big),$$

with $\Lambda(l)$ taken for ε and $\Lambda(l')$ for ε'. It is linear in x.

LEAN `HCParam.phase`

Definition 2.10 (steep weights)**.** Let Ω and Ξ be finite sets, $\varepsilon\colon \Omega \to \{\pm 1\}$ signs, $l \in \mathbb{Z}^{\Omega}$, B a real number, M a natural number and $e\colon \Xi \sqcup \{0, \dots, M-1\} \to \Omega$ a bijection. We say that $\mathrm{Steep}(l; B, e)$ *holds for* ε if

(S1) $\varepsilon_{e(\xi)} = 1$ for every $\xi \in \Xi$, and $\varepsilon_{e(n)} = (-1)^n$ for $0 \le n < M$;
(S2) $\Lambda(l)_{e(\xi)} = 0$ for every $\xi \in \Xi$;
(S3) the map $n \mapsto \Lambda(l)_{e(n)}$ is strictly decreasing on $\{0, \dots, M-1\}$, and $-3B \le \Lambda(l)_{e(n)} \le -B$ for $0 \le n < M$.

LEAN `IsSteep`

Definition 2.11 (the pair $j-1$)**.** Let S be a group, ε and ε' signs on finite sets Ω and Ω', $j\colon S \to \mathrm{U}(\varepsilon) \times \mathrm{U}(\varepsilon')$ a homomorphism of groups with components $j = (j_1, j_2)$, and $\sigma \in S$. The formal text abbreviates the pair of matrices $(j_1(\sigma) - 1,\ j_2(\sigma) - 1)$; this note writes the pair out.

LEAN subOnePair

Definition 2.12 (the integrand)**.** Let S be a group, ε and ε' signs on finite sets Ω and Ω', $l \in \mathbb{Z}^{\Omega}$, $l' \in \mathbb{Z}^{\Omega'}$, $j = (j_1, j_2) \colon S \to \mathrm{U}(\varepsilon) \times \mathrm{U}(\varepsilon')$ a homomorphism of groups, $\phi \colon S \to \mathbb{C}$ a function and $\sigma \in S$. Then

$$\theta^{j,\phi}_{l,l'}(\sigma) = \psi_l(j_1(\sigma)) \cdot \overline{\psi_{l'}(j_2(\sigma))} \cdot \exp(-\phi(\sigma)).$$

LEAN HCParam.laplaceIntegrand

Definition 2.13 (two-factor data)**.** Let Ω, Ω', Ξ be finite sets, S a group, ε and ε' signs on Ω and Ω', $l \in \mathbb{Z}^{\Omega}$, $l' \in \mathbb{Z}^{\Omega'}$, $j = (j_1, j_2) \colon S \to \mathrm{U}(\varepsilon) \times \mathrm{U}(\varepsilon')$ a homomorphism of groups, $e' \colon \Xi \to \Omega'$ a map, B a real number, M a natural number, $e \colon \Xi \sqcup \{0, \dots, M-1\} \to \Omega$ a bijection and $\phi \colon S \to \mathbb{C}$ a function. We say that $\mathrm{TwoFactor}(l, l'; j, \phi; e, e'; B)$ *holds* if $\mathrm{Steep}(l; B, e)$ holds for ε and

(T1) $\mathrm{ph}_{l,l'}(j_1(\sigma) - 1,\ j_2(\sigma) - 1) = \phi(\sigma)$ for every $\sigma \in S$;
(T2) $\mathrm{tr}(j_1(\sigma) - 1) = \mathrm{tr}(j_2(\sigma) - 1)$ for every $\sigma \in S$;
(T3) $\phi(\sigma\sigma') = \phi(\sigma) + \phi(\sigma')$ for all $\sigma, \sigma' \in S$;
(T4) $\mathrm{Re}\,\phi(\sigma) = 0$ for every $\sigma \in S$;
(T5) $\varepsilon'_{e'(\xi)} = 1$ and $\Lambda(l')_{e'(\xi)} = 0$ for every $\xi \in \Xi$;
(T6) $0 \le \Lambda(l')_i$ for every $i \in \Omega'$, and $\Lambda(l')_i \le 3B$ for every $i \in \Omega'$ such that $\varepsilon'_i \ne \varepsilon'_{i'}$ for some $i' \in \Omega'$;
(T7) for every unitary $\Xi \times \Xi$ matrix u there is $\sigma \in S$ with

$$j_1(\sigma) - 1 = \mathrm{in}_{e|_\Xi}\,(u - 1)\,\mathrm{in}^*_{e|_\Xi} \qquad \text{and} \qquad j_2(\sigma) - 1 = \mathrm{in}_{e'}\,(u - 1)\,\mathrm{in}^*_{e'}\,.$$

LEAN IsTwoFactor

In (T6) the second bound is asked of the indices whose sign differs from some other sign; if ε' is constant, as for a compact second factor, it is empty. Condition (T7) says that $j(S)$ contains the unitary group of $\mathbb{C}^{\Xi}$, placed in the block $e(\Xi)$ of the first factor and along e' in the second. No topology on S is part of the definition.

The shape and its Bessel subgroup

The objects of the next definition belong to the background library and to the units of Chapter VI on the real place; the letters are those of the notes of these units.

Definition 2.14 (shape)**.** A *shape* $\mathfrak{s}$ consists of a natural number $\varrho(\mathfrak{s})$, a finite set $\kappa(\mathfrak{s})$ and an element of $\kappa(\mathfrak{s})$. Its *basis index set* $\mathrm{Ind}(\mathfrak{s})$ is the disjoint union of $\{0, \dots, \varrho(\mathfrak{s}) - 1\}$, of $\kappa(\mathfrak{s}) \sqcup \{z\}$ and of a second copy of $\{0, \dots, \varrho(\mathfrak{s}) - 1\}$; the standard basis of $\mathbb{C}^{\mathrm{Ind}(\mathfrak{s})}$ is written

$$e_0, \dots, e_{\varrho(\mathfrak{s})-1}; \qquad x_i\ (i \in \kappa(\mathfrak{s})),\ z; \qquad e^*_0, \dots, e^*_{\varrho(\mathfrak{s})-1}.$$

Put $n(\mathfrak{s}) = \varrho(\mathfrak{s}) + 1$ and $r(\mathfrak{s}) = |\mathrm{Ind}(\mathfrak{s})| = 2\varrho(\mathfrak{s}) + |\kappa(\mathfrak{s})| + 1$.

(a) $J_1(\mathfrak{s})$ is the Gram matrix of the hermitian form on $\mathbb{C}^{\mathrm{Ind}(\mathfrak{s})}$ for which e_a and e^*_a span a hyperbolic plane with product 1, the x_i are orthonormal, z has square -1, and all other products of basis vectors vanish. $G_1(\mathfrak{s})$ is its unitary group, with the Haar measure dg of the library.

(b) The *level* of a basis index is given by

$$\mathrm{lev}(e_a) = a, \qquad \mathrm{lev}(x_i) = \mathrm{lev}(z) = \varrho(\mathfrak{s}), \qquad \mathrm{lev}(e_a^*) = 2\varrho(\mathfrak{s}) - a.$$

A matrix X is *block upper triangular for the levels* if $X_{ii'} = 0$ whenever $\mathrm{lev}(i') < \mathrm{lev}(i)$. The *frame* is $f_a = e_a$ for $a < \varrho(\mathfrak{s})$ and $f_{\varrho(\mathfrak{s})} = z$.

(c) $\mathcal{U}(\mathfrak{s})$ is the subgroup of the $u \in G_1(\mathfrak{s})$ with $u_{ii'} = \delta_{ii'}$ whenever $\mathrm{lev}(i') \le \mathrm{lev}(i)$, a unipotent group; $\mathrm{sd}(g) = \sum_{a<\varrho(\mathfrak{s})} g_{f_a, f_{a+1}}$ is the sum of the entries of g just above the diagonal on the frame.

(d) $\mathrm{U}(1)$, for the signs 1 on $\kappa(\mathfrak{s})$, is the group of the unitary matrices of $\mathbb{C}^{\kappa(\mathfrak{s})}$, and $\mathrm{emb}_{\mathfrak{s}}(h)$, for $h \in \mathrm{U}(1)$, is the element of $G_1(\mathfrak{s})$ that acts as h on the span of the x_i and fixes the other basis vectors. $\mathcal{H}(\mathfrak{s})$ is the image of $\mathrm{emb}_{\mathfrak{s}}(\,\cdot\,)$, and $\mathcal{Q}(\mathfrak{s})$, the *Bessel subgroup*, is the group $\mathcal{U}(\mathfrak{s})\,\mathcal{H}(\mathfrak{s})$; it normalizes $\mathcal{U}(\mathfrak{s})$.

(e) $\eta_{\mathfrak{s}}$ is the Haar measure on $\mathcal{U}(\mathfrak{s})$ that the library fixes, and $\dot\eta_{\mathfrak{s}}$ is the measure on $G_1(\mathfrak{s})$ of the library that computes integrals over $\mathcal{U}(\mathfrak{s})\backslash G_1(\mathfrak{s})$: for measurable $f \ge 0$ on $G_1(\mathfrak{s})$, $\int f(g)\,dg = \iint f(u\,y)\,d\eta_{\mathfrak{s}}(u)\,d\dot\eta_{\mathfrak{s}}(y)$.

(f) For $x \in G_1(\mathfrak{s})$, $\mathrm{size}_{\mathcal{Q}}(x) = \inf_{q\in\mathcal{Q}(\mathfrak{s})} \|q\,x\|_{\mathrm{op}} + 1$.

(g) Let I be a finite set and rep a homomorphism from $\mathrm{U}(1)$ to the unitary $I \times I$ matrices. A function $W\colon G_1(\mathfrak{s}) \to \mathbb{C}^I$ is a *Bessel function for* rep if it is bounded and strongly measurable, $W(u\,y) = e^{\mathrm{sd}(u) - \overline{\mathrm{sd}(u)}}\,W(y)$ for $u \in \mathcal{U}(\mathfrak{s})$, and $W(\mathrm{emb}_{\mathfrak{s}}(h)\,y) = \mathrm{rep}(h)\,W(y)$ for $h \in \mathrm{U}(1)$, for all $y \in G_1(\mathfrak{s})$.

The signature of $J_1(\mathfrak{s})$ is $(\varrho(\mathfrak{s}) + |\kappa(\mathfrak{s})|,\ \varrho(\mathfrak{s}) + 1)$. The function $\mathrm{sd}(\,\cdot\,)$ is additive on $\mathcal{Q}(\mathfrak{s})$ and vanishes on $\mathcal{H}(\mathfrak{s})$ (Input (I13)), so $u \mapsto e^{\mathrm{sd}(u) - \overline{\mathrm{sd}(u)}}$ is a unitary character of $\mathcal{U}(\mathfrak{s})$, the generic character of this note. Item (f) is a definition of the unit on the second moment (Filling), item (g) one of the library.

Definition 2.15 (frames)**.** Let $\mathfrak{s}$ be a shape and $\varepsilon\colon \mathrm{Ind}(\mathfrak{s}) \to \{\pm 1\}$ signs. The set $\mathrm{Frm}(\mathfrak{s}, \varepsilon)$ of the *frames of* $\mathfrak{s}$ *for* ε consists of the unitary $\mathrm{Ind}(\mathfrak{s}) \times \mathrm{Ind}(\mathfrak{s})$ matrices p with $\mathsf{p}^*\,J_1(\mathfrak{s})\,\mathsf{p} = \mathrm{diag}(\varepsilon)$. For such p, $g \mapsto \mathsf{p}^{-1} g\,\mathsf{p}$ is an isomorphism of topological groups from $G_1(\mathfrak{s})$ to $\mathrm{U}(\varepsilon)$ that preserves the Haar measures.

LEAN `Frame`

In the formal text a frame is a structure that the declarations below take as an instance argument, and the signs are determined by it; here a statement names its frame. The columns of p are an orthonormal basis of $\mathbb{C}^{\mathrm{Ind}(\mathfrak{s})}$ that is also orthogonal for $J_1(\mathfrak{s})$.

Definition 2.16 (the Bessel graph)**.** Let $\mathfrak{s}$ be a shape. Then $\mathsf{HU}(\mathfrak{s})$ is the set of the pairs $(g, h) \in G_1(\mathfrak{s}) \times \mathrm{U}(1)$, $\mathrm{U}(1)$ the unitary group of $\mathbb{C}^{\kappa(\mathfrak{s})}$, with $\mathrm{emb}_{\mathfrak{s}}(h)^{-1} g \in \mathcal{U}(\mathfrak{s})$. It is a subgroup, because $\mathrm{emb}_{\mathfrak{s}}(h)$ normalizes $\mathcal{U}(\mathfrak{s})$. An element is written $\sigma = (\sigma_1, \sigma_2)$.

LEAN `Shape.besselGraph`

The first projection maps $\mathsf{HU}(\mathfrak{s})$ isomorphically onto $\mathcal{H}(\mathfrak{s})\,\mathcal{U}(\mathfrak{s}) = \mathcal{Q}(\mathfrak{s})$; the graph records the factor in $\mathrm{U}(1)$, which the second coefficient of the integrand needs.

Definition 2.17 (the chart)**.** Let $\mathfrak{s}$ be a shape. Then $\mathrm{ch}_{\mathfrak{s}}$ is the homeomorphism from $\mathcal{U}(\mathfrak{s}) \times \mathrm{U}(1)$ to $\mathsf{HU}(\mathfrak{s})$ that sends (u, h) to $(\mathrm{emb}_{\mathfrak{s}}(h)\,u,\ h)$; its inverse sends (g, h) to $(\mathrm{emb}_{\mathfrak{s}}(h)^{-1} g,\ h)$.

LEAN `Shape.besselGraphChart`

Definition 2.18 (the measure of the Bessel graph)**.** Let $\mathfrak{s}$ be a shape. Then $\nu_{\mathfrak{s}}$ is the image under the chart $\mathrm{ch}_{\mathfrak{s}}$ of the product of the measure $\eta_{\mathfrak{s}}$ on $\mathcal{U}(\mathfrak{s})$ and the Haar measure of total mass one on $\mathrm{U}(1)$.

LEAN `Shape.besselGraphMeasure`

Definition 2.19 (the pair homomorphism)**.** Let $\mathfrak{s}$ be a shape, $\varepsilon\colon \mathrm{Ind}(\mathfrak{s}) \to \{\pm 1\}$ signs, $\mathsf{p} \in \mathrm{Frm}(\mathfrak{s}, \varepsilon)$ a frame and $x \in G_1(\mathfrak{s})$. For $\sigma = (\sigma_1, \sigma_2) \in \mathsf{HU}(\mathfrak{s})$,

$$\jmath_x(\sigma) = \left(\mathsf{p}^{-1} x^{-1} \sigma_1 \, x \, \mathsf{p},\ \sigma_2\right) \ \in\ \mathrm{U}(\varepsilon) \times \mathrm{U}(1).$$

The map $\jmath_x(\,\cdot\,)$ is a homomorphism of groups. It depends on $\mathfrak{s}$ and on the frame p, which the notation does not show (they are implicit in the formal text).

LEAN `besselPairHom`

Definition 2.20 (peak)**.** Let $\mathfrak{s}$ be a shape, $\varepsilon\colon \mathrm{Ind}(\mathfrak{s}) \to \{\pm 1\}$ signs, $\mathsf{p} \in \mathrm{Frm}(\mathfrak{s}, \varepsilon)$ a frame, I a finite set, $l \in \mathbb{Z}^{\mathrm{Ind}(\mathfrak{s})}$, rep a homomorphism from $\mathrm{U}(1)$ (of $\mathbb{C}^{\kappa(\mathfrak{s})}$) to the unitary $I \times I$ matrices, $W\colon G_1(\mathfrak{s}) \to \mathbb{C}^I$ a function, $x \in G_1(\mathfrak{s})$ and P a real number. We say that $\mathrm{Peak}_{l,\mathrm{rep}}(W; x, P)$ *holds* if W is a Bessel function for rep (Definition 2.14(g)) and

(P1) $W(y) = \int_{G_1(\mathfrak{s})} \deg(l)\, \psi_l(\mathsf{p}^{-1} g^{-1} \mathsf{p})\, W(y\,g)\, dg$ for every $y \in G_1(\mathfrak{s})$;
(P2) $W(x) \neq 0$;
(P3) $\|W(y)\| \le \int_{\mathsf{HU}(\mathfrak{s})} \left|\psi_l(\mathsf{p}^{-1} x^{-1} \sigma_1\, y\, \mathsf{p})\right| d\nu_{\mathfrak{s}}(\sigma)$ for every $y \in G_1(\mathfrak{s})$;
(P4) for every $y \in G_1(\mathfrak{s})$,

$$\int_{\mathsf{HU}(\mathfrak{s})} \left(\frac{2}{\|\mathsf{p}^{-1} x^{-1} \sigma_1\, y\, \mathsf{p}\|_{\mathrm{op}}}\right)^{8\, r(\mathfrak{s})^2 + 1} d\nu_{\mathfrak{s}}(\sigma) \ \le\ P \cdot \|W(x)\|.$$

The integrals in (P3) and (P4) are integrals of non-negative functions, with values in $[0, \infty]$, and $\|\cdot\|$ is the Euclidean norm of $\mathbb{C}^I$.

LEAN `HasPeakAt`

Condition (P1) says that W is fixed by the right convolution with the function $g \mapsto \deg(l)\, \psi_l(\mathsf{p}^{-1} g\, \mathsf{p})$, the kernel of the projection to the discrete series of l (Input (I12)). In the formal text the predicate has the five arguments l, rep, W, x, P, and $\mathfrak{s}$, ε, p, I are implicit.

Definition 2.21 (type of a Bessel system)**.** Let $\mathfrak{s}$ be a shape, $\varepsilon\colon \mathrm{Ind}(\mathfrak{s}) \to \{\pm 1\}$ signs, $\mathsf{p} \in \mathrm{Frm}(\mathfrak{s}, \varepsilon)$ a frame, b a real number, f a natural number, and $\mathcal{B}$ a Bessel system for $(\mathfrak{s}, b, f)$ in the sense of the unit on the second moment: among its components are a finite index set $I(\mathcal{B})$, a continuous function $W_{\mathcal{B}}\colon G_1(\mathfrak{s}) \to \mathbb{C}^{I(\mathcal{B})}$ called its full Bessel function, and a positive number $\deg \mathcal{B}$. Let $l \in \mathbb{Z}^{\mathrm{Ind}(\mathfrak{s})}$ and let w be a dominant weight of the unitary group of $\mathbb{C}^{\kappa(\mathfrak{s})}$, in the encoding of the library. We say that $\mathrm{BesType}(\mathcal{B}; l, w)$ *holds* if

(B1) $\deg \mathcal{B} = \prod_{i,i'\colon l_{i'} < l_i} \tfrac{1}{2}(l_i - l_{i'}) \big/ \prod_{n < r(\mathfrak{s})} n!$, the value of Weyl's dimension polynomial at l;

(B2) there are a matrix Y with rows indexed by the standard tableaux of the weight w for $\kappa(\mathfrak{s})$ and columns indexed by $I(\mathcal{B})$, and a matrix Y' with $YY' = 1$ and $Y'Y = 1$, such that $YW_{\mathcal{B}}(\mathrm{emb}_{\mathfrak{s}}(h)\, y) = \mathbb{S}_w(h)\, YW_{\mathcal{B}}(y)$ for all $h \in \mathrm{U}(1)$ and $y \in G_1(\mathfrak{s})$, where $\mathbb{S}_w(h)$ is the matrix of h in the Schur module of weight w in the basis of the tableaux, as the library defines it;

(B3) $\int_{\mathrm{U}(\varepsilon)} \deg(l)\, \overline{\psi_l(g)}\, W_{\mathcal{B}}(y\, \mathsf{p}\, g\, \mathsf{p}^{-1})_i \, dg = W_{\mathcal{B}}(y)_i$ for every $y \in G_1(\mathfrak{s})$ and $i \in I(\mathcal{B})$.

LEAN `Filling.IsBessel`

This predicate records that a Bessel system has the degree, the U(1)-type and the discrete series type that belong to l and w. The unit proves nothing about it; the units on the second moment and on the real place use it (§11). In (B2) the formal text writes the Schur matrix as the transpose of the Schur functor applied to the transpose of h; we have not unfolded the library's Schur functor further than this.

Adelic unitary groups

Theorem 6 uses the following objects of the background library, in its coordinates. Let E be a CM field with a CM type: for every infinite place v an embedding $E \to \mathbb{C}$ that induces v is chosen, and $|a|_v$ is the absolute value of the image of $a \in E$ under it. The number of infinite places is f_E. Let Ω be a finite set and ε signs on Ω.

Definition 2.22 (the adelic group and a space with a point)**.** (a) $G_\infty(E, \varepsilon)$ is the product of copies of $\mathrm{U}(\varepsilon)$, one for each infinite place v, with the product of the Haar measures; $G_{\mathrm{f}}(E, \Omega)$ is the unitary group of the identity matrix over the finite adeles of E; and $G(E, \varepsilon) = G_\infty(E, \varepsilon) \times G_{\mathrm{f}}(E, \Omega)$, with components $g = (g_\infty, g_{\mathrm{f}})$ and the product Haar measure of the library. For an integer $a_0 \neq 0$ of E, $K_\Omega(a_0)$ is the open compact subgroup of the $k \in G_{\mathrm{f}}(E, \Omega)$ with integral entries such that all entries of $k - 1$ are multiples of a_0 by integral finite adeles.

(b) A *hermitian space with a point over E for ε* is a pair $V = (J_V, x_V)$ of an $\Omega \times \Omega$ matrix J_V over E and an invertible adelic matrix x_V with $x_V^* J_V x_V$ equal to $\mathrm{diag}(\varepsilon)$ at the infinite places and to 1 at the finite places. Then $\Gamma(V) = x_V^{-1}\, \mathrm{U}(J_V)(E)\, x_V$ is a discrete subgroup of $G(E, \varepsilon)$, and ϖ_V is the measure of the library on $G(E, \varepsilon)$ that computes integrals over $\Gamma(V) \backslash G(E, \varepsilon)$. For $x \in G(E, \varepsilon)$, $V \cdot x = (J_V, x_V x)$.

(c) The *lattice* of V consists of the vectors $\eta \in E^\Omega$ for which $x_V^{-1}\eta$ is integral at all finite places; the *length* $\|\eta\|_V$ is the Euclidean norm of the vector of the components of $x_V^{-1}\eta$ at the infinite places, a vector of $\mathbb{C}^{f_E|\Omega|}$; and $\lambda_1(V)$ is the infimum of 1 and of the lengths of the non-zero lattice vectors.

(d) $\langle 1 \rangle$ is the positive line: the space with matrix (1) and point 1 on the index set $\{*\}$; $V \oplus \langle 1 \rangle$ has block-diagonal matrix and point, for the signs $\varepsilon \oplus 1$ on $\Omega \sqcup \{*\}$; and $g \mapsto g \oplus 1$ is the block-diagonal embedding of $G(E, \varepsilon)$ into $G(E, \varepsilon \oplus 1)$, of $\mathrm{U}(\varepsilon)$ into $\mathrm{U}(\varepsilon \oplus 1)$, and of $G_\infty(E, \varepsilon)$ into $G_\infty(E, \varepsilon \oplus 1)$.

(e) A function $F\colon G(E, \varepsilon) \to \mathbb{C}$ is *automorphic for V* if it is continuous, invariant under left translations by $\Gamma(V)$, and invariant under right translations by some open subgroup of $G_{\mathrm{f}}(E, \Omega)$. It *has level K*, for a subgroup K of $G_{\mathrm{f}}(E, \Omega)$, if $F(g \cdot (1, k)) = F(g)$ for all g and all $k \in K$.

The formal text requires of E, for the positive line to exist as a space with a point, that it contains a primitive root of unity of order 5040.

3 What is taken for granted

The proofs below use the following results of other units. All but (I14) are proved in the background library of the artifact (the unit numbers are those of the reading project's ledger); they are stated as the dossier of the unit reads them, and the statements (I7), (I9), (I10), (I11) and (I14) were compared here with their Lean text. A reference is given where the second reader of the project or the catalogue of the write-up has one; "reference to be confirmed" means that none is recorded. Facts of Mathlib (Sylvester's identity $\det(1+XY)=\det(1+YX)$, the spectral theorem for hermitian matrices, Fubini's and Tonelli's theorems, Markov's inequality) are used without number.

Inputs. In (I1) to (I4), X, Y, D are $\Omega\times\Omega$ matrices; in (I7) to (I10), $\varepsilon\colon\Omega\to\{\pm1\}$ are signs, $l\in\mathbb{Z}^{\Omega}$ and $g\in\mathrm{U}(\varepsilon)$.

(I1) *Hilbert–Schmidt norm* (unit 112, linear algebra; elementary). For matrices X, Y: $|\mathrm{tr}(XY)|\le\|X\|_{\mathrm{HS}}\|Y\|_{\mathrm{HS}}$; $\mathrm{tr}(X^*X)=\|X\|_{\mathrm{HS}}^2$; $\|uX\|_{\mathrm{HS}}=\|X\|_{\mathrm{HS}}$ if $u^*u=1$; $\|pX\|_{\mathrm{HS}}\le\|X\|_{\mathrm{HS}}$ for an orthogonal projection p; replacing entries of X by zero does not increase $\|X\|_{\mathrm{HS}}$.

(I2) *Inverses near* 1 (unit 112; elementary). If $\|X\|_{\mathrm{HS}}\le c\le\frac12$ then $1+X$ is invertible and $\|(1+X)^{-1}-1\|_{\mathrm{HS}}\le 2c$.

(I3) *The logarithm* (unit 112; an integrated form of Jacobi's formula, elementary). If $\|X\|_{\mathrm{HS}}\le\frac12$, then $s\mapsto\mathrm{tr}(D(1+sX)^{-1}X)$ is integrable on $[0,1]$ with integral $\mathrm{tr}(D\,\lg(X))$; $\det(1+X)=\exp\mathrm{tr}\,\lg(X)$; and $\|\lg(X)\|_{\mathrm{HS}}\le 2\|X\|_{\mathrm{HS}}$.

(I4) *Principal minors* (unit 112; elementary). (a) If $X_{ii'}=0$ or $Y_{i'i''}=0$ whenever $i,i''\in\Pi$ and $i'\notin\Pi$, then $\Delta_\Pi(XY)=\Delta_\Pi(X)\,\Delta_\Pi(Y)$. (b) If $o\colon\Omega\to\mathbb{Z}$ is injective and $X_{ii'}=0$ whenever $o_{i'}<o_i$, then $\det X=\prod_i X_{ii}$. (c) For invertible X, the library's product of principal minors of weight w is the number $\mathfrak{m}_w(X)$ of Definition 2.6.

(I5) *Lagrange weights* (unit 112; Lagrange interpolation, elementary). For integers $\tau_0>\tau_1>\cdots>\tau_{n_0-1}$ there are real numbers $c_n\ge1$ with $c_n^2=(-1)^n\prod_{n'\ne n}(\tau_n-\tau_{n'})$ such that for every real polynomial π of degree less than n_0 the coefficient of degree n_0-1 of π is $\sum_n(-1)^n\pi(\tau_n)/c_n^2$.

(I6) *Witt's theorem* (unit 120). Let J be an invertible hermitian $\Omega\times\Omega$ matrix and Y_1, Y_2 matrices with Ω rows, the same set of columns, and left inverses, such that $Y_1^*JY_1=Y_2^*JY_2$. Then $gY_1=Y_2$ for some g with $g^*Jg=J$. Reference: U. First, *Witt's extension theorem for quadratic spaces over semiperfect rings*, Remark 2.1 and Corollary 4.9; relation: the theorem itself, for a field with a non-trivial involution (second reader's verdict).

(I7) *Flensted-Jensen's formula and its invariance* (units 116 and 112). Suppose that l has gap H for ε for some real H. Call a matrix Y *accretive* if $Y+Y^*$ is positive definite; this holds if $\|Y-1\|_{\mathrm{HS}}<1$. (a) If a is an $\Omega\times\Omega$ matrix commuting with $\mathrm{diag}(\varepsilon)$ such that $a^{-1}g$ is accretive, then $\psi_l(g)=\mathrm{FJ}_l(a,a^{-1}g)$. (b) If u' is a matrix with $u'_{ii'}=0$ unless $l_i<l_{i'}$ and $\varepsilon_i=\varepsilon_{i'}$, and Y and $(1-u')Y$ are accretive, then $\mathrm{FJ}_l(1,((1-u')Y)^*)=\overline{\mathrm{FJ}_l(1,Y)}$. Reference for (a): M. Flensted-Jensen, *Discrete series for semisimple symmetric spaces*, Ann. of Math. 111 (1980); place and relation to be confirmed. For (b): reference to be confirmed.

(I8) *Elementary properties of the coefficient* (units 116, 100, 18). (a) $\|\mathfrak{m}_w\|^2_{\mathrm{K}(\varepsilon)} > 0$ for every integer vector w, and $\psi_l(\,\cdot\,)$ is continuous on $\mathrm{U}(\varepsilon)$. Suppose that l has gap H for ε for some H. Then: (b) $\psi_l(h) = \mathfrak{m}_{\Lambda(l)}(h)$ for $h \in \mathrm{K}(\varepsilon)$; (c) $\psi_{-l}(g) = \overline{\psi_l(g)}$ and $\Lambda(-l) = -\Lambda(l)$; (d) $\Upsilon(l)_i < \Upsilon(l)_{i'}$ if and only if $l_i < l_{i'}$.

(I9) *Decay* (unit 116). If l has gap H for ε and c is a natural number with $c \le H$, then $|\psi_l(g)|^2\,|\det \mathrm{bd}_\varepsilon(g)|^c \le 1$; if moreover $\Omega \neq \emptyset$, then $|\psi_l(g)| \le (2/\|g\|_{\mathrm{op}})^c$. In particular $|\psi_l(g)| \le 1$. The catalogue relates the first bound to Flensted-Jensen (1980) and to B. Sun, *Matrix coefficients of cohomologically induced representations*, Compositio Math. 143 (2007), and does not identify it with a published statement: reference to be confirmed.

(I10) *The coefficient near* 1 (unit 116; elementary from a Lipschitz bound of the library). Let $\Omega \neq \emptyset$, let l have large gap A_0 for ε for some natural A_0, and $|l_i| \le L$ for all i. (a) If $\beta_0 \ge |\Omega| + L + 1$ and $\|g - 1\|_{\mathrm{op}} \le (200\beta_0^5)^{-1}$, then $|\psi_l(g)| \ge \frac{1}{2}$. (b) If ν is a right-invariant measure on a group S with measurable translations, $j\colon S \to \mathrm{U}(\varepsilon)$ a homomorphism with $|\psi_l(j(\,\cdot\,))|^2$ integrable, and $R \ge 1$, then $\nu\{\|j(\sigma)\|_{\mathrm{op}} \le R\} \le (2000(|\Omega| + L + 1)^6 R^2)^{2|\Omega|^2} \int 4|\psi_l(j(\sigma))|^2 d\nu$.

(I11) *Balls and decay on a group over* $\mathrm{U}(\varepsilon)$ (unit 115, Haar measure; the catalogue cites Hua, Macdonald and Gross for the volume growth and has no place: reference to be confirmed). Let $\Omega \neq \emptyset$, let S be a group with a topology, its Borel σ-algebra and measurable translations, ν a right-invariant measure on S, $j\colon S \to \mathrm{U}(\varepsilon)$ a homomorphism and $y \in \mathrm{U}(\varepsilon)$. (a) For real $D, R \ge 1$ and $K \ge 3DR^2$: $\nu\{\|j(\sigma)y\|_{\mathrm{op}} \le R\} \le K^{2|\Omega|^2}\,\nu\{\|j(\sigma) - 1\|_{\mathrm{op}} \le D^{-1}\}$. (b) If j is proper and ν is finite on compact sets, then $\nu\{\|j(\sigma) - 1\|_{\mathrm{op}} \le R\} < \infty$ for every R, and for every natural $c > 8|\Omega|^2$ the function $\sigma \mapsto (2/\|j(\sigma)y\|_{\mathrm{op}})^c$ is ν-integrable with $\int (2/\|j(\sigma)y\|_{\mathrm{op}})^c\, d\nu \le 16^c\,\nu\{\|j(\sigma) - 1\|_{\mathrm{op}} \le 1\}$. (c) If ν is positive on non-empty open sets and $f\colon S \to \mathbb{R}$ is continuous with $f(1) < R$, then $\nu\{f \le R\} \neq 0$. (d) $\|g\|_{\mathrm{op}} \ge 1$ and $\|g^{-1}\|_{\mathrm{op}} = \|g\|_{\mathrm{op}}$ for $g \in \mathrm{U}(\varepsilon)$, and $g \mapsto g \oplus 1$ is a proper map from $\mathrm{U}(\varepsilon)$ to $\mathrm{U}(\varepsilon \oplus 1)$ (unit 43).

(I12) *Schur orthogonality* (units 112 and 117). Let $\mathfrak{s}$ be a shape, p a frame of $\mathfrak{s}$ for signs ε, and let $l \in \mathbb{Z}^{\mathrm{Ind}(\mathfrak{s})}$ have large gap A for ε, for some natural A. Then $k(g) = \deg(l)\,\psi_l(\mathsf{p}^{-1} g\, \mathsf{p})$ is continuous and integrable on $G_1(\mathfrak{s})$, $\deg(l) > 0$, and $\int k(g^{-1})\,k(y_1\, g\, y_2)\, dg = k(y_1)\,k(y_2)/k(1)$ for all $y_1, y_2 \in G_1(\mathfrak{s})$. Reference: Harish-Chandra, *Discrete series for semisimple Lie groups II*, Acta Math. 116 (1966), pp. 87–88, for the orthogonality of square-integrable coefficients with the formal degree; relation unresolved (second reader's verdict): the cited pages do not give the integrability for the explicit gap nor the normalisation of the measure, which the library proves.

(I13) *The Bessel subgroup* (units 1, 118, 119). Let $\mathfrak{s}$ be a shape. (a) $\mathrm{sd}(q\,q') = \mathrm{sd}(q) + \mathrm{sd}(q')$ and $\mathrm{sd}(q^{-1}) = -\mathrm{sd}(q)$ for $q, q' \in \mathcal{Q}(\mathfrak{s})$, and $\mathrm{sd}(q) = 0$ for $q \in \mathcal{H}(\mathfrak{s})$. (b) Let $C_{\mathfrak{s}}$ be the *chain matrix*: the $\mathrm{Ind}(\mathfrak{s}) \times \mathrm{Ind}(\mathfrak{s})$ matrix with $C_{\mathfrak{s}} e_a = f_{a+1}$ for $a < \varrho(\mathfrak{s})$, $C_{\mathfrak{s}} x_i = 0$, $C_{\mathfrak{s}} z = -e^*_{\varrho(\mathfrak{s})-1}$ (if $\varrho(\mathfrak{s}) \ge 1$), $C_{\mathfrak{s}} e^*_{a+1} = e^*_a$ and $C_{\mathfrak{s}} e^*_0 = 0$. Then $\mathrm{tr}(C_{\mathfrak{s}}\, g) = \mathrm{sd}(g) + \overline{\mathrm{sd}(g^{-1})}$ for $g \in G_1(\mathfrak{s})$. (c) Conjugation by an element of $\mathcal{Q}(\mathfrak{s})$ and right translation by an element of $\mathcal{U}(\mathfrak{s})$ preserve $\eta_{\mathfrak{s}}$. (d) The formula of Definition 2.14(e) for $\dot{\eta}_{\mathfrak{s}}$; it is an instance of Weil's formula for the integral over a quotient, which the library proves for a measure with a cutoff function. Reference: Kallenberg, Probab. Theory Related Fields 139 (2007), as cited by Gentner and Last; relation unresolved (second reader's verdict:

the library's statement is more general than the cited one and follows from Tonelli's theorem and the two invariances).

(I14) *A bound for the size* (unit 132 of the argument, the complex chart at the real place). Let $\mathfrak{s}$ be a shape, $x \in G_1(\mathfrak{s})$, $M_0 \geq 1$ real, and $V_0, V_1, V_2, \ldots$ vectors of $\mathbb{C}^{\mathrm{Ind}(\mathfrak{s})}$ with $V_a = x^{-1} f_a$ for $a \leq \varrho(\mathfrak{s})$, with Euclidean norms $\|V_a\| \leq M_0$ for $a \leq 2\varrho(\mathfrak{s})$, and with $V_a^* J_1(\mathfrak{s}) V_{a'} = -1$ if $a + a' = 2\varrho(\mathfrak{s})$ and $= 0$ if $a + a' < 2\varrho(\mathfrak{s})$. Then $\mathrm{size}_{\mathcal{Q}}(x) \leq 8\,(\varrho(\mathfrak{s}) + 1)\, M_0$.

The adelic inputs of Theorem 6 are listed in §10, where they are used.

4 Determinants of compressions, to second order

In this section and the next, norms without mention are Hilbert–Schmidt norms. The first result compares $\log\det(1 + pX)$, for an orthogonal projection p, with $\mathrm{tr}(q\,\lg(X))$ for a second projection q close to p. To first order the two agree when $q = p$; the second-order term is a quarter of the square of the commutator, and the error is cubic: one factor $\|X\|_{\mathrm{HS}}$ and two factors of the size of commutators.

Proposition 4.1 (compressed determinants)**.** *Let Ω be a finite set, let p, q, X, Y be $\Omega \times \Omega$ matrices and δ, c real numbers. Suppose that p and q are orthogonal projections ($p = p^2 = p^*$ and $q = q^2 = q^*$), that $\delta \leq \frac{1}{2}$, and that*

$$\|X\|_{\mathrm{HS}} \leq \delta, \qquad \mathrm{tr}([p,q]\,Y) = 0, \qquad \|[p,X] - Y\|_{\mathrm{HS}} \leq \delta\, c,$$
$$\|Y\|_{\mathrm{HS}} \leq c, \qquad \|q - p\|_{\mathrm{HS}} \leq c.$$

Then there is a complex number η with

$$\det(1 + p\,X) = \exp\Big(\mathrm{tr}(q\,\lg(X)) - \tfrac{1}{4}\,\mathrm{tr}(Y^2) + \eta\Big) \qquad \text{and} \qquad |\eta| \leq 100\,\delta\, c^2.$$

Proof. The numbers δ and c are non-negative, being bounds of norms. Put $A = [p, X]$ and $D = q - p$; then $\|A\|_{\mathrm{HS}} \leq \|Y\|_{\mathrm{HS}} + \|A - Y\|_{\mathrm{HS}} \leq c + \delta c \leq \frac{3}{2} c$. Since $\|pX\|_{\mathrm{HS}} \leq \delta \leq \frac{1}{2}$ by (I1), Input (I3) gives $\det(1 + pX) = \exp \mathrm{tr}\,\lg(pX)$, and it writes $\mathrm{tr}\,\lg(pX)$ and $\mathrm{tr}(q\,\lg(X))$ as integrals over $s \in [0,1]$. With $\frac{1}{4} = \int_0^1 \frac{s}{2}\, ds$ the claimed identity holds for

$$\eta = \int_0^1 \mathcal{J}(s)\, ds, \qquad \mathcal{J}(s) = \mathrm{tr}(V p X) - \mathrm{tr}(q\,W X) + \tfrac{s}{2}\,\mathrm{tr}(Y^2),$$

where $W = (1 + sX)^{-1}$ and $V = (1 + s\,pX)^{-1}$, and it suffices to show $|\mathcal{J}(s)| \leq 100\,\delta c^2$ for each $s \in [0,1]$. Fix s. By (I2), $\|W - 1\|_{\mathrm{HS}} \leq 2\delta$ and $\|V - 1\|_{\mathrm{HS}} \leq 2\delta$, so $\|W\|_{\mathrm{op}} \leq 2$ and $\|V\|_{\mathrm{op}} \leq 2$. We split $\mathcal{J}(s) = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3$ with

$$\mathcal{J}_1 = \mathrm{tr}(V p X) - \mathrm{tr}(p\,W X) + \tfrac{s}{2}\,\mathrm{tr}(A^2), \qquad \mathcal{J}_2 = -\,\mathrm{tr}(D\,W X),$$
$$\mathcal{J}_3 = -\tfrac{s}{2}\,\mathrm{tr}\big((A - Y)(A + Y)\big),$$

using $\mathrm{tr}(A^2 - Y^2) = \mathrm{tr}((A - Y)(A + Y))$. At once $|\mathcal{J}_3| \leq \frac{1}{2}\,\delta c\,(\frac{3}{2} c + c) = \frac{5}{4} \delta c^2$.

The term $\mathcal{J}_1$: one projection. We use

$$pAp = 0 = (1 - p)A(1 - p), \qquad pA = pX(1 - p),$$
$$Ap = -(1 - p)Xp, \qquad \mathrm{tr}(pA^2) = \tfrac{1}{2}\,\mathrm{tr}(A^2),$$

the last because $\operatorname{tr}(pA^2) = \operatorname{tr}((1-p)A^2)$. From $p(1+sX) - (1+s\,pX)p = s\,pA$ we get, multiplying by V on the left and by W on the right, $Vp - pW = s\,VpA\,W$, so

$$\operatorname{tr}(VpX) - \operatorname{tr}(p\,WX) = s\,\operatorname{tr}\big(pA\cdot WXV\big) = s\,\operatorname{tr}\big(pA\,(1-p)\,WXVp\big).$$

Now $[p,W] = -s\,WAW$, so $(1-p)W = W(1-p) + s\,WAW$; and $Vp = pVp$, because $(1-p)V = 1-p$ by $V = 1 - s\,pXV$. Therefore

$$(1-p)\,WXVp = W(1-p)Xp\,Vp + s\,WAWXVp = -\,WA\,Vp + s\,WAWXVp,$$

and, writing $WAV - A = (W-1)AV + A(V-1)$,

$$\mathcal{J}_1 = -s\,\operatorname{tr}\big(pA\,((W-1)AV + A(V-1))\,p\big) + s^2\,\operatorname{tr}\big(pA\cdot WAW\cdot XVp\big).$$

By (I1) and the bounds for W and V, $|\mathcal{J}_1| \le \|A\|_{\mathrm{HS}}^2(2\delta\cdot 2 + 2\delta) + \|A\|_{\mathrm{HS}}\cdot 4\|A\|_{\mathrm{HS}}\cdot\delta\cdot 2 = 14\,\delta\|A\|_{\mathrm{HS}}^2 \le \frac{63}{2}\,\delta c^2$.

The term $\mathcal{J}_2$: the change of projection. A computation with $p^2 = p$, $q^2 = q$ shows that D^2 commutes with p and that $pDp = -pD^2$, $(1-p)D(1-p) = (1-p)D^2$. So $D = D_0 + D_1$ with

$$D_0 = (1-2p)\,D^2, \qquad D_1 = pq(1-p) + (1-p)qp,$$
$$\text{while} \quad [p,q] = pq(1-p) - (1-p)qp.$$

For every matrix Z, since $(1-p)Zp = -(1-p)[p,Z]p$ and $pZ(1-p) = p\,[p,Z](1-p)$, the cyclicity of the trace gives $\operatorname{tr}(D_1 Z) = -\operatorname{tr}\big([p,q]\,[p,Z]\big)$. For $Z = WX$ we have $[p,WX] = WA - s\,WAWX$, and $\operatorname{tr}([p,q]\,Y) = 0$ by hypothesis, so

$$\mathcal{J}_2 = -\operatorname{tr}(D_0\,WX) + \operatorname{tr}\Big([p,q]\,\big((A-Y) + (W-1)A - s\,WAWX\big)\Big).$$

Here $\|D_0\|_{\mathrm{HS}} = \|D^2\|_{\mathrm{HS}} \le c^2$, because $1-2p$ is unitary; $\|WX\|_{\mathrm{HS}} \le 2\delta$; $\|[p,q]\|_{\mathrm{HS}} = \|D_1\|_{\mathrm{HS}} \le \|D\|_{\mathrm{HS}} \le c$; and the second factor has norm at most $\delta c + 2\delta\|A\|_{\mathrm{HS}} + 4\|A\|_{\mathrm{HS}}\delta \le 10\,\delta c$. Hence $|\mathcal{J}_2| \le 2\,\delta c^2 + 10\,\delta c^2$.

Together $|\mathcal{J}(s)| \le (\frac{63}{2} + 12 + \frac{5}{4})\,\delta c^2 \le 100\,\delta c^2$. □

LEAN `exists_det_compress_eq_exp_logOneAdd`

The hypothesis $\operatorname{tr}([p,q]Y) = 0$ removes the only term of second order that the change from p to q could produce: the pairing of the off-diagonal part of $q-p$ with the off-diagonal part of X.

The second result applies this to Flensted-Jensen's formula. For $g \in \mathrm{U}(\varepsilon)$ near 1 it finds a matrix y that may replace g in the formula and whose principal minors, conjugated by any element of the maximal compact subgroup, are Gaussian in $g-1$ with the *same* sign for compact and non-compact pairs. For g itself the signs differ, because $X = g-1$ is, to first order, anti-hermitian on the compact pairs and hermitian on the non-compact ones; the passage from g to y removes the triangular parts of the compact block of X, which costs nothing in the formula and flips the sign.

Proposition 4.2 (principal minors for Flensted-Jensen's formula)**.** *Let Ω be a finite set, $\varepsilon\colon \Omega \to \{\pm 1\}$ signs, H a real number, $l \in \mathbb{Z}^{\Omega}$ a vector that has gap H for ε, and $g \in \mathrm{U}(\varepsilon)$. Put $X = g - 1$ and suppose that $\|X\|_{\mathrm{HS}} < \frac{1}{1000}$. Then there is an $\Omega \times \Omega$*

matrix y with $\psi_l(g) = \mathrm{FJ}_l(1, y)$ such that for every subset $\Pi \subseteq \Omega$ with $l_i < l_{i'}$ for all $i \in \Pi$ and $i' \notin \Pi$, and for every $h \in \mathrm{K}(\varepsilon)$, there is a complex number η with

$$\Delta_\Pi(h^*\, y\, h) = \exp\Big(\mathrm{tr}(p_\Pi\ \lg(X)) + \tfrac{1}{4}\|[p_\Pi, X]\|_{\mathrm{HS}}^2 + \eta\Big),$$
$$|\eta| \le 200000\, \|X\|_{\mathrm{HS}} \left(\|[p_\Pi, X]\|_{\mathrm{HS}}^2 + \|[p_\Pi, h]\|_{\mathrm{HS}}^2\right).$$

Proof. For an $\Omega \times \Omega$ matrix Z let $Z^\uparrow$ be the matrix that keeps the entries $Z_{ii'}$ with $\varepsilon_i = \varepsilon_{i'}$ and $l_i < l_{i'}$ and replaces the others by zero, and $Z^\downarrow$ the same with $l_i > l_{i'}$. Put $\delta_0 = \|X\|_{\mathrm{HS}}$ and

$$y = (1 - X^\uparrow)\, g\, (1 - X^\downarrow).$$

With $X_1 = (1 - X^\uparrow)g - 1 = X - X^\uparrow - X^\uparrow X$ we have $y - 1 = X_1 - X^\downarrow - X_1 X^\downarrow$, and by (I1) $\|X^\uparrow\|_{\mathrm{HS}}, \|X^\downarrow\|_{\mathrm{HS}} \le \delta_0$, so

$$\|X_1\|_{\mathrm{HS}} \le 2\delta_0 + \delta_0^2, \qquad \|y - 1\|_{\mathrm{HS}} \le 3\delta_0 + 3\delta_0^2 + \delta_0^3 \le 4\,\delta_0 < \tfrac{1}{2}. \tag{2}$$

The formula. The matrices g, $(1 - X^\uparrow)g$ and y are within Hilbert–Schmidt distance less than 1 of 1, hence accretive, as are their adjoints. By (I7)(a) with $a = 1$, $\psi_l(g) = \mathrm{FJ}_l(1, g)$. By (I7)(b) for $u' = X^\uparrow$ and $Y = g$, $\mathrm{FJ}_l(1, ((1 - X^\uparrow)g)^*) = \overline{\mathrm{FJ}_l(1, g)}$. The matrix $u' = (X^\downarrow)^*$ also has its non-zero entries at compact pairs with $l_i < l_{i'}$, and $(1 - u')\,\overline{((1 - X^\uparrow)g)^*} = y^*$; so (I7)(b) for this u' and $Y = ((1 - X^\uparrow)g)^*$ gives $\mathrm{FJ}_l(1, y) = \overline{\mathrm{FJ}_l(1, ((1 - X^\uparrow)g)^*)} = \mathrm{FJ}_l(1, g)$.

Upper sets. Let $\Pi' \subseteq \Omega$ be such that $l_{i'} < l_i$ for all $i \in \Pi'$, $i' \notin \Pi'$. Then

$$\Delta_{\Pi'}(y) = \Delta_{\Pi'}(g). \tag{3}$$

Indeed $(X^\uparrow)_{ii'} = 0$ for $i \in \Pi'$, $i' \notin \Pi'$, and $(X^\downarrow)_{i'i} = 0$ for such indices, so by (I4)(a) the minor is multiplicative on the three factors of y; and $\Delta_{\Pi'}(1 - X^\uparrow) = 1 = \Delta_{\Pi'}(1 - X^\downarrow)$ by (I4)(b), these being determinants of matrices that are triangular for the injective function l, or $-l$, with diagonal entries 1. In particular $\det y = \det g$.

Now fix Π and h as in the statement, and write $p = p_\Pi$, $A = [p, X]$, $\alpha_0 = \|A\|_{\mathrm{HS}}$, $\kappa_0 = \|[p, h]\|_{\mathrm{HS}}$. The map $Z \mapsto [p, Z]$ multiplies the entry at (i, i') by 1, -1 or 0; so it commutes with the passage to $Z^\uparrow$, $Z^\downarrow$ and $\mathrm{bd}_\varepsilon(Z)$, and it kills diagonal entries. Let $A_c = \mathrm{bd}_\varepsilon(A) = A^\uparrow + A^\downarrow$ be the compact part of A and $A_n = A - A_c$ its non-compact part; they have norms at most α_0, and $\mathrm{tr}(A_c A_n) = 0$.

The commutator of y. From the formulas for $y - 1$ and X_1,

$$[p, y-1] = A_n - R, \qquad R = G + X_1 A^\downarrow + (A - A^\uparrow - G)\, X^\downarrow, \qquad G = A^\uparrow X + X^\uparrow A, \tag{4}$$

and $\|R\|_{\mathrm{HS}} \le 2\delta_0\alpha_0 + (2\delta_0 + \delta_0^2)\alpha_0 + (2\alpha_0 + 2\delta_0\alpha_0)\delta_0 \le 7\,\delta_0\alpha_0$.

Three expansions. Let $p' = 1 - p = p_{\Omega\setminus\Pi}$ and $q = h\,p\,h^*$, an orthogonal projection with $\|p - q\|_{\mathrm{HS}} = \|[p, h]h^*\|_{\mathrm{HS}} = \kappa_0$. We apply Proposition 4.1 three times; the six slots are (p, q, X, Y, δ, c).

(i) With $(p', p', X, -A, \delta_0, \alpha_0)$: here $[p', X] = -A$. So

$$\det(1 + p'X) = \exp(\mathrm{tr}(p'\lg(X)) - \tfrac{1}{4}\,\mathrm{tr}(A^2) + \eta_1) \qquad \text{with} \quad |\eta_1| \le 100\,\delta_0\alpha_0^2.$$

(ii) With $(p', p', y - 1, -A_n, 4\delta_0, 4\alpha_0)$: by (4), $[p', y - 1] + A_n = R$ has norm at most $16\,\delta_0\alpha_0$. So $\det(1 + p'(y - 1)) = \exp(\mathrm{tr}(p'\lg(y - 1)) - \tfrac{1}{4}\,\mathrm{tr}(A_n^2) + \eta_2)$ with $|\eta_2| \le 6400\,\delta_0\alpha_0^2$.

(iii) With $(q, p, y-1, A_n, 4\delta_0, 4(\alpha_0+\kappa_0))$: the matrices h and p commute with $\mathrm{diag}(\varepsilon)$, so $[q,p]$ is block diagonal, and its trace against A_n, which is supported on the non-compact pairs, vanishes; and $[q, y-1] - A_n = [q-p, y-1] - R$ has norm at most $2\kappa_0 \cdot 4\delta_0 + 7\delta_0\alpha_0 \le 16\,\delta_0(\alpha_0+\kappa_0)$. So $\det(1+q(y-1)) = \exp(\mathrm{tr}(p\,\lg(y-1)) - \frac{1}{4}\,\mathrm{tr}(A_n^2) + \eta_3)$ with $|\eta_3| \le 6400\,\delta_0(\alpha_0+\kappa_0)^2$.

The left sides are minors: $\det(1+p'X) = \Delta_{\Omega\setminus\Pi}(g)$, $\det(1+p'(y-1)) = \Delta_{\Omega\setminus\Pi}(y)$, and by Sylvester's identity $\det(1+q(y-1)) = \det(1+p\,h^*(y-1)h) = \Delta_\Pi(h^*yh)$. The complement of Π is an upper set, so the first two are equal by (3); and $\exp\mathrm{tr}\,\lg(y-1) = \det y = \det g = \exp\mathrm{tr}\,\lg(X)$ by (I3). Multiplying (iii) by (ii), using $p+p'=1$, and dividing by (i), which is not zero, we find

$$\Delta_\Pi(h^*yh) = \exp\Big(\mathrm{tr}(p\,\lg(X)) + \tfrac{1}{4}\,\mathrm{tr}(A^2) - \tfrac{1}{2}\,\mathrm{tr}(A_n^2) + \eta_2 + \eta_3 - \eta_1\Big). \tag{5}$$

The sign. Put $V = g^{-1} - 1$ and $E = VA + AV + VAV$. The relation $g^* = \mathrm{diag}(\varepsilon)g^{-1}\,\mathrm{diag}(\varepsilon)$ gives $X^* = \mathrm{diag}(\varepsilon)V\mathrm{diag}(\varepsilon)$, and

$$[p, V] = -g^{-1}A\,g^{-1} = -(A+E), \qquad A^* = -[p, X^*] = \mathrm{diag}(\varepsilon)(A+E)\,\mathrm{diag}(\varepsilon).$$

Conjugation by $\mathrm{diag}(\varepsilon)$ fixes A_c and changes the sign of A_n, so

$$\|A\|_{\mathrm{HS}}^2 = \mathrm{tr}(A^*A) = \mathrm{tr}\big((A+E)(A_c - A_n)\big) = \mathrm{tr}(A_c^2) - \mathrm{tr}(A_n^2) + \mathrm{tr}\big(E(A_c - A_n)\big).$$

With $\mathrm{tr}(A^2) = \mathrm{tr}(A_c^2) + \mathrm{tr}(A_n^2)$ this turns the exponent of (5) into $\mathrm{tr}(p\,\lg(X)) + \frac{1}{4}\|A\|_{\mathrm{HS}}^2 + \eta$ for

$$\eta = \eta_2 + \eta_3 - \eta_1 + \tfrac{1}{4}\,\mathrm{tr}\big(E\,(A_n - A_c)\big).$$

By (I2), $\|V\|_{\mathrm{HS}} \le 2\delta_0$, so $\|E\|_{\mathrm{HS}} \le (4\delta_0 + 4\delta_0^2)\alpha_0 \le 5\,\delta_0\alpha_0$, and $\|A_n - A_c\|_{\mathrm{HS}} = \alpha_0$. Therefore

$$|\eta| \le \delta_0\big(6400\,\alpha_0^2 + 6400\,(\alpha_0+\kappa_0)^2 + 100\,\alpha_0^2 + 2\,\alpha_0^2\big) \le 19302\,\delta_0\,(\alpha_0^2 + \kappa_0^2),$$

which is less than the bound claimed. □

LEAN `exists_principalMinor_eq_exp`

5 The Gaussian shape: proof of Theorem 1

For an integer vector w on Ω write $|w|_1 = \sum_i |w_i|$; then $-|w|_1 \le w_i \le |w|_1$ for all i, so that Definition 2.6 applies with $c_1 = -|w|_1$ and $c_2 = |w|_1$.

Lemma 5.1 (sums over the level sets)**.** *Let Ω be a finite set, w an integer vector on Ω and X an $\Omega\times\Omega$ matrix. Then, the sums being over the integers τ with $-|w|_1 < \tau \le |w|_1$,*

$$\sum_\tau \|[p_{\{w\ge\tau\}}, X]\|_{\mathrm{HS}}^2 = \mathrm{gw}_w(X) \qquad \textit{and} \qquad \sum_\tau \mathrm{tr}(p_{\{w\ge\tau\}}\,X) = \sum_{i\in\Omega}(w_i + |w|_1)\,X_{ii}.$$

Proof. The entry of $[p_{\{w\ge\tau\}}, X]$ at (i,i') is $X_{ii'}$ times the difference of the indicators of $w_i \ge \tau$ and $w_{i'} \ge \tau$. This difference is ± 1 for the integers τ with $\min(w_i, w_{i'}) < \tau \le \max(w_i, w_{i'})$, which are $|w_i - w_{i'}|$ in number and lie in the range of summation, and it is 0 for the others. This gives the first identity. For the second, the diagonal entry of $p_{\{w\ge\tau\}}X$ at i is X_{ii} for the $w_i + |w|_1$ values $\tau \le w_i$ of the range, and 0 for the others. □

LEAN `sum_norm_lie_coordProj_sq`

Lemma 5.2 (the products of minors decay on the compact group)**.** *Let Ω be a finite set, $\varepsilon\colon \Omega \to \{\pm 1\}$ signs, w an integer vector on Ω and $h \in \mathrm{K}(\varepsilon)$. Then $|\mathfrak{m}_w(h)|^2 \le \exp(-\frac{1}{2}\,\mathrm{gw}_w(h))$.*

Proof. Since $|\det h| = 1$, Definition 2.6 and Lemma 5.1 reduce the claim to one factor: for a subset $\Pi \subseteq \Omega$ and $p = p_\Pi$ we show $|\Delta_\Pi(h)|^2 \le \exp(-\frac{1}{2}\|[p,h]\|_{\mathrm{HS}}^2)$. By Sylvester's identity $\Delta_\Pi(h) = \det(1 - p(1-h)) = \det Z$ with $Z = (1-p) + p\,h\,p$. A positive semidefinite matrix has determinant at most the exponential of its trace minus its size, by the inequality $a \le e^{a-1}$ for its eigenvalues a; for Z^*Z this reads

$$|\det Z|^2 \le \exp\bigl(\mathrm{tr}(Z^*Z) - |\Omega|\bigr) = \exp\bigl(\|php\|_{\mathrm{HS}}^2 - \mathrm{tr}\,p\bigr).$$

As h is unitary, $\|ph\|_{\mathrm{HS}}^2 = \mathrm{tr}\,p = \|hp\|_{\mathrm{HS}}^2$, hence $\|[p,h]\|_{\mathrm{HS}}^2 = \|ph(1-p)\|_{\mathrm{HS}}^2 + \|(1-p)hp\|_{\mathrm{HS}}^2 = 2\,(\mathrm{tr}\,p - \|php\|_{\mathrm{HS}}^2)$, and the exponent is $-\frac{1}{2}\|[p,h]\|_{\mathrm{HS}}^2$. □

LEAN `norm_weightMinors_sq_le_exp`

Proof of Theorem 1. Put $X = g - 1$, so $\|X\|_{\mathrm{HS}} \le \delta \le 10^{-12}$ because $T \ge 1$, and abbreviate $\Lambda = \Lambda(l)$, $\Upsilon = \Upsilon(l)$, $c_\Lambda = \|\mathfrak{m}_{\Lambda(l)}\|_{\mathrm{K}(\varepsilon)}^2$ and

$$E = \mathrm{tr}\bigl(\mathrm{diag}(\Lambda)\,\lg(X)\bigr), \qquad q_0 = \tfrac{1}{4}\,\mathrm{gw}_{-\Upsilon}(X) \;\ge 0, \qquad \kappa_{\mathrm{G}} = e^{-q_0} \in (0,1].$$

The vector l has a gap, so Proposition 4.2 gives a matrix y with $\psi_l(g) = \mathrm{FJ}_l(1,y)$. By Definition 2.7, with the factor $\mathfrak{m}_\Lambda(h)\overline{\mathfrak{m}_\Lambda(h)}$ in the numerator,

$$\psi_l(g) = \frac{1}{c_\Lambda}\int_{\mathrm{K}(\varepsilon)} \frac{|\mathfrak{m}_\Lambda(h)|^2}{\mathfrak{m}_{-\Upsilon}(h^*y\,h)}\,dh. \tag{6}$$

The denominator. Fix $h \in \mathrm{K}(\varepsilon)$. For an integer τ the set $\{-\Upsilon \ge \tau\}$ satisfies the condition of Proposition 4.2: if $-\Upsilon_i \ge \tau > -\Upsilon_{i'}$ then $\Upsilon_i < \Upsilon_{i'}$, so $l_i < l_{i'}$ by (I8)(d). Write $p_\tau = p_{\{-\Upsilon \ge \tau\}}$. The proposition gives numbers η_τ with

$$\Delta_{\{-\Upsilon\ge\tau\}}(h^*yh) = \exp\Bigl(\mathrm{tr}(p_\tau \lg(X)) + \tfrac{1}{4}\|[p_\tau, X]\|_{\mathrm{HS}}^2 + \eta_\tau\Bigr),$$

$$|\eta_\tau| \le 2\cdot 10^5\,\delta\,(\|[p_\tau,X]\|_{\mathrm{HS}}^2 + \|[p_\tau,h]\|_{\mathrm{HS}}^2),$$

and, for the set Ω itself, where the commutators vanish, $\det(h^*yh) = \exp\mathrm{tr}\,\lg(X)$. We insert this into Definition 2.6 for the weight $-\Upsilon$ and sum over τ by Lemma 5.1: the traces add up to $-\mathrm{tr}(\mathrm{diag}(\Upsilon)\lg(X)) + |\Upsilon|_1\,\mathrm{tr}\,\lg(X)$, the squares to $4q_0$. Hence

$$\mathfrak{m}_{-\Upsilon}(h^*yh) = \exp(q_0 - E - r(h)), \qquad r(h) = \mathrm{tr}\bigl(\mathrm{diag}(\Upsilon - \Lambda)\lg(X)\bigr) - \sum_\tau \eta_\tau. \tag{7}$$

To bound $r(h)$ we use (1), the bound $|\lg(X)_{ii}| \le \|\lg(X)\|_{\mathrm{HS}} \le 2\delta$ of (I3), Lemma 5.1 for X and for h, and, again by (1), $\mathrm{gw}_{-\Upsilon}(h) \le \mathrm{gw}_\Lambda(h) + 2|\Omega|\|h\|_{\mathrm{HS}}^2 = \mathrm{gw}_\Lambda(h) + 2|\Omega|^2$:

$$|r(h)| \le 2|\Omega|^2\delta + 2\cdot 10^5\,\delta\,(4q_0 + \mathrm{gw}_\Lambda(h) + 2|\Omega|^2) \le 10^6\,\delta\,(G_0(h) + |\Omega|^2), \tag{8}$$

where $G_0(h) = q_0 + \frac{1}{2}\,\mathrm{gw}_\Lambda(h)$.

The comparison. By (6) and (7), and since $\int|\mathfrak{m}_\Lambda(h)|^2dh = c_\Lambda$ and dh has mass one,

$$\frac{\psi_l(g)}{e^E} - \kappa_{\mathrm{G}} = \frac{1}{c_\Lambda}\int_{\mathrm{K}(\varepsilon)} |\mathfrak{m}_\Lambda(h)|^2\,e^{-q_0}\bigl(e^{r(h)} - 1\bigr)\,dh,$$

$$\frac{\kappa_{\mathrm{G}}}{10} + e^{-100T} = \frac{1}{c_\Lambda}\int_{\mathrm{K}(\varepsilon)} \Bigl(\frac{\kappa_{\mathrm{G}}}{10}|\mathfrak{m}_\Lambda(h)|^2 + e^{-100T}c_\Lambda\Bigr)dh.$$

So it suffices to show, for every $h \in \mathrm{K}(\varepsilon)$,

$$|\mathfrak{m}_\Lambda(h)|^2\, e^{-q_0}\left|e^{r(h)}-1\right| \;\le\; \frac{\kappa_{\mathrm{G}}}{10}\,|\mathfrak{m}_\Lambda(h)|^2 + e^{-100T}c_\Lambda. \tag{9}$$

If $|r(h)| < \frac{1}{20}$, then $|e^{r(h)}-1| \le 2|r(h)| < \frac{1}{10}$ and (9) holds without the last term. Suppose $|r(h)| \ge \frac{1}{20}$. Since $10^6\delta \le 10^{-6}/T \le 10^{-6}$ and $|\Omega|^2 \le T$, (8) gives $\frac{1}{20} \le 10^{-6}(G_0(h)+T)/T$, that is $G_0(h) \ge (5\cdot 10^4 - 1)\,T$, and also $|r(h)| \le 10^{-6}(G_0(h)+T)$. By Lemma 5.2 and $|e^r-1| \le e^{|r|}$ the left side of (9) is at most

$$\exp\bigl(-G_0(h)+|r(h)|\bigr) \le \exp\bigl(-(1-10^{-6})\,G_0(h) + 10^{-6}T\bigr) \le e^{-49998\,T}.$$

It remains to bound c_Λ from below. Apply (I10)(b) to the compact group $S = \mathrm{K}(\varepsilon)$ with its Haar measure, the inclusion j and $R = 1$: every $h \in \mathrm{K}(\varepsilon)$ has $\|h\|_{\mathrm{op}} = 1$, and $\psi_l(h) = \mathfrak{m}_\Lambda(h)$ by (I8)(b), so

$$\begin{aligned} 1 &\le \bigl(2000\,(|\Omega|+L+1)^6\bigr)^{2|\Omega|^2}\cdot 4\,c_\Lambda,\\ \log c_\Lambda &\ge -\log 4 - 2|\Omega|^2\log 2000 - 12|\Omega|^2\log(|\Omega|+L+1) \ge -30\,T, \end{aligned}$$

by the three lower bounds for T. Hence $e^{-49998T} \le e^{-100T}c_\Lambda$, which proves (9), and the theorem with $\kappa_{\mathrm{G}} = e^{-q_0}$. □

LEAN `dsCoeff_gaussian_shape` · `gapWeightedNormSq`

6 Two-factor data: the integrand near the block-diagonal matrices

Norms without mention are Hilbert–Schmidt norms in this section too. The aim is Proposition 6.7: where both $j_1(\sigma)$ and $j_2(\sigma)$ are close to block-diagonal matrices, the point σ can be moved, without changing the integrand, to a point where both are close to 1.

Lemma 6.1 (a trace inequality)**.** *Let Ω be a finite set, $\varepsilon\colon \Omega \to \{\pm 1\}$ signs, $g \in \mathrm{U}(\varepsilon)$, let c and C be real numbers and v, w integer vectors on Ω such that, for all $i, i' \in \Omega$,*

$$c\,v_i^2 \le w_i, \qquad \text{and} \qquad w_i \le C \;\text{ if }\; \varepsilon_i \ne \varepsilon_{i'}.$$

Then $c\,\|\mathrm{diag}(v)(g-1)\|_{\mathrm{HS}}^2 + 2\,\mathrm{Re}\,\mathrm{tr}\bigl(\mathrm{diag}(w)(g-1)\bigr) \le 2C\,\|g - \mathrm{bd}_\varepsilon(g)\|_{\mathrm{HS}}^2$.

Proof. Since $g^{-1} = \mathrm{diag}(\varepsilon)g^*\,\mathrm{diag}(\varepsilon)$ and $\mathrm{diag}(\varepsilon)$ commutes with $\mathrm{diag}(w)$, the number $\mathrm{tr}(\mathrm{diag}(w)(g^{-1}-1)) = \mathrm{tr}(\mathrm{diag}(w)(g^*-1))$ is the complex conjugate of $\mathrm{tr}(\mathrm{diag}(w)(g-1))$. As $(g-1)(1-g^{-1}) = (g-1)+(g^{-1}-1)$, this gives $2\,\mathrm{Re}\,\mathrm{tr}(\mathrm{diag}(w)(g-1)) = \mathrm{Re}\,\mathrm{tr}\bigl(\mathrm{diag}(w)(g-1)(1-g^{-1})\bigr) = \sum_{i,i'} w_i\,\mathrm{Re}\bigl((g-1)_{ii'}(1-g^{-1})_{i'i}\bigr)$. Now $(1-g^{-1})_{i'i} = \delta_{ii'} - \varepsilon_i\varepsilon_{i'}\overline{g_{ii'}}$: for a compact pair the term is $-w_i|(g-1)_{ii'}|^2$, for a non-compact pair it is $w_i|g_{ii'}|^2$. Hence the left side of the claim is

$$\sum_{\varepsilon_i=\varepsilon_{i'}} (c\,v_i^2 - w_i)\,|(g-1)_{ii'}|^2 + \sum_{\varepsilon_i\ne\varepsilon_{i'}} (c\,v_i^2 + w_i)\,|g_{ii'}|^2.$$

The first sum is at most 0, and in the second $c\,v_i^2 + w_i \le 2w_i \le 2C$, while $\|g - \mathrm{bd}_\varepsilon(g)\|_{\mathrm{HS}}^2 = \sum_{\varepsilon_i\ne\varepsilon_{i'}}|g_{ii'}|^2$. □

LEAN `norm_sq_add_re_trace_le`

Lemma 6.2 (a unitary block)**.** *Let Ω, Ξ and Ξ' be finite sets, $\varepsilon\colon \Omega \to \{\pm 1\}$ signs, u a $\Xi \times \Xi$ matrix, and $e\colon \Xi \sqcup \Xi' \to \Omega$ a bijection such that $\varepsilon_{e(\xi)} = 1$ for every $\xi \in \Xi$. Suppose that $u^*u = 1$. Then there is $g \in \mathrm{U}(\varepsilon)$ with $g - 1 = \mathrm{in}_{e|_\Xi}\,(u-1)\,\mathrm{in}_{e|_\Xi}^*$.*

Proof. Put $Q = \mathrm{in}_{e|_\Xi}$. Then $Q^*Q = 1$ because e is injective, and $\mathrm{diag}(\varepsilon)Q = Q$ because $\varepsilon = 1$ on $e(\Xi)$. For $g = 1 + Q(u-1)Q^*$ this gives $g^*\,\mathrm{diag}(\varepsilon)g = \mathrm{diag}(\varepsilon) + Q\big((u^*-1) + (u-1) + (u^*-1)(u-1)\big)Q^* = \mathrm{diag}(\varepsilon) + Q(u^*u - 1)Q^* = \mathrm{diag}(\varepsilon)$. □

LEAN `exists_unitary_sub_one_eq_inclMatrix`

Lemma 6.3 (a block of weight zero does not change the coefficient)**.** *Let Ω and Ξ be finite sets, $\varepsilon\colon \Omega \to \{\pm 1\}$ signs, $l \in \mathbb{Z}^\Omega$, u a $\Xi \times \Xi$ matrix, and $\iota\colon \Xi \to \Omega$ a map with $\varepsilon_{\iota(\xi)} = 1$ and $\Lambda(l)_{\iota(\xi)} = 0$ for every $\xi \in \Xi$. Let $h, g, \tilde g \in \mathrm{U}(\varepsilon)$ with $h - 1 = \mathrm{in}_\iota\, u\, \mathrm{in}_\iota^*$ and $\tilde g = h\,g$. Then*

(1) $\mathrm{diag}(\Lambda(l))\,(h-1) = 0$;
(2) $\psi_l(\tilde g) = \psi_l(g)$;
(3) $\|\tilde g - 1\|_{\mathrm{HS}}^2 + 2\,\mathrm{Re}\,\mathrm{tr}(\tilde g - 1) \le 2\,\|g - \mathrm{bd}_\varepsilon(g)\|_{\mathrm{HS}}^2$.

Proof. The matrix $h-1$ is supported on $\iota(\Xi) \times \iota(\Xi)$, where $\Lambda(l) = 0$ and $\varepsilon = 1$. This gives (1), and it shows that h commutes with $\mathrm{diag}(\varepsilon)$; so $h \in \mathrm{K}(\varepsilon)$, h is unitary, and $\mathrm{bd}_\varepsilon(hg) = h\,\mathrm{bd}_\varepsilon(g)$.

For (2) we show $\mathfrak{m}_{\Lambda(l)}(h\,y) = \mathfrak{m}_{\Lambda(l)}(y)$ for every invertible y. Let $\tau \in \mathbb{Z}$ and $\Pi = \{\Lambda(l) \ge \tau\}$. An entry $h_{ii'}$ with $i \ne i'$ vanishes unless $i, i' \in \iota(\Xi)$, and then $\Lambda(l)_i = 0 = \Lambda(l)_{i'}$; so no such entry joins Π to its complement, and $\Delta_\Pi(hy) = \Delta_\Pi(h)\Delta_\Pi(y)$ by (I4)(a). If $\tau \le 0$ then $\iota(\Xi) \subseteq \Pi$ and $\Delta_\Pi(h) = \det h$; if $\tau > 0$ then Π does not meet $\iota(\Xi)$ and $\Delta_\Pi(h) = 1$. In the product of Definition 2.6, taken with $c_1 \le 0 \le c_2$, the factors $\det h$ are $-c_1$ in number and cancel against $(\det h)^{c_1}$. Now $\mathrm{bd}_\varepsilon(\tilde g)^{-1}\tilde g = \mathrm{bd}_\varepsilon(g)^{-1}g$, and in the integral that defines $\psi_l(\tilde g) = \mathrm{FJ}_l(h\,\mathrm{bd}_\varepsilon(g), \mathrm{bd}_\varepsilon(g)^{-1}g)$ the factor $\mathfrak{m}_{\Lambda(l)}(h\,\mathrm{bd}_\varepsilon(g)\,h_1)$, with h_1 the variable of integration, equals $\mathfrak{m}_{\Lambda(l)}(\mathrm{bd}_\varepsilon(g)\,h_1)$. This is (2).

For (3) apply Lemma 6.1 to $\tilde g$ with $c = C = 1$ and $v = w = 1$; its right side is $2\|\tilde g - \mathrm{bd}_\varepsilon(\tilde g)\|_{\mathrm{HS}}^2 = 2\|h(g - \mathrm{bd}_\varepsilon(g))\|_{\mathrm{HS}}^2 = 2\|g - \mathrm{bd}_\varepsilon(g)\|_{\mathrm{HS}}^2$ by (I1). □

LEAN `dsCoeff_mul_eq_of_zero_weight_block`

Lemma 6.4 (decay away from the block-diagonal matrices)**.** *Let Ω be a finite set, $\varepsilon\colon \Omega \to \{\pm 1\}$ signs, $l \in \mathbb{Z}^\Omega$ and A a natural number such that l has large gap A for ε, and let $g \in \mathrm{U}(\varepsilon)$. Then $\mathrm{bd}_\varepsilon(g)$ is invertible and*

$$|\psi_l(g)|^4 \le \big(1 + \|g - \mathrm{bd}_\varepsilon(g)\|_{\mathrm{HS}}^2\big)^{-A}.$$

Proof. Write $J = \mathrm{diag}(\varepsilon)$, $g_0 = \mathrm{bd}_\varepsilon(g) = \tfrac12(g + JgJ)$ and $X = g - g_0 = \tfrac12(g - JgJ)$. Expanding and using $g^*Jg = J$ and $J^2 = 1$ one finds $g_0^*g_0 = \tfrac14(g^*g + Jg^*gJ + 2)$ and $X^*X = \tfrac14(g^*g + Jg^*gJ - 2)$, so

$$g_0^*\,g_0 = 1 + X^*X.$$

Hence g_0 is invertible, and $|\det g_0|^2 = \det(1 + X^*X) \ge 1 + \mathrm{tr}(X^*X) = 1 + \|X\|_{\mathrm{HS}}^2$, the eigenvalues of X^*X being non-negative. By (I9) with the natural number $A \le 500|\Omega|^3 + A$, $|\psi_l(g)|^2 \le |\det g_0|^{-A}$; squaring gives the claim. □

LEAN `norm_dsCoeff_pow_four_le`

Setting 6.5 (two-factor data)**.** Ω, Ω' and Ξ are finite sets, S is a group, $\varepsilon\colon \Omega \to \{\pm 1\}$ and $\varepsilon'\colon \Omega' \to \{\pm 1\}$ are signs, $l \in \mathbb{Z}^{\Omega}$, $l' \in \mathbb{Z}^{\Omega'}$, $j = (j_1, j_2)\colon S \to \mathrm{U}(\varepsilon) \times \mathrm{U}(\varepsilon')$ is a homomorphism of groups, $e'\colon \Xi \to \Omega'$ is a map, B is a real number, M a natural number, $e\colon \Xi \sqcup \{0, \dots, M-1\} \to \Omega$ a bijection and $\phi\colon S \to \mathbb{C}$ a function, and $\mathrm{TwoFactor}(l, l'; j, \phi; e, e'; B)$ holds.

Lemma 6.6 (the phase of the logarithms)**.** *In Setting 6.5, let $\sigma \in S$ with $\|j_1(\sigma) - 1\|_{\mathrm{HS}} \le \frac{1}{2}$ and $\|j_2(\sigma) - 1\|_{\mathrm{HS}} \le \frac{1}{2}$. Then*

$$\mathrm{ph}_{l,l'}(\lg(j_1(\sigma) - 1),\ \lg(j_2(\sigma) - 1)) = \phi(\sigma).$$

Proof. Pairs of matrices are added and multiplied componentwise, and for $\sigma' \in S$ we write $Y_{\sigma'} = (j_1(\sigma') - 1,\ j_2(\sigma') - 1)$. Since j is a homomorphism, $Y_\sigma(1 + Y_{\sigma'}) = Y_{\sigma\sigma'} - Y_{\sigma'}$. We claim that for every natural number n and every $\sigma' \in S$

$$\mathrm{ph}_{l,l'} Y_\sigma^n\, Y_\sigma\,(1 + Y_{\sigma'}) = \phi(\sigma) \ \text{ if } n = 0, \qquad \text{and} \qquad = 0 \ \text{ if } n \ge 1.$$

For $n = 0$ the left side is $\phi(\sigma\sigma') - \phi(\sigma') = \phi(\sigma)$ by (T1), the linearity of the phase, and (T3). For the step, $Y_\sigma^{n+1} Y_\sigma(1 + Y_{\sigma'}) = Y_\sigma^n Y_\sigma(1 + Y_{\sigma\sigma'}) - Y_\sigma^n Y_\sigma(1 + Y_{\sigma'})$, and the two terms have the same phase by the claim for n. With $\sigma' = 1$ the claim says that the phase of Y_σ is $\phi(\sigma)$ and the phase of every higher power of Y_σ is 0. Now for $s \in [0, 1]$ the Neumann series $(1 + sY_\sigma)^{-1} Y_\sigma = \sum_{n \ge 0} (-s)^n Y_\sigma^{n+1}$ converges in both components, the norms being at most $\frac{1}{2}$; so the phase of $(1 + sY_\sigma)^{-1} Y_\sigma$ is $\phi(\sigma)$ for every s. Integrating over s and using (I3) for each of the two traces gives the lemma. □

LEAN `IsTwoFactor.phase_logOneAdd` · `HCParam.phase` · `subOnePair`

Proposition 6.7 (moving a point near the identity)**.** *In Setting 6.5, let δ be a real number with $1 \le B$, $0 < \delta$ and $\delta \le \frac{1}{100}$, and let $\sigma \in S$ be such that*

$$\|j_1(\sigma) - \mathrm{bd}_\varepsilon(j_1(\sigma))\|_{\mathrm{HS}}^2 + \|j_2(\sigma) - \mathrm{bd}_{\varepsilon'}(j_2(\sigma))\|_{\mathrm{HS}}^2 \le \delta^2.$$

Then there is $\sigma' \in S$ with

$$\theta_{l,l'}^{j,\phi}(\sigma) = \theta_{l,l'}^{j,\phi}(\sigma'), \qquad \|j_1(\sigma') - 1\|_{\mathrm{HS}} \le 10\,\delta, \qquad \|j_2(\sigma') - 1\|_{\mathrm{HS}} \le 10\,\delta.$$

Proof. Write $g = j_1(\sigma)$, $g' = j_2(\sigma)$, $\Lambda = \Lambda(l)$, $\Lambda' = \Lambda(l')$, and let χ be the integer vector on Ω that is 0 on $e(\Xi)$ and 1 elsewhere. The matrix $c_0 = \mathrm{diag}(\chi)(g - 1)$ consists of the rows of $g - 1$ outside $e(\Xi)$.

Step 1: $\|c_0\|_{\mathrm{HS}}^2 \le 6\,\delta^2$. We use Lemma 6.1 three times; each time we give its data (c, C, v, w). For g with $(B, 3B, \chi, -\Lambda)$: the hypothesis holds by (S2) on $e(\Xi)$ and by (S3) elsewhere, and the lemma gives

$$B\|c_0\|_{\mathrm{HS}}^2 - 2\,\mathrm{Re}\,\mathrm{tr}(\mathrm{diag}(\Lambda)(g - 1)) \le 6B\,\|g - \mathrm{bd}_\varepsilon(g)\|_{\mathrm{HS}}^2.$$

For g' with $(0, 3B, 0, \Lambda')$: the hypothesis is (T6), and $2\,\mathrm{Re}\,\mathrm{tr}(\mathrm{diag}(\Lambda')(g' - 1)) \le 6B\,\|g' - \mathrm{bd}_{\varepsilon'}(g')\|_{\mathrm{HS}}^2$. By (T1) and (T4) the two real parts are equal. Adding, $B\|c_0\|_{\mathrm{HS}}^2 \le 6B\,\delta^2$, and $B \ge 1$. For g with $(-1, 0, \chi, -\chi)$ the lemma gives

$$-2\,\mathrm{Re}\,\mathrm{tr}\,c_0 \le \|c_0\|_{\mathrm{HS}}^2. \tag{10}$$

Step 2: a unitary matrix. We write $\Omega \times \Omega$ matrices in blocks for the partition of Ω into $e(\Xi)$, identified with Ξ by e, and its complement Ω_1:

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad c_0 = \begin{pmatrix} 0 & 0 \\ g_{21} & g_{22} - 1 \end{pmatrix}, \qquad \operatorname{diag}(\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & J_2 \end{pmatrix}.$$

By Step 1, $\|\frac{1}{2}c_0\|_{\mathrm{HS}} \le \frac{1}{2}$; so by (I2) the matrix $W = (1 + \frac{1}{2}c_0)^{-1}$ exists with $\|W - 1\|_{\mathrm{HS}} \le \|c_0\|_{\mathrm{HS}}$, the block $1 + g_{22}$ is invertible, and

$$W = \begin{pmatrix} 1 & 0 \\ -(1+g_{22})^{-1} g_{21} & 2\,(1+g_{22})^{-1} \end{pmatrix}.$$

Let u be the $\Xi \times \Xi$ matrix with $u^* = g_{11} - g_{12}(1+g_{22})^{-1} g_{21}$. It is unitary: for $v \in \mathbb{C}^{\Xi}$ put $v' = -(1+g_{22})^{-1} g_{21} v \in \mathbb{C}^{\Omega_1}$; then g maps the vector (v, v') to $(u^* v,\, -v')$, and since g preserves the form $\|v\|^2 + v'^* J_2 v'$, we get $\|u^* v\| = \|v\|$.

Step 3: the new point. By (T7) there is $\sigma_0 \in S$ with $j_1(\sigma_0) - 1 = \mathrm{in}_{e|_\Xi}(u-1)\,\mathrm{in}_{e|_\Xi}^*$ and $j_2(\sigma_0) - 1 = \mathrm{in}_{e'}(u-1)\,\mathrm{in}_{e'}^*$; put $\sigma' = \sigma_0 \sigma$, $T = j_1(\sigma') - 1$ and $T' = j_2(\sigma') - 1$. Lemma 6.3 applies to $(\iota, u, h, g, \tilde{g}) = (e|_\Xi,\, u-1,\, j_1(\sigma_0),\, g,\, j_1(\sigma'))$ by (S1) and (S2), and to $(e',\, u-1,\, j_2(\sigma_0),\, g',\, j_2(\sigma'))$ by (T5). Its part (1) and (T1) give $\phi(\sigma_0) = 0$, so $\phi(\sigma') = \phi(\sigma)$ by (T3); with part (2) for both factors, the integrand has the same value at σ' and at σ. Part (3) and (T2) give

$$\|T\|_{\mathrm{HS}}^2 + 2\,\mathrm{Re}\,\mathrm{tr}\,T \le 2\,\|g - \mathrm{bd}_\varepsilon(g)\|_{\mathrm{HS}}^2, \qquad \|T'\|_{\mathrm{HS}}^2 + 2\,\mathrm{Re}\,\mathrm{tr}\,T \le 2\,\|g' - \mathrm{bd}_{\varepsilon'}(g')\|_{\mathrm{HS}}^2. \tag{11}$$

Step 4: the trace of T. In blocks $j_1(\sigma_0)$ is u on the first block and 1 on the second, and $u\,g_{11} = 1 + u\,g_{12}(1+g_{22})^{-1} g_{21}$ by the definition of u. Hence

$$T - c_0 = \begin{pmatrix} u\,g_{12}(1+g_{22})^{-1} g_{21} & u\,g_{12} \\ 0 & 0 \end{pmatrix},$$

$$\mathrm{tr}\big((T - c_0)(W-1)\big) = -\,\mathrm{tr}\big(u\,g_{12}(1+g_{22})^{-1} g_{21}\big) = \mathrm{tr}\,c_0 - \mathrm{tr}\,T,$$

and by (I1), $\mathrm{Re}\,\mathrm{tr}\,c_0 - \mathrm{Re}\,\mathrm{tr}\,T \le (\|T\|_{\mathrm{HS}} + \|c_0\|_{\mathrm{HS}})\,\|c_0\|_{\mathrm{HS}}$.

Step 5: the bounds. By (10) and Step 4, $-2\,\mathrm{Re}\,\mathrm{tr}\,T \le \|c_0\|_{\mathrm{HS}}^2 + 2(\|T\|_{\mathrm{HS}} + \|c_0\|_{\mathrm{HS}})\|c_0\|_{\mathrm{HS}}$. With (11) and the hypothesis on σ this gives $(\|T\|_{\mathrm{HS}} - \|c_0\|_{\mathrm{HS}})^2 \le 2\delta^2 + 4\|c_0\|_{\mathrm{HS}}^2 \le 26\,\delta^2$, so $\|T\|_{\mathrm{HS}} \le (\sqrt{6} + \sqrt{26})\,\delta < 7.6\,\delta$, and then $\|T'\|_{\mathrm{HS}}^2 \le 2\delta^2 + 6\delta^2 + 2\,(7.6 + 2.5)\cdot 2.5\,\delta^2 < 59\,\delta^2$. Both norms are at most $10\,\delta$. □

LEAN `IsTwoFactor.exists_near_one` · `HCParam.laplaceIntegrand`

The unitary u of Step 2 is the transfer matrix of the matrix g with its second block closed by the reflection -1; the formal proof writes it as $1 + \mathrm{in}_{e|_\Xi}^*\, W^*(g-1)^*\,\mathrm{in}_{e|_\Xi}$ and verifies its unitarity, and the identity of Step 4, by linear combinations of the defining relations.

7 Laplace's method: proof of Theorem 2

Throughout the proof the data are those of Theorem 2, and we abbreviate

$$\mathsf{n} = (|\Omega| + |\Omega'|)^2, \qquad T = \mathsf{n}\,\log\beta, \qquad \gamma(\sigma) = \frac{2}{\|j_1(\sigma)\|_{\mathrm{op}}} \qquad (\sigma \in S).$$

Since $\beta \geq 10^{12}$ we have $\log\beta \geq 1$, so $T \geq \mathsf{n} \geq 4$. The map j_1 is continuous, being proper. By (T4), $|\theta^{j,\phi}_{l,l'}(\sigma)| = |\psi_l(j_1(\sigma))|\,|\psi_{l'}(j_2(\sigma))|$.

Proof of Theorem 2. Step 1: decay. For every natural number $c \leq 18\,\mathsf{n}$ and every $\sigma \in S$,

$$|\theta^{j,\phi}_{l,l'}(\sigma)| \leq \gamma(\sigma)^c. \tag{12}$$

Indeed $|\psi_{l'}(\,\cdot\,)| \leq 1$ and $|\psi_l(g)| \leq (2/\|g\|_{\mathrm{op}})^c$ by (I9), since $c \leq 18\,\mathsf{n} \leq A$ by hypothesis (d), and A is at most the gap of l.

Step 2: the real part. For every $\sigma \in S$,

$$4\,|\theta^{j,\phi}_{l,l'}(\sigma)| \;\leq\; 8\left(\operatorname{Re}\theta^{j,\phi}_{l,l'}(\sigma) + 16\,\beta^{-50\mathsf{n}}\,\gamma(\sigma)^{8\mathsf{n}}\right). \tag{13}$$

Note first that an inequality $4|z| \leq 8(\operatorname{Re} z + 16\,\epsilon)$ for a complex number z follows from $|z| \leq \epsilon$, because $\operatorname{Re} z \geq -|z|$. If $\gamma(\sigma)^{8\mathsf{n}} \leq \beta^{-50\mathsf{n}}$, then $\beta^{-50\mathsf{n}}\gamma(\sigma)^{8\mathsf{n}} \geq \gamma(\sigma)^{16\mathsf{n}} \geq |\theta^{j,\phi}_{l,l'}(\sigma)|$ by (12), and (13) holds. Suppose now that $\gamma(\sigma)^{8\mathsf{n}} \geq \beta^{-50\mathsf{n}}$. Then $\beta^{-50\mathsf{n}}\gamma(\sigma)^{8\mathsf{n}} \geq \beta^{-100\mathsf{n}} = e^{-100T}$, and it suffices to show

$$4\,|\theta^{j,\phi}_{l,l'}(\sigma)| \leq 8\,(\operatorname{Re}\theta^{j,\phi}_{l,l'}(\sigma) + 16\,e^{-100T}). \tag{14}$$

Put $\delta = 10^{-13}/T$, so $0 < \delta \leq \frac{1}{100}$, and let

$$\operatorname{dist}(\sigma)^2 = \|j_1(\sigma) - \operatorname{bd}_{\varepsilon}(j_1(\sigma))\|^2_{\mathrm{HS}} + \|j_2(\sigma) - \operatorname{bd}_{\varepsilon'}(j_2(\sigma))\|^2_{\mathrm{HS}}$$

be the squared distance of $j(\sigma)$ from the block-diagonal matrices.

Far from the block-diagonal matrices: $\operatorname{dist}(\sigma)^2 \geq \delta^2$. By Lemma 6.4 for both factors, $|\theta^{j,\phi}_{l,l'}(\sigma)|^4 \leq (1+\operatorname{dist}(\sigma)^2)^{-A} \leq (1+\delta^2)^{-A}$. As $\log(1+\delta^2) \geq \frac{1}{2}\delta^2$ and $A \geq 10^{30}T^3$ by (d), $A\log(1+\delta^2) \geq 5000\,T \geq 400\,T$. So $|\theta^{j,\phi}_{l,l'}(\sigma)| \leq e^{-100T}$, which gives (14).

Near them: $\operatorname{dist}(\sigma)^2 \leq \delta^2$. By (a), $B \geq 1$ and Proposition 6.7 there is $\sigma' \in S$ with $\theta^{j,\phi}_{l,l'}(\sigma) = \theta^{j,\phi}_{l,l'}(\sigma')$ and $\|g_1 - 1\|_{\mathrm{HS}}, \|g_2 - 1\|_{\mathrm{HS}} \leq 10\,\delta$, where $g_1 = j_1(\sigma')$ and $g_2 = j_2(\sigma')$. Theorem 1 applies to (l, g_1) with the numbers $(10\delta, T, L, A)$: indeed $10\delta \cdot T = 10^{-12}$, $|\Omega|^2 \leq T$, and $|\Omega|^2\log(|\Omega|+L+1) \leq \mathsf{n}\log\beta$ because $|\Omega|+L+1 \leq |\Omega|+|\Omega'|+L \leq \beta$. It applies in the same way to $(-l', g_2)$, for the signs ε': the vector $-l'$ has large gap A and entries bounded by L. With

$$E_1 = \operatorname{tr}(\operatorname{diag}(\Lambda(l))\,\lg(g_1 - 1)), \qquad E_2 = \operatorname{tr}(\operatorname{diag}(\Lambda(l'))\,\lg(g_2 - 1)),$$

and $\Lambda(-l') = -\Lambda(l')$, $\psi_{-l'}(g_2) = \overline{\psi_{l'}(g_2)}$ from (I8)(c), we get $\kappa_1, \kappa_2 \in [0,1]$ with

$$\left|\frac{\psi_l(g_1)}{e^{E_1}} - \kappa_1\right| \leq \frac{\kappa_1}{10} + e^{-100T}, \qquad \left|\frac{\overline{\psi_{l'}(g_2)}}{e^{-E_2}} - \kappa_2\right| \leq \frac{\kappa_2}{10} + e^{-100T}.$$

By Lemma 6.6, $\phi(\sigma') = E_1 - E_2$: the phase of the integrand cancels the two phases of the coefficients. So the complex number $z = \theta^{j,\phi}_{l,l'}(\sigma')$ is $(\kappa_1 + z_1)(\kappa_2 + z_2)$ with complex numbers $|z_n| \leq \frac{1}{10}\kappa_n + \epsilon$, $\epsilon = e^{-100T} \leq 1$. Expanding, $\operatorname{Re} z \geq 0.79\,\kappa_1\kappa_2 - 3.2\,\epsilon$

and $|z| \le 1.21\,\kappa_1\kappa_2 + 3.2\,\epsilon$, whence $8(\operatorname{Re} z + 16\epsilon) \ge 6.32\,\kappa_1\kappa_2 + 102\,\epsilon \ge 4|z|$. This is (14), and Step 2 is complete.

Step 3: integrability. By (I11)(b) for the proper homomorphism j_1, the point $y = 1$ and the exponent $8\,\mathsf{n} > 8|\Omega|^2$, the number $\mathsf{m} = \nu\{\sigma : \|j_1(\sigma) - 1\|_{\mathrm{op}} \le 1\}$ is finite, and

$$\int_S \gamma(\sigma)^{8\mathsf{n}}\, d\nu(\sigma) \le 16^{\,8\mathsf{n}}\,\mathsf{m}. \tag{15}$$

The function $\theta^{j,\phi}_{l,l'}(\,\cdot\,)$ is continuous: the coefficients are continuous by (I8)(a), j_1 and j_2 are continuous, and ϕ is continuous by (T1). By (12) it is dominated by $\gamma^{8\mathsf{n}}$, so it is ν-integrable.

Step 4: a small ball. Let U_0 be the set of the $\sigma \in S$ with $\|j_1(\sigma) - 1\|_{\mathrm{op}} \le (200\beta^5)^{-1}$ and $\|j_2(\sigma) - 1\|_{\mathrm{op}} \le (200\beta^5)^{-1}$. By (I10)(a), with $\beta_0 = \beta$, both coefficients have absolute value at least $\frac12$ on U_0, so $4|\theta^{j,\phi}_{l,l'}(\sigma)| \ge 1$ for $\sigma \in U_0$. The right side of (13) is therefore a non-negative integrable function that is at least 1 on U_0, and Markov's inequality with (15) gives

$$\nu(U_0) \le 8\,\operatorname{Re}\int_S \theta^{j,\phi}_{l,l'}(\sigma)\, d\nu(\sigma) + 128\,\beta^{-50\mathsf{n}} \int_S \gamma^{8\mathsf{n}}\, d\nu \le 8\,\mathsf{J} + 128\,\beta^{-50\mathsf{n}}\,16^{\,8\mathsf{n}}\,\mathsf{m},$$

where $\mathsf{J} = \bigl|\int_S \theta^{j,\phi}_{l,l'}(\sigma)\, d\nu(\sigma)\bigr|$.

Step 5: covering. The map $\sigma \mapsto j_1(\sigma) \oplus j_2(\sigma)$ is a homomorphism from S to the unitary group of the signs $\varepsilon \oplus \varepsilon'$ on $\Omega \sqcup \Omega'$, and U_0 is the set where it is within operator distance $(200\beta^5)^{-1}$ of 1. By (I11)(a) with $y = 1$, $D = 200\beta^5$, $R = \beta + 1$ and $K = \beta^8 \ge 3DR^2$,

$$\nu\{\sigma : \|j_1(\sigma) \oplus j_2(\sigma)\|_{\mathrm{op}} \le \beta + 1\} \le \beta^{16\mathsf{n}}\,\nu(U_0).$$

The set on the left contains $\{\sigma : \|j_1(\sigma) - 1\|_{\mathrm{op}} \le 1\}$, because there $\|j_1(\sigma)\|_{\mathrm{op}} \le 2$ and $\|j_2(\sigma)\|_{\mathrm{op}} \le \beta + 1$ by (e). Hence

$$\mathsf{m} \le 8\,\beta^{16\mathsf{n}}\,\mathsf{J} + 128\cdot 16^{\,8\mathsf{n}}\,\beta^{-34\mathsf{n}}\,\mathsf{m} \le \tfrac12\,\beta^{20\mathsf{n}}\,\mathsf{J} + \tfrac12\,\mathsf{m},$$

using $\beta \ge 16$ twice. Since m is finite, this is the theorem. □

LEAN `IsTwoFactor.laplace_method` · `IsTwoFactor` · `IsSteep`

8 The restriction to a conjugate subgroup: proof of Theorem 3

The data are those of Theorem 3. Write $\Lambda = \Lambda(l)$ for ε and $\Lambda' = \Lambda(l')$ for $\varepsilon \oplus 1$; since l' has a gap, $\Lambda(-l') = -\Lambda'$ by (I8)(c). Operator norms are used.

Proof of Theorem 3. Nodes. Define integers $\tau_0, \dots, \tau_{2M}$ by $\tau_n = -\Lambda'_{\tilde e(M-n)}$ for $0 \le n \le M$ and $\tau_{M+1+n} = \Lambda_{e(n)}$ for $0 \le n < M$. By (S3) for l' the first $M+1$ of them decrease strictly and lie in $[B, 3B]$; by (S3) for l the last M decrease strictly and lie in $[-3B, -B]$. So the sequence is strictly decreasing, and $|\tau_n - \tau_{n'}| \le 6B$ for all n, n'. Let $c_0, \dots, c_{2M}$ be the weights of (I5) for these $2M+1$ nodes. For $0 \le a \le M$ put

$$d_a = \frac{(-1)^a}{c_a} \prod_{n<M} (\tau_a - \tau_{M+1+n}), \qquad \pi_a(\tau) = d_a \prod_{n \le M,\, n \ne a} (\tau - \tau_n),$$

$$\rho_{na} = \frac{\pi_a(\tau_{M+1+n})}{c_{M+1+n}} \quad (n < M).$$

The polynomial π_a has degree M and leading coefficient d_a, and by the formula for c_a^2, $\pi_a(\tau_{a'})/c_{a'} = \delta_{aa'}$ for $a, a' \le M$. For $a, a' \le M$ the product $\pi_a\pi_{a'}$ has degree $2M$ and leading coefficient $d_a d_{a'}$, so (I5) gives $\sum_{n\le 2M}(-1)^n \pi_a(\tau_n)\pi_{a'}(\tau_n)/c_n^2 = d_a d_{a'}$. The terms with $n \le M$ add up to $(-1)^a\delta_{aa'}$; in the others $(-1)^{M+1+n} = -(-1)^n$, because M is even. Hence

$$\sum_{n<M} (-1)^n \rho_{na}\,\rho_{na'} + d_a\,d_{a'} = (-1)^a\,\delta_{aa'} \qquad (a, a' \le M). \tag{16}$$

The element ω. Let (b_i) be the standard basis of $\mathbb{C}^{\Omega\sqcup\{*\}}$, and let ω be the matrix with

$$\omega\, b_{\tilde e(\xi)} = b_{e(\xi)} \quad (\xi \in \Xi), \qquad \omega\, b_{\tilde e(a)} = \sum_{n<M} \rho_{n,M-a}\, b_{e(n)} + d_{M-a}\, b_* \quad (0 \le a \le M).$$

The signs $\varepsilon \oplus 1$ are $+1$ at $e(\xi)$, $\tilde e(\xi)$ and $*$, $(-1)^n$ at $e(n)$ and $(-1)^a$ at $\tilde e(a)$, by (S1) for l and for l'. So the product of the columns $\tilde e(a)$ and $\tilde e(a')$ of ω for the form $\mathrm{diag}(\varepsilon \oplus 1)$ is the left side of (16) for $(M-a, M-a')$, that is $(-1)^{M-a}\delta_{aa'} = (-1)^a\delta_{aa'}$; the columns $\tilde e(\xi)$ are distinct basis vectors of sign $+1$ that do not occur in the other columns. Therefore $\omega^*\,\mathrm{diag}(\varepsilon\oplus 1)\,\omega = \mathrm{diag}(\varepsilon\oplus 1)$, and $\omega \in \mathrm{U}(\varepsilon\oplus 1)$.

Property (i). As $c_n \ge 1$ and $6B \ge 1$, we have $|d_a| \le (6B)^M$ and $|\rho_{na}| \le (6B)^{2M}$; so every entry of ω has absolute value at most $(6B)^{2M+1}$, and the operator norm of a square matrix of size $|\Omega|+1$ is at most $(|\Omega|+1)^2$ times the largest entry.

Property (iii). $\omega^{-1}(g\oplus 1)\omega - 1 = \omega^{-1}((g\oplus 1) - 1)\omega$.

Property (ii). The condition $\mathrm{Steep}(l; B, e)$ is a hypothesis. For (T1) we show that

$$\text{the rows of } \ \omega\,\mathrm{diag}(-\Lambda')\,\omega^{-1} \ \text{ with index in } \Omega \text{ are those of } \ (\mathrm{diag}(\Lambda) \mid v) \tag{17}$$

for a column v at $*$. Then $\mathrm{tr}\big(\mathrm{diag}(-\Lambda')\,\omega^{-1}(Z\oplus 0)\,\omega\big) = \mathrm{tr}(\mathrm{diag}(\Lambda)Z)$ for every $\Omega\times\Omega$ matrix Z, and with (iii) the phase $\mathrm{ph}_{l,-l'}(g-1,\ j_2^\omega(g)-1)$ vanishes for every g, as (T1) asks for the function $\phi = 0$. To see (17), let $v_{e(\xi)} = 0$ and $v_{e(n)} = -\prod_{a\le M}(\tau_{M+1+n} - \tau_a)/c_{M+1+n}$, and compare the entries of $\omega\,\mathrm{diag}(-\Lambda')$ and of $(\mathrm{diag}(\Lambda)\mid v)\,\omega$ in the rows of Ω. In the column $\tilde e(a)$, where $-\Lambda' = \tau_{M-a}$, and the row $e(n)$, where $\Lambda = \tau_{M+1+n}$, the difference is

$$(\tau_{M-a} - \tau_{M+1+n})\,\rho_{n,M-a} - v_{e(n)}\,d_{M-a} = 0,$$

because $(\tau_{a'} - \tau)\,\pi_{a'}(\tau) = -d_{a'}\prod_{a\le M}(\tau - \tau_a)$. In the rows $e(\xi)$ and in the columns $\tilde e(\xi)$ both sides vanish, as $\Lambda = 0$ on $e(\Xi)$ and $\Lambda' = 0$ on $\tilde e(\Xi)$ by (S2).

Conditions (T3) and (T4) hold for the function 0, and (T2) because a conjugation keeps the trace. For (T5) and (T6): the signs $\varepsilon\oplus 1$ are $+1$ on $\tilde e(\Xi)$, and $\Lambda(-l') = -\Lambda'$ is 0 there and lies in $[B, 3B]$ at the other indices, by (S1) to (S3) for l'. For (T7), let u be a unitary $\Xi\times\Xi$ matrix. Lemma 6.2 gives $g \in \mathrm{U}(\varepsilon)$ with $g - 1 = \mathrm{in}_{e|_\Xi}(u-1)\,\mathrm{in}_{e|_\Xi}^*$. The matrix ω maps $b_{\tilde e(\xi)}$ to $b_{e(\xi)}$, and its row at $e(\xi)$ is the unit row at $\tilde e(\xi)$; so $\omega^{-1}((g-1)\oplus 0)\omega = \mathrm{in}_{\tilde e|_\Xi}(u-1)\,\mathrm{in}_{\tilde e|_\Xi}^*$, which by (iii) is the second equation of (T7). □

LEAN `IsSteep.exists_isTwoFactor_restriction`

9 The Bessel graph: proofs of Theorems 4 and 5

Lemma 9.1 (the Bessel graph and its measure)**.** *Let $\mathfrak{s}$ be a shape. Then $\mathsf{HU}(\mathfrak{s})$ is a closed subset of $G_1(\mathfrak{s}) \times \mathrm{U}(1)$, and the measure $\nu_{\mathfrak{s}}$ is positive on non-empty open subsets of $\mathsf{HU}(\mathfrak{s})$, finite on its compact subsets, and invariant under right translations.*

Proof. The condition $\mathrm{emb}_{\mathfrak{s}}(h)^{-1} g \in \mathcal{U}(\mathfrak{s})$ says that certain entries of $\mathrm{emb}_{\mathfrak{s}}(h)^{-1} g$ are those of the identity matrix, which is a closed condition on (g, h). The chart $\mathrm{ch}_{\mathfrak{s}}$ is a homeomorphism, and the product of the Haar measure $\eta_{\mathfrak{s}}$ with the Haar measure of the compact group $\mathrm{U}(1)$ is positive on open sets and finite on compact sets; so is its image. For the invariance let $\sigma' = (\mathrm{emb}_{\mathfrak{s}}(h_0) u_0,\, h_0) \in \mathsf{HU}(\mathfrak{s})$. Then

$$\begin{aligned} \mathrm{ch}_{\mathfrak{s}}(u, h) \cdot \sigma' &= \big(\mathrm{emb}_{\mathfrak{s}}(h)\, u\, \mathrm{emb}_{\mathfrak{s}}(h_0)\, u_0,\ h\, h_0\big) \\ &= \mathrm{ch}_{\mathfrak{s}}\big(\mathrm{emb}_{\mathfrak{s}}(h_0)^{-1} u\, \mathrm{emb}_{\mathfrak{s}}(h_0) \cdot u_0,\ h\, h_0\big). \end{aligned}$$

In the chart, right translation by σ' is therefore the product of two maps: on $\mathcal{U}(\mathfrak{s})$ the conjugation by $\mathrm{emb}_{\mathfrak{s}}(h_0)^{-1} \in \mathcal{Q}(\mathfrak{s})$ followed by the right translation by u_0, which preserve $\eta_{\mathfrak{s}}$ by (I13)(c); and on $\mathrm{U}(1)$ the right translation by h_0. □

LEAN `Shape.besselGraph` · `Shape.besselGraphChart` · `Shape.besselGraphMeasure` · `Shape.besselGraphMeasure_finiteOnCompacts` · `Shape.besselGraphMeasure_isMulRightInvariant`

The closedness and the positivity on open sets are two unnamed instances of the formal text.

Lemma 9.2. *Let $\mathfrak{s}$ be a shape and $\sigma = (\sigma_1, \sigma_2) \in \mathsf{HU}(\mathfrak{s})$. Then $\sigma_1 \in \mathcal{Q}(\mathfrak{s})$.*

Proof. $\sigma_1 = \mathrm{emb}_{\mathfrak{s}}(\sigma_2) \cdot \mathrm{emb}_{\mathfrak{s}}(\sigma_2)^{-1} \sigma_1$ is the product of an element of $\mathcal{H}(\mathfrak{s})$ and one of $\mathcal{U}(\mathfrak{s})$. □

LEAN `besselGraph_fst_mem_besselS`

Lemma 9.3 (unfolding along the Bessel graph)**.** *Let $\mathfrak{s}$ be a shape, $x \in G_1(\mathfrak{s})$, and let f be a function from $G_1(\mathfrak{s})$ to $[0, \infty]$ that is measurable. Then*

$$\int_{G_1(\mathfrak{s})} f(g)\, dg = \int_{G_1(\mathfrak{s})} \int_{\mathsf{HU}(\mathfrak{s})} f(x\, \sigma_1\, y)\, d\nu_{\mathfrak{s}}(\sigma)\, d\dot{\eta}_{\mathfrak{s}}(y).$$

Proof. By the definition of $\nu_{\mathfrak{s}}$ the inner integral is

$$\int_{\mathrm{U}(1)} \int_{\mathcal{U}(\mathfrak{s})} f(x\, \mathrm{emb}_{\mathfrak{s}}(h)\, u\, y)\, d\eta_{\mathfrak{s}}(u)\, dh.$$

By Tonelli's theorem and Definition 2.14(e), applied to $g \mapsto f(x\, \mathrm{emb}_{\mathfrak{s}}(h)\, g)$ for each h, the right side is $\int_{\mathrm{U}(1)} \int_{G_1(\mathfrak{s})} f(x\, \mathrm{emb}_{\mathfrak{s}}(h)\, g)\, dg\, dh$. The inner integral is $\int f(g)\, dg$ by the left invariance of the Haar measure, and dh has mass one. □

LEAN `lintegral_unfold_besselGraphMeasure`

Lemma 9.4 (traces against the graph)**.** *Let $\mathfrak{s}$ be a shape, $\sigma = (\sigma_1, \sigma_2) \in \mathsf{HU}(\mathfrak{s})$, and let Z be an $\mathrm{Ind}(\mathfrak{s}) \times \mathrm{Ind}(\mathfrak{s})$ matrix that is block upper triangular for the levels. Then $\mathrm{tr}\big(Z\,(\sigma_1 - \mathrm{emb}_{\mathfrak{s}}(\sigma_2))\big) = 0$.*

Proof. With $u = \mathrm{emb}_{\mathfrak{s}}(\sigma_2)^{-1} \sigma_1 \in \mathcal{U}(\mathfrak{s})$ the trace is $\mathrm{tr}\big(Z\, \mathrm{emb}_{\mathfrak{s}}(\sigma_2)\,(u - 1)\big)$. The matrix $Z\, \mathrm{emb}_{\mathfrak{s}}(\sigma_2)$ is block upper triangular, since $\mathrm{emb}_{\mathfrak{s}}(\sigma_2)$ differs from 1 only inside the middle level; and $(u - 1)_{i'i} = 0$ whenever $\mathrm{lev}(i) \le \mathrm{lev}(i')$. So in $\sum_{i,i'} (Z\, \mathrm{emb}_{\mathfrak{s}}(\sigma_2))_{ii'} (u - 1)_{i'i}$ every term vanishes. □

LEAN `besselGraph_trace_mul_sub_eq_zero`

Lemma 9.5 (properness)**.** *Let $\mathfrak{s}$ be a shape, ε signs on $\mathrm{Ind}(\mathfrak{s})$, $\mathsf{p} \in \mathrm{Frm}(\mathfrak{s}, \varepsilon)$ a frame and $x \in G_1(\mathfrak{s})$. Then the map $\sigma \mapsto \jmath_x(\sigma)_1$ from $\mathsf{HU}(\mathfrak{s})$ to $\mathrm{U}(\varepsilon)$ is proper.*

Proof. It is the first projection of the closed subset $\mathsf{HU}(\mathfrak{s})$ of $G_1(\mathfrak{s}) \times \mathrm{U}(1)$, which is proper because $\mathrm{U}(1)$ is compact, followed by the homeomorphism $g \mapsto \mathsf{p}^{-1} x^{-1} g\, x\, \mathsf{p}$. □

LEAN `isProperMap_fst_besselPairHom · besselPairHom · Frame`

Proof of Theorem 4. Write $\varrho = \varrho(\mathfrak{s})$ and $\Lambda = \Lambda(l)$, and let p_i be the column of p at $i \in \mathrm{Ind}(\mathfrak{s})$; these columns are orthonormal, and $\mathsf{p}_i^* J_1(\mathfrak{s}) \mathsf{p}_{i'} = \varepsilon_i \delta_{ii'}$.

Nodes and vectors. Put $\tau_0 = 0$ and $\tau_{n+1} = \Lambda_{e(n)} - a$ for $0 \le n \le 2\varrho + 1$. By (S3) these $2\varrho + 3$ integers decrease strictly, and $|\tau_{n+1}| \le 3B + a \le 2b^2$, so $\tau_{n+1}^2 \le 4b^4 \le b^5$. Let $c_0, \dots, c_{2\varrho+2}$ be the weights of (I5), and define vectors of $\mathbb{C}^{\mathrm{Ind}(\mathfrak{s})}$

$$V^{(0)} = c_0 \sum_{n \le 2\varrho+1} \frac{1}{c_{n+1}} \mathsf{p}_{e(n)}, \qquad V^{(k)} = \sum_{n \le 2\varrho+1} \frac{\tau_{n+1}^k}{c_{n+1}} \mathsf{p}_{e(n)} \quad (k \ge 1).$$

Since $\varepsilon_{e(n)} = (-1)^n$, the products of these vectors for $J_1(\mathfrak{s})$ are, up to the factors c_0, the sums

$$\sum_{n \le 2\varrho+1} (-1)^n \frac{\tau_{n+1}^{k+k'}}{c_{n+1}^2}.$$

Input (I5) for the polynomial $\tau^{k+k'}$, of degree at most $2\varrho + 2$, evaluates them: the node $\tau_0 = 0$ contributes only for $k + k' = 0$, and the coefficient of degree $2\varrho + 2$ is 1 or 0. The result is, for $k, k' \ge 1$ with $k + k' \le 2\varrho + 2$ and for $1 \le k'' \le 2\varrho + 1$,

$$\begin{aligned} &V^{(0)*} J_1(\mathfrak{s}) V^{(0)} = 1, \qquad V^{(0)*} J_1(\mathfrak{s}) V^{(k'')} = 0, \\ &V^{(k)*} J_1(\mathfrak{s}) V^{(k')} = \begin{cases} -1 & \text{if } k + k' = 2\varrho + 2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{18}$$

The vectors $\mathsf{p}_{e(\xi)}$, $\xi \in \Xi$, are orthonormal for $J_1(\mathfrak{s})$ with squares $+1$ by (S1), and orthogonal to all $V^{(k)}$.

The point x. Consider the two families of vectors

$$(\mathsf{p}_{e(\xi)}\ (\xi \in \Xi),\ V^{(0)},\ V^{(1)}, \dots, V^{(\varrho+1)}) \qquad \text{and} \qquad (x_{\tilde{e}(\xi)}\ (\xi \in \Xi),\ x_{\tilde{e}(*)},\ f_0, \dots, f_\varrho).$$

By (18) and Definition 2.14 they have the same Gram matrix for $J_1(\mathfrak{s})$: the identity on the first $|\Xi| + 1$ vectors, and -1 at the last vector, 0 elsewhere. The second family consists of distinct basis vectors. The first is linearly independent, because its matrix of products with the vectors $\mathsf{p}_{e(\xi)}$, $V^{(0)}$, $V^{(2\varrho+1)}, \dots, V^{(\varrho+1)}$ is triangular with diagonal entries ± 1, again by (18). By Witt's theorem (I6) there is $x \in G_1(\mathfrak{s})$ with

$$x\, \mathsf{p}_{e(\xi)} = x_{\tilde{e}(\xi)}, \qquad x\, V^{(0)} = x_{\tilde{e}(*)}, \qquad x\, V^{(a+1)} = f_a \quad (0 \le a \le \varrho).$$

Its size. Apply (I14) to x and the vectors $V_a = V^{(a+1)}$, $a \ge 0$. For $a \le \varrho$, $V_a = x^{-1} f_a$. For $a + a' \le 2\varrho$ the product $V_a^* J_1(\mathfrak{s}) V_{a'}$ is -1 if $a + a' = 2\varrho$ and 0 otherwise, by (18). As p is unitary and $c_{n+1} \ge 1$, for $a \le 2\varrho$ we have $\|V_a\|^2 \le (2\varrho + 2)\, b^{5(2\varrho+1)} \le b^{10\varrho+6}$. So $\mathrm{size}_Q(x) \le 8(\varrho + 1)\, b^{5\varrho+3} \le b^{5\varrho+4} \le b^{5n(\mathfrak{s})}$.

The conjugated weight matrix. Put $q = x\, \mathsf{p}$, so that $q^* J_1(\mathfrak{s}) q = \mathrm{diag}(\varepsilon)$ and $\jmath_x(\sigma) = (q^{-1} \sigma_1 q,\ \sigma_2)$, and let $D = q\, \mathrm{diag}(\Lambda)\, q^{-1}$. Because $\Lambda_{e(\xi)} = 0$ and $\Lambda_{e(n)} = \tau_{n+1} + a$,

the matrix $\mathsf{p}\,\mathrm{diag}(\Lambda)\,\mathsf{p}^{-1}$ kills $\mathsf{p}_{e(\xi)}$, sends $V^{(k)}$ to $a\,V^{(k)}+V^{(k+1)}$ for $k\ge 1$, and $V^{(0)}$ to $a\,V^{(0)}+c_0V^{(1)}$. Applying x:

$$D\,x_{\tilde e(\xi)}=0,\qquad D\,x_{\tilde e(*)}=a\,x_{\tilde e(*)}+c_0\,f_0,\qquad D\,f_k=a\,f_k+f_{k+1}\quad(k<\varrho).\tag{19}$$

Moreover $q^{-1}=\mathrm{diag}(\varepsilon)q^{*}J_1(\mathfrak{s})$, so $J_1(\mathfrak{s})D=J_1(\mathfrak{s})q\,\mathrm{diag}(\Lambda)\,\mathrm{diag}(\varepsilon)\,q^{*}J_1(\mathfrak{s})$ is hermitian: $J_1(\mathfrak{s})D=D^{*}J_1(\mathfrak{s})$.

The matrix $D-C_{\mathfrak{s}}$ is block upper triangular for the levels, where $C_{\mathfrak{s}}$ is the chain matrix of (I13)(b). By (19), $(D-C_{\mathfrak{s}})e_k=a\,e_k$, and $(D-C_{\mathfrak{s}})x_i=D\,x_i$ lies in the span of $x_{\tilde e(*)}$ and f_0, of levels at most ϱ. The rows of level higher than that of a column z or e_b^{*} are rows e_c^{*}, and the coefficient of e_c^{*} in a vector w is $e_c^{*}J_1(\mathfrak{s})w$. By the symmetry of D and (19), $e_c^{*}J_1(\mathfrak{s})Dw=(a\,e_c+f_{c+1})^{*}J_1(\mathfrak{s})w$. For $w=z$ this is -1 if $c=\varrho-1$ and 0 for the other $c<\varrho$; for $w=e_b^{*}$ and $c<b$ it is 1 if $c=b-1$ and 0 otherwise. These are the coefficients of e_c^{*} in $C_{\mathfrak{s}}z$ and in $C_{\mathfrak{s}}e_b^{*}$.

Conditions (T1) to (T7), for $j=\jmath_x(\,\cdot\,)$ and $\phi=\phi_{\mathfrak{s}}$; $\mathrm{Steep}(l;B,e)$ is a hypothesis. Let $\sigma\in\mathsf{HU}(\mathfrak{s})$. By Lemma 9.4, $\mathrm{tr}\big((D-C_{\mathfrak{s}})(\sigma_1-\mathrm{emb}_{\mathfrak{s}}(\sigma_2))\big)=0$. By (I13)(a),(b) and Lemma 9.2, $\mathrm{tr}(C_{\mathfrak{s}}\sigma_1)=\mathrm{sd}(\sigma_1)-\overline{\mathrm{sd}(\sigma_1)}$ and $\mathrm{tr}(C_{\mathfrak{s}}\,\mathrm{emb}_{\mathfrak{s}}(\sigma_2))=0$. And by (19) the block of D on the vectors x_i is diagonal, with entry a at $\tilde e(*)$ and 0 elsewhere, which is $\mathrm{diag}(\Lambda(l'))$ by hypothesis; so $\mathrm{tr}\big(D(\mathrm{emb}_{\mathfrak{s}}(\sigma_2)-1)\big)=\mathrm{tr}\big(\mathrm{diag}(\Lambda(l'))(\sigma_2-1)\big)$. Adding the three facts,

$$\mathrm{ph}_{l,l'}(q^{-1}(\sigma_1-1)q,\,\sigma_2-1)=\mathrm{tr}\big(D(\sigma_1-1)\big)-\mathrm{tr}\big(\mathrm{diag}(\Lambda(l'))(\sigma_2-1)\big)=\mathrm{sd}(\sigma_1)-\overline{\mathrm{sd}(\sigma_1)},$$

which is (T1). For (T2), $\mathrm{tr}(q^{-1}(\sigma_1-1)q)=\mathrm{tr}(\sigma_1-1)=\mathrm{tr}(\mathrm{emb}_{\mathfrak{s}}(\sigma_2)-1)=\mathrm{tr}(\sigma_2-1)$, by Lemma 9.4 for $Z=1$. (T3) holds by (I13)(a) and Lemma 9.2, and (T4) is clear. (T5) and (T6) hold because the signs on $\kappa(\mathfrak{s})$ are $+1$ and $\Lambda(l')$ is 0 on $\tilde e(\Xi)$ and $a\ge 0$ at $\tilde e(*)$. For (T7) let u be a unitary $\Xi\times\Xi$ matrix. By Lemma 6.2, for the bijection $\tilde e$ and the signs 1, there is $h\in\mathrm{U}(1)$ with $h-1=\mathrm{in}_{\tilde e|_\Xi}(u-1)\,\mathrm{in}_{\tilde e|_\Xi}^{*}$. Take $\sigma=\mathrm{ch}_{\mathfrak{s}}(1,h)=(\mathrm{emb}_{\mathfrak{s}}(h),h)$. Its second component gives the second equation of (T7). For the first, $\mathrm{emb}_{\mathfrak{s}}(h)-1$ is $u-1$ placed in the block of the vectors $x_{\tilde e(\xi)}$; the matrix q maps the basis vector at $e(\xi)$ to $x_{\tilde e(\xi)}$, and $q^{-1}=\mathrm{diag}(\varepsilon)q^{*}J_1(\mathfrak{s})$ has the unit row at $x_{\tilde e(\xi)}$ in its row $e(\xi)$; hence $q^{-1}(\mathrm{emb}_{\mathfrak{s}}(h)-1)q=\mathrm{in}_{e|_\Xi}(u-1)\,\mathrm{in}_{e|_\Xi}^{*}$. □

LEAN `IsSteep.exists_isTwoFactor_besselPairHom`

Proof of Theorem 5. Write $\psi_l^{\mathsf{p}}(g)=\psi_l(\mathsf{p}^{-1}g\,\mathsf{p})$ for $g\in G_1(\mathfrak{s})$, $\mathsf{c}=8\,r(\mathfrak{s})^2+1$, and $\mathsf{m}=\nu_{\mathfrak{s}}\{\sigma:\|\jmath_x(\sigma)_1-1\|_{\mathrm{op}}\le 1\}$. For $y\in G_1(\mathfrak{s})$ and $\sigma\in\mathsf{HU}(\mathfrak{s})$ let

$$F(y,\sigma)=\psi_l^{\mathsf{p}}(x^{-1}\sigma_1 y)\,e^{-\phi_{\mathfrak{s}}(\sigma)}\,\mathrm{rep}(\sigma_2)^{-1}v_0\ \in\mathbb{C}^{I},\qquad W(y)=\int_{\mathsf{HU}(\mathfrak{s})}F(y,\sigma)\,d\nu_{\mathfrak{s}}(\sigma).$$

Sizes. From the hypothesis on v_0 at $h=1$ and $\psi_{l'}(1)=1$ (the integral of Definition 2.7 at $g=z=1$ is the normalising constant), $\|v_0\|=1$. As rep is unitary and $\phi_{\mathfrak{s}}$ is imaginary, $\|F(y,\sigma)\|=|\psi_l^{\mathsf{p}}(x^{-1}\sigma_1 y)|$. The map $\sigma\mapsto\jmath_x(\sigma)_1$ is a proper homomorphism (Lemma 9.5), $\nu_{\mathfrak{s}}$ is right invariant and finite on compact sets (Lemma 9.1), and $\jmath_x(\sigma)_1\cdot\mathsf{p}^{-1}x^{-1}y\,\mathsf{p}=\mathsf{p}^{-1}x^{-1}\sigma_1 y\,\mathsf{p}$. So (I11)(b), with the exponent $\mathsf{c}>8|\mathrm{Ind}(\mathfrak{s})|^2$, shows that m is finite and that for every y

$$\int_{\mathsf{HU}(\mathfrak{s})}\Big(\frac{2}{\|\mathsf{p}^{-1}x^{-1}\sigma_1 y\,\mathsf{p}\|_{\mathrm{op}}}\Big)^{\mathsf{c}}d\nu_{\mathfrak{s}}(\sigma)\le 16^{\mathsf{c}}\,\mathsf{m}.\tag{20}$$

By (I9), with $\mathsf{c} \le 500\, r(\mathfrak{s})^3 + A$, the integrand of (20) dominates $\|F(y,\sigma)\|$. Hence $F(y,\,\cdot\,)$ is integrable, F being continuous, and

$$\|W(y)\| \le \int_{\mathsf{HU}(\mathfrak{s})} \left|\psi_l^{\mathsf{p}}(x^{-1}\sigma_1 y)\right| d\nu_{\mathfrak{s}}(\sigma) \le 16^{\mathsf{c}}\,\mathsf{m} \qquad \text{for every } y. \tag{21}$$

W is a Bessel function for rep. It is bounded by (21) and measurable. Let $\sigma' \in \mathsf{HU}(\mathfrak{s})$. Replacing σ by $\sigma\sigma'^{-1}$ in the integral for $W(\sigma_1' y)$, which does not change the measure, and using $\phi_{\mathfrak{s}}(\sigma\sigma'^{-1}) = \phi_{\mathfrak{s}}(\sigma) - \phi_{\mathfrak{s}}(\sigma')$ from (I13)(a), we get

$$W(\sigma_1'\, y) = e^{\phi_{\mathfrak{s}}(\sigma')}\, \mathrm{rep}(\sigma_2')\, W(y).$$

For $\sigma' = \mathrm{ch}_{\mathfrak{s}}(u,1) = (u,1)$ and for $\sigma' = \mathrm{ch}_{\mathfrak{s}}(1,h) = (\mathrm{emb}_{\mathfrak{s}}(h),h)$, where $\phi_{\mathfrak{s}}(\sigma') = 0$, these are the two transformation laws of Definition 2.14(g).

The value at x. Since $\langle v_0, \mathrm{rep}(h)^{-1} v_0\rangle = \overline{\langle v_0, \mathrm{rep}(h) v_0\rangle} = \overline{\psi_{l'}(h)}$,

$$\begin{aligned}\langle v_0, W(x)\rangle &= \int_{\mathsf{HU}(\mathfrak{s})} \psi_l(\mathsf{p}^{-1}x^{-1}\sigma_1 x\,\mathsf{p})\, \overline{\psi_{l'}(\sigma_2)}\, e^{-\phi_{\mathfrak{s}}(\sigma)}\, d\nu_{\mathfrak{s}}(\sigma) \\ &= \int_{\mathsf{HU}(\mathfrak{s})} \theta_{l,l'}^{\jmath_x(\,\cdot\,),\phi_{\mathfrak{s}}}(\sigma)\, d\nu_{\mathfrak{s}}(\sigma).\end{aligned}$$

So the hypothesis of the theorem and the Cauchy–Schwarz inequality give

$$\mathsf{m} \le b^{N}\, \|W(x)\|. \tag{22}$$

The four conditions of Definition 2.20. (P3) is the first inequality of (21). (P4) follows from (20), (22) and $16 \le b$, with $P = b^{\mathsf{c}+N}$. (P2): the set in the definition of m contains a neighbourhood of 1, and $\nu_{\mathfrak{s}}$ is positive on open sets, so $\mathsf{m} > 0$ by (I11)(c), and $W(x) \ne 0$ by (22). (P1): by Fubini's theorem, which (21) and the integrability of the kernel k of (I12) allow,

$$\begin{aligned}\int_{G_1(\mathfrak{s})} k(g^{-1})\, W(y\,g)\, dg = \int_{\mathsf{HU}(\mathfrak{s})} \Big(\int_{G_1(\mathfrak{s})} k(g^{-1})\, \psi_l^{\mathsf{p}}(x^{-1}\sigma_1 y\, g)\, dg \Big) \\ \cdot\, e^{-\phi_{\mathfrak{s}}(\sigma)}\, \mathrm{rep}(\sigma_2)^{-1} v_0\, d\nu_{\mathfrak{s}}(\sigma),\end{aligned}$$

and the inner integral is $\psi_l^{\mathsf{p}}(x^{-1}\sigma_1 y)$ by (I12) with $y_2 = 1$, because $k = \deg(l)\,\psi_l^{\mathsf{p}}$ and $k(1) = \deg(l)$. □

LEAN exists_isBesselFunction · HasPeakAt

10 The period of a Poincaré series of deep level: proof of Theorem 6

The data are those of Theorem 6. We abbreviate $G = G(E,\varepsilon)$, $G_\infty = G_\infty(E,\varepsilon)$, $V^+ = V \oplus \langle 1\rangle$, $\Gamma^+ = \Gamma(V^+)$, $\mathsf{K} = K_\Omega(a_0)$ and $\mathsf{K}^+ = K_{\Omega\sqcup\{*\}}(a_0)$, write b_* for the basis vector of the added line, and

$$k_\gamma(y) = \Psi(X_\gamma(y)_\infty)\, \mathbf{1}_{\mathsf{K}^+}(X_\gamma(y)_{\mathrm{f}}) \qquad (\gamma \in \Gamma^+,\ y \in G).$$

Norms of matrices are operator norms. Besides (I9) and (I11) the proof uses these results of the background library on adelic unitary groups (units 114, 120, 121, 122; all elementary except the first):

(A1) *Unfolding.* If f is integrable on G for the Haar measure, then $y \mapsto f(\delta y)$ is ϖ_V-integrable for every $\delta \in \Gamma(V)$, and $\sum_{\delta\in\Gamma(V)} \int f(\delta y)\, d\varpi_V(y) = \int_G f(y)\, dy$. (Weil's formula; reference and relation as for (I13)(d).)

(A2) *The first minimum.* $\lambda_1(V) \le \|\eta\|_V$ for every non-zero lattice vector η of V; for spaces V_1, V_2 and points g_1, g_2 of their groups, $\min(\lambda_1(V_1 \cdot g_1), \lambda_1(V_2 \cdot g_2)) \le \lambda_1((V_1 \oplus V_2) \cdot (g_1 \oplus g_2))$; and $\lambda_1 = 1$ for a positive line.

(A3) *Separated points.* If points f_n of a real normed space of dimension d_0 have norms and mutual distances at least $m_0 > 0$, and $c \ge d_0 + 2$ is an integer, then $\sum_n \|f_n\|^{-c} \le (5/m_0)^c$.

(A4) *Structure.* For $a_0 \neq 0$, K is an open subgroup of finite Haar measure, and a_0 is invertible in the finite adeles. An element $g_1 \oplus g_2$ lies in $\Gamma(V_1 \oplus V_2)$ if and only if $g_1 \in \Gamma(V_1)$ and $g_2 \in \Gamma(V_2)$; and an element of the unitary group of a block-diagonal form $J_1 \oplus J_2$, with J_2 invertible, that fixes the vectors of the second block is of the form $g_1 \oplus 1$.

Proof of Theorem 6. Step 0: the decay of Ψ. Let $g \in G_\infty(E, \varepsilon\oplus 1)$ and $t_0 > 0$ be such that $t_0 \le \prod_v \|g_v(r_v \oplus 1)\|_{\mathrm{op}}$ for every $r \in G_\infty$. Then $r \mapsto \Psi(g\,(r \oplus 1))$ is integrable on G_∞ and

$$\int_{G_\infty} \big|\Psi(g\,(r \oplus 1))\big|\, dr \le L_*\, t_0^{-A}. \tag{23}$$

Indeed, for $h \in \mathrm{U}(\varepsilon \oplus 1)$ and a place v, (I9) with the exponent $2A$, which is at most the gap of l_v, and $\|h\|_{\mathrm{op}} \le \|\alpha_v\|_{\mathrm{op}}\, \|\alpha_v^{-1} h\, \omega_v\|_{\mathrm{op}}\, \|\omega_v\|_{\mathrm{op}}$ from (I11)(d) give

$$\big|\psi_{l_v}(\alpha_v^{-1} h\, \omega_v)\big| \le \Big(\frac{2\|\alpha_v\|_{\mathrm{op}}\|\omega_v\|_{\mathrm{op}}}{\|h\|_{\mathrm{op}}}\Big)^{2A} = (2(\|\alpha_v\|_{\mathrm{op}}\|\omega_v\|_{\mathrm{op}})^2)^A \Big(\frac{2}{\|h\|_{\mathrm{op}}}\Big)^A \|h\|_{\mathrm{op}}^{-A}.$$

Take $h = g_v(r_v \oplus 1)$ and multiply over v: the last factors give at most t_0^{-A}, and the integral over G_∞ of the product of the middle factors is the product over v of the integrals $\int_{\mathrm{U}(\varepsilon)} (2/\|g_v(r \oplus 1)\|_{\mathrm{op}})^A dr$. By (I11)(d) and the invariance of dr under inversion, such an integral equals $\int (2/\|(r \oplus 1) g_v^{-1}\|_{\mathrm{op}})^A dr$, which by (I11)(b), for the proper homomorphism $r \mapsto r \oplus 1$ and the exponent $A > 8(|\Omega| + 1)^2$, is at most $16^A \,\mathrm{vol}\{r : \|r - 1\|_{\mathrm{op}} \le 1\}$. This proves (23). Since $\|h\|_{\mathrm{op}} \ge 1$ on unitary groups, it applies with $t_0 = 1$ to every g; for $g = 1$ this is the first assertion of the theorem.

Step 1: the vector of a class. Let $\gamma \in \Gamma^+$, say $\gamma = x_{V^+}^{-1} \gamma_0\, x_{V^+}$ with a rational isometry γ_0, and put $\eta_\gamma = \gamma_0\, b_* \in E^{\Omega \sqcup \{*\}}$. As the point of the positive line is 1 and $y \oplus 1$ fixes b_*, the adelic vector of η_γ at the point of $V^+ \cdot (x \oplus 1)$ is

$$\big(x_{V^+}(x \oplus 1)\big)^{-1} \eta_\gamma = X_\gamma(y)\, b_* \qquad \text{for every } y \in G :$$

the last column of $X_\gamma(y)$, which does not depend on y. Let $f(\gamma) \in \mathbb{C}^{f_E(|\Omega|+1)}$ be the vector of its components at the infinite places; its Euclidean norm is the length of η_γ for $V^+ \cdot (x \oplus 1)$.

Step 2: representatives. By (A4), $\Gamma(V) \oplus 1$ is the subgroup of the elements of Γ^+ of the form $g \oplus 1$. Choose a set $\mathcal{C} \subseteq \Gamma^+$ of representatives of the cosets $\gamma\,(\Gamma(V) \oplus 1)$, with $1 \in \mathcal{C}$. Every element of Γ^+ is $\gamma(\delta \oplus 1)$ for unique $\gamma \in \mathcal{C}$ and $\delta \in \Gamma(V)$, and $X_{\gamma(\delta\oplus 1)}(y) = X_\gamma(\delta y)$. As F is invariant under $\Gamma(V)$,

$$\sum_{\gamma' \in \Gamma^+} k_{\gamma'}(y)\, F(y) = \sum_{\gamma\in\mathcal{C}} \sum_{\delta\in\Gamma(V)} k_\gamma(\delta y)\, F(\delta y), \tag{24}$$

as soon as one side converges absolutely. Let $\mathcal{W}$ be the set of the $\gamma \in \mathcal{C}$ for which $X_\gamma(y)_{\mathrm{f}} \in \mathsf{K}^+$ for some y; for the other γ the function k_γ vanishes. For $\gamma \in \mathcal{W}$ fix such a point y_γ, with $y_1 = x$, for which $X_1(x) = 1$.

Step 3: one class. Let $\gamma \in \mathcal{W}$. Since $X_\gamma(y_\gamma y) = X_\gamma(y_\gamma)\,(y \oplus 1)$, since K^+ is a group, and since $y_{\mathrm{f}} \oplus 1 \in \mathsf{K}^+$ if and only if $y_{\mathrm{f}} \in \mathsf{K}$,

$$k_\gamma(y_\gamma\, y) = \Psi\big(X_\gamma(y_\gamma)_\infty\,(y_\infty \oplus 1)\big) \cdot \mathbf{1}_{\mathsf{K}}(y_{\mathrm{f}}) \qquad (y \in G).$$

By Step 0 and the left invariance of the Haar measure, k_γ is integrable on G, and

$$\int_G |k_\gamma(y)|\,dy = \mathrm{vol}(\mathsf{K}) \int_{G_\infty} \big|\Psi\big(X_\gamma(y_\gamma)_\infty (r \oplus 1)\big)\big|\,dr. \tag{25}$$

For $\gamma = 1$, the same substitution $y \mapsto x\,y$ and the level of F give the main term:

$$\begin{aligned} \int_G k_1(y)\,F(y)\,dy &= \int_{G_\infty} \int_{\mathsf{K}} \Psi(r \oplus 1)\,F\big(x \cdot (r,k)\big)\,dk\,dr \\ &= \mathrm{vol}(\mathsf{K}) \int_{G_\infty} \Psi(r \oplus 1)\,F\big(x \cdot (r,1)\big)\,dr. \end{aligned} \tag{26}$$

Step 4: the vectors $f(\gamma)$, $\gamma \in \mathcal{W}$, are pairwise at distance at least Q^2. Let $\gamma \neq \gamma'$ in $\mathcal{W}$. The matrices $X_\gamma(y_\gamma)_{\mathrm{f}}$ and $X_{\gamma'}(y_{\gamma'})_{\mathrm{f}}$ lie in K^+, so their last columns, which by Step 1 are the finite components of the adelic vectors of η_γ and $\eta_{\gamma'}$, are congruent to b_* modulo a_0. Hence, a_0 being invertible in the finite adeles, $\eta_\gamma - \eta_{\gamma'} = a_0\,\eta$ for a lattice vector η of $V^+ \cdot (x \oplus 1)$. It is not zero: otherwise $\gamma_0^{-1}\gamma_0'$ would fix b_* and by (A4) be of the form $g \oplus 1$, so that γ and γ' would represent the same coset. At an infinite place v the components of $a_0\eta$ are those of η times the image of a_0 under the embedding chosen for v, of absolute value $|a_0|_v \geq \rho \geq 0$. With (A2), hypothesis (f), and $\lambda_1(V \cdot x) \leq 1$:

$$\begin{aligned} \|f(\gamma) - f(\gamma')\| \;&\geq\; \rho\,\|\eta\|_{V^+ \cdot (x \oplus 1)} \;\geq\; \rho\,\lambda_1\big(V^+ \cdot (x \oplus 1)\big) \\ &\geq\; \rho\,\min\big(\lambda_1(V \cdot x), 1\big) = \rho\,\lambda_1(V \cdot x) \;\geq\; Q^2. \end{aligned}$$

Step 5: the lattice sum. The vector $f(1)$ has the component b_* at each infinite place, so $\|f(1)\| = \sqrt{f_E} \leq f_E \leq Q$. By Step 4, $\|f(\gamma)\| \geq Q^2 - Q \geq \frac{1}{2}Q^2$ for $\gamma \in \mathcal{W} \setminus \{1\}$. The real dimension of $\mathbb{C}^{f_E(|\Omega|+1)}$ is $2f_E(|\Omega|+1) \leq A - 2$, so by (A3) with $m_0 = \frac{1}{2}Q^2$

$$\sum_{\gamma \in \mathcal{W} \setminus \{1\}} \|f(\gamma)\|^{-A} \leq \Big(\frac{10}{Q^2}\Big)^A. \tag{27}$$

Step 6: the other classes. Let $\gamma \in \mathcal{W} \setminus \{1\}$, and let $g = X_\gamma(y_\gamma)_\infty$. For $r \in G_\infty$ the matrix $g_v(r_v \oplus 1)$ is the component at v of $X_\gamma(y_\gamma \cdot (r,1))$, whose last column is the component $f(\gamma)_v$ of $f(\gamma)$. Hence $\|f(\gamma)_v\| \leq \|g_v(r_v \oplus 1)\|_{\mathrm{op}}$, and, these norms being at least 1,

$$\|f(\gamma)\| \leq \sqrt{f_E} \prod_v \|g_v(r_v \oplus 1)\|_{\mathrm{op}} \leq Q \prod_v \|g_v(r_v \oplus 1)\|_{\mathrm{op}}.$$

So (23) applies with $t_0 = \|f(\gamma)\|/Q$, and with (25) and $|F| \leq C$,

$$\int_G |k_\gamma(y)\,F(y)|\,dy \leq C \cdot \mathrm{vol}(\mathsf{K}) \cdot L_* \cdot Q^A\,\|f(\gamma)\|^{-A}. \tag{28}$$

Step 7: conclusion. For $\gamma \in \mathcal{W}$ the function $k_\gamma F$ is integrable on G, so by (A1) the family $\delta \mapsto \int k_\gamma(\delta y) F(\delta y)\, d\varpi_V(y)$ sums to $\int_G k_\gamma F$, and the same holds for the absolute values. By (28) and (27) the numbers $\int_G |k_\gamma F|$, $\gamma \in \mathcal{W}$, have a finite sum. So the double family of (24) is absolutely summable in the norm of $L^1(\varpi_V)$, the sum and the integral may be exchanged, and

$$\int \sum_{\gamma' \in \Gamma^+} k_{\gamma'}(y)\, F(y)\, d\varpi_V(y) = \sum_{\gamma \in \mathcal{W}} \int_G k_\gamma(y)\, F(y)\, dy.$$

The term $\gamma = 1$ is the main term, by (26). The others are bounded by (28) and (27): their sum has absolute value at most

$$C \cdot \mathrm{vol}(\mathsf{K}) \cdot L_* \cdot Q^A \Big(\frac{10}{Q^2}\Big)^A = C \cdot \mathrm{vol}(\mathsf{K}) \cdot L_* \Big(\frac{10}{Q}\Big)^A \le C \cdot \mathrm{vol}(\mathsf{K}) \cdot L_* \cdot \frac{10}{Q},$$

because $10/Q \le 1$ and $A \ge 1$. □

LEAN `norm_poincare_period_sub_le`

11 Where the results are used

The six theorems are consumed by three other units of the argument; the places were found by a search of the source for the formal names, and the statements of the consumers are those of their own notes.

- *Theorem C, the exchange step* (unit 147). The proof that every row pairs with a restriction of explicit level uses Theorem 6 for the unfolding of a period, and Theorem 2 for the data of Theorem 3, placed at every infinite place, to bound the main term of that period from below. There the exponent A of Theorem 2 is a power of the rank r of the rows.
- *Theorem B, the Bessel systems at the real places* (unit 148). A *real place* is an infinite place of the maximal totally real subfield of the CM field of the data; the completion of the CM field above it is $\mathbb{C}$, the unitary group of the rows there is $G_1(\mathfrak{s})$, and the notes on unit 132 treat it as the place $\mathbb{C}$ of their formalism and call it the complex place. The existence of a Bessel system at such a place uses Theorem 4 for a point x of size at most $b^{5n(\mathfrak{s})}$, Theorem 2 on the Bessel graph, with the measure of Definition 2.18 and Lemmas 9.1 and 9.5, for the parameter $\beta = b^3$, and Theorem 5 for the Bessel function. There $\mathfrak{s}$ is the shape of the admissible data D of the overview, with $\varrho(\mathfrak{s}) = m(D)/2 - 1$ and $|\kappa(\mathfrak{s})| = k(D)$, so that $r(\mathfrak{s}) = r(D)$, and b is the third scale $b(D)$. The predicate of Definition 2.21 is the conclusion of that statement.
- *Theorem B, the cut of a Bessel function* (unit 132). A peaked Bessel function, in the sense of Definition 2.20 with a point of size at most $b^{5n(\mathfrak{s})}$, is cut to a pair of a full and a cut Bessel function; the proof integrates by Lemma 9.3.

Theorem 1 is used only inside this unit, in the proof of Theorem 2.

12 Notes on the formal text

Two norms. The formal text uses two norms on matrices through scoped instances, and a statement does not show which one it means. The files on two-factor data and on minors use the Hilbert–Schmidt norm throughout; the files with Theorems 2, 3, 6 and with the second half of the Bessel statements use the operator norm; the file with Theorem 1 opens both. That the hypothesis $\|g - 1\|_{\mathrm{HS}} \le \delta$ of Theorem 1 is a Hilbert–Schmidt bound is forced by the text: the proof of Theorem 2 passes the conclusion of Proposition 6.7, which is elaborated with the Hilbert–Schmidt norm, to Theorem 1 as this hypothesis, and the proof of Theorem 1 passes it on to Proposition 4.2. The second reader of the project reached the same conclusion. Nothing was compiled for this note, so this is a reading of the source, not Lean's answer.

Statements. (1) In Theorem 1 the last three hypotheses carry default proofs in the formal text, so that a caller may omit them; they are hypotheses all the same. (2) The formal Theorem 2 asserts only the inequality, with the measure of the ball read as a real number; that the ball has finite measure and that the integrand is integrable is shown in its proof and added to the statement here. (3) In Theorem 3 the hypothesis that M is even follows from the other hypotheses: e and $\tilde{e}$ are bijections, so the numbers of positive signs counted through them, $|\Xi| + \lceil M/2 \rceil + 1$ and $|\Xi| + \lceil (M+1)/2 \rceil$, agree, which forces M to be even. The gap of l' enters only through $\Lambda(-l') = -\Lambda(l')$, and its size H is arbitrary. (4) In Proposition 4.1 and Proposition 4.2 the constants 100 and 200000 are not sharp; the proofs above give 45 and 19302. (5) In Lemma 6.4 the formal text takes the gap as an instance argument, and it does not state the invertibility of the block-diagonal part. (6) In Lemma 9.3 the point x is an implicit argument. (7) In Definition 2.13 the formal text writes (T6) as a statement about all pairs of indices; the form above is equivalent. (8) In Theorem 6 the eight auxiliary objects (Ψ, its restriction, L_*, the sum of spaces, the embedding, the volume, X_γ and the summand) are bound by `let` inside the statement; the final step loses a factor $(10/Q)^{A-1}$. The sum over γ is Lean's unconditional sum, which is 0 where the family is not summable; by Step 7 this happens at most on a null set. (9) The definitions of a frame, of a peak and of the type of a Bessel system are structures; the frame is a type class, and the signs are an output parameter of it.

The arguments as the census reads them. The reading project's census records the arguments of each declaration from the text. For ten declarations of this unit its list differs from the one Lean must have, because a name bound inside the statement coincides with a variable of the file, or because a `let` or an arrow of homomorphisms was misread: the pair $j-1$, two-factor data, Lemmas 6.1, 5.1 and 9.3, Proposition 4.2, the pair homomorphism, the peak, and Theorems 4 and 5. The statements above follow Lean's text, as the first reader's notes describe the expected lists; a confirmation by Lean itself (`#check`) is pending. In particular the measure of the Bessel graph and the peak do not depend on an element of the graph, and Theorem 4 does not take a point x as an argument: it provides one.

The dossier. The first reader's text for the right invariance of the measure of the Bessel graph named only the conjugation; the second reader pointed out the right translation by u_0, and Lemma 9.1 has both. In the proof of Theorem 2 the source leaves the two quantities of the case distinction to unification; they are $\gamma(\sigma)^{8\mathfrak{n}}$ and $\beta^{-50\mathfrak{n}}$, which is the only choice for which the next rewriting steps of the source apply (the second reader's message makes the arithmetic explicit in a slightly different way; both lead to the same statement).

How this note was written. From the dossier of the unit, after its program check passed: all 17 structured proofs were read, and all five Lean files of the unit were read in full. Line by line, against the structured proofs, we read the proofs of Propositions 4.1, 4.2 and 6.7 and of Theorems 1, 2, 3, 4 and 5. For Theorem 6 the statement was read in full and the proof once; in its steps on the choice of representatives and on the exchange of sum and integral we follow the first reader's reconstruction and the second reader's verdict, and did not trace every rewriting. The four statements that the census cuts at a `let` (Lemma 5.1, Proposition 4.2, Theorems 3 and 6) were read in full in the source. The definitions of the background library that the statements rest on (parameters, Flensted-Jensen's function, the shape and its groups, frames, Bessel functions, the size) were read in the source; those of the adelic groups are taken from the dossier's readings. No Lean was run.

Several arguments here are shorter than, or different from, the formal ones. In Proposition 4.1 the formal proof verifies one identity between traces by a linear combination of the hypotheses; the three terms $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ above are a derivation of that identity. In Proposition 6.7 the formal proof checks the unitarity of u and the trace identity of Step 4 by linear combinations; the block form and the argument with the vector (v, v') are ours. Both identities, and the two of Proposition 4.1, were also tested numerically on random matrices, which is a check of the transcription and no proof. The numerical estimates in Theorems 1 and 2 and in Proposition 4.2 are done by hand here with other intermediate constants than the formal ones. In Theorem 4 the vectors $V^{(k)}$ and the description of the chain matrix are our reading of the matrices of the formal proof. The program check of the unit's dossier covered 235 leaf steps of 17 proofs.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Definition 2.15	`Frame`	BesselExistence.lean	16–17
Definition 2.21	`Filling.IsBessel`	BesselExistence.lean	21–26
Definition 2.16	`Shape.besselGraph`	BesselExistence.lean	34–37
Definition 2.17	`Shape.besselGraphChart`	BesselExistence.lean	45–47
Definition 2.18	`Shape.besselGraphMeasure`	BesselExistence.lean	51–51
Lemma 9.1	`Shape.besselGraphMeasure_finiteOnCompacts`	BesselExistence.lean	56–57
Lemma 9.1	`Shape.besselGraphMeasure_isMulRightInvariant`	BesselExistence.lean	59–63
Lemma 9.2	`besselGraph_fst_mem_besselS`	BesselExistence.lean	67–68
Lemma 9.3	`lintegral_unfold_besselGraphMeasure`	BesselExistence.lean	73–77
Lemma 9.4	`besselGraph_trace_mul_sub_eq_zero`	BesselExistence.lean	83–87
Definition 2.19	`besselPairHom`	BesselExistence.lean	91–93
Definition 2.20	`HasPeakAt`	BesselExistence.lean	98–103
Theorem 4	`IsSteep.exists_isTwoFactor_besselPairHom`	BesselExistence.lean	108–173

In this note	Lean declaration	File	Lines
Lemma 9.5	`isProperMap_fst_besselPairHom`	BesselExistence.lean	177–179
Theorem 5	`exists_isBesselFunction`	BesselExistence.lean	185–225
Definition 2.8	`gapWeightedNormSq`	GaussianShape.lean	16–16
Lemma 5.1	`sum_norm_lie_coordProj_sq`	GaussianShape.lean	18–25
Lemma 5.2	`norm_weightMinors_sq_le_exp`	GaussianShape.lean	27–41
Theorem 1	`dsCoeff_gaussian_shape`	GaussianShape.lean	48–99
Proposition 4.1	`exists_det_compress_eq_exp_logOneAdd`	MinorExpansion.lean	18–44
Proposition 4.2	`exists_principalMinor_eq_exp`	MinorExpansion.lean	49–109
Theorem 6	`norm_poincare_period_sub_le`	RestrictionLemma.lean	25–124
Theorem 2	`IsTwoFactor.laplace_method`	RestrictionLemma.lean	134–192
Theorem 3	`IsSteep.exists_isTwoFactor_restriction`	RestrictionLemma.lean	200–243
Definition 2.9	`HCParam.phase`	TwoFactorSetup.lean	19–20
Definition 2.10	`IsSteep`	TwoFactorSetup.lean	24–29
Definition 2.11	`subOnePair`	TwoFactorSetup.lean	33–33
Definition 2.12	`HCParam.laplaceIntegrand`	TwoFactorSetup.lean	37–37
Definition 2.13	`IsTwoFactor`	TwoFactorSetup.lean	42–50
Lemma 6.2	`exists_unitary_sub_one_eq_inclMatrix`	TwoFactorSetup.lean	54–57
Lemma 6.1	`norm_sq_add_re_trace_le`	TwoFactorSetup.lean	59–68
Lemma 6.6	`IsTwoFactor.phase_logOneAdd`	TwoFactorSetup.lean	72–84
Lemma 6.3	`dsCoeff_mul_eq_of_zero_weight_block`	TwoFactorSetup.lean	88–100
Lemma 6.4	`norm_dsCoeff_pow_four_le`	TwoFactorSetup.lean	102–114
Proposition 6.7	`IsTwoFactor.exists_near_one`	TwoFactorSetup.lean	119–148

Legend. The formal names of the variables, and the letters of this note. Index types `n`, `m`, `X`: Ω, Ω', Ξ. Signs `ε`, `n`: ε, ε'. Parameters `l`, `k`: l, l'; in the Bessel statements `u` is l'. The map `τ` of two-factor data: e'; the bijections `e'` of Theorem 3 and `τ` of Theorem 4: $\tilde{e}$. Elements `s`, `t` of S: σ, σ'. The radius `r`: δ. Gaps `h`: H if real, A if natural. Projections `P`, `Q`: p, q; errors `r`: η; elements `k` of the maximal compact subgroup: h. The frame `p`: $\mathfrak{p}$; the vector `v`: v_0; in Theorem 5 the exponent `n`: N. In Theorem 6, `n`, `s`: Ω, ε; `S`, `a`, `R`, `d`: C, a_0, ρ, Q; places `w`: v; `φ`: Ψ; `L`: L_*; `h`: y. Objects of the background library: `USign`, `maxCompactU`, `blockDiagPart`, `inclMatrix`, `logOneAdd`, `coordProj`, `principalMinor` are $\mathrm{U}(\varepsilon)$, $\mathrm{K}(\varepsilon)$, $\mathrm{bd}_\varepsilon(\cdot)$, in_ι, $\lg(\cdot)$, p_Π, $\Delta_\Pi(\cdot)$; `paramAddRho`, `lowestKType`, `weightMinors`, `flenstedJensen`, `dsCoeff`, `formalDegree` are $\Upsilon(l)$, $\Lambda(l)$, $\mathfrak{m}_w(\cdot)$, $\mathrm{FJ}_l(\cdot,\cdot)$, $\psi_l(\cdot)$, $\deg(l)$; `besselU`, `besselH`, `besselS`, `embedH`, `simpleRootSum`, `besselHaar`, `besselQuotMeasure`, `besselSize` are $\mathcal{U}(\mathfrak{s})$, $\mathcal{H}(\mathfrak{s})$, $\mathcal{Q}(\mathfrak{s})$, $\mathrm{emb}_\mathfrak{s}(\cdot)$, $\mathrm{sd}(\cdot)$, $\eta_\mathfrak{s}$, $\dot{\eta}_\mathfrak{s}$, $\mathrm{size}_Q(\cdot)$.

Two statements, as the artifact has them. Theorem 1:

```
lemma dsCoeff_gaussian_shape [Nonempty n]
    (hg : l.HasLargeGap A) (hL : ∀ a, |l a| ≤ L) {g : USign ε} (hZ : ‖g.1.1
  - 1‖ ≤ δ) (hT : 1 ≤ T) (a : card n ^ 2 ≤ T := by nlinarith) (b : card n ^ 2 * Real.log (card n +
  L + 1) ≤ T := by bound) (hδ : δ * T ≤ 1e-12 := by bound) : ∃ κ ∈ Set.Icc (0 : ℝ) 1, ‖l.dsCoeff g /
  (intDiagonal l.lowestKType * logOneAdd (g.1.1 - 1)).trace.exp - κ‖ ≤ κ / 10 + (-100 * T).exp
```

Theorem 2:

```
lemma IsTwoFactor.laplace_method {n m X S} [Fintype n] [Fintype m] [Fintype X] [Group S]
  [TopologicalSpace S] [MeasurableSpace S] [MeasurableMul S] [BorelSpace S] (v : Measure S)
  [v.IsMulRightInvariant] [IsFiniteMeasureOnCompacts v] {ε l n k τ} {B β : ℝ} {L M h : ℕ} [Nonempty
  n] [Nonempty m] {e : X ⊕ Fin M ≃ n} {j : S →* USign ε × _} {φ} (H : IsTwoFactor l k j τ B e φ)
  (hg : l.HasLargeGap h) (hg' : @HasLargeGap m _ n k h) (hB : 1 ≤ B) (hL : ∀ a, |l a| ≤ L) (hK :
  ∀ a, |k a| ≤ L) (hβ : 10 ^ 12 ≤ β) (hn : card n + card m + L ≤ β) (hh : 10 ^ 30 * ((card n + card
  m) ^ 2 * β.log) ^ 3 ≤ h) (hj : IsProperMap (j · |>.1)) (hj' : Continuous (j · |>.2)) (lip :
  ∀ s, ‖(subOnePair j s).1‖ ≤ 1 → ‖(j s).2.1.1‖ ≤ β + 1) : v.real {s | ‖(subOnePair j s).1‖ ≤ 1} ≤
  β ^ (20 * (card n + card m) ^ 2) * ‖∫ s, l.laplaceIntegrand k j φ s ∂v‖
```

Notation

Ω, Ω', Ξ, Ξ'; i, i', i''; ξ finite index sets; elements of Ω or Ω'; an element of Ξ

Ω_1, Π, Π' subsets of Ω (the complement of $e(\Xi)$; lower and upper sets)

ε, ε', $\varepsilon \oplus 1$, 1 families of signs; ε extended by $+1$ at $*$; the constant family

X, Y, Z, D; X^*, $[X, Y]$ matrices; conjugate transpose; commutator

$\|X\|_{\mathrm{HS}}$, $\|X\|_{\mathrm{op}}$, $\|v\|$, tr Hilbert–Schmidt norm, operator norm, Euclidean norm of a vector; trace

$\mathrm{diag}(w)$, $\mathrm{bd}_\varepsilon(X)$, in_ι, ι diagonal matrix; block-diagonal part; the matrix of a map $\iota\colon \Xi \to \Omega$

$\mathrm{U}(\varepsilon)$, $\mathrm{K}(\varepsilon)$; g, $\tilde{g}$, g_0, g_1, g_2; h, h_0, h_1 unitary group and its maximal compact subgroup; their elements

dg, dh, dr, dk, vol Haar measures; volume

$\lg(X)$, s the logarithm of $1 + X$; a real variable of integration in $[0, 1]$ (not the complex variable of the write-up)

p, q, p', p_Π, p_τ; $\Delta_\Pi(y)$ orthogonal projections; coordinate projections; principal minor

w, v; $|w|_1$; τ, c_1, c_2 integer vectors (in Definition 2.21, w is a weight); sum of absolute values; integers

l, l', l_v; λ, ρ, ρ_c, μ integer vectors, read as doubled Harish-Chandra parameters; $l/2$, half sums of roots, and the name of $\Upsilon(l)$ in the source (§2 only)

$\Upsilon(l)$, $\Lambda(l)$; Υ, Λ, Λ' $\lambda + \rho$ and the lowest K-type; their values in a proof

H, A, A_0, L a real gap; natural numbers: large gaps, a bound for the entries of l

$\mathfrak{m}_w(g)$, $\|\mathfrak{m}_w\|^2_{\mathrm{K}(\varepsilon)}$, c_Λ product of principal minors; its square norm on $\mathrm{K}(\varepsilon)$

$\mathrm{FJ}_l(g, z)$, $\psi_l(g)$, $\deg(l)$, ψ_l^{p} Flensted-Jensen's integral, his function, the formal degree; the function in a frame

$\mathrm{gw}_w(X)$ the gap-weighted square norm

$\mathrm{ph}_{l,l'}\, x$; $x = (x_1, x_2)$ the phase of a pair of matrices

$\mathrm{Steep}(l; B, e)$; B, M, e steep weights; a real number, a natural number, a bijection onto Ω

S; σ, σ', σ_0; $j = (j_1, j_2)$ a group; its elements (in §9, $\sigma = (\sigma_1, \sigma_2)$); a homomorphism

ϕ, $\phi_{\mathfrak{s}}$, e', $\tilde{e}$ phase functions; a map $\Xi \to \Omega'$; a bijection extending such a map

$\theta^{j,\phi}_{l,l'}(\sigma)$ the integrand of two coefficients

$\mathrm{TwoFactor}(l, l'; j, \phi; e, e'; B)$ two-factor data

ν, β, β_0, T, δ, δ_0, κ_{G}, κ_0, κ_1, κ_2 a measure; real parameters; a radius; Gaussian factors (κ_0 a norm in §4); the bare letter κ occurs only in $\kappa(\mathfrak{s})$

c, C, R, K, D, P, t_0, m_0, d_0, n_0 real numbers and bounds, natural numbers, as each statement says

η, η_1, η_2, η_3, η_τ, $r(h)$, ϵ error terms (in §10, η, η_γ are vectors)

$\mathcal{J}(s)$, $\mathcal{J}_n$, W, V, A, A_c, A_n, E, R, G, X_1, $X^\uparrow$, $X^\downarrow$, α_0 in §4: integrands, inverses, commutators and their parts, a norm

E, q_0, $G_0(h)$, a in §5: the exponent of the phase, the exponent of κ_{G}, a sum; an eigenvalue

χ, c_0, J, J_2, $g_{11}, \dots, g_{22}$, u, T, T', Q, Y_σ, v' in §6: an indicator, rows of $g - 1$, sign matrices, blocks, a unitary matrix, $j(\sigma') - 1$, $\mathrm{in}_{e|_\Xi}$, a pair

n, $\gamma(\sigma)$, $\mathrm{dist}(\sigma)$, m, J, U_0, z, z_n, E_1, E_2 in §7: $(|\Omega| + |\Omega'|)^2$, $2/\|j_1(\sigma)\|_{\mathrm{op}}$, a distance, the measure of the ball, the integral, a small ball, complex numbers

τ_n, c_n, π, π_a, d_a, ρ_{na}, ω, j^ω, b_i, b_*, v nodes, Lagrange weights, polynomials and their values; the element of Theorem 3 and its homomorphism; standard basis vectors; a column

$\mathfrak{s}$; $\varrho(\mathfrak{s})$, $\kappa(\mathfrak{s})$, $n(\mathfrak{s})$, $r(\mathfrak{s})$, $\mathrm{Ind}(\mathfrak{s})$, ϱ a shape and its numbers and index sets

e_a, x_i, z, e^*_a, f_a; a, a', b, c, k basis and frame of a shape; indices of basis vectors (also: a a weight, b a real number)

$J_1(\mathfrak{s})$, $G_1(\mathfrak{s})$, $\mathrm{lev}(i)$, $\mathcal{U}(\mathfrak{s})$, $\mathcal{H}(\mathfrak{s})$, $\mathcal{Q}(\mathfrak{s})$, $\mathrm{emb}_{\mathfrak{s}}(h)$, $\mathrm{sd}(g)$ Gram matrix, unitary group, level, unipotent group, embedded compact group, Bessel subgroup, embedding, sum above the diagonal

$\eta_{\mathfrak{s}}$, $\dot{\eta}_{\mathfrak{s}}$, $\mathrm{size}_{\mathcal{Q}}(x)$, $C_{\mathfrak{s}}$ Haar measure of $\mathcal{U}(\mathfrak{s})$, quotient measure, size, chain matrix

$\mathrm{Frm}(\mathfrak{s}, \varepsilon)$, p, p_i, q frames; a frame, its columns; in §9, $q = x\,\mathsf{p}$

$\mathsf{HU}(\mathfrak{s})$, $\mathrm{ch}_{\mathfrak{s}}$, $\nu_{\mathfrak{s}}$, $\jmath_x(\sigma)$ the Bessel graph, its chart, its measure, the pair homomorphism

I, rep, W, v_0, x, y, N an index set, a unitary representation, a Bessel function, a unit vector, points of $G_1(\mathfrak{s})$, the exponent of a peak (the same letter as in the note on the complex place)

n an index of summation or of a node; $n(\mathfrak{s}) = \varrho(\mathfrak{s}) + 1$

$m(D)$, $k(D)$, $r(D)$, $b(D)$ numbers and third scale of the admissible data D of the overview (only in §11)

d, λ, K, f, ρ local to this note and different from the letters of the overview: values d_a and bounds d_0; $l/2$ and the first minimum λ_1; bounds and compact groups; functions and f_E; a half sum of roots (§2) and a real number (Theorem 6)

$\mathrm{Peak}_{l,\mathrm{rep}}(W; x, P)$ peak

$\mathrm{BesType}(\mathcal{B};l,w)$, $\mathcal{B}$, $I(\mathcal{B})$, $W_{\mathcal{B}}$, $\mathbb{S}_w$, f type of a Bessel system; its components; a Schur matrix

$V^{(k)}$, V_a, M_0, D, $F(y,\sigma)$, $k(g)$, c, y_1, y_2 in §9: vectors, a bound, a conjugated weight matrix, an integrand, a kernel, an exponent, points

E, f_E, v, $|a|_v$, a_0 in Theorem 6: a CM field, its number of infinite places, such a place, its absolute value, the level

$G(E,\varepsilon)$, G_∞, G_{f}, $K_\Omega(a_0)$, K, K^+ adelic groups and congruence subgroups

V, J_V, x_V, $V\cdot x$, V^+, $\langle 1\rangle$, $\Gamma(V)$, Γ^+, ϖ_V, λ_1, $\|\eta\|_V$ a space with a point, its parts, its translates and sum, discrete groups, quotient measure, first minimum, length

F, C, ρ, Q, α, ω, Ψ, L_*, X_γ, k_γ, γ, γ_0, δ, r, k in Theorem 6: an automorphic form, bounds, positions, the test function, a constant, group elements

$\mathcal{C}$, $\mathcal{W}$, y_γ, $f(\gamma)$, f_n, η_γ, g, o representatives, points, vectors; in (I4), o is an ordering function

References

References are given as the second reader of the project and the catalogue of the write-up record them; none was typed from memory, and the places not marked as confirmed there are to be confirmed.

- U. A. First, *Witt's extension theorem for quadratic spaces over semiperfect rings*, J. Pure Appl. Algebra 219 (2015); Remark 2.1 and Corollary 4.9 (preprint arXiv:1408.0522). For (I6).
- M. Flensted-Jensen, *Discrete series for semisimple symmetric spaces*, Ann. of Math. 111 (1980). For (I7); place to be confirmed.
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- Harish-Chandra, *Discrete series for semisimple Lie groups II*, Acta Math. 116 (1966), pp. 87–88. For (I12); relation unresolved.
- O. Kallenberg, *Invariant measures and disintegrations with applications to Palm and related kernels*, Probab. Theory Related Fields 139 (2007), 285–310; D. Gentner and G. Last, *Palm pairs and the general mass-transport principle*, §2. For (I13)(d) and (A1); relation unresolved.
- The header of the formal text of the unit names as related work: P. Nelson and A. Venkatesh, *The orbit method and analysis of automorphic forms* (2021); H. He, *On the Gan–Gross–Prasad conjecture for* $\mathrm{U}(p,q)$ (2017); B. Sun (2007), as above. We have not compared them with the statements of this note.

Chapter VII

Theorem C: integral forms are long

- **The rows: target forms on an indefinite U(r), the volume of the chain of lifts as a product of lift volumes, and the height of each lift volume** (unit 143)
  read: note built, structured proofs checked. 219 declarations, 1\,454 proof lines.

- **The exchange identity: theta integrals of complementary rank agree up to a constant, and cruder forms complete a family to a basis in the lower bound** (unit 147)
  read: note built, structured proofs checked. 305 declarations, 1\,947 proof lines.

- **Good idempotents, labels and the transfer of traces between unitary groups** (unit 140)
  read: note built, structured proofs checked. 46 declarations, 163 proof lines.

- **The finite test datum of an auxiliary pair and the theta lift as a conjugate descent** (unit 134)
  read: note built, structured proofs checked. 16 declarations, 56 proof lines.

- **Vanishing of theta descents: reflection sums at the level, and climbing the theta tower** (unit 146)
  read: note built, structured proofs checked. 8 declarations, 186 proof lines.

- **Degree-uniform decay of Heisenberg matrix coefficients of homogeneous Fock vectors** (unit 142)
  read: note built, structured proofs checked. 1 declarations, 19 proof lines.

- **At most one primitive label per Hecke eigensystem at a chosen place (via Galois representations of lifts)** (unit 139)
  read: note built, structured proofs checked. 2 declarations, 33 proof lines.

The volume of the chain of theta lifts as a product of lift volumes

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The rows of the proof are the good automorphic forms on an indefinite unitary group of rank r, and the chain of lifts sends them, through m theta lifts, to algebraic modular forms on a definite unitary group. This note proves that the volume of the chain on the rows, the determinant of its Gram operator, is a constant to the power of the dimension of the rows times the product of the volumes of m single lifts, each taken at a definite group of fixed rank and at the matching power of the twist. The constant depends on the twist, but only through quantities whose ratio to the untwisted one has controlled height. The proof compares Gram forms: by the artifact's inner product formula, a coset-sum form of the Rallis inner product formula, each Gram form is a constant times a Hecke operator, and the same Hecke operators occur on both sides.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles with Lean v4.35.0 against a pinned version of Mathlib and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note is the first of four on one unit of the artifact, the folder of the rows; it treats 8 of the unit's 219 declarations, those of the file on the volume of the chain. Theorem A and Lemma 3.1 are proved in full from Inputs 1 to 7, which belong to other units and are stated in a box and not proved here; Lemmas 3.2 to 3.4 are proved in full from two further facts of Chapter VIII, which their proofs name. Every proof of this note is complete relative to the inputs it names. Every statement below was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 8 declarations.

1 The statement

Theorem C of the handout bounds a height from below by comparing the chain of lifts at the trivial twist with the chain at well-chosen twists. It needs to know how the volume of the chain depends on the twist. The answer has two parts: a factorisation, which is Theorem A here, and a bound for each factor, which is the subject of the note *The height of a ratio of lift volumes*. The objects are those of Setting 2.1 and Definition 2.2.

Theorem A (the volume of the chain is a product). *In Setting 2.1, for all parameters P and all admissible data D for P, with number of lifts m, there is a function c from the m-fold twists of D to $\mathbb{C}$ with the following two properties.*

(a) *For every family $\theta = (\theta_{\ell,\iota})$ of m-fold twists of D that is fit for D, the family $(\ell, \iota) \mapsto |c(\theta_{\ell,\iota})/c(1)|^{-2}$ has height at most $m^2 \cdot 2\,t(D)\,x^{999}$, where x is the first scale of D and $t(D) = 5r$.*

(b) *For every m-fold twist ϑ of D,*

$$\mathfrak{z}_D(\vartheta) \neq 0 \qquad \text{and} \qquad \mathfrak{z}_D(\vartheta) = |c(\vartheta)|^{\dim \mathrm{Row}(D)} \cdot \prod_{j=1}^{m} \mathrm{vol}_D(\vartheta, j).$$

LEAN `Data.volume_iota_eq_prod_liftVolume` · property (a) is the formal text's "c costs $m \cdot m$ squares" (Input 7)

Here $\mathfrak{z}_D(\vartheta)$ is the volume of the chain $\Theta_{D,\vartheta}$ of m lifts on the rows, and $\mathrm{vol}_D(\vartheta, j)$ is the volume of the j-th single lift, formed with the twist ϑ^j, on the good forms of its own source. The exponent $\dim \mathrm{Row}(D)$ appears because all the volumes are determinants on spaces of this one dimension. Since $\dim \mathrm{Row}(D) > 0$ (Theorem B of the handout), property (b) gives $c(\vartheta) \neq 0$ for every m-fold twist ϑ, so that the family in (a) is defined. The formal text does not need this remark, since it reads a quotient by zero as zero.

2 The setting

Conventions. Inner product spaces are complex, and an inner product $\langle y', y\rangle$ is conjugate-linear in y' and linear in y. For a linear map u between finite-dimensional inner product spaces, u^* is its adjoint. The implicit arguments of the declarations of this note are the parameters P (in all of them) and, in Lemmas 3.1 and 3.2, an inner product space U and, in Lemma 3.1, a second one U' and the scalar a; in Lemma 3.2 also the homomorphism ϱ and the tuple $\mathbf{c}$; in Lemmas 3.3 and 3.4 the tame character ϑ and the lift index j, in the first of them the element ξ and in the second the subspace $\mathcal{N}$. Each of them is introduced again in the statement that uses it.

Setting 2.1 (data, forms and lifts; all taken from other units)**.** *Data.* $P = (N_0, d)$ are parameters and D is admissible data for P (Setting 1 of the overview of the handout; the notes on Chapter III). Attached to D are the CM field $E(D)$ with its ring of integers $\mathcal{O}$, the number m of lifts, the rank r, the step $t(D) = 5r$, the first scale x, the second level $g(D) = g_0 g_1$ and the level $N(D)$, a prime (Setting 2 of the overview), the finite set of *chosen primes* of $E(D)$, and the finite group $\mathsf{G}(D)$, the product of the groups GL_r of the residue fields of the chosen primes. A *lift index* is an integer j with $1 \le j \le m$.

Spaces. $Y_0(D), \dots, Y_m(D)$ are hermitian spaces over $E(D)$, the space $Y_i(D)$ of rank $r + i\,t(D)$; $Y_0(D)$ is indefinite and $Y_m(D)$ is definite. For a lift index j, $W_j(D)$ is a diagonal hermitian space of rank r.

Forms of principal level. For a hermitian space V of rank r over $E(D)$, $\mathcal{M}_D(V)$ is the space of automorphic forms on the unitary group of V of a fixed type at the infinite

places and fixed by the principal congruence subgroup at the chosen primes. It is an inner product space under the Petersson product, the group algebra $\mathbb{C}[\mathsf{G}(D)]$ acts on it, and it is finite-dimensional for $V = Y_0(D)$ and for $V = W_j(D)$.

Good forms and rows. A *tuple of labels* is a family of monic polynomials of degree r, one over the residue field of each chosen prime. Each tuple $\mathbf{c}$ has a central idempotent $\mathrm{idem}(\mathbf{c})$ of $\mathbb{C}[\mathsf{G}(D)]$; idempotents of different tuples are orthogonal. A finite set of tuples is called *good*, and $e_0(D)$ is the sum of $\mathrm{idem}(\mathbf{c})$ over the good tuples $\mathbf{c}$; it is a central idempotent. The *good forms* on V are $\mathcal{M}_D^\circ(V) = e_0(D)\,\mathcal{M}_D(V)$, and the *rows* are $\mathrm{Row}(D) = \mathcal{M}_D^\circ(Y_0(D))$.

Hecke operators. $\mathbb{T}_D$ is the Hecke algebra of D: an algebra whose elements T act on every $\mathcal{M}_D(V)$, by an endomorphism T_V that preserves $\mathcal{M}_D^\circ(V)$; it contains the Hecke operators at the primes away from the level. A *Hecke eigensystem* is a function p that assigns a complex number to each of these Hecke operators, and $\mathcal{M}_D(V)[p]$ is the space of the forms on which each of them acts by the value of p.

Twists. A *tame character* of D is a character ϑ of the norm-one ideles of $E(D)$, trivial on the norm-one elements of $E(D)$ and at the infinite places, at most tamely ramified, with a finite set $Q(\vartheta)$ of primes outside of which it is unramified; $Q(\vartheta)$ contains the complex conjugate of none of its members, and the norms of its members are prime to $g(D)N(D)$. It is an *m-fold twist* if ϑ^n is ramified at every prime of $Q(\vartheta)$ for $1 \le n \le m$. The trivial character 1 is an m-fold twist with $Q(1) = \emptyset$, and every power ϑ^n is a tame character with $Q(\vartheta^n) = Q(\vartheta)$. For an integer $e \ge 0$, $\mathfrak{m}_D(\vartheta, e)$ is the ideal of $E(D)$ equal to the e-th power of the product of the chosen primes times the product of the primes in $Q(\vartheta)$, and $|\mathcal{O}/\mathfrak{m}_D(\vartheta, e)|$ is the number of elements of its residue ring.

Single lifts. For a tame character ϑ and a lift index j, the *auxiliary pair* $O_D(\vartheta, j)$ consists of $W_j(D)$, of a definite hermitian space of rank $r + t(D)$ and of a character built from ϑ; $\mathcal{F}(O_D(\vartheta, j))$ is its space of algebraic modular forms on the definite space, a finite-dimensional inner product space on which $\mathbb{C}[\mathsf{G}(D)]$ acts (the note *Auxiliary dual pairs, determinant polynomials and the carrier*). The *adjoint lift*

$$\mathrm{lift}_D^*(\vartheta, j)\colon \mathcal{M}_D(W_j(D)) \longrightarrow \mathcal{F}(O_D(\vartheta, j))$$

is the adjoint of a theta lift from the definite space to $W_j(D)$.

The chain. For an m-fold twist ϑ, the linear map $\mathrm{chain}_D(\vartheta)\colon \mathcal{M}_D(Y_0(D)) \to \mathcal{A}_{D,\vartheta}$ is the composite of m theta lifts $Y_0(D) \to Y_1(D) \to \ldots \to Y_m(D)$, the i-th formed with ϑ^i, with values in a finite-dimensional inner product space $\mathcal{A}_{D,\vartheta}$ of algebraic modular forms on the definite space $Y_m(D)$.

Heights and fit families. A family $Z = (Z_{\ell,\iota})$ of complex numbers, indexed by the pairs of a prime ℓ and a field isomorphism ι from $\mathbb{C}$ to an algebraic closure of $\mathbb{Q}_\ell$, has *height at most h*, for a real number h, if there are $H \in \mathbb{C}$ with $H \neq 0$, a real number α and real numbers η_ℓ with $|\sigma(H)| \le e^\alpha$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, $|\iota(H\,Z_{\ell,\iota})|_\ell \le e^{\eta_\ell}$ for all (ℓ, ι), and such that α plus the sum of the η_ℓ over the primes ℓ below any sufficiently large bound is at most h. A family $\theta = (\theta_{\ell,\iota})$ of m-fold twists of D, with the same index set, may be *fit for D*; the definition belongs to Chapter VIII and enters this note only through Input 7.

Definition 2.2 (volumes)**.** For a linear map u between finite-dimensional inner product spaces, the *volume factor* $\mathrm{vf}(u)$ is the real number ≥ 0 with $\mathrm{vf}(u)^2 = \det(u^* u)$; it vanishes exactly when u is not injective. In Setting 2.1:

(i) for an m-fold twist ϑ of D and a lift index j, the *lift volume* is

$$\mathrm{vol}_D(\vartheta, j) = \mathrm{vf}(u)^2 = \det(u^* u),$$

where $u \colon \mathcal{M}^\circ_D(W_j(D)) \to \mathcal{F}(O_D(\vartheta^j, j))$ is the restriction of $\mathrm{lift}^*_D(\vartheta^j, j)$, the adjoint lift for the twist ϑ^j, to the good forms;

(ii) for an m-fold twist ϑ of D, the *chain of lifts on the rows* is the restriction

$$\Theta_{D,\vartheta} = \mathrm{chain}_D(\vartheta)\big|_{\mathrm{Row}(D)} \colon \ \mathrm{Row}(D) \to \mathcal{A}_{D,\vartheta};$$

(iii) for an m-fold twist ϑ of D, the *volume of the chain* is

$$\mathfrak{z}_D(\vartheta) = \mathrm{vf}(\Theta_{D,\vartheta})^2 = \det(\Theta^*_{D,\vartheta}\, \Theta_{D,\vartheta}).$$

Both volumes are real numbers ≥ 0, regarded as complex numbers.

LEAN `Data.liftVolume` · `Data.θ` · `Data.𝔷`

Inputs from other units. The seven results below are taken for granted. They are stated as the dossier of this unit reads them; the notes on their units prove them. In all of them P and D are as in Setting 2.1.

Input 1 (unit *Exchange*; the inner product formula for one lift). For every lift index j there is $A_j \in \mathbb{C}$, $A_j \neq 0$, such that for every tame character ϑ of D the map $\mathrm{lift}^*_D(\vartheta, j)$ is injective and, for all $y', y \in \mathcal{M}_D(W_j(D))$,

$$\langle \mathrm{lift}^*_D(\vartheta, j)\, y', \mathrm{lift}^*_D(\vartheta, j)\, y \rangle = A_j\, \delta_D(\vartheta) \sum_{\gamma} R_D(\vartheta, j, W_j(D); y', y; \gamma).$$

Here the terms $R_D(\vartheta, j, V; y', y; \gamma)$, defined for every hermitian space V of rank r and all $y', y \in \mathcal{M}_D(V)$, are those of the artifact's coset-sum form of the Rallis inner product formula (a matrix coefficient of a translation at a coset γ, weighted by a local factor that depends on ϑ and j; this note calls them Rallis terms), the series converges unconditionally, and $\delta_D(\vartheta) \neq 0$ is the *Gram factor*: a ratio of indices and orders of finite groups that depends only on the chosen primes and on $Q(\vartheta)$.

Input 2 (unit *Exchange*; the first lift of each row of the triangle). For every m-fold twist ϑ there are a family of inner product spaces of forms $F_\vartheta(i, n)$, for $i + n \leq m$, with linear maps $a_\vartheta(i, n) \colon F_\vartheta(i, n) \to F_\vartheta(i, n+1)$, which are theta lifts, isometric embeddings $\jmath_i \colon \mathcal{M}_D(Y_0(D)) \to F_\vartheta(i, 0)$, and positive integers $N_n(\vartheta)$, for $0 \leq n \leq m$. For every $i < m$ there is $C_i \in \mathbb{C}$ such that for every m-fold twist ϑ and all $z, y \in \mathcal{M}_D(Y_0(D))$

$$N_0(\vartheta)\, \langle a_\vartheta(i,0)\jmath_i z, a_\vartheta(i,0)\jmath_i y \rangle = C_i \sum_{\gamma} R_D(\vartheta^{i+1}, i+1, Y_0(D); z, y; \gamma).$$

Input 3 (unit *Exchange*; the Gram form of the weighted chain). Let $u_\vartheta \colon F_\vartheta(0,0) \to F_\vartheta(0,m)$ be the composite of the maps $N_n(\vartheta)\, a_\vartheta(0,n)$, $0 \leq n < m$. Suppose that for every m-fold twist ϑ and every $i < m$ an endomorphism $S(\vartheta, i)$ of $\mathrm{Row}(D)$ is given with $N_0(\vartheta)\langle a_\vartheta(i,0)\jmath_i z, a_\vartheta(i,0)\jmath_i y\rangle = \langle z, S(\vartheta,i) y\rangle$ for all $z \in \mathcal{M}_D(Y_0(D))$ and $y \in \mathrm{Row}(D)$. Then there is a function κ from the m-fold twists to $\mathbb{C}$ that costs $m \cdot m$ squares (Input 7), and for every m-fold twist ϑ there is an endomorphism $T(\vartheta)$ of $\mathrm{Row}(D)$ with $\det T(\vartheta) = \prod_{i<m} \det S(\vartheta, i)$ and

$$\langle u_\vartheta \jmath_0 z, u_\vartheta \jmath_0 y \rangle = \kappa(\vartheta)\, \frac{N_0(\vartheta)\, N_1(\vartheta)^m}{N_m(\vartheta)}\, \langle z, T(\vartheta) y \rangle$$

for all $z \in \mathcal{M}_D(Y_0(D))$ and $y \in \mathrm{Row}(D)$.

Inputs from other units, continued.

Input 4 (unit *Exchange*; the chain at its level). For every m-fold twist ϑ the map $\mathrm{chain}_D(\vartheta)$ is injective. Let $I(\vartheta)$ be the index of the level of $\mathcal{A}_{D,\vartheta}$ in the stabiliser of the standard lattice of $Y_m(D)$, and let ω be the dimension of the weight of $\mathcal{A}_{D,\vartheta}$, a positive integer that does not depend on ϑ. Then $\langle \mathrm{chain}_D(\vartheta)z, \mathrm{chain}_D(\vartheta)y\rangle = I(\vartheta)\,\omega\,\langle u_\vartheta \jmath_0 z, u_\vartheta \jmath_0 y\rangle$ for all $z, y \in \mathcal{M}_D(Y_0(D))$. Moreover there are a number G and an integer l with

$$I(\vartheta) = G \cdot |\mathcal{O}/\mathfrak{m}_D(\vartheta,0)|^{\,l} \cdot \delta_D(\vartheta)^m \cdot \frac{N_m(\vartheta)}{N_0(\vartheta)\,N_1(\vartheta)^m} \qquad \text{for every } m\text{-fold twist } \vartheta.$$

Input 5 (unit *Trace*; one Hecke element for two Rallis series). Let ϑ be a tame character of D and j a lift index, let V and V' be hermitian spaces of rank r over $E(D)$ whose signatures satisfy the gap condition of that unit (the proof below takes $V = Y_0(D)$ and $V' = W_{i+1}(D)$), let $A, A' \in \mathbb{C}$, and let B and B' be complex functions of two forms. If $A \sum_\gamma R_D(\vartheta, j, V; z, y; \gamma) = B(z,y)$ for all $z, y \in \mathcal{M}_D(V)$ and $A' \sum_\gamma R_D(\vartheta, j, V'; z, y; \gamma) = B'(z,y)$ for all $z, y \in \mathcal{M}_D(V')$, the series converging unconditionally, then there is $T \in \mathbb{T}_D$ with $B(z,y) = A\langle z, T_V y\rangle$ and $B'(z,y) = A'\langle z, T_{V'} y\rangle$ for all such z and y.

Input 6 (unit *Trace*; good forms on different spaces). For every lift index j and every $T \in \mathbb{T}_D$, the determinants of T on $\mathcal{M}^\circ_D(Y_0(D))$ and on $\mathcal{M}^\circ_D(W_j(D))$ are equal, and $\dim \mathrm{Row}(D) = \dim \mathcal{M}^\circ_D(W_j(D))$.

Input 7 (unit *Exchange*; the cost of squares). For an integer $n \geq 0$, a function c from the m-fold twists of D to $\mathbb{C}$ *costs n squares* if, for every family θ of m-fold twists that is fit for D, the family $(\ell, \iota) \mapsto |c(\theta_{\ell,\iota})/c(1)|^{-2}$ has height at most $n \cdot 2\,t(D)\,x^{999}$. (i) If c costs n squares, c' costs n' squares and $n + n' \leq n''$, then the product $c\,c'$ costs n'' squares. (ii) If there are a real number G', an integer $e \geq 0$ and an integer l' with $|c(\vartheta)|^2 = G' \cdot |\mathcal{O}/\mathfrak{m}_D(\vartheta, e)|^{\,l'}$ for every m-fold twist ϑ, then c costs 0 squares.

Property (a) of Theorem A says that c costs $m \cdot m$ squares, and the function κ of Input 3 costs $m \cdot m$ squares. Inputs 1 and 2 rest on the artifact's coset-sum (doubling) form of the Rallis inner product formula for a theta lift L in the stable range,

$$\langle Ly', Ly\rangle = A\,c_F \sum_\gamma \rho(\gamma)\,\langle y', R(\gamma)y\rangle,$$

with a constant $A \neq 0$, the mass c_F of the finite datum, a right translation $R(\gamma)$ and a matrix coefficient $\rho(\gamma)$ of the vacuum. For it the catalogue of the handout cites S. Rallis (*Injectivity properties of liftings associated to Weil representations*, Compositio Math. 52 (1984), §1 and Theorem 1.6) and J.-S. Li (*Non-vanishing theorems for the cohomology of certain arithmetic quotients*, J. reine angew. Math. 428 (1992), §2, Theorem 2.1, eq. (26), p. 184). The second reading classes these as analogues: that the formal statement is a special case of the cited ones is not supported by the citations, and the uniform constant, the measures, the evaluation as a coset sum and the injectivity would each need an argument. So the note does not call Inputs 1 and 2 instances of the classical formula. In Input 2 the spaces $F_\vartheta(i,n)$ form a triangle whose row $i = 0$ is the chain $Y_0(D) \to \ldots \to Y_m(D)$; the other rows serve the proof of Input 3 only.

3 Proofs

Lemma 3.1 (a Gram form determines the volume)**.** *Let U and U' be finite-dimensional inner product spaces, $u\colon U \to U'$ a linear map, S an endomorphism of U and $a \in \mathbb{C}$. If $\langle uy', uy\rangle = a\,\langle y', Sy\rangle$ for all $y', y \in U$, then*

$$\operatorname{vf}(u)^2 = a^{\dim U} \cdot \det S.$$

Proof. For all $y', y \in U$,

$$\langle y', u^*uy\rangle = \langle uy', uy\rangle = a\,\langle y', Sy\rangle = \langle y', (aS)y\rangle.$$

So $u^*u = aS$, and $\operatorname{vf}(u)^2 = \det(u^*u) = \det(aS) = a^{\dim U}\det S$. □

LEAN `normDet_sq_eq_pow_mul_det`

Proof of Theorem A. Fix P and D, and write $n_0 = \dim \operatorname{Row}(D)$.

The single lifts. Take the numbers $A_j \neq 0$ of Input 1 and the numbers C_i of Input 2. Let ϑ be an m-fold twist and $0 \le i < m$. The power ϑ^{i+1} is a tame character with $Q(\vartheta^{i+1}) = Q(\vartheta)$, so $\delta_D(\vartheta^{i+1}) = \delta_D(\vartheta)$. Input 1, for the tame character ϑ^{i+1} and the lift index $i+1$, and Input 2 give two series of Rallis terms with the same parameters $(\vartheta^{i+1}, i+1)$, one on $W_{i+1}(D)$ with the constant $A_{i+1}\delta_D(\vartheta)$ and one on $Y_0(D)$ with the constant C_i. By Input 5 there is $T_{\vartheta,i} \in \mathbb{T}_D$ with

$$N_0(\vartheta)\,\langle a_\vartheta(i,0)\jmath_i z, a_\vartheta(i,0)\jmath_i y\rangle = C_i\,\langle z, T_{\vartheta,i}\,y\rangle,$$
$$\langle \operatorname{lift}^*_D(\vartheta^{i+1}, i+1)\,z, \operatorname{lift}^*_D(\vartheta^{i+1}, i+1)\,y\rangle = A_{i+1}\,\delta_D(\vartheta)\,\langle z, T_{\vartheta,i}\,y\rangle,$$

the first for all $z, y \in \mathcal{M}_D(Y_0(D))$ and the second for all $z, y \in \mathcal{M}_D(W_{i+1}(D))$, where $T_{\vartheta,i}$ acts on the forms of the space in question. Since $T_{\vartheta,i}$ preserves the good forms, the second identity and Lemma 3.1, applied to the restriction of the adjoint lift to $\mathcal{M}^\circ_D(W_{i+1}(D))$, a space of dimension n_0 by Input 6, give

$$\operatorname{vol}_D(\vartheta, i+1) = \left(A_{i+1}\,\delta_D(\vartheta)\right)^{n_0} \cdot \det\left(T_{\vartheta,i} \mid \mathcal{M}^\circ_D(W_{i+1}(D))\right). \tag{1}$$

The chain. Let $S(\vartheta, i)$ be C_i times the restriction of $T_{\vartheta,i}$ to $\operatorname{Row}(D)$. By the first identity the hypothesis of Input 3 holds. Take κ and the endomorphisms $T(\vartheta)$ from Input 3. With Input 4, for $z, y \in \operatorname{Row}(D)$,

$$\begin{aligned}\langle \Theta_{D,\vartheta} z, \Theta_{D,\vartheta} y\rangle &= I(\vartheta)\,\omega\,\kappa(\vartheta)\,\frac{N_0(\vartheta)N_1(\vartheta)^m}{N_m(\vartheta)}\,\langle z, T(\vartheta)y\rangle \\ &= G\,|\mathcal{O}/\mathfrak{m}_D(\vartheta,0)|^{\,l}\,\delta_D(\vartheta)^m\,\omega\,\kappa(\vartheta)\,\langle z, T(\vartheta)y\rangle.\end{aligned}$$

By Lemma 3.1 and the formula for $\det T(\vartheta)$,

$$\mathfrak{z}_D(\vartheta) = \left(G\,|\mathcal{O}/\mathfrak{m}_D(\vartheta,0)|^{\,l}\,\delta_D(\vartheta)^m\,\omega\,\kappa(\vartheta)\right)^{n_0} \cdot \prod_{i<m} C_i^{n_0}\,\det\left(T_{\vartheta,i} \mid \operatorname{Row}(D)\right). \tag{2}$$

Comparison. By Input 6 the determinants of $T_{\vartheta,i}$ on $\operatorname{Row}(D) = \mathcal{M}^\circ_D(Y_0(D))$ and on $\mathcal{M}^\circ_D(W_{i+1}(D))$ agree. Insert this into (2) and write

$$C_i^{n_0}\,\delta_D(\vartheta)^{n_0} = (C_i/A_{i+1})^{n_0}\,(A_{i+1}\,\delta_D(\vartheta))^{n_0},$$

which is allowed since $A_{i+1} \neq 0$. By the identities (1) for $i < m$,

$$\mathfrak{z}_D(\vartheta) = c(\vartheta)^{n_0} \cdot \prod_{j=1}^{m} \operatorname{vol}_D(\vartheta, j), \qquad c(\vartheta) = \kappa(\vartheta) \cdot \prod_{i<m} \frac{C_i}{A_{i+1}} \cdot \omega \cdot G \cdot |\mathcal{O}/\mathfrak{m}_D(\vartheta,0)|^{\,l}.$$

The volumes are real and ≥ 0. Taking absolute values on both sides gives the formula of (b), with $|c(\vartheta)|^{n_0}$ in the place of $c(\vartheta)^{n_0}$.

Non-vanishing. The map $\mathrm{chain}_D(\vartheta)$ is injective by Input 4, so its restriction $\Theta_{D,\vartheta}$ is injective and $\mathfrak{z}_D(\vartheta) \neq 0$.

The cost. Write $c = \kappa \cdot c_0$. The function c_0 satisfies $|c_0(\vartheta)|^2 = G' \cdot |\mathcal{O}/\mathfrak{m}_D(\vartheta, 0)|^{2l}$ with a real number G' that does not depend on ϑ, so it costs 0 squares by Input 7 (ii). The function κ costs $m \cdot m$ squares. By Input 7 (i), with $m \cdot m + 0 \leq m \cdot m$, the function c costs $m \cdot m$ squares, which is (a). □

LEAN `Data.volume_iota_eq_prod_liftVolume` · the formal proof takes absolute values of the two identities given by Lemma 3.1 before it compares them

The last three results of the file describe the image of one adjoint lift on the good forms. They are not used for Theorem A; the note *The height of a ratio of lift volumes* and Chapter VIII use them (§4).

Lemma 3.2 (a good tuple acts inside the good part)**.** *In Setting 2.1, let U be an inner product space, let $\varrho\colon \mathbb{C}[\mathsf{G}(D)] \to \mathrm{End}_{\mathbb{C}}(U)$ be a homomorphism of $\mathbb{C}$-algebras, and let $\mathbf{c}$ be a tuple of labels that is good. Then the image of $\varrho(\mathrm{idem}(\mathbf{c}))$ is contained in the image of $\varrho(e_0(D))$.*

Proof. The idempotents of the tuples are orthogonal and $e_0(D)$ is their sum over the good tuples, of which $\mathbf{c}$ is one. So $\mathrm{idem}(\mathbf{c}) = \mathrm{idem}(\mathbf{c})\, e_0(D) = e_0(D)\, \mathrm{idem}(\mathbf{c})$, the second equality because $e_0(D)$ is central. Hence $\varrho(\mathrm{idem}(\mathbf{c})) = \varrho(e_0(D)) \circ \varrho(\mathrm{idem}(\mathbf{c}))$, and the image of a composite lies in the image of its outer map. □

LEAN `Data.range_tupleIdem_le`

Lemma 3.3 (the adjoint lift commutes with the finite group)**.** *In Setting 2.1, let ϑ be a tame character of D, j a lift index and $\xi \in \mathbb{C}[\mathsf{G}(D)]$. For every $y \in \mathcal{M}_D(W_j(D))$,*

$$\mathrm{lift}^*_D(\vartheta, j)(\xi \cdot y) = \xi \cdot \mathrm{lift}^*_D(\vartheta, j)(y),$$

where ξ acts on the left through the action of $\mathbb{C}[\mathsf{G}(D)]$ on $\mathcal{M}_D(W_j(D))$ and on the right through its action on $\mathcal{F}(O_D(\vartheta, j))$.

Proof. Both actions are linear in ξ, and ξ is a finite combination of elements of $\mathsf{G}(D)$. For a group element the identity is the equivariance of the adjoint lift, which the unit of Chapter VIII that defines the adjoint lift proves. □

LEAN `Data.groupAlgAction_auxLift`

Lemma 3.4 (the image of the adjoint lift on the good forms)**.** *In Setting 2.1, let ϑ be a tame character of D, j a lift index and $\mathcal{N}$ a linear subspace of $\mathcal{F}(O_D(\vartheta, j))$. Suppose that for every Hecke eigensystem p, every $y \in \mathcal{M}^\circ_D(W_j(D)) \cap \mathcal{M}_D(W_j(D))[p]$ and every good tuple of labels $\mathbf{c}$: if $\mathrm{lift}^*_D(\vartheta, j)(y)$ lies in the image of the action of $\mathrm{idem}(\mathbf{c})$ on $\mathcal{F}(O_D(\vartheta, j))$, then $\mathrm{lift}^*_D(\vartheta, j)(y) \in \mathcal{N}$. Then*

$$\mathrm{lift}^*_D(\vartheta, j)(\mathcal{M}^\circ_D(W_j(D))) \subseteq \mathcal{N}.$$

Proof. The space $\mathcal{M}_D(W_j(D))$ is the sum of the spaces $\mathcal{M}_D(W_j(D))[p]$ over the Hecke eigensystems p (a fact of the unit of Chapter VIII that defines them). The good forms are the forms $e_0(D)\, y$. So it suffices to show, for every p and every $y \in \mathcal{M}_D(W_j(D))[p]$, that the lift of $e_0(D)\, y$ lies in $\mathcal{N}$. Now $e_0(D)\, y = \sum_{\mathbf{c}} \mathrm{idem}(\mathbf{c})\, y$, the sum over the good

tuples. Fix one. The form $y_{\mathbf{c}} = \mathrm{idem}(\mathbf{c})\, y$ is good by Lemma 3.2, applied to the action on $\mathcal{M}_D(W_j(D))$, and it lies in $\mathcal{M}_D(W_j(D))[p]$, because the Hecke operators commute with the action of $\mathsf{G}(D)$. By Lemma 3.3, $\mathrm{lift}^*_D(\vartheta, j)(y_{\mathbf{c}}) = \mathrm{idem}(\mathbf{c}) \cdot \mathrm{lift}^*_D(\vartheta, j)(y)$ lies in the image of the action of $\mathrm{idem}(\mathbf{c})$. The hypothesis gives $\mathrm{lift}^*_D(\vartheta, j)(y_{\mathbf{c}}) \in \mathcal{N}$, and the sum over $\mathbf{c}$ lies in $\mathcal{N}$. □

LEAN `Data.range_goodLift_le`

4 Where the results are used

Theorem A is one of the two inputs of Theorem C of the handout that the head of the argument names; the unit *Endgame* applies it twice, once to bound the height of $\mathfrak{z}_D(1)^2/\mathfrak{z}_D(\theta_{\ell,\iota})^2$ for a fit family θ, and once for the non-vanishing in (b). The other input bounds each ratio $\mathrm{vol}_D(1, j)/\mathrm{vol}_D(\theta_{\ell,\iota}, j)$; it is proved in the note *The height of a ratio of lift volumes*, which uses Lemmas 3.2 and 3.4 to describe the image of a single lift, and Lemma 3.3 is used in the note *Glued lattices, product forms and the integral comparison forms*. Chapter VIII uses Lemmas 3.2 and 3.4 in the proof that fit families exist.

5 Notes on the formal text

1. *The volume factor.* The formal text uses the library's volume factor of a linear map and its two properties quoted in Definition 2.2: its square is $\det(u^* u)$, and it vanishes exactly when the kernel is not zero. The library's definition was not read; the definition above is by these properties. The lift volume and the volume of the chain are defined as squares of this real number, cast to $\mathbb{C}$.

2. *The constant is explicit.* The formal proof gives c by the formula at the end of the proof above. It does not say that c is unique, and the statement does not use more than its existence.

3. *Two identifications made by Lean.* The proof applies Input 5 to the series of Input 2, for the twist ϑ and the row i, and to the series of Input 1, for the tame character ϑ^{i+1} and the lift index $i + 1$. That the terms of the first series are the Rallis terms with the parameters $(\vartheta^{i+1}, i + 1)$, and that the Gram factor of the pair $O_D(\vartheta^{i+1}, i + 1)$ is the one of $O_D(\vartheta, 1)$, is nowhere stated as a lemma: Lean accepts the application because the definitions unfold to the same terms. Input 2 above is worded with these identifications made.

4. *Arguments that the formal statements leave to Lean.* In the definition of the volume of the chain the twist is written without its type, and in the definition of the lift volume the index j is; Lean infers that the first is an m-fold twist and the second a lift index, from the maps they are given to. In Lemma 3.3 the form y and in Lemmas 3.3 and 3.4 the twist ϑ are written without types as well; that ϑ is a tame character, not necessarily a twist, was read off the definition of the adjoint lift.

5. *Hypotheses of Lemma 3.1.* The formal statement asks that both spaces be finite-dimensional; U' could be arbitrary once the adjoint exists, but the library's volume factor is defined for finite-dimensional spaces.

6. *What an assembly assembles.* Theorem A carries no analysis of its own: the convergence of the Rallis series, the triangle of lifts and its square laws, the rigidity of the labels behind Input 6 and the index computation of Input 4 all lie in the inputs.

How this note was written. From the dossier of the unit: the structured proofs of Theorem A (eight steps), of Lemma 3.1 and of Lemmas 3.2 and 3.4. I read the whole Lean file of these eight declarations, and the Lean statements of the inputs in their files: the cost of squares and its two rules, the inner product formula for one lift, the formula for the first lift of a row, the statement about the Gram form of a weighted chain in a triangle together with the definition of a triangle, the index of the last level and the norm of the chain at its level, the common Hecke element, and the two statements about good forms on different spaces; also the Lean definitions of tame characters, twists, m-fold twists, forms of principal level, good forms, rows, heights and fit families. I did not read the proofs of the inputs. The reader of the dossier notes that no second reader had looked at it. After the dossier was frozen, the second reader of the write-up compared the readings of all statements of the unit with the Lean text and sampled its structured proofs (her message to the writer of 26 November 2026); for this note she confirms the two statements about volumes and that the last two lemmas (Lemmas 3.3 and 3.4) take a tame character, as stated here. The proof of Theorem A above follows the formal one step by step. The program check of the unit's dossier covered 634 leaf steps of 85 proofs.

Revision. This text was revised after the report of an independent referee (26 November 2026), who re-derived the proof of Theorem A and compared the inputs with their Lean statements. The changes: Inputs 1 and 2 are no longer called instances of the Rallis inner product formula, and the references are those of the catalogue with the second reader's verdict; the note says that the constant c does not vanish; the levels and the ring of integers are introduced in the Setting; the implicit arguments of the last three lemmas are listed correctly.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Lemma 3.1	`normDet_sq_eq_pow_mul_det`	RowVolume.lean	15–19
Definition 2.2	`Data.liftVolume`	RowVolume.lean	23–23
Definition 2.2	`Data.θ`	RowVolume.lean	27–27
Definition 2.2	`Data.ϛ`	RowVolume.lean	30–30
Theorem A	`Data.volume_iota_eq_prod_liftVolume`	RowVolume.lean	37–58
Lemma 3.2	`Data.range_tupleIdem_le`	RowVolume.lean	62–65
Lemma 3.3	`Data.groupAlgAction_auxLift`	RowVolume.lean	67–70
Lemma 3.4	`Data.range_goodLift_le`	RowVolume.lean	75–82

Theorem A and the three definitions as they are written in the artifact.

```
lemma Data.volume_iota_eq_prod_liftVolume : ∃ c, D.CostsSquares (D.m * D.m) c ∧ ∀ θ, D.ϛ θ ≠ 0 ∧
  D.ϛ θ = ‖c θ‖ ^ D.rowDim * ∏ j ∈ Icc 1 D.m, D.liftVolume θ j

noncomputable def Data.liftVolume (θ : D.MFoldTwist) j := ((D.goodLift (θ.pow j) j).normDet : ℂ) ^ 2

noncomputable abbrev Data.θ (θ : D.MFoldTwist) := D.iotaAtLevel θ ∘ι D.rows.subtype
```

```
noncomputable def Data.ʒ ϑ := ((D.ϑ ϑ).normDet : ℂ) ^ 2
```

In these texts `D.MFoldTwist` is the type of the m-fold twists of D, `D.goodLift` the adjoint lift restricted to the good forms, `D.iotaAtLevel` the map $\mathrm{chain}_D(\vartheta)$, `D.rows` the space $\mathrm{Row}(D)$, `D.rowDim` its dimension, `normDet` the volume factor, and `D.CostsSquares` the notion of Input 7.

Notation

$P = (N_0, d)$, D parameters; admissible data for P

$E(D)$ the CM field of the data

m, r, $t(D)$, x number of lifts; rank of the rows; the step $5r$; the first scale

$g(D)$, $N(D)$ the two levels of the data

$\mathsf{G}(D)$ the finite group of the chosen primes

j, i, n, e a lift index; a row index $< m$; integers ≥ 0

$Y_i(D)$, $W_j(D)$ the hermitian spaces of the chain; of the j-th single lift

V, V' hermitian spaces of rank r

$\mathcal{M}_D(V)$, $\mathcal{M}^\circ_D(V)$ forms of principal level on V; the good ones

$\mathrm{Row}(D)$, n_0 the rows; their dimension

$e_0(D)$, $\mathrm{idem}(\mathbf{c})$, $\mathbf{c}$ the good idempotent; the idempotent of a tuple of labels; a tuple

$\mathbb{T}_D$, T, T_V, $T_{\vartheta,i}$ the Hecke algebra; an element; its action on $\mathcal{M}_D(V)$; the element of the proof

p, $\mathcal{M}_D(V)[p]$ a Hecke eigensystem; its eigenspace

ϑ, $Q(\vartheta)$ a tame character or twist; its set of ramified primes

$\mathfrak{m}_D(\vartheta, e)$, $\mathcal{O}$ the ideal of chosen primes to the power e times $Q(\vartheta)$; the ring of integers of $E(D)$

$O_D(\vartheta, j)$, $\mathcal{F}(O_D(\vartheta, j))$ the auxiliary pair; its space of algebraic forms

$\mathrm{lift}^*_D(\vartheta, j)$ the adjoint lift

$\mathrm{chain}_D(\vartheta)$, $\mathcal{A}_{D,\vartheta}$ the chain of m lifts; its target

$\Theta_{D,\vartheta}$ the chain restricted to the rows

$\mathrm{vf}(u)$, u^* the volume factor of a linear map u; its adjoint

$\mathrm{vol}_D(\vartheta, j)$, $ʒ_D(\vartheta)$ the lift volume; the volume of the chain

L in "L-function": the Dirichlet series attached to a character; in the remark after the inputs a theta lift

ℓ, ι, σ a prime; a field isomorphism from $\mathbb{C}$ to an algebraic closure of $\mathbb{Q}_\ell$; an automorphism of $\mathbb{C}$

$\theta = (\theta_{\ell,\iota})$ a family of m-fold twists

Z, h, H, α, η_ℓ a family of complex numbers; the objects of a height bound

c, c', c_0, κ functions on the m-fold twists

A_j, C_i, G, G', l, l', ω the constants of the inputs

$\delta_D(\vartheta)$, $I(\vartheta)$, $N_n(\vartheta)$ the Gram factor; the index of the last level; the weights of the triangle

$R_D(\vartheta, j, V; y', y; \gamma)$, γ a Rallis term; a coset

c_F, $\rho(\gamma)$, $R(\gamma)$ in the remark after the inputs: the mass of a finite datum, a matrix coefficient, a right translation

$F_\vartheta(i, n)$, $a_\vartheta(i, n)$, $\jmath_i$, u_ϑ the triangle: spaces, lifts, embeddings; the weighted chain

$S(\vartheta, i)$, $T(\vartheta)$, S endomorphisms of the rows; an endomorphism

U, U', u, a, y, y', z inner product spaces, a linear map, a scalar, vectors or forms

A, A', B, B' constants and forms of Input 5

ϱ, ξ, $\mathcal{N}$, $y_{\mathbf{c}}$ a homomorphism of algebras; an element of the group algebra; a subspace; a component

n', n'', j' further integers and a further lift index

Auxiliary dual pairs, determinant polynomials and the carrier: the explicit matching of theta kernels

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. This note sets up the objects with which the rows of the proof are compared with definite groups: the record of an *auxiliary dual pair*, with its level, its space of algebraic forms, its theta descent and its Hecke test; the determinants of a constant matrix stacked over a matrix of variables; and the *carrier*, a fixed holomorphic theta lift of the constant 1. It then proves three identities between theta kernels. The *explicit matching* (Theorem A) says that a theta kernel summed over the classes of a definite unitary group equals a theta kernel integrated over a unitary group of smaller rank, with explicit determinant polynomials at the infinite places on both sides, up to a constant whose absolute value is given by Hecke L-values. Two consequences follow: a class sum of a kernel of two definite groups of equal rank is a constant of this kind (Theorem B), and a twisted class sum against an explicit form is a constant multiple of the carrier (Theorem C), so that a twist survives only as a number.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles with Lean v4.35.0 against a pinned version of Mathlib and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note is the second of four on one unit of the artifact, the folder of the rows; it treats 73 of the unit's 219 declarations, those of five files. All definitions are given. The lemmas of linear algebra, Proposition 3.2 and Theorems A and B are proved in full from results of the background library and from one result of another unit, the exchange theorem (Inputs 1 to 9); these are stated and not proved here. The proofs of Proposition 3.5 and of Theorem C, which rests on it, each describe one computation without carrying it out: the exact form of the translation rule in the first, the bookkeeping of the characters in the second. They say so where they stand. The archimedean step, Proposition 3.2, was read line by line in the Lean text. Every statement below was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 73 declarations.

1 The statements

The objects are defined in §2: the theta kernel $\theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Psi,P](g,h)$ of two hermitian spaces, the class sum $\sum^{\mathrm{cl}}_{h \bmod \mathbf{K}_{\mathsf{a}}}$ over a definite unitary group, tame characters and the product $\Pi_{\mathfrak{m}}(\nu,d)$ of L-values in Setting 2.1; the determinant polynomials $\Delta(e;\eta)$ and $Q_{f,j}(\eta)$ in Definitions 2.15 and 2.20; the carrier in Definition 2.19.

Theorem A (explicit matching). *In Setting 2.1, let* $\mathsf{n},\mathsf{m},\mathsf{a},\mathsf{b},\mathsf{p},\mathsf{l}$ *be finite index sets,* s *a sign vector on* n *and* W *a hermitian space over* E *with index set* n *and signs* s*. Let* $f\colon \mathsf{p}\sqcup\mathsf{l}\to\mathsf{b}$ *be a bijection and* V' *a hermitian space with index set* b *and signs* $\operatorname{sgn}_f(\cdot)$*. Let* $\sigma\colon \mathsf{b}\sqcup\mathsf{a}\to\mathsf{n}$*,* $e\colon \mathsf{p}\sqcup\mathsf{a}\to\mathsf{m}$ *and* $j\colon \mathsf{m}\sqcup\mathsf{l}\to\mathsf{n}$ *be bijections with* $s=\operatorname{sgn}_j(\cdot)$*, let* $d\ge 0$ *be an integer with* $|\mathsf{a}|=|\mathsf{b}|+2d$*, suppose* $2|\mathsf{l}|+2\le|\mathsf{n}|$*, and let* $\iota=(\iota_\tau)$ *be a family of finite sets indexed by the infinite places of* E*. Then there is a real number* C *such that for every modulus* $\mathfrak{m}$*, every character* α*, weight* γ *and character* χ *of* $\mathcal{O}_E/\mathfrak{m}$ *for which* α *is tame for* $(\mathfrak{m},\gamma,\chi)$ *and* $\gamma_\tau=|\mathsf{l}|-|\iota_\tau|$ *for all* τ*, there is* $c\in\mathbb{C}$ *with*

$$|c|^2=\frac{C}{\Pi_{\mathfrak{m}}(\alpha^{-1}\upsilon^{|\mathsf{b}|+d},d)}$$

such that for every family $\eta=(\eta_{\tau k})$ *of complex* $\mathsf{p}\times\mathsf{m}$ *matrices,* $k\in\iota_\tau$*, for all partners* $\check\Psi$*,* Ψ *for* (σ,χ^{-1})*, every character* β *and every* $g\in\mathrm{U}(W)(\mathbb{A})$*:*

$$c\cdot\sum_{h\bmod\mathbf{K}_{\mathsf{a}}}^{\mathrm{cl}}\theta^{W,I_{\mathsf{a}}}_{\alpha,\beta;\mathfrak{m}}[\check\Psi,\Delta(e;\eta)^j](g,h)=\int_{[\mathrm{U}(V')]}\theta^{V',W}_{\alpha/\beta,\alpha;\mathfrak{m}}[\Psi,Q_{f,j}(\eta)](x,g)\,dx.$$

Here $\Delta(e;\eta)^j$ *is the polynomial* $\Delta(e;\eta)$ *with the column index* $c\in\mathsf{m}$ *of its variables replaced by* $j(c)\in\mathsf{n}$*.*

LEAN `explicit_matching`

Theorem B (a class sum of a theta kernel as Hecke values). *In Setting 2.1, let $m \geq 1$ be an even integer and let c assign an integer $c_\tau \geq 0$ to each infinite place τ of E. Then there is a real number C such that for every modulus $\mathfrak{m}$, every character α and every character χ of $\mathcal{O}_E/\mathfrak{m}$ with α tame for $(\mathfrak{m}, -c, \chi)$ there is $\gamma \in \mathbb{C}$ with $|\gamma|^2 = C/\Pi_{\mathfrak{m}}(\alpha^{-1}\upsilon^{m/2}, m/2)$ and, for every $g \in \mathrm{U}(I_m)(\mathbb{A})$,*

$$\gamma \cdot \sum_{h \bmod \mathbf{K}_m}^{\mathrm{cl}} \theta^{I_m, I_m}_{\alpha,\alpha;\mathfrak{m}}[\chi \circ \det, \textstyle\prod_\tau \det(x_\tau)^{c_\tau}](g,h) = 1.$$

Theorem C (the carrier: one fixed form under all twists). *In Setting 2.1, let $m \geq 1$ be an even integer. Then there is a real number C such that for every modulus $\mathfrak{m}$, every character c and every character χ of $\mathcal{O}_E/\mathfrak{m}$ with c tame for $(\mathfrak{m}, 0, \chi)$ there is $\gamma \in \mathbb{C}$ with*

$$|\gamma|^2 = \frac{C}{\Pi_{\mathfrak{m}}(c^{-1}\upsilon^m, m) \cdot \Pi_{\mathfrak{m}}(c\,\upsilon^{m/2}, m/2)}$$

and, for every $x \in \mathrm{U}(I_{2m,m})(\mathbb{A})$,

$$\gamma \cdot \sum_{h \bmod \mathbf{K}_{3m}}^{\mathrm{cl}} \Xi^\circ_{m,c}(h) \cdot \theta^{I_{2m,m}, I_{3m}}_{\upsilon^m, c;\mathfrak{m}}[\chi \circ \det, 1](x,h) = \mathfrak{C}_{E,m}(x).$$

LEAN `stdClassSum_det_modKernel_eq` · `stdClassSum_carrierXi_eq_carrierForm` · in Theorems B and C the integer m is the Lean variable; it is not the number of lifts of the data, which is its value where the theorems are applied

In Theorem C the left side depends on the character c, which in the applications carries the twist, and the right side does not: the twist is seen only through the number γ, of known absolute value. In Theorems A to C the constant C may be zero; then the corresponding number is zero and the identity says that the right side vanishes. That the carrier is not zero is proved in the note *The height of a ratio of lift volumes*.

2 The setting

Conventions. Finite index sets are written in sans-serif, with the letters of the formal text: $\mathsf{r}, \mathsf{n}, \mathsf{N}, \mathsf{m}, \mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{p}, \mathsf{l}, \mathsf{L}, \mathsf{k}$; $|\mathsf{n}|$ is the number of elements of n, and an $\mathsf{n} \times \mathsf{r}$ matrix has rows indexed by n and columns by r. For a matrix z, z^{T} is its transpose and z^{H} its conjugate transpose. The ambient objects of the declarations of this note, their implicit and instance arguments, are: the commutative rings R and R' of §2; the finite set Pl of places and the finite index sets of each statement; and, from Definition 2.7 on, a field E that is a CM field with a CM type and with Weil data, as in Setting 2.1. Each statement names the objects it uses. In the proofs, when a statement of the formal text has implicit arguments that a later hypothesis determines (a modulus $\mathfrak{m}$, characters, signs, a polynomial), they are introduced with that hypothesis.

Setting 2.1 (the background: fields, characters, spaces, kernels, weights)**.** All of this is defined in the background library of the artifact; the catalogue of the handout gives references.

(a) The field. E is a CM field with a CM type (for each infinite place τ a complex embedding that induces it) and with *Weil data*: the choices that fix the Weil representation, among them a character υ of the norm-one ideles of E, the restriction of a splitting character. Finite places are written w, with residue field k_w, norm q_w and complex conjugate $\bar{w}$; $\widehat{\mathcal{O}}_E$ is the ring of integral finite adeles, $\mathbb{A}$ the adeles and $\mathbb{A}_f$ the finite adeles of the totally real subfield.

(b) Moduli. A *modulus* is a non-zero ideal $\mathfrak{m}$ of $\mathcal{O}_E$ divisible by at most one of w, $\bar{w}$ for each w. It has a chosen reduction $\widehat{\mathcal{O}}_E \to \mathcal{O}_E/\mathfrak{m}$ extending the one of $\mathcal{O}_E$ and a chosen *generator*, an integral idele equal to 1 at the places not dividing $\mathfrak{m}$ that generates the kernel of the reduction. *Level places* are a pair Σ of finite families of finite places, the *chosen places* and the *twist places*, such that all these places are distinct and none is the conjugate of another; $\mathfrak{M}_\Sigma$ and $\mathfrak{Q}_\Sigma$ are the products of the primes of the two families, so that $\mathfrak{M}_\Sigma^c\mathfrak{Q}_\Sigma$ and $\mathfrak{M}_\Sigma^c$ are moduli for every integer $c \geq 0$.

(c) Characters. A *character* is a continuous character of the norm-one ideles of E trivial on the norm-one elements of E. It has *weight* e, an integer e_τ for each τ, and *tame part* $\psi = (\psi_w)$, a character of $k_w^\times$ for each w, almost all trivial, if its value at u is $\prod_\tau u_\tau^{e_\tau} \prod_w \psi_w(u \bmod w)$ whenever the finite part of u is integral. It is *tame for* $(\mathfrak{m}, e, \chi)$, for a modulus $\mathfrak{m}$ and a multiplicative character χ of $\mathcal{O}_E/\mathfrak{m}$, if $\mathfrak{m} \neq (1)$, it has weight e and a tame part that is non-trivial exactly at the primes dividing $\mathfrak{m}$, and χ is the product of the ψ_w. For a set T of finite places, a character ν and an integer $d \geq 0$, with $L^T(z, \nu)$ the partial Hecke L-function of the base change $x \mapsto \nu(x/\bar{x})$ of ν to the ideles of E, without its Euler factors at the places of T and at their complex conjugates,

$$\Pi_T(\nu, d) = \Big(\prod_{w \in T} q_w\Big)^{d^2} \cdot \prod_{z=1}^{d} \big|L^T(z, \nu)\big|^2,$$

and $\Pi_{\mathfrak{m}}(\nu, d)$ is this number for the set of primes dividing $\mathfrak{m}$.

(d) Spaces and groups. For a finite index set n and a sign vector $s\colon \mathsf{n} \to \{\pm 1\}$, a *hermitian space with index set* n *and signs* s is a hermitian matrix over E together with an adelic point that brings it to $\mathrm{diag}(s)$ at the infinite places and to 1 at the finite places. I_{n} is the standard positive definite one, I_m the one with index set $\{1, \dots, m\}$, $I_{2m,m}$ the diagonal space with $2m$ signs $+1$ and m signs -1, S^- the space S with the opposite form and $S \oplus T$ the orthogonal sum. $\mathrm{U}(S)(\mathbb{A})$ is the adelic unitary group, $[\mathrm{U}(S)]$ its automorphic quotient with its invariant measure, and $G_\infty(s)$ the product over the infinite places of the real unitary groups of $\mathrm{diag}(s)$. For the definite space, $\mathbf{K}_{\mathsf{n}} = \mathrm{U}(I_{\mathsf{n}})(\widehat{\mathcal{O}}_E)$ is the stabiliser of the standard lattice, and a *level* is a subgroup of finite index in an open subgroup of it. For a level Γ and a function F on $\mathrm{U}(I_{\mathsf{n}})(\mathbb{A}_f)$, the *class sum* is

$$\sum_{h \bmod \Gamma}^{\mathrm{cl}} F(h) = \sum_{\mathfrak{c}} \frac{F(h_{\mathfrak{c}})}{\#\big(\mathrm{U}(I_{\mathsf{n}})(E) \cap h_{\mathfrak{c}} \Gamma h_{\mathfrak{c}}^{-1}\big)},$$

the sum over the double cosets $\mathfrak{c}$ of $\mathrm{U}(I_{\mathsf{n}})(E)\backslash\mathrm{U}(I_{\mathsf{n}})(\mathbb{A}_f)/\Gamma$ with chosen representatives $h_{\mathfrak{c}}$.

(e) Theta kernels. Let S and T be hermitian spaces with index sets n and n' and α, β characters. The Weil representation $\omega_{\alpha,\beta}$ of $\mathrm{U}(S)(\mathbb{A}) \times \mathrm{U}(T)(\mathbb{A})$ acts on the Schwartz functions on $\mathrm{Hom}(S,T)$. At the infinite places it is realised in a Fock model: a complex polynomial P in the variables $x_{\tau,i,c}$, with τ an infinite place, $i \in \mathsf{n}'$ and $c \in \mathsf{n}$, has a *Fock vector* v_P, the Fock vectors are dense in the Fock space, and the groups G_∞ act through the *real Weil representation* $\omega^{\infty}_{\gamma,\gamma'}$ with integer weights γ, γ'. For a modulus $\mathfrak{m}$, a complex function Ψ on the $\mathsf{n}' \times \mathsf{n}$ matrices over $\mathcal{O}_E/\mathfrak{m}$ and a polynomial P, the *theta kernel* is

$$\theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Psi,P](g,h) = \theta\big(\omega_{\alpha,\beta}(g,h)\,(v_P \otimes \Psi)\big), \qquad g \in \mathrm{U}(S)(\mathbb{A}),\ h \in \mathrm{U}(T)(\mathbb{A}),$$

where θ is the theta functional and $v_P \otimes \Psi$ the Schwartz function with archimedean part v_P and finite part the function Ψ of the reduction modulo $\mathfrak{m}$ at the primes dividing $\mathfrak{m}$, applied to the vacuum vector elsewhere. When $T = I_{\mathsf{n}'}$ and h is a finite adelic point, the same notation is used with the infinite component of h equal to 1. Functions $\check{\Psi}$ on $\mathsf{a} \times \mathsf{n}$ matrices and Ψ on $\mathsf{n} \times \mathsf{b}$ matrices over a commutative ring are *partners for* (σ,χ), with $\sigma\colon \mathsf{b} \sqcup \mathsf{a} \to \mathsf{n}$ a bijection and χ a multiplicative character of the ring, if $\check{\Psi}$ vanishes off the matrices formed by the rows $\sigma(\mathsf{a})$ of k^{-1} and Ψ off those formed by the columns $\sigma(\mathsf{b})$ of k, for invertible $\mathsf{n} \times \mathsf{n}$ matrices k, and the value of $\check{\Psi}$ at the first is $\chi(\det k)$ times the value of Ψ at the second.

(f) Weights and polynomials. Pl is a finite set, in the applications the set of infinite places. A *weight* κ assigns to each $\tau \in \mathrm{Pl}$ and each height $h \geq 0$ a number $\kappa_\tau(h) \geq 0$ of columns. For a finite index set n, the *standard fillings* of weight κ with rows n index a basis b_l of the polynomials in matrix variables that are semi-invariant of weight κ under the Borel subgroup: products of minors. With the integer Gram matrix G_κ of this basis, a complex coordinate vector ξ indexed by the standard fillings has the polynomial $\mathrm{pol}_\kappa(\xi) = \sum_l (G_\kappa \xi)_l\, b_l$ and the squared norm $\|\xi\|^2_\kappa = \mathrm{Re}(\xi^{\mathsf{H}} G_\kappa \xi)$. The *Fock pairing* of two polynomials is $\langle P, Q\rangle = P(\partial)Q(0)$; the *hook number* h_κ is the pairing with itself of the product of leading minors of weight κ; the *Schur matrix* $S_\kappa(z)$ of matrices z_τ expresses the polynomials $b_l(zy)$ in the basis.

Matrices with prescribed columns

Definition 2.2 (stabiliser and column datum)**.** Let R and R' be commutative rings, r and N finite index sets, c an index set, $\chi\colon R \to \mathbb{C}$ a multiplicative character (zero on the non-units), $\varphi\colon R \to R'$ a ring homomorphism, w a $\mathsf{c} \times \mathsf{N}$ matrix over R' and $e\colon \mathsf{r} \hookrightarrow \mathsf{N}$ an embedding. Let I_e be the $\mathsf{N} \times \mathsf{r}$ matrix whose column i is the standard basis vector of index $e(i)$.

(i) For an $\mathsf{N} \times \mathsf{N}$ matrix z over R, $\mathrm{Fix}_e(z)$ is the proposition $z\,(1 - I_e I_e^{\mathsf{T}}) = 1 - I_e I_e^{\mathsf{T}}$: the columns of z with index outside $e(\mathsf{r})$ are those of the identity matrix.

(ii) $\mathrm{Stab}(\varphi, w, e)$ is the subgroup of the $A \in \mathrm{GL}_{\mathsf{N}}(R)$ with $\mathrm{Fix}_e(A)$ and $w\,\varphi(A) = w$.

(iii) For an $\mathsf{N} \times \mathsf{r}$ matrix x over R, the *column datum* is

$$\mathrm{cd}_{\chi,\varphi,w,e}(x) = \begin{cases} \chi\big(\det(I_e^{\mathsf{T}} x)\big) & \text{if } w\,\varphi(x) = w\,I_e, \\ 0 & \text{otherwise,} \end{cases}$$

where $I_e^{\mathsf{T}} x$ consists of the rows $e(\mathsf{r})$ of x.

LEAN FixesColsOff · rowStabilizer · columnDatum

Lemma 2.3 (on the stabiliser the datum is a product)**.** *Let R and R' be commutative rings, r and N finite index sets, a and c index sets, χ a multiplicative character of R, $\varphi\colon R \to R'$ a ring homomorphism, w a $\mathsf{c} \times \mathsf{N}$ matrix over R' and $e\colon \mathsf{r} \hookrightarrow \mathsf{N}$ an embedding. Let F be a complex function on the $\mathsf{N} \times \mathsf{a}$ matrices over R with $F(Ay) = F(y)$ for all $A \in \mathrm{Stab}(\varphi, w, e)$ and all y. Then for every $\mathsf{N} \times \mathsf{a}$ matrix y and every $\mathsf{N} \times \mathsf{N}$ matrix z over R*

$$\mathrm{cd}_{\chi,\varphi,w,e}(zI_e) \cdot [\mathrm{Fix}_e(z)]\, F(y) = [z \in \mathrm{Stab}(\varphi, w, e)]\, \chi(\det z)\, F(z^{-1}y),$$

where a bracket is 1 if the proposition in it holds and 0 if not.

Proof. Suppose $\mathrm{Fix}_e(z)$. Then $z = 1 + (z-1)I_e I_e^{\mathsf{T}}$, so

$$\det z = \det(1 + I_e^{\mathsf{T}}(z-1)I_e) = \det(I_e^{\mathsf{T}} z I_e),$$

since $I_e^{\mathsf{T}} I_e = 1$; and $w\varphi(z) = w$ holds if and only if $w\varphi(z)I_e = wI_e$, because the columns outside $e(\mathsf{r})$ agree on both sides. So $\mathrm{cd}_{\chi,\varphi,w,e}(zI_e)$ is $\chi(\det z)$ if $w\varphi(z) = w$, and 0 if not. If z is not invertible, then $\chi(\det z) = 0$ and z is in no subgroup of $\mathrm{GL}_{\mathsf{N}}(R)$: both sides vanish. If z is invertible, the left side is $\chi(\det z)F(y)$ when z lies in the stabiliser and 0 otherwise, and for z in the stabiliser $F(y) = F(z^{-1}y)$ by the invariance of F under z^{-1}. □

LEAN columnDatum_mul_ite_eq

Weights and norms of products

Definition 2.4 (weight with a norm bound)**.** Let Pl and n be finite, p a proposition, κ a weight, F a polynomial and b a real number. $\mathrm{WN}_p(\kappa, F, b)$ means: there is a coordinate vector ξ of weight κ with rows n such that $\mathrm{pol}_\kappa(\xi) = F$ and, if p holds, $\|\xi\|_\kappa^2 \le b$.

LEAN HasWeightNormLe

Lemma 2.5 (products)**.** *Let* Pl *and* n *be finite, p a proposition, s a finite set and, for $i \in s$, let κ_i be a weight, F_i a polynomial and b_i a real number. If $\mathrm{WN}_p(\kappa_i, F_i, b_i)$ for all $i \in s$, then*

$$\mathrm{WN}_p\Big(\textstyle\sum_{i\in s} \kappa_i, \prod_{i\in s} F_i, \prod_{i\in s} b_i\Big).$$

Proof. By induction on s. For the empty set the polynomial 1 is the polynomial of the coordinate vector of weight 0 with entry 1, of squared norm 1. For the step it suffices to treat two factors $F = \mathrm{pol}_\kappa(\xi)$ and $F' = \mathrm{pol}_{\kappa'}(\xi')$. The product FF' is semi-invariant of weight $\kappa + \kappa'$, so by Input 1 it is $\mathrm{pol}_{\kappa+\kappa'}(\xi'')$ for some ξ'', and by Input 2 $\|\xi''\|^2_{\kappa+\kappa'} \le \|\xi\|^2_\kappa \, \|\xi'\|^2_{\kappa'}$. The squared norms are ≥ 0, so the bounds multiply. □

LEAN HasWeightNormLe.prod

Auxiliary pairs

Definition 2.6 (the finite groups of level places)**.** In Setting 2.1, let Σ be level places of E, r and n finite index sets and $u\colon \mathsf{r} \hookrightarrow \mathsf{n}$ an embedding.

(i) $\mathsf{G}(\Sigma, \mathsf{r})$ is the product of the groups $\mathrm{GL}_{\mathsf{r}}(k_w)$ over the chosen places w of Σ.

(ii) $\mathrm{blk}_{\Sigma,u}(g)$, for $g = (g_w) \in \mathsf{G}(\Sigma, \mathsf{r})$, is the family in the product of the groups $\mathrm{GL}_{\mathsf{n}}(k_w)$ over all places of Σ equal to $\mathrm{diag}(g_w, 1)$ at the chosen places, the block g_w on the indices $u(\mathsf{r})$, and to 1 at the twist places; it is a homomorphism in g.

(iii) $\mathsf{H}(\Sigma, u)$ is the subgroup of the product of the groups $\mathrm{GL}_{\mathsf{n}}(k_w)$ over all places of Σ formed by the families that fix the basis vectors of index in $u(\mathsf{r})$ at the chosen places and whose rows of index in $u(\mathsf{r})$ vanish outside the columns $u(\mathsf{r})$ at the twist places.

LEAN LevelPlaces.ChosenGL · firstBlockHom · auxLevel

Definition 2.7 (auxiliary pair)**.** In Setting 2.1, let r and n be finite index sets. An *auxiliary pair* for $(E, \mathsf{r}, \mathsf{n})$, an element O of $\mathrm{Aux}(E; \mathsf{r}, \mathsf{n})$, consists of: level places of E, with products $\mathfrak{M}_O$ and $\mathfrak{Q}_O$; a sign vector s_O on r and a hermitian space W_O with index set r and signs s_O; an embedding $e_O \colon \mathsf{r} \hookrightarrow \mathsf{n}$; a numbering $\mathrm{ord}_O \colon \mathsf{r} \to \mathbb{N}$; two weights κ_O^+ and κ_O^-; and a character μ_O with a weight and a tame part ψ_O such that $\psi_{O,w} = 1$ at every finite place w that is not a place of the level places; with $|\mathsf{r}| < |\mathsf{n}|$. The second space of the pair is I_{n}.

LEAN AuxPair

Definition 2.8 (the objects of an auxiliary pair)**.** In Setting 2.1, let O be an auxiliary pair for $(E, \mathsf{r}, \mathsf{n})$.

(1) *Level.* Γ_O is the level in $\mathbf{K}_{\mathsf{n}}$ of the elements whose reductions at the places of O form an element of $\mathsf{H}(O, e_O)$. The character $\bar{\chi}_O(g)$ of $g \in \mathsf{H}(O, e_O)$ is the product over the twist places w of $\psi_{O,w}^{-1}$ at the determinant of the block of g_w with rows and columns $e_O(\mathsf{r})$, and χ_O is the character of Γ_O obtained by reduction.

(2) *Datum at the level.* For a family $X = (X_w)$ of $\mathsf{n} \times \mathsf{r}$ matrices over the residue fields at the places of O, $\mathrm{res}_O(X)$ is the product over the twist places w of $\psi_{O,w}$ at the determinant of the rows $e_O(\mathsf{r})$ of X_w if $X_w = I_{e_O}$ at every chosen place, and 0 otherwise.

(3) *Weight and forms.* κ_O is the weight and $\epsilon_O(\tau)$ the integer given by

$$\kappa_O \colon (\tau, i) \mapsto \kappa_{O,\tau}^+(i) + \kappa_{O,\tau}^-(|\mathsf{n}| - i), \qquad \epsilon_O(\tau) = -\sum_{1 \le i < |\mathsf{n}|} \kappa_{O,\tau}^-(i).$$

$\mathcal{F}(O)$ is the space of algebraic modular forms on I_{n} of weight $(\kappa_O, \epsilon_O(\cdot))$ on which Γ_O acts through χ_O. $\mathrm{cork}(O) = |\mathsf{n}| - |\mathsf{r}|$.

(4) *The finite group.* For a complex representation π of $\mathrm{U}(I_{\mathsf{n}})(\mathbb{A}_f)$, $\mathrm{rep}_O(\pi)$ is the representation of $\mathsf{G}(O, \mathsf{r})$ on the vectors of π on which Γ_O acts through χ_O, in which g acts by $\prod_w \psi_{O,w}(\det g_w)$, the product over the chosen places, times π of a lift of $\mathrm{blk}_{O,e_O}(g)$. For the forms it is the representation act_O of $\mathsf{G}(O, \mathsf{r})$ on $\mathcal{F}(O)$.

(5) *Polynomials.* tp_O is the linear map from the complex functions on pairs of standard fillings, of weights κ_O^+ and κ_O^- with rows n, to polynomials that sends the indicator of a pair (l, l') to the product of the polynomial of the indicator of l, of weight κ_O^+, written in the positive letters and of the polynomial of the indicator of l', of weight κ_O^-, written in the negative letters: a variable (τ, i, h)

of the first is replaced by the sum of the variables $x_{\tau,i,c}$ over the $c \in \mathsf{r}$ with $s_O(c) = +1$ and $\mathrm{ord}_O(c) = h$, and likewise for the second with $s_O(c) = -1$. The *harmonic polynomial* of a coordinate vector ξ of weight κ_O is $P^O(\xi)$, where P^O is tp_O composed with the integral Cartan embedding of the weight κ_O into the tensor product of the weights κ_O^+ and κ_O^-.

(6) *Kernel.* $\mathbf{a}_O = v^{r_-}$, with r_- the number of signs -1 of s_O, and $\mathbf{b}_O = \mu_O\, v^{\lfloor |\mathsf{n}|/2 \rfloor}$. $\mathcal{K}_O$ is the theta kernel of the pair (W_O, I_{n}) with these two characters: the Weil representation $\omega_{\mathbf{a}_O,\mathbf{b}_O}$ with the theta functional. Υ_O is the finite datum of the function $x \mapsto \mathrm{res}_O((x \bmod w)_w)$ of the $\mathsf{n} \times \mathsf{r}$ matrices modulo $\mathfrak{M}_O^2\mathfrak{Q}_O$, and $\Upsilon_O[\xi]$ is the Schwartz function with archimedean part the Fock vector of $P^O(\xi)$ and finite part Υ_O.

(7) *Descent.* For $g \in \mathrm{U}(W_O)(\mathbb{A})$ and $h \in \mathrm{U}(I_{\mathsf{n}})(\mathbb{A}_f)$, $\bar\theta_O(g,h)$ is the linear form on coordinate vectors $\xi \mapsto \overline{\theta(\omega_{\mathbf{a}_O,\mathbf{b}_O}(g,h)\,\Upsilon_O[\bar\xi])}$. The *descent* of a form $\Xi \in \mathcal{F}(O)$ is the function on $\mathrm{U}(W_O)(\mathbb{A})$

$$\mathrm{lift}_O(\Xi)(g) = \sum_{h \bmod \Gamma_O}^{\mathrm{cl}} \bar\theta_O(g,h)(\Xi(h)).$$

(8) *Hecke test.* Let w be a finite place with $w \neq \bar w$ such that neither w nor $\bar w$ is a place of O. For $X \in \mathbb{C}$, the operator $H_{O,w}(X) = \sum_{i=0}^{|\mathsf{n}|} (-X)^i q_w^{i(i-1)/2} T_{w,i}$ acts on $\mathcal{F}(O)$, the $T_{w,i}$ being the Hecke operators at w. The *test value* is $z_O(w) = \big(\nu_{\mathbf{a}_O}(w)\, q_w^{\mathrm{cork}(O)-1}\big)^{-1}$, where $\nu_\alpha(w)$ denotes the Frobenius value at w of a character α in the rank at hand. A form $y \in \mathcal{F}(O)$ *lifts with parameters* z, where z assigns to each finite place a sequence of complex numbers, if at every finite place w as above y is an eigenvector of all $H_{O,w}(X)$, with eigenvalue

$$\prod_{0 \le l < \mathrm{cork}(O)} (1 - q_w^l\, \nu_{\mathbf{a}_O}(w) X) \cdot \sum_{i=0}^{|\mathsf{r}|} \Big(-\frac{q_w^{\mathrm{cork}(O)}\, \nu_{\mathbf{a}_O}(w)}{\nu_{\mathbf{b}_O}(w)}\, X \Big)^i q_w^{i(i-1)/2} z(w)_i.$$

LEAN AuxPair.levelGroup · AuxPair.auxLevelChar · AuxPair.levelChar · AuxPair.resDetDatum · AuxPair.κ · AuxPair.detExp · AuxPair.FormV · AuxPair.rankDiff · AuxPair.chosenGLRep · AuxPair.formAction · AuxPair.tensorPoly · AuxPair.harmonicPoly · AuxPair.a · AuxPair.b · AuxPair.pairThetaKernel · AuxPair.auxDatum · AuxPair.fockDatum · AuxPair.conjKernel · AuxPair.θ · AuxPair.heckePolyOnForms · AuxPair.testValue · AuxPair.Lifts

Remark 2.9 (two small facts)**.** (i) For coordinate vectors a and b of weights κ_O^+ and κ_O^-, the map tp_O sends the function $(l, l') \mapsto a_l\, b_{l'}$ to the product of the polynomial of a in the positive letters and the polynomial of b in the negative letters; this is the definition, extended bilinearly. (ii) If a form y in $\mathcal{F}(O)$ is, at a place w as in Definition 2.8 (8), an eigenvector of the Hecke polynomial of the right regular representation with eigenvalue $p(X)$, then $H_{O,w}(X)y = p(X)y$: the operator on the forms is the restriction of that Hecke polynomial.

LEAN AuxPair.tensorPoly_vecMulVec · AuxPair.heckePolyOnForms_eq_smul

The auxiliary pairs of the proof are the pairs (W_j, V_j) of the single lifts: the adjoint lift of the note *The volume of the chain of theta lifts as a product of lift volumes* is the adjoint of the descent of Definition 2.8 (7), and the Hecke test enters the notion of a fit family of twists.

Residue rings and the divided splitting

Definition 2.10 (moduli of level places)**.** In Setting 2.1, let Σ be level places of E. $\mathfrak{M}_\Sigma^2$ is the modulus $\mathfrak{M}_\Sigma^2$; for $x \in \mathcal{O}_E/\mathfrak{M}_\Sigma^2\mathfrak{Q}_\Sigma$, $\mathrm{red}_0(x)$ is its image under the reduction to $\mathcal{O}_E/\mathfrak{M}_\Sigma^2$; and $(1)_E$ is the modulus whose ideal is $\mathcal{O}_E$.

LEAN `LevelPlaces.chosenModulusSq` · `reduceToR₀` · `unitModulus`

Definition 2.11 (generator and divided splitting)**.** In Setting 2.1, let O be an auxiliary pair for $(E, \mathsf{r}, \mathsf{n})$.

(i) ϖ_O is the chosen generator of the modulus $\mathfrak{M}_O$: an integral idele, equal to 1 at the places that are not chosen places of O; $\hat{\varpi}_O$ is the same element, regarded in $\widehat{\mathcal{O}}_E$.

(ii) Let N, m, a be finite index sets and $z\colon \mathsf{r} \hookrightarrow \mathsf{m}$ an embedding. A *divided splitting*, an element $\mathcal{V}$ of $\mathrm{Div}(\mathsf{N}, \mathsf{m}, \mathsf{a}; O, z)$, consists of a bijection $\varepsilon\colon \mathsf{N} \sqcup \mathsf{m} \to \mathsf{n}$ with $\varepsilon(z(i)) = e_O(i)$ for $i \in \mathsf{r}$, of two embeddings $\mathsf{r} \hookrightarrow \mathsf{N}$ and $\mathsf{a} \hookrightarrow \mathsf{N}$, the *fixed* and the *divided* embedding, and of the conditions that $|\mathsf{N}| \le |\mathsf{n}|$ and κ_O^+ has no column of height h with $|\mathsf{N}| \le h < |\mathsf{n}|$, and the same for m and κ_O^-.

(iii) For a divided splitting $\mathcal{V}$, $\bar{\varpi}_\mathcal{V}$ is the image of $\hat{\varpi}_O$ in $\mathcal{O}_E/\mathfrak{M}_O^2$; it depends on O only.

LEAN `AuxPair.genIdele` · `AuxPair.genIntAdele` · `DividedSplitting` · `DividedSplitting.genResidue`

Lemma 2.12 (reduction modulo a power of the chosen primes)**.** *In Setting 2.1, let O be an auxiliary pair for $(E, \mathsf{r}, \mathsf{n})$, $c \ge 0$ an integer and $x \in \widehat{\mathcal{O}}_E$. Then x reduces to 0 modulo $\mathfrak{M}_O^c$ if and only if $\varpi_O^{-c}\, x \in \widehat{\mathcal{O}}_E$.*

Proof. The kernel of the reduction modulo $\mathfrak{M}_O^c$ consists of the integral adeles of valuation at least c at every chosen place. The idele ϖ_O has valuation 1 at the chosen places, which are prime factors of multiplicity one of $\mathfrak{M}_O$, and is 1 elsewhere. So $\varpi_O^{-c}x$ is integral at every place exactly when x has valuation at least c at the chosen places. □

LEAN `reduction_levelModulus_eq_zero_iff`

Lemma 2.13 (multiplying by the generator tests congruence modulo the chosen primes)**.** *In Setting 2.1, let O be an auxiliary pair for $(E, \mathsf{r}, \mathsf{n})$, N, m finite index sets, a an index set, $z\colon \mathsf{r} \hookrightarrow \mathsf{m}$ and $\mathcal{V}$ a divided splitting in $\mathrm{Div}(\mathsf{N}, \mathsf{m}, \mathsf{a}; O, z)$. Let k, c be index sets, Z a $\mathsf{k} \times \mathsf{c}$ matrix over $\mathcal{O}_E/\mathfrak{M}_O^2\mathfrak{Q}_O$ and $u\colon \mathsf{c} \to \mathsf{k}$ a map, with I_u the matrix whose column i is the standard basis vector of index $u(i)$. Then*

$$\bar{\varpi}_\mathcal{V} \cdot \mathrm{red}_0(Z) = \bar{\varpi}_\mathcal{V} \cdot I_u \quad \Longleftrightarrow \quad Z \equiv I_u \text{ modulo every chosen place of } O,$$

where $\mathrm{red}_0(Z)$ is Z reduced entry by entry.

Proof. Write each entry of $Z - I_u$ as the reduction of an integral adele, and let X be the matrix of these adeles. The left side says that $\hat{\varpi}_O X$ reduces to 0 modulo $\mathfrak{M}_O^2$. By Lemma 2.12, for $c = 2$ and for $c = 1$, this holds if and only if X reduces to 0 modulo $\mathfrak{M}_O$. Since $\mathfrak{M}_O$ is a product of distinct primes, two matrices over $\mathcal{O}_E/\mathfrak{M}_O$ are equal if and only if they are equal modulo each of these primes (Input 3). This is the right side. □

LEAN `DividedSplitting.genResidue_smul_eq_iff`

Remark 2.14 (bookkeeping lemmas of the file)**.** Five further lemmas of the same file are used in the other notes of this unit and need no argument. For level places Σ: the ideal $\mathfrak{M}_\Sigma^2\mathfrak{Q}_\Sigma$ is contained in $\mathfrak{M}_\Sigma^2$; the primes dividing $\mathfrak{M}_\Sigma^c$, $c \geq 1$, are the chosen places; the chosen lift to $\mathbf{K}_{\mathsf{n}}$ of an invertible matrix over $\mathcal{O}_E/\mathfrak{M}_\Sigma^2$ has component 1 outside the chosen places and their conjugates, and an element with this property has component 1 at every twist place; and reduction modulo a ring homomorphism of a block diagonal element of $\mathbf{K}_{\mathsf{n}}$, along a bijection $\mathsf{N} \sqcup \mathsf{m} \to \mathsf{n}$, is the block diagonal matrix of the reductions. For an auxiliary pair O: ϖ_O has component 1 at every finite place that is not a chosen place.

LEAN `levelModulus_ideal_le` · `places_levelModulus_false` · `liftMod_mem_supportedOn` · `atPlace_twistPlace_eq_one` · `map_block_eq_blockGL` · `genIdele_apply_of_notMem`

Stacked determinants

Definition 2.15 (stacked determinants, substitutions, coordinates)**.** Let Pl be a finite set.

(i) Let I and n' be finite index sets, n an index set, $\tau \in \mathrm{Pl}$, η a complex $\mathsf{I} \times \mathsf{n}$ matrix and $j \colon \mathsf{I} \sqcup \mathsf{n}' \to \mathsf{n}$ a map. The *stacked determinant* $\Delta_\tau(\eta; j)$ is the determinant of the square matrix, with rows and columns indexed by $\mathsf{I} \sqcup \mathsf{n}'$, whose entry at (i, c) is $\eta_{i,j(c)}$ for $i \in \mathsf{I}$ and the variable $x_{\tau,i,j(c)}$ for $i \in \mathsf{n}'$: the columns j of η stacked over the matrix x_τ of variables. It is a complex polynomial in the variables $x_{\tau,i,c}$, $i \in \mathsf{n}'$, $c \in \mathsf{n}$.

(ii) Let a, N, L be finite index sets, $e \colon \mathsf{a} \sqcup \mathsf{N} \to \mathsf{L}$ a bijection, $(\iota_\tau)_{\tau \in \mathrm{Pl}}$ a family of finite sets and $M = (M_{\tau k})$ a family of complex $\mathsf{a} \times \mathsf{L}$ matrices, $k \in \iota_\tau$. The *product of stacked determinants* is

$$\Delta(e; M) = \prod_{\tau \in \mathrm{Pl}} \prod_{k \in \iota_\tau} \Delta_\tau(M_{\tau k}; e),$$

a polynomial in the variables $x_{\tau,i,c}$, $i \in \mathsf{N}$, $c \in \mathsf{L}$.

(iii) For families $c = (c_\tau)$ of complex $\mathsf{N} \times \mathsf{N}$ matrices and $g = (g_\tau)$ of complex $\mathsf{L} \times \mathsf{L}$ matrices, $\mathrm{sub}_{c,g}$ is the substitution $x_\tau \mapsto c_\tau^{\mathsf{T}} x_\tau g_\tau^{\mathsf{T}}$ on the polynomials in these variables: $x_{\tau,a,b} \mapsto \sum_{i,j} c_\tau(i,a)\, g_\tau(b,j)\, x_{\tau,i,j}$.

(iv) Let $p \geq 0$ be an integer. For a polynomial F in the variables $x_{\tau,a,b}$, with $a \in \mathsf{N}$ and b in the disjoint union of $\{0, \ldots, p-1\}$ and L, $\mathrm{col}(F)$ is F regarded as a polynomial in the variables of the first p columns, with coefficients in the polynomials in the variables of the columns L.

(v) For a weight κ, such a polynomial F and a standard filling l of weight κ with rows N, the *l-th coordinate of F in the first columns* is

$$\mathrm{coord}_\kappa(F)_l = h_\kappa^{-1} \cdot \langle \mathrm{col}(F), b_l \rangle,$$

the Fock pairing taken in the variables of the first p columns; it is a polynomial in the variables of the columns L.

LEAN `stackedDet` · `stackedDetProd` · `twoSidedSubst` · `inFirstCols` · `firstColsCoord`

Lemma 2.16 (a determinant with fixed columns)**.** *Let* a, N, L *be finite index sets,* $e\colon \mathsf{L} \to \mathsf{a} \sqcup \mathsf{N}$ *a bijection and* Z *a complex* $\mathsf{L} \times \mathsf{N}$ *matrix. Then there are a complex* $\mathsf{a} \times \mathsf{L}$ *matrix* Y *and* $\gamma \in \mathbb{C}$ *such that for every complex* $\mathsf{L} \times \mathsf{a}$ *matrix* X

$$\det[X \mid Z]_e = \gamma \cdot \det((YX)^{\mathsf{T}}), \qquad \text{and} \qquad |\gamma|^2 \cdot |\det(YY^{\mathsf{H}})| \le |\det(Z^{\mathsf{H}}Z)|,$$

where $[X \mid Z]_e$ *is the square matrix with columns* X *and* Z*, its columns indexed by* L *through* e*.*

Proof. By Input 4 there is an invertible $\mathsf{L} \times \mathsf{L}$ matrix whose columns, indexed through e, split as $[U_1 \mid U_2]$ with $Z^{\mathsf{H}}U_1 = 0$, that is $U_1^{\mathsf{H}}Z = 0$. For matrices A, X of size $\mathsf{L} \times \mathsf{a}$ and B of size $\mathsf{L} \times \mathsf{N}$,

$$\overline{\det[A \mid B]_e} \cdot \det[X \mid Z]_e = \det \begin{pmatrix} A^{\mathsf{H}}X & A^{\mathsf{H}}Z \\ B^{\mathsf{H}}X & B^{\mathsf{H}}Z \end{pmatrix}.$$

For $A = U_1$ and $B = U_2$ the upper right block vanishes, so

$$\overline{\det[U_1 \mid U_2]_e} \cdot \det[X \mid Z]_e = \det(U_1^{\mathsf{H}}X)\det(U_2^{\mathsf{H}}Z)\,:$$

the first assertion holds with $Y = U_1^{\mathsf{H}}$ and $\gamma = \det(U_2^{\mathsf{H}}Z)/\overline{\det[U_1 \mid U_2]_e}$. For $A = X = U_1$ and $B = Z$ the same identity gives

$$|\det[U_1 \mid Z]_e|^2 = \det(YY^{\mathsf{H}})\det(Z^{\mathsf{H}}Z),$$

and the left side is $|\gamma|^2 \det(YY^{\mathsf{H}})^2$. If $\det(YY^{\mathsf{H}}) \neq 0$, divide by it; otherwise the inequality is trivial. □

LEAN `exists_det_fromCols`

Lemma 2.17 (the substitution law)**.** *Let* Pl, a, N, L *be finite,* $e\colon \mathsf{a} \sqcup \mathsf{N} \to \mathsf{L}$ *a bijection,* (ι_τ) *a family of finite sets,* $M = (M_{\tau k})$ *a family of complex* $\mathsf{a} \times \mathsf{L}$ *matrices and* $c = (c_\tau)$*,* $g = (g_\tau)$ *families of complex* $\mathsf{N} \times \mathsf{N}$ *and* $\mathsf{L} \times \mathsf{L}$ *matrices. Then*

$$\mathrm{sub}_{c,g}\Big(\Delta(e; (M_{\tau k}\, g_\tau^{\mathsf{T}})_{\tau,k})\Big) = \prod_{\tau \in \mathrm{Pl}} \big(\det c_\tau \cdot \det g_\tau\big)^{|\iota_\tau|} \cdot \Delta(e; M).$$

Proof. The substitution replaces the matrix $M_{\tau k}g_\tau^{\mathsf{T}}$ stacked over x_τ by $M_{\tau k}g_\tau^{\mathsf{T}}$ stacked over $c_\tau^{\mathsf{T}}x_\tau g_\tau^{\mathsf{T}}$, which is $\mathrm{diag}(1, c_\tau^{\mathsf{T}})$ times the matrix $M_{\tau k}$ stacked over x_τ, times g_τ^{T}. Determinants are multiplicative, and the columns are ordered by the same bijection e on both sides. Each of the $|\iota_\tau|$ factors at τ contributes $\det c_\tau \det g_\tau$. □

LEAN `twoSidedSubst_stackedDetProd`

Lemma 2.18 (coordinates in the first columns)**.** *Let* Pl, N, L *be finite,* $p \ge 0$ *an integer,* $c = (c_\tau)$ *and* $g = (g_\tau)$ *families of complex* $\mathsf{N} \times \mathsf{N}$ *and* $\mathsf{L} \times \mathsf{L}$ *matrices,* F *and* F' *polynomials as in Definition 2.15 (iv) and* $s \in \mathbb{C}$*. Suppose* $\mathrm{sub}_{c,\mathrm{diag}(1,g)}(F') = s \cdot F$*, where* $\mathrm{diag}(1, g)$ *is the family of the block matrices with the identity on the first* p *columns and* g_τ *on* L*. Then for every weight* κ *and every standard filling* l *of weight* κ *with rows* N

$$\mathrm{sub}_{c,g}\Big(\sum_{l'} S_\kappa(c)_{l,l'} \cdot \mathrm{coord}_\kappa(F')_{l'}\Big) = s \cdot \mathrm{coord}_\kappa(F)_l.$$

Proof. Apply col to the hypothesis. On the variables of the first p columns the substitution $\mathrm{sub}_{c,\mathrm{diag}(1,g)}$ is the row substitution $y \mapsto c^{\mathsf{T}} y$, and on the coefficients it is $\mathrm{sub}_{c,g}$. So $s \cdot \mathrm{col}(F)$ is obtained from $\mathrm{col}(F')$ by applying $\mathrm{sub}_{c,g}$ to the coefficients and the row substitution by c^{T} to the variables. Pair with b_l. A ring homomorphism applied to the coefficients commutes with the Fock pairing, and the row substitution by c^{T} on one side is the row substitution by c on the other (Input 5): the pairing becomes $\mathrm{sub}_{c,g}$ of $\langle \mathrm{col}(F'), b_l(c\,y)\rangle$. By the definition of the Schur matrix $b_l(cy) = \sum_{l'} S_\kappa(c)_{l,l'} b_{l'}(y)$. Dividing by h_κ gives the claim. □

LEAN `twoSidedSubst_firstColsCoord`

The carrier

Definition 2.19 (the carrier)**.** In Setting 2.1, let $m \geq 0$ be an integer.

(i) The *carrier* $\mathfrak{C}_{E,m}$ is the theta lift of the constant function 1 from the definite group $\mathrm{U}(I_m^-)$ to the unitary group of signature $(2m, m)$: for $x \in \mathrm{U}(I_{2m,m})(\mathbb{A})$,

$$\mathfrak{C}_{E,m}(x) = \int_{[\mathrm{U}(I_m^-)]} \theta^{I_m^-, I_{2m,m}}_{\upsilon^m, 1; (1)_E}[1, \textstyle\prod_\tau \det(y_\tau^-)^{2m}](g, x)\, dg,$$

where the product runs over the infinite places τ and y_τ^- is the $m \times m$ matrix of the variables $x_{\tau,i,c}$ with i in the negative block of the index set of $I_{2m,m}$.

(ii) For a character c, an index set n' and $h \in \mathrm{U}(I_{\mathsf{n}'})(\mathbb{A}_f)$,

$$\Xi^\circ_{m,c}(h) = \sum_{u \bmod \mathbf{K}_m}^{\mathrm{cl}} \theta^{I_m, I_{\mathsf{n}'}}_{c, 1; (1)_E}[1, 1](u, h),$$

the class sum over $\mathrm{U}(I_m)$ of the theta kernel with constant datum and polynomial 1.

LEAN `carrierForm` · `carrierXi`

The archimedean matching

Definition 2.20 (block signs, the positive block, the matched polynomial)**.** Let E be a number field and $\mathsf{n}, \mathsf{m}, \mathsf{a}, \mathsf{b}, \mathsf{p}, \mathsf{l}$ finite index sets.

(i) For a bijection $f\colon \mathsf{p} \sqcup \mathsf{l} \to \mathsf{b}$ and $i \in \mathsf{b}$, the sign $\mathrm{sgn}_f(i)$ is $+1$ if $i \in f(\mathsf{p})$ and -1 if $i \in f(\mathsf{l})$.

(ii) For a bijection $j\colon \mathsf{m} \sqcup \mathsf{l} \to \mathsf{n}$ and $k \in G_\infty(1)$, the definite group on m, the element $\mathrm{pos}_j(k)$ of $G_\infty(\mathrm{sgn}_j(\cdot))$ is, at each infinite place τ, the matrix with block k_τ on $j(\mathsf{m})$ and the identity on $j(\mathsf{l})$; it is a homomorphism in k.

(iii) For bijections f and j as in (i) and (ii), a family (ι_τ) of finite sets indexed by the infinite places and a family $\eta = (\eta_{\tau k})$ of complex $\mathsf{p} \times \mathsf{m}$ matrices, $k \in \iota_\tau$, the *matched polynomial* is

$$Q_{f,j}(\eta) = \prod_\tau \det(y_\tau^{--})^{|\iota_\tau| + |\mathsf{n}| - 2|\mathsf{l}|} \cdot \prod_\tau \prod_{k \in \iota_\tau} \det(\eta_{\tau k}\, y_\tau^{++}),$$

a polynomial in the variables $x_{\tau,i,c}$, $i \in \mathsf{n}$, $c \in \mathsf{b}$, where y_τ^{--} is the block of the variables with $i \in j(\mathsf{l})$, $c \in f(\mathsf{l})$ and y_τ^{++} the block with $i \in j(\mathsf{m})$, $c \in f(\mathsf{p})$. The exponent is a truncated difference of natural numbers.

(iv) For a bijection j as in (ii), fv_j is the linear map from the polynomials in the variables $x_{\tau,i,c}$, $i \in \mathsf{a}$, $c \in \mathsf{m}$, to the Fock space of the pair of signs $\mathrm{sgn}_j(\cdot)$ on n and of the definite space on a: replace the column index c by $j(c)$ and take the Fock vector.

LEAN `blockSigns` · `posBlockEmb` · `matchedPoly` · `posFockVector`

Remark 2.21. Two computations about these objects are used without comment below: $\mathrm{pos}_j(k)$ is block diagonal for the signs $\mathrm{sgn}_j(\cdot)$, its inverse is $\mathrm{pos}_j(k^{-1})$, its minor on the positive block is $\det k_\tau$ and its minor on the negative block is 1; and the number of $i \in \mathsf{b}$ with $\mathrm{sgn}_f(i) = -1$ is $|\mathsf{I}|$.

LEAN `posBlockEmb_spec` · `card_neg_blockSigns`

Inputs. The results below are taken for granted. Inputs 1 to 8 are proved in the background library of the artifact; the catalogue of the handout records, for each, whether a reference with its place has been confirmed. Input 9 belongs to the unit *Exchange* of this chapter.

Input 1 (the image of pol_{κ}). Over $\mathbb{C}$, the image of $\xi \mapsto \mathrm{pol}_{\kappa}(\xi)$ is the space of all polynomials that are semi-invariant of weight κ; the product of semi-invariants of weights κ and κ' is semi-invariant of weight $\kappa + \kappa'$. (Part of the standard basis theorem for Schur modules; reference to be confirmed.)

Input 2 (norms of products). If $\mathrm{pol}_{\kappa+\kappa'}(\xi'') = \mathrm{pol}_{\kappa}(\xi)\,\mathrm{pol}_{\kappa'}(\xi')$, then $\|\xi''\|^2_{\kappa+\kappa'} \le \|\xi\|^2_{\kappa}\,\|\xi'\|^2_{\kappa'}$. (Reference to be confirmed.)

Input 3 (the Chinese remainder theorem for matrices). If the ideal of a modulus is a product of distinct primes, two matrices over its residue ring are equal if and only if their reductions to the residue fields of these primes are equal.

Input 4 (clearing columns; the Hermite normal form over a field). For a complex $\mathsf{N} \times \mathsf{L}$ matrix M and an embedding $\mathsf{N} \hookrightarrow \mathsf{L}$ there is an invertible $\mathsf{L} \times \mathsf{L}$ matrix U such that the columns of MU with index outside the image of the embedding vanish. (The library proves it for valuation rings; a place for the reference is not confirmed.)

Input 5 (the Fock pairing). (a) For matrices z_{τ}, the row substitutions by z and by z^{T} are adjoint: $\langle p(zy), q\rangle = \langle p, q(z^{\mathsf{T}}y)\rangle$. (b) A ring homomorphism applied to the coefficients of two polynomials commutes with their Fock pairing. (c) For a non-zero complex polynomial q, with $\bar{q}$ the polynomial with conjugated coefficients, $\langle \bar{q}, q\rangle > 0$.

Input 6 (the real Weil representation on Fock vectors). Let s, s' be sign vectors of signatures (p_0, q_0) and (p_0', q_0'), and let $x \in G_{\infty}(s)$, $y \in G_{\infty}(s')$ be block diagonal, with blocks $x_{\tau}^{\pm}$, $y_{\tau}^{\pm}$. Then for every polynomial P

$$\omega^{\infty}_{\gamma,\gamma'}(x,y)\,v_P = \prod_{\tau} (\det y_{\tau}^{+})^{\gamma_{\tau}-q_0} (\det y_{\tau}^{-})^{\gamma_{\tau}-p_0} (\det x_{\tau}^{+})^{q_0'-\gamma_{\tau}'} (\det x_{\tau}^{-})^{p_0'-\gamma_{\tau}'} \cdot v_{P'},$$

where P' is obtained from P by the substitution $X \mapsto y^{\mathsf{T}} X\,(x^{-1})^{\mathsf{T}}$ on the blocks of variables whose row and column have the same sign and $X \mapsto y^{-1} X\,x$ on the other blocks.

Input 7 (polarisation). Let $\mathcal{S}$ be a commutative $\mathbb{C}$-algebra, M a linear form on $\mathcal{S}$, $Y = (Y_{\tau})$ matrices over $\mathcal{S}$ of size $\mathsf{n} \times \mathsf{a}$, q a complex polynomial in the entries of a generic family (X'_{τ}) of $\mathsf{n} \times \mathsf{a}$ matrices, K a complex polynomial in the entries of a generic family (X_{τ}) of $\mathsf{n} \times \mathsf{n}$ matrices, and $r \colon \mathsf{a} \to \mathsf{n}$ a map. If $M(q(u^{\mathsf{T}}Y)) = K(u)$ for every family $u = (u_{\tau})$ of unitary $\mathsf{n} \times \mathsf{n}$ matrices, then for every family $\eta = (\eta_{\tau k})$, $k \in \iota_{\tau}$, of complex $\mathsf{a} \times \mathsf{n}$ matrices

$$\Big\langle q,\ \prod_{\tau} \det(X'^{\,r}_{\tau})^{|\iota_{\tau}|} \Big\rangle \cdot M\Big(\prod_{\tau,k} \det(\eta_{\tau k} Y_{\tau})\Big) = \Big\langle K,\ \prod_{\tau,k} \det(\eta_{\tau k} X^{r}_{\tau}) \Big\rangle,$$

where $X'^{\,r}_{\tau}$ is the $\mathsf{a} \times \mathsf{a}$ block of X'_{τ} in the rows r and X^{r}_{τ} the $\mathsf{n} \times \mathsf{a}$ block of X_{τ} in the columns r. (Reference to be confirmed.)

Inputs, continued.

Input 8 (calculus of theta kernels). In Setting 2.1: (a) *decay*: if $\mathfrak{m} \neq (1)$ and $\Psi(y) \neq 0$ only for matrices y with a left inverse, then for every polynomial P and every h the function $g \mapsto \theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Psi, P](g,h)$ decreases rapidly on $\mathrm{U}(S)(\mathbb{A})$; this is Lemma 3.3 below, a direct consequence of the decay estimate of the library and of the finiteness of the moments of v_P. (b) *invariance*: the theta functional is invariant under the rational points of both groups. (c) *twisting*: for characters μ, μ', $\theta^{S,T}_{\alpha\mu,\beta\mu';\mathfrak{m}}[\Psi,P](g,h) = \mu(\det h)\,\mu'(\det g)^{-1}\,\theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Psi,P](g,h)$. (d) *translation by integral points*: right translation of g or of h by an element k of the stabiliser of the standard lattice can be moved into the datum: it replaces Ψ by a character value at $\det k$ times the translate of Ψ by the reduction of k modulo $\mathfrak{m}$; for a tame character the value is that of its character χ at the determinant of the reduction. (e) *seesaw*: for an orthogonal sum in the second place, the kernel of a product of data on the two blocks of rows and of a product of polynomials is the product of the kernels, $\theta^{S,T\oplus V}_{\alpha,\beta\beta';\mathfrak{m}}[\Psi\Psi', PP'](g,(h,h')) = \theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Psi,P](g,h)\cdot\theta^{S,V}_{\alpha,\beta';\mathfrak{m}}[\Psi',P'](g,h')$, and likewise for a sum in the first place. (f) *change of modulus*: if $\mathfrak{m} \subseteq \mathfrak{n}$, the kernel modulo $\mathfrak{m}$ of a function that factors through the matrices modulo $\mathfrak{n}$ is the kernel modulo $\mathfrak{n}$ of that function. (g) *negation*: there is a number of absolute value 1, depending on S and T only, by which the kernel of (S^-, T), with the inverse characters times powers of υ, the conjugated polynomial and the conjugated datum composed with multiplication by an element of norm -1, differs from the complex conjugate of the kernel of (S,T). (h) *definite groups*: for a function on $[\mathrm{U}(I_{\mathsf{a}})]$ that is invariant under G_∞ and under $\mathbf{K}_{\mathsf{a}}$, the integral is the class sum at level $\mathbf{K}_{\mathsf{a}}$; a class sum at a smaller level Γ is the class sum at level $\mathbf{K}_{\mathsf{a}}$ of the sum of the translates by representatives of $\mathbf{K}_{\mathsf{a}}/\Gamma$.

Input 9 (unit *Exchange*; the exchange theorem and block partners). (a) Let W, V, V' be hermitian spaces with index sets n, a, b and $\sigma\colon \mathsf{b}\sqcup\mathsf{a}\to\mathsf{n}$ a bijection; suppose that the number q_W of negative signs of W is the sum of those of V and V', that $|\mathsf{a}| = |\mathsf{b}| + 2d$, that $2q_W + 2 \le |\mathsf{n}|$, and let μ be a weight with $|\mathsf{n}| \le 2\mu_\tau + 2q_W$ for all τ and v a vector of the Fock space of (V', W) with finite moments. Then there are a real number C_0 and a vector x_0 of the Fock space of (W, V) with finite moments such that for every modulus $\mathfrak{m}$, character ν and character χ of $\mathcal{O}_E/\mathfrak{m}$ with ν tame for $(\mathfrak{m},\mu,\chi)$ there is $c_0 \neq 0$ with $|c_0|^2 = C_0\cdot\Pi_{\mathfrak{m}}(\nu,d)$ and, for all partners $\check{\Psi}$, Ψ for (σ,χ) and all $g\in\mathrm{U}(W)(\mathbb{A})$, with $\alpha_0 = \nu^{-1}\upsilon^{\lfloor|\mathsf{n}|/2\rfloor}$,

$$\int_{[\mathrm{U}(V)]} \theta(\omega_{\alpha_0,1}(g,h)(x_0\otimes\check{\Psi}))\,dh = c_0\cdot\int_{[\mathrm{U}(V')]}\theta(\omega_{\alpha_0,\alpha_0}(h,g)(v\otimes\Psi))\,dh.$$

(b) For a multiplicative character χ of a commutative ring and any function f, the functions $\check{\Psi}(x_1 \mid x_2) = \chi^{-1}(\det x_2)\,f(x_2^{-1}x_1)$ and $\Psi\left(\begin{smallmatrix} y_1 \\ y_2\end{smallmatrix}\right) = \chi^{-1}(\det y_1)\,f(-y_2y_1^{-1})$, with x_2 and y_1 square, are partners for the identity bijection and χ; and partners $\check{\Psi}$, Ψ for (σ,χ) satisfy $\chi(\det k)\,\check{\Psi}(kx) = \check{\Psi}(x)$ for invertible k.

3 Proofs

The archimedean step

Lemma 3.1 (equivariance of the product of stacked determinants)**.** *Let E be a number field, $\mathsf{n},\mathsf{m},\mathsf{a},\mathsf{p},\mathsf{l}$ finite index sets, $j\colon \mathsf{m}\sqcup\mathsf{l}\to\mathsf{n}$ and $e\colon \mathsf{p}\sqcup\mathsf{a}\to\mathsf{m}$ bijections, (ι_τ) a family of finite sets indexed by the infinite places, γ the weight with $\gamma_\tau=|\mathsf{l}|-|\iota_\tau|$, and $\eta=(\eta_{\tau k})$ a family of complex $\mathsf{p}\times\mathsf{m}$ matrices. For all $k\in G_\infty(1)$, the definite group on m, and all u in the definite group on a,*

$$\omega^\infty_{\gamma,0}(\mathrm{pos}_j(k),u)\ \ \mathrm{fv}_j\Big(\Delta\big(e;(\eta_{\tau k'}\,(k_\tau^{-1})^{\mathsf{T}})_{\tau,k'}\big)\Big)=\prod_\tau(\det k_\tau)^{-|\iota_\tau|}\cdot\mathrm{fv}_j(\Delta(e;\eta)).$$

Proof. By Input 6 for the pair of signs $\mathrm{sgn}_j(\cdot)$, of signature $(|\mathsf{m}|,|\mathsf{l}|)$, and of the definite signs on a: the element $(\mathrm{pos}_j(k),u)$ multiplies a Fock vector by $\prod_\tau(\det u_\tau)^{\gamma_\tau-|\mathsf{l}|}=\prod_\tau(\det u_\tau)^{-|\iota_\tau|}$ and substitutes $x_\tau\mapsto u_\tau^{\mathsf{T}}x_\tau(k_\tau^{-1})^{\mathsf{T}}$ in the variables of the positive columns, the only ones that occur. By Lemma 2.17, with $c=u$ and $g=k^{-1}$, this substitution multiplies the polynomial on the left by $\prod_\tau(\det u_\tau\det k_\tau^{-1})^{|\iota_\tau|}$ and turns it into $\Delta(e;\eta)$. The powers of $\det u_\tau$ cancel. □

LEAN `realPairWeil_stackedDetProd`

Proposition 3.2 (the archimedean matching)**.** *Let E be a number field, $\mathsf{n},\mathsf{m},\mathsf{a},\mathsf{b},\mathsf{p},\mathsf{l}$ finite index sets, $f\colon\mathsf{p}\sqcup\mathsf{l}\to\mathsf{b}$, $j\colon\mathsf{m}\sqcup\mathsf{l}\to\mathsf{n}$ and $e\colon\mathsf{p}\sqcup\mathsf{a}\to\mathsf{m}$ bijections, (ι_τ) a family of finite sets indexed by the infinite places τ of E, and γ the weight with $\gamma_\tau=|\mathsf{l}|-|\iota_\tau|$ for all τ. Let $\mathcal{H}_1$ be the Fock space of the pair of signs $\mathrm{sgn}_j(\cdot)$ on n and of the definite signs on a, and write v_Q also for the Fock vector of a polynomial Q in the variables $x_{\tau,i,c}$, $i\in\mathsf{n}$, $c\in\mathsf{b}$, in the Fock space of the pair of signs $\mathrm{sgn}_f(\cdot)$ and $\mathrm{sgn}_j(\cdot)$. Then there is such a polynomial Q_0 with the following property. For every $v\in\mathcal{H}_1$ there is $\varkappa\in\mathbb{C}$ such that for every family $\lambda=(\lambda_g)$ of continuous linear forms on $\mathcal{H}_1$, indexed by $g\in G_\infty(\mathrm{sgn}_j(\cdot))$, and every linear form λ' on the polynomials in these variables with*

- (H1) *$\lambda_g(\omega^\infty_{\gamma,0}(x)\,w)=\lambda_{g\,x_1}(w)$ for all g, all $x=(x_1,x_2)$ in the product of the two groups and all $w\in\mathcal{H}_1$, and*
- (H2) *$\lambda_x(v)=\lambda'(Q)$ for all x in $G_\infty(\mathrm{sgn}_j(\cdot))$ and all polynomials Q such that $\omega^\infty_{\gamma,0}(1,x)\,v_{Q_0}=v_Q$,*

one has, for every family $\eta=(\eta_{\tau k})$ of complex $\mathsf{p}\times\mathsf{m}$ matrices, $k\in\iota_\tau$,

$$\varkappa\cdot\lambda_1(\mathrm{fv}_j(\Delta(e;\eta)))=\lambda'(Q_{f,j}(\eta)).$$

LEAN `real_matching`

In the application $\lambda_g(w)$ is a theta integral over the definite group with archimedean vector w, evaluated at g, and $\lambda'(Q)$ is an integral of a theta kernel with polynomial Q over the smaller group. Hypothesis (H2) is then the exchange identity for one vector v about which nothing explicit is known, and the conclusion is the same identity for explicit polynomials, at the price of the constant $\varkappa$.

Proof. Write $d_\tau=|\iota_\tau|$, let η^0 be the family with $\eta^0_{\tau k}$ the $\mathsf{p}\times\mathsf{m}$ matrix whose row i is the standard basis vector of index $e(i)$, for all τ and k, and put $Q_0=Q_{f,j}(\eta^0)$ and $v^0=\mathrm{fv}_j(\Delta(e;\eta^0))$. Let $r\colon\mathsf{a}\to\mathsf{n}$ be the embedding $i\mapsto j(e(i))$. Developing along

the rows of η^0, the stacked determinant of $\eta^0_{\tau k}$ is, up to a sign, the determinant of the $\mathsf{a}\times\mathsf{a}$ block of x_τ in the columns $e(\mathsf{a})$; so $v^0=\pm v_{q_0}$ with $q_0=\prod_\tau \det(x_\tau^r)^{d_\tau}$, where x_τ^r is the block of the variables in the columns r.

Step 1: the constant. Let $\mathcal{L}$ be the set of the continuous linear forms λ on $\mathcal{H}_1$ with $\lambda(\omega^\infty_{\gamma,0}(1,u)w)=\lambda(w)$ for all u in the definite group on a and all w. For a complex number z of absolute value 1 let $t(z)$ be the element of the definite group on m that is, at every place, diagonal with entries z^{-2} on $e(\mathsf{p})$ and z^{-1} on $e(\mathsf{a})$, let $o(z)$ be the scalar z^{-1} in the definite group on a, and for $y\in\mathcal{H}_1$ put

$$\Omega(y,z)=\prod_\tau \det(o(z)_\tau)^{d_\tau}\cdot\omega^\infty_{\gamma,0}(\mathrm{pos}_j(t(z)),o(z))\,y.$$

Claim: for every $y\in\mathcal{H}_1$ there is $\varkappa(y)\in\mathbb{C}$ with $\frac{1}{2\pi}\int_0^{2\pi}\lambda(\Omega(y,e^{i\vartheta}))\,d\vartheta=\varkappa(y)\,\lambda(v^0)$ for all $\lambda\in\mathcal{L}$. The left side, as a function of λ, depends continuously on y for the topology of pointwise convergence, because $k\mapsto\mathrm{pos}_j(k)$ is continuous, and the multiples of the function $\lambda\mapsto\lambda(v^0)$ form a closed set of functions on $\mathcal{L}$. So the set of the y for which the claim holds is closed, and since the Fock vectors of polynomials are dense it suffices to prove the claim for $y=v_P$. By Input 6, as in the proof of Lemma 3.1, the element $(\mathrm{pos}_j(t(z)),o(z))$ multiplies v_P by $\prod_\tau\det(o(z)_\tau)^{-d_\tau}$, which cancels the factor in front, and multiplies the variable $x_{\tau,i,c}$ by $z^{-1}\cdot z^2=z$ if the column c lies in $j(e(\mathsf{p}))$, by $z^{-1}\cdot z=1$ if it lies in $r(\mathsf{a})$, and by z if it is a negative column. So $\Omega(v_P,z)=v_{P_z}$, where P_z is P with every variable of a column outside $r(\mathsf{a})$ multiplied by z. The function $T(z)=\lambda(v_{P_z})$ is a polynomial in z, and its mean over the unit circle is $T(0)=\lambda(v_q)$, where q is P with the variables outside the columns r put to zero. For a family u of unitary matrices, Input 6 gives $\omega^\infty_{\gamma,0}(1,u)v_q=\prod_\tau(\det u_\tau)^{-d_\tau}v_{q(u^{\mathsf{T}}x^r)}$, so the invariance of λ gives $\lambda(v_{q(u^{\mathsf{T}}x^r)})=\lambda(v_q)\prod_\tau(\det u_\tau)^{d_\tau}$. This is the hypothesis of Input 7 for the index set $\mathsf{n}=\mathsf{a}$, the identity map r, the linear form $F\mapsto\lambda(v_F)$, the matrices $Y_\tau=x_\tau^r$, the polynomial q and $K=\lambda(v_q)\prod_\tau\det(X_\tau)^{d_\tau}$. Its conclusion, for the family of identity matrices, is $\langle q,q_0\rangle\cdot\lambda(v_{q_0})=\lambda(v_q)\cdot\langle q_0,q_0\rangle$. Since $\langle q_0,q_0\rangle>0$ by Input 5 (c) and $v_{q_0}=\pm v^0$, the claim holds for $y=v_P$ with $\varkappa(v_P)=\pm\langle q,q_0\rangle/\langle q_0,q_0\rangle$, which does not depend on λ.

Step 2: the identity for the families η^0k^{T}. Fix v, put $\varkappa=\varkappa(v)$, and let λ, λ' satisfy (H1) and (H2). Let k be in the definite group on m. By (H1) with $x=(1,u)$, the form $\lambda_{\mathrm{pos}_j(k)}$ lies in $\mathcal{L}$. By the claim and by (H1),

$$\varkappa\cdot\lambda_{\mathrm{pos}_j(k)}(v^0)=\frac{1}{2\pi}\int_0^{2\pi}z^{-|\mathsf{a}|\sum_\tau d_\tau}\,\lambda_{\mathrm{pos}_j(k\,t(z))}(v)\,d\vartheta,\qquad z=e^{i\vartheta}.$$

By Input 6 for the pair of signs $\mathrm{sgn}_f(\cdot)$ and $\mathrm{sgn}_j(\cdot)$, the element $(1,\mathrm{pos}_j(k\,t(z)))$ sends v_{Q_0} to v_Q with $Q=\prod_\tau\det(k_\tau t(z)_\tau)^{-d_\tau}\cdot Q_{f,j}((\eta^0_{\tau k'}\,t(z)_\tau^{\mathsf{T}}k_\tau^{\mathsf{T}})_{\tau,k'})$: the block y^{--} is unchanged and y^{++} is multiplied on the left by $(k\,t(z))^{\mathsf{T}}$. Since $\eta^0_{\tau k'}t(z)_\tau=z^{-2}\eta^0_{\tau k'}$ and $\det t(z)_\tau=z^{-2|\mathsf{p}|-|\mathsf{a}|}$, this is $Q=z^{|\mathsf{a}|\sum_\tau d_\tau}\prod_\tau(\det k_\tau)^{-d_\tau}\cdot Q_{f,j}(\eta^0k^{\mathsf{T}})$. By (H2) the integrand is constant, and

$$\varkappa\cdot\lambda_{\mathrm{pos}_j(k)}(v^0)=\prod_\tau(\det k_\tau)^{-d_\tau}\cdot\lambda'(Q_{f,j}(\eta^0k^{\mathsf{T}})).$$

On the other hand Lemma 3.1, for k^{-1} and $u=1$, and (H1) give

$$\lambda_1(\mathrm{fv}_j(\Delta(e;\eta^0k^{\mathsf{T}})))=\prod_\tau(\det k_\tau)^{d_\tau}\cdot\lambda_{\mathrm{pos}_j(k)}(v^0).$$

Together: the identity of the proposition holds for $\eta = \eta^0 k^{\mathsf{T}}$, for every unitary k.

Step 3: all η. Both sides are values of linear forms at products of determinants $\det(\eta_{\tau k} Y_\tau)$. For the right side this is the definition of the matched polynomial, with $Y_\tau = y_\tau^{++}$ and the fixed factor $\prod_\tau \det(y_\tau^{--})^{d_\tau + |\mathsf{n}| - 2|\mathsf{l}|}$. For the left side, a polynomial is determined by its values, and at every point ζ of the space of variables Lemma 2.16 gives matrices $Z_\tau(\zeta)$ and numbers $c_\tau(\zeta)$ with $\Delta_\tau(\eta_{\tau k}; e)(\zeta) = c_\tau(\zeta) \det(\eta_{\tau k} Z_\tau(\zeta)^{\mathsf{T}})$ for all η. So the difference of the two sides is $M(\prod_{\tau,k} \det(\eta_{\tau k} Y_\tau))$ for one linear form M on the algebra of pairs of a function of ζ and a polynomial, with Y_τ the pair $(Z_\tau^{\mathsf{T}}, y_\tau^{++})$. By Step 2, M vanishes on these products for the families $\eta^0 u^{\mathsf{T}}$, u unitary, which are the values $q_1(u^{\mathsf{T}} Y)$ of the polynomial $q_1 = \prod_\tau \det(X_\tau^e)^{d_\tau}$, with X_τ^e the rows $e(\mathsf{p})$ of a generic $\mathsf{m} \times \mathsf{p}$ matrix. Input 7, for the index sets m and p in the places of n and a, the map e on p and $K = 0$, gives $\langle q_1, q_1 \rangle \cdot M(\prod_{\tau,k} \det(\eta_{\tau k} Y_\tau)) = 0$ for all η, and $\langle q_1, q_1 \rangle > 0$ by Input 5 (c). □

Kernels

Lemma 3.3 (rapid decay of the kernel)**.** *In Setting 2.1, let S and V be hermitian spaces with finite index sets n and b, a and β characters, $\mathfrak{m}$ a modulus with at least one prime divisor, Ψ a function on the $\mathsf{b} \times \mathsf{n}$ matrices over $\mathcal{O}_E/\mathfrak{m}$ such that every y with $\Psi(y) \neq 0$ has a left inverse, P a polynomial and $h \in \mathrm{U}(V)(\mathbb{A})$. Then $g \mapsto \theta^{S,V}_{a,\beta;\mathfrak{m}}[\Psi, P](g, h)$ decreases rapidly on $\mathrm{U}(S)(\mathbb{A})$: its product with every power of the height is bounded.*

Proof. The decay estimate of the library bounds the kernel, times any power of the height of g, by a constant times a moment of the archimedean vector and a power of the height of h, under exactly these hypotheses on $\mathfrak{m}$ and Ψ; and all moments of a Fock vector v_P are finite. □

LEAN `isRapid_modKernel`

Lemma 3.4 (from the stabiliser to the full level)**.** *In Setting 2.1, let n, b, r be finite index sets, c an index set, R' a commutative ring, $\mathfrak{m}$ a modulus, W a hermitian space with index set $\mathsf{n} \sqcup \mathsf{b}$, α and β characters, γ a weight and χ a character of $\mathcal{O}_E/\mathfrak{m}$ such that α is tame for $(\mathfrak{m}, \gamma, \chi)$. Let P be a polynomial with $\omega^\infty_{\gamma,0}(1, u) v_P = v_P$ for all u in the definite group on b, let $\varphi \colon \mathcal{O}_E/\mathfrak{m} \to R'$ be a ring homomorphism, w a $\mathsf{c} \times \mathsf{b}$ matrix over R', $e \colon \mathsf{r} \hookrightarrow \mathsf{b}$ an embedding, and F a complex function on the $\mathsf{b} \times \mathsf{n}$ matrices over $\mathcal{O}_E/\mathfrak{m}$ with $F(Ay) = F(y)$ for all $A \in \mathrm{Stab}(\varphi, w, e)$ and all y. Write a $\mathsf{b} \times (\mathsf{n} \sqcup \mathsf{b})$ matrix as $x = (x' \mid x'')$, and let Γ be the level of the elements of $\mathbf{K}_\mathsf{b}$ whose reduction modulo $\mathfrak{m}$ lies in $\mathrm{Stab}(\varphi, w, e)$. Then for every $g \in \mathrm{U}(W)(\mathbb{A})$*

$$\sum_{h \bmod \Gamma}^{\mathrm{cl}} \theta^{W, I_\mathsf{b}}_{\alpha,\beta;\mathfrak{m}}[\Psi_1, P](g, h) = \sum_{h \bmod \mathbf{K}_\mathsf{b}}^{\mathrm{cl}} \theta^{W, I_\mathsf{b}}_{\alpha,\beta;\mathfrak{m}}[\Psi_0, P](g, h),$$

with $\Psi_1(x) = \mathrm{cd}_{\chi,\varphi,w,e}(x'' I_e) \cdot [\mathrm{Fix}_e(x'')]\, F(x')$ and $\Psi_0(x) = \chi(\det x'')\, F(x''^{-1} x')$.

Proof. The kernel is invariant under left translation of h by rational points: the theta functional is (Input 8 (b)), and the archimedean component of a rational point fixes v_P by the hypothesis on P. By Input 8 (h) the left side is the class sum at level $\mathbf{K}_\mathsf{b}$ of the sum over the cosets q of the stabiliser in $\mathrm{GL}_\mathsf{b}(\mathcal{O}_E/\mathfrak{m})$ of the translates by lifts of q. By Input 8 (d) and the tameness of α, the translate by a lift of q is

the kernel with datum $x \mapsto \chi(\det q)\Psi_1(q^{-1}x)$. By Lemma 2.3, with $z = q^{-1}x''$ and $y = q^{-1}x'$,

$$\chi(\det q)\,\Psi_1(q^{-1}x) = [q^{-1}x'' \in \mathrm{Stab}(\varphi, w, e)]\,\chi(\det x'')\,F(x''^{-1}x').$$

If x'' is invertible, exactly one coset q contains x''; if not, $\chi(\det x'') = 0$. So the sum over q of the data is Ψ_0, and the kernel is linear in its datum. □

LEAN `stdClassSum_rowStabilizer_eq`

Proposition 3.5 (matching summed against a form on a square block)**.** *In Setting 2.1, let* n *and* b *be finite index sets,* V *a hermitian space with index set* b *and* U *one with index set* n*,* $\mathfrak{m}$ *a modulus,* a*,* β*,* β' *characters,* e *a weight and* χ *a character of* $\mathcal{O}_E/\mathfrak{m}$ *such that* $\beta\beta'$ *is tame for* $(\mathfrak{m}, e, \chi)$*. Let* Ξ *be a function on* $\mathrm{U}(I_{\mathsf{b}})(\mathbb{A}_f)$*,* F *a function on* $\mathrm{U}(V)(\mathbb{A})$ *and* $\varkappa \in \mathbb{C}$ *with*

$$\varkappa \cdot \sum_{h \bmod \mathbf{K}_{\mathsf{b}}}^{\mathrm{cl}} \Xi(h)\,\theta^{V,I_{\mathsf{b}}}_{a,\beta;\mathfrak{m}}[\chi\circ\det, 1](x,h) = F(x) \qquad \text{for all } x \in \mathrm{U}(V)(\mathbb{A}). \tag{1}$$

Let $\mathfrak{n}$ *be a modulus with* $\mathfrak{m} \subseteq \mathfrak{n}$*,* f *a function on the* $\mathsf{n} \times \mathsf{b}$ *matrices over* $\mathcal{O}_E/\mathfrak{n}$*,* P *a polynomial,* $g \in \mathrm{U}(U)(\mathbb{A})$*,* $\varkappa' \in \mathbb{C}$*, and let* K *assign to a function* $\check{\Psi}$ *and a point* $h \in \mathrm{U}(I_{\mathsf{b}})(\mathbb{A}_f)$ *a function* $K(\check{\Psi}, h)$ *on* $\mathrm{U}(I_{\mathsf{n}})(\mathbb{A}_f)$ *such that, for all partners* $\check{\Psi}$*,* Ψ *for the identity of* $\mathsf{b} \sqcup \mathsf{n}$ *and* χ^{-1} *and all* h*,*

$$\varkappa' \cdot \sum_{u \bmod \mathbf{K}_{\mathsf{n}}}^{\mathrm{cl}} K(\check{\Psi}, h)(u) = \int_{[\mathrm{U}(V)]} \theta^{V,I_{\mathsf{b}}\oplus U}_{a,\beta\beta';\mathfrak{m}}[\Psi, P](x,(h,g))\,dx, \tag{2}$$

the polynomial P *being read in the variables of the rows of* U*. Then, with*

$$\check{\Psi}_0(x' \mid x'') = \chi(\det x'') \sum_{k \in \mathrm{GL}_{\mathsf{b}}(\mathcal{O}_E/\mathfrak{n})} f((x''^{-1}x' \bmod \mathfrak{n})\,k),$$

one has

$$\varkappa' \cdot \frac{\varkappa}{|\mathrm{GL}_{\mathsf{b}}(\mathcal{O}_E/\mathfrak{n})|} \cdot \sum_{u \bmod \mathbf{K}_{\mathsf{n}}}^{\mathrm{cl}} \sum_{h \bmod \mathbf{K}_{\mathsf{b}}}^{\mathrm{cl}} \Xi(h)\,K(\check{\Psi}_0, h)(u) = \int_{[\mathrm{U}(V)]} F(x)\cdot\theta^{V,U}_{a,\beta';\mathfrak{n}}[f,P](x,g)\,dx.$$

Proof (*the translation rule is described, not recomputed*)*.* The proof is complete but for the exact form of the translation rule of Input 8 (d), which the formal text applies by simplification. For a matrix y with square upper block y_1 and lower block y_2 let $\Psi(y) = \chi(\det y_1)\,f(y_2 \bmod \mathfrak{n})$, and let

$$\Psi_0 = \sum_k \chi(\det \tilde{k})^{-1}\,\Psi(\,\cdot\,\tilde{k}),$$

the sum over $k \in \mathrm{GL}_{\mathsf{b}}(\mathcal{O}_E/\mathfrak{n})$ with a lift $\tilde{k}$ of k to $\mathrm{GL}_{\mathsf{b}}(\mathcal{O}_E/\mathfrak{m})$. By Input 9 (b), applied to the function $\sum_k f(\,\cdot\,k)$ and after a change of the summation index, $\check{\Psi}_0$ and Ψ_0 are partners for χ^{-1}. So (2) holds for them. A matrix y with $\Psi(y) \neq 0$ has y_1 invertible, hence a left inverse, so by Lemma 3.3 the kernels with datum Ψ are integrable over $[\mathrm{U}(V)]$. By Input 8 (d) the kernel with datum $\chi(\det\tilde{k})^{-1}\Psi(\,\cdot\,\tilde{k})$ is the kernel with datum Ψ at the point x translated by an integral point, and the integral over $[\mathrm{U}(V)]$ does not change under such a translation. So the right side of (2) is $|\mathrm{GL}_{\mathsf{b}}(\mathcal{O}_E/\mathfrak{n})|$

times the integral of the kernel with datum Ψ, which by the seesaw (Input 8 (e)) and the change of modulus (Input 8 (f)) is

$$\theta^{V,I_{\mathsf{b}}}_{a,\beta;\mathfrak{m}}[\chi\circ\det,1](x,h)\cdot\theta^{V,U}_{a,\beta';\mathfrak{n}}[f,P](x,g).$$

Multiply by $\varkappa\,\Xi(h)$, take the class sum over h, which is a finite sum and commutes with the integral, and use (1). □

LEAN explicit_matching_of_sum

Proofs of the theorems

Proof of Theorem A. The signs of W are $\mathrm{sgn}_j(\cdot)$, so W has $|\mathsf{l}|$ negative signs, as has V', and I_{a} has none. Take Q_0 and the function $\varkappa$ of Proposition 3.2 for f, j, e, (ι_τ). Apply Input 9 (a) to W, $V=I_{\mathsf{a}}$, V', σ and the vector $v=v_{Q_0}$, whose moments are finite; its numerical hypotheses hold by the assumptions on $|\mathsf{a}|$, $|\mathsf{l}|$, $|\mathsf{n}|$ and by the weight of the character below. It gives C_0 and x_0. Put $C=|\varkappa(x_0)|^2/C_0$.

Let $\mathfrak{m}$, α, γ, χ be as in the statement. The character $\nu=\alpha^{-1}\upsilon^{|\mathsf{b}|+d}$ is tame for $\mathfrak{m}$ and χ^{-1}, and $\nu^{-1}\upsilon^{\lfloor|\mathsf{n}|/2\rfloor}=\alpha$, because $|\mathsf{n}|=|\mathsf{a}|+|\mathsf{b}|=2(|\mathsf{b}|+d)$. Input 9 (a) gives $c_0\neq0$ with $|c_0|^2=C_0\,\Pi_{\mathfrak{m}}(\nu,d)$; put $c=\varkappa(x_0)/c_0$, so that $|c|^2$ is as stated. Fix η, partners $\check\Psi$, Ψ for (σ,χ^{-1}), β and g. For $g'\in G_\infty(\mathrm{sgn}_j(\cdot))$, regarded as an adelic point with trivial finite part, and a vector w of the Fock space, let

$$\lambda_{g'}(w)=\int_{[\mathrm{U}(I_{\mathsf{a}})]}\theta(\omega_{\alpha,1}(gg',h)(w\otimes\check\Psi))\,dh,$$
$$\lambda'(Q)=c_0\int_{[\mathrm{U}(V')]}\theta^{V',W}_{\alpha,\alpha;\mathfrak{m}}[\Psi,Q](x,g)\,dx.$$

The forms $\lambda_{g'}$ are continuous, and λ' is defined because the kernels are rapidly decreasing (Lemma 3.3; a matrix in the support of Ψ consists of columns of an invertible matrix). Hypothesis (H1) holds because the theta integral intertwines the Weil representation at the infinite places with right translation. Hypothesis (H2), for $v=x_0$: if $\omega^\infty_{\gamma,0}(1,g')v_{Q_0}=v_Q$, then by the exchange identity at the point gg'

$$\begin{aligned}\lambda_{g'}(x_0)&=c_0\int_{[\mathrm{U}(V')]}\theta(\omega_{\alpha,\alpha}(x,gg')(v_{Q_0}\otimes\Psi))dx\\&=c_0\int_{[\mathrm{U}(V')]}\theta(\omega_{\alpha,\alpha}(x,g)(v_Q\otimes\Psi))dx=\lambda'(Q).\end{aligned}$$

Proposition 3.2 gives $\varkappa(x_0)\,\lambda_1(\mathrm{fv}_j(\Delta(e;\eta)))=\lambda'(Q_{f,j}(\eta))$. The vector $\mathrm{fv}_j(\Delta(e;\eta))$ is fixed by the definite group at infinity (Lemma 3.1 with $k=1$), and the kernel is invariant under right translation of h by $\mathbf{K}_{\mathsf{a}}$: by Input 8 (d) the translation acts on the datum through χ, and the partner $\check\Psi$ absorbs it (Input 9 (b)). So by Input 8 (h) the integral λ_1 is the class sum, and after division by c_0

$$c\cdot\sum_{h\bmod\mathbf{K}_{\mathsf{a}}}^{\mathrm{cl}}\theta^{W,I_{\mathsf{a}}}_{\alpha,1;\mathfrak{m}}[\check\Psi,\Delta(e;\eta)^j](g,h)=\int_{[\mathrm{U}(V')]}\theta^{V',W}_{\alpha,\alpha;\mathfrak{m}}[\Psi,Q_{f,j}(\eta)](x,g)\,dx.$$

Twisting by β in the second place on the left and by β^{-1} in the first place on the right multiplies both sides by $\beta(\det g)^{-1}$ (Input 8 (c)), which gives the statement. □

Proof of Theorem B. Apply Theorem A with $\mathsf{n} = \mathsf{m} = \mathsf{a} = \{1, \dots, m\}$, $\mathsf{p} = \mathsf{l} = \mathsf{b} = \emptyset$, $W = I_m$, V' the zero space, $d = m/2$ and ι_τ a set of c_τ elements. The conditions $|\mathsf{a}| = 2d$ and $2 \le m$ hold since m is even and $m \ge 1$, and $\gamma_\tau = -c_\tau$. Take the family η in which each of the c_τ matrices $\eta_{\tau k}$, $k \in \iota_\tau$, is the empty $0 \times m$ matrix: a matrix with no rows stacked over x_τ is x_τ, so $\Delta(e;\eta) = \prod_\tau \det(x_\tau)^{c_\tau}$, and the matched polynomial is 1. Take the partners of Input 9 (b) for $f = 1$: $\check{\Psi} = \chi \circ \det$ and $\Psi = 1$. Take $\beta = \alpha$. The right side of Theorem A is the integral of the constant 1 over the quotient of the trivial group, a positive number ϱ_0. So Theorem B holds with $\gamma = c/\varrho_0$ and the constant C/ϱ_0^2. □

Proof of Theorem C (the characters are not recomputed). Three facts are combined; the characters in the seesaw and in the negation of the first space are followed as the formal text follows them, by simplification, and were not recomputed. Write $J = I_m$ and $S_0 = I_{2m,m}$. First, by Input 8 (g) there is n_0 with $|n_0| = 1$ relating the kernels of (J^-, J) to the conjugates of those of (J, J). Second, Theorem B with $c_\tau = 0$ gives a constant C_1 and, for the character c^{-1}, which is tame of weight 0 for $(\mathfrak{m}, \chi^{-1})$, a number γ_1 with $|\gamma_1|^2 = C_1/\Pi_{\mathfrak{m}}(c\,\upsilon^{m/2}, m/2)$ and

$$\gamma_1 \sum_{h \bmod \mathbf{K}_m}^{\mathrm{cl}} \theta^{J,J}_{c^{-1},c^{-1};\mathfrak{m}}[\chi^{-1} \circ \det, 1](g,h) = 1.$$

Conjugating and passing to J^- gives (1) for $V = J^-$, the functions $\Xi = 1$ and $F = 1$, and $\varkappa = \bar{\gamma}_1 n_0/\chi(\epsilon)^m$, where ϵ is the element of norm -1 of Input 8 (g) and $|\chi(\epsilon)| = 1$. Third, Theorem A for $W = J \oplus S_0$, with $4m$ indices of which m are negative, $V' = J^-$, $|\mathsf{a}| = 3m$, $d = m$ and empty sets ι_τ, so that $\gamma_\tau = m$, gives C_2 and, for the character $c\,\upsilon^m$, tame of weight m, a number γ_2 with $|\gamma_2|^2 = C_2/\Pi_{\mathfrak{m}}(c^{-1}\upsilon^m, m)$; its matched polynomial is $\prod_\tau \det(y_\tau^{--})^{2m}$, the polynomial of the carrier, and its identity is (2) with $\varkappa' = \gamma_2$, $U = S_0$ and $K(\check{\Psi}, h)(u)$ the kernel of $(J \oplus S_0, I_{3m})$ at $((h,x),u)$. Proposition 3.5, with the modulus $\mathfrak{n} = (1)_E$, whose general linear group has one element, and $f = 1$, gives

$$\begin{aligned}\gamma_2\,\varkappa \cdot \sum_{u \bmod \mathbf{K}_{3m}}^{\mathrm{cl}} \sum_{h \bmod \mathbf{K}_m}^{\mathrm{cl}} K(\check{\Psi}_0, h)(u) &= \int_{[\mathrm{U}(J^-)]} \theta^{J^-,S_0}_{\upsilon^m,1;(1)_E}[1, \textstyle\prod_\tau \det(y_\tau^-)^{2m}](x', x)\,dx' \\ &= \mathfrak{C}_{E,m}(x),\end{aligned}$$

with $\check{\Psi}_0(x' \mid x'') = \chi(\det x'')$. By the seesaw in the first place (Input 8 (e)) the kernel $K(\check{\Psi}_0, h)(u)$ is the product of the kernel of (J, I_{3m}) with constant datum at (h,u) and of the kernel of (S_0, I_{3m}) with datum $\chi \circ \det$ at (x,u); the class sum of the first over h is $\Xi^\circ_{m,c}(u)$. This is Theorem C with $\gamma = \gamma_2\varkappa$ and $C = C_1 C_2$. □

LEAN explicit_matching · stdClassSum_det_modKernel_eq · stdClassSum_carrierXi_eq_carrierForm

4 Where the results are used

The auxiliary pairs, their forms and their descent are used throughout Chapters VII and VIII: the single lifts of the chain are adjoint to descents of auxiliary pairs. The divided splitting, the stabiliser and the column datum serve the glued lattices of the note *Glued lattices, product forms and the integral comparison forms.* Theorem A, Lemma 3.4, Proposition 3.5 and Theorem C are the inputs of the identity "the twist survives only as a constant", which is Theorem C of the note *The height of a ratio of*

lift volumes; Lemma 2.18 serves the line multipliers of that note, and Lemma 2.5 gives there the size of the polynomials.

5 Notes on the formal text

1. *Arguments that only a compiler can settle.* The arguments of the declarations were read from the text. For six definitions and two results, a lemma and a proposition, this reading lists a letter that the definition binds itself, as the index of a product or as the variable of a quantifier, and that Lean, by its rule for section variables, presumably does not take as an argument: the place τ in the product of stacked determinants, in the substitution $\mathrm{sub}_{c,g}$, in col, in the matched polynomial, in the carrier, in Lemma 2.17 and in Proposition 3.2; the map j in $\mathrm{sub}_{c,g}$; the family η in Proposition 3.2, where it is quantified in the conclusion; and the condition $w \neq \bar{w}$ in the definition of "lifts with parameters", where it is quantified inside. The uses of these declarations in the files confirm the shorter lists (the carrier is written with the two arguments E and m, the matched polynomial with f, j, η). This note states the declarations with the arguments that their uses show; it was not confirmed by asking Lean. For this reason these six definitions carry no notation made by the build.

2. *Instances listed twice.* In the lemmas of Remark 2.14 and in Lemmas 2.12 and 2.13 the text reading lists the assumptions on E twice, once from a variable line that a later one shadows.

3. *An argument that is not used.* In Lemma 2.13 the divided splitting enters only through the element $\bar{\varpi}_{\mathcal{V}}$, which depends on O alone; in Definition 2.11 (iii) its argument is unnamed in the formal text.

4. *Constants that may vanish.* Theorems A to C assert the existence of a real number C without sign or non-vanishing. If $C = 0$, the identities say that their right sides vanish. In Theorem A the formal constant is $|\varkappa(x_0)|^2/C_0$, with C_0 from the exchange theorem.

5. *Truncated subtraction.* The exponent $|\iota_\tau| + |\mathsf{n}| - 2|\mathsf{l}|$ of the matched polynomial, the rank difference $|\mathsf{n}| - |\mathsf{r}|$ and the exponent in the test value are differences of natural numbers, which Lean truncates at zero; under the hypotheses of the statements they are not negative.

6. *Index sets of Theorem C.* The second space of the kernel in Theorem C has the index set of $I_{2m,m}$, the disjoint union of $\{1, \dots, 2m\}$ and $\{1, \dots, m\}$; it is written I_{3m} above.

7. *What was not reconstructed.* In the proof of Theorem C the bookkeeping of the characters in the seesaw and in the negation of the first space, and in the proof of Proposition 3.5 the exact form of the translation rule, are carried out in the formal text by simplification; they are described here and were not recomputed. In Step 3 of the proof of Proposition 3.2 the linear form M is defined in the formal text on pairs, after multiplication by the pair of the product of the numbers $c_\tau(\zeta)^{d_\tau}$ and of the fixed factor of the matched polynomial; the proof above says this in words.

8. *Letters that differ from the formal text.* The partner on the large side is a capital phi in the formal text and $\check{\Psi}$ here, as in the notes on the unit *Exchange*; the

finite set of places is a capital phi there and Pl here; the level places are S there and Σ here; the second ring of §2 and of Lemma 3.4 is S there and R' here; the matrix of Lemma 2.13 is S there and Z here; the matrix of the column datum is σ there and x here; the constants of Propositions 3.2 and 3.5 are κ, κ' there and $\varkappa$, $\varkappa'$ here. The capital phi and the letters S, σ and κ have other meanings in this write-up.

How this note was written. From the dossier of the unit, whose reader calls the step claims of the long proofs a first reading and notes that no second reader had looked at them. After the dossier was frozen, the second reader of the write-up compared the readings of all statements of the unit with the Lean text and sampled its structured proofs (her message to the writer of 26 November 2026); her two remarks on this part, the rank of the second space in Theorem C and the direction of the space of homomorphisms behind the Fock vectors of Definition 2.20, agree with the statements above. I read the five Lean files of these declarations completely. The proof of the archimedean matching (Proposition 3.2) I read line by line in the Lean text and recomputed its character identities by hand from Input 6: the cancellation of the determinant factor in Ω, the exponents of z in Steps 1 and 2, and the two uses of polarisation. These computations are mine; the formal proof obtains them by simplification, and I could not see its intermediate goals. I read the Lean statements of the exchange theorem, of the polarisation lemma, of the rule for the real Weil representation on Fock vectors, of the notions of a tame character, of partners and of the product of L-values, and the docstrings of the other definitions of Setting 2.1; I did not read the proofs of the inputs. The proofs of Lemmas 2.3, 2.16 and 2.17 are written out by hand and are shorter than the formal ones. The program check of the unit's dossier covered 634 leaf steps of 85 proofs.

Revision. This text was revised after the report of an independent referee (26 November 2026), who compared the statements with the Lean text and recomputed the proofs of Proposition 3.2 and of Theorems A to C. The changes: the partial L-function is that of the base change of the character, without the Euler factors at the conjugate places as well; Input 7 is stated with the sizes of its matrices; the box says which two computations are described and not carried out; several letters were changed, among them the partner on the large side, now $\check{\Psi}$ as in the notes on the unit *Exchange*.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Definition 2.2	`FixesColsOff`	PartnerCut.lean	16–17
Definition 2.2	`rowStabilizer`	PartnerCut.lean	21–23
Definition 2.2	`columnDatum`	PartnerCut.lean	27–28
Lemma 2.3	`columnDatum_mul_ite_eq`	PartnerCut.lean	33–43
Definition 2.4	`HasWeightNormLe`	SemiInvariantNorm.lean	11–11
Lemma 2.5	`HasWeightNormLe.prod`	SemiInvariantNorm.lean	13–20
Definition 2.6	`LevelPlaces.ChosenGL`	StackedDeterminants.lean	33–33
Definition 2.6	`firstBlockHom`	StackedDeterminants.lean	40–41
Definition 2.6	`auxLevel`	StackedDeterminants.lean	45–47
Definition 2.7	`AuxPair`	StackedDeterminants.lean	58–66

In this note	Lean declaration	File	Lines
Definition 2.8	`AuxPair.levelGroup`	StackedDeterminants.lean	76–76
Definition 2.8	`AuxPair.resDetDatum`	StackedDeterminants.lean	80–81
Definition 2.8	`AuxPair.κ`	StackedDeterminants.lean	84–84
Definition 2.8	`AuxPair.detExp`	StackedDeterminants.lean	88–88
Definition 2.8	`AuxPair.auxLevelChar`	StackedDeterminants.lean	92–94
Definition 2.8	`AuxPair.levelChar`	StackedDeterminants.lean	98–98
Definition 2.8	`AuxPair.FormV`	StackedDeterminants.lean	102–102
Definition 2.8	`AuxPair.chosenGLRep`	StackedDeterminants.lean	106–111
Definition 2.8	`AuxPair.formAction`	StackedDeterminants.lean	115–115
Definition 2.8	`AuxPair.tensorPoly`	StackedDeterminants.lean	119–121
Remark 2.9	`AuxPair.tensorPoly_vecMulVec`	StackedDeterminants.lean	124–127
Definition 2.8	`AuxPair.harmonicPoly`	StackedDeterminants.lean	131–131
Definition 2.8	`AuxPair.a`	StackedDeterminants.lean	134–134
Definition 2.8	`AuxPair.b`	StackedDeterminants.lean	138–138
Definition 2.8	`AuxPair.pairThetaKernel`	StackedDeterminants.lean	142–142
Definition 2.8	`AuxPair.auxDatum`	StackedDeterminants.lean	146–147
Definition 2.8	`AuxPair.fockDatum`	StackedDeterminants.lean	151–151
Definition 2.8	`AuxPair.conjKernel`	StackedDeterminants.lean	155–156
Definition 2.8	`AuxPair.θ`	StackedDeterminants.lean	160–161
Definition 2.8	`AuxPair.heckePolyOnForms`	StackedDeterminants.lean	167–170
Remark 2.9	`AuxPair.heckePolyOnForms_eq_smul`	StackedDeterminants.lean	173–175
Definition 2.8	`AuxPair.rankDiff`	StackedDeterminants.lean	178–178
Definition 2.8	`AuxPair.testValue`	StackedDeterminants.lean	182–182
Definition 2.8	`AuxPair.Lifts`	StackedDeterminants.lean	186–188
Definition 2.10	`LevelPlaces.chosenModulusSq`	StackedDeterminants.lean	199–199
Remark 2.14	`map_block_eq_blockGL`	StackedDeterminants.lean	201–203
Remark 2.14	`levelModulus_ideal_le`	StackedDeterminants.lean	205–206
Remark 2.14	`places_levelModulus_false`	StackedDeterminants.lean	208–210
Definition 2.10	`reduceToR₀`	StackedDeterminants.lean	213–214
Remark 2.14	`liftMod_mem_supportedOn`	StackedDeterminants.lean	216–217
Remark 2.14	`atPlace_twistPlace_eq_one`	StackedDeterminants.lean	224–225
Definition 2.11	`DividedSplitting`	StackedDeterminants.lean	229–232
Definition 2.11	`AuxPair.genIdele`	StackedDeterminants.lean	236–236
Definition 2.11	`AuxPair.genIntAdele`	StackedDeterminants.lean	240–240
Remark 2.14	`genIdele_apply_of_notMem`	StackedDeterminants.lean	244–245
Lemma 2.12	`reduction_levelModulus_eq_zero_iff`	StackedDeterminants.lean	247–250
Definition 2.11	`DividedSplitting.genResidue`	StackedDeterminants.lean	256–256
Lemma 2.13	`DividedSplitting.genResidue_smul_eq_iff`	StackedDeterminants.lean	258–266
Definition 2.15	`stackedDet`	StackedDeterminants.lean	277–277
Definition 2.10	`unitModulus`	StackedDeterminants.lean	280–280
Definition 2.19	`carrierForm`	StackedDeterminants.lean	289–291
Definition 2.19	`carrierXi`	StackedDeterminants.lean	297–298
Lemma 2.16	`exists_det_fromCols`	StackedDeterminants.lean	304–313
Definition 2.15	`twoSidedSubst`	StackedDeterminants.lean	317–318
Definition 2.15	`inFirstCols`	StackedDeterminants.lean	322–323
Definition 2.15	`stackedDetProd`	StackedDeterminants.lean	327–327
Lemma 2.17	`twoSidedSubst_stackedDetProd`	StackedDeterminants.lean	332–337
Definition 2.15	`firstColsCoord`	StackedDeterminants.lean	342–343
Lemma 2.18	`twoSidedSubst_firstColsCoord`	StackedDeterminants.lean	345–353
Definition 2.20	`blockSigns`	RealMatching.lean	20–20
Definition 2.20	`posBlockEmb`	RealMatching.lean	24–25
Remark 2.21	`posBlockEmb_spec`	RealMatching.lean	28–31
Remark 2.21	`card_neg_blockSigns`	RealMatching.lean	33–34
Definition 2.20	`matchedPoly`	RealMatching.lean	38–39
Definition 2.20	`posFockVector`	RealMatching.lean	43–44

In this note	Lean declaration	File	Lines
Lemma 3.1	`realPairWeil_stackedDetProd`	RealMatching.lean	52–56
Proposition 3.2	`real_matching`	RealMatching.lean	61–113
Lemma 3.3	`isRapid_modKernel`	Carrier.lean	42–45
Proposition 3.5	`explicit_matching_of_sum`	Carrier.lean	52–75
Lemma 3.4	`stdClassSum_rowStabilizer_eq`	Carrier.lean	80–103
Theorem A	`explicit_matching`	Carrier.lean	112–146
Theorem B	`stdClassSum_det_modKernel_eq`	Carrier.lean	159–171
Theorem C	`stdClassSum_carrierXi_eq_carrierForm`	Carrier.lean	178–199

The three theorems and the archimedean matching as they are written in the artifact.

```
lemma explicit_matching {E} [Field E] [CMType E] [WeilData E] {n m a b p l} [Fintype n] [Fintype m]
  [Fintype a] [Fintype b] [Fintype p] [Fintype l] {s} (W : Space E n s) (f : p ⊕ l ≃ b) (V' : Space
  E _ (blockSigns f)) σ e (j : m ⊕ l ≃ n) (hj : s = blockSigns j) {d} (hd : card a = card b + 2 * d)
  (hn : 2 * card l + 2 ≤ card n) ι [∀ τ, Fintype (ι τ)] : ∃ C, ∀ {m α γ x},
  (∀ τ, γ τ = card l - card (ι τ)) → ∃ c, ‖c‖ ^ 2 = C / lValueProd m.places
  (α⁻¹ * upsilonU1 ^ (card b + d)) d ∧ ∀ n {Φ Ψ}, IsPartner τ σ x⁻¹ Φ Ψ → ∀ β g, c * stdClassSum
  (Kf E a) (modKernelI W α β m Φ (renameRowsCols id (j ∘ .inl) (stackedDetProd e n)) g ·) = ∫ x,
  modKernel V' W (α / β) α m Ψ (matchedPoly (ι := ι) f j n) x g ∂V'.quotMeasure

lemma stdClassSum_det_modKernel_eq (he : Even m) (c : _ → ℕ) : ∃ C, ∀ {m α x},
  IsTameFor m α (-c ·) x → ∃ γ, ‖γ‖ ^ 2 = C / lValueProd
  m.places (α⁻¹ * upsilonU1 ^ (m / 2)) (m / 2) ∧ ∀ g, γ * stdClassSum (Kf E (Fin m))
  (modKernelI (stdSpace ..) α α m (x ·.det) (∏ τ, letterMinor τ id id ^ c τ) g) = 1

lemma stdClassSum_carrierXi_eq_carrierForm (he : Even m) : ∃ C, ∀ {m c x}, IsTameFor m c 0 x →
  ∃ γ, ‖γ‖ ^ 2 = C / (lValueProd m.places (c⁻¹ * upsilonU1 ^ m) m * lValueProd m.places
  (c * upsilonU1 ^ (m / 2)) (m / 2)) ∧ ∀ x, γ * stdClassSum (Kf ..) (fun h ↦ carrierXi m c h *
  modKernelI (signedSpace ..) (upsilonU1 ^ m) c m (x ·.det) 1 x h) = carrierForm E m x

lemma real_matching : ∃ Q₀, ∀ v, ∃ κ, ∀ (ℓ : _ → _ →L[ℂ] ℂ) (ℓ' : _ →ₗ[ℂ] ℂ), (∀ g x w,
  ℓ g (realPairWeil γ 0 x w) = ℓ (g * x.1) w) → (∀ x Q, realPairWeil (s := blockSigns f) γ 0 (1,
  x) (polyVector _ Q₀) = polyVector _ Q → ℓ x v = ℓ' Q) → ∀ n,
  κ * ℓ 1 (posFockVector j (stackedDetProd e n)) = ℓ' (matchedPoly (ι := ι) f j n)
```

In these texts `modKernel` and `modKernelI` are the theta kernels, `stdClassSum` the class sum, `Kf` the stabiliser of the standard lattice, `IsTameFor` tameness, `IsPartner` the partners, `lValueProd` the product of L-values, `upsilonU1` the character υ, `realPairWeil` the real Weil representation and `polyVector` the Fock vector of a polynomial.

Notation

E, $\mathcal{O}_E$, $\widehat{\mathcal{O}}_E$, $\mathbb{A}$, $\mathbb{A}_f$ a CM field; its integers; integral finite adeles; adeles; finite adeles

τ, w, $\bar{w}$, k_w, q_w an infinite place; a finite place, its conjugate, residue field and norm

υ the character of the Weil data

$\mathsf{r}, \mathsf{n}, \mathsf{N}, \mathsf{m}, \mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{p}, \mathsf{l}, \mathsf{L}, \mathsf{k}, \mathsf{n}'$ finite index sets; $|\mathsf{n}|$ the number of elements

$\mathfrak{m}$, $\mathfrak{n}$ moduli

Σ, $\mathfrak{M}_\Sigma$, $\mathfrak{Q}_\Sigma$ level places; the products of the chosen and of the twist primes

$\alpha, \beta, \beta', \mu, \mu', \nu, a, c$ characters of the norm-one ideles

γ, γ', e, μ (weights) integer weights at the infinite places

ψ, χ a tame part; a multiplicative character of a residue ring

$\Pi_T(\nu, d)$, $L^T(z, \nu)$ the product of L-values; the partial Hecke L-function of the base change of ν, without the Euler factors at T and at the conjugate places

s, s' sign vectors

S, T, U, V, V', W, J, S_0 hermitian spaces

I_n, I_m, $I_{2m,m}$, S^- the standard definite space; the signed space; the opposite space

$\mathrm{U}(S)(\mathbb{A})$, $[\mathrm{U}(S)]$, $G_\infty(s)$ adelic unitary group; automorphic quotient; archimedean group

$\mathbf{K}_\mathsf{n}$, Γ the stabiliser of the standard lattice; a level

$\sum^{\mathrm{cl}}_{h \bmod \Gamma}$, $\mathfrak{c}$, $h_{\mathfrak{c}}$ the class sum; a class; its representative

$\omega_{\alpha,\beta}$, $\omega^{\infty}_{\gamma,\gamma'}$, θ the Weil representation; its archimedean part; the theta functional

v_P, $x_{\tau,i,c}$ the Fock vector of a polynomial P; a variable

$\theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Psi,P](g,h)$ the theta kernel

$\check{\Psi}$, Ψ, Ψ_0, Ψ_1, $\check{\Psi}_0$, f (datum) functions on matrices over a residue ring; $\check{\Psi}$, Ψ partners

Pl a finite set, in the applications the set of infinite places

σ a bijection $\mathsf{b} \sqcup \mathsf{a} \to \mathsf{n}$

κ, κ', b_l, G_κ, h_κ, $S_\kappa(z)$ weights; basis polynomials; Gram matrix; hook number; Schur matrix

$\mathrm{pol}_\kappa(\xi)$, $\|\xi\|_\kappa$, ξ, ξ', ξ'' the polynomial and the norm of a coordinate vector

$\langle P,Q\rangle$ the Fock pairing

R, R', φ, w (matrix), I_e, I_u rings, a homomorphism, a matrix; inclusion matrices

$\mathrm{Fix}_e(z)$, $\mathrm{Stab}(\varphi,w,e)$, $\mathrm{cd}_{\chi,\varphi,w,e}(x)$ Definition 2.2

$\mathrm{WN}_p(\kappa,F,b)$ Definition 2.4

$\mathsf{G}(\Sigma,\mathsf{r})$, $\mathrm{blk}_{\Sigma,u}(g)$, $\mathsf{H}(\Sigma,u)$ Definition 2.6

O, $\mathrm{Aux}(E;\mathsf{r},\mathsf{n})$ an auxiliary pair; the set of them

$\mathfrak{M}_O$, $\mathfrak{Q}_O$, s_O, W_O, e_O, ord_O, $\kappa^{\pm}_O$, μ_O, ψ_O its components

Γ_O, χ_O, $\bar{\chi}_O(g)$, $\mathrm{res}_O(X)$ level, characters, datum

κ_O, $\epsilon_O(\tau)$, $\mathcal{F}(O)$, $\mathrm{cork}(O)$ weight, determinant exponent, forms, rank difference

$\mathrm{rep}_O(\pi)$, act_O representations of $\mathsf{G}(O,\mathsf{r})$

tp_O, P^O the tensor polynomial; the harmonic polynomial

$\mathbf{a}_O$, $\mathbf{b}_O$, r_-, $\mathcal{K}_O$ the characters of the kernel; the number of negative signs; the kernel

Υ_O, $\Upsilon_O[\xi]$, $\bar{\theta}_O(g,h)$, $\mathrm{lift}_O(\Xi)(g)$ finite datum, test datum, conjugate kernel, descent

$H_{O,w}(X)$, $T_{w,i}$, $z_O(w)$, $\nu_\alpha(w)$, z (parameters) Hecke test

$\mathfrak{M}^2_\Sigma$, $\mathrm{red}_0(x)$, $(1)_E$ Definition 2.10

ϖ_O, $\hat{\varpi}_O$, $\bar{\varpi}_{\mathcal{V}}$ the generator of $\mathfrak{M}_O$; its class

$\mathcal{V}$, $\mathrm{Div}(\mathsf{N},\mathsf{m},\mathsf{a};O,z)$, ε a divided splitting; the set of them; its bijection

$\Delta_\tau(\eta;j)$, $\Delta(e;M)$, $\Delta(e;\eta)^j$ stacked determinant; the product; with renamed columns

$\mathrm{sub}_{c,g}$, col, $\mathrm{coord}_\kappa(F)_l$ substitution; first columns; coordinates

η, η^0, M, ι_τ, d_τ families of matrices; the index sets and their sizes

$\mathfrak{C}_{E,m}$, $\Xi^\circ_{m,c}(h)$, y^-_τ the carrier; the class sum of Definition 2.19

$\mathrm{sgn}_f(i)$, $\mathrm{pos}_j(k)$, fv_j block signs; the positive block; Fock vectors

$Q_{f,j}(\eta)$, y^{++}_τ, y^{--}_τ the matched polynomial; blocks of variables

f, j, e (bijections) bijections of index sets

d, m, p, h, i, k, l integers, indices, fillings; h, g, x, u also points of groups

C, C_0, C_1, C_2, c, c_0, γ, γ_1, γ_2, $\varkappa$, $\varkappa'$, n_0, ϱ_0 constants

$\mathcal{H}_1$, $\mathcal{L}$, λ, λ_g, λ', Q_0, Q, v, v^0, x_0 Fock space, invariant forms, linear forms, polynomials, vectors of Proposition 3.2

$t(z)$, $o(z)$, $\Omega(y,z)$, $T(z)$, P_z, q, q_0, q_1, r, ϑ objects of its proof; ϑ an angle

X, X', Y, Z, U_1, U_2, A, B, K, F, F', Ξ, ζ, ϵ matrices, polynomials, functions, points

$\mathcal{S}$, M (linear form), π, y, a, b, u, g, x, x', x'', y_1, y_2, $\tilde{k}$ an algebra; a linear form; a representation; vectors, matrices, blocks

L in "L-function" and "L-value"

Glued lattices, product forms and the integral comparison forms

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. To bound the volume of a single theta lift at a twist against the volume at the trivial twist, the proof needs forms on the definite side that can be compared for all twists at once. This note constructs them: for each lift index j and each twist ϑ a linear map from a fixed space of *tables* to the algebraic forms of the auxiliary pair of (ϑ, j), such that the adjoint lift carries all these maps to one and the same surjection onto the good forms, up to a non-zero number $\gamma(\vartheta)$; the forms are integral away from the conductor of the twist, small at the trivial twist, and orthogonal to what the congruence bound of Chapter VIII requires (Theorem A). The forms are sums of delta functions at the points of *glued lattices*, with values products of two coordinate vectors, and their descents are averages of products of two theta kernels. Along the way the note proves crude integrality and size bounds for theta kernels of definite pairs, the vanishing of theta periods for forms that are not lifts, and that one explicit jet of a theta function is no more divisible than predicted.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles with Lean v4.35.0 against a pinned version of Mathlib and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note is the third of four on one unit of the artifact, the folder of the rows; it treats 57 of the unit's 219 declarations, those of six files. Every declaration is stated. The proofs are given at the level at which they were read. Complete, relative to the inputs they name, are the proofs of the algebra of glued lattices (Lemmas 3.1, 3.2, 3.4, 3.5, 3.6 and Proposition 3.8), of Corollary 3.14, of Lemmas 3.16 (a), 3.21, 3.22 (a) and 3.23, and the assembly of Theorem A. All other proofs are accounts of the formal proofs: they say which inputs combine and do not verify the estimates. Each of them is headed as an account where it stands, and the proof of Proposition 3.19 is an outline. The inputs belong to other units and to the background library and are named and not proved here. Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 57 declarations.

1 The statement

Theorem A (comparison forms). *In Settings 2.1 and 2.2, let P be parameters, D admissible data for P and j a lift index. Then there are a function γ from the twists of D to $\mathbb{C}$, a linear map $A\colon \mathrm{Tab}_D(j) \to \mathcal{M}_D^\circ(W_j(D))$ and, for every twist ϑ of D, a linear map $\mathrm{gl}_\vartheta\colon \mathrm{Tab}_D(j) \to \mathcal{F}(O_D(\vartheta, j))$ with the following properties, in which u_ϑ denotes the restriction of the adjoint lift $\mathrm{lift}_D^*(\vartheta, j)$ to $\mathcal{M}_D^\circ(W_j(D))$ and u_ϑ^* its adjoint.*

(i) *For every twist ϑ: $\gamma(\vartheta) \neq 0$ and $|\gamma(\vartheta)/\gamma(1)|^2 = \mathrm{HV}_D(j,\vartheta)/\mathrm{HV}_D(j,1)$.*

(ii) *For every twist ϑ: $u_\vartheta^* \circ \mathrm{gl}_\vartheta = \gamma(\vartheta) \cdot A$.*

(iii) *A is surjective.*

(iv) *For every twist ϑ and every $y \in \mathcal{M}_D^\circ(W_j(D))$, the form $\mathrm{gl}_\vartheta(A^*y)$ is tested against the orthogonal complement of the image of u_ϑ for the good idempotent $e_0(D)$.*

(v) *For every twist ϑ, every table ζ, every prime ℓ prime to the norms of the primes in $Q(\vartheta)$ and every embedding $\varphi\colon \mathbb{C} \to \mathbb{C}_\ell$: if, for every index a of the table, the conjugate vector $\overline{\zeta_a}$ has gauge at most 1 at the representative of its class for the embedding $\varphi \circ \mathrm{conj}$, then $\mathrm{gl}_\vartheta(\zeta)$ is integral at φ.*

(vi) *For every table ζ and every $\sigma \in \mathrm{Aut}(\mathbb{C})$:*

$$\mathsf{h}_D(j) \cdot \|\mathrm{gl}_1(\zeta)\|_\sigma^2 \le \exp(2\,r\,B_0(D)) \cdot \sum_a \|\sigma(\zeta_a)\|^2.$$

LEAN `exists_comparison_forms`

Here $\mathrm{Tab}_D(j)$ is the space of tables (Definition 2.5), $\mathrm{HV}_D(j,\vartheta)$ is a product of Hecke L-values at twisted characters (Setting 2.2), $B_0(D)$ is the budget for congruences, and "tested against a subspace for an idempotent", "gauge" and "integral" are recalled in Setting 2.1 (d). Theorem A without further words is this theorem, not Theorem A of the handout. In (i) the twist enters only through $\gamma(\vartheta)$: this is what "the comparison forms see the twist through one number" means. The note *The height of a ratio of lift volumes* deduces from (i) to (vi) the height bound for a ratio of lift volumes.

2 The setting

Conventions. Finite index sets are written in sans-serif with the letters of the formal text, as in the note *Auxiliary dual pairs, determinant polynomials and the carrier*; $|\mathsf{n}|$ is the number of elements. The ambient objects of the declarations are: a field E that is a CM field with a CM type and, where theta kernels occur, with Weil data; the finite index sets $\mathsf{r}, \mathsf{n}, \mathsf{N}, \mathsf{m}, \mathsf{a}, \mathsf{l}, \mathsf{L}, \mathsf{c}$ of each statement; an auxiliary pair O for $(E, \mathsf{r}, \mathsf{n})$ and an embedding $z\colon \mathsf{r} \hookrightarrow \mathsf{m}$, where a divided splitting occurs; in the statements about data the parameters P and the data D; a prime p with an embedding $\iota_p\colon \mathbb{C} \to \mathbb{C}_p$, where gauges occur. In statements whose formal text introduces further implicit objects (characters, moduli, vectors, points), these are introduced in the statement itself.

Setting 2.1 (background and auxiliary pairs; from the background library and the sister note)**.** *(a)* E is a CM field with a CM type and Weil data; υ, moduli, characters

with weight and tame part, hermitian spaces, the groups $\mathrm{U}(S)(\mathbb{A})$ and $G_\infty(s)$, the stabiliser $\mathbf{K}_{\mathsf{n}}$ of the standard lattice of I_{n}, levels, class sums $\sum^{\mathrm{cl}}_{h \bmod \Gamma}$, the theta kernel $\theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Psi,P](g,h)$, weights κ, standard fillings, the polynomial $\mathrm{pol}_\kappa(\xi)$, the Gram norm $\|\xi\|_\kappa$, the Fock pairing and the Schur matrix $S_\kappa(z)$ are as in the background Setting of the sister note. $\langle \xi, \xi' \rangle_\kappa = \xi^{\mathsf{H}} G_\kappa \xi'$ is the Gram form.

(b) An auxiliary pair O for $(E, \mathsf{r}, \mathsf{n})$, with its products of chosen and of twist primes $\mathfrak{M}_O$, $\mathfrak{Q}_O$, its space W_O with signs s_O, its embedding e_O, its numbering ord_O, its weights κ^+_O, κ^-_O, κ_O, its determinant exponent ϵ_O, its character μ_O with tame part ψ_O, its characters $\mathbf{a}_O$, $\mathbf{b}_O$, its level Γ_O with character χ_O, its forms $\mathcal{F}(O)$, its harmonic polynomial $P^O(\xi)$, its descent $\mathrm{lift}_O(\Xi)(g)$, its Hecke test operators $H_{O,w}(X)$ and test values $z_O(w)$, the generator ϖ_O of $\mathfrak{M}_O$, a divided splitting $\mathcal{V}$ in $\mathrm{Div}(\mathsf{N}, \mathsf{m}, \mathsf{a}; O, z)$ with its bijection $\varepsilon \colon \mathsf{N} \sqcup \mathsf{m} \to \mathsf{n}$, its fixed embedding $\phi_{\mathcal{V}} \colon \mathsf{r} \hookrightarrow \mathsf{N}$ and its divided embedding $\mathsf{a} \hookrightarrow \mathsf{N}$, the class $\bar{\varpi}_{\mathcal{V}}$ of the generator in $\mathcal{O}_E/\mathfrak{M}_O^2$, the reduction red_0 from $\mathcal{O}_E/\mathfrak{M}_O^2\mathfrak{Q}_O$ to $\mathcal{O}_E/\mathfrak{M}_O^2$, the stabiliser $\mathrm{Stab}(\varphi, w, e)$ and the column datum $\mathrm{cd}_{\chi,\varphi,w,e}(x)$, and the inclusion matrices I_e are those of the sister note. For a weight, the polynomial of a coordinate vector *in the positive letters* (or *negative letters*) of O is pol_κ with each variable (τ, i, h) replaced by the sum of the $x_{\tau,i,c}$ over the $c \in \mathsf{r}$ with sign $+1$ (or -1) and $\mathrm{ord}_O(c) = h$; it is written $\mathrm{pol}^{\pm}_{O,\kappa}(\xi)$.

(c) For level places with places w, $\tilde{A}$ denotes the chosen lift to $\mathbf{K}_{\mathsf{n}}$ of an invertible matrix A over a residue ring; it has component 1 outside the primes of the modulus and their conjugates.

(d) Forms, gauges, tests. An algebraic form of weight (κ, ϵ) on I_{n} has values coordinate vectors of weight κ. For $\sigma \in \mathrm{Aut}(\mathbb{C})$ the number $\|\phi\|^2_\sigma$ is the class sum of the squared Gram norms of the vectors $\sigma(\phi(g))$, and $\langle \phi, \phi' \rangle$ is the class sum of the Gram forms of the values. For a prime p, an embedding $\iota_p \colon \mathbb{C} \to \mathbb{C}_p$, a finite adelic point g and a coordinate vector ξ, the *gauge* $\mathrm{gauge}^{\iota_p}_\kappa(\epsilon; g, \xi)$ is the largest p-adic absolute value of a coordinate of $g^{-1} \cdot \iota_p(\xi)$, the action being that of weight (κ, ϵ); a form is *integral at* ι_p if each value has gauge at most 1 at its point. The *delta form* at a point x with value v is the form supported on the class of x that the level and the rational points generate from v. For an idempotent e_0 of the group algebra of the finite group of the chosen places and a subspace $\mathcal{Z}$ of $\mathcal{F}(O)$, a form is *tested against* $\mathcal{Z}$ *for* e_0 if it is orthogonal to every form of $\mathcal{Z}$ that lies in the image of e_0 and is killed by $H_{O,w}(z_O(w))$ for some finite place $w \neq \bar{w}$ with w, $\bar{w}$ not among the places of O.

Setting 2.2 (the data; from other units and from the sister note on heights)**.** P are parameters and D admissible data for P, with field $E(D)$, number of lifts m, ranks $k(D)$ and r, step $t(D) = 5r$, scales x, x_1, b, second level $g(D) = g_0 g_1$, the integer $\bar{\pi}_g$ of $E(D)$ with $\bar{\pi}_g \overline{\bar{\pi}_g} = g(D)$, whose prime divisors are the chosen primes, the finite group $\mathsf{G}(D)$ and its good idempotent $e_0(D)$, and $B_0(D) = \mathrm{budget}(D)/(m\,k(D))$. A lift index is an integer j with $1 \le j \le m$. A *twist* of D is a tame character whose tame part is non-trivial at every prime of its set $Q(\vartheta)$; the trivial character 1 is one. For a twist ϑ and a lift index j: $O_D(\vartheta, j)$ is the auxiliary pair of the j-th single lift, for $(E(D), \mathsf{r}, \mathsf{n})$ with $|\mathsf{r}| = r$ and $|\mathsf{n}| = r + t(D)$, with space $W_j(D)$, chosen primes those of D and twist primes $Q(\vartheta)$; $\mathcal{M}^\circ_D(W_j(D))$ is the space of good forms on $W_j(D)$; $\mathrm{lift}^*_D(\vartheta, j)$ is the adjoint lift, whose adjoint pairing is $\langle \mathrm{lift}^*_D(\vartheta, j) y, \Xi \rangle = \langle y, \mathrm{lift}_{O_D(\vartheta,j)}(\Xi) \rangle$ (Petersson product on the right).

From the note *The height of a ratio of lift volumes*: the index sets $\mathsf{B}_D(j)$ (of size $r+9m+1$), S_D and C_D (of size $3m$), with $\mathsf{n} = \mathsf{B}_D(j) \sqcup \mathsf{S}_D$ up to a fixed bijection; the finite set $\mathrm{Cl}^\circ(D)$ of classes of $\mathrm{U}(I_{\mathsf{S}_D})$ at the principal level $\bar{\pi}_g^2$, with representatives $h_{\mathfrak{c}}$; the finite set $\mathrm{Mem}_D(j)$ of *choices*, a choice i having a first component i_1, a $\mathsf{C}_D \times \mathsf{B}_D(j)$ matrix over $\mathcal{O}_E/\bar{\pi}_g^2$; the character $\mathbf{b}''_D(\vartheta, j)$; the *line multiplier* $\mathrm{lm}_D(\vartheta, j, i, h)$, a coordinate vector of weight κ^+ with rows $\mathsf{B}_D(j)$, for a choice i and $h \in \mathrm{U}(I_{\mathsf{B}_D(j)})(\mathbb{A}_f)$; the function $F_D^\natural(j, i)$ on $\mathrm{U}(W_j(D))(\mathbb{A})$; for a function ζ from $\mathrm{Cl}^\circ(D)$ to coordinate vectors of weight κ^- with rows S_D, the function

$$\mathrm{tt}_D(j,\zeta)(g) = \sum_{\mathfrak{c} \in \mathrm{Cl}^\circ(D)} \overline{\theta^{W_j(D), I_{\mathsf{S}_D}}_{\mathbf{a}, v^{m/2-j}; \bar{\pi}_g^2}[\mathbf{1}_I, \mathrm{pol}^-(\overline{\zeta(\mathfrak{c})})](g, h_{\mathfrak{c}})}$$

on $\mathrm{U}(W_j(D))(\mathbb{A})$, with $\mathbf{a}$ the first character of $O_D(1,j)$, $\mathbf{1}_I$ the indicator of the inclusion matrix of $\mathsf{r} \hookrightarrow \mathsf{S}_D$ and the polynomial in the negative letters; and the positive real number $\mathrm{HV}_D(j, \vartheta)$, a product of products of Hecke L-values at three characters twisted by $\vartheta^{\pm 1}$. Here $\kappa^\pm$ are the weights of $O_D(1,j)$, which do not depend on the twist.

Definitions of this note

Definition 2.3 (levels, kernel vectors and point forms)**.** In Setting 2.1:

(i) For level places Σ, finite index sets r, m, an index set c, a $\mathsf{c} \times \mathsf{m}$ matrix w over $\mathcal{O}_E/\mathfrak{M}_\Sigma^2$ and an embedding $e \colon \mathsf{r} \hookrightarrow \mathsf{m}$, the level $\Gamma(\Sigma; w, e)$ consists of the elements of $\mathbf{K}_{\mathsf{m}}$ whose reduction modulo $\mathfrak{M}_\Sigma^2 \mathfrak{Q}_\Sigma$ lies in $\mathrm{Stab}(\mathrm{red}_0, w, e)$: its columns outside $e(\mathsf{r})$ are those of 1, and its reduction A modulo $\mathfrak{M}_\Sigma^2$ satisfies $wA = w$.

(ii) For an auxiliary pair O', a finite index set m, a character b, a modulus $\mathfrak{m}$, a function Ψ on matrices modulo $\mathfrak{m}$, a weight κ, points g, h and a standard filling i of weight κ with rows m, the *kernel vector* has the coordinates

$$\mathrm{kv}_{O',b;\mathfrak{m}}[\Psi;\kappa](g,h)_i = \overline{\theta^{W_{O'}, I_{\mathsf{m}}}_{\mathbf{a}_{O'}, b; \mathfrak{m}}[\Psi, b_i^-](g,h)},$$

where b_i^- is the basis polynomial b_i written in the negative letters of O'.

(iii) For an auxiliary pair O', a finite index set m, a character b, an index set c, a $\mathsf{c} \times \mathsf{m}$ matrix w over $\mathcal{O}_E/\mathfrak{M}_{O'}^2$, an embedding e into m, a function y from $\mathrm{U}(I_{\mathsf{m}})(\mathbb{A}_f)$ to coordinate vectors of weight $\kappa_{O'}^+$ and a point g of $\mathrm{U}(W_{O'})(\mathbb{A})$, the *coset kernel sum* is

$$\mathrm{cks}_{O',b}(w,e;y)(g) = \sum_{h \bmod \Gamma(O';w,e)}^{\mathrm{cl}} \theta^{W_{O'}, I_{\mathsf{m}}}_{\mathbf{a}_{O'}, b; \mathfrak{M}^2\mathfrak{Q}}[\mathrm{cd}_{\chi, \mathrm{red}_0, w, e}, \mathrm{pol}^+(y(h))](g,h),$$

where $\mathfrak{M}^2\mathfrak{Q}$ is the modulus of O', χ the character of its residue ring given by the tame part $\psi_{O'}$, and the polynomial of $y(h)$ is in the positive letters of O'. The formal text lists a further point h among the arguments; see §5.

(iv) For an auxiliary pair O for $(E, \mathsf{r}, \mathsf{n})$, a point $h \in \mathrm{U}(I_{\mathsf{n}})(\mathbb{A}_f)$, a family $Y = (Y_w)$ of $\mathsf{n} \times \mathsf{r}$ matrices with linearly independent columns over the residue fields of the chosen places, an embedding $e \colon \mathsf{r} \hookrightarrow \mathsf{n}$ and a coordinate vector v of weight κ_O, the *point form* $\mathrm{pt}_{h,Y,e}(v) \in \mathcal{F}(O)$ is the delta form with value v at the point $h\, k_{Y,e}$, where $k_{Y,e} \in \mathbf{K}_{\mathsf{n}}$ is the chosen element of Lemma 3.1. It is linear in v.

LEAN rowFixingLevel · kernelVector · cosetKernelSum · pointForm

Definition 2.4 (divided lattices, glued lattices, product forms)**.** In Setting 2.1, let O be an auxiliary pair for $(E, \mathsf{r}, \mathsf{n})$.

(i) A *divided lattice*, an element L of $\mathrm{Lat}(O)$, is a point $\gamma_L \in \mathrm{U}(I_{\mathsf{n}})(\mathbb{A}_f)$ with component 1 outside the chosen places of O and their conjugates such that every entry of $\varpi_O\,\gamma_L^{-1}$ is integral.

(ii) Its *reduced inverse* G_L is the $\mathsf{n} \times \mathsf{n}$ matrix over $\mathcal{O}_E/\mathfrak{M}_O^2$ of the reductions of the entries of $\varpi_O\gamma_L^{-1}$, and its *Weil scalar* is $\beta_L = \chi(\delta)/|\delta|^{|\mathsf{r}|/2}$, where δ is the idele equal to $\det\gamma_L$ at the chosen places and to 1 elsewhere, $|\delta|$ its idele norm and χ the Hecke character attached to $\mathbf{a}_O$ in rank $|\mathsf{r}|$.

Let further N, m, a be finite index sets, $z\colon \mathsf{r} \hookrightarrow \mathsf{m}$ an embedding, $\mathcal{V}$ a divided splitting in $\mathrm{Div}(\mathsf{N}, \mathsf{m}, \mathsf{a}; O, z)$ and s an $\mathsf{a} \times \mathsf{N}$ matrix over $\mathcal{O}_E/\mathfrak{M}_O^2$.

(iii) $[s;\bar\varpi]_{\mathcal{V}}$ is the $(\mathsf{a} \sqcup \mathsf{N}) \times \mathsf{N}$ matrix with s stacked over $\bar\varpi_{\mathcal{V}} \cdot 1$.

(iv) $\mathsf{C}(\mathcal{V})$ is the group of the $A \in \mathrm{GL}_{\mathsf{r}}(\mathcal{O}_E/\mathfrak{M}_O^2)$ with $A \cdot \bar\varpi_{\mathcal{V}} = \bar\varpi_{\mathcal{V}}$, that is $A \equiv 1$ modulo the chosen primes (Input M (a)).

(v) A divided lattice L is *glued for* s, written $\mathrm{Glued}_{\mathcal{V}}(s, L)$, if (1) for every $\mathsf{n} \times \mathsf{r}$ matrix x over $\mathcal{O}_E/\mathfrak{M}_O^2$ and every $c \in \mathbb{C}$,

$$[G_L\, x = \bar\varpi_{\mathcal{V}} I_{e_O}]\, c = \sum_{A \in \mathsf{C}(\mathcal{V})} [(xA)_{\mathsf{m}} = I_z \text{ and } [s;\bar\varpi]_{\mathcal{V}}\,(xA)_{\mathsf{N}} = [s;\bar\varpi]_{\mathcal{V}}\, I_{\phi_{\mathcal{V}}}]\, c,$$

where $(xA)_{\mathsf{m}}$ and $(xA)_{\mathsf{N}}$ are the rows $\varepsilon(\mathsf{m})$ and $\varepsilon(\mathsf{N})$ of xA and a bracket is 1 or 0; and (2) there is $J \in \mathrm{GL}_{\mathsf{N}}(\mathcal{O}_E/\mathfrak{M}_O^2)$ such that $\bar\varpi_{\mathcal{V}}$ times the columns $\varepsilon(\mathsf{N})$ of G_L, multiplied by J, vanishes in every column outside the image of the divided embedding.

(vi) For a divided lattice L, a pair $h = (h_1, h_2)$ of points of $\mathrm{U}(I_{\mathsf{N}})(\mathbb{A}_f)$ and $\mathrm{U}(I_{\mathsf{m}})(\mathbb{A}_f)$ and a coordinate vector y of weight κ_O^+ with rows N, the *product form* is the linear map from coordinate vectors η of weight κ_O^- with rows m to $\mathcal{F}(O)$

$$\mathrm{pr}_{\mathcal{V},L}(h; y)(\eta) = \mathrm{pt}_{\mathrm{diag}_\varepsilon(h_1,h_2)\,\gamma_L,\, I_{e_O},\, \varepsilon\circ\phi_{\mathcal{V}}}(y \cdot \eta),$$

where $\mathrm{diag}_\varepsilon$ is the block diagonal embedding along ε and $y \cdot \eta$, the *Cartan product*, is the coordinate vector of weight κ_O obtained from $y \otimes \eta$ by an integral left inverse of the Cartan embedding.

(vii) The *loss* of $\mathcal{V}$ is $\mathrm{loss}(\mathcal{V}) = \sum_\tau \big(|\mathsf{a}| \sum_{h<|\mathsf{N}|} \kappa^+_{O,\tau}(h) + |\mathsf{m}| \sum_{h<|\mathsf{m}|} \kappa^-_{O,\tau}(h)\big)$.

LEAN DividedLattice · DividedLattice.reducedInverse · DividedLattice.weilScalar · DividedSplitting.stackOverGenScalar · DividedSplitting.congruenceGroup · DividedSplitting.IsGlued · DividedSplitting.productForm · DividedSplitting.loss

Definition 2.5 (tables and comparison forms)**.** In Settings 2.1 and 2.2, let j be a lift index and ϑ a twist of D.

(i) $\mathsf{h}_D(j)$ is the hook number of the weight κ^- of $O_D(1, j)$ in the rank $|\mathsf{S}_D|$.

(ii) $\mathcal{W}_D(j)$ is the space of coordinate vectors of weight κ^- with rows S_D, with the inner product of the Gram form; $\mathrm{Tab}_D(j)$ is the orthogonal sum of copies of $\mathcal{W}_D(j)$, one for each pair $a = (i, \mathfrak{c})$ of a choice $i \in \mathrm{Mem}_D(j)$ and a class $\mathfrak{c} \in \mathrm{Cl}^\circ(D)$. Its elements, the *tables*, are families $\zeta = (\zeta_a)$.

(iii) $\mathcal{V}_D(j,\vartheta)$ is the divided splitting, in $\mathrm{Div}(\mathsf{B}_D(j),\mathsf{S}_D,\mathsf{C}_D;O_D(\vartheta,j),z)$ with z the inclusion of r in S_D, given by the fixed bijection $\mathsf{B}_D(j)\sqcup\mathsf{S}_D\to\mathsf{n}$, the inclusion of r in $\mathsf{B}_D(j)$ as fixed embedding and a chosen embedding of C_D in $\mathsf{B}_D(j)$ as divided embedding.

(iv) For a choice $i\in\mathrm{Mem}_D(j)$, $L_D(j,\vartheta,i)$ is a chosen divided lattice of $O_D(\vartheta,j)$ glued for i_1 and $\mathcal{V}_D(j,\vartheta)$; its point is the one chosen for the trivial twist, so it does not depend on ϑ.

(v) $\beta_D^{\mathrm{line}}(j)$ is a chosen complex number $\neq 0$ with the properties of Input L (c).

(vi) The *comparison form* of a table ζ is

$$\mathrm{gl}_{D,j,\vartheta}(\zeta)=g(D)^{\mathrm{loss}(\mathcal{V}_D(j,\vartheta))}\sum_{a=(i,\mathfrak{c})}\sum_{h\bmod\Gamma_i}^{\mathrm{cl}}\mathrm{pr}_{\mathcal{V}_D(j,\vartheta),L_D(j,\vartheta,i)}((h,h_{\mathfrak{c}});y_i(h))(\zeta_a),$$

$$y_i(h)=\overline{\beta_D^{\mathrm{line}}(j)\cdot\mathrm{lm}_D(\vartheta,j,i,h)},\qquad \Gamma_i=\Gamma(O_D(\vartheta,j);[i_1;\bar\varpi]_{\mathcal{V}_D(j,\vartheta)},\phi),$$

with ϕ the fixed embedding. It is linear in ζ and lies in $\mathcal{F}(O_D(\vartheta,j))$.

LEAN `smallHookNumber` · `SmallFormValues` · `Data.Tables` · `splittingV` · `glueLattice` · `lineScale` · `comparisonForm` · the three instances that make $\mathcal{W}_D(j)$ a finite-dimensional inner product space are unnamed

Inputs from the other notes of this unit.

Input L (note *The height of a ratio of lift volumes*). In Setting 2.2, for a lift index j: (a) for every finite index set a there are a real number C and numbers $c(\vartheta)\in\mathbb{C}$, for the twists ϑ of D, with $|c(\vartheta)|^2=C/\,\mathrm{HV}_D(j,\vartheta)$ such that for every twist ϑ, every choice $i\in\mathrm{Mem}_D(j)$, every point g, every matrix u' with columns a and every matrix w with rows a and columns $\mathsf{B}_D(j)$, both over $\mathcal{O}_E/\bar\pi_g^2$, with $u'w=i_1$:

$$F_D^\natural(j,i)(g)=c(\vartheta)\cdot\mathrm{cks}_{O_D(\vartheta,j),\mathfrak{b}''_D(\vartheta,j)}(w,\phi;\mathrm{lm}_D(\vartheta,j,i,\cdot))(g);$$

this note uses it for $\mathsf{a}=\mathsf{C}_D\sqcup\mathsf{B}_D(j)$; (b) a good form $F\in\mathcal{M}_D^\circ(W_j(D))$ with $\int_{[\mathrm{U}(W_j(D))]}\overline{F(g)\,F_D^\natural(j,i)(g)}\;\mathrm{tt}_D(j,\zeta)(g)\,dg=0$ for all choices i and all ζ is zero; for every good F and choice i the product $F\cdot F_D^\natural(j,i)$ decreases rapidly, so that these integrals converge; (c) there is $c\neq0$ in $\mathbb{C}$ such that for every choice i, twist ϑ and point h the vector $y=\overline{c\cdot\mathrm{lm}_D(\vartheta,j,i,h)}$ has gauge at most 1 at h for every prime p prime to the norms of the primes in $Q(\vartheta)$ and every ι_p, and, if $Q(\vartheta)=\emptyset$, $\|\sigma(y)\|^2\le\exp(r\,B_0(D))$ for all $\sigma\in\mathrm{Aut}(\mathbb{C})$. (d) The rank bounds, from the unit *Exchange*: $6r\le k(D)+m\,(t(D)+1)$, $100m+9<k(D)$, $r+1=k(D)+m$, $t(D)=5r$, $|\mathsf{r}|=r$ and $|\mathsf{B}_D(j)|=r+9m+1$.

Input M (note *Auxiliary dual pairs, determinant polynomials and the carrier*). (a) For a divided splitting $\mathcal{V}$ and a matrix Z over $\mathcal{O}_E/\mathfrak{M}_O^2\mathfrak{Q}_O$: $\bar\varpi_{\mathcal{V}}\,\mathrm{red}_0(Z)=\bar\varpi_{\mathcal{V}}I_u$ if and only if $Z\equiv I_u$ modulo every chosen place. (b) An integral adele x reduces to 0 modulo $\mathfrak{M}_O^c$ if and only if $\varpi_O^{-c}x$ is integral. (c) The descent lift_O is the class sum of the conjugate kernels of the test datum Υ_O with the harmonic polynomial. (d) The bookkeeping lemmas on chosen lifts and block matrices.

Input V (note *The volume of the chain of theta lifts*). The adjoint lift commutes with the actions of the group algebra of $\mathsf{G}(D)$.

Inputs from other units and from the background library. Each is used where it is named in the proofs; the notes on its unit, or the catalogue of the handout, state it in full.

Unit *Forms*. The conjugate kernel of an auxiliary pair is equivariant: at $(g, \gamma x k)$, applied to $\chi_O(k)\rho(\gamma)v$, it equals its value at (g, x) applied to v, for rational γ and k in the level. The descent of a form is a multiple of the conjugate of a theta lift of its conjugate. **Unit *SiegelWeil*.** (a) If $\psi_O \neq 1$ at each twist place, the reflection sums of all non-zero rows at the chosen places kill Ξ and $\mathrm{lift}_O(\Xi) = 0$, then the descents of the matrix coefficients of Ξ from $\mathrm{U}(I_{\mathsf{n}})$ to $\mathrm{U}(W_O)$ vanish for all data that are the vacuum outside the places of O and their complex conjugates. (b) If the descents of a function to $\mathrm{U}(W)$ vanish and it is killed by the Hecke polynomial at a split place at the test value, then its descents to $\mathrm{U}(W \oplus \mathbb{H}^p)$ vanish, when $|\mathsf{n}| > 2|\mathsf{r}| + 2p + 3$. (c) The representation of the finite group on a level vector is translation by a lift times a character value. **Unit *Exchange*.** The pairs $O_D(\vartheta, j)$ fit the parameter of the forms; the adjoint lift is adjoint to the descent; the datum of a function modulo $\mathfrak{m}$, integral away from q, times a power of an integer N with $qN \in \mathfrak{m}$, is a standard datum at primes not dividing q, and it is an admissible combination of translates of the vacuum of size $|\mathcal{O}_E/\mathfrak{m}|^{|\mathsf{n}||\mathsf{n}'|}$ times a bound for the conjugates of the function; $|\mathcal{O}_E/\mathfrak{m}| \leq k^{2f}$ if $k \in \mathfrak{m}$; the trivial lift of an L-value datum and its normalised value. **Unit *Curve*.** Integrality of jets of standard data; conjugates of jets under $\mathrm{Gal}(\mathbb{C}/K)$, with the semilinear maps and their bounds; tight frames and the finite data they produce; the limit to Katz's divisibility; Frobenius packings and large splits; a non-zero value of the integral vacuum; the period factor. **Units *Data*, *Columns*, *Trace*, *Seesaw*.** Levels inside the chosen level in rank at most a fixed bound are free, and the orders of class stabilisers divide the chosen group order; the number of classes of a definite unitary group at a level containing a principal level is at most an exponential in the scales; comparison of monomials in the scales; $e_0(D)$ is a central projection whose coefficients are algebraic integers up to the order of the group, and it kills the reflection sums; the chain of lifts intertwines the finite group with a right translation; $\dim \mathrm{Row}(D) = \dim \mathcal{M}^{\circ}_D(W_j(D)) \neq 0$.

Inputs from the background library. The references are those of the handout's catalogue, with the relation that its second reading allows.

The theorem on periods (the authors call it "Fourier coefficients of theta lifts as periods"): a continuous function on $\mathrm{U}(I_{\mathsf{n}})(\mathbb{A})$, $\mathsf{n} = \mathsf{n}_1 \sqcup \mathsf{n}_2$, left invariant under the rational points and fixed by the part of $\mathbf{K}_{\mathsf{n}}$ above a set T of places and their conjugates, all of whose theta descents to the split group $\mathrm{U}(\mathbb{H}^{|\mathsf{n}_1|})$, for the character $v^{|\mathsf{n}_1|}$ and the data that are the vacuum outside T and its conjugates, vanish, has vanishing period over $[\mathrm{U}(I_{\mathsf{n}_2})]$ at every point of the first block. (The catalogue gives P. Walls, *The theta correspondence, periods of automorphic forms and special values of standard automorphic L-functions*, Proposition 2.1.1, eq. (2.1.5), pp. 5–6, and S. Rallis, *On the Howe duality conjecture*, Compositio Math. 51 (1984), pp. 333–335; both concern the symplectic–orthogonal pair, so they are statements for a different dual pair, and the unitary vanishing is proved in the artifact.) Further: the Hermite normal form over a valuation ring; the Schrödinger-model formula at split places; the seesaw for $I_{\mathsf{n}} = I_{\mathsf{n}_1} \oplus I_{\mathsf{n}_2}$; the translation rules of kernels; Schur matrices are multiplicative and contract the sup norm on integral matrices; the Cartan product and the mixed Cartan embedding; duality for the Gram norm; the gauge of a delta form is at most the gauge of its value; a finite Fourier expansion of a translated vacuum; the p-adic logarithm of the theta function of a CM curve (a result of the artifact, without a reference in the catalogue); references to be confirmed where the catalogue has none.

Status of the proofs. A proof headed "Proof" is complete relative to the inputs it names. A proof headed "an account of the formal proof" says which results combine, in the order of the formal proof, and does not verify the estimates and the exponents; an "outline" does less and says what it leaves out.

3 Proofs

Point forms

Lemma 3.1 (a point with prescribed reductions)**.** *In Setting 2.1, let O be an auxiliary pair for $(E, \mathsf{r}, \mathsf{n})$, $Y = (Y_w)$ a family of $\mathsf{n} \times \mathsf{r}$ matrices over the residue fields of the chosen places with linearly independent columns, and $e \colon \mathsf{r} \hookrightarrow \mathsf{n}$ an embedding. Then there is $k \in \mathbf{K}_{\mathsf{n}}$, with component 1 outside the places of O and their conjugates, such that $(k \bmod w)\, I_{e_O} = Y_w$ at every chosen place w and $(k \bmod w)^{\mathsf{T}} I_e = I_{e_O}$ at every twist place w.*

Proof. A matrix with independent columns over a field extends to an invertible matrix with these columns at the indices $e_O(\mathsf{r})$; this gives A_w with $A_w I_{e_O} = Y_w$ at the chosen places and, applied to I_{e_O} with the embedding e, matrices B_w with $B_w I_e = I_{e_O}$ at the twist places. The reduction from the elements of $\mathbf{K}_{\mathsf{n}}$ supported at the places of O to the product of the groups $\mathrm{GL}_{\mathsf{n}}(k_w)$ is surjective; take k with reductions A_w and B_w^{T}. □

LEAN `exists_reduceAt_mul_incl_eq`

Lemma 3.2 (pairing with a point form)**.** *In Setting 2.1, let O be an auxiliary pair for $(E, \mathsf{r}, \mathsf{n})$, $Y = (Y_w)$ a family of matrices with linearly independent columns and e an embedding as in Lemma 3.1, and $h \in \mathrm{U}(I_{\mathsf{n}})(\mathbb{A}_f)$. For every form $\phi \in \mathcal{F}(O)$ and every coordinate vector v of weight κ_O, $\langle \mathrm{pt}_{h,Y,e}(v), \phi\rangle = \langle v, \phi(h\, k_{Y,e})\rangle_{\kappa_O}$.*

Proof. The inner product is the class sum of the Gram forms of the values. The Gram form is invariant under the weight representation of the rational points, which are unitary at infinity, and the character of the level has absolute value 1. So the summand is constant on the class of $hk_{Y,e}$, where the delta form has the value v, and it vanishes on the other classes. □

LEAN `inner_pointForm`

Lemma 3.3 (the descent of a point form)**.** *In Setting 2.1, let O be an auxiliary pair for $(E, \mathsf{r}, \mathsf{n})$, $Y = (Y_w)$ a family of matrices with linearly independent columns, e an embedding and h a point as in Lemma 3.2, v a coordinate vector of weight κ_O and $g \in \mathrm{U}(W_O)(\mathbb{A})$. Then*

$$\mathrm{lift}_O(\mathrm{pt}_{h,Y,e}(v))(g) = \overline{\theta^{W_O, I_{\mathsf{n}}}_{\mathbf{a}_O, \mathbf{b}_O; \mathfrak{M}_O^2 \mathfrak{Q}_O}[\Psi_{Y,e}, P^O(\bar{v})](g,h)},$$

where $\Psi_{Y,e}(T)$ is the product over the twist places w of $\psi_{O,w}(\det(I_e^{\mathsf{T}} T \bmod w))$ if $T \equiv Y_w$ at every chosen place w, and 0 otherwise. The formal statement names a further character b, which does not occur in it.

Proof (an account of the formal proof). The descent is the class sum of the conjugate kernels at $(g, \cdot)$ applied to the values (Input M (c)). By the equivariance of the conjugate kernel (unit *Forms*) the class sum for a delta form at $hk_{Y,e}$ is the kernel at $(g, hk_{Y,e})$ applied to v. The integral point $k_{Y,e}$ is moved into the finite datum, which becomes the translate of the datum of Υ_O by the reductions of $k_{Y,e}$; by the two properties of Lemma 3.1 this translate is $\Psi_{Y,e}$. □

LEAN `theta_pointForm`

Glued lattices

Lemma 3.4 (conjugating by a divided lattice)**.** *In Setting 2.1, let O be an auxiliary pair for $(E, \mathsf{r}, \mathsf{n})$, $L \in \mathrm{Lat}(O)$ and $j \in \mathbf{K}_{\mathsf{n}}$ with $j \equiv 1$ modulo $\mathfrak{M}_O^2$. Then $\gamma_L^{-1} j \gamma_L$ lies in $\mathbf{K}_{\mathsf{n}}$ and reduces to 1 at every chosen place of O.*

Proof. Write $\gamma = \gamma_L$ and $\varpi = \varpi_O$, and let ϖ^* be the complex conjugate of the idele ϖ (not the residue class $\bar{\varpi}_{\mathcal{V}}$). Since γ is unitary, $\gamma = (\gamma^{-1})^{\mathsf{H}}$, so $\varpi^*\gamma$ is integral together with $\varpi\gamma^{-1}$. The entries of $\varpi^{-2}(j-1)$ are integral by Input M (b), and so are those of $(\varpi^*)^{-1}\varpi^{-2}(j-1)$: at a chosen place ϖ^* is a unit, at the conjugate of a chosen place $\varpi = 1$ and $j - 1$ is the conjugate transpose of $j^{-1} - 1$, which is divisible by ϖ^2 at the chosen place. Hence

$$\varpi^{-1}(\gamma^{-1} j \gamma - 1) = (\varpi\gamma^{-1}) \cdot (\varpi^*)^{-1}\varpi^{-2}(j-1) \cdot (\varpi^*\gamma)$$

is integral. So $\gamma^{-1} j \gamma$ is integral, hence in $\mathbf{K}_{\mathsf{n}}$, and it is 1 modulo each chosen prime. □

LEAN `DividedLattice.reduceAt_conj_eq_one`

Lemma 3.5 (a divided lattice from a factorisation)**.** *In Setting 2.1, let O be an auxiliary pair for $(E, \mathsf{r}, \mathsf{n})$ and let C, G be $\mathsf{n} \times \mathsf{n}$ matrices over $\widehat{\mathcal{O}}_E$ with $CG = \varpi_O \cdot 1$. Then there is $L \in \mathrm{Lat}(O)$ with $G_L = G$ modulo $\mathfrak{M}_O^2$.*

Proof. Let c be the element of $\mathrm{GL}_{\mathsf{n}}(\mathbb{A}_f)$ equal to C at the chosen places, with inverse $\varpi_O^{-1}G$ there, and to 1 elsewhere. Its conjugate transpose c^{H} is supported at the conjugates of the chosen places, a disjoint set, so c and c^{H} commute and $\gamma = (c^{\mathsf{H}})^{-1}c$ is unitary. At a chosen place $\varpi_O\gamma^{-1} = G$; at its conjugate $\varpi_O = 1$ and $\gamma^{-1} = c^{\mathsf{H}}$ is integral; elsewhere $\gamma = 1$. □

LEAN `DividedLattice.exists_reducedInverse_eq` · the formal statement lists a lattice L among its arguments, which it then binds itself

Lemma 3.6 (the levels of a divided splitting)**.** *In Setting 2.1, let O be an auxiliary pair for $(E, \mathsf{r}, \mathsf{n})$, N, m, a finite index sets, $z\colon \mathsf{r} \hookrightarrow \mathsf{m}$, $\mathcal{V}$ a divided splitting in $\mathrm{Div}(\mathsf{N}, \mathsf{m}, \mathsf{a}; O, z)$ and s an $\mathsf{a} \times \mathsf{N}$ matrix over $\mathcal{O}_E/\mathfrak{M}_O^2$.*

(a) *Every element of $\Gamma(O; [s; \bar{\varpi}]_{\mathcal{V}}, \phi_{\mathcal{V}})$ reduces to 1 at every chosen place of O.*

(b) *If Y is an $\mathsf{r} \times \mathsf{r}$ matrix over $\mathcal{O}_E/\mathfrak{M}_O^2$ with $\bar{\varpi}_{\mathcal{V}}Y = 0$, then there is $A \in \mathsf{C}(\mathcal{V})$ with $A^{-1} = 1 - Y$.*

Proof. (a) The reduction A modulo $\mathfrak{M}_O^2$ of such an element fixes the stacked matrix, in particular its lower block: $\bar{\varpi}_{\mathcal{V}}A = \bar{\varpi}_{\mathcal{V}} \cdot 1$. By Input M (a), $A \equiv 1$ at every chosen place. (b) The entries of Y lie in the image of $\mathfrak{M}_O$, so $Y^2 = 0$, the matrix $1 - Y$ is invertible and $\bar{\varpi}_{\mathcal{V}}(1 - Y) = \bar{\varpi}_{\mathcal{V}}$; so $1 - Y$ and its inverse lie in $\mathsf{C}(\mathcal{V})$. □

LEAN `rowFixingLevel_le_levelOf` · `DividedSplitting.exists_congruenceGroup_inv_eq_one_sub`

Remark 3.7. By Input M (a), the chosen lift $\tilde{A}$ of an element $A \in \mathsf{C}(\mathcal{V})$ reduces to 1 at every chosen place. For the data, the level Γ_i of Definition 2.5 (vi) is *free*: the stabiliser in the rational unitary group of each of its classes is trivial, by part (a) of Lemma 3.6, the rank bounds of Input L (d) and the unit *Data*. So a class sum at this level is a plain sum over the classes.

LEAN `DividedSplitting.reduceAt_liftMod_congruenceGroup` · `classAut_rowFixingLevel_eq_bot`

Proposition 3.8 (glued lattices exist)**.** *In Setting 2.1, let O be an auxiliary pair for $(E, \mathsf{r}, \mathsf{n})$, N, m, a finite index sets, $z\colon \mathsf{r} \hookrightarrow \mathsf{m}$, $\mathcal{V}$ a divided splitting in $\mathrm{Div}(\mathsf{N}, \mathsf{m}, \mathsf{a}; O, z)$ and s an $\mathsf{a} \times \mathsf{N}$ matrix over $\mathcal{O}_E/\mathfrak{M}_O^2$. Then there is a divided lattice $L \in \mathrm{Lat}(O)$ with $\mathrm{Glued}_{\mathcal{V}}(s, L)$.*

Proof. Lift s to a matrix t over $\widehat{\mathcal{O}}_E$. By the Hermite normal form over $\widehat{\mathcal{O}}_E$ (library), the matrix with t stacked over $\varpi_O \cdot 1$ is $U_{\mathsf{N}}D$, where U is invertible over $\widehat{\mathcal{O}}_E$, U_{N} are its last columns and D is a square matrix; the lower block gives $U_{22}D = \varpi_O \cdot 1$. Along ε, let

$$C = \begin{pmatrix} U_{22} & I_\phi I_z^{\mathsf{T}} \\ 0 & \Delta \end{pmatrix}, \qquad G = \begin{pmatrix} D & -D\, I_\phi I_z^{\mathsf{T}} \\ 0 & \Delta' \end{pmatrix},$$

with ϕ the fixed embedding, Δ the diagonal matrix with entries 1 on $z(\mathsf{r})$ and ϖ_O elsewhere, and Δ' the one with ϖ_O on $z(\mathsf{r})$ and 1 elsewhere. Then $CG = \varpi_O \cdot 1$, and Lemma 3.5 gives L with $G_L = G$ modulo $\mathfrak{M}_O^2$. For condition (1): the lower block of G_L $x = \bar{\varpi}_{\mathcal{V}} I_{e_O}$ says that the rows $\varepsilon(\mathsf{m})$ of x outside $z(\mathsf{r})$ vanish and that the rows $z(\mathsf{r})$ form a matrix $1 - Y$ with $\bar{\varpi}_{\mathcal{V}}Y = 0$; by Lemma 3.6 (b) there is exactly one $A \in \mathsf{C}(\mathcal{V})$ with $(xA)_{\mathsf{m}} = I_z$. For this A the upper block says $\bar{D}\,(xA)_{\mathsf{N}} = \bar{D}\, I_\phi$, with $\bar{D}$ the reduction of D, and this is equivalent to the same equation with the stacked matrix in the place of $\bar{D}$, because the stacked matrix is $\bar{U}_{\mathsf{N}}\bar{D}$ and U is invertible. So if the left side of (1) is c, exactly one term on the right is c and the others vanish. Conversely, let $A \in \mathsf{C}(\mathcal{V})$ satisfy the two conditions on the right. Then the rows $\varepsilon(\mathsf{m})$

of x are $I_z A^{-1}$, and $\bar{\varpi}_{\mathcal{V}} A^{-1} = \bar{\varpi}_{\mathcal{V}}$ gives the lower block of $G_L\, x = \bar{\varpi}_{\mathcal{V}} I_{e_O}$; the upper block is $\bar{D}((xA)_{\mathsf{N}} - I_\phi)A^{-1} = 0$. So the left side is c as well, the matrix A is the one found before, and both sides of (1) are c or both are 0. For condition (2): the Hermite normal form over the residue ring gives an invertible J such that sJ vanishes outside the columns of the divided embedding; then $\bar{\varpi}_{\mathcal{V}} \bar{D} J$ vanishes there too, since $\bar{D}$ is a combination of the rows of s and of $\bar{\varpi}_{\mathcal{V}} \cdot 1$, and $\bar{\varpi}_{\mathcal{V}}^2 = 0$. □

LEAN `DividedSplitting.exists_isGlued` · the formal statement lists a lattice L among its arguments, which it binds itself

The descent of a product form

Lemma 3.9 (the descent of a point form at a divided lattice)**.** *In Setting 2.1, let O be an auxiliary pair for $(E, \mathsf{r}, \mathsf{n})$, N, m finite index sets, a an index set, $z\colon \mathsf{r} \hookrightarrow \mathsf{m}$, $\mathcal{V}$ a divided splitting in $\mathrm{Div}(\mathsf{N}, \mathsf{m}, \mathsf{a}; O, z)$, $L \in \mathrm{Lat}(O)$, $e\colon \mathsf{r} \hookrightarrow \mathsf{n}$ an embedding, v a coordinate vector of weight κ_O, $g \in \mathrm{U}(W_O)(\mathbb{A})$ and $h \in \mathrm{U}(I_{\mathsf{n}})(\mathbb{A}_f)$. Then*

$$\mathrm{lift}_O(\mathrm{pt}_{h\gamma_L, I_{e_O}, e}(v))(g) = \overline{\beta_L \cdot \theta^{W_O, I_{\mathsf{n}}}_{\mathbf{a}_O, \mathbf{b}_O; \mathfrak{M}_O^2 \mathfrak{Q}_O}[\Psi_{L,e}, P^O(\bar{v})](g, h)},$$

where $\Psi_{L,e}(S)$ is the product of the numbers $\psi_{O,w}(\det(I_e^{\mathsf{T}} S \bmod w))$ over the twist places w if

$$G_L \,\mathrm{red}_0(S) = \bar{\varpi}_{\mathcal{V}} I_{e_O},$$

and 0 otherwise. The formal statement names a character b, which does not occur in it.

Proof (an account of the formal proof). By Lemma 3.3 at the point $h\gamma_L$. The point γ_L is supported at the chosen places and their conjugates, where the unitary group is a general linear group, and the Schrödinger-model formula (library) moves it into the datum: the function $x \mapsto \Psi(x)$ becomes $x \mapsto \Psi(\gamma_L^{-1} x)$, with the factor $\chi(\delta)|\delta|^{-|\mathsf{r}|/2} = \beta_L$. The condition "$\gamma_L^{-1} x$ is integral and congruent to I_{e_O} at the chosen places" is, by Input M (a) and (b), the condition on G_L. □

LEAN `DividedSplitting.theta_pointForm`

Proposition 3.10 (the descent of a product form)**.** *In Setting 2.1, let O be an auxiliary pair for $(E, \mathsf{r}, \mathsf{n})$, N, m, a finite index sets, $z\colon \mathsf{r} \hookrightarrow \mathsf{m}$, $\mathcal{V}$ a divided splitting in $\mathrm{Div}(\mathsf{N}, \mathsf{m}, \mathsf{a}; O, z)$, s an $\mathsf{a} \times \mathsf{N}$ matrix over $\mathcal{O}_E/\mathfrak{M}_O^2$, $L \in \mathrm{Lat}(O)$ with $\mathrm{Glued}_{\mathcal{V}}(s, L)$, and b, b' characters with $b\,b' = \mathbf{b}_O$. Write $w_s = [s; \bar{\varpi}]_{\mathcal{V}}$, $\phi = \phi_{\mathcal{V}}$, and $\Theta^-_\eta(g, h') = \theta^{W_O, I_{\mathsf{m}}}_{\mathbf{a}_O, b'; \mathfrak{M}_O^2}[\mathbf{1}_{I_z}, \mathrm{pol}^-_{O, \kappa_O^-}(\bar{\eta})](g, h')$.*

(a) *For points h of $\mathrm{U}(I_{\mathsf{N}})(\mathbb{A}_f)$ and h' of $\mathrm{U}(I_{\mathsf{m}})(\mathbb{A}_f)$, coordinate vectors y of weight κ_O^+ and η of weight κ_O^- and $g \in \mathrm{U}(W_O)(\mathbb{A})$:*

$$\mathrm{lift}_O(\mathrm{pr}_{\mathcal{V},L}((h, h'); y)(\eta))(g) = \overline{\beta_L \sum_{A \in \mathsf{C}(\mathcal{V})} \Theta^+_y(g\tilde{A}, h) \cdot \Theta^-_\eta(g\tilde{A}, h')},$$

$$\Theta^+_y(g, h) = \theta^{W_O, I_{\mathsf{N}}}_{\mathbf{a}_O, b; \mathfrak{M}_O^2 \mathfrak{Q}_O}[\mathrm{cd}_{\chi, \mathrm{red}_0, w_s, \phi}, \mathrm{pol}^+_{O, \kappa_O^+}(\bar{y})](g, h).$$

(b) *For a point h', $c \in \mathbb{C}$, a function y from $\mathrm{U}(I_{\mathsf{N}})(\mathbb{A}_f)$ to coordinate vectors of weight κ_O^+, a vector η and g as before:*

$$\mathrm{lift}_O\Big(\sum_{h \bmod \Gamma(O;w_s,\phi)}^{\mathrm{cl}} \mathrm{pr}_{\mathcal{V},L}((h,h');\overline{c\,y(h)})(\eta)\Big)(g) = \overline{\beta_L\, c}\cdot \sum_{A\in \mathsf{C}(\mathcal{V})} \overline{\mathrm{cks}_{O,b}(w_s,\phi;y)(g\tilde{A})\cdot \Theta_\eta^-(g\tilde{A},h')}.$$

Proof (*an account of the formal proof*). (a) Apply Lemma 3.9 to the vector $y\cdot\eta$ and to the point $\mathrm{diag}_\varepsilon(h,h')$. By condition (1) of a glued lattice the datum $\Psi_{L,e}$ is the sum over $A \in \mathsf{C}(\mathcal{V})$ of the translates by A of the product of the indicator of I_z on the rows m and of the column datum on the rows N; the factors $\psi_{O,w}$ at the twist places are the character χ of the column datum, because $\tilde{A}$ is 1 there. The translation by A is moved to the first variable (library), which gives $g\tilde{A}$. The harmonic polynomial of a Cartan product is the product of the polynomial of y in the positive letters and of η in the negative letters, the columns N and m carrying no columns of the other weight by the conditions of a divided splitting (library: the mixed Cartan embedding). The seesaw for $I_{\mathsf{n}} = I_{\mathsf{N}} \oplus I_{\mathsf{m}}$, with $\mathbf{b}_O = b\,b'$, splits the kernel into the product, and the second factor only depends on its datum modulo $\mathfrak{M}_O^2$. (b) The descent is linear and commutes with the finite class sum; apply (a) to each term and use that kernels are linear in the polynomial. □

LEAN `DividedSplitting.theta_productForm` · `DividedSplitting.theta_stdClassSum_productForm`

Forms that are not lifts have vanishing theta periods

Setting 3.11 (a form that is not a lift)**.** In Setting 2.1: r is not empty, N is a finite index set, O is an auxiliary pair for $(E,\mathsf{r},\mathsf{n})$ that fits a parameter l_0 of forms on W_O in the sense of the unit *Exchange*, with $\psi_{O,w} \neq 1$ at every twist place w and $2|\mathsf{r}| + 2|\mathsf{N}| + 3 < |\mathsf{n}|$. Ξ is a form in $\mathcal{F}(O)$ such that $\mathrm{lift}_O(\Xi) = 0$; such that for every chosen place w_0 and every family v of rows over the residue fields of the chosen places with $v_{w_0} \neq 0$ the tensor product of the reflection sums of ψ_O and v acts on Ξ by zero; and such that $H_{O,w}(z_O(w))\,\Xi = 0$ for a finite place w with $w \neq \bar{w}$ and w, $\bar{w}$ not among the places of O.

Lemma 3.12 (descents to a sum with hyperbolic planes)**.** *In Setting 3.11, for every coordinate vector u of weight κ_O, the complex conjugate of the matrix coefficient $h \mapsto \langle u, \Xi(h)\rangle_{\kappa_O}$ has vanishing theta lifts from $\mathrm{U}(I_{\mathsf{n}})$ to the unitary group of $W_O \oplus \mathbb{H}^{|\mathsf{N}|}$, the sum of W_O with $|\mathsf{N}|$ hyperbolic planes, for the character $\mathbf{a}_O v^{|\mathsf{N}|}$ and all data that are the vacuum outside the places of O and their conjugates.*

Proof (*an account of the formal proof*). By the unit *SiegelWeil* (a) the descents of the matrix coefficients to $\mathrm{U}(W_O)$ vanish, for all data that are the vacuum outside the places of O and their conjugates: its hypotheses are those of Setting 3.11, the vanishing of $\mathrm{lift}_O(\Xi)$ being translated by the unit *Forms* into the vanishing of a theta lift of the conjugate coefficient against the highest weight vector, and extended to all u because the highest weight vector is cyclic. By the unit *SiegelWeil* (b), with $p = |\mathsf{N}|$, the descents to $\mathrm{U}(W_O \oplus \mathbb{H}^{|\mathsf{N}|})$ vanish: the coefficient is spherical at w and killed by the Hecke polynomial at the test value, since Ξ is. □

LEAN `descentsVanish_hypSum`

Proposition 3.13 (vanishing periods)**.** *In Setting 3.11, let* I *be a finite index set,* $\varepsilon\colon \mathsf{N}\sqcup\mathsf{I}\to\mathsf{n}$ *a bijection,* κ *a weight,* $Y=(Y_w)$ *matrices with independent columns as in Lemma 3.1, and* $e'\colon \mathsf{r}\hookrightarrow\mathsf{n}$ *an embedding of the form* $\varepsilon\circ u$ *with* $u\colon \mathsf{r}\to\mathsf{N}$*. Let* C *be a conjugate-linear map from the coordinate vectors of weight* κ *with rows* I *to those of weight* κ_O *that is equivariant for the definite group on* I *embedded as the block* $\operatorname{diag}_\varepsilon(1,y)$*. Let* b *be a character,* $\mathfrak{m}$ *a modulus whose primes are places of* O*,* Ψ *a function on matrices modulo* $\mathfrak{m}$*,* g *and* h_1 *points,* $x\in \mathrm{U}(I_{\mathsf{n}})(\mathbb{A}_f)$ *a point with component* 1 *outside the chosen places and their conjugates,* Γ' *a level of* $\mathrm{U}(I_{\mathsf{I}})$*, and let*

$$f(h)=\Big\langle \mathrm{pt}_{\operatorname{diag}_\varepsilon(h_1,h)\,x,Y,e'}\big(C(\mathrm{kv}_{O,b;\mathfrak{m}}[\Psi;\kappa](g,h).)\big),\ \Xi\Big\rangle.$$

Suppose that every $k\in\Gamma'$ *satisfies* $\Psi((k \bmod \mathfrak{m})^{-1}T)=\Psi(T)$ *for all* T*, and that the element* $x^{-1}\operatorname{diag}_\varepsilon(1,k)\,x$ *lies in* $\mathbf{K}_{\mathsf{n}}$ *and its reduction fixes every* Y_w*. Then* $\sum^{\mathrm{cl}}_{h \bmod \Gamma'} f(h)=0$*.*

Proof (*an account of the formal proof*)*.* By Lemma 3.2, $f(h)$ is the Gram form of C(kernel vector at (g,h)) with the value of Ξ at $\operatorname{diag}_\varepsilon(h_1,h)\,x\,k_{Y,e'}$. Consider on $\mathrm{U}(I_{\mathsf{n}})(\mathbb{A})$ the function

$$F(h)=\sum_i \theta(\omega(g,h)\varphi_i)\cdot\overline{\langle C(\delta_i),\Xi(h\,x\,k_{Y,e'})\rangle_{\kappa_O}},$$

where i runs over the standard fillings of weight κ, δ_i is the i-th basis vector, and φ_i is the product, for the decomposition $I_{\mathsf{n}}=I_{\mathsf{N}}\oplus I_{\mathsf{I}}$, of a fixed translate v of the vacuum on the first summand and of the datum of Ψ with the i-th basis polynomial in the negative letters on the second. It is left invariant under the rational points and right invariant under a level. On the block $\operatorname{diag}_\varepsilon(h_1,(y,h))$, by the seesaw in the second space, by the equivariance of C and by the rule for the real Weil representation on Fock vectors, $F=\theta(\omega(g,h_1)v)\cdot\overline{f(h)}$. The theorem on periods (library) is applied to F and to the splitting $\mathsf{n}=\mathsf{N}\sqcup\mathsf{I}$; its complementary group is the split group $\mathrm{U}(\mathbb{H}^{|\mathsf{N}|})$. The theta descents of F to that group are, term by term and by the seesaw in the first space, the lifts of Lemma 3.12 to $\mathrm{U}(W_O\oplus\mathbb{H}^{|\mathsf{N}|})$, evaluated at the point g on the summand W_O and translated by $x\,k_{Y,e'}$; the data φ_i are the vacuum outside the places of O and their conjugates. So these descents vanish, and the period of F over $\mathrm{U}(I_{\mathsf{I}})$ vanishes: $\theta(\omega(g,h_1)v)\cdot\overline{\int f}=0$. The translate v can be chosen with $\theta(\omega(g,h_1)v)\neq 0$, because the theta functional does not vanish on all translates of the vacuum. Finally the integral of f over the definite group is its class sum at level Γ', since f is right invariant under Γ': a translation by $k\in\Gamma'$ is absorbed by Ψ in the kernel vector, by the hypothesis on Ψ, and by the level of Ξ in its value, by the hypothesis on $x^{-1}\operatorname{diag}_\varepsilon(1,k)x$ and the unit *SiegelWeil* (c). □

LEAN `stdClassSum_period_eq_zero`

Corollary 3.14 (product forms are orthogonal to forms that are not lifts)**.** *In Setting 3.11, let* I *be a finite index set,* a *an index set,* $z\colon \mathsf{r}\hookrightarrow\mathsf{I}$*,* $\mathcal{V}$ *a divided splitting in* $\mathrm{Div}(\mathsf{N},\mathsf{I},\mathsf{a};O,z)$*, and suppose that* $\mathfrak{M}^2_O$ *is generated by an element* $\varpi_2\neq 0$ *of* $\mathcal{O}_E$*. Let* $L\in\mathrm{Lat}(O)$*,* x *a point of* $\mathrm{U}(I_{\mathsf{N}})(\mathbb{A}_f)$*,* y *a coordinate vector of weight* κ^+_O*,* b *a character,* Ψ *a function on matrices modulo* $\mathfrak{M}^2_O$ *and* g *a point. Then, with* $\Gamma(\varpi_2)$ *the principal level of* $\mathrm{U}(I_{\mathsf{I}})$ *modulo* ϖ_2*,*

$$\sum^{\mathrm{cl}}_{h \bmod \Gamma(\varpi_2)}\Big\langle \mathrm{pr}_{\mathcal{V},L}((x,h);y)\Big(\overline{\mathrm{kv}_{O,b;\mathfrak{M}^2_O}[\Psi;\kappa^-_O](g,h).}\Big),\ \Xi\Big\rangle=0.$$

Proof. This is Proposition 3.13 for ε of $\mathcal{V}$, $Y_w = I_{e_O}$, $e' = \varepsilon \circ \phi_{\mathcal{V}}$, the point γ_L, the map $C(\zeta) = y \cdot \bar{\zeta}$ and the level $\Gamma(\varpi_2)$. The map C is equivariant because the weight representation of a block diagonal element on a Cartan product is the Cartan product of the representations (library). An element k of $\Gamma(\varpi_2)$ is 1 modulo $\mathfrak{M}_O^2$, so it fixes Ψ, and by Lemma 3.4 the conjugate of $\mathrm{diag}_\varepsilon(1,k)$ by γ_L is integral and reduces to 1 at the chosen places. □

LEAN `DividedSplitting.stdClassSum_inner_productForm_eq_zero`

Integrality of product forms

Lemma 3.15 (the loss of the Cartan product)**.** *In Setting 2.1, let O be an auxiliary pair for $(E, \mathsf{r}, \mathsf{n})$, N, m, l finite index sets, a an index set, $z\colon \mathsf{r} \hookrightarrow \mathsf{m}$, $\mathcal{V}$ a divided splitting in $\mathrm{Div}(\mathsf{N}, \mathsf{m}, \mathsf{a}; O, z)$, p a prime with an embedding $\iota_p\colon \mathbb{C} \to \mathbb{C}_p$, and $M \geq 1$ an integer in the ideal $\mathfrak{M}_O$.*

(a) *Let $L \in \mathrm{Lat}(O)$, κ a weight without columns of height h for $|\mathsf{l}| \leq h < |\mathsf{n}|$, with $|\mathsf{l}| \leq |\mathsf{n}|$, let $f\colon \mathsf{l} \to \mathsf{n}$ be a map, S a set of columns, g and h finite adelic points and v a coordinate vector of weight κ. Suppose that some invertible J over $\mathcal{O}_E/\mathfrak{M}_O^2$ makes $\bar{\varpi}_{\mathcal{V}}$ times the columns f of G_L, multiplied by J, vanish outside the columns S, and that the columns f of g^{-1} are $I_f h^{-1}$. Then the sup norm of the Schur matrix of the columns f of $(g\gamma_L)^{-1}$ applied to $\iota_p(v)$ is at most $|M|_p^{-\nu} \cdot \mathrm{gauge}_\kappa^{\iota_p}(0; h, v)$, with $\nu = \sum_\tau |S| \sum_{i<|\mathsf{l}|} \kappa_\tau(i)$.*

(b) *Let s be an $\mathsf{a} \times \mathsf{N}$ matrix over $\mathcal{O}_E/\mathfrak{M}_O^2$, with a finite, $L \in \mathrm{Lat}(O)$ with $\mathrm{Glued}_{\mathcal{V}}(s, L)$, $h = (h_1, h_2)$ a pair of points and y, η coordinate vectors of weights κ_O^+ and κ_O^-. Then*

$$\mathrm{gauge}_{\kappa_O}^{\iota_p}(\epsilon_O; \mathrm{diag}_\varepsilon(h_1, h_2)\gamma_L, y \cdot \eta) \leq |M|_p^{-\mathrm{loss}(\mathcal{V})} \cdot \mathrm{gauge}_{\kappa_O^+}^{\iota_p}(0; h_1, y) \cdot \mathrm{gauge}_{\kappa_O^-}^{\iota_p \circ \mathrm{conj}}(0; h_2, \bar{\eta}).$$

Proof (an account of the formal proof). (a) Lift J to an integral point k; replacing h by hk^{-1} does not change the gauge. The matrix $\varpi_O \gamma_L^{-1}$ is integral, so the columns f of $(g\gamma_L)^{-1}$, multiplied by k and by the diagonal matrix with entries M on S and 1 elsewhere, form an integral matrix: on S because $M \in \mathfrak{M}_O$, and outside S by the hypothesis and Input M (b). Schur matrices are multiplicative and contract the sup norm on integral matrices (library), and the Schur matrix of the inverse diagonal matrix has norm at most $|M|_p^{-\nu}$. (b) The weight representation of a block diagonal element on a Cartan product is the Cartan product of the representations, and an integral left inverse of the Cartan embedding has norm at most 1. Apply (a) to the two blocks: to κ_O^+ on N with S the image of the divided embedding, by condition (2) of a glued lattice, and to κ_O^- on m with S everything, for the conjugate embedding, since the block on m enters through the inverse transpose. The two exponents add to $\mathrm{loss}(\mathcal{V})$. □

LEAN `DividedSplitting.norm_schur_inv_cols_le_weylGauge` · `DividedSplitting.weylGauge_cartanProduct_le`

Crude denominators and conjugate bounds for theta jets

The next results concern pairs of *definite* standard spaces over the fields of the parameters. The notions are those of the units *Data* and *Curve*: for parameters

$P = (N_0, d)$ with $K = \mathbb{Q}(\sqrt{-d})$ and an integer $\mu \geq 0$, the *exponent*, $E(P, \mu)$ is the CM field of the parameters with exponent μ (for data D with exponent $\mu(D)$, $E(D)$ is the field $E(P, \mu(D))$), $\mathrm{Pl}(P, \mu)$ its set of infinite places, $N_1(P, \mu)$ its conductor, $\Omega(P, \mu) > 0$ its period constant, $\mathrm{per}(P, \mu)$ its jet scale, $\kappa_0(P, \mu)$ its vacuum scale and $C(P)$ the constant of the parameters; for data D, $N_1(D)$, per and κ_0 are those of $E(D)$. For a Schwartz datum f of a pair of definite standard spaces and $c \in \mathbb{C}$, $T_c(f)$ is the *scaled Taylor series* of f: a formal power series in variables $X_{\tau,i,i'}$ whose coefficients are, up to the scale c, the Taylor coefficients at the origin of the theta function of f in the coordinates of the CM curve. The *jet* of a power series F along a polynomial Q is $\mathrm{jet}_Q(F) = Q(\partial)F(0)$, and $\mathrm{jet}^\kappa(F)$ is the matrix whose product with a coordinate vector ξ of weight κ has as coordinates the jets of F along the coefficient polynomials of $z \mapsto S_\kappa(z)\xi$. An *admissible combination of size B* of translates of the integral vacuum is a finite combination of Heisenberg translates of the integral vacuum by finite adelic vectors whose coefficients have, under every element of $\mathrm{Gal}(\mathbb{C}/K)$, absolute values of sum at most B. The *boxes* of a weight κ below N are $\sum_\tau \sum_{h<N} h\,\kappa_\tau(h)$, and $\lambda_\kappa(\tau, i) = \sum_{i<h<N} \kappa_\tau(h)$ is its row length.

Lemma 3.16 (small facts)**.** (a) *For parameters P and every $\sigma \in \mathrm{Aut}(\mathbb{C})$ there is $\sigma' \in \mathrm{Gal}(\mathbb{C}/K)$ with $|\sigma(z)| = |\sigma'(z)|$ for all $z \in \mathbb{C}$.* (b) *Let L and N be finite index sets, L not empty, e a family of integers and l a family of integer vectors, indexed by N and by $\mathsf{N} \times \mathsf{L}$, and κ a weight such that l is twice the Harish-Chandra parameter of (κ, e). If $\beta \geq 0$ is an integer with $|l(w,i)| \leq \beta$ for all w, i, then the boxes of κ below $|\mathsf{L}|$ are at most $|\mathsf{N}|\,|\mathsf{L}|\,\beta$, and $|e(w)| \leq \beta$ for all w.* (c) *Let* Pl *and n be finite, s a sign vector on n, κ a weight, e a family of integers indexed by* Pl*, c a real number and l a family of integer vectors on n indexed by* Pl*. If every l_w has gap c for the signs s, in the sense of the unit on discrete series, and $e(w) \leq l(w,i)/2 + |\mathsf{n}|/2$ for all w, i, with integer division, then there is a weight f such that l is twice the Harish-Chandra parameter of $(f,\ w \mapsto e(w) + f_w(|\mathsf{n}|))$ and $f_w(k) = \kappa_w(k)$ for all w and all $k > |\mathsf{n}|$.* (d) *In Setting 2.1, let L, N be finite index sets, a and b characters with weights e, e' and tame parts ψ, ψ', $\mathfrak{m}$ a modulus, Ψ a function on matrices modulo $\mathfrak{m}$, Q a polynomial, γ and γ' rational points of $\mathrm{U}(I_\mathsf{L})$ and $\mathrm{U}(I_\mathsf{N})$, and g, h finite adelic points. Then*

$$\theta^{I_\mathsf{L}, I_\mathsf{N}}_{a,b;\mathfrak{m}}[\Psi, Q](\gamma g, \gamma' h) = \prod_\tau \tau(\det \gamma')^{-e_\tau} \tau(\det \gamma)^{e'_\tau} \cdot \theta^{I_\mathsf{L}, I_\mathsf{N}}_{a,b;\mathfrak{m}}[\Psi, Q'](g, h),$$

where τ also denotes the embedding of the CM type at τ and Q' is Q after the substitution $x_\tau \mapsto \tau(\gamma'^{-1})^{\mathsf{T}} x_\tau\, \tau(\gamma)^{\mathsf{T}}$.

Proof ((a)*: complete;* (b) *to* (d)*: an account of the formal proof*)*.* (a) The restriction of σ to the imaginary quadratic field K is the identity or complex conjugation. Take $\sigma' = \sigma$ in the first case and $\sigma' = \mathrm{conj} \circ \sigma$ in the second: it is the identity on K, and $|\sigma'(z)| = |\overline{\sigma(z)}| = |\sigma(z)|$ for all z. (The order matters: $|\sigma(\bar{z})|$ need not be $|\sigma(z)|$ for a discontinuous σ.) (b) The boxes are the sum of the row lengths over the $|\mathsf{N}|\,|\mathsf{L}|$ pairs (w, i), and each row length and each $|e(w)|$ is recovered from two values of l_w (library), hence at most β. (c) By induction on the values of l_w in increasing order, defining f_w at one more height in each step so that the next value is matched; the gap makes the differences positive. (d) The theta functional is invariant under rational points, and at the infinite places a rational point acts on the Fock vector of Q through the real Weil representation, which substitutes the matrices into Q and multiplies by

the determinant characters (Input 6 of the sister note); the weights of a and b give the exponents. □

LEAN `Parameters.exists_algEquiv_norm_eq` · `boxes_le_of_isHCParam` · `exists_isHCParam_of_le` · `modKernel_embFin_mul` · with $\kappa = 0$ and e the minimum of the values, (c) gives: every family with a gap is twice a Harish-Chandra parameter, `exists_isHCParam_of_hasGap` · the two bookkeeping lemmas `FitsRanks.sum_diagramWeight_eq_boxes` (the sum of the row lengths is the number of boxes) and `sum_mem_integralAbove` (a block sum of points integral above p is integral above p) need no argument

Proposition 3.17 (conjugates of jets)**.** *Let P be parameters, $\mu \geq 0$ an integer, n and n' finite index sets, a and b characters of $E(P,\mu)$ with weights e_a, e_b and tame parts ψ, ψ', B a real number and f a datum of the pair $(I_{\mathsf{n}}, I_{\mathsf{n}'})$ that is an admissible combination of size B of translates of the integral vacuum. Let $\sigma \in \mathrm{Aut}(\mathbb{C})$, $c \in \mathbb{C}$, and let g, h be finite adelic points of the two unitary groups. Put $u = \log(|\sigma(c)|^2/\Omega(P,\mu))$,*

$$M(L,w) = B \cdot \exp\big(C(P)\,(N_1(P,\mu)\,|\mathsf{n}'|\,|\mathsf{n}|+1)^2 + |\operatorname{Pl}(P,\mu)|\,|\mathsf{n}'|\,|\mathsf{n}|\,\log(2L+2) + w\big),$$

and $F = T_{c\,\mathrm{per}(P,\mu)}(\omega_{a,b}(g,h)f)$. Then:

(i) *for every polynomial Q, every integer $L \geq 0$, every real w and every real $A \geq 0$: if $|\sigma(Q(z))| \leq A\,e^{w+LR}$ for all points z and all real R with $\sum_i |\sigma(z_i)|^2 \leq e^u(2L+2)R$, then $|\sigma(\mathrm{jet}_Q(F))| \leq M(L,w)\cdot A$;*

(ii) *for every integer $L \geq 1$, every weight (κ, e) that is a lift weight for $(\mathsf{n}, \mathsf{n}', e_b)$ and every coordinate vector ξ:*

$$\|\sigma(\mathrm{jet}^{\kappa}(F)\xi)\|_{\kappa} \leq M(L,w_{\kappa})\cdot\|\sigma(\xi)\|_{\kappa},$$
$$w_{\kappa} = \tfrac{1}{2}\sum_{\tau}\sum_{i<|\mathsf{n}|}\Big(\frac{\lambda_{\kappa}(\tau,i)}{L} + \log(\lambda_{\kappa}(\tau,i)!) + \lambda_{\kappa}(\tau,i)\,u\Big).$$

Proof (*an account of the formal proof*). (i) By Lemma 3.16 (a) one may take $\sigma \in \mathrm{Gal}(\mathbb{C}/K)$. The datum is a combination of translates of the vacuum with coefficients of total size B under σ, so it suffices to bound each term. For a translate of the vacuum, the conjugate of a jet is the jet of the image under a semilinear map, of norm at most $\exp(C(P)(N_1(P,\mu)|\mathsf{n}'||\mathsf{n}|+1)^2)$ (unit *Curve*), and a jet of a scaled Taylor series is the theta functional of the datum integrated against the Fock vector of the scaled polynomial (library): a Gaussian integral of Q against a translated vacuum. The hypothesis on Q on balls and the value of the Gaussian integral with parameter $(2L+2)^{-1}$ give the factor $(2L+2)^{|\operatorname{Pl}(P,\mu)||\mathsf{n}'||\mathsf{n}|}A\,e^{w}$. (ii) By duality for the Gram norm (library) it suffices to bound $|\sigma(\lambda\cdot\mathrm{jet}^{\kappa}(F)\xi)|$ for the vectors λ of dual norm at most 1. This number is the jet of F along the polynomial $z \mapsto \lambda\cdot S_{\kappa}(z)\xi$, to which (i) applies: the Gram norm of $S_{\kappa}(z)\xi$ is at most the product of the squared singular values of z to the row lengths times the norm of ξ (library), and $x^l \leq \exp(\log l! + lv + xe^{-v})$ turns each power into the admissible growth, with $v = u + 1/L$. □

LEAN `Parameters.norm_jet_le_of_isAdmissibleComb`

Proposition 3.18 (crude denominators and sizes of kernels of definite pairs)**.** *Let L and N be finite index sets, P parameters, D admissible data for P and $\mathsf{d} \geq 0$ an integer. Then there is $c \in \mathbb{C}$, $c \neq 0$, such that for all characters a, b of $E(D)$ with weights and tame parts, every modulus $\mathfrak{m}$, every integer $q \geq 0$ with $q\,g(D)^2 \in \mathfrak{m}$, every function Ψ on matrices modulo $\mathfrak{m}$ whose values are integral at every embedding*

into $\mathbb{C}_\ell$ with $\ell \nmid q$, every polynomial Q that is homogeneous of degree d, and all finite adelic points g, h, the number $v = c \cdot \theta^{I_{\mathsf{L}}, I_{\mathsf{N}}}_{a,b;\mathfrak{m}}[\Psi, Q](g,h)$ satisfies:

(i) *for every prime p prime to q and every embedding $\iota_p \colon \mathbb{C} \to \mathbb{C}_p$: if g and h are integral at the primes above p and the coefficients of Q are ι_p-integral, then $|\iota_p(v)|_p \le 1$;*

(ii) *if $|\mathsf{L}| \le 40\,k(D)$, $|\mathsf{N}| \le 40\,k(D)$, $\mathsf{d} \le g(D)$ and $q = 1$, then for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, every integer $M \ge 0$ and every real B such that $|\sigma'(\Psi(x))| \le M$ for all $\sigma' \in \mathrm{Aut}(\mathbb{C})$ and all x, and $|\sigma(Q(z))| \le B$ for all z with $|\sigma(z_i)| \le g(D)^4$ for all i: $|\sigma(v)| \le e^{x_1}\, M\, B$.*

The formal text lists a point h and a character b among the arguments of this statement; they are those quantified inside it.

Proof (an account of the formal proof). The constant c is the product of the scales that make c times the kernel equal to the jet along Q of the scaled Taylor series of $\omega_{a,b}(g,h)$ applied to a multiple of the datum of Ψ; it depends on D, d and the index sets only, because Q is homogeneous of degree d. (i) By the unit *Exchange*, a power of $g(D)$ times the datum of Ψ is a standard datum at p, since $q\,g(D)^2 \in \mathfrak{m}$, Ψ is integral away from q and $p \nmid q$; the Weil representation at points integral above p preserves standard data, and jets of standard data along ι_p-integral polynomials are ι_p-integral (unit *Curve*). (ii) The datum is an admissible combination of size $|\mathcal{O}_E/\mathfrak{m}|^{|\mathsf{L}||\mathsf{N}|}M$ (unit *Exchange*). Apply Proposition 3.17 (i) with $L = \mathsf{d}$ and $w = 0$: by homogeneity the bound for Q on the polydisc of radius $g(D)^4$ gives the growth hypothesis with $A = B$. The remaining factors, $|\mathcal{O}_E/\mathfrak{m}| \le g(D)^{4f}$, the exponential of $C(P)(N_1(D)|\mathsf{L}||\mathsf{N}| + 1)^2$ and the power of $2\mathsf{d} + 2$, are at most e^{x_1} by the inequalities between the scales (unit *Data*). □

LEAN `exists_modKernel_integral_bound`

A jet that is no more divisible than predicted

Proposition 3.19 (the depth of an indicator jet)**.** *Let P be parameters and D admissible data for P, with nominal weights $w_D(\tau)$ at the infinite places τ of $E(D)$, and let $\mathbf{m} = (\mathbf{m}_\tau)$ be integers with $w_D(\tau) \le \mathbf{m}_\tau \le w_D(\tau) + x^{42}$ for all τ. Then there is $t_{\mathbf{m}} \in \mathbb{C}$, $t_{\mathbf{m}} \ne 0$, with $|\sigma(t_{\mathbf{m}})| \le \exp\big(f(D)\,m\,b/k(D)\big)$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, such that for every modulus $\mathfrak{m}$ with ideal generated by $\bar\pi_g$, every prime p and every embedding $\iota_p \colon \mathbb{C} \to \mathbb{C}_p$ there are an exponent α, a family of integers $\alpha(\tau, i) \ge 0$ for $1 \le i \le 12$, with $\sum_i \alpha(\tau,i) = \mathbf{m}_\tau$ for all τ, and a 12×1 matrix x_0 modulo $\mathfrak{m}$ with*

$$p^{-F_D(p)} \le \Big|\iota_p\Big(\frac{\kappa_0^{12}}{t_{\mathbf{m}}} \cdot \prod_\tau \mathbf{m}_\tau! \cdot \mathrm{coeff}_\alpha\; T_{\bar\pi_g\,\mathrm{per}}(\mathbf{1}_{x_0})\Big)\Big|_p .$$

Here $\mathbf{1}_{x_0}$ is the datum of (I_1, I_{12}) given by the indicator of x_0 modulo $\mathfrak{m}$ and the vacuum, κ_0 and per *are the vacuum scale and the jet scale of $E(D)$, the scale of the Taylor series is $\bar\pi_g\,$*per*, in which the integer $\bar\pi_g$, an element of the imaginary quadratic field K, is read as a complex number through $K \subset \mathbb{C}$, and $F_D(p)$ is the* forced part*: the sum over τ of the divisibility by p that a derivative of order $w_D(\tau)$ at τ is sure to have (unit* Curve*).*

Outline of the proof. The outline follows the formal proof and names its steps and the results that they use. It does not reproduce the comparisons of exponents and of

valuations, which the formal proof closes by simplification and linear arithmetic, nor the exact statements of the inputs from the unit *Curve*, which are taken from their docstrings.

The number $t_{\mathbf{m}}$. Let $V = \mathrm{Hom}(I_1, I_{12})$. Put $t_{\mathbf{m}} = g(D)^{24 f(D)} \prod_{\tau} \lambda(\mathbf{m}_\tau) \cdot y_0$, where $\lambda(l)$ is the *depth loss* of the unit *Curve*, the square of a product of $\lceil 60b/k(D) \rceil$ consecutive integers ending at l, and y_0 is a non-zero value of the integral vacuum of V at a finite vector y. The bound for the conjugates of $t_{\mathbf{m}}$ follows from Lemma 3.16 (a), the bound for the integral vacuum, $\mathbf{m}_\tau \le w_D(\tau) + x^{42}$ and the inequalities between the scales.

The series. Fix $\mathfrak{m}$, p and ι_p. A tight frame for p (unit *Curve*) gives vectors w_i with multiplicities $c_i \ge 3 \sum_\tau \mathbf{m}_\tau$, one of them of unit size at every conjugate. With the period factor S of $E(D)$ let ϱ_i be the linear form $\tau \mapsto \iota_p(S \overline{\tau(w_i)})$, and let $\Theta = \prod_i \vartheta_p(\varrho_i)^{c_i}$, where ϑ_p is the p-adic Taylor series of the theta function of the CM curve. In the ring generated by the series of the translated derivative of the Weierstrass function at the ϱ_i, choose for each exponent α an exponent $e(\alpha)$, of degree at most that of α, such that the product Θ_α of the generators to the powers $e(\alpha)_i$ with Θ has the largest coefficient at α among all products of an element of the ring with Θ (library). Let $N(\alpha)$ be the absolute value of this coefficient. By construction every product $H\Theta$, H in the ring, is *carried by one copy of N*: its coefficient at α is at most $N(\alpha)$. The inverse Θ^{-1} is carried by five copies: by the p-adic logarithm of the theta function (library; a result of the artifact, without a reference in the catalogue), $\vartheta_p = \exp(-\Lambda)$ with Λ of order two, so that, for four p-adic integers j_b with $\sum_b j_b^2 = -2$, which exist because every natural number is a sum of four squares, the natural numbers are dense in the p-adic integers and the set of sums of four squares of p-adic integers is closed, Θ^{-1} is a unit of the ring times Θ times the four rescalings of Θ by the j_b. Hence every $H\Theta \cdot \Theta^{-1} = H$ is carried by six copies.

A large split. The claim is that $\mathbf{m}$ is a sum of twelve exponents α_b with

$$\prod_b N(\alpha_b) \;\ge\; p^{-F_D(p) - \sum_\tau v_p(\lambda(\mathbf{m}_\tau)) + \sum_\tau v_p(\mathbf{m}_\tau!)}.$$

If p splits in K, this is the limit to Katz's divisibility (unit *Curve*): for each order there is a series of the ring with a coefficient of controlled valuation; apply it to $\mathbf{m}$ and to 0, each on six copies, at the frame vector of unit size. If p does not split in K, then p is prime to $g(D)$ and the period factor is a p-adic unit. A Frobenius packing of the orders (unit *Curve*) offers two alternatives, and a large split from pairs gives exponents carried by six copies; in the first alternative an element of the ring with lowest term a monomial with coefficient a power of $\iota_p(\bar{S} S\, g(D))$ is built by induction on the monomial from the frame identity, in the second the Katz bound is used again; the valuations of the factorials involved are compared by the inequality of the packing.

From the series to a datum. The degrees satisfy $3e(\alpha_b)_i \le c_i$, so the tight frame produces (unit *Curve*) a finite datum of V whose scaled Taylor series is, after ι_p, the product over b of the series Θ_{α_b} in twelve separate sets of variables; its coefficient at the exponent α assembled from the α_b is $\prod_b N(\alpha_b)$ in absolute value. This datum is a combination with ι_p-integral coefficients of translates of the vacuum, and each translate is a finite sum of indicator data modulo $\mathfrak{m}$ with coefficients of absolute

value 1 (library). By the ultrametric inequality one indicator datum $\mathbf{1}_{x_0}$ has a coefficient at α at least as large. Inserting the large split and the definition of t_{m} gives the inequality. □

LEAN `Data.depth_indicator_jet`

Cleared values, good projections, a monomial vector

Lemma 3.20. *In Setting 2.2:*

(a) *Let p be a pair of D in the sense of the unit* Exchange*, a character of weight $-c$ with an integer z, let n be a finite index set for which p is flat, that is $c_\tau = z$ for all τ and $|\mathsf{n}| = 2z \geq 4$, and $\sigma \in \mathrm{Aut}(\mathbb{C})$. Then the normalised value of p at the trivial twist, the squared norm of the trivial lift cleared of its denominators, has $|\sigma(\,\cdot\,)| \leq \exp(x^{34})$.*

(b) *Let ℓ be a prime, $\iota\colon \mathbb{C} \to \mathbb{C}_\ell$ an embedding, ϑ an m-fold twist of D, $a \in \mathbb{C}$, z a form of principal level on $Y_0(D)$ such that a times its image under the chain of lifts for ϑ is integral at ι, and $s \in \mathsf{G}(D)$. Then a times the image of $|\mathsf{G}(D)|\, e_0(D)\, s\, z$ under the chain is integral at ι.*

(c) *Let $\sigma \in \mathrm{Aut}(\mathbb{C})$, z a form of principal level on $Y_0(D)$, $s \in \mathsf{G}(D)$ and B a real number such that every value of the image of z under the chain of lifts for the trivial twist has $\|\sigma(\,\cdot\,)\|^2 \leq e^B$. Then the image ϕ of $|\mathsf{G}(D)|\, e_0(D)\, s\, z$ under the chain has $\|\phi\|_\sigma^2 \leq \exp(x^{99} + B)$.*

(d) *Let n be a finite index set, X an L-value datum of $(E(D), \mathsf{n})$ in the sense of the unit* Exchange*, with auxiliary pair O_X and jet degrees $d_X(\tau)$, $r' \geq 0$ an integer, $e\colon \{1, \dots, r'\} \hookrightarrow \mathsf{n}$ an embedding and α an exponent with $\sum_i \alpha(\tau, i) = d_X(\tau)$ for all τ. Then there is an integer coordinate vector v with $P^{O_X}(\bar{v}) = (\prod_\tau d_X(\tau)! / \prod \alpha!) \cdot x^\alpha$, the monomial being read in the rows e, such that for every field R of characteristic zero and every family g of invertible $\mathsf{n} \times \mathsf{n}$ matrices over R whose columns $e(a)$ are those of the identity, the weight representation of g fixes the vector $w = v$, read in R.*

Proof (*an account of the formal proof*). (a) The squared norm of the trivial lift is a class sum; each value is a matrix of jets of a vacuum datum applied to a vector, bounded by Proposition 3.17 (ii) with $L = 1$; the number of classes and the orders of the residue rings are bounded by the units *Columns* and *Exchange*, and the scales absorb everything. (b), (c) By the unit *Seesaw* the chain intertwines the finite group with a right translation by integral points, so the value of the image at g is $\sum_{t \in \mathsf{G}(D)} |\mathsf{G}(D)|\, e_0(t)$ times values of the image of z at translates of g. The numbers $|\mathsf{G}(D)|\, e_0(t)$ are algebraic integers (unit *Trace*) and translation by integral points keeps the gauge, which gives (b) by the ultrametric inequality. For (c) the sum of their conjugates' absolute values, the order of the group and the number of classes are bounded by the scales. (d) Let v_k be the coefficient of the monomial in the integral basis polynomial b_k. The monomial is semi-invariant of the mixed weight, hence the polynomial of a coordinate vector (Input 1 of the sister note), and by Fock duality that vector has the coordinates $\alpha!\, v_k / h_\kappa$; the hook number is $\prod_\tau d_X(\tau)!$ (unit *Exchange*). The coordinates of the weight representation of g applied to w are Fock pairings of the basis with the monomial after the row substitution by g^{T}, and the monomial, which involves the rows e only, is fixed by this substitution. □

LEAN `Data.Pair.norm_conj_intAlgPart_le` · `Data.goodProjTranslate_mem_integralForms` · `Data.normSqAt_goodProjTranslate_le` · `exists_int_vector_harmonicPoly_eq` · the formula for the value of the image at a point is `Data.iota_goodProjTranslate_apply`

The comparison forms

Lemma 3.21 (the adjoint of the lift on a comparison form)**.** *In Settings 2.1 and 2.2, let j be a lift index, ϑ a twist of D and $\kappa \in \mathbb{C}$ such that, for every choice $i \in \mathrm{Mem}_D(j)$ and every point g,*

$$F_D^\natural(j,i)(g) = \kappa \cdot \mathrm{cks}_{O_D(\vartheta,j),\mathbf{b}''_D(\vartheta,j)}([i_1;\bar{\varpi}]_{\mathcal{V}_D(j,\vartheta)},\phi;\mathrm{lm}_D(\vartheta,j,i,\cdot))(g).$$

Then for every $F \in \mathcal{M}^\circ_D(W_j(D))$ and every table ζ, with u_ϑ as in Theorem A,

$$\bar{\kappa}\cdot\langle F, u_\vartheta^*\ \mathrm{gl}_{D,j,\vartheta}(\zeta)\rangle = |\mathsf{C}(\mathcal{V}_D(j,1))|\cdot g(D)^{\mathrm{loss}(\mathcal{V}_D(j,1))}\cdot\overline{\beta_D^{\mathrm{line}}(j)}\cdot\sum_{i\in\mathrm{Mem}_D(j)}\overline{\beta_{L_D(j,1,i)}}$$
$$\cdot\int_{[\mathrm{U}(W_j(D))]}\overline{F(g)\,F_D^\natural(j,i)(g)}\cdot\mathrm{tt}_D(j,\zeta_{(i,\cdot)})(g)\,dg,$$

the integral standing under the sum over i.

Proof. By adjointness (unit *Exchange*) the left inner product is the Petersson product of F with the descent of the comparison form. By Proposition 3.10 (b), for the characters $b = \mathbf{b}''_D(\vartheta,j)$ and $b' = \upsilon^{m/2-j}$, whose product is the second character of the pair, the descent of the term $a = (i,\mathfrak{c})$ is the conjugate of the Weil scalar times $\beta_D^{\mathrm{line}}(j)$ times the sum over $A \in \mathsf{C}(\cdot)$ of the conjugate of the coset kernel sum of the line multipliers times the kernel $\Theta^-_{\zeta_a}(\,\cdot\,,h_{\mathfrak{c}})$, both at $g\tilde{A}$. By hypothesis the coset kernel sum is $F_D^\natural(j,i)/\kappa$. The form F is invariant under $\tilde{A}$, which reduces to 1 at the chosen places (Remark 3.7), and the integral over $[\mathrm{U}(W_j(D))]$ is invariant under right translation, so the terms of the sum over A are equal; the integrals converge by Input L (b). The sum over $\mathfrak{c}$ of the conjugates of the kernels is $\mathrm{tt}_D(j,\zeta_{(i,\cdot)})$. The splitting, its loss, its congruence group and the glued lattices do not depend on the twist. □

LEAN `inner_adjoint_comparisonForm`

Lemma 3.22 (integrality and size)**.** *In Settings 2.1 and 2.2, let j be a lift index and ζ a table.*

(a) *Let ϑ be a twist of D, p a prime prime to the norms of the primes in $Q(\vartheta)$ and $\varphi\colon \mathbb{C}\to\mathbb{C}_p$ an embedding. If $\mathrm{gauge}^{\varphi\circ\mathrm{conj}}_{\kappa^-}(0;h_{\mathfrak{c}},\overline{\zeta_a}) \le 1$ for every index $a=(i,\mathfrak{c})$, then $\mathrm{gl}_{D,j,\vartheta}(\zeta)$ is integral at φ.*

(b) *For every $\sigma \in \mathrm{Aut}(\mathbb{C})$: $\mathsf{h}_D(j)\cdot\|\mathrm{gl}_{D,j,1}(\zeta)\|_\sigma^2 \le \exp(2\,r\,B_0(D))\cdot\sum_a\|\sigma(\zeta_a)\|^2$.*

Proof ((a)*: complete;* (b)*: an account of the formal proof*)*.* (a) By Remark 3.7 the comparison form is a finite sum of point forms, that is of delta forms, and the gauge of a delta form is at most the gauge of its value at its point (library). By Lemma 3.15 (b), with $M = g(D)$, which lies in the ideal of the chosen primes because $g(D) = \bar{\pi}_g\overline{\bar{\pi}_g}$, the gauge of the value $g(D)^{\mathrm{loss}}\,(y\cdot\zeta_a)$ is at most the product of the gauges of y and of $\overline{\zeta_a}$, the powers of $|g(D)|_p$ cancelling. The first is at most 1 by Input L (c), and the second by hypothesis. (b) The squared norm is a class sum of squared norms of sums of values of delta forms. The norm of a Cartan product is

at most the product of the norms (library), $\|\sigma(y)\|^2 \le \exp(rB_0(D))$ by Input L (c) at the trivial twist, and the orders of class stabilisers divide the chosen group order (unit *Data*). The numbers of classes at the levels involved (unit *Columns*), the order of the residue ring, the hook number and $g(D)^{2\,\mathrm{loss}}$ are together at most $\exp(rB_0(D))$ by the inequalities between the scales. □

LEAN `comparisonForm_mem_integralForms` · `normSqAt_comparisonForm_le`

Lemma 3.23 (the adjoint of the comparison map on a tested form)**.** *In Settings 2.1 and 2.2, let j be a lift index, ϑ a twist of D, and let $\Xi \in \mathcal{F}(O_D(\vartheta, j))$ lie in the image of $e_0(D)$, in the kernel of $H_{O,w}(z_O(w))$ for the pair $O = O_D(\vartheta, j)$ and a finite place w with $w \neq \bar{w}$ and w, $\bar{w}$ not places of O, and in the orthogonal complement of the image of u_ϑ. Then for every choice $i \in \mathrm{Mem}_D(j)$ and every point g, with ζ^Ξ the image of Ξ under the adjoint of $\mathrm{gl}_{D,j,\vartheta}(\cdot)$, $\mathrm{tt}_D(j, \zeta^\Xi_{(i,\cdot)})(g) = 0$.*

Proof. The form Ξ satisfies Setting 3.11, with $\mathsf{N} = \mathsf{B}_D(j)$: the pair fits its parameter (unit *Exchange*); the tame part of the pair at a prime of $Q(\vartheta)$ is that of ϑ, which is not trivial; the reflection sums kill the image of $e_0(D)$ (unit *Trace*); $\mathrm{lift}_O(\Xi) = 0$ because, for every form y of principal level, $\langle y, \mathrm{lift}_O(\Xi)\rangle = \langle \mathrm{lift}^*_D(\vartheta, j)y, \Xi\rangle$ and, Ξ being in the image of the self-adjoint idempotent $e_0(D)$, this is $\langle \mathrm{lift}^*_D(\vartheta, j)(e_0(D)y), \Xi\rangle = 0$ by Input V and the orthogonality to the lifts of good forms, and $\mathrm{lift}_O(\Xi)$ is itself a form of principal level on $W_j(D)$ (unit *Selmer*), so that taking $y = \mathrm{lift}_O(\Xi)$ gives $\mathrm{lift}_O(\Xi) = 0$; and the rank condition holds by Input L (d). Now $\mathrm{tt}_D(j, \zeta^\Xi_{(i,\cdot)})(g)$ is the inner product of ζ^Ξ with the table that has the conjugate kernel vector at $(g, h_{\mathfrak{c}})$ in the place $(i, \mathfrak{c})$ and zero at the other choices, hence the inner product of Ξ with the comparison form of this table. That comparison form is a sum, over the classes of the level Γ_i, of sums over $\mathfrak{c}$ of product forms as in Corollary 3.14, the sum over the classes $\mathfrak{c}$ being a class sum at the principal level $\bar{\pi}_g^2$, which is free. Each inner sum pairs to zero with Ξ. □

LEAN `tableTheta_adjoint_eq_zero`

Proof of Theorem A. Take C and the numbers $c(\vartheta)$ of Input L (a) for the index set $\mathsf{a} = \mathsf{C}_D \sqcup \mathsf{B}_D(j)$, and apply it with $u' = (1 \mid 0)$ and $w = [i_1; \bar{\varpi}]_{\mathcal{V}_D(j,\vartheta)}$, so that $u'w = i_1$: the hypothesis of Lemma 3.21 holds with $\kappa = c(\vartheta)$. Write $R(F, \zeta)$ for the right side of Lemma 3.21; it does not depend on ϑ. *The numbers $c(\vartheta)$ are not zero*: otherwise all $F^\natural_D(j, i)$ vanish, and by Input L (b) every good form on $W_j(D)$ is zero, while $\dim \mathcal{M}^\circ_D(W_j(D)) \neq 0$. Put

$$\gamma(\vartheta) = \overline{c(1)/c(\vartheta)}, \qquad \mathrm{gl}_\vartheta = \mathrm{gl}_{D,j,\vartheta}(\cdot), \qquad A = u_1^* \circ \mathrm{gl}_1 .$$

(i) $\gamma(\vartheta) \neq 0$, $\gamma(1) = 1$ and $|\gamma(\vartheta)|^2 = |c(1)|^2/|c(\vartheta)|^2 = \mathrm{HV}_D(j, \vartheta)/\mathrm{HV}_D(j, 1)$. (ii) For all F and ζ, Lemma 3.21 for ϑ and for 1 gives

$$\overline{c(\vartheta)}\,\langle F, u^*_\vartheta\,\mathrm{gl}_\vartheta\,\zeta\rangle = R(F, \zeta) = \overline{c(1)}\,\langle F, A\zeta\rangle,$$

so $u^*_\vartheta\,\mathrm{gl}_\vartheta\,\zeta = \gamma(\vartheta)A\zeta$. (iii) Let F be orthogonal to the image of A. For a choice i and a function ζ' on $\mathrm{Cl}^\circ(D)$, the table equal to ζ' at the indices $(i, \mathfrak{c})$ and to zero elsewhere gives $0 = R(F, \zeta)$, which is a non-zero constant times $\int \overline{F F^\natural_D(j, i)} \cdot \mathrm{tt}_D(j, \zeta')$: the group order, $g(D)$, $\beta^{\mathrm{line}}_D(j)$ and the Weil scalar are not zero. By Input L (b), $F = 0$. So A is surjective. (iv) Let $y \in \mathcal{M}^\circ_D(W_j(D))$ and let Ξ be as in Lemma 3.23. Then $\langle \mathrm{gl}_\vartheta(A^*y), \Xi\rangle = \langle y, A\,\zeta^\Xi\rangle$, and $\overline{c(1)}\langle y, A\zeta^\Xi\rangle = R(y, \zeta^\Xi)$ vanishes because every

$\mathrm{tt}_D(j, \zeta^{\Xi}_{(i,\cdot)})$ does. So $\mathrm{gl}_\vartheta(A^*y)$ is orthogonal to every such Ξ: it is tested against the orthogonal complement of the image of u_ϑ for $e_0(D)$. (v) and (vi) are Lemma 3.22. □

LEAN `exists_comparison_forms`

4 Where the results are used

Theorem A is the input of the height bound for a ratio of lift volumes in the note *The height of a ratio of lift volumes*: (ii) and (iii) express the Gram determinant of the adjoint lift through the comparison forms, (iv) makes the congruence bound of a fit family applicable, (v) and (vi) bound it at the primes and at the conjugates, and (i) turns the twist into a ratio of Hecke values. Proposition 3.18 is used there for the integrality and size of the line multipliers, Proposition 3.19, Lemma 3.20 (a), (d) and Proposition 3.17 for the heights of the Hecke value ratios, and Lemma 3.20 (b), (c) in the unit *Endgame*, where crude rows complete a family of rows to a basis.

5 Notes on the formal text

1. *Arguments that only a compiler can settle.* The arguments were read from the text. In the following declarations this reading lists a letter that the statement binds itself or that occurs only in another declaration of the same section, and Lean presumably does not take it: a point h in the coset kernel sum and a point h and a character b in Proposition 3.18; a character b in Lemmas 3.3 and 3.9; a lattice L in Lemma 3.5 and Proposition 3.8; a twist ϑ in Theorem A, where it is quantified inside. Conversely, for Proposition 3.18 the reading does not list the two index sets L, N, which its uses supply as its first two arguments. The statements above follow the uses; none of this was confirmed by asking Lean. For this reason the coset kernel sum carries no notation made by the build.
2. *Statements cut at a local definition.* The formal statements of Propositions 3.17 and 3.18 and of Proposition 3.10 (b) contain local definitions (of u, M, F, of v, of the translated point); the tools of this write-up read such a statement only up to the first of them. The statements above were written from the Lean text.
3. *What is chosen.* The point $k_{Y,e}$ of a point form, the glued lattice, the divided embedding of the splitting, the representatives of classes and the number $\beta^{\mathrm{line}}_D(j)$ are chosen by the axiom of choice; the statements hold for every choice. The glued lattice for a twist is the one for the trivial twist, transported: this is used in Lemma 3.21.
4. *Theorem A is stated for twists.* The maps gl_ϑ and the number $\gamma(\vartheta)$ are defined for every twist; the height bound later uses them at the powers of m-fold twists.
5. *Hypotheses not used as stated.* In Lemma 3.16 (b) the formal statement asks that L be non-empty. In Lemma 3.15 (a) the divided splitting enters only through $\bar{\varpi}_{\mathcal{V}}$.
6. *Truncated subtraction.* Differences of natural numbers in the formal text, as in the depth loss and the exponents of v, are truncated at zero or are integers as the text specifies; the note writes $v^{m/2-j}$ for the integer exponent $\lfloor m/2 \rfloor - j$.
7. *The scale in Proposition 3.19.* The data contain an integer π_g of K of norm $g(D)$, and $\bar{\pi}_g$ is its complex conjugate, regarded as an integer of $E(D)$. The formal scale of the Taylor series is the jet scale times the complex conjugate of the image

of π_g in $\mathbb{C}$: it is $\bar{\pi}_g$, read in $\mathbb{C}$ through K, and not π_g. At the primes above $g(D)$, which split in K, the two have different valuations.

8. *Letters that differ from the formal text.* The exponent of the parameters is s in the formal text and μ here; the number of Proposition 3.19 is t there and t_{m} here; the degree in Proposition 3.18 is d there and d here; the level of Proposition 3.13 is K there and Γ' here; the matrices of condition (2) of a glued lattice and of Lemma 3.15 (a) are K there and J here; the matrix u of Input L (a) is written u'. The letters s, t, d, K and u have other meanings in this write-up.

How this note was written. From the dossier of the unit, whose reader calls the step claims of the long analytic proofs a first reading and notes that no second reader had looked at them. After the dossier was frozen, the second reader of the write-up compared the readings of all statements of the unit with the Lean text and sampled its structured proofs (her message to the writer of 26 November 2026); where her corrections concern this note they are followed: in Lemma 3.12 the data are the vacuum *outside* the places of the pair (the dossier had the set reversed; the Lean definition of vanishing descents confirms her reading), the growth hypothesis of Proposition 3.17 (i) is the implication stated there, and the coefficients of the good projector in Lemma 3.20 are algebraic integers after multiplication by the group order. I read the six Lean files of these declarations completely. The proof of Proposition 3.19 (the depth of an indicator jet) I read line by line in the Lean text. I could follow its architecture: the number t, the series Θ and the carrying by copies of N, the four squares, the case split on the splitting of p, the passage from the series to an indicator datum. I could not reconstruct the comparisons of exponents that the formal proof closes by simplification and linear arithmetic, nor the exact statements of the inputs from the unit *Curve* beyond their docstrings; the proof above is an outline of the argument, not a verification of its estimates. The proofs of Propositions 3.17, 3.18, 3.10 and 3.13, of Lemmas 3.3, 3.9, 3.12, 3.15, 3.16 (b) to (d), 3.20 and 3.22 (b) are accounts in the same sense. The proofs of Lemmas 3.1, 3.2 and 3.4 to 3.6, of Proposition 3.8, of Corollary 3.14, of Lemmas 3.16 (a), 3.21, 3.22 (a) and 3.23 and of Theorem A are complete relative to the inputs; I rewrote them from the Lean text, and they are shorter than the formal ones. I read the Lean statements of the three results of Input L and the definition of "tested"; the other inputs are stated from the dossier's reading of their docstrings. The program check of the unit's dossier covered 634 leaf steps of 85 proofs.

Revision. This text was revised after the report of an independent referee (26 November 2026), who compared the statements with the Lean text and re-derived the complete proofs. The changes: in Proposition 3.19 the scale is $\bar{\pi}_g$ per, not its conjugate; in Lemma 3.16 (a) the automorphism is $\mathrm{conj} \circ \sigma$; the input from the unit *SiegelWeil* states the vacuum condition of the corrected dossier; the theorem on periods and the p-adic logarithm of the theta function are cited as the catalogue and the second reader allow; the proofs of Proposition 3.8 and Lemma 3.23 are completed; Input L names its index set and its rank bounds; "tested" names its idempotent; several letters were changed; and each proof says where it stands whether it is complete, an account or an outline.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Lemma 3.16	`FitsRanks.sum_diagramWeight_eq_boxes`	MemberKernels.lean	31–35
Lemma 3.16	`Parameters.exists_algEquiv_norm_eq`	MemberKernels.lean	37–40
Proposition 3.17	`Parameters.norm_jet_le_of_isAdmissibleComb`	MemberKernels.lean	47–88
Lemma 3.16	`boxes_le_of_isHCParam`	MemberKernels.lean	92–96
Lemma 3.16	`exists_isHCParam_of_le`	MemberKernels.lean	98–107
Lemma 3.16	`exists_isHCParam_of_hasGap`	MemberKernels.lean	111–113
Lemma 3.16	`modKernel_embFin_mul`	MemberKernels.lean	118–125
Lemma 3.16	`sum_mem_integralAbove`	MemberKernels.lean	127–130
Proposition 3.18	`exists_modKernel_integral_bound`	MemberKernels.lean	143–167
Lemma 3.1	`exists_reduceAt_mul_incl_eq`	MemberKernels.lean	172–179
Definition 2.3	`pointForm`	MemberKernels.lean	183–184
Lemma 3.2	`inner_pointForm`	MemberKernels.lean	186–191
Lemma 3.3	`theta_pointForm`	MemberKernels.lean	194–203
Definition 2.3	`kernelVector`	MemberKernels.lean	207–208
Definition 2.3	`rowFixingLevel`	MemberKernels.lean	212–214
Definition 2.3	`cosetKernelSum`	MemberKernels.lean	218–221
Proposition 3.19	`Data.depth_indicator_jet`	GluedFormTheta.lean	33–143
Definition 2.4	`DividedLattice`	GluedFormTheta.lean	147–151
Definition 2.4	`DividedLattice.reducedInverse`	GluedFormTheta.lean	159–160
Definition 2.4	`DividedLattice.weilScalar`	GluedFormTheta.lean	164–165
Lemma 3.4	`DividedLattice.reduceAt_conj_eq_one`	GluedFormTheta.lean	172–191
Lemma 3.5	`DividedLattice.exists_reducedInverse_eq`	GluedFormTheta.lean	194–208
Definition 2.4	`DividedSplitting.stackOverGenScalar`	GluedFormTheta.lean	212–213
Lemma 3.6	`rowFixingLevel_le_levelOf`	GluedFormTheta.lean	219–222
Definition 2.4	`DividedSplitting.congruenceGroup`	GluedFormTheta.lean	226–227
Remark 3.7	`DividedSplitting.reduceAt_liftMod_congruenceGroup`	GluedFormTheta.lean	230–232
Definition 2.4	`DividedSplitting.IsGlued`	GluedFormTheta.lean	236–242
Definition 2.4	`DividedSplitting.productForm`	GluedFormTheta.lean	246–248
Lemma 3.6	`DividedSplitting.exists_congruenceGroup_inv_eq_one_sub`	GluedFormTheta.lean	251–256
Proposition 3.8	`DividedSplitting.exists_isGlued`	GluedFormTheta.lean	259–294
Lemma 3.9	`DividedSplitting.theta_pointForm`	GluedFormTheta.lean	308–336
Proposition 3.10	`DividedSplitting.theta_productForm`	GluedFormTheta.lean	343–366
Proposition 3.10	`DividedSplitting.theta_stdClassSum_productForm`	GluedFormTheta.lean	372–378
Lemma 3.20	`Data.Pair.norm_conj_intAlgPart_le`	MonomialCoefficientVector.lean	20–36
Lemma 3.20	`Data.iota_goodProjTranslate_apply`	MonomialCoefficientVector.lean	38–41
Lemma 3.20	`Data.goodProjTranslate_mem_integralForms`	MonomialCoefficientVector.lean	47–54
Lemma 3.20	`Data.normSqAt_goodProjTranslate_le`	MonomialCoefficientVector.lean	58–70
Lemma 3.20	`exists_int_vector_harmonicPoly_eq`	MonomialCoefficientVector.lean	80–94
Lemma 3.12	`descentsVanish_hypSum`	MemberOrthogonality.lean	35–53
Proposition 3.13	`stdClassSum_period_eq_zero`	MemberOrthogonality.lean	62–120
Corollary 3.14	`DividedSplitting.stdClassSum_inner_productForm_eq_zero`	MemberOrthogonality.lean	126–138
Definition 2.4	`DividedSplitting.loss`	CartanIntegrality.lean	19–20
Lemma 3.15	`DividedSplitting.norm_schur_inv_cols_le_weylGauge`	CartanIntegrality.lean	29–51
Lemma 3.15	`DividedSplitting.weylGauge_cartanProduct_le`	CartanIntegrality.lean	56–69
Definition 2.5	`smallHookNumber`	ComparisonForms.lean	17–17
Definition 2.5	`SmallFormValues`	ComparisonForms.lean	21–22
Definition 2.5	`Data.Tables`	ComparisonForms.lean	32–32
Definition 2.5	`splittingV`	ComparisonForms.lean	38–42

In this note	Lean declaration	File	Lines
Remark 3.7	`classAut_rowFixingLevel_eq_bot`	ComparisonForms.lean	46–49
Definition 2.5	`glueLattice`	ComparisonForms.lean	53–54
Definition 2.5	`lineScale`	ComparisonForms.lean	59–59
Definition 2.5	`comparisonForm`	ComparisonForms.lean	70–75
Lemma 3.21	`inner_adjoint_comparisonForm`	ComparisonForms.lean	81–111
Lemma 3.22	`comparisonForm_mem_integralForms`	ComparisonForms.lean	116–126
Lemma 3.22	`normSqAt_comparisonForm_le`	ComparisonForms.lean	131–163
Lemma 3.23	`tableTheta_adjoint_eq_zero`	ComparisonForms.lean	170–192
Theorem A	`exists_comparison_forms`	ComparisonForms.lean	200–227

The definitions of a glued lattice and of the comparison form as they are written in the artifact.

```
structure DividedSplitting.IsGlued where
  indicator_eq_sum (x : Matrix n r O.chosenModulusSq.ResRing) (c : ℂ) : ite (L.reducedInverse * x =
    V.genResidue • inclMatrix _ O.incl) c 0 = ∑ Y : V.congruenceGroup, ite
    ((x * Y.1.1).submatrix (V.ε ∘ inr) id = inclMatrix _ z ∧ V.stackOverGenScalar s *
    (x * Y.1.1).submatrix (V.ε ∘ inl) id = V.stackOverGenScalar s * inclMatrix _ V.fixedEmb) c 0
  exists_cols_eq_zero : ∃ K : GL N O.chosenModulusSq.ResRing, ∀ x, ∀ y ∉ univ.map V.dividedEmb,
    V.genResidue * (L.reducedInverse.submatrix id (V.ε ∘ inl) * K.1) x y = 0

@[simps -isSimp] def comparisonForm : Tables j →ₗ[ℂ] (D.auxPair θ j).FormV where
  toFun ζ := (D.gProd ^ (splittingV j θ).loss : ℂ) • ∑ b, stdClassSum (rowFixingLevel _ ((splittingV
    j θ).stackOverGenScalar b.1.1) (splittingV j θ).fixedEmb) fun h ↦ (splittingV j θ).productForm
    (glueLattice j θ b.1) (h, b.2.out) (star ((lineScale j).1 • lineMultiplier θ j b.1 h)) (ζ b)
  map_add' := by simp [↓classAut_rowFixingLevel_eq_bot]
  map_smul' a := by simp [↓classAut_rowFixingLevel_eq_bot, smul_comm a, ← smul_sum]
```

The formal text of Theorem A contains a character, the sign of the orthogonal complement, that the fonts of this note lack; it is therefore not quoted, and its row in the table above leads to it. In these texts `Tables j` is the space of tables, `D.goodForms` the good forms, `D.goodLift` the map u_ϑ, `Data.heckeValueProd` the number $\mathrm{HV}_D(j,\vartheta)$, `AuxPair.IsTested` the notion "tested", `weylGauge` the gauge, `LevelForm.integralForms` the integral forms, `normSqAt` the number $\|\cdot\|_\sigma^2$ and `weylNormSq` the squared Gram norm.

Notation

E, $\mathcal{O}_E$, $\widehat{\mathcal{O}}_E$, $\mathbb{A}$, $\mathbb{A}_f$ a CM field; its integers; integral finite adeles; adeles; finite adeles

τ, w, $\bar{w}$, k_w an infinite place; a finite place, its conjugate, its residue field

υ the character of the Weil data

$\mathsf{r}, \mathsf{n}, \mathsf{N}, \mathsf{m}, \mathsf{a}, \mathsf{l}, \mathsf{L}, \mathsf{c}, \mathsf{n}', \mathsf{n}_1, \mathsf{n}_2$ finite index sets

$\mathfrak{m}$, Σ, $\mathfrak{M}$, $\mathfrak{Q}$ a modulus; level places; products of chosen and twist primes

$\mathrm{U}(S)(\mathbb{A})$, $G_\infty(s)$, $\mathbf{K}_\mathsf{n}$, Γ, Γ', $\Gamma(\varpi_2)$, ϖ_2 unitary groups; stabiliser of the standard lattice; levels; a generator of $\mathfrak{M}_O^2$

$\sum^{\mathrm{cl}}_{h \bmod \Gamma}$ the class sum

$\theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Psi,P](g,h)$, ω, θ the theta kernel; the Weil representation; the theta functional

κ, b_i, b_i^-, G_κ, h_κ, $S_\kappa(z)$, $\lambda_\kappa(\tau,i)$ a weight; basis polynomials; Gram matrix; hook number; Schur matrix; row lengths

$\mathrm{pol}_\kappa(\xi)$, $\mathrm{pol}^\pm_{O,\kappa}(\xi)$, $\|\xi\|_\kappa$, $\langle\xi,\xi'\rangle_\kappa$ the polynomial of a coordinate vector, also in letters; Gram norm and form

O, O', O_X, W_O, s_O, e_O, $\kappa_O^\pm$, κ_O, ϵ_O, μ_O, ψ_O, $\mathbf{a}_O$, $\mathbf{b}_O$, Γ_O, χ_O an auxiliary pair and its components

$\mathcal{F}(O)$, $P^O(\xi)$, Υ_O, $\mathrm{lift}_O(\Xi)(g)$, $H_{O,w}(X)$, $z_O(w)$ forms, harmonic polynomial, test datum, descent, Hecke test

ϖ_O, ϖ, ϖ^*, $\bar{\varpi}_\mathcal{V}$, red_0, I_e, $\tilde{A}$ generator, its complex conjugate, its class, reduction, inclusion matrix, chosen lift

$\mathcal{V}$, ε, $\phi_\mathcal{V}$, ϕ, z a divided splitting, its bijection, its fixed embedding; the embedding of r

$\mathrm{Stab}(\varphi,w,e)$, $\mathrm{cd}_{\chi,\varphi,w,e}(x)$ stabiliser and column datum of the sister note

$\mathrm{gauge}^{\iota_\mathcal{V}}_\kappa(\epsilon;g,\xi)$, $\|\phi\|^2_\sigma$, $\langle\phi,\phi'\rangle$ gauge; squared norm at σ; inner product of forms

p, ℓ, ι_p, ι, φ, σ, σ', $\mathbb{C}_p$ primes; embeddings; automorphisms of $\mathbb{C}$

$\Gamma(\Sigma; w, e)$, $\mathrm{kv}_{O',b;\mathfrak{m}}[\Psi;\kappa](g,h)_i$ Definition 2.3

$\mathrm{cks}_{O',b}(w,e;y)(g)$, $\mathrm{pt}_{h,Y,e}(v)$, $k_{Y,e}$ coset kernel sum; point form; its point

L, $\mathrm{Lat}(O)$, γ_L, G_L, β_L, δ a divided lattice and its objects

$[s;\bar{\varpi}]_{\mathcal{V}}$, $\mathsf{C}(\mathcal{V})$, $\mathrm{Glued}_{\mathcal{V}}(s,L)$ Definition 2.4

$\mathrm{pr}_{\mathcal{V},L}(h;y)(\eta)$, $y\cdot\eta$, $\mathrm{loss}(\mathcal{V})$ product form; Cartan product; loss

P, D, $E(D)$, m, $k(D)$, r, $t(D)$, x, x_1, b, $g(D)$, $f(D)$ parameters and data: field, number of lifts, ranks, step, scales, second level, number of infinite places

$\bar{\pi}_g$, π_g, $\mathsf{G}(D)$, $e_0(D)$, $B_0(D)$, $w_D(\tau)$, $F_D(p)$ the integer of norm $g(D)$ and its conjugate, an integer of K; the finite group; the good idempotent; the budget for congruences; nominal weight; forced part

j, ϑ, $Q(\vartheta)$, $O_D(\vartheta,j)$, $W_j(D)$, $Y_0(D)$ lift index; twist; its primes; the auxiliary pair; spaces

$\mathcal{M}^{\circ}_D(W_j(D))$, $\mathrm{lift}^{*}_D(\vartheta,j)$, u_ϑ good forms; the adjoint lift; its restriction

$\mathsf{B}_D(j)$, S_D, C_D, $\mathrm{Cl}^{\circ}(D)$, $\mathfrak{c}$, $h_{\mathfrak{c}}$, $\mathrm{Mem}_D(j)$, i, i_1 index sets; classes and representatives; choices

$\mathbf{b}''_D(\vartheta,j)$, $\mathbf{a}$, $\mathrm{lm}_D(\vartheta,j,i,h)$, $F^{\natural}_D(j,i)$, $\mathrm{tt}_D(j,\zeta)$, $\mathrm{HV}_D(j,\vartheta)$ objects of the sister note on heights

$\mathsf{h}_D(j)$, $\mathcal{W}_D(j)$, $\mathrm{Tab}_D(j)$, ζ, ζ_a, a hook number; table values; tables; a table; an index

$\mathcal{V}_D(j,\vartheta)$, $L_D(j,\vartheta,i)$, $\beta^{\mathrm{line}}_D(j)$, $\mathrm{gl}_{D,j,\vartheta}(\zeta)$, Γ_i Definition 2.5

γ, A, gl_ϑ, $c(\vartheta)$, C, u', $R(F,\zeta)$, ζ^{Ξ} objects of Theorem A, of Input L and of the proof

K, μ, $\mu(D)$, $E(P,\mu)$, $\mathrm{Pl}(P,\mu)$, $N_1(P,\mu)$, $N_1(D)$, $\Omega(P,\mu)$, $\mathrm{per}(P,\mu)$, $\kappa_0(P,\mu)$, per, κ_0, $C(P)$, N_0, d objects of the parameters: the imaginary quadratic field, the exponent, the CM field, its infinite places, conductor, period constant, jet scale and vacuum scale; the constant

$T_c(f)$, $\mathrm{jet}_Q(F)$, $\mathrm{jet}^{\kappa}(F)$, $M(L,w)$, u, w_κ, B scaled Taylor series; jets; the bound of Proposition 3.17

$\mathbf{m}$, $t_{\mathbf{m}}$, α, x_0, $\mathbf{1}_{x_0}$, V, y_0, $\lambda(l)$, v_p objects of Proposition 3.19; v_p the p-adic valuation

Θ, ϑ_p, Λ, ϱ_i, w_i, c_i, S (period factor), $N(\alpha)$, $e(\alpha)$, Θ_α, H, j_b objects of its proof

Pl a finite set, in the applications the set of infinite places

Ξ, ϕ, ϕ', F, y, η, v, ξ, ξ', ζ', φ_i, δ_i forms, functions, coordinate vectors

g, h, h', h_1, h_2, x, k, γ', T, X, Y, Z, U, D (matrix), G, C, Δ, Δ' points and matrices

a, b, b', α, β, e, e', e_a, e_b, ψ, ψ', χ characters, weights, tame parts; e, e' also embeddings

Ψ, $\Psi_{Y,e}$, $\Psi_{L,e}$, $\mathbf{1}_I$, $\Theta^{\pm}$, w_s, $y_i(h)$ functions on matrices; kernels; the scaled multiplier

l_0, l, q, M, R, A (bound), s, z (point), c, d, ν, β (bound), f, ρ, Q, J, t numbers, parameters, matrices, auxiliary objects of the statements

$\mathbb{H}$, $\mathcal{Z}$, $d_X(\tau)$, r', R (field), $\mathbb{Q}$ the hyperbolic plane; a subspace; jet degrees; an integer; a field; the rationals

L (integer, also in "L-function") an integer parameter of Proposition 3.17

The height of a ratio of lift volumes: twisted theta forms on one line and Hecke values

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. For a fit family of twists, the volume of a single theta lift at the trivial twist divided by its volume at the twist has height at most the dimension of the rows times $k(D) \cdot (\log b \cdot B_0(D) - \mathsf{U}(D))$ (Theorem A): each lift repays the cost $\mathsf{U}(D)$ of one direction, $k(D)$ times, for every row. The proof has three parts. First, an explicit theta form $F^\natural$ on the indefinite group of the j-th lift is, for every twist, a constant multiple of a class sum of twisted theta kernels, and the constant has absolute value a product of Hecke L-values: the twisted forms lie on one line (Theorem C). Second, the products of $F^\natural$ with theta kernels of a complementary definite block detect every good form (Theorem D); with the comparison forms of the sister note this turns the volume of a lift into a determinant with integral entries times a ratio of Hecke values. Third, the inverse of each ratio of a twisted to an untwisted Hecke value has height at most $6B_0(D) - \mathsf{U}(D)$ (Theorem B): its valuations are bounded by test forms and the congruence bound of a fit family, and its size by the norm of an untwisted theta lift.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles with Lean v4.35.0 against a pinned version of Mathlib and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note is the last of four on one unit of the artifact, the folder of the rows; it treats 77 of the unit's 219 declarations, those of seven files. Every declaration is stated. Theorem A is proved in full from Theorem B with Corollary 3.14, from Lemma 3.15 and Proposition 3.17, from the comparison forms of the sister note and from the definition of a fit family; the proofs of Corollary 3.14, of Lemmas 3.12 and 3.15, of Proposition 3.6 and of Proposition 3.17 (b) are complete as well. All other proofs are accounts of the formal proofs: they say which results combine and do not verify the estimates and the normalising constants. Each of them is headed as an account where it stands, and the proof of Proposition 3.9 is an outline. Theorems B, C and D and Proposition 3.17 (a), on which Theorem A rests, are among the accounts. The inputs from other units and from the background library are named and not proved here. Every statement was compared with its Lean text, argument by argument (for the definitions of the line see the last section). The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 77 declarations.

1 The statements

Theorem A (the height of a ratio of lift volumes). *In Setting 2.1, let P be parameters, D admissible data for P, $\theta = (\theta_{\ell,\iota})$ a family of m-fold twists of D that is fit for D, and j a lift index. Then the family*

$$(\ell, \iota) \longmapsto \frac{\mathrm{vol}_D(1, j)}{\mathrm{vol}_D(\theta_{\ell,\iota}, j)}$$

has height at most $\dim \mathrm{Row}(D) \cdot \big(k(D) \cdot (\log b \cdot B_0(D) - \mathsf{U}(D))\big)$.

Theorem B (each ratio of Hecke values repays one direction). *In Setting 2.1, let θ be a family of m-fold twists of D that is fit for D, j a lift index and p a pair of D that is of rank one for j. Then the family $(\ell, \iota) \mapsto \rho_p(\theta^j_{\ell,\iota}, 1)^{-1}$ of inverse ratios for the sign 1 has height at most $6B_0(D) - \mathsf{U}(D)$.*

LEAN `Data.heightLe_liftVolume_ratio` · `Data.heightLe_inv_ratio`

Theorem C (twisted theta forms lie on one line). *In Setting 2.1, let a be a finite index set and j a lift index. Then there are a real number C and a function c from the twists of D to $\mathbb{C}$ such that for every twist ϑ: $|c(\vartheta)|^2 = C / \mathrm{HV}_D(j, \vartheta)$, and for every matrix u with columns a over $\mathcal{O}_E/\bar{\pi}_g^2$, every choice $i \in \mathrm{Mem}_D(j)$, every point $g \in \mathrm{U}(W_j(D))(\mathbb{A})$ and every matrix w with rows a and columns $\mathsf{B}_D(j)$ over $\mathcal{O}_E/\bar{\pi}_g^2$ with $uw = i_1$:*

$$F_D^\natural(j, i)(g) = c(\vartheta) \cdot \mathrm{cks}_{O_D(\vartheta,j),\mathrm{b}''_D(\vartheta,j)}(w, \phi; \mathrm{lm}_D(\vartheta, j, i, \cdot)_\cdot)(g),$$

with ϕ the inclusion of the row indices in $\mathsf{B}_D(j)$.

Theorem D (products of theta forms detect all good forms). *In Setting 2.1, let j be a lift index and $F \in \mathcal{M}^\circ_D(W_j(D))$ a good form such that*

$$\int_{[\mathrm{U}(W_j(D))]} \overline{F(g)\, F_D^\natural(j, i)(g)} \cdot \mathrm{tt}_D(j, \zeta)(g)\, dg = 0$$

for every choice $i \in \mathrm{Mem}_D(j)$ and every function ζ from $\mathrm{Cl}^\circ(D)$ to coordinate vectors of weight κ^-. Then $F = 0$.

LEAN `Data.cosetKernelSum_eq_fNatural` · `Data.eq_zero_of_integral_fNatural_tableTheta`

Here $\mathrm{vol}_D(\vartheta, j)$ is the volume of the j-th single lift at the twist ϑ^j (the note *The volume of the chain of theta lifts as a product of lift volumes*), heights are those of Chapter IV, $B_0(D) = \mathrm{budget}(D)/(m\,k(D))$ is the budget for congruences and $\mathsf{U}(D)$ the cost of one direction (Setting 2.1); $\rho_p(\vartheta, s)$ is the ratio of a twisted to an untwisted Hecke L-value of the pair p, for the tame character ϑ and the sign s (Setting 2.1); the function $F_D^\natural(j, i)(\cdot)$, the line multiplier and the product $\mathrm{HV}_D(j, \vartheta)$ of Hecke values are defined in §2. In Theorem C the left side does not depend on the twist. The lettered theorems are those of this note; "Theorem C of the handout" is the theorem of the overview. Since $h_{\mathrm{main}}(D) = m\,k(D)\,\mathsf{U}(D)$, the sum over $j = 1, \dots, m$ of the bounds of Theorem A, which is the bound for the product of the m families, is $\dim \mathrm{Row}(D) \cdot (\log b \cdot \mathrm{budget}(D) - h_{\mathrm{main}}(D))$: with the factorisation of the volume of

the chain in Theorem A of the first note of this unit, this gives the second height bound in the proof of Theorem C of the handout.

2 The setting

Conventions. Finite index sets are written in sans-serif, as in the other notes of this unit; $|\mathsf{n}|$ is the number of elements. The ambient objects of the declarations are the parameters P and the data D in the statements about data; a number field or CM field E with a CM type and Weil data, finite non-empty index sets and sign vectors in the general statements; a prime ℓ and an embedding into $\mathbb{C}_\ell$ where gauges occur; and finite-dimensional inner product spaces in §3. Objects that the formal text introduces as implicit arguments of a statement are introduced in the statement.

Setting 2.1 (the data and what other units and the sister notes attach to it)**.** The general notions are those of the background Setting of the note *Auxiliary dual pairs, determinant polynomials and the carrier*: a CM field E with CM type and Weil data, the character υ, moduli, characters with weight and tame part, tameness, the partial L-functions $L^T(z,\nu)$ (of the base change of ν, without the Euler factors at the places of T and at their complex conjugates) and the products $\Pi_T(\nu,d)$ of their values, hermitian spaces, class sums, theta kernels $\theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Psi,P](g,h)$, weights, coordinate vectors with their polynomials $\mathrm{pol}_\kappa(\xi)$ and Gram norms, auxiliary pairs O with their forms $\mathcal{F}(O)$, and the carrier $\mathfrak{C}_{E,m}$ with the class sum $\Xi^\circ_{m,c}(h)$. Gauges, integrality at an embedding, the numbers $\|\phi\|^2_\sigma$, "tested against a subspace for an idempotent" and the coset kernel sum $\mathrm{cks}_{O,b}(w,e;y)(g)$ are those of the note *Glued lattices, product forms and the integral comparison forms*.

Data. $P=(N_0,d)$ are parameters, with $K=\mathbb{Q}(\sqrt{-d})$, and D is admissible data for P: the field $E(D)$ with its infinite places τ, the scales x, x_1, b, the number m of lifts, an even number, the ranks $k(D)$ and r with $r+1=k(D)+m$, the step $t(D)=5r$, the number $f(D)$ of infinite places, the level $g(D)=g_0g_1$ and the integer $\bar\pi_g$ with $\bar\pi_g\overline{\bar\pi_g}=g(D)$, whose prime divisors are the chosen primes, the nominal weights $w_D(\tau)$, the prime sums $\mathrm{Ask}_D(z,\tau)$, the weights $w_j(\tau)$ of the lifts for $0\le j\le m$, the gap $\mathsf{g}(D)$ and the reduced half gap $\mathsf{g}'(D)$, the fixed characters χ_D, of weight 0 and tame part ω_D, and $\mu^\flat_D$, and $\mu^0_j=\mu^\flat_D(\upsilon^{-\mathsf{g}(D)}\chi_D)^j$. Further

$$B_0(D)=\frac{\mathrm{budget}(D)}{m\,k(D)},\qquad \mathrm{js}_D(n)=\log n!-n\log\frac{\Omega}{g(D)},$$
$$\mathsf{U}(D)=\sum_\tau\Big(\mathrm{js}_D(w_0(\tau))-\mathrm{Ask}_D(w_D(\tau),\tau)\Big),$$

with Ω the period constant of $E(D)$; $\mathsf{U}(D)$ is the *cost of one direction* and js_D the *jet size*. A lift index is an integer j with $1\le j\le m$. Tame characters and m-fold twists with their sets $Q(\vartheta)$, heights, the good idempotent $e_0(D)$ and the idempotents $\mathrm{idem}(\mathbf{c})$ of the tuples $\mathbf{c}$ of labels, the spaces $W_j(D)$, the good forms $\mathcal{M}^\circ_D(W_j(D))$, the rows, the auxiliary pairs $O_D(\vartheta,j)$, with index sets r of size r and n of size $r+t(D)$, the adjoint lifts $\mathrm{lift}^*_D(\vartheta,j)$ and the lift volumes $\mathrm{vol}_D(\vartheta,j)$ are as in the first note of this unit. A *twist* is a tame character whose tame part is non-trivial at every prime of $Q(\vartheta)$ (the Setting on the data of the note *Glued lattices, product forms and the integral comparison forms*); the powers ϑ^j, $1\le j\le m$, of an m-fold twist are twists.

For a tame character ϑ and an integer $c \geq 0$, $\mathfrak{m}_D(\vartheta, c)$ is the modulus $\bar{\pi}_g^c$ times the primes of $Q(\vartheta)$.

Pairs. A *pair* of D (unit *Exchange*) is a character ν_p of weight $-c_p$, with tame part ψ_p, together with an integer z_p. It is *flat* for a finite index set n if $c_{p,\tau} = z_p$ for all τ and $|\mathsf{n}| = 2z_p \geq 4$; *of rank one* for a lift index j if $m < z_p \leq r$, its tame part is ω_D^j and $w_j(\tau) + r - t(D) \leq c_{p,\tau} \leq w_j(\tau)$; *ramified* if its tame part is non-trivial exactly at the chosen primes. A pair with a suitable index set and a twist ϑ has an *L-value datum*, with an auxiliary pair of a definite group $\mathrm{U}(1)$ and a definite space, jet degrees $c_{p,\tau} - z_p$, kernel forms, and a *trivial lift* $\Xi_p(\vartheta)$, the theta lift of the trivial character of $\mathrm{U}(1)$; its *normalised value* $I_p(\vartheta)$ is the squared norm of the trivial lift cleared of its denominators. The labels of such an auxiliary pair have degree 1, and $\mathrm{idem}(X-1)$, the *label idempotent*, is the idempotent of its tuple of labels all equal to $X-1$. For a pair p, a tame character ϑ and a sign s, the *ratio* of the twisted to the untwisted Hecke L-value of the pair is

$$\rho_p(\vartheta, s) = \left|L^{T_\vartheta}(z_p, (\nu_p \vartheta^s)^{-1})\right| \Big/ \left|L^{T_1}(z_p, \nu_p^{-1})\right|,$$

with T_ϑ the set of the chosen primes and of the primes of $Q(\vartheta)$, so that T_1 is the set of the chosen primes: the sign is the exponent of the twist. The ratio is real and not zero, and $\rho_p(\vartheta, 1) = I_p(\vartheta)/I_p(1)$ for an index set that fits p (unit *Exchange*).

Denominators. Let O be an auxiliary pair whose first space has rank r_0, $\mathcal{Z}$ a linear subspace of $\mathcal{F}(O)$, e an idempotent of the group algebra of the finite group of O, (ℓ, ι) a prime with a field isomorphism from $\mathbb{C}$ to an algebraic closure of $\mathbb{Q}_\ell$, and β a real number. The subspace $\mathcal{Z}$ has *denominators at most β at ι for e* if, for all families (y_a), (y'_b) of $\dim \mathcal{Z}$ forms, the y_a integral at every embedding $\mathbb{C} \to \mathbb{C}_\ell$ and the y'_b integral at ι and tested against $\mathcal{Z}^\perp$ for e, the determinant of the numbers $\langle y_a, \mathrm{pr}_{\mathcal{Z}}\, y'_b \rangle$ has $|\iota(\,\cdot\,)|_\ell \leq \exp(\dim \mathcal{Z} \cdot r_0\, \beta)$; here $\mathrm{pr}_{\mathcal{Z}}$ is the orthogonal projection to $\mathcal{Z}$.

Fitness. A family θ of m-fold twists is *fit for D* if: (1) for every (ℓ, ι), the prime ℓ is prime to the norms of the primes in $Q(\theta_{\ell,\iota})$; (2) for every lift index j there are real numbers β_ℓ, whose sum over every finite set of primes is at most $B_0(D)$, such that for every (ℓ, ι) the image of the adjoint lift for $\theta_{\ell,\iota}^j$ on the good forms has denominators at most β_ℓ at ι for the good idempotent $e_0(D)$, and, for every pair p of rank one for j, the line $\mathbb{C}\,\Xi_p(\theta_{\ell,\iota}^j)$ has denominators at most β_ℓ at ι for the label idempotent $\mathrm{idem}(X-1)$; (3) for every lift index j, every ramified pair p and every sign s, the family $(\ell, \iota) \mapsto \rho_p(\theta_{\ell,\iota}^j, s)^{-1}$ has height at most x^{999}. In (3) the sign stands in the third slot of the ratio: the twist enters the numerator as $(\theta_{\ell,\iota}^j)^s$.

Definitions: index sets and characters

Definition 2.2 (index sets of the j-th lift)**.** In Setting 2.1, let j be a lift index.

(1) $\mathfrak{m}_D^\circ$ is the function $c \mapsto \mathfrak{m}_D(1, c)$, the modulus $\bar{\pi}_g^c$; for a tame character ϑ and $x \in \mathcal{O}_E/\mathfrak{m}_D(\vartheta, 2)$, $\mathrm{fg}_\vartheta(x)$ is its reduction in $\mathcal{O}_E/\bar{\pi}_g^2$; $O_D^\circ(j) = O_D(1, j)$ is the auxiliary pair at the trivial twist, with weights κ^+, κ^-.

(2) C_D is the disjoint union of $\{1, \dots, 2m\}$ and $\{1, \dots, m\}$; F_D is $\{1, \dots, 6m+1\}$; S_D is the disjoint union of the row index set r and a set of $t(D) - 9m - 1 - r$ elements; $\mathrm{Cl}^\circ(D)$ is the set of classes of $\mathrm{U}(I_{\mathsf{S}_D})$ at the principal level modulo $\bar{\pi}_g^2$.

(3) $\mathsf{P}_D(j)$ is the disjoint union of $\{1,\dots,j\}$ and a set of $3m-j$ elements; $\mathsf{B}_D(j)$ is the disjoint union of r, $\mathsf{P}_D(j)$ and F_D; $\mathsf{L}^+_D(j)$ is the disjoint union of a set of $3m-j$ elements, F_D and C_D. The sign vector $\mathrm{ps}_D(j)$ on $\mathsf{r}\sqcup\mathsf{P}_D(j)$ has the signs of $W_j(D)$ on r, -1 on the first j padding indices and $+1$ on the others.

(4) $\mathrm{bv}_D(i)$, for an integer $i\ge 0$, is the row index with number $\min(i,r-1)$ in the numbering of r. For $0\le i<m$, the index $\mathrm{neg}_{D,j}(i)$ of $\mathsf{r}\sqcup\mathsf{P}_D(j)$ is the i-th of the first padding indices if $i<j$, and $\mathrm{bv}_D(k(D)+i-1)$ otherwise. For integers $h\ge 0$ and $0\le i<2m$, $\mathrm{pos}_{D,j}(h,i)$ is $\mathrm{bv}_D(h+i)$ if $h+i<k(D)+j-1$, and the i-th of the other padding indices if not.

(5) $\mathrm{rg}_D(j)$ is the bijection that regroups the summands, and $\mathrm{pe}_D(j)$ is a chosen bijection:

$$\mathrm{rg}_D(j)\colon\ \mathsf{r}\sqcup\{1,\dots,j\}\sqcup\mathsf{L}^+_D(j)\ \longrightarrow\ \mathsf{C}_D\sqcup\mathsf{B}_D(j),$$
$$\mathrm{pe}_D(j)\colon\ \{1,\dots,2m\}\sqcup\mathsf{B}_D(j)\ \longrightarrow\ \{1,\dots,k(D)+j-1\}\sqcup\mathsf{L}^+_D(j).$$

LEAN Data.untwistedModulus · Data.forgetTwist · Data.untwistedAuxPair · Data.CarrierIndex · Data.FlatIndex · Data.SmallIndex · Data.SmallClasses · Data.PadIndex · Data.BigIndex · Data.LPlusIndex · Data.padSigns · Data.basisVec · Data.negIndex · Data.posRowIndex · Data.regroupEquiv · Data.posEquiv · the fact that $m\neq 0$, an unnamed instance of the same file, is used throughout

The space of the j-th auxiliary pair on the definite side has the index set $\mathsf{n}=\mathsf{B}_D(j)\sqcup\mathsf{S}_D$: a big block V'', which contains copies of the indices of $W_j(D)$, of a padding and of a flat block $I^\flat$ of rank $6m+1$, and a small block V°. The carrier lives on a space of signature $(2m,m)$ with index set C_D.

Definition 2.3 (characters)**.** In Setting 2.1, let j be a lift index and ϑ a tame character of D.

(i) $\chi_D^{(j)}[\vartheta]=\chi_D^j\,\vartheta$.
(ii) $\mathsf{x}_D(j)=j+\lfloor(t(D)+k(D)-1)/2\rfloor$, and $\mathbf{b}''_D(\vartheta,j)=\mu_j^0\,\vartheta\,\upsilon^{\mathsf{x}_D(j)}$.
(iii) $\mathbf{c}_D(j)=\mu_D^\flat\,\upsilon^{m+\mathsf{x}_D(j)-j\,\mathsf{g}(D)}$.

LEAN Data.twistedChi · Data.bigCharExp · Data.bigChar · Data.carrierChar · $\chi_D^{(j)}[\vartheta]$ has weight 0 and tame part ω_D^j times that of ϑ, Data.twistedChi_hasWeight · $\mathbf{c}_D(j)$ has weight $\tau\mapsto m+\mathsf{x}_D(j)-w_j(\tau)$ and trivial tame part, Data.carrierChar_hasWeight · the ideal of $\mathfrak{m}^\circ_D(c)$ is generated by $\bar\pi_g^c$, Data.ideal_untwistedModulus_eq_span

Definitions: the line

Definition 2.4 (choices, determinants, the line multiplier, the function F-natural)**.** In Setting 2.1, let j be a lift index.

(1) A *line choice*, an element of $\mathrm{LC}(D)$, is a pair $i=(i_c,i_l)$ of a class i_c of $\mathrm{U}(I_{\mathsf{F}_D})$ at the principal level modulo $\bar\pi_g^2$ and, for each infinite place τ, a family i_l of $\mathsf{g}'(D)$ lists of $2m$ indices in F_D. For a line choice i, $\mathrm{sr}_D(i)$ is the transpose of a chosen invertible matrix over $E(D)$ whose columns lie in the lattice of the class i_c and whose entries are small at every complex embedding (Input 3).

(2) A *choice*, an element of $\mathrm{Mem}_D(j)$, is a pair $i=(i_1,i_2)$ of a $\mathsf{C}_D\times\mathsf{B}_D(j)$ matrix i_1 over $\mathcal{O}_E/\bar\pi_g^2$ and a line choice i_2.

(3) For an infinite place τ, $\mathrm{Det}_D(j,\tau)$ is the index set with one element (h,c) for every $h < k(D)+j-1$ and every $c < \kappa^+_\tau(h)$, and $\mathsf{g}'(D)$ further elements.

(4) The *line polynomial* is

$$\Lambda_D(j) = \prod_\tau \det(y^-_\tau)^{|\mathrm{Det}_D(j,\tau)|+|\mathsf{B}_D(j)|+m} \cdot \prod_\tau \prod_{h<k(D)+j-1} \det(y_{\tau,h})^{\kappa^+_\tau(h)},$$

where y^-_τ is the minor of the matrix of variables at τ in the rows $\mathrm{neg}_{D,j}(\cdot)$ and the negative columns, and $y_{\tau,h}$ the minor in the rows $\mathrm{pos}_{D,j}(h,\cdot)$ and the positive columns.

(5) For a line choice i, a matrix Y over $E(D)$, by default $Y = \mathrm{sr}_D(i)$, an infinite place τ and $l \in \mathrm{Det}_D(j,\tau)$, $\eta_D(j,i,Y,\tau)_l$ is the complex matrix with rows $\{1,\dots,2m\}$ and columns $\{1,\dots,k(D)+j-1\} \sqcup \mathsf{L}^+_D(j)$ that is, for $l=(h,c)$, the transposed inclusion matrix whose row a is the basis vector $h+a$ of the first block if $h+a < k(D)+j-1$ and the vector a of the block of $3m-j$ indices otherwise, and, for the other l, the rows $i_l(\tau,l)$ of $\tau(Y)$, placed in the block F_D.

(6) The *line coordinates* $\mathrm{lc}_D(j,i,Y)$ are the coordinates in the first columns, in the weight κ^+, of the product of stacked determinants of the matrices $\eta_D(j,i,Y,\tau)_l$ along $\mathrm{pe}_D(j)$ (the sister note on auxiliary pairs, the definition of stacked determinants): for each standard filling l of weight κ^+ with rows $\mathsf{B}_D(j)$, a polynomial in variables indexed by $\mathsf{L}^+_D(j)$.

(7) For a tame character ϑ, a matrix u over $\mathcal{O}_E/\bar\pi_g^2$ and a $\mathsf{B}_D(j)\times\mathsf{L}^+_D(j)$ matrix z over $\mathcal{O}_E/\mathfrak{m}_D(\vartheta,2)$, $\mathrm{ld}_D(\vartheta,j;u)(z)$ is the sum, over the invertible $3m\times 3m$ matrices g over $\mathcal{O}_E/\bar\pi_g^2$, of $\psi'(\mathrm{tr}(z'gu))$ if the columns of z outside the last $3m$ are the standard inclusion, and 0 otherwise; here z' consists of the last $3m$ columns of z, reduced by $\mathrm{fg}_\vartheta(\cdot)$, and ψ' is the additive character of the Weil data on $\mathcal{O}_E/\bar\pi_g^2$.

(8) For a tame character ϑ, a choice $i \in \mathrm{Mem}_D(j)$, $h \in \mathrm{U}(I_{\mathsf{B}_D(j)})(\mathbb{A}_f)$ and a standard filling l, the *line multiplier* is

$$\mathrm{lm}_D(\vartheta,j,i,h)_l = \theta^{I_j^-,I_{\mathsf{B}_D(j)}}_{\upsilon^j,\beta;\mathfrak{m}}[\mathbf{1}_I,1](1,h)\cdot \sum_{c \bmod \mathbf{K}_{3m}}{}^{\mathrm{cl}}\ \Xi^\circ_{m,\chi_D^{(j)}[\vartheta]}(c)\cdot\Theta^+_l(c,h),$$

$$\Theta^+_l(c,h) = \theta^{L^+,I_{\mathsf{B}_D(j)}}_{\beta,\beta;\mathfrak{m}}[\mathrm{ld}_D(\vartheta,j;i_1)(\cdot),\mathrm{lc}_D(j,i_2,Y)_l]((h_i,c),h),$$

where $\beta = \mathbf{b}''_D(\vartheta,j)$, $\mathfrak{m}=\mathfrak{m}_D(\vartheta,2)$, I_j^- is the negative definite space of rank j, $\mathbf{1}_I$ the indicator of the inclusion matrix, L^+ the definite space with index set $\mathsf{L}^+_D(j)$, $Y=\mathrm{sr}_D(i_2)$, and (h_i,c) the point of L^+ with component 1 on the first block, a representative h_i of the class of i_2 on I_{F_D} and c on the carrier block.

(9) For a choice $i\in\mathrm{Mem}_D(j)$ and $g\in\mathrm{U}(W_j(D))(\mathbb{A})$,

$$F^\natural_D(j,i)(g) = \int_{[\mathrm{U}(S)]} \mathfrak{C}_{E(D),m}(x)\cdot\theta^{S,W'}_{\upsilon^m,\mathbf{c}_D(j);\bar\pi_g^2}[\psi'_{i_1},\Lambda_D(j)\cdot\Delta_{i_2}](x,(g,1,h_i))\,dx,$$

where S is the space of signature $(2m,m)$ with index set C_D, W' is the orthogonal sum of $W_j(D)$, of the padding $I_j^-\oplus I_{3m-j}$ and of I_{F_D}, ψ'_{i_1} is the function $y\mapsto\psi'(\mathrm{tr}(y\,i_1))$, the line polynomial is read in the variables of the first two summands, and Δ_{i_2} is the product of the determinants of the rows $i_l(\tau,l)$ of $\tau(\mathrm{sr}_D(i_2))$ times the block of variables of the third summand.

LEAN Data.LineChoice · Data.shortRows · Data.MemberIndex · Data.DetIndex · Data.linePoly · Data.detRows · Data.linePolyCoords · Data.lPlusDatum · Data.lineMultiplier · Data.fNatural

Definition 2.5 (Hecke values and the theta function of a table)**.** In Setting 2.1:

(i) For a sign s, $\mathsf{e}_D(s)$ is m if $s = +1$ and $m/2$ if $s = -1$; $\mathsf{e}^*(D) = 7m/2 + \lfloor k(D)/2 \rfloor$.

(ii) For a lift index j, the three *line pairs* $\mathrm{lp}_D(j, o)$, for o either absent or a sign, are: for o absent, the character $\mu_j^0\, \upsilon^{n_j}$, $n_j = j + \lfloor (t(D) - 11m - 1)/2 \rfloor$, of weight $\tau \mapsto n_j - w_j(\tau)$, with the integer $\mathsf{e}^*(D)$; for a sign s, the character $\upsilon^{-\mathsf{e}_D(s)} \chi_D^{sj}$, of weight $-\mathsf{e}_D(s)$, with the integer $\mathsf{e}_D(s)$.

(iii) For a lift index j and a twist ϑ, with T_ϑ the set of the chosen primes and of the primes in $Q(\vartheta)$,

$$\mathrm{HV}_D(j, \vartheta) = \prod_o \Pi_{T_\vartheta}((\nu_o\, \vartheta^{\,s_o})^{-1}, z_o),$$

the product over the three line pairs $(\nu_o, z_o) = \mathrm{lp}_D(j, o)$, with $s_o = -1$ for the sign -1 and $s_o = +1$ otherwise.

(iv) For a lift index j, a function ζ from $\mathrm{Cl}^\circ(D)$ to coordinate vectors of weight κ^- with rows S_D and $g \in \mathrm{U}(W_j(D))(\mathbb{A})$,

$$\mathrm{tt}_D(j, \zeta)(g) = \sum_{\mathfrak{c} \in \mathrm{Cl}^\circ(D)} \overline{\theta^{W_j(D), I_{\mathsf{S}_D}}_{\mathbf{a}, \upsilon^{m/2-j}; \bar{\pi}_g^2}[\mathbf{1}_I, \mathrm{pol}^-(\overline{\zeta(\mathfrak{c})})](g, h_{\mathfrak{c}})},$$

with $\mathbf{a}$ the first character of $O_D^\circ(j)$, $h_{\mathfrak{c}}$ a representative of the class and the polynomial of the conjugate vector written in the negative letters.

LEAN Data.eSign · Data.heckeCount · Data.linePair · Data.heckeValueProd · Data.tableTheta

Definition 2.6 (two auxiliary constructions)**.** (i) In Setting 2.1, let q be a pair that is flat for $\{1, \dots, m'\}$, X an L-value datum with auxiliary pair O_X, ϑ a twist and F a function from $\mathrm{U}(I_{m'})(\mathbb{A}_f)$ to $\mathcal{F}(O_X)$. Then

$$\mathrm{xs}_{q,\vartheta}(F) = M_D \cdot \sum_{h \bmod \Gamma}^{\mathrm{cl}} \frac{\delta_q}{N(Q(\vartheta))^{m'-1}} \langle 1, \delta_q\, \Xi_q(\vartheta)(h) \rangle \cdot F(h),$$

where M_D is the chosen group order of D, δ_q the denominator bound of q, $N(Q(\vartheta))$ the product of the norms of the primes in $Q(\vartheta)$, Γ the level of the auxiliary pair of the L-value datum of q at ϑ, and the bracket the Gram form with the vector 1.

(ii) For finite-dimensional inner product spaces T and V, $\varkappa \in \mathbb{C}$, a subspace L of V and a linear map $\mathrm{gl}\colon T \to V$, the *compressed Gram operator* $\mathrm{cg}_\varkappa(L, \mathrm{gl})$ is the endomorphism $z \mapsto \varkappa \cdot \mathrm{pr}_L(\mathrm{gl}(\mathrm{gl}^*\, z))$ of L, with pr_L the orthogonal projection.

LEAN Data.Pair.xiSum · Data.compressedGramOp

Inputs from the other notes of this unit.

Input 1 (note *Glued lattices, product forms and the integral comparison forms*; comparison forms). In Setting 2.1, for a lift index j there are a function γ from the twists of D to $\mathbb{C}$, a finite-dimensional inner product space $\mathrm{Tab}_D(j)$ of tables $\zeta = (\zeta_a)$, the index a a pair of a choice and a class $\mathfrak{c} \in \mathrm{Cl}^\circ(D)$, a linear map $A\colon \mathrm{Tab}_D(j) \to \mathcal{M}_D^\circ(W_j(D))$ and linear maps $\mathrm{gl}_\vartheta\colon \mathrm{Tab}_D(j) \to \mathcal{F}(O_D(\vartheta, j))$ such that, with u_ϑ the restriction of $\mathrm{lift}_D^*(\vartheta, j)$ to $\mathcal{M}_D^\circ(W_j(D))$: (i) $\gamma(\vartheta) \neq 0$ and $|\gamma(\vartheta)/\gamma(1)|^2 = \mathrm{HV}_D(j,\vartheta)/\mathrm{HV}_D(j,1)$; (ii) $u_\vartheta^* \circ \mathrm{gl}_\vartheta = \gamma(\vartheta) A$; (iii) A is surjective; (iv) $\mathrm{gl}_\vartheta(A^* y)$ is tested against the orthogonal complement of the image of u_ϑ for the good idempotent $e_0(D)$, for every $y \in \mathcal{M}_D^\circ(W_j(D))$; (v) if ℓ is prime to the norms of the primes in $Q(\vartheta)$, $\varphi\colon \mathbb{C} \to \mathbb{C}_\ell$ is an embedding and every $\overline{\zeta_a}$ has gauge at most 1 at the representative of its class for $\varphi \circ \mathrm{conj}$, then $\mathrm{gl}_\vartheta(\zeta)$ is integral at φ; (vi) $\mathsf{h}_D(j) \cdot \| \mathrm{gl}_1(\zeta) \|_\sigma^2 \leq \exp(2 r B_0(D)) \sum_a \|\sigma(\zeta_a)\|^2$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, with $\mathsf{h}_D(j)$ the hook number of the weight κ^-. Its proof uses Theorems C and D and Proposition 3.3 of this note; there is no circle, since these do not use it.

Input 2 (the same note). (a) Crude bounds: for finite index sets L, N and an integer $\mathsf{d} \geq 0$ there is $c \neq 0$ such that c times every kernel of the definite pair $(I_\mathsf{L}, I_\mathsf{N})$ over $E(D)$, with a homogeneous polynomial of degree d, a modulus $\mathfrak{m}$ containing $q\, g(D)^2$ and a datum integral away from q, is integral at the primes prime to q at points integral above them, and has conjugates of size at most e^{x_1} times bounds for the datum and the polynomial, when $q = 1$, $\mathsf{d} \leq g(D)$ and the ranks are at most $40k(D)$. (b) The depth of an indicator jet: for orders $\mathbf{m} = (\mathbf{m}_\tau)$ with $w_D(\tau) \leq \mathbf{m}_\tau \leq w_D(\tau) + x^{42}$ there is $t_\mathbf{m} \in \mathbb{C}$, $t_\mathbf{m} \neq 0$, with conjugates at most $\exp(f(D) m b / k(D))$, such that for every prime ℓ and every embedding $\varphi\colon \mathbb{C} \to \mathbb{C}_\ell$ some Taylor coefficient of total orders $\mathbf{m}_\tau$, of an indicator datum of (I_1, I_{12}) modulo $\bar{\pi}_g$, in the Taylor series scaled by $\bar{\pi}_g$ times the jet scale, times $\kappa_0^{12} \prod_\tau \mathbf{m}_\tau! / t_\mathbf{m}$, has $|\varphi(\,\cdot\,)|_\ell \geq \ell^{-F_D(\ell)}$, with $F_D(\ell)$ the forced part and κ_0 the vacuum scale. (c) The conjugates of the normalised value of a flat pair at the trivial twist are at most $\exp(x^{34})$. (d) For an L-value datum and a monomial of the jet degrees there is an integral coordinate vector whose harmonic polynomial is the monomial times a ratio of factorials, fixed by the matrices that fix the columns of the monomial. (e) Point forms, their pairing and descent; the vanishing of periods for forms that are not lifts; the coset kernel sum.

Input 3 (notes *Auxiliary dual pairs, determinant polynomials and the carrier* and *The volume of the chain*). The explicit matching (its Theorem A), the matching summed against a form on a square block, the passage from the stabiliser to the full level, the carrier identity (its Theorem C), the equivariance of the product of stacked determinants, the lemma on a determinant with fixed columns, the product rule for weights and norms, the coordinates in the first columns; the image of the adjoint lift on the good forms is generated tuple by tuple.

Inputs from other units and from the background library. Each is used where it is named.

Unit *Heights*. Heights add in products and powers; a bound may be increased; if each $y'_{\ell,\iota}$ is $y_{\ell,\iota}$ times an integer power of the norm of the modulus of $\theta_{\ell,\iota}$, for a fit family, a height bound for y holds for y' (unit *Exchange*). **Unit *Exchange*.** The pairs $O_D(\vartheta, j)$ fit their parameters; adjoint lifts are injective; $\rho_p(\vartheta, s)$ is real and not zero, and $\rho_p(\vartheta, 1) = I_p(\vartheta)/I_p(1)$; the logarithm of the conjugates of the squared norm of a trivial lift; the pairing of a trivial lift with kernel forms; rank bounds; bounds for the weights of the kernels. **Unit *Selmer*.** The definition of a fit family; for a tested form, the orthogonal projection to a subspace killed by a Hecke test does not increase the norms $\|\cdot\|_\sigma^2$; the lifts of eigenforms are killed by the Hecke test; an auxiliary pair has infinitely many split places off its level; the descent of a form of an auxiliary pair $O_D(\vartheta, j)$ is a form of principal level on $W_j(D)$. The *rank-one bound*: if the line $\mathbb{C}\,\Xi_p(\vartheta)$ of a trivial lift has denominators at most β at ι for $\operatorname{idem}(X-1)$, the class stabilisers of $\mathrm{U}(1)$ at the chosen level are trivial, and Ξ, Ξ' are test forms at ℓ (Lemma 3.13 (c)), then $|\iota(h_{\mathrm{U}}\langle \Xi, \operatorname{pr}\Xi'\rangle)|_\ell \le e^\beta$, with pr the orthogonal projection to the line and h_{U} the number of classes of $\mathrm{U}(1)$ at that level. **Units *Data*, *Columns*, *Trace*, *Filling*.** Comparison of monomials in the scales; free levels and class numbers; $\mathrm{Ask}_D(y, \tau)$ as a sum over primes of local terms; equal dimensions of good forms; the line coordinates as coefficients of the product of determinants in the positive letters.

Inputs from the background library. Each is used where it is named. The references are those of the handout's catalogue, with the relation that its second reading allows.

A cusp form is large only away from the cusps: a non-zero rapidly decreasing holomorphic automorphic form has a point g with $|F| \le 2|F(g)|$ everywhere and first minimum bounded below wherever $|F| \ge |F(g)|$. (The authors call this Siegel's argument. The catalogue gives C. Poor and D. Yuen, *Linear dependence among Siegel modular forms*, Math. Ann. 318 (2000), §1, Theorems 1.2–1.3, as an analogue for Siegel modular forms; the unitary estimate and its constant have not been located in the literature.) The reproducing formula for theta kernels in the stable range is the artifact's own: the catalogue locates the classical basis theorem (S. Böcherer, *Über die Fourier–Jacobi-Entwicklung Siegelscher Eisensteinreihen*, §1, Theorem III, p. 22) and a unitary pairing in the stable range (J.-S. Li, *Non-vanishing theorems for the cohomology of certain arithmetic quotients*, §2, Theorem 2.1, p. 184), and the finite unitary formula used here is distinct from both. Auerbach's lemma, in the form used here (a field with an embedding into a normed field, a compact separating family of linear forms, the coordinates chosen in the family), is a determinant-maximisation extension of the classical lemma (Schlumprecht, course notes for Functional Analysis I, Theorem 1.5.1); this form has not been located in a published source. A short basis of a lattice (W. Banaszczyk, Math. Ann. 296 (1993), Theorems (2.1)–(2.2), pp. 631–632; J. C. Lagarias, H. W. Lenstra and C. P. Schnorr, *Korkin–Zolotarev bases and successive minima of a lattice and its reciprocal lattice*, Theorems 2.4–2.5, pp. 336–337; the formal radius is weaker than the one these give). The Cauchy–Schwarz inequality for conjugated pairings of forms, $|\sigma\langle y, x\rangle| \le \|x\|_\sigma \, \|y\|_{\tilde\sigma}$ with $\tilde\sigma = \mathrm{conj} \circ \sigma \circ \mathrm{conj}$ and $\|x\|_\sigma$ the square root of $\|x\|_\sigma^2$. Holomorphic minors as coefficients of discrete series, their circle mean and their reproducing identity; Poisson summation for theta kernels; the dual isometry of Weyl modules (Fulton and Harris, *Representation theory*, §15.5, as cited in the artifact); the Fock model. References to be confirmed where the catalogue has none.

Status of the proofs. A proof headed "Proof" is complete relative to the inputs it names. A proof headed "an account of the formal proof" says which results combine, in the order of the formal proof, and does not verify the estimates, the exponents and the normalising constants; an "outline" does less and says what it leaves out.

3 Proofs

The line: counts, coordinates, arithmetic

Lemma 3.1 (counts and index maps)**.** *In Setting 2.1, let j be a lift index and τ an infinite place.*

(a) $|\operatorname{Det}_D(j,\tau)| + \mathsf{x}_D(j) = w_j(\tau)$.

(b) $|\operatorname{Det}_D(j,\tau)| = \sum_{h<|\mathsf{B}_D(j)|} \kappa_\tau^+(h)$, *and* $|\operatorname{Det}_D(j,\tau)| \le b_1 + b_2 + \mathsf{g}(D) + m\,\mathsf{g}(D)$, *with b_1, b_2 the auxiliary bounds of the data.*

(c) *The maps* $i \mapsto \mathrm{neg}_{D,j}(i)$ *and, for every* h, $i \mapsto \mathrm{pos}_{D,j}(h,i)$ *are injective, the sign* $\mathrm{ps}_D(j)$ *is* -1 *on the values of the first and* $+1$ *on the values of the second.*

(d) *The integers* $\lfloor t(D)/2 \rfloor - \mathrm{sgn}(\sigma)(\mathrm{rp}_D(\tau_\sigma, q) + w_j(\tau_\sigma))$, $0 \le q < r$, *at the embeddings* σ *of* $E(D)$, *with* τ_σ *the infinite place of* σ, $\mathrm{rp}_D(\tau_\sigma, q)$ *the row parameters of the data and* $\mathrm{sgn}(\sigma) = \pm 1$ *according as* σ *belongs to the CM type or not, satisfy the jump conditions of a row type for* $(r, t(D))$ *of the unit* Filling.

Proof (*an account of the formal proof*). (a) and (b): the number of columns of height below $k(D) + j - 1$ is a telescoping sum of the row parameters of the weight of $W_j(D)$; adding the $\mathsf{g}'(D)$ further elements and the exponent gives the weight, and the bound follows from (a). (c) and (d) are case distinctions on the definitions, with the inequalities between the scales for (d). □

LEAN `Data.card_detIndex_add_bigCharExp` · `Data.card_detIndex` · `Data.negIndex_posRowIndex_spec` · `isRowType_auxPair` · the sign of the conjugate embedding is the opposite one, `cmSign_star`

Lemma 3.2 (the line coordinates)**.** *In Setting 2.1, let* j *be a lift index,* i *a line choice and* Y *a matrix over* $E(D)$ *of the size of* $\mathrm{sr}_D(i)$. *Write* $\Delta(j,i,Y)$ *for the product of the stacked determinants of the matrices* $\eta_D(j,i,Y,\tau)_l$ *along* $\mathrm{pe}_D(j)$, *read as a polynomial in the first columns.*

(a) $\mathrm{pol}_{\kappa^+}(\mathrm{lc}_D(j,i,Y)) = \Delta(j,i,Y)$, *as polynomials in the first columns with polynomial coefficients.*

(b) *Let* $\varphi\colon \mathbb{C} \to \mathbb{C}$ *be a ring homomorphism and* z *a point of the space of the variables indexed by* $\mathsf{L}^+_D(j)$, *and let* y *be the vector of the numbers* $\varphi(\mathrm{lc}_D(j,i,Y)_l(z))$. *Then* $\mathrm{pol}_{\kappa^+}(y)$ *is the image of* $\Delta(j,i,Y)$ *under evaluation at* z *followed by* φ; *and if* $|\varphi(\tau(Y_{ab}))| \le g(D)^4$ *for all* τ, a, b *and* $|\varphi(z_x)| \le g(D)^4$ *for all* x, *then* $\|y\|^2_{\kappa^+} \le (g(D)^9)^{31m\sum_\tau |\mathrm{Det}_D(j,\tau)|}$.

(c) *There are an integer* $\mathsf{d} \ge 0$ *with* $\mathsf{d} \le 31m\sum_\tau |\mathrm{Det}_D(j,\tau)|$ *and an integer* H *with* $0 < H \le (\mathsf{d}^{\mathsf{d}})^3$ *such that for every line choice* i, *every* Y *and every standard filling* l *the polynomial* $Q = H \cdot \mathrm{lc}_D(j,i,Y)_l$ *is homogeneous of degree* d *and has its coefficients in every subring of* $\mathbb{C}$ *that contains all the numbers* $\tau(Y_{ab})$.

Proof (*an account of the formal proof*). (a) Compare coefficients after evaluating the coefficient polynomials at every point: the coordinates in the first columns are defined by Fock duality so that the polynomial of the coordinate vector is the given polynomial, once it is semi-invariant of weight κ^+, which the product of determinants is (unit *Filling*). (b) The first assertion is (a) after evaluation. For the bound, each stacked determinant is, by the lemma on a determinant with fixed columns (Input 3), a number γ times a single minor, whose coordinate vector has norm 1, and $|\gamma|^2$ is at most a Gram determinant of a matrix with entries bounded by $g(D)^4$. By the product rule for weights and norms (Input 3) the norms multiply over the $\sum_\tau |\mathrm{Det}_D(j,\tau)|$ factors. (c) H is the hook number of κ^+. By the dual isometry of Weyl modules (library), H times a coordinate is an integral combination of coefficients of the product of determinants, which are polynomials with integer coefficients in the entries of the $\tau(Y)$; the degree and its bound come from the substitution law for the coordinates (Input 3). □

LEAN `Data.toPoly_linePolyCoords` · `Data.weylNormSq_linePolyCoords_le` · `Data.exists_denom_linePolyCoords`

Proposition 3.3 (the line multipliers are integral and small)**.** *In Setting 2.1, let* j *be a lift index. Then there is* $c \in \mathbb{C}$, $c \neq 0$, *such that for every choice* $i \in \mathrm{Mem}_D(j)$, *every*

tame character ϑ of D and every $h \in \mathrm{U}(I_{\mathsf{B}_D(j)})(\mathbb{A}_f)$, the vector $y = \overline{c \cdot \mathrm{lm}_D(\vartheta, j, i, h)}$. satisfies:

(i) *for every prime p prime to the norms of the primes in $Q(\vartheta)$ and every embedding $\iota_p \colon \mathbb{C} \to \mathbb{C}_p$: $\mathrm{gauge}^{\iota_p}_{\kappa^+}(0; h, y) \le 1$;*
(ii) *if $Q(\vartheta) = \emptyset$, then $\|\varphi(y)\|^2_{\kappa^+} \le \exp(r\, B_0(D))$ for every $\varphi \in \mathrm{Aut}(\mathbb{C})$.*

Moreover, for a tame character ϑ and an integer $c' \ge 0$, the integer $N(Q(\vartheta))\, g(D)^{c'}$ lies in the modulus $\mathfrak{m}_D(\vartheta, c')$.

Proof (an account of the formal proof). The last assertion: $\bar{\pi}_g^{c'}$ divides $g(D)^{c'}$, and the product of the norms of the primes of $Q(\vartheta)$ lies in their product. For the rest, the line multiplier is a class sum of products of three theta kernels of definite pairs: the kernel inside $\Xi^{\circ}_{m, \chi_D^{(j)}[\vartheta]}(c)$, the constant of the negative definite space I_j^-, whose kernel is the conjugate of a kernel of I_j up to a number of absolute value 1, and the kernel of L^+ with the line coordinates. Take d and H from Lemma 3.2 (c) and the three constants of Input 2 (a) for these pairs, with degrees 0, 0 and d; let c be their product with H, with the square of the chosen group order and with the number of absolute value 1. (i) The moduli contain $N(Q(\vartheta))g(D)^2$, the data are integral away from $N(Q(\vartheta))$, and class representatives can be moved by rational points to points integral above p, the kernels changing by the rule for rational points; the coefficients of $H \cdot \mathrm{lc}_D(j, i_2, Y)_l$ are ι_p-integral for a suitable choice of the short basis at p. So each factor is ι_p-integral, and the orders of class stabilisers divide the chosen group order. (ii) If $Q(\vartheta) = \emptyset$, then $q = 1$; the conjugates of the three kernels are bounded by Input 2 (a), with the bound of Lemma 3.2 (b) for the polynomial, the entries of the short basis being bounded at every complex embedding. The numbers of classes and all other factors are at most $\exp(rB_0(D))$ by the inequalities between the scales. □

LEAN `Data.lineMultiplier_integral_and_small` · `Data.norm_mul_gProd_pow_mem_modulus`

Sparse theta sums and isolation

The next results are archimedean. For a number field E, a finite non-empty index set n and a sign vector s, $\|h\|$ is the maximum over the infinite places of the operator norms of the components of $h \in G_\infty(s)$; it is at least 1. For a weight k, a family of integer vectors k_τ on n, mh_k is the product over τ of the holomorphic minors of weight k_τ, a matrix coefficient of a holomorphic discrete series when k is dominant with a gap. A function on the adelic group is *holomorphic of order o and weight k* if it transforms under the maximal compact subgroup and is annihilated by the antiholomorphic derivatives accordingly (library); an order o on n is *positive first* for s if the positive indices come first. The *first minimum* λ_1 of a hermitian space moved by a point is the infimum of 1 and of the lengths of the non-zero vectors of its lattice: the length of a shortest vector, cut off at 1.

Setting 3.4 (weights for isolation)**.** E is a number field with f_E infinite places, n a finite non-empty index set, s a sign vector on n, $\nu \ge 0$ an integer and B a real number with $f_E \cdot 2|\mathsf{n}|^3 \le \nu$, $\nu \le B$ and $10^{13} \le B$; o is an order that is positive first for s; w and k are weights that are dominant for o and have holomorphic gap 300ν, and $|k_\tau(i)| \le B$ for all τ, i.

Lemma 3.5 (integrals against holomorphic minors)**.** *In Setting 3.4, the integral* $\int |\,\mathrm{mh}_k(h)|^2 dh$ *over* $G_\infty(s)$ *is positive; and for every continuous function* A *on* $G_\infty(s)$*, every real number* C *and every integer* $N \le 6\nu$ *such that* $|A(h)| \le C\|h\|^N$ *for all* h*, the product* $A \cdot \mathrm{mh}_w$ *is integrable and*

$$\Big|\int A(h)\,\mathrm{mh}_w(h)\,dh\Big| \le C\,B^{7\nu}\int |\,\mathrm{mh}_k(h)|^2 dh.$$

Proof (*an account of the formal proof*)*.* The holomorphic minors are coefficients of discrete series; their L^2 norms are inverses of formal degrees, hence positive and comparable for w and k up to powers of B, and their decay integrates the polynomial growth of A (library). □

LEAN `integral_mul_holMinorsPi_le` · the bound $1 \le \|h\|$ is `one_le_maxOpNorm`

Proposition 3.6 (isolation)**.** *In Setting 3.4, let* E *be a CM field,* F *a continuous function on the adelic group that is holomorphic of order* o *and weight* k*,* z *a point with* $F(z) \ne 0$ *and* $|F(x)| \le 2|F(z)|$ *for all* x*, let* K_0 *be a function on* $G_\infty(s)$ *with* $|K_0(h)| \le C$ *for all* h*, for a real number* C*, and let* ε *be real. Suppose: if* mh_w *is integrable, then for every* x *with* $|F(z)| \le |F(x)|$ *there is a function* K*, holomorphic of weight* $k-w$*, with* $|K(h) - K_0(h)| \le \|h\|^{6\nu}\varepsilon$ *for all* h *and* $\int \overline{F(x\,(h,1))}\,K(h)\,\mathrm{mh}_w(h)\,dh = 0$*. Then* $|K_0(1)| \le 4\varepsilon B^{7\nu} + \varepsilon$*.*

Proof. By Lemma 3.5, mh_w is integrable and $I = \int |\,\mathrm{mh}_k\,|^2 > 0$. For an angle ϑ let z_ϑ be z times the element $e^{i\vartheta o}$ of the diagonal torus at infinity, with entries $e^{i\vartheta o(i)}$, and let $\chi_\vartheta = \prod_\tau \prod_{i\in\mathsf{n}} e^{i\vartheta\, o(i)\, k_\tau(i)}$, a number of absolute value 1: the conjugate of the factor of the library's circle mean. By holomorphy, $|F(z_\vartheta)| = |F(z)|$, so the hypothesis gives for every angle a function $K^{(\vartheta)}$, holomorphic of weight $k-w$, within $\|h\|^{6\nu}\varepsilon$ of K_0, with

$$\int W_\vartheta(h)\,K^{(\vartheta)}(h)\,dh = 0, \qquad W_\vartheta(h) = \chi_\vartheta\,\overline{F(z_\vartheta(h,1))}\,\mathrm{mh}_w(h);$$

the constant χ_ϑ does not change the vanishing. Hence

$$\int W_\vartheta K^{(0)} = \int W_\vartheta\,(K^{(0)} - K^{(\vartheta)}),$$

and since $|F| \le 2|F(z)|$ and $|K^{(0)} - K^{(\vartheta)}| \le 2\|h\|^{6\nu}\varepsilon$, Lemma 3.5 with $N = 6\nu$ bounds this by $4\varepsilon|F(z)|B^{7\nu}I$, for every angle. On the other hand, by the circle mean for holomorphic functions of weight k (library), the mean over the angles of $\chi_\vartheta\,\overline{F(z_\vartheta(h,1))}$ is $\overline{F(z)}\cdot\overline{\mathrm{mh}_k(h)}$, so that the mean of $W_\vartheta(h)$ is

$$\overline{F(z)}\cdot\overline{\mathrm{mh}_k(h)}\cdot\mathrm{mh}_w(h).$$

The function $K^{(0)}\,\mathrm{mh}_w$ is holomorphic of weight k, so by the reproducing identity (library) its integral against the conjugate of mh_k is its value at 1 times I. Exchanging the two integrals gives $|F(z)|\,|K^{(0)}(1)|\,I \le 4\varepsilon|F(z)|B^{7\nu}I$, that is $|K^{(0)}(1)| \le 4\varepsilon B^{7\nu}$; and $|K_0(1)| \le |K^{(0)}(1)| + \varepsilon$. □

LEAN `isolation`

Lemma 3.7 (first minima and matrix coefficients)**.** *Let* E *be a field with a CM type,* n*,* n' *finite non-empty index sets,* S *and* T *hermitian spaces with index sets* n*,* n'*.*

(a) *If $N \geq 1$ is an integer in the different of E, then, with $c_N = 6N(f_E|\mathfrak{n}|)^2$,*

$$\frac{\lambda_1(S)\,\lambda_1(T)}{c_N\,|\mathfrak{n}'|\,|\mathfrak{n}|} \leq \lambda_1(\mathrm{Hom}(S,T)).$$

(b) *Let E have Weil data, $\mathfrak{m}$ a modulus, t a matrix over $\mathcal{O}_E/\mathfrak{m}$, with trace character $\psi'_t(y) = \psi'(\mathrm{tr}(yt))$, v and w Fock vectors and z an adelic vector of $\mathrm{Hom}(S,T)$. Then the pairing of the datum $v \otimes \psi'_t$ with the Heisenberg translate by z of $w \otimes \psi'_t$ is the value of ψ'_t at the finite part of z times the archimedean matrix coefficient of v, w at the real part of z times the vacuum function at the finite part.*

(c) *Let E have Weil data, a and b characters with weights γ, γ' and tame parts, $\mathfrak{m}$ a modulus, g, h points, $c \in E$ non-zero such that c times the idele of the Weil data times z is integral for every integral adele z that reduces to 0 modulo $\mathfrak{m}$, $\mu > 0$ a real number with $\mu\,|\tau(c)| \leq \sqrt{2\pi\,\mathrm{Im}\,\tau(\jmath)} \cdot \lambda_1(\mathrm{Hom}(S^g, T^h))$ at every infinite place τ, with $\jmath$ the purely imaginary element of the Weil data and S^g, T^h the spaces moved by the points, $q > f_E|\mathfrak{n}'||\mathfrak{n}|$ an integer, and Q a polynomial, homogeneous of degree d for real scalars, with Fock vector $v = v_Q$. Then there is a function Ψ on matrices modulo $\mathfrak{m}$ such that for all $u \in G_\infty(s)$, $u' \in G_\infty(s')$*

$$\Big|\theta^{S,T}_{a,b;\mathfrak{m}}[\Psi, Q](g\,(u,1), h\,(u',1)) - \langle v, \omega^\infty_{\gamma,\gamma'}(u,u')v\rangle\Big| \leq \|u'\|^{3q}\Big(\frac{100(\mathsf{d}+2q)}{\mu^2}\Big)^q |\langle v,v\rangle|\,\|u\|^{3q}.$$

Proof (an account of the formal proof). (a) The first minima are cut off at 1. So it suffices to show that the left side is at most 1 and at most the length of every non-zero vector z of the lattice of $\mathrm{Hom}(S,T)$. The first holds because $\lambda_1(S) \leq 1$, $\lambda_1(T) \leq 1$ and $c_N\,|\mathfrak{n}'|\,|\mathfrak{n}| \geq 1$. For the second, a short basis Y of the lattice of S (library), with vectors of length at most $c_N/\lambda_1(S)$, turns z into the vectors zY_k of T, one of them non-zero, of length at least $\lambda_1(T)$ and at most $|z||Y_k|$; the factor $|\mathfrak{n}'|\,|\mathfrak{n}|$ allows for the comparison of the norms of vectors and of matrices in the formal text. (b) Lift t, write the Heisenberg translate of the named datum as a translate of the character, and compute place by place. (c) Let Ψ be the average over t of $\overline{\theta[\psi'_t, Q](g,h)}\,\psi'_t$. By Poisson summation the kernel is a sum over the lattice of $\mathrm{Hom}(S^g, T^h)$ of archimedean matrix coefficients, computed by (b); the term of the vector 0 is $\langle v, \omega^\infty_{\gamma,\gamma'}(u,u')v\rangle$. The other lattice vectors have length at least the first minimum, hence at least μ in the normalised coordinates, and the Gaussian decay of the matrix coefficients of a Fock vector of degree d, summed over the lattice with $q > f_E|\mathfrak{n}'||\mathfrak{n}|$, gives the error. □

LEAN `firstMinimum_mul_le_hom` · `dataInner_traceChar` · `norm_modKernel_sub_realInner_le`

The carrier and the approximation by $F^\natural$

Lemma 3.8 (the carrier is rapid, holomorphic and not zero)**.** *Let E be a CM field with a CM type and Weil data and $m \geq 1$ an integer.*

(a) *Let m be even, and let $\mathfrak{m}$ be a modulus, c a character and χ a character of $\mathcal{O}_E/\mathfrak{m}$ with c tame for $(\mathfrak{m}, 0, \chi)$. Then $\mathfrak{C}_{E,m}$ is rapidly decreasing and automorphic of every level, and there is an order o, positive first for the signs of $I_{2m,m}$, for*

which $\mathfrak{C}_{E,m}$ is holomorphic of weight 0 on the positive block and $3m$ on the negative block.

(b) $\mathfrak{C}_{E,m} \neq 0$.

Proof (an account of the formal proof). (a) By the carrier identity (Input 3) the carrier is a finite combination of theta kernels of the pair $(I_{2m,m}, I_{3m})$ with datum $\chi \circ \det$, which are rapidly decreasing (the datum is supported on invertible matrices), invariant under every level through the tame character, and holomorphic of the stated weight because the polynomial 1 is a holomorphic datum for a definite second space. (b) If the lift of the constant 1 vanished, then by the adjunction between lift and descent the integral of the constant 1 against the descent of a suitable datum would vanish; for the polynomial $\prod_\tau \det(y_\tau^-)^{2m}$ the archimedean matrix coefficient is a product of holomorphic minors with non-zero value at 1, the datum is sharp, and the volume of the compact group is not zero. □

LEAN carrierForm_spec · carrierForm_ne_zero

Proposition 3.9 (the function F-natural approximates a test function)**.** *In Setting 2.1, let j be a lift index and ν_D the integer ν of the data for which Setting 3.4 holds with $B = 5b$. Then there are a function K_0 on the archimedean group of $W_j(D)$ and real numbers C, ε with*

$$4\varepsilon\,(5b)^{7\nu_D} + \varepsilon < |K_0(1)|, \qquad |K_0(y)| \le C \text{ for all } y,$$

such that for every $g \in \mathrm{U}(W_j(D))(\mathbb{A})$ with $(5b \cdot b^{23})^{-1/2} \le \lambda_1(W_j(D)^g)$ there are complex numbers c_a, $a \in \mathrm{Mem}_D(j)$, with

$$\Big|\sum_a c_a\, F_D^\natural(j,a)(g\,(y,1)) - K_0(y)\Big| \le \|y\|^{6\nu_D}\,\varepsilon \qquad \text{for all archimedean } y.$$

Outline of the proof. The outline follows the formal proof step by step and names the result that each step uses. It does not reproduce the estimates: the lower bound for the first minimum of the lattice of homomorphisms, the size of the constant ε and the inequality between $\varepsilon(5b)^{7\nu_D}$ and $|K_0(1)|$ are closed in the formal text by the inequalities between the scales and were not verified here.

The main term. The padded space $W_j(D) \oplus I_j^- \oplus I_{3m-j}$ has m negative signs (Lemma 3.1 (c)). By Lemma 3.8 the carrier is rapidly decreasing, holomorphic of weight $(0, 3m)$ and not zero; by the library's result on near maxima it has a point x with $|\mathfrak{C}_{E(D),m}| \le 2|\mathfrak{C}_{E(D),m}(x)|$ everywhere. The archimedean matrix coefficient

$$C(u,y) = \langle v, \omega^\infty(u,y)v\rangle, \qquad v = v_{\Lambda_D(j)},$$

of the Fock vector of the line polynomial is a product of holomorphic minors in each variable, with weights determined by the exponents of the line polynomial and Lemma 3.1 (a). Put

$$K_0(y) = \int \mathfrak{C}_{E(D),m}(x\,u)\ \mathrm{mh}_{w^\circ}(u)\, C(u,y)\, du,$$

the integral over the archimedean group of signature $(2m, m)$, with w° the weight $(-\mathsf{g}'(D), 6m+1)$ on the two blocks. It is bounded, by Lemma 3.5 with $N = 0$. At $y = 1$ the integrand is a holomorphic function times the conjugate of a holomorphic

minor of its weight, and the integral is its value at $u = 1$ times $\int |\,\mathrm{mh}\,|^2 > 0$: $K_0(1)$ is a positive multiple of $\mathfrak{C}_{E(D),m}(x) C(1,1) \neq 0$.

The approximation. Let g be as in the statement. By Lemma 3.7 (a), the bound for the first minimum at x from the near maximum and the hypothesis on g, the lattice of Hom between the carrier space at xu and the padded space at $g(y,1)$ has first minimum bounded below. Lemma 3.7 (c), for the pair of the carrier space and the padded space, the polynomial $\Lambda_D(j)$ and $q = 2\nu_D$, gives a function Ψ modulo $\bar{\pi}_g^2$ whose kernel differs from $C(u,y)$ by at most $\|u\|^{6\nu_D}\|y\|^{6\nu_D}$ times a small constant. The reproducing formula (library), for the pair of the carrier space and I_{F_D}, expresses the pairing of the rapidly decreasing form $\mathfrak{C}_{E(D),m}\,\mathrm{mh}_{w^\circ}$ with theta kernels as a finite combination, with coefficients $c(i_2)$, of the kernels with the determinant polynomials of the short rows of the line choices i_2. The product of Ψ with the indicator of an inclusion matrix is a finite combination of trace characters ψ'_{i_1}, with coefficients $f(i_1)$ (finite Fourier analysis on $\mathcal{O}_E/\bar{\pi}_g^2$). With $c_a = c(i_2) f(i_1)$ for $a = (i_1, i_2)$, the seesaw for the three summands of W' identifies $\sum_a c_a F_D^\natural(j,a)(g(y,1))$ with the integral of the carrier against the kernel of Ψ; replacing this kernel by $C(u,y)$ costs, by Lemma 3.5 with $N = 6\nu_D$ and $|\mathfrak{C}_{E(D),m}| \le 2|\mathfrak{C}_{E(D),m}(x)|$, at most $\|y\|^{6\nu_D}\varepsilon$, with ε a multiple of the small constant. The inequality between $\varepsilon(5b)^{7\nu_D}$ and $|K_0(1)|$ is one of the inequalities between the scales. □

LEAN `Data.exists_fNatural_approx`

Theorems C and D

Proof of Theorem C (an account of the formal proof). Fix j. The padded space $W_j(D) \oplus I_j^- \oplus I_{3m-j} \oplus I_{\mathsf{F}_D} \oplus S$, regrouped by $\mathrm{rg}_D(j)$, has the block signs of a bijection built from $\mathrm{neg}_{D,j}(\cdot)$ and the row indices (Lemma 3.1 (c)), and for every twist ϑ the characters satisfy $\mathbf{b}''_D(\vartheta, j)\,\upsilon^m = \chi_D^{(j)}[\vartheta]\,\mathbf{c}_D(j)$, by the definitions. Two results of Input 3 give constants that do not depend on the twist. The explicit matching, for the pair of $W' \oplus S$ and L^+-type blocks, with the bijection $\mathrm{pe}_D(j)$ and the index sets $\mathrm{Det}_D(j,\tau)$, whose sizes are the required weights by Lemma 3.1 (a), gives C_1 and, for the tame character $\chi_D^{(j)}[\vartheta]\mathbf{c}_D(j)$, a number $c_1(\vartheta)$ with $|c_1(\vartheta)|^2$ equal to C_1 divided by a product of L-values, and an identity between a class sum of kernels with the product of stacked determinants of the matrices $\eta_D(j, i_2, Y, \tau)$ and an integral over $[\mathrm{U}(S)]$ of kernels with the matched polynomial; the matched polynomial is $\Lambda_D(j)$ times the determinants of the short rows, as one sees by comparing the minors. The carrier identity gives C_2 and, for the tame character $\chi_D^{(j)}[\vartheta]$ of weight 0, a number $c_2(\vartheta)$ with $c_2(\vartheta) \sum^{\mathrm{cl}}_{c \bmod \mathbf{K}_{3m}} \Xi^\circ_{m,\chi_D^{(j)}[\vartheta]}(c) \cdot \theta(x,c) = \mathfrak{C}_{E(D),m}(x)$. Now apply the matching summed against a form on a square block (Input 3) with the function $\Xi = \Xi^\circ_{m,\chi_D^{(j)}[\vartheta]}(\cdot)$, the function $F = \mathfrak{C}_{E(D),m}$, the modulus $\mathfrak{n} = \bar{\pi}_g^2$, the datum ψ'_{i_1} and the polynomial of $F_D^\natural(j,i)(\cdot)$: its right side is $F_D^\natural(j,i)(g)$, and its left side is a double class sum of kernels with the datum $\check{\Psi}_0(x' \mid x'') = \chi(\det x'') \sum_k \psi'_{i_1}((x''^{-1}x' \bmod \bar{\pi}_g^2)k)$. By the passage from the stabiliser to the full level (Input 3), for the matrix w and the function $y \mapsto \sum_k \psi'_{i_1}(yk)$, which is invariant under the stabiliser of w because $uw = i_1$, the class sum at the full level is the class sum at the level of the stabiliser with the column datum. The seesaw for the three blocks of L^+ and the expression of the product of determinants through its coordinates in the positive letters (unit *Filling*) turn the inner

sums into $\sum_l \mathrm{lm}_D(\vartheta,j,i,h)_l$ times the l-th basis polynomial, that is into the kernel with the polynomial of the line multiplier in the positive letters: the coset kernel sum. So the identity of the theorem holds with $c(\vartheta)=c_1(\vartheta)\,c_2(\vartheta)/|\mathrm{GL}_{3m}(\mathcal{O}_E/\bar\pi_g^2)|$, by the formula of the matching summed against a form on a square block, and $|c(\vartheta)|^2$ is a constant divided by a product of products of L-values at the characters $(\chi_D^{(j)}[\vartheta]\mathbf{c}_D(j))^{-1}\upsilon^{\cdots}$, $\chi_D^{(j)}[\vartheta]^{-1}\upsilon^{m}$ and $\chi_D^{(j)}[\vartheta]\upsilon^{m/2}$; unfolding the characters, these are the three factors of $\mathrm{HV}_D(j,\vartheta)$. □

Lemma 3.10 (levels and holomorphy)**.** *In Setting 2.1, let j be a lift index and $F\in\mathcal{M}_D^\circ(W_j(D))$.*

(a) *For every integer a of $E(D)$ divisible by $\bar\pi_g$, F is rapidly decreasing and invariant under the principal level modulo a.*

(b) *For every choice $i\in\mathrm{Mem}_D(j)$, the product $F\cdot F_D^\natural(j,i)(\cdot)$ is rapidly decreasing and invariant under the principal level modulo $\bar\pi_g^2$, and $F_D^\natural(j,i)(\cdot)$ is holomorphic, for a fixed order that is positive first for the signs of $W_j(D)$, of weight the difference of the weight of $\bar F$ and the weight of the kernels of $\mathrm{tt}_D(j,\cdot)(\cdot)$.*

Proof (*an account of the formal proof*). (a) Good forms are cusp forms of principal level at the chosen primes. (b) By Theorem C with $\mathsf{a}=\mathsf{C}_D$, $u=1$ and $w=i_1$, $F_D^\natural(j,i)(\cdot)$ is a finite combination of theta kernels of the pair of $W_j(D)$ and the big block with polynomials in the positive letters; such kernels are holomorphic, the Fock vector being a highest weight vector (library), and the product of a rapidly decreasing form with a kernel of this kind is rapidly decreasing of the level of the modulus. □

LEAN `Data.goodForms_mem_rapidOfLevel` · `Data.mul_fNatural_mem_rapidOfLevel_and_hol`

Proof of Theorem D (*an account of the formal proof*). Suppose $F\neq 0$. Its conjugate $\bar F$ is holomorphic, of the weight k of the kernels of the adjoint lift, because F is in the image of the descent of the trivial twist (the adjoint lift is injective). By the library's result on near maxima there is a point g_0 with $\bar F(g_0)\neq 0$, $|F|\le 2|F(g_0)|$ everywhere and first minimum at least $(5b\cdot b^{23})^{-1/2}$ wherever $|F|\ge|F(g_0)|$. Take K_0, C, ε from Proposition 3.9. The claim is that the hypothesis of Proposition 3.6 holds for $\bar F$, the point g_0, the weight w of the kernels of $\mathrm{tt}_D(j,\cdot)(\cdot)$ and $B=5b$; its conclusion $|K_0(1)|\le 4\varepsilon(5b)^{7\nu_D}+\varepsilon$ then contradicts Proposition 3.9. Let x be a point with $|F(g_0)|\le|F(x)|$, so that Proposition 3.9 applies at x, and put $K(y)=\sum_a c_a F_D^\natural(j,a)(x(y,1))$; it is holomorphic of weight $k-w$ by Lemma 3.10 (b) and within $\|y\|^{6\nu_D}\varepsilon$ of K_0. The integral $\int F(x(y,1))K(y)\,\mathrm{mh}_w(y)\,dy$ is, by the reproducing formula for the pair of $W_j(D)$ and the small block V° (library), applied to the rapidly decreasing function $F\cdot\sum_a c_a F_D^\natural(j,a)(\cdot)$, a finite combination of the integrals over $[\mathrm{U}(W_j(D))]$ of $\overline{F F_D^\natural(j,a)(\cdot)}$ against kernels of the small block with basis polynomials in the negative letters at the class representatives. Each of these is, after writing a basis polynomial as the polynomial of a coordinate vector v, the integral of the hypothesis for the function ζ equal to $\bar v$ at one class and zero elsewhere, hence zero. □

LEAN `Data.eq_zero_of_integral_fNatural_tableTheta`

Hecke values: Theorem B

Lemma 3.11 (flat pairs and the untwisted norm)**.** *In Setting 2.1:*

(a) *Let* n *be a finite index set with* $4 \le |\mathsf{n}| \le t(D)$ *and* $|\mathsf{n}|$ *even, and* χ *a character of weight* 0 *whose tame part is trivial outside the chosen primes. Then there is a pair* p *that is flat for* n*, with* $\nu_p \upsilon^{|\mathsf{n}|/2} = \chi$ *and with the tame part of* χ*.*

(b) *Let* j *be a lift index and* p *a pair of rank one for* j*. Then the constant family* $(\ell, \iota) \mapsto \|\Xi_p(1)\|^2$ *has height at most* $B_0(D) - \sum_\tau \mathrm{js}_D(w_0(\tau))$*.*

Proof (*an account of the formal proof*). (a) Take $\nu_p = \chi \upsilon^{-|\mathsf{n}|/2}$ and $z_p = |\mathsf{n}|/2$; the conditions on a pair hold by the inequalities between the scales. (b) Take a flat pair q for the index set of p by (a) and the witness $H = I_q(1)/\|\Xi_p(1)\|^2$: then $H \cdot \|\Xi_p(1)\|^2 = I_q(1)$ is an algebraic integer, so the ℓ-adic exponents are 0, and the conjugates of H are bounded by the unit *Exchange*, which computes the logarithm of the conjugates of the squared norm of a trivial lift as a sum of jet sizes up to terms that the scales bound by $B_0(D)$. □

LEAN `Data.exists_isFlat` · `Data.heightLe_norm_trivialLift_sq`

Lemma 3.12 (two pigeonholes)**.** (a) *Let* E *be a CM field with a CM type,* ℓ *a prime,* $\iota\colon \mathbb{C} \to \mathbb{C}_\ell$ *a ring homomorphism,* $w = (w_i)$ *a family of finite places, all in a set* S *of finite places,* s *a complex function on* $\mathrm{U}(I_1)(\mathbb{A}_f)$*, and* Γ*,* K *subgroups such that* Γ *contains the rational points,* K *contains the principal level at the places* w_i*, the double coset space is finite and all class stabilisers are trivial. Then there are a point* z *and a character* χ *of weight* 0 *with tame part trivial outside* S *such that* $|\iota(s(1))|_\ell$ *is at most the* ℓ*-adic absolute value of the mean over the classes of* $g \mapsto \chi(\det(gz))\, s(gz)$*.* (b) *Let* G *be a finite group,* ι *a ring homomorphism from* $\mathbb{C}$ *to a normed field,* a *and* b *complex functions on* G *and* $V \in \mathbb{C}$ *with* $V^{|G|} = \prod_{\chi \in G} a(\chi) b(\chi)$*. Then for every* $\psi \in G$ *there are* $\phi, \chi \in G$ *with* $\phi\chi = \psi$ *and* $|\iota(V)| \le |\iota(a(\phi) b(\chi))|$*.*

Proof. (a) The class group of $\mathrm{U}(1)$ at this level is a finite abelian group, and every character of it is a character of weight 0 with tame part trivial outside S (library). By orthogonality of characters, $s(1)$ is the sum over the characters of the means in question, for z the inverse of a representative of the trivial class; by the ultrametric inequality one of them is at least as large. (b) Choose χ maximising $|\iota(a(\psi\chi^{-1}) b(\chi))|$; the product over χ of these numbers is $|\iota(V)|^{|G|}$, after the change of variable $\chi \mapsto \psi\chi^{-1}$ in a. □

LEAN `exists_twisted_classSum_ge` · `Data.exists_factorisation_le`

Lemma 3.13 (test forms)**.** *In Setting 2.1, let* ℓ *be a prime with an embedding* $\iota\colon \mathbb{C} \to \mathbb{C}_\ell$*.*

(a) *Let* q *be a pair that is flat for* $\{1, \dots, m'\}$*,* X *an* L*-value datum,* ϑ *a twist with* ℓ *prime to* $N(Q(\vartheta))$*,* $\mathcal{N}$ *a module over the valuation ring of* ι *inside* $\mathcal{F}(O_X)$*, and* F *a function with values in* $\mathcal{N}$*. Then* $\mathrm{xs}_{q,\vartheta}(F) \in \mathcal{N}$*.*

(b) *Let* q*,* X*,* ϑ*,* F *be as in Definition 2.6 (i),* g *a point,* $A, B \in \mathbb{C}$ *and* Λ *a continuous conjugate-linear form on* $\mathcal{F}(O_X)$ *such that, in every coordinate and for all* h*,*

$$\Lambda(F(h)) = B \cdot (A \cdot \text{the kernel form of } q \text{ at } (g, h)).$$

Then $h_{\mathrm{U}}\, \kappa_0^{m'} \cdot \Lambda(\mathrm{xs}_{q,\vartheta}(F)) = B \cdot I_q(\vartheta) \cdot A$*, with* κ_0 *the vacuum scale and* h_{U} *the class sum of the constant* 1 *over* $\mathrm{U}(1)$ *at the chosen level of the data: the sum*

over the classes of the inverses of the orders of their stabilisers. That level is free (unit Data*), so the stabilisers are trivial and h_{U} is the number of classes.*

(c) *Let p be a pair of rank one for a lift index j, let q, q_2, q_3 be pairs flat for $\{1,\dots,2z_p-20\}$, $\{1,\dots,4\}$ and $\{1,\dots,4\}$, ϑ a twist of D with ℓ prime to $N(Q(\vartheta))$, x_0 a 12×1 matrix modulo $\bar\pi_g$, α an exponent whose total orders are the jet degrees of the L-value datum X of p at ϑ, and b a character whose product with the second characters of the data of q_2 at ϑ, of q_3 at 1 and of q at 1 is the second character of X. Then there is a form $\Xi\in\mathcal{F}(O_X)$ that is a* test form at ℓ*, integral at every embedding into $\mathbb{C}_\ell$ and tested against the kernel of the descent for the label idempotent* $\mathrm{idem}(X-1)$*, with, for every point g of* $\mathrm{U}(1)$*,*

$$\langle \Xi, \textit{kernel form of } X \textit{ at } g\rangle\cdot h_{\mathrm{U}}^{3}\kappa_0^{2z_p-12}=\frac{\prod_\tau d_\tau!}{\alpha!}\cdot I_q(1)\,I_{q_3}(1)\,I_{q_2}(\vartheta)\cdot\theta^{I_1,I_{12}}_{1,b;\bar\pi_g}[\mathbf{1}_{x_0},x^\alpha](g,1),$$

with h_{U} as in (b) and d_τ the jet degrees.

Proof (*an account of the formal proof*). (a) The class sum is an integral combination, the orders of class stabilisers dividing M_D; the factor $\delta_q/N(Q(\vartheta))^{m'-1}$ is ι-integral since ℓ is prime to $N(Q(\vartheta))$, and the Gram form of 1 with δ_q times the trivial lift is ι-integral by the definition of the denominator bound. (b) Expand the class sum and use that the pairing of the trivial lift with the kernel forms is the theta function of the trivial lift (unit *Exchange*); the sum of the squares of its values is the squared norm, whose normalisation is $I_q(\vartheta)$. (c) Let v be the integral vector of Input 2 (d) for the monomial x^α, and let $F(h_1,h_2,h_3)$ be the point form (Input 2 (e)) with value v at the block diagonal point $(1,h_2,h_3,h_1)$ of the index set $12+4+4+(2z_p-20)$, with the reductions prescribed by x_0 and inclusion matrices. Put $\Xi=\mathrm{xs}_{q_2,\vartheta}(h_2\mapsto \mathrm{xs}_{q_3,1}(h_3\mapsto \mathrm{xs}_{q,1}(F(\cdot,h_2,h_3))))$. The formula follows from (b), applied three times, once the pairing of the point form with a kernel form is computed: by the descent of a point form and the seesaw for the four blocks it is the product of the kernel of the rank-12 block with the monomial, by Input 2 (d), and of the kernel forms of the three flat blocks, of jet degree 0. Integrality: by (a) three times, and because the point form of the integral vector v has gauge at most 1 at its point, v being fixed by the matrices that fix the columns of the monomial. Tested: a form in the kernel of the descent, in the image of the label idempotent and killed by a Hecke test satisfies the hypotheses of the vanishing of periods (Input 2 (e)), for the splitting into the first three blocks and the last one, and the class sum over the last block that defines Ξ pairs to zero with it. □

LEAN `Data.Pair.xiSum_mem` · `Data.Pair.smul_apply_xiSum` · `Data.exists_test_form_inner_kernelForm`

Proof of Theorem B (*an account of the formal proof*). The argument is that of the formal proof; the powers of the class number, of the vacuum scale and of the denominator bounds are not recomputed (§5). Write $\vartheta_{\ell,\iota}=\theta^{j}_{\ell,\iota}$, a twist, and let G be the finite group of the characters of weight 0 with tame part trivial outside the chosen primes that occur in Lemma 3.12 (a) for $\mathrm{U}(1)$ at the chosen level. That level is free (unit *Data*), so the class sum h_{U} of the constant 1 is the number of classes.

Choices. For every $\chi\in G$ take pairs $q_1(\chi)$, flat for $\{1,\dots,2z_p-20\}$, and $q_3(\chi)$, flat for $\{1,\dots,4\}$, attached to χ by Lemma 3.11 (a), and let q_2 be the flat pair for $\{1,\dots,4\}$

attached to χ_D; it is ramified. The orders $\mathbf{m}_\tau = c_{p,\tau} - z_p$, the jet degrees of p, satisfy $w_D(\tau) \le \mathbf{m}_\tau \le w_D(\tau) + x^{42}$, because p is of rank one; take $t_{\mathbf{m}}$ from Input 2 (b). Let V be a complex number with

$$V^{|G|} = \prod_{\chi \in G} I_{q_3(\chi)}(1) \cdot \big(I_{q_1(\chi)}(1) \cdot t_{\mathbf{m}} \cdot I_{q_2}(1)\big).$$

Then $V \neq 0$, and $|\sigma(V)| \le e^{B_0(D)}$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$: by Lemma 3.12 (b) for σ and $\psi = 1$, $|\sigma(V)|$ is at most the absolute value of a conjugate of a product of three normalised values and of $t_{\mathbf{m}}$, which Input 2 (b), (c) and the scales bound.

The valuation. Claim: for every index (ℓ, ι) of the family and every embedding $\varphi\colon \mathbb{C} \to \mathbb{C}_\ell$, with $\vartheta = \vartheta_{\ell,\iota}$ the twist of the index, there is a test form Ξ at ℓ for the L-value datum of p at ϑ with

$$|\varphi(V)|_\ell \cdot \ell^{-F_D(\ell)} \le \big|\varphi\big(h_{\mathrm{U}}^2 \,\langle \Xi, \Xi_p(\vartheta)\rangle / \rho_{q_2}(\vartheta, 1)\big)\big|_\ell. \tag{1}$$

In this claim the index fixes the twist and the embedding varies. Of the index the proof uses only that ℓ is prime to $N(Q(\vartheta))$, which is clause (1) of fitness; Input 2 (b) and Lemma 3.12 hold for every embedding. Indeed, Input 2 (b), for the modulus $\bar{\pi}_g$ and the embedding φ, gives α and x_0 such that the Taylor coefficient $s(1)$, where $s(g)$ is the kernel of (I_1, I_{12}) with the indicator of x_0 and the monomial x^α at g, satisfies

$$\ell^{-F_D(\ell)} \le \Big|\varphi\Big(\kappa_0^{12} \prod_\tau \mathbf{m}_\tau! \cdot s(1)/t_{\mathbf{m}}\Big)\Big|_\ell.$$

Lemma 3.12 (a) gives z and $\psi \in G$ with $|\varphi(s(1))|_\ell$ at most the absolute value of the mean of $(\psi s)(gz)$, and Lemma 3.12 (b) gives $\psi_3\psi_1 = \psi$ with

$$|\varphi(V)|_\ell \le \big|\varphi\big(I_{q_3(\psi_3)}(1)\, I_{q_1(\psi_1)}(1)\, t_{\mathbf{m}}\, I_{q_2}(1)\big)\big|_\ell.$$

Lemma 3.13 (c), for the pairs $q_1(\psi_1)$, q_2, $q_3(\psi_3)$ and the character b determined by them and ψ, gives Ξ. Its pairing with the trivial lift, which is the class sum over $\mathrm{U}(1)$ of the kernel forms, is by the formula of Lemma 3.13 (c) the mean of $(\psi s)(gz)$ times $I_{q_1(\psi_1)}(1)\, I_{q_3(\psi_3)}(1)\, I_{q_2}(\vartheta) \prod_\tau \mathbf{m}_\tau!/\alpha!$ and powers of h_{U} and κ_0. Since $I_{q_2}(\vartheta)/\rho_{q_2}(\vartheta, 1) = I_{q_2}(1)$ and $|1/\alpha!|_\ell \ge 1$, the three inequalities combine to (1).

The height. Consider the family

$$y'_{\ell,\iota} = \big(\rho_{q_2}(\vartheta_{\ell,\iota}, 1) \big/ \|\Xi_p(\vartheta_{\ell,\iota})\|\big)^2 \big/ h_{\mathrm{U}}^3.$$

It has height at most $2B_0(D) + B_0(D) + \sum_\tau \mathrm{Ask}_D(w_D(\tau), \tau)$, with the witness $H = V\bar{V}$. The conjugates of H are at most $e^{2B_0(D)}$, because $\sigma(\bar{V})$ is the image of V under the automorphism $\sigma \circ \mathrm{conj}$. Fix an index (ℓ, ι) and write $\vartheta = \vartheta_{\ell,\iota}$. The claim (1) for this index, with $\varphi = \iota$ and with $\varphi = \iota \circ \mathrm{conj}$, that is twice for the same twist ϑ, gives test forms Ξ, Ξ' with

$$|\iota(V\bar{V})|_\ell \, \ell^{-2F_D(\ell)} \le \big|\iota\big(h_{\mathrm{U}} \,\langle \Xi, \mathrm{pr}\, \Xi'\rangle\big)\big|_\ell \cdot \big|\iota\big(h_{\mathrm{U}}^3 \|\Xi_p(\vartheta)\|^2 / \rho_{q_2}(\vartheta, 1)^2\big)\big|_\ell,$$

where pr is the orthogonal projection to the line $\mathbb{C}\,\Xi_p(\vartheta)$, so that

$$\langle \Xi, \mathrm{pr}\, \Xi'\rangle = \langle \Xi, \Xi_p(\vartheta)\rangle \cdot \langle \Xi_p(\vartheta), \Xi'\rangle \big/ \|\Xi_p(\vartheta)\|^2;$$

here one uses that the ratio is real.

The first factor is at most e^{β_ℓ} by the rank-one bound of the unit *Selmer*: by clause (2) of fitness the line $\mathbb{C}\,\Xi_p(\vartheta)$ has denominators at most β_ℓ at ι for $\mathrm{idem}(X-1)$, the class stabilisers of $\mathrm{U}(1)$ are trivial, and Ξ, Ξ' are test forms at ℓ. Clause (2) cannot be applied to Ξ and Ξ' directly: a test form is tested against the kernel of the descent, not against the orthogonal complement of the line. The rank-one bound applies it to Ξ and to the sum of the translates of Ξ' by the twisted central operators at representatives of the h_{U} classes of $\mathrm{U}(1)$. This sum is integral at ι and tested against the orthogonal complement of the line, the trivial lift is fixed by these operators, and the sum is where the factor h_{U} comes from. So

$$|\iota(Hy'_{\ell,\iota})|_\ell \le \exp(\beta_\ell + 2F_D(\ell)\log\ell),$$

and the sum of $2F_D(\ell)\log\ell$ over the primes below a bound $\ge x_1$ is $\sum_\tau \mathrm{Ask}_D(w_D(\tau),\tau)$ (unit *Columns*).

The family $\rho_{q_2}(\vartheta_{\ell,\iota},1)^{-2}$ has height at most $2x^{999}$ by clause (3) of fitness, for the ramified pair q_2 and the sign 1, and the constant family $\|\Xi_p(1)\|^2$ has height at most $B_0(D)-\sum_\tau \mathrm{js}_D(w_0(\tau))$ by Lemma 3.11 (b). The product of the three families is

$$\|\Xi_p(1)\|^2 \big/ (h_{\mathrm{U}}^3\,\|\Xi_p(\vartheta_{\ell,\iota})\|^2),$$

and $\rho_p(\vartheta_{\ell,\iota},1)^{-1}$ is this number times a natural number, a power of h_{U} times a power of $N(Q(\vartheta_{\ell,\iota}))$, by the definition of the normalised values; a height bound passes to a family whose ℓ-adic absolute values are not larger (unit *Heights*). Heights add: $4B_0(D)+2x^{999}-\mathsf{U}(D)\le 6B_0(D)-\mathsf{U}(D)$, since $x^{999}\le B_0(D)$. □

LEAN `Data.heightLe_inv_ratio`

Corollary 3.14 (the product of Hecke values)**.** *In Setting 2.1, let θ be a family of m-fold twists of D that is fit for D and j a lift index. Then the family*

$$(\ell,\iota)\longmapsto \Big(\frac{\mathrm{HV}_D(j,\theta^j_{\ell,\iota})}{\mathrm{HV}_D(j,1)}\Big)^{-1}$$

has height at most $10\,m\,x^{999}+k(D)\,(6B_0(D)-\mathsf{U}(D))$.

Proof. Let ϑ be a twist. By its definition $\mathrm{HV}_D(j,\vartheta)$ is the product, over the three line pairs $(\nu_o,z_o)=\mathrm{lp}_D(j,o)$ with their signs s_o, of a power of the norm of T_ϑ and of the squares $|L^{T_\vartheta}(z,(\nu_o\vartheta^{s_o})^{-1})|^2$, $1\le z\le z_o$. Since T_1 is the set of the chosen primes, the ratio to the untwisted product is, up to an integer power of $N(Q(\vartheta))$, the square of the product of the ratios

$$\rho_{(\nu_o,z)}(\vartheta,s_o),\qquad 1\le z\le z_o,$$

over the three line pairs. The sign of each ratio is the sign s_o of its line pair: the line pair of the sign -1 contributes the ratios $\rho_{(\nu_o,z)}(\vartheta,-1)$, and not those for the sign 1. For the first line pair, where $s_o=1$, and $7m/2<z\le \mathsf{e}^*(D)$, the pairs (ν_o,z) are of rank one for j, and there are $\lfloor k(D)/2\rfloor$ of them. All the other pairs are ramified, and there are $7m/2+m+m/2=5m$ of them. Now take $\vartheta=\theta^j_{\ell,\iota}$. Theorem B for the first, clause (3) of fitness with the sign s_o for the others, the rules for products and squares of heights, and the rule for powers of $N(Q(\vartheta))$ (unit *Heights*) give the bound $2\,(\lfloor k(D)/2\rfloor\,(6B_0(D)-\mathsf{U}(D))+5m\,x^{999})$, and $2\lfloor k(D)/2\rfloor=k(D)$ because $k(D)$ is even. □

LEAN `Data.heightLe_inv_heckeValueProd_div`

The volume of a lift: Theorem A

Lemma 3.15 (the compressed Gram operator)**.** *Let T, U, V be finite-dimensional inner product spaces,* $\mathrm{gl}\colon T \to V$*,* $u\colon U \to V$ *and* $A\colon T \to U$ *linear maps and* $\gamma, \varkappa \in \mathbb{C}$*.*

(a) *If* $u^* \circ \mathrm{gl} = \gamma A$*, then* $\mathrm{gl}^* \circ u = \bar{\gamma} A^*$*.*

(b) *With* L *the image of* u*:* $\mathrm{vf}(u)^2 \cdot \det \mathrm{cg}_{\varkappa}(L, \mathrm{gl}) = \varkappa^{\dim L} \cdot \mathrm{vf}(\mathrm{gl}^* \circ u)^2$*, where* vf *is the volume factor.*

Proof. (a) Take adjoints. (b) If u is not injective, both sides vanish: $\mathrm{vf}(u) = 0$, and $\mathrm{gl}^* \circ u$ is not injective, so $\mathrm{vf}(\mathrm{gl}^* \circ u) = 0$. Let u be injective; it is then a bijection from U onto L. The operator is $\varkappa$ times $w^* w$ for $w = \mathrm{gl}^*|_L$, since on L

$$\langle z, \mathrm{pr}_L \, \mathrm{gl}\, \mathrm{gl}^* z' \rangle = \langle \mathrm{gl}^* z, \mathrm{gl}^* z' \rangle;$$

and $\mathrm{gl}^* \circ u = w \circ u$, with u regarded as a bijection onto L, so that

$$\mathrm{vf}(w \circ u)^2 = \det(w^* w) \cdot \mathrm{vf}(u)^2 \qquad \text{and} \qquad \varkappa^{\dim L} \, \mathrm{vf}(\mathrm{gl}^* \circ u)^2 = \det(\varkappa \, w^* w) \cdot \mathrm{vf}(u)^2.$$

□

LEAN `Data.adjoint_comp_eq_conj_smul` · `Data.det_compressedGramOp`

Setting 3.16 (a comparison map)**.** In Setting 2.1: j is a lift index, ϑ a twist of D, $\mathrm{gl}\colon \mathrm{Tab}_D(j) \to \mathcal{F}(O_D(\vartheta, j))$ a linear map, L a linear subspace of $\mathcal{F}(O_D(\vartheta, j))$ and β a real number, such that gl^* is injective on L and, for every $z \in L$, the form $\mathrm{gl}(\mathrm{gl}^* z)$ is tested against $L^{\perp}$ for the good idempotent $e_0(D)$.

Proposition 3.17 (the determinant at the primes and at the conjugates)**.** *In Setting 3.16:*

(a) *Let* (ℓ, ι) *be a prime with a field isomorphism from* $\mathbb{C}$ *to an algebraic closure of* $\mathbb{Q}_\ell$*. Suppose that for every table* ζ *and every embedding* $\varphi\colon \mathbb{C} \to \mathbb{C}_\ell$*: if every* $\overline{\zeta_a}$ *has gauge at most* 1 *at the representative of its class for* $\varphi \circ \mathrm{conj}$*, then* $\mathrm{gl}(\zeta)$ *is integral at* φ*; and that* L *has denominators at most* β *at* ι *for* $e_0(D)$*. Then*

$$\left|\iota\left(\det \mathrm{cg}_{\mathsf{h}_D(j)}(L, \mathrm{gl})\right)\right|_\ell \le \exp\left(\dim L \cdot (r\,\beta)\right).$$

(b) *Suppose that* L *is the image of* u_ϑ *and that, for all tables* ζ *and all* $\sigma \in \mathrm{Aut}(\mathbb{C})$*,*

$$\mathsf{h}_D(j) \, \|\mathrm{gl}(\zeta)\|_\sigma^2 \le e^{\beta} \sum_a \|\sigma(\zeta_a)\|^2.$$

Then for every $\sigma \in \mathrm{Aut}(\mathbb{C})$*:*

$$\left|\sigma\left(\det \mathrm{cg}_{\mathsf{h}_D(j)}(L, \mathrm{gl})\right)\right| \le \exp(\dim L \cdot \beta).$$

Proof ((a)*: an account of the formal proof;* (b)*: complete*)*.* (a) Move the class representatives of $\mathrm{Cl}^\circ(D)$ by rational points $\delta_{\mathfrak{c}}$ to points integral above ℓ (library). For an index $a = (i, \mathfrak{c})$ and a coordinate e let $f_{a,e}$ be the table that is zero outside a and whose entry at a is the translate by $\delta_{\mathfrak{c}}$ of the vector dual to the coordinate e for the Fock pairing of the integral bases; then $\langle f_{a,e}, \zeta \rangle$ is $\mathsf{h}_D(j)$ times the conjugate of the coordinate e of the translate of $\overline{\zeta_a}$. The linear forms $z \mapsto \langle f_{a,e}, \mathrm{gl}^* z \rangle$ on L have no common zero, because gl^* is injective on L. By the extension of Auerbach's lemma (library), for the absolute value $|\iota(\,\cdot\,)|_\ell$, some $\dim L$ of them, with indices $c_1, c_2, \dots,$

are the coordinates of a basis (z_b) of L on which all the forms have $|\iota(\,\cdot\,)|_\ell \le 1$. In this basis the matrix of the compressed Gram operator has the entries $\langle y_a, \mathrm{pr}_L\, y'_b\rangle$ with $y_a = \mathrm{gl}(f_{c_a})$ and $y'_b = \mathrm{gl}(\mathrm{gl}^*(\mathsf{h}_D(j) z_b))$. The forms y_a are integral at every embedding into $\mathbb{C}_\ell$: the Fock pairings of the integral bases are integers, and the transport by $\delta_{\mathfrak{c}}$ and the complex conjugation in the definition of the tables move the gauge to a representative that is integral above ℓ, where a vector with integer coordinates has gauge at most 1. The forms y'_b are integral at ι, because by the estimate of Auerbach's lemma the conjugates of the entries of $\mathsf{h}_D(j)\,\mathrm{gl}^*\, z_b$ have gauge at most 1 after the same transport, and they are tested by Setting 3.16. The definition of denominators at most β gives the bound, the rank of the first space being r. (b) The determinant is the product of the $\dim L$ eigenvalues, so it suffices to show $|\sigma(\alpha)| \le e^\beta$ for an eigenvalue α with eigenvector $z \in L$. Let $\zeta = \mathrm{gl}^*\, z$, which is not zero, put $\tilde\sigma = \mathrm{conj} \circ \sigma \circ \mathrm{conj}$, and let ζ^σ be the table with the entries $\zeta^\sigma_a = \overline{\sigma^{-1}(\overline{\sigma(\zeta_a)})}$. It is chosen so that

$$\tilde\sigma(\zeta^\sigma_a) = \sigma(\zeta_a) \quad \text{for all } a, \qquad \text{and} \qquad \sigma\langle \zeta^\sigma, \zeta\rangle = \sum_a \|\sigma(\zeta_a)\|^2,$$

without any assumption that σ commutes with complex conjugation. Pairing $\alpha z = \mathsf{h}_D(j)\,\mathrm{pr}_L\,\mathrm{gl}(\zeta)$ with $\mathrm{gl}(\zeta^\sigma)$ and applying σ gives

$$|\sigma(\alpha)| \sum_a \|\sigma(\zeta_a)\|^2 = \mathsf{h}_D(j)\,\big|\sigma\langle \mathrm{gl}(\zeta^\sigma), \mathrm{pr}_L\,\mathrm{gl}(\zeta)\rangle\big|.$$

The norms $\|\cdot\|_\sigma$ do not satisfy a Cauchy–Schwarz inequality at σ alone when σ does not commute with complex conjugation. The inequality of the library is $|\sigma\langle y, x\rangle| \le \|x\|_\sigma \, \|y\|_{\tilde\sigma}$; apply it to $y = \mathrm{gl}(\zeta^\sigma)$ and $x = \mathrm{pr}_L\,\mathrm{gl}(\zeta)$. The size bound at $\tilde\sigma$ for the table ζ^σ gives $\mathsf{h}_D(j)\,\|\mathrm{gl}(\zeta^\sigma)\|^2_{\tilde\sigma} \le e^\beta \sum_a \|\tilde\sigma(\zeta^\sigma_a)\|^2 = e^\beta \sum_a \|\sigma(\zeta_a)\|^2$. The size bound at σ for ζ gives the same bound for $\mathsf{h}_D(j)\,\|\mathrm{pr}_L\,\mathrm{gl}(\zeta)\|^2_\sigma$, because the projection to L does not increase $\|\cdot\|^2_\sigma$ on the tested form $\mathrm{gl}(\zeta) = \mathrm{gl}(\mathrm{gl}^*\, z)$ (unit *Selmer*; L is killed by the Hecke test at a split place off the level, which exists, since the lifts of good eigenforms are, and L is generated by such lifts, tuple by tuple, by Input 3). So the right side is at most the geometric mean of the two bounds, $e^\beta \sum_a \|\sigma(\zeta_a)\|^2$, and the sum is positive. □

LEAN `Data.norm_det_compressedGramOp_padic_le` · `Data.norm_det_compressedGramOp_arch_le`

Proof of Theorem A. Write $n_0 = \dim \mathrm{Row}(D) = \dim \mathcal{M}^\circ_D(W_j(D))$, $\vartheta_{\ell,\iota} = \theta^j_{\ell,\iota}$, and note that $1^j = 1$, so $\mathrm{vol}_D(1, j) = \mathrm{vf}(u_1)^2$ and $\mathrm{vol}_D(\theta_{\ell,\iota}, j) = \mathrm{vf}(u_{\vartheta_{\ell,\iota}})^2$. Take γ, A and the maps gl_ϑ of Input 1, and the numbers β_ℓ of clause (2) of fitness for j. For a twist ϑ let L_ϑ be the image of u_ϑ, of dimension n_0 since u_ϑ is injective, and

$$\Delta(\vartheta) = \det \mathrm{cg}_{\mathsf{h}_D(j)}(L_\vartheta, \mathrm{gl}_\vartheta).$$

By Input 1 (ii) and Lemma 3.15,

$$\mathrm{vf}(u_\vartheta)^2 \cdot \Delta(\vartheta) = \mathsf{h}_D(j)^{n_0}\ \mathrm{vf}(\bar\gamma(\vartheta) A^*)^2 = \mathsf{h}_D(j)^{n_0}\,|\gamma(\vartheta)|^{2n_0}\ \mathrm{vf}(A^*)^2. \tag{2}$$

The right side is not zero: $\gamma(\vartheta) \neq 0$, and A^* is injective because A is surjective. So $\Delta(\vartheta) \neq 0$; hence the compressed Gram operator is invertible and gl^*_ϑ is injective on L_ϑ. For $z = u_\vartheta y \in L_\vartheta$, $\mathrm{gl}_\vartheta(\mathrm{gl}^*_\vartheta\, z) = \bar\gamma(\vartheta)\,\mathrm{gl}_\vartheta(A^* y)$ is tested against $L^\perp_\vartheta$ for $e_0(D)$ by Input 1 (iv). So Setting 3.16 holds for every twist. Dividing (2) for 1 by (2) for $\vartheta_{\ell,\iota}$ and using Input 1 (i),

$$\frac{\mathrm{vol}_D(1, j)}{\mathrm{vol}_D(\theta_{\ell,\iota}, j)} = \frac{\Delta(\vartheta_{\ell,\iota})}{\Delta(1)} \cdot \Big(\Big(\frac{\mathrm{HV}_D(j, \vartheta_{\ell,\iota})}{\mathrm{HV}_D(j, 1)}\Big)^{-1}\Big)^{n_0}.$$

The first family has height at most $n_0 \cdot 3rB_0(D)$, with the witness $H = \Delta(1)$: its conjugates are at most $\exp(n_0 \cdot 2rB_0(D))$ by Proposition 3.17 (b) and Input 1 (vi); and $|\iota(\Delta(\vartheta_{\ell,\iota}))|_\ell \le \exp(n_0\, r\, \beta_\ell)$ by Proposition 3.17 (a), Input 1 (v), clause (1) of fitness and clause (2) for the image of the adjoint lift; the sum of the β_ℓ is at most $B_0(D)$. The second family has height at most $n_0\,(10mx^{999} + k(D)(6B_0(D) - \mathsf{U}(D)))$ by Corollary 3.14. Heights add, and

$$3rB_0(D) + 10mx^{999} + 6k(D)B_0(D) \le k(D)\log b \cdot B_0(D)$$

is one of the inequalities between the scales. □

LEAN `Data.heightLe_liftVolume_ratio`

4 Where the results are used

Theorem A is the second of the two inputs of Theorem C of the handout named in the head of the argument; the unit *Endgame* multiplies the m families for $j = 1, \dots, m$, adds their bounds, and combines the result with the factorisation of the volume of the chain (Theorem A of the first note). Theorems C and D and Proposition 3.3 are the inputs of the comparison forms in the sister note. Lemma 3.8 and the definitions of §2 are used in this unit only.

5 Notes on the formal text

1. *Statements cut at a local definition.* The formal statements of Lemma 3.2 (b) and (c), of Proposition 3.3 and of Lemma 3.13 (c) contain a local definition; the tools of this write-up read such a statement only up to it and therefore list no argument for Proposition 3.3 and only the point z for Lemma 3.2 (b). The statements above were written from the Lean text: the first takes the lift index, the second the lift index, the line choice, the matrix Y and the homomorphism φ besides z.

2. *Arguments that only a compiler can settle.* The text reading lists letters that the statements bind themselves or that belong to other declarations of the same section, and that Lean presumably does not take: a natural number h in the definitions of $\mathrm{neg}_{D,j}(i)$, $\mathrm{Det}_D(j,\tau)$, $\Lambda_D(j)$, $\mathbf{c}_D(j)$ and $\eta_D(j,i,Y,\tau)$ and in Lemma 3.1 (b); a complex number k in Theorem A; the line choice and the matrix Y in Lemma 3.2 (c), where they are quantified inside. The five definitions therefore carry no notation made by the build, and the note writes them with the arguments that their uses show. None of this was confirmed by asking Lean.

3. *Constants that may vanish.* In Theorem C the constant C is not shown to be non-zero; that the numbers $c(\vartheta)$ are not zero is proved in the sister note from Theorem D.

4. *Truncated and integer exponents.* $t(D) - 9m - 1 - r$, $2z_p - 20$ and $3m - j$ are differences of natural numbers, truncated at zero in the formal text; $m/2$ is an integer because m is even; the exponent of v in $\mathrm{tt}_D(j,\zeta)(g)$ is the integer $m/2 - j$, and that in $\mathbf{c}_D(j)$ is an integer that may be negative.

5. *Theorem A.* The formal bound is the product of $\dim \mathrm{Row}(D)$ and of $k(D) \cdot (\log b \cdot B_0(D) - \mathsf{U}(D))$, with $\log b$ a field of the data. The family is indexed by the pairs

of a prime and a field isomorphism from $\mathbb{C}$ to an algebraic closure of $\mathbb{Q}_\ell$; the embeddings into $\mathbb{C}_\ell$ of the statements about gauges are obtained from them.

6. *What was not reconstructed.* In the proof of Theorem B the exact normalising factors (the powers of the class number, of the vacuum scale and of the denominator bounds) are balanced in the formal text by simplification; the proof above tracks the absolute values that matter and says where a factor is harmless. In Proposition 3.9 the function K_0 is not written in the formal text, Lean infers it; the formula above is my reading of what it must be. The referee of this note compared it with the terms that the formal proof constructs and found it in agreement.

7. *The sign of the ratio.* The formal ratio of a pair takes three arguments: the pair, the tame character and a sign, which has the default value 1 and multiplies the exponent of the tame character in the numerator. Theorem B and the claim in its proof are written in the formal text with the default; the note writes the sign 1 in the third slot. Clause (3) of fitness is stated for every sign, and the proof of Corollary 3.14 uses it with the sign of each line pair, -1 for the third.

8. *The index and the embedding in the proof of Theorem B.* The formal claim behind (1) is stated for an index of the family, a prime with a field isomorphism, and separately for an embedding into $\mathbb{C}_\ell$; the twist is that of the index. It is applied twice to one index, at the embedding of the isomorphism and at its composite with complex conjugation.

9. *Letters that differ from the formal text.* The scalar of the compressed Gram operator is k in the formal text and $\varkappa$ here; the number of the indicator jet is t there and t_{m} here; the class sum of the constant 1 over $\mathrm{U}(1)$ is N there and h_{U} here; the degrees of Lemmas 3.2 and 3.7 are d there and d here. The letters k, t, N and d have other meanings in this write-up.

How this note was written. From the dossier of the unit, whose reader calls the step claims of the long analytic proofs a first reading and notes that no second reader had looked at them. After the dossier was frozen, the second reader of the write-up compared the readings of all statements of the unit with the Lean text and sampled its structured proofs (her message to the writer of 26 November 2026); her corrections are followed where they concern this note: the first minimum is cut off at 1 (confirmed in the Lean definition); in the proof of isolation the approximant at angle zero is kept apart from the bounded target; the second family in Proposition 3.17 (a) is $\mathrm{gl}(\mathrm{gl}^*(\mathsf{h}_D(j)z_b))$ and the tables are transported before integrality is read off; the auxiliary table in Proposition 3.17 (b) carries the two complex conjugations; the size bound of Proposition 3.3 (ii) is stated for every tame character with empty set of primes, not only for the trivial one; and in the proof of Theorem A the power of the conductor has exponent zero and the local exponents carry no factor $\log \ell$. Her exact identity for the ratio of the volumes is the one derived above. I read completely the Lean files of the height of the lift volume, of the Hecke values, of the separation and of the approximation, and the statements and docstrings of the other three. The definitions of the line (Definitions 2.2 to 2.5) I took from the dossier's reading and compared with the docstrings; the referee of this note then compared them with the Lean text of the two files (the sizes of the index sets, the two index maps, the

index set of the determinants, the exponent of the line polynomial, the rows of the determinants, the datum, the line multiplier, the function $F^\natural$, the exponent and the characters, the line pairs, and the signs in the product of Hecke values) and found them in agreement. The proofs of Theorem B and of Proposition 3.9 I read line by line in the Lean text. For Theorem B I could reconstruct the argument: the choice of V, the two pigeonholes, the test form, the use of the three clauses of fitness and the bookkeeping of heights; the powers of the normalising constants I did not recompute. For Proposition 3.9 I could reconstruct the architecture (the near maximum of the carrier, the main term as an integral against holomorphic minors, the sparse theta sum, the reproducing formula and the finite Fourier expansion), not the estimates, which the formal proof closes with the inequalities between the scales. The proof of Theorem A is complete relative to its inputs and follows the formal one; I checked the identity of the volumes and the count of the heights by hand. I read the Lean definitions of a fit family, of denominators, of heights, of the budget for congruences and of the cost of one direction. The program check of the unit's dossier covered 634 leaf steps of 85 proofs.

Revision. This text was revised after the report of an independent referee (26 November 2026), who compared the statements with the Lean text and re-derived the complete proofs. The changes: the ratio of a pair carries its sign, as the formal definition does, in fitness, in Theorem B and in Corollary 3.14, where the sign -1 is needed; the partial L-function is that of the base change, without the Euler factors at the conjugate places as well; in the proof of Theorem B the claim on valuations separates the index from the embedding and the rank-one bound of the unit *Selmer* is used where clause (2) of fitness does not apply directly; in Proposition 3.17 (b) the Cauchy–Schwarz inequality is the conjugated one; the proofs of Lemmas 3.15 (b) and 3.7 (a) and of Proposition 3.6 are completed; "tested" and "denominators" name their idempotent; the references of the background results are those of the catalogue with the second reader's verdicts; and each proof says where it stands whether it is complete, an account or an outline.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Lemma 3.5	`one_le_maxOpNorm`	SparseThetaSum.lean	21–22
Lemma 3.5	`integral_mul_holMinorsPi_le`	SparseThetaSum.lean	30–56
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The four theorems as they are written in the artifact.

```
lemma heightLe_liftVolume_ratio {θ} (O : D.IsFitFamily θ) (j : D.LiftIndex) :
  HeightLe (D.rowDim * (D.k * (D.logB * D.B₀ - D.dirCost))) fun ι ↦
  D.liftVolume 1 j / D.liftVolume (θ ι) j

lemma heightLe_inv_ratio {θ} (o : D.IsFitFamily θ) (j : D.LiftIndex) (p : D.Pair) (hE :
  p.IsRankOne j) : HeightLe (6 * D.B₀ - D.dirCost) fun ι ↦ (p.ratio ((θ ι).powTwist j))⁻¹

lemma Data.cosetKernelSum_eq_fNatural {P} {D : Data P} {a} [Fintype a] j : ∃ C, ∃ c : D.Twist → ℂ,
  ∀ θ, ‖c θ‖ ^ 2 = C / heckeValueProd j θ ∧ ∀ (υ : Matrix _ a (D.untwistedModulus 2).ResRing) i g
  (w : Matrix a _ (D.untwistedModulus 2).ResRing), υ * w = i.1 → fNatural j i g = c θ *
  cosetKernelSum (D.auxPair θ j) (bigChar θ j) w (.trans .inl .inl) (lineMultiplier θ j i) g

lemma Data.eq_zero_of_integral_fNatural_tableTheta {P} {D : Data P} {j : D.LiftIndex} (F :
  D.goodForms (D.spaceW j)) (h : ∀ i ζ, ∫ g, conj (F.1.1 g * fNatural j i g) * tableTheta j ζ
  g ∂(D.spaceW j).quotMeasure = 0) : F = 0
```

In these texts `HeightLe` is "height at most", `D.IsFitFamily` fitness, `D.rowDim` the dimension of the rows, `D.logB` the number $\log b$, `D.B₀` the budget for congruences, `D.dirCost` the cost of one direction, `D.liftVolume` the lift volume, `p.ratio` the ratio $\rho_p(\vartheta, s)$, whose sign s is 1 where the formal text does not write it, `cosetKernelSum` the coset kernel sum and `D.goodForms` the good forms.

Notation

$P = (N_0, d)$, K, D, $E(D)$, $\mathcal{O}_E$ parameters; the imaginary quadratic field; data; its field; integers

x, x_1, b, m, $k(D)$, r, $t(D)$, $f(D)$, $g(D)$ scales; number of lifts; ranks; step; number of infinite places; second level

$\bar{\pi}_g$, M_D, b_1, b_2, $\mathfrak{g}(D)$, $\mathfrak{g}'(D)$ the integer of norm $g(D)$; the chosen group order; auxiliary bounds; the gap and the reduced half gap

$w_D(\tau)$, $w_j(\tau)$, $\mathrm{Ask}_D(z, \tau)$, $F_D(\ell)$ nominal weight; weight of the j-th lift; prime sum; forced part

$B_0(D)$, $\mathsf{U}(D)$, $\mathrm{js}_D(n)$, Ω, $\mathrm{budget}(D)$, $h_{\mathrm{main}}(D)$ budget for congruences; cost of one direction; jet size; period constant; budget; main amount

χ_D, ω_D, $\mu_D^\flat$, μ_j^0, υ the fixed characters; the character of the Weil data

τ, σ, ℓ, p (prime), ι, ι_p, φ, $\mathbb{C}_\ell$ an infinite place; automorphisms and embeddings; primes

j, ϑ, $Q(\vartheta)$, $N(Q(\vartheta))$, $\mathfrak{m}_D(\vartheta, c)$, θ lift index; twist; its primes and their norm; a modulus; a fit family

β_ℓ, β, $\mathcal{Z}$, r_0, $\mathrm{pr}_{\mathcal{Z}}$, pr denominator exponents; a subspace; a rank; orthogonal projections

$e_0(D)$, $\mathrm{idem}(\mathbf{c})$, $\mathbf{c}$, $\mathrm{idem}(X-1)$, e the good idempotent; the idempotent of a tuple of labels; the label idempotent; an idempotent

$W_j(D)$, $\mathcal{M}_D^\circ(W_j(D))$, $O_D(\vartheta, j)$, $\mathrm{lift}_D^*(\vartheta, j)$, u_ϑ, $\mathrm{vol}_D(\vartheta, j)$ objects of the first note

r, n, a, L, N, n' finite index sets

$\mathrm{U}(S)(\mathbb{A})$, $\mathbb{A}$, $\mathbb{A}_f$, $G_\infty(s)$, $\mathbf{K}_{3m}$, $\sum^{\mathrm{cl}}_{h \bmod \Gamma}$, Γ groups; adeles; levels; class sums

$\theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Psi, P](g, h)$, $\Pi_T(\nu, d)$, $L^T(z, \nu)$, T_ϑ theta kernel; product of L-values; partial L-function of the base change, without the Euler factors at T and at the conjugate places; the chosen primes and those of $Q(\vartheta)$

$\kappa^\pm$, κ, $\mathrm{pol}_\kappa(\xi)$, pol^-, $\|\xi\|_\kappa$ weights; polynomials and norms of coordinate vectors

$\mathcal{F}(O)$, $\mathrm{gauge}^\iota_\kappa(\epsilon; g, \xi)$, $\|\phi\|^2_\sigma$, $\mathrm{cks}_{O,b}(w, e; y)(g)$ forms; gauge; norm at σ; coset kernel sum

$\mathfrak{C}_{E,m}$, $\Xi^\circ_{m,c}(h)$ the carrier; its class sum

p, q, q_1, q_2, q_3 (pairs), ν_p, c_p, z_p, ψ_p pairs and their components

$\rho_p(\vartheta, s)$, s (sign), $\Xi_p(\vartheta)$, $I_p(\vartheta)$, δ_q, X, O_X ratio of the pair p for the tame character ϑ and the sign s; trivial lift; normalised value; denominator bound; an L-value datum and its pair

$\mathfrak{m}^\circ_D$, $\mathrm{fg}_\vartheta(x)$, $O^\circ_D(j)$ Definition 2.2 (1)

C_D, F_D, S_D, $\mathrm{Cl}^\circ(D)$, $\mathfrak{c}$, $h_\mathfrak{c}$ index sets; classes and representatives

$\mathsf{P}_D(j)$, $\mathsf{B}_D(j)$, $\mathsf{L}^+_D(j)$, $\mathrm{ps}_D(j)$ index sets of the j-th lift; signs

$\mathrm{bv}_D(i)$, $\mathrm{neg}_{D,j}(i)$, $\mathrm{pos}_{D,j}(h,i)$, $\mathrm{rg}_D(j)$, $\mathrm{pe}_D(j)$ indices and bijections

$\chi^{(j)}_D[\vartheta]$, $\mathsf{x}_D(j)$, $\mathbf{b}''_D(\vartheta, j)$, $\mathbf{c}_D(j)$ characters

$\mathrm{LC}(D)$, $\mathrm{sr}_D(i)$, $\mathrm{Mem}_D(j)$, i, i_1, i_2, i_c, i_l line choices, short rows, choices

$\mathrm{Det}_D(j,\tau)$, $\Lambda_D(j)$, $\eta_D(j,i,Y,\tau)$, $\mathrm{lc}_D(j,i,Y)$, $\Delta(j,i,Y)$, Δ_{i_2} determinants and coordinates of the line

$\mathrm{ld}_D(\vartheta, j; u)(z)$, $\mathrm{lm}_D(\vartheta, j, i, h)_l$, $F^\natural_D(j,i)(g)$, ψ', ψ'_t datum, line multiplier, $F^\natural$; additive characters

S, W', L^+, I^-_j, $I^\flat$, V'', V°, $\mathbf{1}_I$, $\mathbf{1}_{x_0}$, Θ^+_l spaces of the construction; indicators; a kernel

$\mathsf{e}_D(s)$, $\mathsf{e}^*(D)$, $\mathrm{lp}_D(j,o)$, $\mathrm{HV}_D(j,\vartheta)$, $\mathrm{tt}_D(j,\zeta)(g)$ Definition 2.5

$\mathrm{xs}_{q,\vartheta}(F)$, $\mathrm{cg}_\varkappa(L, \mathrm{gl})$, $\varkappa$, vf Definition 2.6; a complex scalar; the volume factor

$\mathrm{Tab}_D(j)$, ζ, ζ_a, γ, A, gl_ϑ, $\mathsf{h}_D(j)$ objects of Input 1

$\|h\|$, mh_k, λ_1, ν, ν_D, B, f_E, o, w, k (weights) objects of Setting 3.4

K_0, K, C, ε, $C(u,y)$, w°, c_a, Ψ, μ, q (integer), c_N, $\jmath$ objects of Propositions 3.6 and 3.9 and Lemma 3.7

G, V, h_U, $s(g)$, $\mathbf{m}$, $\mathbf{m}_\tau$, $t_\mathbf{m}$, α, x_0, κ_0, d_τ, Ξ, Ξ', y' objects of the proof of Theorem B

$\Delta(\vartheta)$, L_ϑ, n_0, H, T, U, u objects of the proof of Theorem A; a witness of a height; spaces and maps

z_ϑ, χ_ϑ, W_ϑ, $K^{(\vartheta)}$, I, n_j objects of the proof of isolation, with ϑ an angle; an exponent

$\mathrm{rp}_D(\tau, q)$, τ_σ, $\mathrm{sgn}(\sigma)$, $\tilde\sigma$ row parameters; the place and the sign of an embedding; the automorphism $\mathrm{conj}\circ\sigma\circ\mathrm{conj}$

N, t, d an integer in Lemmas 3.5 and 3.7; a matrix in Lemma 3.7; a degree

$c(\vartheta)$, c_1, c_2, C_1, C_2, u, w (matrices), ϕ, $\breve\Psi_0$, $\mathfrak{n}$ objects of Theorem C

F, g_0, y, z, a, e, l, h, d, c, c', v, Y, Q, H (integer), A, B (numbers), Λ, $\mathcal{N}$, m', $\delta_\mathfrak{c}$, $f_{a,e}$, z_b, ζ^σ, ψ, χ, ϵ further forms, points, indices, vectors, constants

L in "L-value" and "L-function"; also a subspace

$\mathbb{Q}$, $\mathbb{Q}_\ell$ the rationals; the ℓ-adic numbers

READBACK NOTE 16

Partner data, vanishing on standard data, and the exchange identity at the top of the tower

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. Let X be a hermitian space over a CM field and let V, V' be hermitian spaces whose ranks add up to the rank of X. The *exchange identity* says that the theta integrals of the pairs (X, V) and $(\mathsf{X}, \mathsf{V}')$, formed with data that are *partners* at a set of split places, are proportional, and that the square of the absolute value of the constant is a fixed number times a product of Hecke L-values. This note proves the identity at the top of a tower of spaces, for standard data (Theorem 1). The difference of the two integrals is an equivariant functional of the partner datum alone; such a functional vanishes on all standard data once certain Fourier coefficients of one of its values vanish (Theorem 2), and these coefficients are found by unfolding (Theorem 3). The note also proves two norm formulas for theta lifts to definite groups (Theorems 4 and 5).

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, attributed by its authors to L. Alpöge with the AI system Claude, whose final theorems are the Riemann hypothesis and its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles and that Lean's kernel accepted the proofs; the handout's record of checks says who checked what. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note is the first of three on one unit of the artifact, the unit of the exchange identity, which belongs to the proof of Theorem C of the handout (integral forms are long). It treats 47 of the unit's 305 declarations; the notes *The exchange theorem, one square of lifts, and the triangle of lifts and descents* and *The grid of lifts and descents, and the crude rows* treat the others. Every statement of the note is proved here from the structured proofs of the unit's dossier, which a program compared with the Lean text; what the proofs take from other units is recalled in §2 and listed as Inputs, and it is not proved here. The five theorems, the definitions and the statements of §3 and §4 were compared with their Lean statements by the writer; §9 says what was not. A second reader compared all statements of the unit and a passage of every structured proof with the Lean text after the dossier was made; where her reading differs from the dossier's, this note follows hers (§9). The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 47 declarations.

1 The statements

The notions used in the five theorems are recalled or defined in §2: hermitian spaces and their splittings, real data, split tame characters $\mathfrak{c}$ with their local characters $\chi_{\mathfrak{c}}$, the theta integral on the large side, the standard data $\check{\Psi}^{\mathrm{st}}_{\mathsf{e},\chi,\mathsf{z}}$ and $\Psi^{\mathrm{st}}_{\chi,\mathsf{z}}$, the products $\Pi(\mathfrak{c}, e)$ of L-values, and equivariant lifts. Four notions are defined later: the theta integral on the small side (Definition 4.1), the theta functional on real data (Definition 5.1), and the equal-rank ratio and the standard exchange identity (Definition 5.6).

Theorem 1 (the exchange identity at the top of the tower). *Let E be a CM field with a CM type and Weil data, and let X, W_1, W, V' be hermitian spaces over E with index sets $\mathbf{m}$, $\mathbf{n}$, $\mathbf{a}$, $\mathbf{b}$. Let $\mathfrak{S}$ be a base-adapted splitting of X over W_1 along $\mathbf{b}$ and $\mathfrak{S}_1$ a base-adapted splitting of W_1 over W along one line. Suppose that $|\mathbf{n}|$ is even. Let e' be an identification $\mathbf{b} \sqcup (\mathbf{b} \sqcup \mathbf{n}) \simeq \mathbf{m}$, let μ be weights, and let $\mathfrak{v}$ and $\mathfrak{v}_1$ be rapid real data for the pairs $(\mathsf{W}_1, \mathsf{V}')$ and $(\mathsf{W}_1, \mathsf{W}_1)$. Then there are real data $\mathfrak{x}$ for $(\mathsf{X}, \mathsf{V}' \oplus \mathsf{W}_1)$ and $\mathfrak{x}'$ for $(\mathsf{X}, \mathsf{V}')$ such that the slice of $\mathfrak{x}'$ at 0 for the splitting opposite to $\mathfrak{S}$ is $\mathfrak{v}$, and such that*

$$\mathrm{Exch}(\mathsf{X}, \mathsf{V}' \oplus \mathsf{W}_1, \mathsf{V}'; \mu, \mathsf{e}'; \mathfrak{x}, \mathfrak{x}'; \mathfrak{c} \mapsto \varrho(\mathsf{W}_1; \mathfrak{c}, \mathfrak{v}_1))$$

holds: $\mathfrak{x}$ and $\mathfrak{x}'$ are rapid, and there is a real number C such that every split tame character $\mathfrak{c}$ of weight μ has a complex number c with $\mathrm{C}\,|\mathrm{c}|^2 = \Pi(\mathfrak{c}, |\mathbf{n}|/2) \cdot \varrho(\mathsf{W}_1; \mathfrak{c}, \mathfrak{v}_1)$ and

$$\theta^{\mathrm{lg}}_{\mathsf{X},\mathsf{V}'\oplus\mathsf{W}_1}(\mathfrak{c}; \mathfrak{x}, \check{\Psi}^{\mathrm{st}}_{\mathsf{e}',\chi_{\mathfrak{c}},\mathsf{z}}) = \mathrm{c} \cdot \theta^{\mathrm{sm}}_{\mathsf{X},\mathsf{V}'}(\mathfrak{c}; \mathfrak{x}', \Psi^{\mathrm{st}}_{\chi_{\mathfrak{c}},\mathsf{z}})$$

for every integral $\mathbf{b} \times \mathbf{m}$ matrix z over the product of the completions of E at the places of $\mathfrak{c}$.

LEAN `exists_stdExchange_top`

Theorem 2 (vanishing on standard data). *Let E be a CM field with a CM type and Weil data, with the element $\jmath$; let X, W_1, W, V' be hermitian spaces over E with index sets $\mathbf{m}$, $\mathbf{n}$, $\mathbf{a}$, $\mathbf{b}$; let $\mathfrak{c}$ be a split tame character and χ' a multiplicative character of the product of the completions of E at the places of $\mathfrak{c}$. Let $(\mathcal{M}, \mathcal{D})$ be an equivariant lift for $\mathfrak{c}$, X and χ', let $\mathfrak{S}$ be a base-adapted splitting of X over W_1 along $\mathbf{b}$ and $\mathfrak{S}_1$ a base-adapted splitting of W_1 over W along one line. Write $\Psi_0 = \Psi^{\mathrm{st}}_{\chi_{\mathfrak{c}}, \mathfrak{A}'_3}$ for the standard datum of the block $\mathfrak{A}'_3$ of the frame of $\mathfrak{S}$ adapted to the places of $\mathfrak{c}$. Suppose that Ψ_0 lies in $\mathcal{M}$, and that the centre coefficient*

$$\mathrm{coef}_{\mathfrak{S}}(\jmath T; \mathcal{D}(\Psi_0))(\mathrm{rad}_{\mathfrak{S}}(\mathsf{c}, \mathsf{w})\ \mathrm{lev}_{\mathfrak{S}}(\mathsf{q}, \mathsf{g}))$$

vanishes for all T, c, w, q, g, where T is a $\mathbf{b} \times \mathbf{b}$ matrix over E and $\mathrm{rad}_{\mathfrak{S}}(\mathsf{c}, \mathsf{w})$, $\mathrm{lev}_{\mathfrak{S}}(\mathsf{q}, \mathsf{g})$ are a radical and a Levi element of the parabolic subgroup of $\mathfrak{S}$, such that one of the following holds:

(i) *T is the Gram matrix $\mathrm{J}_{\mathsf{V}'}$ of V', and there are an adelic $\mathbf{b} \times \mathbf{b}$ matrix Y and $\mathsf{k} \in \mathrm{GL}_{\mathbf{b}}(\hat{\mathcal{O}}_E)$ with $\mathsf{q} = \mathrm{sc}_E\, \mathrm{x}_{\mathsf{V}'}\, Y$ such that $Y_f\, \mathsf{k}$ is unitary for the identity form;*

(ii) *T is invertible and, as an adelic matrix, is not represented by the signs of V'.*

Then for every identification $\mathsf{e}' : \mathbf{b} \sqcup (\mathbf{b} \sqcup \mathbf{n}) \simeq \mathbf{m}$ and every integral z, that is, every $\mathbf{b} \times \mathbf{m}$ matrix z with entries in the integers of the product of the completions at the places of $\mathfrak{c}$: $\mathcal{D}(\Psi^{\mathrm{st}}_{\chi_{\mathfrak{c}}, \mathsf{z}}) = 0$.

LEAN `IsEquivariantLift.apply_stdColDatum_eq_zero`

Theorem 3 (the exchange identity from a period condition). *Let E be a CM field with a CM type and Weil data; let X, W_1, W, V' be hermitian spaces over E with index sets $\mathbf{m}$, $\mathbf{n}$, $\mathbf{a}$, $\mathbf{b}$, let $\mathfrak{S}$ be a base-adapted splitting of X over W_1 along $\mathbf{b}$ and $\mathfrak{S}_1$ a base-adapted splitting of W_1 over W along one line, and suppose that $|\mathbf{n}|$ is even. Let μ be weights, e' an identification $\mathbf{b} \sqcup (\mathbf{b} \sqcup \mathbf{n}) \simeq \mathbf{m}$, and let $\mathfrak{v}$ and $\mathfrak{v}'$ be rapid real data for $(\mathsf{X}, \mathsf{V}' \oplus \mathsf{W}_1)$ and $(\mathsf{X}, \mathsf{V}')$. Let l be a complex function on the split tame characters of weight μ, and Ω a real function on the real points of $\mathrm{U}(\mathsf{V}' \oplus \mathsf{W}_1)$ which is non-negative and does not vanish near 1. Assume:*
(the period condition) for every split tame character $\mathfrak{c}$ of weight μ, with $\omega = \omega_{\mathfrak{c}}^{-1} \upsilon^{|\mathbf{n}|/2}$, and every real point h of $\mathrm{U}(\mathsf{V}' \oplus \mathsf{W}_1)$ there is a real point p of $\mathrm{U}(\mathsf{V}') \times \mathrm{U}(\mathsf{W}_1)$ such that, for all $B \in \mathrm{GL}_{\mathbf{b}}(\mathbb{A}_E)$, with $\det_\infty B$ the idele that is $\det B$ at the infinite places and 1 at the finite ones, all characters ω', finite data F, adelic matrices x and points g, y:

$$\int_{[\mathrm{U}(\mathsf{W}_1)]} \theta^{\infty}_{\mathsf{W}_1, \mathsf{V}' \oplus \mathsf{W}_1}(\omega, \omega\omega'; F \boxtimes F_{\mathfrak{c}}, \tbinom{\mathsf{x}}{0}, (\mathsf{g}, \mathsf{y} \oplus \mathsf{u})) \Big(\omega_\infty(1, \mathsf{ph})\, \mathrm{sl}_{\mathfrak{S}}((\mathsf{ph})^{-1}\tbinom{B}{0}; \mathfrak{v})\Big)\, d\mathsf{u}$$
$$= \frac{\Omega(\mathsf{h})\, \mathrm{l}(\mathfrak{c})}{\omega_{\mathfrak{c}}^{\mathrm{bc}}(\det_\infty B)\, |\det_\infty B|^{|\mathbf{n}|/2}}\, \theta^{\infty}_{\mathsf{W}_1, \mathsf{V}'}(\omega, \omega'; F, \mathsf{x}, (\mathsf{g}, \mathsf{y}))(\mathrm{sl}_{\mathfrak{S}}(B; \mathfrak{v}'));$$

here $\omega_{\mathfrak{c}}^{\mathrm{bc}}$ is the base change of $\omega_{\mathfrak{c}}$, a character of the ideles of E, and $|\cdot|$ the idele absolute value; $F_{\mathfrak{c}}$ is the finite datum for $(\mathsf{W}_1, \mathsf{W}_1)$ named by $\Psi^{\mathrm{st}}_{\chi_{\mathfrak{c}}, 1}$ at the places of $\mathfrak{c}$, ω_∞ is the real Weil representation of weights $|\mathbf{n}|/2 - \mu$ and 0, and $\mathrm{sl}_{\mathfrak{S}}$ is the real slice;
(the support or sign condition) either every non-zero real slice $\mathrm{sl}_{\mathfrak{S}}(\mathsf{x}; \mathfrak{v})$ has a Gram matrix $\mathsf{x}^* \operatorname{diag}(\mathsf{V}' \oplus \mathsf{W}_1)\mathsf{x}$, with the diagonal form of the signs of $\mathsf{V}' \oplus \mathsf{W}_1$, that is represented by the signs of V'; or V' is negative definite, W_1 has a positive sign, and $2\,\mathrm{neg}(\mathsf{W}_1) \le |\mathbf{n}|$.
Then

$$\mathrm{Exch}(\mathsf{X}, \mathsf{V}' \oplus \mathsf{W}_1, \mathsf{V}'; \mu, \mathsf{e}'; \mathfrak{v}, \mathfrak{v}'; \mathfrak{c} \mapsto |\mathrm{l}(\mathfrak{c})|^2 / \Pi(\mathfrak{c}, |\mathbf{n}|/2))$$

holds.

LEAN `stdExchange_of_periods`

Theorem 4 (the inner product formula for the adjoint lift). *Let P be parameters, D admissible data for P and i a lift index, $1 \le i \le m$. There is a complex number $A \neq 0$ such that for every tame character ϑ of D the adjoint lift $\theta^*_{D,\vartheta,i}$, from the forms of principal level on $\mathsf{W}_i(D)$ to the algebraic forms on the definite space of the i-th auxiliary pair $\mathfrak{a}_D(\vartheta, i)$, is injective, and for all forms y', y of principal level on $\mathsf{W}_i(D)$*

$$\langle \theta^*_{D,\vartheta,i}\, y',\ \theta^*_{D,\vartheta,i}\, y \rangle = \sum_{\bar{\mathsf{g}}} A \cdot \mathrm{gram}(\mathfrak{a}_D(\vartheta, i)) \cdot \mathrm{ral}_{D,\vartheta,i}(\mathsf{W}_i(D); y', y; \bar{\mathsf{g}}),$$

the sum over the cosets $\bar{\mathsf{g}}$ of Definition 8.5 being unconditionally convergent. The number A does not depend on ϑ.

LEAN `Data.auxLift_injective_and_hasSum_inner`

Theorem 5 (the norm of the lift of the trivial character). *Let E be a CM field with a CM type and Weil data, $\mathbf{n}$ a finite index set and $\mathfrak{l}$ L-value data for E and $\mathbf{n}$, with character $\omega_{\mathfrak{l}}$, with chosen places and twist places, and with the set $\Sigma(\mathfrak{l})$ of all these places. Let e be a natural number with $2e = |\mathbf{n}|$ and $e > 1$. Put*

$$\mathrm{C} = \frac{\mathrm{vol}[\mathrm{U}(\mathsf{l}_{\mathbf{n}})] \cdot \mathrm{vol}[\mathrm{U}(\mathsf{l}_1)]}{\prod_{\mathfrak{p}} \left(1 - \mathrm{sg}(\mathfrak{p})\, \mathrm{N}\mathfrak{p}^{-e}\right)^{-1}},$$

the product over all finite places $\mathfrak{p}$ of E, with the signs $\mathrm{sg}(\mathfrak{p})$ of Setting 8.1. Then $\mathrm{C} \neq 0$, and the lift $\Xi(\mathfrak{l})$ of the trivial character from $\mathrm{U}(\mathsf{l}_1)$ to $\mathrm{U}(\mathsf{l}_{\mathbf{n}})$ satisfies

$$\langle \Xi(\mathfrak{l}), \Xi(\mathfrak{l}) \rangle = \mathrm{C} \cdot \prod_{\mathfrak{p}\ \text{chosen}} (\mathrm{N}\mathfrak{p} - 1) \cdot \mathrm{hook}(\mathfrak{l}) \cdot \mathrm{weyl}(\mathfrak{l}) \cdot L^{\Sigma(\mathfrak{l}) \cup \overline{\Sigma(\mathfrak{l})}}(e, (\omega_{\mathfrak{l}}^{-1})^{\mathrm{bc}}) \cdot \Big(\prod_{\mathfrak{q}\ \text{twist}} \mathrm{N}\mathfrak{q} \Big)^{|\mathbf{n}|-1},$$

the products over the chosen places and over the twist places of $\mathfrak{l}$. The L-value is that of §2: the Hecke L-function of the base change of $\omega_{\mathfrak{l}}^{-1}$ without the Euler factors at the places of $\mathfrak{l}$ and at their conjugates, at the point e. The number C depends only on E and $|\mathbf{n}|$; $\mathrm{hook}(\mathfrak{l})$ is the hook number of the weight of $\mathfrak{l}$ and $\mathrm{weyl}(\mathfrak{l})$ the Weyl index of Setting 8.1.

LEAN `LValueData.inner_trivialLift_self_eq_L`

Theorem 1 is the input of the second note of the unit, which removes hyperbolic planes from X to reach the exchange theorem for arbitrary complementary spaces and arbitrary partners. Theorems 2 and 3 are its proof. Theorem 4 is used in Chapter VIII (the determinants that measure congruences) and Theorem 5 in the second note (the norm of the lift from $\mathrm{U}(1)$ is an L-value).

2 The setting

Conventions. The natural numbers are $0, 1, 2, \ldots$. The letters of the handout keep their meaning: P are parameters, D admissible data, m the number of lifts, ϑ a tame character of D, τ an infinite place. The objects of this note that are local to the theory of theta series are set in other alphabets. Finite index sets are bold ($\mathbf{a}$, $\mathbf{b}$, $\mathbf{m}$, $\mathbf{n}$, $\mathbf{r}$, and $\mathbf{n}'$, $\mathbf{r}'$), $|\mathbf{n}|$ is the number of elements of $\mathbf{n}$, and an $\mathbf{a} \times \mathbf{n}$ matrix has rows indexed by $\mathbf{a}$ and columns by $\mathbf{n}$; $\mathrm{M}_{\mathbf{a}\times\mathbf{n}}(\mathsf{R})$ is the set of such matrices over a ring R, and $\mathrm{GL}_{\mathbf{n}}(\mathsf{R})$ the group of invertible $\mathbf{n} \times \mathbf{n}$ matrices. Sans-serif letters are a semilocal field F with its subring O, a commutative ring R, hermitian spaces, and matrices and points of groups; fraktur letters are real data ($\mathfrak{v}$, $\mathfrak{x}$), places ($\mathfrak{p}$), split tame character ($\mathfrak{c}$), splittings ($\mathfrak{S}$), integral frames ($\mathfrak{A}$), auxiliary pairs ($\mathfrak{a}$) and L-value data ($\mathfrak{l}$); upright letters (c, C) are constants. The implicit arguments of the declarations of the unit are of the following kinds, and each statement below introduces the ones it uses: a semilocal field with its subring; finite index sets, one of them non-empty where a statement says so; a commutative ring; a CM field E with a CM type and, where a statement says so, Weil data; signs and hermitian spaces; splittings, which are base-adapted where a statement says so; weights and characters;

natural numbers; real data, finite data and functions; sets of places and of matrices; matrices and points; forms; parameters and admissible data with a lift index.

Matrices over a semilocal field. A *semilocal field* is what the background library calls so: a locally compact commutative topological ring F with a Haar measure, a compact open subring O of volume 1 and an invertible, topologically nilpotent element $\varpi \in \mathsf{O}$, such that (O, F) has the elementary divisor property of a product of valuation rings. The example is the product of the completions of E at a finite set of finite places, with its integers. A matrix over F is *integral* if its entries lie in O; a matrix $\mathsf{y} \in \mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{F})$ *has a left inverse* if $\mathsf{s}\mathsf{y} = 1$ for some s, and is *primitive* if it is integral with an integral left inverse. A square matrix is primitive exactly when it lies in $\mathrm{GL}_{\mathbf{n}}(\mathsf{O})$. For a set $\mathcal{U}$ of matrices, $\mathcal{S}(\mathcal{U})$ is the space of locally constant, compactly supported complex functions that vanish outside $\mathcal{U}$ (Schwartz–Bruhat functions supported in $\mathcal{U}$); $\mathcal{S}^{\mathrm{inj}}$ is this space for the matrices with a left inverse and $\mathcal{S}^{\mathrm{sur}}$ for the matrices whose transpose has one.

Let $\mathsf{e}\colon \mathbf{b} \sqcup \mathbf{a} \simeq \mathbf{n}$ be an identification and $\mathsf{k} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{R})$. Then $\mathrm{col}_{\mathsf{e}}(\mathsf{k}) \in \mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{R})$ consists of the columns of k indexed by $\mathbf{b}$, and $\mathrm{row}_{\mathsf{e}}(\mathsf{k}) \in \mathrm{M}_{\mathbf{a}\times\mathbf{n}}(\mathsf{R})$ of the rows of k^{-1} indexed by $\mathbf{a}$; so $\mathrm{row}_{\mathsf{e}}(\mathsf{k})\,\mathrm{col}_{\mathsf{e}}(\mathsf{k}) = 0$. Let χ be a multiplicative character of R and $\mathsf{O}' \subseteq \mathsf{R}$ a subring. Complex functions $\check{\Psi}$ on $\mathrm{M}_{\mathbf{a}\times\mathbf{n}}(\mathsf{R})$ and Ψ on $\mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{R})$ are *partners for χ over O' with respect to* e if $\check{\Psi}$ vanishes off the matrices $\mathrm{row}_{\mathsf{e}}(\mathsf{k})$ and Ψ off the matrices $\mathrm{col}_{\mathsf{e}}(\mathsf{k})$, $\mathsf{k} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{O}')$, and

$$\check{\Psi}(\mathrm{row}_{\mathsf{e}}(\mathsf{k})) = \chi(\det \mathsf{k})\,\Psi(\mathrm{col}_{\mathsf{e}}(\mathsf{k})) \qquad \text{for all } \mathsf{k} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{O}').$$

The kernel of a surjective $\mathbf{a}\times\mathbf{n}$ matrix is the column space of an injective $\mathbf{n}\times\mathbf{b}$ matrix, and partners are functions that correspond under this duality of Grassmannians, twisted by χ.

For a multiplicative character χ of F let $\chi^\flat$ be the function equal to χ^{-1} on the units of O and to 0 elsewhere. The character χ is *tame* if there are a finite ring $\bar{\mathsf{R}}$, a surjective homomorphism $\mathsf{O} \to \bar{\mathsf{R}}$ with kernel $\varpi\mathsf{O}$ and a primitive multiplicative character $\bar{\chi}$ of $\bar{\mathsf{R}}$ such that $\chi^\flat$ is $\bar{\chi}$ composed with the reduction $\mathsf{z} \mapsto \bar{\mathsf{z}}$. Here a multiplicative character $\bar{\chi}$ of a finite commutative ring $\bar{\mathsf{R}}$ (extended by 0 to the non-units) is *primitive* if for every $\mathsf{a} \neq 0$ in $\bar{\mathsf{R}}$ there is c with $\bar{\chi}(1+\mathsf{c}\mathsf{a}) \notin \{0,1\}$, and an additive character $\bar{\mathrm{e}}$ of $\bar{\mathsf{R}}$ is *primitive* if for every $\mathsf{a} \neq 0$ there is c with $\bar{\mathrm{e}}(\mathsf{c}\mathsf{a}) \neq 1$, that is, if it is non-trivial on every non-zero ideal. For an integral $\mathsf{s} \in \mathrm{M}_{\mathbf{b}\times\mathbf{n}}(\mathsf{F})$ the *standard column datum* and, for an identification $\mathsf{e}\colon \mathbf{b} \sqcup \mathbf{a} \simeq \mathbf{n}$, the *standard row datum* are the functions

$$\Psi^{\mathrm{st}}_{\chi,\mathsf{s}}(\mathsf{y}) = \chi^\flat(\det(\mathsf{s}\mathsf{y})), \qquad \check{\Psi}^{\mathrm{st}}_{\mathsf{e},\chi,\mathsf{s}}(\mathsf{x}) = \chi^\flat\Big(\det \tbinom{\mathsf{s}}{\mathsf{x}}\Big)$$

of an integral $\mathsf{y} \in \mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{F})$ and of an $\mathsf{x} \in \mathrm{M}_{\mathbf{a}\times\mathbf{n}}(\mathsf{F})$ for which the square matrix with the rows of s and of x, ordered by e, is integral; they are 0 at the other matrices. They are partners for χ over O with respect to e (Input 1). An *integral frame* $\mathfrak{A}$ for $(\mathbf{m},\mathbf{n},\mathbf{b})$ is an integral invertible $\mathbf{m} \times (\mathbf{b} \sqcup \mathbf{n} \sqcup \mathbf{b})$ matrix with integral inverse, cut into column blocks $(\mathfrak{A}_1 \mid \mathfrak{A}_2 \mid \mathfrak{A}_3)$, the inverse being cut into row blocks $\mathfrak{A}'_1$, $\mathfrak{A}'_2$, $\mathfrak{A}'_3$; for a function G, the function $\mathrm{st}_{\mathfrak{A}}(G)$ of an $\mathbf{a}\times\mathbf{m}$ matrix x is $G(\mathsf{x}\,(\mathfrak{A}_1 \mid \mathfrak{A}_2))$ if x is integral and 0 otherwise. An additive character e_{F} of F *has conductor* O if it is continuous and is trivial on $\mathsf{z}\mathsf{O}$ exactly when $\mathsf{z} \in \mathsf{O}$, for elements z of F.

Hermitian spaces, data and theta integrals. E is a CM field, E^+ its maximal totally real subfield, $\mathbb{A}_E$ its ring of adeles and $\hat{\mathcal{O}}_E$ the ring of integral finite adeles. A *CM type* chooses one complex embedding above each infinite place τ. *Weil data* for E consist of: the hypothesis that E contains a primitive root of unity of order $5040 = 7!$ (the formal text calls such an E *stably unramified*; then $\sqrt{-1} \in E$, so $E = E^+(\sqrt{-1})$ is unramified over E^+ at every odd place); an element $\jmath \in E$ with $\bar{\jmath} = -\jmath$, with positive imaginary part at every embedding of the CM type and with $(1 - \jmath^{-1})/2$ integral; a finite idele that makes the integral adeles self-dual for the standard additive character $\mathrm{e}_{\mathbb{A}}$ twisted by $\jmath$ (from it come the scale idele sc_E and the idele of the inverse different); and a Hecke character υ of E (a continuous unitary character of the ideles, trivial on $E^\times$), the *splitting character*, with $\upsilon(\mathsf{x}) = \prod_\tau \mathsf{x}_\tau/|\mathsf{x}_\tau|$ when the finite part of the idele x is a unit, with $\upsilon(\bar{\mathsf{x}}) = \upsilon(\mathsf{x})^{-1}$, and with $\upsilon(\varpi_{\mathfrak{p}}) = -1$ for a uniformiser $\varpi_{\mathfrak{p}}$ at every finite place $\mathfrak{p} = \bar{\mathfrak{p}}$. The same letter υ denotes its restriction to the norm-one ideles $\mathrm{U}(1)(\mathbb{A})$; a power of υ next to a character of $\mathrm{U}(1)(\mathbb{A})$ is a power of the restriction, and the local character of a power of υ (below) is formed with the Hecke character. A *hermitian space* W over E with index set $\mathbf{n}$ and signs (a function from $\mathbf{n}$ to $\{\pm 1\}$) is given by a hermitian Gram matrix $\mathrm{J}_{\mathsf{W}} \in \mathrm{GL}_{\mathbf{n}}(E)$ and an adelic frame $\mathrm{x}_{\mathsf{W}} \in \mathrm{GL}_{\mathbf{n}}(\mathbb{A}_E)$ in which the form is the diagonal form of the signs; $\mathrm{neg}(\mathsf{W})$ is the number of negative signs, $\mathrm{U}(\mathsf{W})$ the unitary group, $[\mathrm{U}(\mathsf{W})] = \mathrm{U}(\mathsf{W})(E^+)\backslash \mathrm{U}(\mathsf{W})(\mathbb{A})$, and $\mathcal{K}_f$ the stabilizer of the integral lattice in the finite adelic points. The measure on $[\mathrm{U}(\mathsf{W})]$ is the one induced by the following Haar measure of $\mathrm{U}(\mathsf{W})(\mathbb{A})$: the product of the measure of the formal text at the infinite places (it calls it the compact-dual measure) and of the Haar measure of the finite adelic points that gives $\mathcal{K}_f$ the mass 1; integrals over the quotient are computed with a cut-off for the counting measure on the rational points. $\mathrm{vol}[\mathrm{U}(\mathsf{W})]$ is the total mass. All constants of this note (R in the proof of Theorem 3, the volumes in Theorem 5 and in Input 6) refer to this normalisation. $\mathsf{I}_{\mathbf{n}}$ is the standard positive definite space on $\mathbf{n}$, and $\mathsf{V}' \oplus \mathsf{W}$ the orthogonal sum.

For hermitian spaces W, V and characters ω, ω' of $\mathrm{U}(1)(\mathbb{A})$ the Weil representation $\omega_{\mathsf{W},\mathsf{V}}(\omega, \omega')$ of $\mathrm{U}(\mathsf{W})(\mathbb{A}) \times \mathrm{U}(\mathsf{V})(\mathbb{A})$ acts on the space of *data* of the pair, the restricted tensor product of the models of the Heisenberg representation of $\mathrm{Hom}(\mathsf{W}, \mathsf{V})$. A datum of the form $\mathfrak{v} \otimes F$ has a *real datum* $\mathfrak{v}$, a vector of the Fock-type model at the infinite places, and a *finite datum* F; $\mathfrak{v}$ is *rapid* if all its weighted norms are finite. With the theta functional θ, the *theta kernel* is $\theta_{\mathsf{W},\mathsf{V}}(\omega, \omega'; f)(\mathsf{g}, \mathsf{h}) = \theta(\omega_{\mathsf{W},\mathsf{V}}(\omega, \omega')(\mathsf{g}, \mathsf{h}) f)$, and the *theta integral* of a datum f is the function on $\mathrm{U}(\mathsf{W})(\mathbb{A})$

$$\mathrm{I}_{\mathsf{W},\mathsf{V}}(\omega, \omega'; f)(\mathsf{g}) = \int_{[\mathrm{U}(\mathsf{V})]} \theta_{\mathsf{W},\mathsf{V}}(\omega, \omega'; f)(\mathsf{g}, \mathsf{h})\, d\mathsf{h}.$$

A *split tame character* $\mathfrak{c}$ of weight μ ($\mu(\tau) > 0$ an integer for each infinite place τ) is a character $\omega_{\mathfrak{c}}$ of $\mathrm{U}(1)(E^+)\backslash \mathrm{U}(1)(\mathbb{A})$ of weight μ whose tame part is non-trivial exactly at the places of a non-empty finite set $\Sigma(\mathfrak{c})$ of finite places of E that contains the conjugate of none of its members. Its places split in E/E^+, and $\mathsf{F}_{\mathfrak{c}} = \prod_{\mathfrak{p} \in \Sigma(\mathfrak{c})} E_{\mathfrak{p}}$ is a semilocal field. The *base change* of a character ω of $\mathrm{U}(1)(\mathbb{A})$ is the Hecke character $\omega^{\mathrm{bc}}(\mathsf{x}) = \omega(\mathsf{x}/\bar{\mathsf{x}})$ of E; we write $\omega_{\mathfrak{c}}^{\mathrm{bc}}$ for that of $\omega_{\mathfrak{c}}$. For a Hecke character ξ of E, a finite set Σ of finite places and an integer e, the *local character of ξ in rank e* is the multiplicative character of $\mathsf{F}_\Sigma = \prod_{\mathfrak{p} \in \Sigma} E_{\mathfrak{p}}$ that is $\mathsf{z} \mapsto \xi(\mathsf{z})\,|\mathsf{z}|^{e/2}$ on the invertible z, read as ideles supported on Σ, with $|\mathsf{z}|$ the product of the normalised absolute

values, and 0 on the others. Then $\chi_{\mathfrak{c}}$ is the local character of $\omega_{\mathfrak{c}}^{\mathrm{bc}}$ in rank 0 at $\Sigma(\mathfrak{c})$, and $\chi_{\mathfrak{c}}^{(e)}(\mathsf{z}) = \chi_{\mathfrak{c}}(\mathsf{z})\,|\mathsf{z}|^{e/2}$ is its local character in rank e. All $\chi_{\mathfrak{c}}^{(e)}$ are tame, and they agree with $\chi_{\mathfrak{c}}$ on the units of the integers of $\mathsf{F}_{\mathfrak{c}}$. For a function $\check{\Psi}$ on matrices over $\mathsf{F}_{\mathfrak{c}}$ and a real datum $\mathfrak{v}$, the *local datum* $\mathfrak{v} \otimes [\check{\Psi}]$ has real part $\mathfrak{v}$, is the function $\check{\Psi}$ of the components at $\Sigma(\mathfrak{c})$, and is the vacuum at the other finite places; $\mathfrak{v} \otimes [\check{\Psi}]^{c}$ is the same with the components at the conjugate places, read through complex conjugation. For hermitian spaces W (index set $\mathbf{n}$) and V (index set $\mathbf{a}$) and $\check{\Psi}$ on $\mathrm{M}_{\mathbf{a}\times\mathbf{n}}(\mathsf{F}_{\mathfrak{c}})$,

$$\theta^{\mathrm{lg}}_{\mathsf{W},\mathsf{V}}(\mathfrak{c}; \mathfrak{v}, \check{\Psi}) = \mathrm{I}_{\mathsf{W},\mathsf{V}}(\omega_{\mathfrak{c}}^{-1}\upsilon^{\lfloor|\mathbf{n}|/2\rfloor}, \omega_{\mathfrak{c}}^{-1}\upsilon^{\lfloor|\mathbf{n}|/2\rfloor}; \mathfrak{v} \otimes [\check{\Psi}])$$

is the theta integral *on the large side*.

The partial L-function. For a set Σ of finite places of E, with $\bar{\Sigma}$ the set of the conjugate places, and a character ω of $\mathrm{U}(1)(\mathbb{A})$, $L^{\Sigma\cup\bar{\Sigma}}(s, \omega^{\mathrm{bc}})$ is the Hecke L-function of the base change ω^{bc} without the Euler factors at the places of $\Sigma \cup \bar{\Sigma}$: for real $s > 1$ the Euler product over the finite places $\mathfrak{p} \notin \Sigma \cup \bar{\Sigma}$ of $(1 - \omega^{\mathrm{bc}}(\varpi_{\mathfrak{p}})\,\mathrm{N}\,\mathfrak{p}^{-s})^{-1}$, with $\varpi_{\mathfrak{p}}$ the chosen uniformiser at $\mathfrak{p}$ read as an idele, and at $s = 1$ its limit from the right. This is the only L-function of the three notes of the unit, and it is always written so. For a finite Σ and a natural number e put

$$\Pi(\Sigma, \omega, e) = \Big(\prod_{\mathfrak{p}\in\Sigma} \mathrm{N}\,\mathfrak{p}\Big)^{e^2} \prod_{i=1}^{e} \big|L^{\Sigma\cup\bar{\Sigma}}(i, \omega^{\mathrm{bc}})\big|^2$$

(the norms run over Σ only; both products are 1 for $e = 0$), and for an integer e

$$\Pi(\mathfrak{c}, e) = \Pi(\Sigma(\mathfrak{c}), \omega_{\mathfrak{c}}, e) \quad (e \geq 0), \qquad \Pi(\mathfrak{c}, e) = \Pi(\Sigma(\mathfrak{c}), \omega_{\mathfrak{c}}, -e)^{-1} \quad (e < 0).$$

Splittings and centre coefficients. A *splitting* $\mathfrak{S}$ of X over W_1 along $\mathbf{b}$ writes X as the orthogonal sum of $|\mathbf{b}|$ hyperbolic planes, spanned by two isotropic frames in duality indexed by $\mathbf{b}$, and of a copy of W_1; it is *base-adapted* when its frames are the standard ones in the adelic frames of the two spaces. The *opposite* splitting exchanges the two isotropic frames. The stabilizer of the first frame is a parabolic subgroup with radical elements $\mathrm{rad}_{\mathfrak{S}}(\mathsf{c}, \mathsf{w})$ (w in the centre of the unipotent radical, a space of hermitian $\mathbf{b} \times \mathbf{b}$ matrices) and Levi elements $\mathrm{lev}_{\mathfrak{S}}(\mathsf{q}, \mathsf{g})$, $\mathsf{q} \in \mathrm{GL}_{\mathbf{b}}(\mathbb{A}_E)$, $\mathsf{g} \in \mathrm{U}(\mathsf{W}_1)(\mathbb{A})$. For a function f on $\mathrm{U}(\mathsf{X})(\mathbb{A})$ and a $\mathbf{b} \times \mathbf{b}$ matrix T, the *centre coefficient* $\mathrm{coef}_{\mathfrak{S}}(T; f)$ is the Fourier coefficient of f along the centre of the radical at the character $\mathsf{w} \mapsto \mathrm{e}_{\mathbb{A}}(\mathrm{tr}(T\mathsf{w}))$. In the mixed model of $\mathfrak{S}$ the theta kernel of (X, V) is a sum, over the rational matrices v that are the images of the first frame, of *theta terms*, and the datum enters a term through its *real slice* $\mathrm{sl}_{\mathfrak{S}}(\mathsf{x}; \mathfrak{v})$, a real datum of the smaller pair. A reading of the components at a set of places Σ attaches to a base-adapted $\mathfrak{S}$ an integral frame, the *adapted frame*. An adelic matrix T is *represented by* signs t if at every infinite place it has the form $\mathsf{c}^{*}\,\mathrm{diag}(\mathsf{t})\,\mathsf{c}$.

Equivariant lifts. Let $\mathfrak{c}$ be a split tame character, X a hermitian space with index set $\mathbf{m}$ and χ' a multiplicative character of $\mathsf{F}_{\mathfrak{c}}$. A pair $(\mathcal{M}, \mathcal{D})$ of a space $\mathcal{M}$ of complex functions on $\mathrm{M}_{\mathbf{m}\times\mathbf{b}}(\mathsf{F}_{\mathfrak{c}})$ and a linear map $\mathcal{D}$ from such functions to functions on $\mathrm{U}(\mathsf{X})(\mathbb{A})$ is an *equivariant lift* for $\mathfrak{c}$, X and χ' if $\mathcal{M} \subseteq \mathcal{S}^{\mathrm{inj}}$; each $\mathcal{D}(\Psi)$, $\Psi \in \mathcal{M}$, is an automorphic function; for every finite adelic point g supported at $\Sigma(\mathfrak{c})$ and $\Psi \in \mathcal{M}$ the translate $\Psi(\mathsf{g}_{\Sigma}\,\cdot\,)$ lies in $\mathcal{M}$ and $\mathcal{D}(\Psi(\mathsf{g}_{\Sigma}\,\cdot\,))(\mathsf{x}\mathsf{g}) = \chi'(\det \mathsf{g}_{\Sigma})\,\mathcal{D}(\Psi)(\mathsf{x})$; and every $\mathcal{D}(\Psi)$, $\Psi \in \mathcal{M}$, is invariant under the elements of $\mathcal{K}_f$ that are trivial at $\Sigma(\mathfrak{c})$.

Inputs. The proofs below take the following for granted. The numbers are the units of the reading; the first eight are units of the background library, whose results this write-up quotes, the last four are units of the argument. References are those of the handout's catalogue; none of their places has been confirmed yet.

Input 1 (unit 103, Gauss sums and standard data). The standard row and column data of an integral s are partners; the blocks of an integral frame satisfy $\mathfrak{A}_1'\mathfrak{A}_1 = 1$, $\mathfrak{A}_2'\mathfrak{A}_2 = 1$, $\mathfrak{A}_3'\mathfrak{A}_3 = 1$ and the six mixed products vanish; the *matrix Gauss sum*: for a primitive multiplicative character $\bar{\chi}$ of a finite ring $\bar{\mathsf{R}}$ and a primitive additive character $\bar{\mathrm{e}}$ and every square matrix $\bar{T}$ over $\bar{\mathsf{R}}$,

$$\sum_{\bar{\mathsf{z}}} \bar{\chi}(\det \bar{\mathsf{z}})\, \bar{\mathrm{e}}(\mathrm{tr}(\bar{T}\bar{\mathsf{z}})) = \bar{\chi}^{-1}(\det \bar{T}) \sum_{\bar{\mathsf{z}}} \bar{\chi}(\det \bar{\mathsf{z}})\, \bar{\mathrm{e}}(\mathrm{tr}\, \bar{\mathsf{z}})$$

(relation to the literature unresolved: T. Kondo, J. Math. Soc. Japan 15 (1963), pp. 244–245, has the law over a finite field at invertible $\bar{T}$; the statement over a finite ring and at all $\bar{T}$ is wider than the source); invariance of functionals under shears.

Input 2 (units 112, 120 and 105, linear algebra, Witt's theorem and a fact on compact groups). A matrix is primitive exactly when it is $\mathrm{col}_{\mathsf{e}}(\mathsf{k})$ for some $\mathsf{k} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{O})$; Jacobi's identity for complementary minors, $\det\left(\begin{smallmatrix}\mathsf{s}\\ \mathrm{row}_{\mathsf{e}}(\mathsf{k})\end{smallmatrix}\right) \det \mathsf{k} = \det(\mathsf{s}\ \mathrm{col}_{\mathsf{e}}(\mathsf{k}))$; a matrix with a left inverse becomes primitive after right multiplication by an invertible matrix; Witt's extension theorem over a field with involution and over $\hat{\mathcal{O}}_E$ (U. First, J. Pure Appl. Algebra 219 (2015); the theorem, and a special case); for stably unramified E (a hypothesis that is part of the Weil data), a unimodular hermitian matrix b over $\hat{\mathcal{O}}_E$ becomes the identity after a change of basis: $\mathsf{k}^*\mathsf{b}\,\mathsf{k} = 1$ for some $\mathsf{k} \in \mathrm{GL}_{\mathbf{b}}(\hat{\mathcal{O}}_E)$ (R. Jacobowitz, Amer. J. Math. 84 (1962); a special case; without the hypothesis the statement fails at places where E/E^+ is ramified); the integral over a compact group of a locally constant family of elements of a space of functions is a finite combination of its members (unit 105).

Input 3 (units 43, 38 and 37, dual pairs and the mixed model). Theta integrals of target-bounded data are automorphic and depend linearly on the datum; local data of rapid real data and Schwartz–Bruhat functions on surjective matrices are target-bounded; $\mathrm{I}_{\mathsf{W},\mathsf{V}}(\omega,\omega';\omega_{\mathsf{W},\mathsf{V}}(\omega,\omega')(\mathsf{g},\mathsf{h})f) = \mathrm{I}_{\mathsf{W},\mathsf{V}}(\omega,\omega';f)(\cdot\,\mathsf{g})$; the action of $\mathcal{K}_f$ and of points supported at Σ on local data; the centre coefficient of a theta integral at $\jmath T$ is the sum over the rational v with $\mathsf{v}^*\mathsf{J}\mathsf{v} = T$ of the integrals of the theta terms, and for invertible T an integral over the quotient by the stabilizer of one v; the laws of theta terms under radical and Levi elements; existence of sections in the mixed model; the linear functional "centre coefficient of $\mathcal{D}(\Psi)$" of an equivariant lift and its behaviour under shears.

Inputs (continued).

Input 4 (units 42, 98, 10 and 123, singular forms). *Freitag's lemma*: under invariance, radical and base conditions, the centre coefficient of an automorphic function at an invertible index vanishes at points supported at Σ once it does at unimodular indices (E. Freitag, LNM 1487 (1991); J.-S. Li, Amer. J. Math. 119 (1997); not a literal theorem of these sources); *Landherr's theorem* for spaces with a point and *weak approximation* for unitary groups (W. Landherr, Abh. Math. Sem. Hamburg 11 (1936); M. Kneser, TIFR lectures 47 (1969), §5.5; special cases); a continuous periodic function on adelic matrices with vanishing Fourier coefficients is zero.

Input 5 (units 131, 17 and 114, the tower). The *tower property*: the constant term of a theta integral on the large side along an isotropic line is a multiple of the theta integral of the smaller pair (after S. Rallis, Compositio Math. 51 (1984); an analogue); the *constancy criterion*, a form of strong approximation: let W be a hermitian space with a splitting $\mathfrak{S}_1$ over a space W_0 along one line, Σ a non-empty set of finite places, and f a continuous automorphic function on $\mathrm{U}(\mathsf{W})(\mathbb{A})$ that is invariant on the right under the finite adelic points supported at Σ and under the elements of $\mathcal{K}_f$ that are trivial at Σ; suppose that, if f is invariant on the left under all radical elements $\mathrm{rad}_{\mathfrak{S}_1}(\mathsf{c},\mathsf{w})$, then $f(\mathrm{lev}_{\mathfrak{S}_1}(\mathsf{q},\mathsf{g}))$ does not depend on the torus coordinate q; then f is constant (M. Kneser, J. reine angew. Math. 218 (1965), §1 and Satz 2; a weaker form, a consequence of strong approximation for the special unitary group with the hypothesis on the torus); *unfolding in stages*: let $\mathcal{G}_1 \times \mathcal{G}_2 \to \mathcal{G}$ be a closed embedding of a product of unimodular groups in a unimodular group, written $(\mathsf{l},\mathsf{a}) \mapsto \mathsf{la}$, let $\mathcal{G}_0 \subset \mathcal{G}_2$ be a discrete subgroup and Ω a non-negative function on $\mathcal{G}$ that does not vanish near some point; then there is $\mathrm{R} \neq 0$ such that $\int_{\mathcal{G}_0\backslash\mathcal{G}} f = \mathrm{R}\,\mathrm{c}$ for every continuous integrable function f on $\mathcal{G}$, invariant on the left under $\mathcal{G}_0$, and every number c with the following property: for every $\mathsf{g} \in \mathcal{G}$ there is a shift $\mathsf{k} \in \mathcal{G}_1 \times \mathcal{G}_2$ with $\int_{\mathcal{G}_1}\int_{\mathcal{G}_0\backslash\mathcal{G}_2} f(\mathsf{l}\,\mathsf{a}\,\mathsf{k}\,\mathsf{g})\,d\mathsf{a}\,d\mathsf{l} = \Omega(\mathsf{g})\,\mathrm{c}$ (reference to be confirmed); the averaging criterion for the vanishing of an integral over a quotient; $\Pi(\mathfrak{c}, e) > 0$.

Inputs (continued).

Input 6 (units 26 and 27, the inner product formula). The inner product formula for theta lifts (the *Rallis inner product formula*) in two forms. Doubling form: for $\operatorname{rk} \mathsf{V} > 2 \operatorname{rk} \mathsf{W}$, a clean datum f, any datum f' and rapidly decreasing automorphic y, y',

$$\langle \theta_{f'} y', \theta_f y \rangle = \operatorname{vol}[\mathrm{U}(\mathsf{V})] \int_{\mathrm{U}(\mathsf{W})(\mathbb{A})} \langle f', \omega(\mathsf{g},1) f \rangle \, \langle y', R(\mathsf{g}) y \rangle_{\mathrm{Pet}} \, d\mathsf{g}$$

(relation to the literature unresolved: the identity and its rank range are those of J.-S. Li, J. reine angew. Math. 428 (1992), §2, Thm 2.1, (24)–(27), p. 184, for cuspidal vectors and Schwartz data; the test spaces and the normalisation of the volume differ; S. Rallis, LNM 1245 (1987), is background). Euler form: for an edge and $2 \operatorname{rk} \mathsf{W} < \operatorname{rk} \mathsf{V}$ there is $A_0 \neq 0$ such that for all characters of the weights of the edge and every finite datum F that is sharp of a level with constant c_F, the lift of forms of that level is injective and $\langle \theta y', \theta y \rangle = \sum_{\bar{\mathsf{g}}} A_0 \, \mathrm{c}_F \, \mathrm{ct}(\bar{\mathsf{g}})$, a sum of coset terms (relation to the literature unresolved: S. Rallis, Compositio Math. 52 (1984), §1 and Thm 1.6, treats another dual pair and is an analogue only; the unitary inner product in this rank range is J.-S. Li, loc. cit., §2, Thm 2.1, eq. (26), p. 184; the constant uniform in the characters and the sharp data, the evaluation as a sum of coset terms and the injectivity go beyond both). Sharp data: a datum $F \otimes$vacuum whose function F modulo a modulus has a fixed left inverse on its support is sharp, with constant $\sum |F|^2$ divided by the numbers of matrices and of invertible matrices modulo the modulus; the local doubling sum at one place; partial integrals.

Input 7 (units 44, 25, 96, 104 and 102, Hecke transfer, lifts and forms). *Hecke transfer*: an equivariant map from data to functions sends the unramified vacuum at a split place to a spherical vector, and the Hecke polynomials of the two sides are related by $T \mapsto \mathrm{tr}_{\mathfrak{p},e}(\mathrm{a}_{\mathfrak{p}} T) \cdot \mathrm{pol}(\mathrm{N}\,\mathfrak{p}^e \mathrm{a}_{\mathfrak{p}} T / \mathrm{b}_{\mathfrak{p}})$, with the Frobenius values $\mathrm{a}_{\mathfrak{p}}$, $\mathrm{b}_{\mathfrak{p}}$ of the two characters (relation to the literature unresolved: A. Mínguez, Ann. Sci. Éc. Norm. Sup. 41 (2008), Thm 1 and Cor. 6.3, gives the correspondence of the parameters at split places; the identity of operators and its normalisation by the splitting character need a further comparison); an edge gives arrows in both directions, and the adjoint of the lift along an arrow is the conjugate of the descent; Euler products converge for exponents > 1; a matrix over the residue ring of a product of distinct places is the family of its reductions at these places (unit 104); a form is spherical with a given Hecke polynomial exactly when its algebraic form is, and the Petersson product is invariant under translation (unit 102).

Input 8 (units 134 Forms, 138 Selmer, 140 Trace, 141 Seesaw, and the second note of this unit). Auxiliary pairs, their data and class-sum lifts, and the facts on them that Setting 8.1 and the proofs of §8 name; the chain of lifts and its arrows; L-value data and the lift $\Xi(\mathfrak{l})$ with its elementary properties (the second note).

3 Partners at the split places

In this section F is a semilocal field with subring O. The functions on the two sides of the exchange identity live on $\mathbf{a} \times \mathbf{n}$ and on $\mathbf{n} \times \mathbf{b}$ matrices; they are compared through their averages over $\mathrm{GL}_{\mathbf{a}}(\mathsf{F})$ and $\mathrm{GL}_{\mathbf{b}}(\mathsf{F})$, which for want of a convergent integral are defined by a choice of coset representatives.

Lemma 3.1 (coset representatives)**.** *Let F be a semilocal field with subring O and let $\mathbf{b}$, $\mathbf{n}$ be finite index sets. There are a linear functional $\mathcal{L}_0$ on the space of all complex functions on $\mathrm{GL}_{\mathbf{b}}(\mathsf{F})$ and a set $\mathcal{R} \subset \mathrm{GL}_{\mathbf{b}}(\mathsf{F})$ of representatives of the cosets of $\mathrm{GL}_{\mathbf{b}}(\mathsf{F})$ modulo $\mathrm{GL}_{\mathbf{b}}(\mathsf{O})$ such that*

(1) *for every finite subset $\mathcal{R}_0 \subseteq \mathcal{R}$ and every function G that vanishes on $\mathcal{R} \setminus \mathcal{R}_0$: $\mathcal{L}_0(G) = \sum_{\mathsf{r} \in \mathcal{R}_0} G(\mathsf{r})$;*
(2) *for every $\mathsf{y} \in \mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{F})$ with a left inverse there is exactly one $\mathsf{r} \in \mathcal{R}$ such that $\mathsf{y}\,\mathsf{r}$ is primitive, and exactly one $\mathsf{r} \in \mathcal{R}$ such that $\mathsf{y}\,\mathsf{r}^{-1}$ is primitive.*

Proof. By the elementary divisor property, every $\mathsf{g} \in \mathrm{GL}_{\mathbf{b}}(\mathsf{F})$ has a $\mathsf{k} \in \mathrm{GL}_{\mathbf{b}}(\mathsf{O})$ with $\mathsf{g}^{\mathsf{T}} = \mathsf{k}^{-\mathsf{T}}\mathsf{g}\,\mathsf{k}$, where $^{-\mathsf{T}}$ is the inverse transpose (Input 2); that is, $(\mathsf{g}\mathsf{k})^{\mathsf{T}} = \mathsf{g}\mathsf{k}$: the matrix $\mathsf{g}\mathsf{k}$ is symmetric. Choose an element g of each coset and such a k for it, and let $\mathcal{R}$ consist of the products $\mathsf{g}\mathsf{k}$; so every $\mathsf{r} \in \mathcal{R}$ is symmetric, and $\mathsf{r}^{-1} = \mathsf{r}^{-\mathsf{T}}$. The functional "sum of the values on $\mathcal{R}$" is defined on the functions that vanish at all but finitely many points of $\mathcal{R}$; extend it linearly to all functions. This gives (1). For (2): a matrix with a left inverse becomes primitive after right multiplication by some $\mathsf{u} \in \mathrm{GL}_{\mathbf{b}}(\mathsf{F})$, and if $\mathsf{y}\mathsf{u}$ is primitive then $\mathsf{y}\mathsf{u}\mathsf{u}'$ is primitive exactly when $\mathsf{u}' \in \mathrm{GL}_{\mathbf{b}}(\mathsf{O})$ (Input 2). So the u with $\mathsf{y}\mathsf{u}$ primitive form one coset, which contains one element of $\mathcal{R}$. The same holds for the inverses: the inverse transpose is an automorphism of $\mathrm{GL}_{\mathbf{b}}(\mathsf{F})$ that preserves $\mathrm{GL}_{\mathbf{b}}(\mathsf{O})$, so it permutes the cosets, and the matrices $\mathsf{r}^{-\mathsf{T}} = \mathsf{r}^{-1}$, $\mathsf{r} \in \mathcal{R}$, are again representatives of all cosets, one for each. □

LEAN `exists_common_coset_reps`

Definition 3.2 (local section)**.** Let F be a semilocal field, $\mathbf{b}$, $\mathbf{n}$ finite index sets, χ a multiplicative character of F and $\mathsf{y} \in \mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{F})$. With the functional $\mathcal{L}_0$ and the set $\mathcal{R}$ of Lemma 3.1, chosen once for $(\mathsf{F}, \mathbf{b}, \mathbf{n})$, $\mathrm{sec}_\chi(\mathsf{y})$ is the linear functional

$$\mathrm{sec}_\chi(\mathsf{y})(\Psi) = \mathcal{L}_0(\mathsf{g} \mapsto \chi(\det \mathsf{g})\,\Psi(\mathsf{y}\,\mathsf{g}))$$

on the complex functions Ψ on $\mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{F})$. By Lemma 3.1 (1) it equals the sum $\sum_{\mathsf{r}\in\mathcal{R}} \chi(\det \mathsf{r})\Psi(\mathsf{y}\,\mathsf{r})$ whenever only finitely many of its terms are non-zero: it stands for the integral of $\chi(\det \mathsf{g})\Psi(\mathsf{y}\mathsf{g})$ over $\mathrm{GL}_{\mathbf{b}}(\mathsf{F})$.

LEAN `localSection`

Definition 3.3 (matched pairs)**.** Let F be a semilocal field, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ finite index sets, χ a multiplicative character of F and $\mathsf{e} \colon \mathbf{b} \sqcup \mathbf{a} \simeq \mathbf{n}$ an identification. $\mathrm{Match}(\chi, \mathsf{e})$ is the space of the pairs $(\check{\Psi}, \Psi) \in \mathcal{S}^{\mathrm{sur}} \times \mathcal{S}^{\mathrm{inj}}$ of Schwartz–Bruhat functions, on $\mathrm{M}_{\mathbf{a}\times\mathbf{n}}(\mathsf{F})$ and on $\mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{F})$, such that for every $\mathsf{k} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{F})$

$$\mathrm{sec}_\chi(\mathrm{row}_{\mathsf{e}}(\mathsf{k})^{\mathsf{T}})(\check{\Psi} \circ {}^{\mathsf{T}}) = \chi(\det \mathsf{k}) \cdot \mathrm{sec}_\chi(\mathrm{col}_{\mathsf{e}}(\mathsf{k}))(\Psi).$$

On the left stands the local section for the index sets $(\mathbf{a}, \mathbf{n})$, applied to the function $\mathsf{x} \mapsto \check{\Psi}(\mathsf{x}^{\mathsf{T}})$ of $\mathbf{n} \times \mathbf{a}$ matrices.

LEAN `matchingPairs`

Lemma 3.4 (translation)**.** *Let* F *be a semilocal field and* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ *finite index sets.*

(a) *For an identification* $\mathsf{e}\colon \mathbf{b} \sqcup \mathbf{a} \simeq \mathbf{n}$ *and* $\mathsf{g}, \mathsf{k} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{F})$*:* $\mathsf{g}\,\mathrm{col}_{\mathsf{e}}(\mathsf{k}) = \mathrm{col}_{\mathsf{e}}(\mathsf{gk})$.

(b) *Let* χ *be a multiplicative character of* F *and* $\mathsf{e}\colon \mathbf{b} \sqcup \mathbf{a} \simeq \mathbf{n}$*. If the pair* $(\check{\Psi}, \Psi)$ *lies in* $\mathrm{Match}(\chi, \mathsf{e})$ *and* $\mathsf{g} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{F})$*, then the pair* $(\chi(\det \mathsf{g})\,\check{\Psi}(\,\cdot\,\mathsf{g}),\ \Psi(\mathsf{g}^{-1}\,\cdot\,))$ *lies in* $\mathrm{Match}(\chi, \mathsf{e})$.

Proof. (a) holds by definition. (b) Translation by an invertible matrix is a homeomorphism that preserves the matrices with a left inverse, so both functions stay Schwartz–Bruhat with the required supports. Since $\mathrm{row}_{\mathsf{e}}(\mathsf{g}^{-1}\mathsf{k}) = \mathrm{row}_{\mathsf{e}}(\mathsf{k})\,\mathsf{g}$ and, by (a), $\mathrm{col}_{\mathsf{e}}(\mathsf{g}^{-1}\mathsf{k}) = \mathsf{g}^{-1}\,\mathrm{col}_{\mathsf{e}}(\mathsf{k})$, the two sides of the matching identity of the new pair at k are $\chi(\det \mathsf{g})$ times those of the old pair at $\mathsf{g}^{-1}\mathsf{k}$, the factor $\chi(\det(\mathsf{g}^{-1}\mathsf{k}))$ becoming $\chi(\det \mathsf{k})$. □

LEAN `mul_firstCols` · `matchingPairs.translate_mem`

Lemma 3.5 (local sections of primitive supports)**.** *Let* F *be a semilocal field,* $\mathbf{b}$, $\mathbf{n}$ *finite index sets,* χ *a multiplicative character of* F *and* $\mathsf{y} \in \mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{F})$ *a matrix that has a left inverse. There is* $\mathsf{g} \in \mathrm{GL}_{\mathbf{b}}(\mathsf{F})$ *such that* yg *is primitive and, for every function* Ψ *that vanishes off the primitive matrices,* $\mathrm{sec}_{\chi}(\mathsf{y})(\Psi) = \chi(\det \mathsf{g})\,\Psi(\mathsf{yg})$.

Proof. Take for g the unique $\mathsf{r} \in \mathcal{R}$ with yr primitive (Lemma 3.1 (2)). The function $\mathsf{h} \mapsto \chi(\det \mathsf{h})\Psi(\mathsf{yh})$ vanishes at the other elements of $\mathcal{R}$, so Lemma 3.1 (1) with the set $\{\mathsf{r}\}$ gives the value. □

LEAN `localSection_of_primitive`

Lemma 3.6 (primitive matrices form a clopen set)**.** *Let* F *be a semilocal field,* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ *finite index sets,* χ *a multiplicative character of* F *that is tame, and* $\mathsf{e}\colon \mathbf{b} \sqcup \mathbf{a} \simeq \mathbf{n}$*. Then the set of primitive matrices in* $\mathrm{M}_{\mathbf{n}\times\mathbf{a}}(\mathsf{F})$ *is closed and open.*

Proof. It is the image of the compact group $\mathrm{GL}_{\mathbf{n}}(\mathsf{O})$ under the continuous map that takes the columns indexed by $\mathbf{a}$ (Input 2), hence closed. Let y_0 be primitive with integral left inverse s. The standard datum $\Psi^{\mathrm{st}}_{\chi,\mathsf{s}}$ is locally constant, its value at y_0 is $\chi^{\flat}(1) = 1$, and every matrix at which it does not vanish is primitive (Input 1). So a neighbourhood of y_0 consists of primitive matrices. The tame character is needed only for the standard datum to be locally constant. □

LEAN `TameLocalChar.isClopen_primitive`

Proposition 3.7 (the large side factors through the partner)**.** *Let* F *be a semilocal field,* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ *finite index sets,* χ *a multiplicative character of* F *that is tame,* $\mathsf{e}\colon \mathbf{b} \sqcup \mathbf{a} \simeq \mathbf{n}$*, and* $\mathcal{X}$ *a set. Let* $\mathcal{L}$ *be a linear map from the complex functions on* $\mathrm{M}_{\mathbf{a}\times\mathbf{n}}(\mathsf{F})$ *to the complex functions on* $\mathcal{X}$ *such that* $\chi(\det \mathsf{g}) \cdot \mathcal{L}(\check{\Psi}(\mathsf{g}\,\cdot)) = \mathcal{L}(\check{\Psi})$ *for every* $\mathsf{g} \in \mathrm{GL}_{\mathbf{a}}(\mathsf{F})$ *and every* $\check{\Psi} \in \mathcal{S}^{\mathrm{sur}}$*. Then there is a linear map* $\mathcal{L}'$ *from the complex functions on* $\mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{F})$ *to the complex functions on* $\mathcal{X}$ *with* $\mathcal{L}'(\Psi) = \mathcal{L}(\check{\Psi})$ *for every pair* $(\check{\Psi}, \Psi) \in \mathrm{Match}(\chi, \mathsf{e})$.

Proof. A matrix $\mathsf{x} \in \mathrm{M}_{\mathbf{a}\times\mathbf{n}}(\mathsf{F})$ has a primitive transpose exactly when $\mathsf{x} = \mathrm{row}_{\mathsf{e}}(\mathsf{k})$ for some $\mathsf{k} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{O})$ (Input 2); choose such a $\mathsf{k}(\mathsf{x})$ for each of them. Define $\mathcal{L}'(\Psi)$ as $\mathcal{L}$ of the function that is $\chi(\det \mathsf{k}(\mathsf{x})) \cdot \mathrm{sec}_{\chi}(\mathrm{col}_{\mathsf{e}}(\mathsf{k}(\mathsf{x})))(\Psi)$ at the x with primitive transpose and 0 elsewhere. It is linear in Ψ.

Let $(\check{\Psi}, \Psi)$ be a matched pair, and let $\mathcal{R}$ be the set of Lemma 3.1 for $(\mathbf{a}, \mathbf{n})$. For $\mathsf{r} \in \mathcal{R}$ let $\check{\Psi}_{\mathsf{r}}$ be the product of $\check{\Psi}$ with the indicator of the set of the x for which $\mathsf{x}^{\mathsf{T}}\mathsf{r}^{-1}$ is primitive. This set is closed and open by Lemma 3.6, so $\check{\Psi}_{\mathsf{r}} \in \mathcal{S}^{\mathrm{sur}}$. Every x in the support of $\check{\Psi}$ has a transpose with a left inverse, so it lies in exactly one of these sets (Lemma 3.1 (2)). The support is compact, hence covered by the sets of a finite subset $\mathcal{R}_0 \subseteq \mathcal{R}$, and $\check{\Psi} = \sum_{\mathsf{r}\in\mathcal{R}_0} \check{\Psi}_{\mathsf{r}}$. By the invariance of $\mathcal{L}$ under the transposes of the elements of $\mathcal{R}_0$,

$$\mathcal{L}(\check{\Psi}) = \sum_{\mathsf{r}\in\mathcal{R}_0} \chi(\det \mathsf{r})\, \mathcal{L}(\check{\Psi}_{\mathsf{r}}(\mathsf{r}^{\mathsf{T}}\,\cdot)) = \mathcal{L}\Big(\mathsf{x} \mapsto [\mathsf{x}^{\mathsf{T}} \text{ primitive}] \sum_{\mathsf{r}\in\mathcal{R}_0} \chi(\det \mathsf{r})\, \check{\Psi}(\mathsf{r}^{\mathsf{T}}\mathsf{x})\Big).$$

If x^{T} is primitive and $\check{\Psi}(\mathsf{r}^{\mathsf{T}}\mathsf{x}) \neq 0$, then $\mathsf{r} \in \mathcal{R}_0$, because r is the representative that belongs to the point $\mathsf{r}^{\mathsf{T}}\mathsf{x}$ of the support. So by Lemma 3.1 (1) the inner sum is $\mathrm{sec}_\chi(\mathsf{x}^{\mathsf{T}})(\check{\Psi} \circ {}^{\mathsf{T}})$, and by the matching identity at $\mathsf{k}(\mathsf{x})$ it equals $\chi(\det \mathsf{k}(\mathsf{x})) \cdot \mathrm{sec}_\chi(\mathrm{col}_{\mathsf{e}}(\mathsf{k}(\mathsf{x})))(\Psi)$. This is the function that defines $\mathcal{L}'(\Psi)$. □

LEAN `matchingPairs.exists_linear_factor`

Lemma 3.8 (partners from a function of a block)**.** *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ *be finite index sets,* R *a commutative ring,* $\mathsf{e}\colon \mathbf{b} \sqcup \mathbf{a} \simeq \mathbf{n}$ *an identification,* χ *a multiplicative character of* R *and* G *a complex function on* $\mathrm{M}_{\mathbf{a}\times\mathbf{b}}(\mathsf{R})$*. Cut an* $\mathbf{a} \times \mathbf{n}$ *matrix* x *by* e *into column blocks* $(\mathsf{x}_1 \mid \mathsf{x}_2)$ *with* x_2 *square, and an* $\mathbf{n} \times \mathbf{b}$ *matrix* y *into row blocks* y_1 *(square) and* y_2*. Then the functions*

$$\check{\Psi}(\mathsf{x}) = \chi^{-1}(\det \mathsf{x}_2)\, G(\mathsf{x}_2^{-1}\mathsf{x}_1), \qquad \Psi(\mathsf{y}) = \chi^{-1}(\det \mathsf{y}_1)\, G(-\mathsf{y}_2\mathsf{y}_1^{-1})$$

are partners for χ *over* R *with respect to* e*.*

Proof. If $\check{\Psi}(\mathsf{x}) \neq 0$ then x_2 is invertible and $\mathsf{x} = \mathrm{row}_{\mathsf{e}}(\mathsf{k})$ for the inverse k of the block matrix with rows $(1\ 0)$ and $(\mathsf{x}_1\ \mathsf{x}_2)$; if $\Psi(\mathsf{y}) \neq 0$ then y_1 is invertible and $\mathsf{y} = \mathrm{col}_{\mathsf{e}}(\mathsf{k})$ for the block matrix k with columns y and $\binom{0}{1}$. For $\mathsf{k} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{R})$ write $\mathrm{row}_{\mathsf{e}}(\mathsf{k}) = (\mathsf{x}_1 \mid \mathsf{x}_2)$ and $\mathrm{col}_{\mathsf{e}}(\mathsf{k}) = \binom{\mathsf{y}_1}{\mathsf{y}_2}$. Jacobi's identity (Input 2, with the rows $(1\ 0)$) gives $\det \mathsf{x}_2 \cdot \det \mathsf{k} = \det \mathsf{y}_1$, and $\mathrm{row}_{\mathsf{e}}(\mathsf{k})\,\mathrm{col}_{\mathsf{e}}(\mathsf{k}) = 0$ gives $\mathsf{x}_1\mathsf{y}_1 + \mathsf{x}_2\mathsf{y}_2 = 0$. So x_2 is invertible exactly when y_1 is; then $\chi^{-1}(\det \mathsf{x}_2) = \chi(\det \mathsf{k})\chi^{-1}(\det \mathsf{y}_1)$ and $\mathsf{x}_2^{-1}\mathsf{x}_1 = -\mathsf{y}_2\mathsf{y}_1^{-1}$, and otherwise both values vanish. □

LEAN `partner_block`

Lemma 3.9 (partners are matched)**.** *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ *be finite index sets,* F *a semilocal field with subring* O*,* χ *a multiplicative character of* F *that is tame,* $\mathsf{e}\colon \mathbf{b} \sqcup \mathbf{a} \simeq \mathbf{n}$*, and let* $\check{\Psi}$*,* Ψ *be partners for* χ *over* O *with respect to* e *such that* Ψ *is locally constant. Then* $(\check{\Psi}, \Psi) \in \mathrm{Match}(\chi, \mathsf{e})$*.*

Proof. Ψ vanishes off the primitive matrices, and so does $\check{\Psi} \circ {}^{\mathsf{T}}$; $\check{\Psi}$ is locally constant because Ψ is and $\mathrm{GL}_{\mathbf{n}}(\mathsf{O})$ is compact. A locally constant function supported on the primitive matrices is a Schwartz–Bruhat function supported on matrices with a left inverse (Input 1). Let $\mathsf{k} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{F})$. By Lemma 3.5 there are g and h with $\mathrm{col}_{\mathsf{e}}(\mathsf{k})\mathsf{g}$ and $\mathrm{row}_{\mathsf{e}}(\mathsf{k})^{\mathsf{T}}\mathsf{h}$ primitive, and the two local sections of the matching identity are $\chi(\det \mathsf{g})\Psi(\mathrm{col}_{\mathsf{e}}(\mathsf{k})\mathsf{g})$ and $\chi(\det \mathsf{h})\check{\Psi}(\mathsf{h}^{\mathsf{T}}\,\mathrm{row}_{\mathsf{e}}(\mathsf{k}))$. Write $\mathrm{col}_{\mathsf{e}}(\mathsf{k})\mathsf{g} = \mathrm{col}_{\mathsf{e}}(\mathsf{j}_1)$ with $\mathsf{j}_1 \in \mathrm{GL}_{\mathbf{n}}(\mathsf{O})$. Then $\mathsf{k}^{-1}\mathsf{j}_1$ is block triangular for e with diagonal blocks g and some f^{-1}, so $\mathrm{row}_{\mathsf{e}}(\mathsf{k}) = \mathsf{f}^{-1}\,\mathrm{row}_{\mathsf{e}}(\mathsf{j}_1)$ and $\det \mathsf{k} \det \mathsf{g} = \det \mathsf{j}_1 \det \mathsf{f}$; as $\mathrm{row}_{\mathsf{e}}(\mathsf{j}_1)^{\mathsf{T}}$ is primitive, $\mathsf{h}^{\mathsf{T}}\mathsf{f}^{-1}$ lies in $\mathrm{GL}_{\mathbf{a}}(\mathsf{O})$. Hence $\mathsf{h}^{\mathsf{T}}\,\mathrm{row}_{\mathsf{e}}(\mathsf{k}) = \mathrm{row}_{\mathsf{e}}(\mathsf{j})$ for some $\mathsf{j} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{O})$ with $\mathrm{col}_{\mathsf{e}}(\mathsf{j}) = \mathrm{col}_{\mathsf{e}}(\mathsf{j}_1)$ and $\det \mathsf{j} \cdot \det(\mathsf{h}^{\mathsf{T}}\mathsf{f}^{-1}) = \det \mathsf{j}_1$ (Input 2). The partner identity at j now gives $\chi(\det \mathsf{h})\check{\Psi}(\mathrm{row}_{\mathsf{e}}(\mathsf{j})) = \chi(\det \mathsf{h} \det \mathsf{j})\Psi(\mathrm{col}_{\mathsf{e}}(\mathsf{j}_1)) = \chi(\det \mathsf{k})\chi(\det \mathsf{g})\Psi(\mathrm{col}_{\mathsf{e}}(\mathsf{k})\mathsf{g})$. □

LEAN `IsPartner.mem_matchingPairs`

Lemma 3.10 (the standard row datum in an integral frame)**.** *Let* F *be a semilocal field,* $\mathbf{b}$*,* $\mathbf{m}$*,* $\mathbf{n}$ *finite index sets,* χ *a multiplicative character of* F *that is tame,* $\mathfrak{A}$ *an integral frame over* F *whose matrix* $(\mathfrak{A}_1 \mid \mathfrak{A}_2 \mid \mathfrak{A}_3)$ *has rows indexed by* $\mathbf{n}$ *and column blocks indexed by* $\mathbf{b}$*,* $\mathbf{m}$*,* $\mathbf{b}$*, and* $\mathsf{e}' : \mathbf{b} \sqcup (\mathbf{b} \sqcup \mathbf{m}) \simeq \mathbf{n}$ *an identification. There is a complex number* c *with* $|\mathrm{c}| = 1$ *and*

$$\check{\Psi}^{\mathrm{st}}_{\mathsf{e}',\chi,\mathfrak{A}'_3} = \mathrm{st}_{\mathfrak{A}}(\mathsf{y} \mapsto \mathrm{c} \cdot \chi^{\flat}(\det \mathsf{y})).$$

Proof. Multiply the square matrix with the rows of $\mathfrak{A}'_3$ and of x by the integral invertible matrix $(\mathfrak{A}_3 \mid \mathfrak{A}_1 \mid \mathfrak{A}_2)$. By the relations $\mathfrak{A}'_3\mathfrak{A}_3 = 1$, $\mathfrak{A}'_3\mathfrak{A}_1 = 0$, $\mathfrak{A}'_3\mathfrak{A}_2 = 0$ the product is block triangular with diagonal blocks 1 and $\mathsf{x}(\mathfrak{A}_1 \mid \mathfrak{A}_2)$; the first matrix is integral exactly when x is, and its determinant is a unit exactly when $\det(\mathsf{x}(\mathfrak{A}_1 \mid \mathfrak{A}_2))$ is. So the two functions agree up to the factor χ of the unit $\det(\mathfrak{A}_3 \mid \mathfrak{A}_1 \mid \mathfrak{A}_2)$, reordered by e', which has absolute value 1. □

LEAN `IntegralFrame.exists_stdRowDatum_eq_stdDatum`

Setting 3.11 (a stable space and a partner)**.** F is a semilocal field with subring O and element ϖ; $\mathbf{a}$, $\mathbf{b}$, $\mathbf{m}$, $\mathbf{n}$ are finite index sets; χ is a multiplicative character of F that is tame; $\mathsf{e} : \mathbf{b} \sqcup \mathbf{a} \simeq \mathbf{n}$ is an identification, $\mathcal{U}$ a set of $\mathbf{n} \times \mathbf{b}$ matrices, and $\mathcal{Q}$ a space of complex functions on $\mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{F})$. The hypotheses are: (i) $\mathcal{Q}$ is stable under translation: $G(\mathsf{g}\,\cdot\,) \in \mathcal{Q}$ for all $G \in \mathcal{Q}$ and $\mathsf{g} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{F})$; (ii) $\mathcal{Q} \subseteq \mathcal{S}(\mathcal{U})$; (iii) Ψ is a locally constant function on $\mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{F})$; (iv) $\check{\Psi}$ and Ψ are partners for χ over O with respect to e.

Lemma 3.12 (a stable space that contains a multiple of a standard datum)**.** *In Setting 3.11, let* $G \in \mathcal{Q}$*, let* $\mathsf{x} \in \mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{F})$ *be integral and* $\mathsf{s} \in \mathrm{M}_{\mathbf{b}\times\mathbf{n}}(\mathsf{F})$ *integral with* $\mathsf{s}\mathsf{x} = 1$*, and let* n *be a function on* $\mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{F})$ *with values in the natural numbers such that* $G(\mathsf{y}) = \mathrm{n}(\mathsf{y}) \cdot \Psi^{\mathrm{st}}_{\chi,\mathsf{s}}(\mathsf{y})$ *for all* y *and* $\mathrm{n}(\mathsf{x}) \neq 0$*. Then* $\Psi \in \mathcal{Q}$*.*

Proof. The partner Ψ is locally constant and vanishes off the compact set of primitive matrices, so there is an exponent i with $\Psi(\mathsf{y} + \varpi^i\mathsf{z}) = \Psi(\mathsf{y})$ for all y and all integral z. Put $\mathsf{d} = \varpi^{-i}(1 - \mathsf{x}(1 - \varpi^i)\mathsf{s})$. It is invertible, with integral inverse $\varpi^i + (1 - \varpi^i)\mathsf{x}\mathsf{s}$, and $\mathsf{s}\mathsf{d} = \mathsf{s}$, $\mathsf{d}\mathsf{x} = \mathsf{x}$. The function

$$G'(\mathsf{y}) = \int_{\mathrm{GL}_{\mathbf{n}}(\mathsf{O})} \Psi(\mathsf{k}^{-1}\mathsf{x})\, G(\mathsf{d}\,\mathsf{k}\,\mathsf{y})\, d\mathsf{k}$$

lies in $\mathcal{Q}$: the integrand is a locally constant family of elements of $\mathcal{Q}$ over a compact group, by (i) and (ii), so the integral is a finite combination of its members (unit 105). We show $G' = \mathrm{K}_0\Psi$ with $\mathrm{K}_0 = \int |G(\mathsf{d}\mathsf{k}\mathsf{x})|\, d\mathsf{k}$, which is positive because the integrand is continuous, non-negative and equal to $\mathrm{n}(\mathsf{x}) \neq 0$ at $\mathsf{k} = 1$. If $G(\mathsf{d}\mathsf{k}\mathsf{y}) \neq 0$ then $\mathsf{d}\mathsf{k}\mathsf{y}$ is integral and $\mathsf{s}\mathsf{k}\mathsf{y} = \mathsf{s}\mathsf{d}\mathsf{k}\mathsf{y}$ has a unit determinant, so y is primitive; hence both sides vanish at a y that is not primitive. Let $\mathsf{y} = \mathrm{col}_{\mathsf{e}}(\mathsf{l})$ and $\mathsf{x} = \mathrm{col}_{\mathsf{e}}(\mathsf{j})$ with $\mathsf{l}, \mathsf{j} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{O})$ (Input 2). Replacing k by $\mathsf{k}\mathsf{j}\mathsf{l}^{-1}$,

$$G'(\mathsf{y}) = \int \Psi(\mathsf{l}\mathsf{j}^{-1}\mathsf{k}^{-1}\mathsf{x})\, G(\mathsf{d}\mathsf{k}\mathsf{x})\, d\mathsf{k}.$$

Where the integrand is non-zero, $\mathsf{g} = \mathsf{s}\mathsf{k}\mathsf{x}$ lies in $\mathrm{GL}_{\mathbf{b}}(\mathsf{O})$ and $\mathsf{k}\mathsf{x} = \mathsf{x}\mathsf{g} + \varpi^i\mathsf{z}$ with z integral, so $\mathsf{l}\mathsf{j}^{-1}\mathsf{k}^{-1}\mathsf{x} = \mathsf{y}\mathsf{g}^{-1}$ up to ϖ^i times an integral matrix. By the choice of i and

the law $\Psi(\mathsf{yg})\chi(\det \mathsf{g}) = \Psi(\mathsf{y})$ of a partner (unit 112), $\Psi(\mathsf{lj}^{-1}\mathsf{k}^{-1}\mathsf{x}) = \chi(\det \mathsf{g})\Psi(\mathsf{y})$, while $G(\mathsf{dkx}) = \mathrm{n}(\mathsf{dkx})\chi^{-1}(\det \mathsf{g})$. The product is $\Psi(\mathsf{y})\,|G(\mathsf{dkx})|$, since $|\chi| = 1$ on units. So $G' = \mathrm{K}_0\Psi$ and $\Psi \in \mathcal{Q}$. □

LEAN `partner_mem_of_stdColDatum`

Proposition 3.13 (a criterion by unimodular indices)**.** *In Setting 3.11, let $\mathfrak{A}$ be an integral frame over F whose matrix $(\mathfrak{A}_1 \mid \mathfrak{A}_2 \mid \mathfrak{A}_3)$ has rows indexed by $\mathbf{n}$ and column blocks indexed by $\mathbf{b}$, $\mathbf{m}$, $\mathbf{b}$, let e_F be an additive character of F that has conductor O, and let $\mathcal{M}$ be a space of complex functions on $\mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{F})$. For a $\mathbf{b}\times\mathbf{b}$ matrix z over F put*

$$S_\mathsf{z}(\mathsf{y}) = \Psi^{\mathrm{st}}_{\chi,\mathfrak{A}'_1}(\mathsf{y} + \mathfrak{A}_3\,\mathsf{z}\,\mathfrak{A}'_1\mathsf{y}), \qquad \text{so that } S_0 = \Psi^{\mathrm{st}}_{\chi,\mathfrak{A}'_1}.$$

Suppose that every S_z lies in $\mathcal{M}$, and that $\mathcal{Q}$ contains every $\Psi' \in \mathcal{M}$ with the following property: $\mathcal{L}(\Psi') = 0$ for every $\mathbf{b}\times\mathbf{b}$ matrix T over F and every linear functional $\mathcal{L}$ such that $\mathcal{L}(S_\mathsf{z}) = \mathrm{e}_\mathsf{F}(\mathrm{tr}(T\mathsf{z}))\,\mathcal{L}(S_0)$ for all z, and such that $\mathcal{L}(S_0) = 0$ if T is primitive. Then $\Psi \in \mathcal{Q}$.

Proof. Let $\bar{\mathsf{R}}$, the reduction $\mathsf{z} \mapsto \bar{\mathsf{z}}$ and $\bar\chi$ be as in the definition of a tame character, and let $\mathcal{Z}$ be a set of integral $\mathbf{b}\times\mathbf{b}$ matrices, containing 1, that maps bijectively to $\mathrm{M}_{\mathbf{b}\times\mathbf{b}}(\bar{\mathsf{R}})$. Since e_F has conductor O, $\bar{\mathrm{e}}(\bar{\mathsf{z}}) = \mathrm{e}_\mathsf{F}(\varpi^{-1}\mathsf{z})$ is a primitive additive character of $\bar{\mathsf{R}}$ (Input 1). Consider the Gauss sum of shears

$$\Gamma_0 = \sum_{\mathsf{z}\in\mathcal{Z}} \bar\chi(\det \bar{\mathsf{z}})\, S_{\varpi^{-1}\mathsf{z}} \in \mathcal{M}.$$

$\Gamma_0 \in \mathcal{Q}$. Let T and $\mathcal{L}$ be as in the hypothesis, and suppose $\mathcal{L}(\Gamma_0) \neq 0$. Then $\mathcal{L}(S_0) \neq 0$, and T is integral: for an integral s and a matrix unit u, $S_{\mathsf{z}+\mathsf{su}} = S_\mathsf{z}$, because the standard datum does not see an integral shear; hence $\mathrm{e}_\mathsf{F}(\mathrm{tr}(T\mathsf{su})) = 1$ for all integral s, and the conductor gives the entries of T in O. Now $\mathcal{L}(S_{\varpi^{-1}\mathsf{z}}) = \mathrm{e}_\mathsf{F}(\mathrm{tr}(T\varpi^{-1}\mathsf{z}))\,\mathcal{L}(S_0) = \bar{\mathrm{e}}(\mathrm{tr}(\bar T\bar{\mathsf{z}}))\,\mathcal{L}(S_0)$, so $\mathcal{L}(\Gamma_0) = (\sum_{\mathsf{z}\in\mathcal{Z}} \bar\chi(\det\bar{\mathsf{z}})\,\bar{\mathrm{e}}(\mathrm{tr}(\bar T\bar{\mathsf{z}})))\mathcal{L}(S_0)$, and by the matrix Gauss sum (Input 1) the sum is $\bar\chi^{-1}(\det \bar T)$ times its value at 1. It is non-zero, so $\det T$ is a unit and T is primitive; but then $\mathcal{L}(S_0) = 0$, a contradiction. So every such $\mathcal{L}$ kills Γ_0, and $\Gamma_0 \in \mathcal{Q}$ by hypothesis.

Conclusion. Apply Lemma 3.12 to the translate $G = \Gamma_0(-\varpi^{-1}\,\cdot\,) \in \mathcal{Q}$, to $\mathsf{x} = \mathfrak{A}_3 - \varpi\mathfrak{A}_1$ and to $\mathsf{s} = \mathfrak{A}'_3$; both are integral, and the frame relations (Input 1) give $\mathfrak{A}'_3\mathsf{x} = \mathfrak{A}'_3\mathfrak{A}_3 - \varpi\mathfrak{A}'_3\mathfrak{A}_1 = 1$. It remains to show that $G = \mathrm{n}\cdot\Psi^{\mathrm{st}}_{\chi,\mathfrak{A}'_3}$ with a function n with values in $\{0,1\}$ and $\mathrm{n}(\mathsf{x}) = 1$. Fix y and write $\mathsf{u} = \mathfrak{A}'_1\mathsf{y}$, $\mathsf{v} = \mathfrak{A}'_2\mathsf{y}$, $\mathsf{w} = \mathfrak{A}'_3\mathsf{y}$. The term of $\mathsf{z}\in\mathcal{Z}$ in $G(\mathsf{y})$ is

$$\bar\chi(\det\bar{\mathsf{z}})\,\Psi^{\mathrm{st}}_{\chi,\mathfrak{A}'_1}(-\varpi^{-1}\mathsf{y} - \varpi^{-2}\mathfrak{A}_3\,\mathsf{z}\,\mathsf{u}).$$

A matrix is integral exactly when its three products with $\mathfrak{A}'_1$, $\mathfrak{A}'_2$, $\mathfrak{A}'_3$ are, the frame being integral with integral inverse. By the frame relations these products of the argument are $-\varpi^{-1}\mathsf{u}$, $-\varpi^{-1}\mathsf{v}$ and $-\varpi^{-1}(\mathsf{w}+\varpi^{-1}\mathsf{zu})$. So the term is non-zero only if $\mathsf{u} = \varpi\mathsf{u}_1$ with $\mathsf{u}_1 \in \mathrm{GL}_\mathbf{b}(\mathsf{O})$, v is ϖ times an integral matrix, and $\mathsf{w} \equiv -\mathsf{zu}_1$ modulo ϖ; then w and y are integral, $\bar{\mathsf{z}} = -\bar{\mathsf{w}}\,\bar{\mathsf{u}}_1^{-1}$, and the term equals

$$\chi^\flat(\det(-\mathsf{wu}_1^{-1}))\,\chi^\flat(\det(-\mathsf{u}_1)) = \chi^\flat(\det\mathsf{w}) = \Psi^{\mathrm{st}}_{\chi,\mathfrak{A}'_3}(\mathsf{y}),$$

because $\bar\chi(\det\bar{\mathsf{z}}) = \chi^\flat(\det\mathsf{z})$ depends only on the reduction of z, and $\chi^\flat$ is multiplicative on the integral elements. At most one $\mathsf{z}\in\mathcal{Z}$ satisfies the congruence. So $G(\mathsf{y}) = \mathrm{n}(\mathsf{y})\,\Psi^{\mathrm{st}}_{\chi,\mathfrak{A}'_3}(\mathsf{y})$, where $\mathrm{n}(\mathsf{y}) \in \{0,1\}$ is the number of non-zero terms. For $\mathsf{y} = \mathsf{x}$ one has $\mathsf{u}_1 = -1$, $\mathsf{v} = 0$, $\mathsf{w} = 1$, and the element $\mathsf{z} = 1$ of $\mathcal{Z}$ contributes: $\mathrm{n}(\mathsf{x}) = 1$. □

LEAN `IntegralFrame.partner_mem_of_unimodular_vanishing`

4 The difference of the two theta integrals

Definition 4.1 (the theta integral on the small side)**.** Let E be a CM field with a CM type and Weil data, W and V hermitian spaces over E with index sets $\mathbf{n}$ and $\mathbf{a}$, $\mathfrak{c}$ a split tame character, $\mathfrak{v}$ a real datum for (W,V) and Ψ a complex function on $\mathrm{M}_{\mathbf{n}\times\mathbf{a}}(\mathsf{F}_{\mathfrak{c}})$. Then

$$\theta^{\mathrm{sm}}_{\mathsf{W},\mathsf{V}}(\mathfrak{c};\mathfrak{v},\Psi)=\mathrm{I}_{\mathsf{W},\mathsf{V}}(\omega_{\mathfrak{c}}\upsilon^{\lfloor|\mathbf{n}|/2\rfloor},\upsilon^{|\mathbf{a}|};\mathfrak{v}\otimes[\Psi\circ{}^{\mathsf{T}}]^{c}),$$

a function on $\mathrm{U}(\mathsf{W})(\mathbb{A})$: the theta integral over $[\mathrm{U}(\mathsf{V})]$ whose finite datum is the function $\mathsf{y}\mapsto\Psi(\mathsf{y}^{\mathsf{T}})$ of the components at the places conjugate to those of $\mathfrak{c}$.

LEAN `thetaSmall` · the integral on the large side is the declaration `thetaLarge` of unit 43

Proposition 4.2 (the difference is an equivariant lift)**.** *Let E be a CM field with a CM type and Weil data; let W, V, V' be hermitian spaces over E with index sets $\mathbf{n}$, $\mathbf{a}$, $\mathbf{b}$; let $\mathfrak{c}$ be a split tame character and $\mathsf{e}\colon\mathbf{b}\sqcup\mathbf{a}\simeq\mathbf{n}$ an identification. Suppose that $|\mathbf{n}|$ is even, let $\mathfrak{v}$ and $\mathfrak{v}'$ be rapid real data for (W,V) and (W,V'), and let c be a complex number. There are a space $\mathcal{M}$ of complex functions on $\mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{F}_{\mathfrak{c}})$ and a linear map $\mathcal{D}$ from such functions to functions on $\mathrm{U}(\mathsf{W})(\mathbb{A})$ such that $(\mathcal{M},\mathcal{D})$ is an equivariant lift for $\mathfrak{c}$, W and the inverse of the local character of $\upsilon^{|\mathbf{b}|}$ in rank $|\mathbf{b}|$, and such that for all partners $\check\Psi$, Ψ for $\chi_{\mathfrak{c}}$ over the integers of $\mathsf{F}_{\mathfrak{c}}$ with respect to e, with Ψ locally constant:*

$$\Psi\in\mathcal{M}\qquad\text{and}\qquad\mathcal{D}(\Psi)=\theta^{\mathrm{lg}}_{\mathsf{W},\mathsf{V}}(\mathfrak{c};\mathfrak{v},\check\Psi)-\mathrm{c}\cdot\theta^{\mathrm{sm}}_{\mathsf{W},\mathsf{V}'}(\mathfrak{c};\mathfrak{v}',\Psi).$$

Proof. Both theta integrals extend to linear functionals of the datum (Input 3). Write $\chi^{\circ}=\chi_{\mathfrak{c}}^{(|\mathbf{n}|)}$ for the local character of $\omega_{\mathfrak{c}}^{\mathrm{bc}}$ in rank $|\mathbf{n}|$ (§2); it is tame. The functional $\mathcal{L}\colon\check\Psi\mapsto\theta^{\mathrm{lg}}_{\mathsf{W},\mathsf{V}}(\mathfrak{c};\mathfrak{v},\check\Psi)$ has the invariance required in Proposition 3.7 for the character χ°, and not for $\chi_{\mathfrak{c}}$: write $\mathsf{g}\in\mathrm{GL}_{\mathbf{a}}(\mathsf{F}_{\mathfrak{c}})$ as the components of a finite adelic point k of $\mathrm{U}(\mathsf{V})$ supported at $\Sigma(\mathfrak{c})$. Moving the datum by k^{-1} on the side of V does not change an integral over $[\mathrm{U}(\mathsf{V})]$, and the Weil representation of k^{-1} sends the local datum of $\check\Psi$ to $\chi^{\circ}(\det\mathsf{g})$ times the local datum of $\check\Psi(\mathsf{g}\cdot)$ (Input 3; here $|\mathbf{n}|$ even makes the character of the integral the base change of $\omega_{\mathfrak{c}}^{-1}$, and the factor $|\det\mathsf{g}|^{|\mathbf{n}|/2}$ of the Weil representation is the reason for the rank $|\mathbf{n}|$). So $\chi^{\circ}(\det\mathsf{g})\,\mathcal{L}(\check\Psi(\mathsf{g}\cdot))=\mathcal{L}(\check\Psi)$, which is the hypothesis of Proposition 3.7 with $\chi=\chi^{\circ}$, and there is a linear $\mathcal{L}'$ with $\mathcal{L}'(\Psi)=\theta^{\mathrm{lg}}_{\mathsf{W},\mathsf{V}}(\mathfrak{c};\mathfrak{v},\check\Psi)$ for every pair $(\check\Psi,\Psi)\in\mathrm{Match}(\chi^{\circ},\mathsf{e})$. Let $\mathcal{M}$ be the space of second components of the pairs in $\mathrm{Match}(\chi^{\circ},\mathsf{e})$, a subspace of $\mathcal{S}^{\mathrm{inj}}$, and $\mathcal{D}=\mathcal{L}'-\mathrm{c}\cdot\theta^{\mathrm{sm}}_{\mathsf{W},\mathsf{V}'}(\mathfrak{c};\mathfrak{v}',\cdot\,)$. Partners for $\chi_{\mathfrak{c}}$ over the integers of $\mathsf{F}_{\mathfrak{c}}$ are partners for χ° over the integers, because the definition of partners over the integers involves the character only at units and the two characters agree there; so, when Ψ is locally constant, they lie in $\mathrm{Match}(\chi^{\circ},\mathsf{e})$ (Lemma 3.9 for χ°), which gives the last assertion. For $\Psi\in\mathcal{M}$, $\mathcal{D}(\Psi)$ is a difference of theta integrals of target-bounded data, hence automorphic (Input 3). For a point g supported at $\Sigma(\mathfrak{c})$ the translated pair lies in $\mathrm{Match}(\chi^{\circ},\mathsf{e})$ (Lemma 3.4), translation of a theta integral is the theta integral of the moved datum, and the Weil representation of g acts on both local data by the translation of the function and a character of $\det\mathsf{g}_{\Sigma}$; with $|\mathbf{n}|=|\mathbf{a}|+|\mathbf{b}|$ the two characters agree and equal the one in the statement, the inverse of the local character

of the Hecke character $v^{|\mathbf{b}|}$ in rank $|\mathbf{b}|$. For $\mathsf{k} \in \mathcal{K}_f$ trivial at $\Sigma(\mathfrak{c})$ both local data are fixed. □

LEAN `exists_isEquivariantLift`

Proof of Theorem 2. Let $\mathfrak{A}$ be the frame of $\mathfrak{S}$ adapted to the places of $\mathfrak{c}$, as in the statement, and $\mathfrak{A}^{\mathrm{op}}$ the adapted frame of the splitting opposite to $\mathfrak{S}$. The opposite splitting exchanges the two isotropic frames, so the first and the third blocks change places: the block $(\mathfrak{A}^{\mathrm{op}})'_1$ is the block $\mathfrak{A}'_3$ of $\mathfrak{A}$, and the standard datum S_0 of Proposition 3.13 for the frame $\mathfrak{A}^{\mathrm{op}}$ is $\Psi_0 = \Psi^{\mathrm{st}}_{\chi_{\mathfrak{c}},\mathfrak{A}'_3}$. Put $\mathcal{Q} = \mathcal{M} \cap \ker \mathcal{D}$. It is stable under translation by $\mathrm{GL}_{\mathbf{m}}(\mathsf{F}_{\mathfrak{c}})$ (Input 3) and lies in $\mathcal{S}^{\mathrm{inj}}$. The standard data $\check{\Psi}^{\mathrm{st}}_{\mathfrak{e}',\chi_{\mathfrak{c}},\mathsf{z}}$ and $\Psi^{\mathrm{st}}_{\chi_{\mathfrak{c}},\mathsf{z}}$ are partners and $\Psi^{\mathrm{st}}_{\chi_{\mathfrak{c}},\mathsf{z}}$ is locally constant. So Proposition 3.13, for the frame $\mathfrak{A}^{\mathrm{op}}$ and the additive character of $\mathsf{F}_{\mathfrak{c}}$, shows $\Psi^{\mathrm{st}}_{\chi_{\mathfrak{c}},\mathsf{z}} \in \mathcal{Q}$, which is the assertion, once its two hypotheses are verified. The shears S_{z} of Ψ_0 lie in $\mathcal{M}$ by Input 3 and the hypothesis $\Psi_0 \in \mathcal{M}$. It remains to take $\Psi' \in \mathcal{M}$ that is killed by every admissible functional and to show $\mathcal{D}(\Psi') = 0$.

The function $\mathcal{D}(\Psi')$ is automorphic. By weak approximation (Input 4) it suffices to show that it vanishes at the points that are trivial at $\Sigma(\mathfrak{c})$, and a continuous function on the adelic hermitian matrices that is periodic under the rational ones vanishes when all its Fourier coefficients do. So it suffices that every centre coefficient $\mathrm{coef}_{\mathfrak{S}}(T'; \mathcal{D}(\Psi'))$, T' a $\mathbf{b} \times \mathbf{b}$ matrix over E, vanishes at such points. For fixed T' and a fixed point, this coefficient is a linear functional $\mathcal{L}$ of $\Psi' \in \mathcal{M}$ which transforms under the shears S_{z} by $\mathrm{e}_{\mathsf{F}}(\mathrm{tr}(T'_{\Sigma}\mathsf{z}))$, T'_{Σ} being the matrix of the components of T' at $\Sigma(\mathfrak{c})$ (Input 3). So $\mathcal{L}$ is admissible, and kills Ψ', provided $\mathcal{L}(\Psi_0) = 0$ whenever T'_{Σ} is primitive.

Let T'_{Σ} be primitive. Then $\det T' \neq 0$, and T' may be taken skew-hermitian, the other coefficients being zero; write $T' = \jmath T$ with T hermitian and invertible, so that the coefficient is the one at $\jmath T$ of the statement. Freitag's lemma (Input 4), applied to $\mathcal{D}(\Psi_0)$ with a base-adapted stage of the composed splitting of X over W, reduces the vanishing of the coefficient to its vanishing at the points $\mathrm{rad}_{\mathfrak{S}}(\mathsf{c}, \mathsf{w})\,\mathrm{lev}_{\mathfrak{S}}(\mathsf{q}, \mathsf{g})$ for which the hermitian matrix built from T and q, divided by the inverse different, is unimodular over $\hat{\mathcal{O}}_E$. For such a point there is $\mathsf{k} \in \mathrm{GL}_{\mathbf{b}}(\hat{\mathcal{O}}_E)$ making this matrix the identity (Input 2; this is where the Weil data must contain the hypothesis that E is stably unramified). If T is represented by the signs of V' at every infinite place, it is the Gram matrix of a space with these signs and a point, and Landherr's theorem (Input 4) gives a rational change of coordinates c_0 with $\mathsf{c}_0^{*}\mathrm{J}_{\mathsf{V}'}\mathsf{c}_0 = T$. Moving the rational Levi element of c_0 through the coefficient turns it into a coefficient at $\jmath\mathrm{J}_{\mathsf{V}'}$ at a point as in alternative (i), with $Y = \mathrm{x}_{\mathsf{V}'}^{-1}\mathsf{c}_0\mathsf{q}$ up to the scale idele. If T is not represented, alternative (ii) applies. In both cases the coefficient vanishes by hypothesis. □

5 Fourier coefficients of theta integrals

Definition 5.1 (the theta functional on real data)**.** Let E be a CM field with a CM type and Weil data, W_1 and V hermitian spaces with index sets $\mathbf{n}$ and $\mathbf{a}$, ω and ω' characters of $\mathrm{U}(1)(\mathbb{A})$, F a finite datum for $(\mathsf{W}_1, \mathsf{V})$, x an adelic $\mathbf{a} \times \mathbf{n}$ matrix and p a point of $\mathrm{U}(\mathsf{W}_1)(\mathbb{A}) \times \mathrm{U}(\mathsf{V})(\mathbb{A})$. Then

$$\theta^{\infty}_{\mathsf{W}_1,\mathsf{V}}(\omega, \omega'; F, \mathsf{x}, \mathsf{p}) \colon \quad \mathfrak{v} \longmapsto \theta\big(\mathrm{heis}(\mathsf{x})\,\omega_{\mathsf{W}_1,\mathsf{V}}(\omega, \omega')(\mathsf{p})(\mathfrak{v} \otimes F)\big)$$

is a continuous linear functional on the real data of the pair; heis(x) is the operator of the Heisenberg group attached to the vector of x.

LEAN thetaOnRealData

Proposition 5.2 (coefficients at indices that are not negative)**.** *Let E be a CM field with a CM type and Weil data, with element $\jmath$ and splitting character υ. Let X, Y, Z be hermitian spaces with index sets $\mathbf{m}$, $\mathbf{n}$, $\mathbf{a}$, and $\mathfrak{S}$ a splitting of X over Z along $\mathbf{b}$. Let ω be a character of $\mathrm{U}(1)(\mathbb{A})$ with weights μ, δ a character, f a datum of the pair (X,Y), g a point of $\mathrm{U}(\mathsf{X})(\mathbb{A})$ and T a $\mathbf{b}\times\mathbf{b}$ matrix over E. Assume: there is an index i of Y with positive sign; $\mu(\tau)$ is smaller than the number of positive signs of Z for every infinite place τ; $|\mathbf{b}|+\mathrm{neg}(\mathsf{Z})=\mathrm{neg}(\mathsf{Y})$; $\delta=\omega\,\upsilon^{|\mathbf{b}|}$; T is invertible; T, as an adelic matrix, is not represented by the signs -1, that is, at some infinite place it is not of the form $-\mathsf{c}^*\mathsf{c}$; and the theta terms of $\mathfrak{S}$ for $(\mathsf{Y},\delta,\delta,f)$ at g are integrable. Then*

$$\mathrm{coef}_{\mathfrak{S}}(\jmath T;\mathrm{I}_{\mathsf{X},\mathsf{Y}}(\delta,\delta;f))(\mathsf{g})=0.$$

Proof. The coefficient is the sum, over the rational $\mathbf{n}\times\mathbf{b}$ matrices v with $\mathsf{v}^*\mathrm{J}_{\mathsf{Y}}\mathsf{v}=T$, of integrals of theta terms (Input 3); if there is no such v it is zero. Fix one, and let $\mathsf{v}'=\mathrm{x}_{\mathsf{Y}}^{-1}\mathsf{v}$ be its coordinates in the adelic frame. As T is invertible, the coefficient is the integral over the quotient of $\mathrm{U}(\mathsf{Y})(\mathbb{A})$ by the stabilizer Stab of v' of the theta term at v' (Input 3). Let τ be an infinite place where T_τ is not of the form $-\mathsf{c}^*\mathsf{c}$, and w_0 the component of v' at τ. The hermitian matrix $\mathsf{w}_0^*\,\mathrm{diag}(\mathsf{Y})\mathsf{w}_0$ is not negative semidefinite, so some vector a_0 has $\mathsf{a}_0^*\mathsf{w}_0^*\,\mathrm{diag}(\mathsf{Y})\mathsf{w}_0\mathsf{a}_0=1$; by Witt's theorem (Input 2) an element u_0 of the real unitary group carries $\mathsf{w}_0\mathsf{a}_0$ to the basis vector of the index i.

No invariant functional. Let $e_0\geq 0$ be the integer with (number of positive signs of Z) $=\mu(\tau)+e_0+1$. We claim that a continuous linear functional $\mathcal{L}_1$ on the Fock model of the pair (Z,Y) at the infinite places vanishes if it is invariant under the real Weil representation, of weights μ and 0, of all elements of the maximal compact subgroup at τ that fix $\mathsf{u}_0\mathsf{w}_0$. It suffices to show $\mathcal{L}_1(\mathrm{heis}(\mathsf{y}_0)\,\mathrm{vac})=0$ for every y_0, the Heisenberg translates of the vacuum being total. Counting dimensions with $|\mathbf{b}|+\mathrm{neg}(\mathsf{Z})=\mathrm{neg}(\mathsf{Y})$ and the positive index i, one finds a unit row vector r_0 supported on the negative indices of Y, orthogonal to the columns of $\mathsf{u}_0\mathsf{w}_0$ and to the negative coordinates of y_0 at τ. For a complex number $\varkappa$ with $|\varkappa|=1$ the rotation $\mathsf{c}_\varkappa=1+(\bar\varkappa-1)\bar{\mathsf{r}}_0\mathsf{r}_0^{\mathsf{T}}$ of the line of r_0 lies in the maximal compact subgroup and fixes $\mathsf{u}_0\mathsf{w}_0$. It moves $\mathrm{heis}(\mathsf{y}_0)\,\mathrm{vac}$ to $\varkappa^{e_0+1}$ times the translate of the vacuum by $\mathsf{y}_0+(\bar\varkappa-1)\mathsf{z}_0$, z_0 the part of y_0 on the line of r_0: the vacuum is multiplied by the minor $\det\mathsf{c}_\varkappa=\bar\varkappa$ to the power given by the weight, which is $\varkappa^{e_0+1}$. So $\mathcal{L}_1(\mathrm{heis}(\mathsf{y}_0)\,\mathrm{vac})$ is the mean over the circle of a function $\varkappa\mapsto\varkappa^{e_0+1}G_0(\varkappa)$ with G_0 holomorphic on the closed disc (the translate depends holomorphically on $\varkappa$, as z_0 lies in the holomorphic coordinates), and by the mean value property this mean is the value at $\varkappa=0$, which is 0 because $e_0+1\geq 1$.

Averaging. Let $\mathcal{C}$ be the intersection of Stab with the conjugate by u_0^{-1}, at τ, of the maximal compact subgroup; it is compact. For $\mathsf{k}\in$ Stab the theta term at v' of the datum moved by k is the theta term moved through the pair (Z,Y) (law of the Levi elements, Input 3, with $\delta=\omega\upsilon^{|\mathbf{b}|}$). The average of the theta term over $\mathcal{C}$, for its Haar probability measure, is therefore a functional of the kind just excluded, applied to the datum: it is zero. Right multiplication by $\mathcal{C}$ preserves the Haar measure of Stab. The averaging criterion (Input 5: if averaging multiplies the integrand by a

number different from the modulus, the integral over the quotient vanishes) gives the result. □

LEAN `centreCoeff_eq_zero_of_not_negative`

Proposition 5.3 (theta terms of standard data)**.** *Let E be a CM field with a CM type and Weil data; let X, W_1, V be hermitian spaces with index sets $\mathbf{m}$, $\mathbf{n}$, $\mathbf{a}$, and $\mathfrak{S}$ a base-adapted splitting of X over W_1 along $\mathbf{b}$. Let Σ be a finite set of finite places whose product of completions F_Σ is not the zero ring, with a reading of the components at Σ, and let $\mathfrak{A}$ be the adapted frame of $\mathfrak{S}$. Let ω' be a character, χ a multiplicative character of F_Σ, c_0 a complex number and n a $(\mathbf{b} \sqcup \mathbf{n}) \times \mathbf{a}$ matrix over F_Σ; let $\check{\Psi}$ be a function with $\check{\Psi} \in \mathcal{S}(\mathcal{U})$ for a set $\mathcal{U}$ and $\check{\Psi} = \mathrm{st}_{\mathfrak{A}}(\mathsf{y} \mapsto \mathrm{c}_0\, \chi^\flat(\det(\mathsf{y}\mathsf{n})))$; let $\mathfrak{v}$ be a rapid real datum; let ω, ω'' be characters with $\omega'' = \omega\, v^{|\mathbf{b}|}$; and let c, w, q, g be the coordinates of a point. Put $f = \mathfrak{v} \otimes [\check{\Psi}]$ and $\mathsf{p} = \mathrm{rad}_{\mathfrak{S}}(\mathsf{c}, \mathsf{w})\ \mathrm{lev}_{\mathfrak{S}}(\mathsf{q}, \mathsf{g})$. Then:*

(1) *the theta terms of $\mathfrak{S}$ for $(\mathsf{V}, \omega'', \omega', f)$ at p are integrable;*

(2) *for every rational $\mathbf{a} \times \mathbf{b}$ matrix v, with $\mathsf{v}' = \mathrm{x}_{\mathsf{V}}^{-1}\mathsf{v}$, every adelic $\mathbf{a} \times \mathbf{b}$ matrix Y with $\mathsf{v}'\mathsf{q} = \mathrm{sc}_E Y$ and every point k of $\mathrm{U}(\mathsf{V})(\mathbb{A})$, with $\mathsf{y} = \mathsf{k}_f^{-1} Y_f$, the theta term at v and k equals*

$$\mathrm{e}_{\mathbb{A}}\big(\tfrac{1}{2}\operatorname{tr}(\mathsf{w}\,\mathsf{v}^*\mathrm{J}_{\mathsf{V}}\mathsf{v})\big)\,\omega'_{\mathbf{a}}(\det \mathsf{q})^{-1}\,|\det \mathsf{q}|^{|\mathbf{a}|/2}\,[\mathsf{y}\ \textit{integral}]\cdot \mathrm{vac}_{\mathfrak{S}}(\mathsf{V}) \cdot\ \theta^{\infty}_{\mathsf{W}_1,\mathsf{V}}(\omega,\omega'; F_{\mathsf{y}}, \mathsf{v}'\mathsf{c}^*\operatorname{diag}(\mathsf{W}_1), (\mathsf{g},\mathsf{k}))(\mathrm{sl}_{\mathfrak{S}}(\mathsf{k}^{-1}Y;\mathfrak{v})),$$

where $\omega'_{\mathbf{a}}$ is the character attached to ω' in rank $|\mathbf{a}|$, $\mathrm{vac}_{\mathfrak{S}}(\mathsf{V})$ the vacuum factor of the mixed model, $\operatorname{diag}(\mathsf{W}_1)$ the diagonal form of the signs of W_1, and F_{y} the finite datum named at Σ by the function $\mathsf{y}' \mapsto \mathrm{c}_0\,\chi^\flat(\det((\bar{\mathsf{y}} \mid \mathsf{y}')\mathsf{n}))$ of the integral y', $\bar{\mathsf{y}}$ being the components of y at Σ;

(3) *for all signs t' on $\mathbf{b}$ and every $\mathbf{b} \times \mathbf{b}$ matrix T over E that, as an adelic matrix, is not represented by t': if every non-zero real slice $\mathrm{sl}_{\mathfrak{S}}(\mathsf{x};\mathfrak{v})$ has a Gram matrix $\mathsf{x}^*\operatorname{diag}(\mathsf{V})\mathsf{x}$ that is represented by t', then $\mathrm{coef}_{\mathfrak{S}}(\jmath T; \mathrm{I}_{\chi,\mathsf{V}}(\omega'',\omega'; f))(\mathsf{p}) = 0$.*

Proof. (1) The datum is named at a set of places disjoint from its conjugate by a Schwartz–Bruhat function, and $\mathfrak{v}$ is rapid; integrability then holds once the matrices at which $\check{\Psi}$ does not vanish give left-invertible transposed coordinate matrices at p (Input 3). Where $\check{\Psi} \neq 0$ the matrix $\mathsf{y}(\mathfrak{A}_1 \mid \mathfrak{A}_2)$ of the standard datum has a unit determinant, and the coordinate matrix of p on the frame of $\mathfrak{S}$ is the adelic frame times a block triangular matrix with diagonal blocks q and an invertible one; this gives the left inverse.

(2) Move the datum through the two factors of p. The radical element contributes the character $\mathrm{e}_{\mathbb{A}}(\frac{1}{2}\operatorname{tr}(\mathsf{w}\,\mathsf{v}^*\mathrm{J}_{\mathsf{V}}\mathsf{v}))$ and the Heisenberg operator of $\mathsf{v}'\mathsf{c}^*\operatorname{diag}(\mathsf{W}_1)$; the Levi element contributes $\omega'_{\mathbf{a}}(\det \mathsf{q})^{-1}|\det \mathsf{q}|^{|\mathbf{a}|/2}$ and replaces v' by $\mathsf{v}'\mathsf{q}$ (Input 3, with $\omega'' = \omega v^{|\mathbf{b}|}$). What remains is the source term of the local datum on the adapted frame: the vacuum factor times, when the finite part is integral, the theta functional of the vector named by the slice of $\check{\Psi}$, applied to the real slice of $\mathfrak{v}$ (Input 3). The slice of the standard datum $\check{\Psi}$ at $\bar{\mathsf{y}}$ is the function named in the statement.

(3) The coefficient is the sum over the rational v with $\mathsf{v}^*\mathrm{J}_{\mathsf{V}}\mathsf{v} = T$ of integrals of theta terms. By (2) each term is a multiple of the functional of Definition 5.1 at the real slice $\mathrm{sl}_{\mathfrak{S}}(\mathsf{k}^{-1}Y;\mathfrak{v})$. If this slice were non-zero, its Gram matrix would be represented by t'; but that Gram matrix is, up to the invertible factor q and the unitary part

of the point, the image of $\mathsf{v}^*\mathrm{J}_{\mathsf{V}}\mathsf{v} = T$, which is not represented by t'. So every term vanishes. □

LEAN `thetaTerm_stdDatum_spec`

Lemma 5.4 (products of data)**.** *Let E be a CM field with a CM type and Weil data, W_1 and V' hermitian spaces with index sets $\mathbf{n}$ and $\mathbf{b}$. Let ω, ω' be characters of $\mathrm{U}(1)(\mathbb{A})$; F' and F finite data for $(\mathsf{W}_1, \mathsf{V}')$ and $(\mathsf{W}_1, \mathsf{W}_1)$; x an adelic $\mathbf{b} \times \mathbf{n}$ matrix; g, u, l points of $\mathrm{U}(\mathsf{W}_1)(\mathbb{A})$, $\mathrm{U}(\mathsf{V}')(\mathbb{A})$, $\mathrm{U}(\mathsf{W}_1)(\mathbb{A})$; and $\mathfrak{v}'$, $\mathfrak{v}$ real data for $(\mathsf{W}_1, \mathsf{V}')$ and $(\mathsf{W}_1, \mathsf{W}_1)$. Then*

$$\theta^{\infty}_{\mathsf{W}_1, \mathsf{V}' \oplus \mathsf{W}_1}(\omega, \omega\omega'; F' \boxtimes F, \binom{\mathsf{x}}{0}, (\mathsf{g}, \mathsf{u} \oplus \mathsf{l}))(\mathfrak{v}' \boxtimes \mathfrak{v}) = \theta^{\infty}_{\mathsf{W}_1, \mathsf{V}'}(\omega, \omega'; F', \mathsf{x}, (\mathsf{g}, \mathsf{u}))(\mathfrak{v}') \cdot \theta_{\mathsf{W}_1, \mathsf{W}_1}(\omega, \omega; \mathfrak{v} \otimes F)(\mathsf{g}, \mathsf{l}).$$

Proof. The model of the pair $(\mathsf{W}_1, \mathsf{V}' \oplus \mathsf{W}_1)$ is the tensor product of the models of $(\mathsf{W}_1, \mathsf{V}')$ and $(\mathsf{W}_1, \mathsf{W}_1)$; the Weil representation of a block-diagonal point, the Heisenberg operator of a vector in the first summand and the theta functional are compatible with this product. Unfolding the definitions gives the identity. □

LEAN `thetaOnRealData_boxProd_eq_mul`

Proposition 5.5 (the equal-rank theta integral is constant)**.** *Let E be a CM field with a CM type and Weil data, X and W hermitian spaces with index sets $\mathbf{m}$ and $\mathbf{a}$, $\mathfrak{S}_1$ a base-adapted splitting of X over W along one line, and $\mathfrak{c}$ a split tame character. Suppose that $|\mathbf{m}|$ is even, and let $\mathfrak{v}$ be a rapid real datum for (X, X). Then for every point g of $\mathrm{U}(\mathsf{X})(\mathbb{A})$*

$$\theta^{\mathrm{lg}}_{\mathsf{X},\mathsf{X}}(\mathfrak{c}; \mathfrak{v}, \Psi^{\mathrm{st}}_{\chi_{\mathfrak{c}},1})(\mathsf{g}) = \theta^{\mathrm{lg}}_{\mathsf{X},\mathsf{X}}(\mathfrak{c}; \mathfrak{v}, \Psi^{\mathrm{st}}_{\chi_{\mathfrak{c}},1})(1).$$

Proof. Apply Proposition 4.2 to (X, X) and the zero space, with constant 0: the partners of the empty frame are $\Psi^{\mathrm{st}}_{\chi_{\mathfrak{c}},1}$ on both sides, so the function in question is $\mathcal{D}(\Psi^{\mathrm{st}}_{\chi_{\mathfrak{c}},1})$ for an equivariant lift. It is therefore automorphic, has level supported at $\Sigma(\mathfrak{c})$ and is invariant under $\mathcal{K}_f$ off $\Sigma(\mathfrak{c})$. By the tower property (Input 5) its constant term along the isotropic line of $\mathfrak{S}_1$ is a theta integral of the smaller pair, so that on the Levi subgroup it depends only on the component in $\mathrm{U}(\mathsf{W})$. The constancy criterion from strong approximation (Input 5) applies. □

LEAN `thetaLarge_eq_apply_one_of_even_rank`

Definition 5.6 (the equal-rank ratio; the standard exchange identity)**.** Let E be a CM field with a CM type and Weil data.

(a) For a hermitian space V with index set $\mathbf{a}$, a split tame character $\mathfrak{c}$ and a real datum $\mathfrak{v}$ for (V, V),

$$\varrho(\mathsf{V}; \mathfrak{c}, \mathfrak{v}) = \frac{\left|\theta^{\mathrm{lg}}_{\mathsf{V},\mathsf{V}}(\mathfrak{c}; \mathfrak{v}, \Psi^{\mathrm{st}}_{\chi_{\mathfrak{c}},1})(1)\right|^2}{\Pi(\mathfrak{c}, \lfloor |\mathbf{a}|/2 \rfloor)}.$$

(b) Let X, V, V' be hermitian spaces with index sets $\mathbf{m}$, $\mathbf{n}$, $\mathbf{b}$, let μ be weights, $\mathsf{e} \colon \mathbf{b} \sqcup \mathbf{n} \simeq \mathbf{m}$ an identification, $\mathfrak{v}$ and $\mathfrak{y}$ real data for (X, V) and $(\mathsf{X}, \mathsf{V}')$, and $\mathcal{N}$ a real function on the split tame characters of weight μ. The statement

$$\mathrm{Exch}(\mathsf{X}, \mathsf{V}, \mathsf{V}'; \mu, \mathsf{e}; \mathfrak{v}, \mathfrak{y}; \mathcal{N})$$

holds if $\mathfrak{v}$ and $\mathfrak{y}$ are rapid and there is a real number C such that every split tame character $\mathfrak{c}$ of weight μ has a complex number c with $C\,|c|^2 = \Pi(\mathfrak{c}, |\mathbf{n}| - \lfloor |\mathbf{m}|/2 \rfloor) \cdot \mathcal{N}(\mathfrak{c})$ and

$$\theta^{\mathrm{lg}}_{X,V}(\mathfrak{c}; \mathfrak{v}, \check{\Psi}^{\mathrm{st}}_{e,\chi_{\mathfrak{c}},z}) = c \cdot \theta^{\mathrm{sm}}_{X,V'}(\mathfrak{c}; \mathfrak{y}, \Psi^{\mathrm{st}}_{\chi_{\mathfrak{c}},z}) \qquad \text{for every integral } z \in M_{\mathbf{b}\times\mathbf{m}}(F_{\mathfrak{c}}).$$

LEAN `equalRankThetaRatio` · `StdExchange`

6 The real pair, and the proofs of Theorems 3 and 1

Proof of Theorem 3. Both data are rapid by assumption. *The constant.* The inclusion of $U(V') \times U(W_1)$ in $U(V' \oplus W_1)$ is a closed embedding. Unfolding in stages (Input 5), with the weight that is Ω at the infinite places on the points whose finite part lies in $\mathcal{K}_f$ up to a block element, and 0 on the other points, gives a number $R \neq 0$. Let $a = \det(\mathrm{sc}_E\, x_{V'})$, an idele, and $k = \mathrm{vac}_{\mathfrak{S}}(V' \oplus W_1)\,\mathrm{vac}_{\mathfrak{S}}(V')^{-1}\,|a|^{|\mathbf{n}|/2}$. Take $C = |R\,k|^{-2}$. Fix a split tame character $\mathfrak{c}$ of weight μ, and let $\mathfrak{A}$ be the frame of $\mathfrak{S}$ adapted to the places of $\mathfrak{c}$. By Lemma 3.10 there is c_1 with $|c_1| = 1$ and $\check{\Psi}^{\mathrm{st}}_{e',\chi_{\mathfrak{c}},\mathfrak{A}'_3} = \mathrm{st}_{\mathfrak{A}}(y \mapsto c_1 \chi^{\flat}_{\mathfrak{c}}(\det y))$. Let $(\mathcal{M}, \mathcal{D})$ be the equivariant lift of Proposition 4.2 for X, $V' \oplus W_1$, V', the data $\mathfrak{v}$, $\mathfrak{v}'$ and the constant

$$c = R\,k\,c_1\,\omega^{\mathrm{bc}}_{\mathfrak{c}}(a)\,l(\mathfrak{c}).$$

Then $C|c|^2 = |l(\mathfrak{c})|^2$, and this is $\Pi(\mathfrak{c}, |\mathbf{b}| + |\mathbf{n}| - |\mathbf{m}|/2)$ times $|l(\mathfrak{c})|^2 / \Pi(\mathfrak{c}, |\mathbf{n}|/2)$ because $|\mathbf{m}| = |\mathbf{n}| + 2|\mathbf{b}|$ and the products of L-values are positive. For an integral z the row and column data of z are partners, so the asserted identity is $\mathcal{D}(\Psi^{\mathrm{st}}_{\chi_{\mathfrak{c}},z}) = 0$. By Theorem 2 it suffices that $\Psi_0 = \Psi^{\mathrm{st}}_{\chi_{\mathfrak{c}},\mathfrak{A}'_3}$ lies in $\mathcal{M}$, which holds because it is the partner of a row datum, and that the centre coefficients of $\mathcal{D}(\Psi_0)$, the integral on the large side of the partner of Ψ_0 minus c times the integral on the small side of Ψ_0, vanish in the two cases of Theorem 2.

Case (i)*: the index* $J_{V'}$. By Proposition 5.3 the theta terms of both pairs are integrable and explicit. The coefficient of the large integral is the integral over the quotient by the stabilizer of the rational frame $\binom{1}{0}$ of V' in $V' \oplus W_1$, which is $U(W_1)$; that of the small integral is the integral of the theta term at 1 over $U(V')(\mathbb{A})$, the stabilizer being trivial (Input 3). By unfolding in stages the first is R times the number c' for which the inner integral over $[U(W_1)]$ of the large theta term at a point h, shifted by a point of $U(V') \times U(W_1)$ that may depend on h, is $\Omega(h)\,c'$ times the small theta term; the shift of Input 5 is the point p of the period condition. If the finite part of the point lies in $\mathcal{K}_f$ up to a block element, insert the explicit terms of Proposition 5.3 (2): for the large pair with the constant c_1 and the matrix 1, for the small one with the frame $\binom{1}{0}$; the standard row datum is fixed by the integral point, the named vector of the large pair is the product of the vector named by the indicator of the integral matrices and of $F_{\mathfrak{c}}$, and the period condition, used with $B = y^{-1}Y$, evaluates the integral over $[U(W_1)]$. The characters are collected as follows: $\omega(\det B)$ splits into its infinite and finite parts; the quotient of the values of $\chi^{\flat}_{\mathfrak{c}}$ at $\det x$ and at its conjugate is the value of $\omega_{\mathfrak{c}}$ at the finite part (unit 38); $|\det y| = 1$ for unitary y. The result is $c' = k\,c_1\,\omega^{\mathrm{bc}}_{\mathfrak{c}}(a)\,l(\mathfrak{c})$. If the finite part does not lie in $\mathcal{K}_f$ up to a block element, the large theta term vanishes: were the finite matrix integral, Witt's theorem over $\hat{\mathcal{O}}_E$ (Input 2) would produce such a block element. So the coefficient of $\mathcal{D}(\Psi_0)$ is $Rc' - c = 0$ times the small coefficient.

Case (ii)*: T invertible and not represented by the signs of V'.* The small coefficient vanishes by Proposition 5.3 (3), the slices of the pair (X, V') having Gram matrices represented by the signs of V'. The large one vanishes by Proposition 5.3 (3) under the first alternative of the support or sign condition, and by Proposition 5.2 under the second: V' is negative definite, W_1 has a positive sign, and the bound on the weights follows from $2\,\mathrm{neg}(W_1) \le |\mathbf{n}|$ and the positivity of the weights of $\mathfrak{c}$. □

Lemma 6.1 (a datum with prescribed slices)**.** *Let E be a CM field with a CM type and Weil data; let X, W_1, V' be hermitian spaces with index sets $\mathbf{m}$, $\mathbf{n}$, $\mathbf{a}$, and $\mathfrak{S}$ a base-adapted splitting of X over W_1 along $\mathbf{b}$. Let Ω_0 be a real function on the space of real coordinates of the slices that is smooth, has compact support, is non-negative and is non-zero at a point y, and let $\mathfrak{v}$ be a rapid real datum for (W_1, V'). Then there are a complex number k and a rapid real datum $\mathfrak{x}'$ for (X, V') such that the slice of $\mathfrak{x}'$ at 0 for the splitting opposite to $\mathfrak{S}$ is $\mathfrak{v}$, and the slice of $\mathfrak{x}'$ at every x, in the mixed model of $\mathfrak{S}$, is $\Omega_0(x)\,k \cdot \omega_\infty(1)\mathfrak{v}$.*

Proof. The integral of Ω_0 is positive; let k be its inverse. By the existence of sections in the mixed model (Input 3: a smooth compactly supported coefficient times a fixed rapid vector) there is a rapid $\mathfrak{x}'$ with the prescribed slices. The slice at 0 for the opposite splitting is the integral over x of the slices (Input 3), which is $(\int \Omega_0)k\,\mathfrak{v} = \mathfrak{v}$; two real data are equal when the integrated data built from their zero terms agree. □

LEAN `exists_rapid_with_realSlice`

Lemma 6.2 (Witt's theorem, smooth local form)**.** *Let $\mathbf{b}$, $\mathbf{m}$ be finite index sets, J a complex $\mathbf{m}\times\mathbf{m}$ matrix with $J^* = J$, and x a complex $\mathbf{m}\times\mathbf{b}$ matrix with $\det(x^*Jx) \neq 0$. There is a map k from the $\mathbf{m}\times\mathbf{b}$ matrices to the unitary group of J such that $y \mapsto k(y)$ is smooth as a map of real vector spaces and, for every y in a neighbourhood of x, $k(y)\,x\,c = y$ for some $\mathbf{b}\times\mathbf{b}$ matrix c.*

Proof. Put $w = (x^*Jx)^{-1}x^*$ and, for an $\mathbf{m}\times\mathbf{m}$ matrix g, $S_0(g) = (gxw - xwg^*)J$, which is skew for J, so that $\exp S_0(g)$ is unitary. The map $g \mapsto \exp(S_0(g))(g - S_0(g))$ has derivative the identity at $g = 1$, so it has a smooth local inverse A near 1 (and a smooth function defined everywhere agrees with it near 1, unit 112). With $p(y) = 1 - (x - y)wJ$ one has $p(y)x = y$ and $p(x) = 1$; put $k(y) = \exp S_0(A(p(y)))$. Near x, $k(y)(Ap(y) - S_0(Ap(y)))x = p(y)x = y$, and $(g - S_0(g))x$ lies in the column space of x, since $gxwJx = gx$. □

LEAN `exists_contDiff_witt`

Proof of Theorem 1. Let J be the diagonal form of $V'\oplus W_1$ at a real place and $x_0 = \binom{1}{0}$ the frame of V', so that $x_0^*Jx_0$ is the diagonal form of V'. Lemma 6.2 gives a smooth k with $k(y)x_0c = y$ near x_0. Choose a smooth bump $\Omega_1 \ge 0$ at x_0 (in the coordinates at every infinite place) with compact support in the neighbourhood where this holds and where the Gram matrix x^*Jx is invertible, and put $f_0(x) = x\,(x^*Jx)^{-1}(x^*Jx_0)$, so $f_0(x_0) = x_0$. Where $\Omega_1(f_0(x)) \neq 0$ one has $x = k(f_0(x))\,x_0\,c$ with $c^*\,\mathrm{diag}(V')c = x^*Jx$.

The small datum. Lemma 6.1, with the weight $x \mapsto \Omega_1(x_0x)\left|\prod_\tau \det(x_\tau)^{|\mathbf{n}|/2+\mu(\tau)}\right|^2$ (smooth, non-negative, compactly supported, non-zero at 1), gives k and a rapid $\mathfrak{x}'$ whose slice at 0 for the opposite splitting is $\mathfrak{v}$.

The large datum. Let $\mathfrak{x}$ be a rapid real datum for $(\mathsf{X}, \mathsf{V}' \oplus \mathsf{W}_1)$ whose slice at x is, with $\mathsf{y} = \mathsf{f}_0(\mathsf{x})$,

$$\Omega_1(\mathsf{y})\,\Omega_1(\mathsf{k}(\mathsf{y})^{-1}\mathsf{x})\,\prod_{\tau}\overline{\det((\mathsf{k}(\mathsf{y})^{-1}\mathsf{x})_{\tau,1})}^{\,2\mu(\tau)}\,\mathsf{k}\cdot\omega_\infty(1,\mathsf{k}(\mathsf{y}))(\mathfrak{v}\boxtimes\mathfrak{v}_1),$$

where $(\,\cdot\,)_{\tau,1}$ is the first block at τ and ω_∞ the real Weil representation of weights $|\mathbf{n}|/2-\mu$. It exists by the existence of sections (Input 3): the coefficient is smooth because k is, it has compact support because its support lies in the image of a product of two compact sets, and the weighted norms of the vector field are bounded by those of the vacuum, of $\mathfrak{v}_1$ and of $\mathfrak{v}$.

The period condition. Apply Theorem 3 to $\mathfrak{x}$, $\mathfrak{x}'$, the weight $\Omega(\mathsf{h}) = \Omega_1(\mathsf{f}_0(\mathsf{h}^{-1}\mathsf{x}_0))$, which is non-negative and non-zero near 1, and $\mathrm{l}(\mathfrak{c}) = \theta^{\mathrm{lg}}_{\mathsf{W}_1,\mathsf{W}_1}(\mathfrak{c};\mathfrak{v}_1,\Psi^{\mathrm{st}}_{\chi_\mathfrak{c},1})(1)$. The support condition holds in its first form by the relation $\mathsf{c}^*\,\mathrm{diag}(\mathsf{V}')\mathsf{c} = \mathsf{x}^*\mathsf{J}\mathsf{x}$. For the period condition let $\mathsf{y} = \mathsf{f}_0(\mathsf{h}^{-1}\mathsf{x}_0)$. If $\Omega_1(\mathsf{y}) = 0$ both sides vanish. Otherwise there are c_τ preserving the form of V' with $\mathsf{k}(\mathsf{y}_\tau)\mathsf{x}_0\mathsf{c}_\tau = (\mathsf{h}^{-1}\mathsf{x}_0)_\tau$, hence elements u_τ of the real unitary group of V'; then $\mathsf{h}_\tau\mathsf{k}(\mathsf{y}_\tau)(\mathsf{u}_\tau\oplus 1)$ fixes x_0, so it is $1\oplus\mathsf{q}_\tau$ with q_τ in the real unitary group of W_1. Use the point $\mathsf{p} = (\mathsf{u},\mathsf{q}^{-1})$. At the slice argument $\mathsf{x} = \mathsf{h}^{-1}\mathsf{x}_0\mathsf{u}^{-1}B$ one checks $\mathsf{f}_0(\mathsf{x}) = \mathsf{y}$, so the slices of $\mathfrak{x}$ and $\mathfrak{x}'$ are those written above. The Weil representation of the point on $\mathfrak{v}\boxtimes\mathfrak{v}_1$ factors by Lemma 5.4; the theta kernel of $(\mathsf{W}_1,\mathsf{W}_1)$ with $\mathfrak{v}_1$, integrated over $[\mathrm{U}(\mathsf{W}_1)]$, is the theta integral on the large side, which is constant (Proposition 5.5, by $\mathfrak{S}_1$ and $|\mathbf{n}|$ even) and so equals $\mathrm{l}(\mathfrak{c})$; the character of $\mathfrak{c}$ at the infinite places is read from its weights (unit 96), which produces $\omega^{\mathrm{bc}}_{\mathfrak{c}}(\det_\infty B)$ and the power of $|\det_\infty B|$. Clearing denominators gives the period condition. So Theorem 3 yields the standard exchange identity with the function $\mathfrak{c}\mapsto|\mathrm{l}(\mathfrak{c})|^2/\Pi(\mathfrak{c},|\mathbf{n}|/2)$, which is $\varrho(\mathsf{W}_1;\mathfrak{c},\mathfrak{v}_1)$ by Definition 5.6. □

7 Determinant vectors and the cone integral

The tower property (Input 5) produces a *cone integral* of the real datum. One floor down in the tower, the second note needs to know that the real data of a special shape are reproduced by it.

Definition 7.1 (determinant vectors)**.** Let E be a CM field with a CM type and Weil data, W and V hermitian spaces with index sets $\mathbf{n}$ and $\mathbf{r}$ and signs sgn_W and sgn_V. Let F be an adelic $\mathbf{n}\times\mathbf{r}$ matrix and u a point of the real coordinate space of the Fock model of the pair (W,V) (coordinates indexed by an infinite place and a pair of indices).

(a) $\mathrm{pos}_{\mathrm{sgn}_\mathsf{W},\mathrm{sgn}_\mathsf{V}}(F,\mathsf{u})$ is the family, over the infinite places, of the square matrices u^+F, where u^+ is the matrix of the coordinates of the projection of u to the coordinates that join basis vectors of equal sign.

(b) For a natural number e, natural numbers $\mu(\tau)$ at the infinite places and $\mathrm{c}\in\mathbb{C}$,

$$\mathrm{dv}^{e,\mu}_{\mathrm{sgn}_\mathsf{W},\mathrm{sgn}_\mathsf{V}}(\mathrm{c},F;\mathsf{u}) = \mathrm{c}\cdot\mathrm{vac}(\mathsf{u})\cdot\overline{\prod_\tau \det{}_+(\mathsf{M}_\tau)^{\mu(\tau)}\,\det{}_-(\mathsf{M}_\tau)^{\mu(\tau)+2e}},$$

where $\mathsf{M} = \mathrm{pos}_{\mathrm{sgn}_\mathsf{W},\mathrm{sgn}_\mathsf{V}}(F,\mathsf{u})$, $\det_\pm$ are the principal minors on the indices of $\mathbf{r}$ of sign ±1, and vac is the Gaussian, the vacuum vector.

(c) $\mathrm{Det}(e,\mu)$ is the set of the vectors of the Fock model of (W,V) that are rapid and equal to $\mathsf{u}\mapsto\mathrm{dv}^{e,\mu}_{\mathrm{sgn}_\mathsf{W},\mathrm{sgn}_\mathsf{V}}(\mathrm{c},F;\mathsf{u})$ for some c and F. The pair (W,V) is an implicit argument; we name it when needed.

LEAN posCoordMatrix · detVectorFn · detVectors

Lemma 7.2 (the real frame of an adapted splitting)**.** (a) *Let E be a CM field with a CM type, W and W' hermitian spaces with index sets $\mathbf{n}$ and $\mathbf{n}'$, and $\mathfrak{S}$ a splitting of W over W' along $\mathbf{r}$ that is adapted at the point 1 with signs t. For every infinite place τ the real frame matrix P of $\mathfrak{S}$ at τ is unitary, and $\operatorname{diag}(\mathsf{W})\,\mathsf{P} = \mathsf{P}\,\mathsf{B}$, where B has the block $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ on $\mathbf{r} \sqcup \mathbf{r}$ and the diagonal form of W' on $\mathbf{n}'$.*

(b) *For a finite index set $\mathbf{n}$ with signs, the projection onto the coordinates of positive sign is $\frac{1}{2}(1 + \operatorname{diag})$ of the signs.*

Proof. (a) Adaptedness gives $\mathsf{P}^*\mathsf{P} = 1$, and P is square, so $\mathsf{P}\mathsf{P}^* = 1$. The Gram matrix of the frame of a splitting is B (unit 38), that is, $\mathsf{P}^* \operatorname{diag}(\mathsf{W})\mathsf{P} = \mathsf{B}$; multiply by P. (b) By definition. □

LEAN frameAt_real_unitary · posProj_eq

Lemma 7.3 (slices of polynomial vectors)**.** *Let E be a CM field with a CM type and Weil data, W and W' hermitian spaces with index sets $\mathbf{n}$ and $\mathbf{n}'$, and $\mathfrak{S}$ a splitting of W over W' along $\mathbf{r}$ that is adapted at 1 with signs t; let L, I, K be the blocks of its mixed model and P_+ the projections to the positive coordinates. For all coordinate vectors a and u there is U with $\mathrm{P}_+\mathsf{U} = \mathsf{L}(-\mathsf{a}) + \mathsf{I}\,\mathrm{P}_+\mathsf{u} + \mathsf{K}(-\mathsf{a})$ such that, for every polynomial in the positive coordinates, with Fock vector $\mathfrak{v}$: the slice of $\mathfrak{v}$ at a, evaluated at u, is $\mathfrak{v}(\mathsf{U})$.*

Proof. Put $\mathsf{J}_0(\mathsf{a},\mathsf{z},\mathsf{b}) = \mathsf{L}\mathsf{a} + \mathsf{I}\mathsf{z} + \mathsf{K}\mathsf{b}$ and $\mathsf{U} = \mathsf{J}_0(-\mathsf{a},\mathsf{u},-\mathsf{a})$. Lemma 7.2 gives $\mathrm{P}_+\mathsf{J}_0(\mathsf{a},\mathsf{z},\mathsf{b}) = \mathsf{J}_0(\frac{1}{2}(\mathsf{a}+\mathsf{b}), \mathrm{P}_+\mathsf{z}, \frac{1}{2}(\mathsf{a}+\mathsf{b}))$ and $\langle \mathsf{J}_0(\mathsf{a},\mathsf{u},\mathsf{a}), \mathsf{J}_0(\mathsf{c},\mathsf{z},\mathsf{c})\rangle = 2\langle \mathsf{a},\mathsf{c}\rangle + \langle \mathsf{u},\mathsf{z}\rangle$, the real frame matrix being orthonormal. Both sides of the asserted identity are continuous and linear in the polynomial vector, so it suffices to compare them on the Heisenberg translates of the vacuum (unit 37), where the slice is explicit and both sides are the same Gaussian exponential by the two formulas. □

LEAN PositivePart.exists_slice_eq

Proposition 7.4 (the cone integral is onto the determinant vectors)**.** *Let E be a CM field with a CM type and Weil data; let W, W', V, V' be hermitian spaces with index sets $\mathbf{n}$, $\mathbf{n}'$, $\mathbf{r}$, $\mathbf{r}'$; let $\mathfrak{A}$ be a base-adapted splitting of W' over W along one line and $\mathfrak{S}$ a base-adapted splitting of V' over V along one line. Let e be a natural number, $\mu(\tau)$ natural numbers at the infinite places, b a parameter of the cone integral and z a real number with $1 \le \mathrm{z}$, and suppose $|\mathbf{n}'| = 2(e + \operatorname{neg}(\mathsf{W}'))$. Then the cone integral with weights $e + \mu$,*

$$\mathfrak{v}' \longmapsto \iint \operatorname{cone}_{\mathfrak{A},\mathfrak{S}}(e+\mu, \mathrm{b}, \mathrm{z}; \mathfrak{v}', \mathsf{p})\, d\mathsf{p}$$

(the integral of the cone integrand of the tower property for the product of two Haar measures) has this property: every element of the set $\operatorname{Det}(e,\mu)$ of the pair (W,V) is the cone integral of an element of the set $\operatorname{Det}(e,\mu)$ of the pair $(\mathsf{W}',\mathsf{V}')$. (Nothing is said about the cone integrals of the other determinant vectors of the larger pair.)

Proof. Let $\mathfrak{v} = \mathrm{dv}^{e,\mu}_{\mathrm{sgn}_{\mathsf{W}},\mathrm{sgn}_{\mathsf{V}}}(\mathrm{c}, F; \cdot)$ be a determinant vector. The line-term expansion of the cone integral (unit 43) provides a splitting, adapted at 1, whose mixed terms are the line terms, and its frame in terms of the blocks of $\mathfrak{A}$ and $\mathfrak{S}$. Let G be the adelic $\mathbf{n}' \times \mathbf{r}'$ matrix that is F on the smaller spaces plus a multiple of the hyperbolic part,

and for $\mathrm{c}' \in \mathbb{C}$ let $\mathrm{R}(\mathrm{c}')$ be the Fock vector of the polynomial $\mathrm{c}' \prod_\tau \Delta_{\tau,+}^{\mu(\tau)} \Delta_{\tau,-}^{\mu(\tau)+2e}$, where $\Delta_{\tau,\pm}$ are determinants of matrices of linear forms with coefficients from G. The real Weil representation of a block-diagonal point acts on a polynomial vector by powers of principal minors and a linear substitution (unit 43); with the exponents fixed by $|\mathbf{n}'| = 2(e + \mathrm{neg}(\mathsf{W}'))$ the minors cancel, and $\mathrm{R}(\mathrm{c}') = \mathrm{dv}^{e,\mu}_{\mathrm{sgn}_{\mathsf{W}'},\mathrm{sgn}_{\mathsf{V}'}}(\mathrm{c}', G; \cdot)$; it is rapid, being a polynomial vector. By Lemma 7.3 the slice of a polynomial vector is its value at an explicit point, and by Lemma 7.2 the matrix $\mathrm{pos}_{\mathrm{sgn}_{\mathsf{W}'},\mathrm{sgn}_{\mathsf{V}'}}(G, \cdot)$ at that point is conjugate, by the unitary frame matrix, to the block sum of the Levi coordinate $\mathfrak{q}_\tau$ on the hyperbolic plane and of $\mathrm{pos}_{\mathrm{sgn}_{\mathsf{W}},\mathrm{sgn}_{\mathsf{V}}}(F, \mathfrak{u})$; so each of its principal minors is $\mathfrak{q}_\tau$ times the corresponding minor of the smaller matrix. Therefore the cone integrand of $\mathrm{R}(\mathrm{c}')$ at a point $\mathfrak{p}$ is $\exp(\mathrm{E}_0(\mathfrak{p}))\, \mathrm{c}' \mathrm{c}^{-1} \mathfrak{v}$ with the real function $\mathrm{E}_0(\mathfrak{p}) = -\sum_\tau |\mathfrak{q}_\tau|^2 + \log(|\mathfrak{q}|^z \prod_\tau |\mathfrak{q}_\tau|^{2(e+\mu(\tau))})$. For $\mathfrak{v} \neq 0$ the function $\exp \mathrm{E}_0$ is integrable with positive integral, and $\mathrm{c}' = \mathrm{c}\,(\int \exp \mathrm{E}_0)^{-1}$ gives a determinant vector with cone integral $\mathfrak{v}$; for $\mathfrak{v} = 0$ take 0. □

LEAN `surjOn_cone_integral_detVectors`

8 Two norm formulas

Setting 8.1 (auxiliary pairs and L-value data)**.** The following objects are those of units 134, 138, 140 and 141 and of the second note of this unit; we recall what the statements need.

(a) E is a CM field with a CM type and Weil data, $\mathbf{r}$ and $\mathbf{n}$ are finite index sets. An *auxiliary pair* $\mathfrak{a}$ for E, $\mathbf{r}$, $\mathbf{n}$ consists of two families of finite places (the chosen places and the twist places), a hermitian space $\mathsf{W}(\mathfrak{a})$ with index set $\mathbf{r}$, an embedding of $\mathbf{r}$ into $\mathbf{n}$ with $|\mathbf{r}| < |\mathbf{n}|$, a weight $\mathrm{wt}(\mathfrak{a})$ of forms on $\mathsf{I}_\mathbf{n}$ and a character. It determines: the characters and the theta kernel of the pair $(\mathsf{W}(\mathfrak{a}), \mathsf{I}_\mathbf{n})$; a finite datum $\Upsilon(\mathfrak{a})$ and, for the highest weight vector of $\mathrm{wt}(\mathfrak{a})$, a real datum $\mathfrak{v}(\mathfrak{a})$; the space $\mathcal{A}(\mathfrak{a})$ of algebraic forms on $\mathsf{I}_\mathbf{n}$ of the level, character and weight of $\mathfrak{a}$, with its inner product $\langle\ ,\ \rangle$; the class-sum lift $\theta_\mathfrak{a}$ from $\mathcal{A}(\mathfrak{a})$ to forms on $\mathsf{W}(\mathfrak{a})$; the residue ring $\mathsf{R}(\mathfrak{a})$ of the product of all places of $\mathfrak{a}$; the level indicator $\mathbb{1}_\mathfrak{a}$, a function on $\mathrm{M}_{\mathbf{n}\times\mathbf{r}}(\mathsf{R}(\mathfrak{a}))$; the number $\mathrm{t}(\mathfrak{a})$, the product over the twist places of the orders of $\mathrm{GL}_\mathbf{r}$ of the residue fields; the level group of $\mathfrak{a}$ and its index $\mathrm{i}(\mathfrak{a})$ in the stabilizer of the integral lattice; and the action of the product of the groups $\mathrm{GL}_\mathbf{r}$ of the residue fields of the chosen places on $\mathcal{A}(\mathfrak{a})$. A parameter on $\mathsf{W}(\mathfrak{a})$ may *fit* $\mathfrak{a}$. A form $F \in \mathcal{A}(\mathfrak{a})$ *lifts* a family $\mathfrak{p}$ of complex numbers $\mathfrak{p}(\mathfrak{p}, n)$, n a natural number, if at every finite place $\mathfrak{p} \neq \bar{\mathfrak{p}}$ off the places of $\mathfrak{a}$ and their conjugates it is spherical with the Hecke polynomial that Hecke transfer (Input 7) attaches to the polynomial of $\mathfrak{p}$ at $\mathfrak{p}$.

(b) P are parameters, D admissible data, ϑ a tame character of D and i a lift index, $1 \le i \le m$. Then $\mathsf{W}_i(D)$ is the i-th space of the tower of the rows, $\mathfrak{a}_D(\vartheta, i)$ the i-th auxiliary pair, the *forms of principal level* on a space are those of the parameter of the rows fixed by the principal level at the chosen places, and the *adjoint lift* $\theta^*_{D,\vartheta,i}$ is the adjoint of $\theta_{\mathfrak{a}_D(\vartheta,i)}$, a linear map from the forms of principal level on $\mathsf{W}_i(D)$ to $\mathcal{A}(\mathfrak{a}_D(\vartheta, i))$. The *eigenpart* of a space for a family $\mathfrak{p}$ consists of the forms of principal level on which each Hecke operator acts by the value of $\mathfrak{p}$. $\Theta^\circ_{D,\vartheta}$ is the chain of the m lifts as a map on all forms of principal level on the first space $\mathsf{Y}_0(D)$ of the chain.

(c) *L-value data* $\mathfrak{l}$ for E and $\mathbf{n}$ consist of finitely many chosen places and twist places (finite places of E; $\Sigma(\mathfrak{l})$ is the set of all of them, and $\overline{\Sigma(\mathfrak{l})}$ the set of their conjugates), jet degrees at the infinite places, and a character $\omega_{\mathfrak{l}}$ of the corresponding weight, unramified off the places, with $|\mathbf{n}| > 1$. They determine an auxiliary pair $\mathfrak{a}(\mathfrak{l})$ for the one-point index set and $\mathbf{n}$, with $\mathsf{W}(\mathfrak{a}(\mathfrak{l})) = \mathsf{I}_1$, and the lift $\Xi(\mathfrak{l}) \in \mathcal{A}(\mathfrak{a}(\mathfrak{l}))$ of the trivial character, a class sum of kernel forms. hook($\mathfrak{l}$) is the hook number of the weight of $\mathfrak{a}(\mathfrak{l})$ in rank $|\mathbf{n}|$, weyl($\mathfrak{l}$) the number of elements of its Weyl index set, and $\mathrm{sg}(\mathfrak{p}) \in \{\pm 1\}$ the sign that is -1 for one place of each pair of conjugate split places and 1 otherwise.

Lemma 8.2 (the adjoint of the class-sum lift is a theta lift)**.** *In Setting 8.1 (a), let* $\mathfrak{a}$ *be an auxiliary pair for* E, $\mathbf{r}$, $\mathbf{n}$ *with* $\mathbf{r}$ *non-empty,* l *a parameter on* $\mathsf{W}(\mathfrak{a})$ *that fits* $\mathfrak{a}$, y *a form on* $\mathsf{W}(\mathfrak{a})$ *of the type of* l*, and* $F \in \mathcal{A}(\mathfrak{a})$ *such that* $\langle F, \Xi\rangle = \langle y, \theta_{\mathfrak{a}}(\Xi)\rangle_{\mathrm{Pet}}$ *for every* $\Xi \in \mathcal{A}(\mathfrak{a})$*. Then* F *is the algebraic form of the theta lift of* y *to* $\mathsf{I}_{\mathbf{n}}$ *with the datum* $\mathfrak{v}(\mathfrak{a}) \otimes \Upsilon(\mathfrak{a})$*.*

Proof. Since l fits $\mathfrak{a}$, the real datum is an edge, so with $\Upsilon(\mathfrak{a})$ it gives an arrow from the type of l to the type on $\mathsf{I}_{\mathbf{n}}$ and, for the swapped pair, an arrow back (unit 134, Input 7). The algebraic form of the lift of y lies in $\mathcal{A}(\mathfrak{a})$. For every Ξ, $\theta_{\mathfrak{a}}(\Xi)$ is a fixed multiple of the conjugate of the descent of $\bar{\Xi}$, and the adjoint of the lift along an arrow is the conjugate of the descent; so the algebraic form of the lift has the same inner products with every Ξ as F. □

LEAN `AuxPair.adjoint_theta_eq_formLift`

Lemma 8.3 (the datum of the level indicator)**.** *In Setting 8.1 (a), let* $\mathfrak{a}$ *be an auxiliary pair,* $\mathfrak{v}$ *a real datum, and* y *a function on* $\mathrm{U}(\mathsf{W}(\mathfrak{a}))(\mathbb{A})$ *that is automorphic and rapidly decreasing and is fixed by the principal level at the chosen places. Then the theta lift of* y *with the datum* $\mathfrak{v} \otimes \Upsilon(\mathfrak{a})$ *equals its theta lift with the datum whose real part is* $\mathfrak{v}$ *and whose finite part is the function* $\mathrm{t}(\mathfrak{a}) \cdot \mathbb{1}_{\mathfrak{a}}$ *modulo the places of* $\mathfrak{a}$*.*

Proof. $\Upsilon(\mathfrak{a})$ is the sum, over the elements A of the product of the groups $\mathrm{GL}_{\mathbf{r}}$ of the residue fields at the twist places, of the translates of the datum of $\mathbb{1}_{\mathfrak{a}}$ by lifts of A^{-1} that reduce to 1 at the chosen places (unit 134). Each such lift lies in the principal level at the chosen places and fixes y; by the equivariance of the theta lift each summand gives the same lift. □

LEAN `AuxPair.lift_auxDatum_eq_lift_modDatum`

Lemma 8.4 (the level indicator)**.** *Let* E *be a CM field with a CM type,* $\mathbf{r}$ *and* $\mathbf{n}$ *finite index sets and* $\mathfrak{a}$ *an auxiliary pair for them. Put* $G = \mathrm{t}(\mathfrak{a}) \cdot \mathbb{1}_{\mathfrak{a}}$*. Then the rows of every* y *with* $G(\mathsf{y}) \neq 0$ *that are indexed by the image of* $\mathbf{r}$ *form the identity matrix, and*

$$\sum_{\mathsf{y}} |G(\mathsf{y})|^2 = \mathrm{t}(\mathfrak{a})^2 \cdot \Big(\prod_{\mathfrak{q}} \mathrm{N}\,\mathfrak{q}\Big)^{(|\mathbf{n}|-|\mathbf{r}|)\,|\mathbf{r}|},$$

the product over the twist places $\mathfrak{q}$ *of* $\mathfrak{a}$*.*

Proof. A matrix over $\mathsf{R}(\mathfrak{a})$ is the family of its reductions at the places of $\mathfrak{a}$, and this is a bijection (unit 104). In these terms $\mathbb{1}_{\mathfrak{a}}$ is the indicator of a product of sets, which are counted place by place. □

LEAN `AuxPair.levelIndicator_support_and_normSq`

Definition 8.5 (the Gram factor and the terms of the inner product)**.** (a) In Setting 8.1 (a), for an auxiliary pair $\mathfrak{a}$,

$$\operatorname{gram}(\mathfrak{a}) = \frac{\mathrm{i}(\mathfrak{a}) \cdot \mathrm{t}(\mathfrak{a})^2 \cdot \left(\prod_{\mathfrak{q}} \mathrm{N}\,\mathfrak{q}\right)^{(|\mathbf{n}|-|\mathbf{r}|)\,|\mathbf{r}|}}{\#\,\mathrm{GL}_{\mathbf{r}}(\mathsf{R}(\mathfrak{a}))},$$

the product over the twist places $\mathfrak{q}$ of $\mathfrak{a}$.

(b) In Setting 8.1 (b), let V be a hermitian space with the index set of the rows, y' and y forms of principal level on V, and $\bar{\mathsf{g}}$ a coset of the finite adelic unitary group of the definite space of the auxiliary pairs modulo its elements that are integral off the places of the modulus $\mathfrak{m}_D(\vartheta, 1)$ of $\mathfrak{a}_D(\vartheta, 1)$ and their conjugates; here $\mathfrak{m}_D(\vartheta, e)$ is, as in the other notes, the product of the chosen places to the power e times the places of ϑ. Then

$$\mathrm{ral}_{D,\vartheta,i}(\mathsf{V}; y', y; \bar{\mathsf{g}}) = \frac{\mathrm{ct}(\bar{\mathsf{g}})}{\#\,\mathrm{M}_{\mathbf{n}\times\mathbf{r}}(\mathcal{O}_E/\mathfrak{m}_D(\vartheta, 1))},$$

where $\mathrm{ct}(\bar{\mathsf{g}})$ is the coset term of the inner product formula (Input 6) for the character of $\mathfrak{a}_D(\vartheta, i)$ on the definite side: the vacuum coefficient at a representative g times $\langle y', R(\mathsf{g})y\rangle_{\mathrm{Pet}}$; here $\mathbf{n}$ and $\mathbf{r}$ are the index sets of the definite space and of the rows.

LEAN `AuxPair.gramFactor` · `Data.rallisTerm`

Lemma 8.6 (the chain of lifts is spherical)**.** *Let P be parameters and D admissible data, and let $\mathfrak{p}$ be a family of complex numbers $\mathfrak{p}(\mathfrak{p}, n)$. There is a family G of functions $G(\mathfrak{p})\colon \mathbb{C} \to \mathbb{C}$, one for each finite place, such that for every form y in the eigenpart of $\mathsf{Y}_0(D)$ for $\mathfrak{p}$ and every finite place $\mathfrak{p} \neq \bar{\mathfrak{p}}$ such that neither $\mathfrak{p}$ nor $\bar{\mathfrak{p}}$ is a chosen place, the form $\Theta^{\circ}_{D,1}(y)$ is spherical at $\mathfrak{p}$ with Hecke polynomial $G(\mathfrak{p})$.*

Proof. A form is spherical with a given Hecke polynomial exactly when its algebraic form is (unit 102). Induct on the length of the chain. A form in the eigenpart is spherical at such $\mathfrak{p}$ with the Hecke polynomial of $\mathfrak{p}$ (unit 138). Each further lift is, up to a scalar, the theta lift by a datum that is an arrow (unit 141), and Hecke transfer for theta lifts (Input 7) applies: the characters are unramified at $\mathfrak{p}$ because their tame parts are trivial off the places of the auxiliary pairs, and the trivial twist has no twist places. □

LEAN `Data.exists_isSphericalWith_iota`

Lemma 8.7 (adjointness)**.** *In Setting 8.1 (b), for parameters P, data D, a tame character ϑ, a form y of principal level on $\mathsf{W}_i(D)$ and $\Xi \in \mathcal{A}(\mathfrak{a}_D(\vartheta, i))$: $\langle \theta^*_{D,\vartheta,i}\, y, \Xi\rangle = \langle y, \theta_{\mathfrak{a}_D(\vartheta,i)}(\Xi)\rangle_{\mathrm{Pet}}$.*

Proof. This is the definition of the adjoint lift (unit 138). □

LEAN `Data.inner_auxLift`

Proof of Theorem 4. The real datum of the auxiliary pair $\mathfrak{a}_D(1, i)$ of the trivial twist is an edge (second note; unit 134), and $2\,\mathrm{rk}\,\mathsf{W}_i(D) < \mathrm{rk}\,\mathsf{I}_{\mathbf{n}}$. The Euler form of the inner product formula (Input 6) gives $A_0 \neq 0$, valid for all characters of the weights of the edge, hence for all ϑ. Put $A = A_0$ times the number of elements of the Weyl index set. Fix ϑ and write $\mathfrak{a} = \mathfrak{a}_D(\vartheta, i)$, $G = \mathrm{t}(\mathfrak{a})\mathbb{1}_{\mathfrak{a}}$. For a form y of principal level, Lemma 8.7 and Lemma 8.2 show that $\theta^*_{D,\vartheta,i}\, y$ is the algebraic form of the lift of y by $\mathfrak{v}(\mathfrak{a}) \otimes \Upsilon(\mathfrak{a})$, which by Lemma 8.3 is the lift by the datum of G. This datum is sharp for the principal level at the places of $\mathfrak{a}$ and their conjugates (Input 6): the

modulus has places (unit 140), the tame part of the character is trivial off them, G has the support property of Lemma 8.4 and $\sum |G|^2 \neq 0$. So the lift is injective, and so is $\theta^*_{D,\vartheta,i}$. The inner product of two algebraic forms is $\mathrm{i}(\mathfrak{a})$ times the number of elements of the Weyl index set times the Petersson product of the corresponding forms (second note). Multiplying the series of Input 6 by this factor and inserting the constant of sharpness, with the value of $\sum |G|^2$ from Lemma 8.4, gives the terms $A \cdot \mathrm{gram}(\mathfrak{a}) \cdot \mathrm{ral}_{D,\vartheta,i}(\mathsf{W}_i(D); y', y; \bar{\mathsf{g}})$. □

Lemma 8.8 (the adjoint lift transfers eigensystems)**.** *In Setting 8.1 (b), let P be parameters, D data, ϑ a tame character, p a family of complex numbers and y a form in the eigenpart of $\mathsf{W}_i(D)$ for p. Then $\theta^*_{D,\vartheta,i}\, y$ lifts p.*

Proof. By Lemmas 8.7 and 8.2 it is the algebraic form of a theta lift along an arrow; y is spherical at the places in question with the Hecke polynomial of p (unit 138), and Hecke transfer (Input 7) gives the polynomial required, the characters being unramified there. □

LEAN `Data.lifts_auxLift`

Corollary 8.9 (injectivity)**.** *In Setting 8.1 (b), for parameters P, data D and a tame character ϑ, the adjoint lift $\theta^*_{D,\vartheta,i}$ is injective.*

Proof. It is the first assertion of Theorem 4. □

LEAN `Data.auxLift_injective`

Lemma 8.10 (the adjoint lift is equivariant at the chosen places)**.** *In Setting 8.1 (b), let P be parameters, D data, ϑ a tame character, s an element of the product of the groups GL of the residue fields of the chosen places, in the rank of the rows, and y a form of principal level on $\mathsf{W}_i(D)$. Then $\theta^*_{D,\vartheta,i}(\mathsf{s} \cdot y) = \mathsf{s} \cdot \theta^*_{D,\vartheta,i}(y)$, for the action of s on forms of principal level by translation and its action on $\mathcal{A}(\mathfrak{a}_D(\vartheta, i))$.*

Proof. Two elements of $\mathcal{A}(\mathfrak{a})$ are equal when their inner products with all $\mathsf{s} \cdot \Xi$ agree, the action being bijective. By Lemma 8.7, $\langle \theta^*(\mathsf{s}y), \mathsf{s}\Xi \rangle = \langle \mathsf{s}y, \theta_{\mathfrak{a}}(\mathsf{s}\Xi) \rangle_{\mathrm{Pet}}$; $\theta_{\mathfrak{a}}(\mathsf{s}\Xi)$ is the translate of $\theta_{\mathfrak{a}}(\Xi)$ by a lift of s (unit 138), and the Petersson product and the inner product of $\mathcal{A}(\mathfrak{a})$ are invariant (units 102, 138). □

LEAN `Data.auxLift_principalRep`

Lemma 8.11 (the lift of the trivial character is spherical)**.** *In Setting 8.1 (c), let E be a CM field with a CM type and Weil data, $\mathbf{n}$ a finite index set and $\mathfrak{l}$ L-value data. Then $\Xi(\mathfrak{l})$ lifts the constant family 1 for the auxiliary pair $\mathfrak{a}(\mathfrak{l})$.*

Proof. Hecke transfer for an equivariant map (Input 7) applies to the map from finite data to forms given by the class sum over $\mathrm{U}(\mathsf{l}_1)$ of the kernel values, and to the datum $\Upsilon(\mathfrak{a}(\mathfrak{l}))$, whose image is $\Xi(\mathfrak{l})$. On $\mathrm{U}(\mathsf{l}_1)$ the representation is trivial, so the Hecke polynomial on that side is the one of the constant family 1. □

LEAN `LValueData.auxPair_lifts_trivialLift`

Proof of Theorem 5. Write $\mathfrak{a} = \mathfrak{a}(\mathfrak{l})$ and $G = \mathrm{t}(\mathfrak{a})\mathbb{1}_{\mathfrak{a}}$. *Reduction to a doubling integral.* The inner product of $\Xi(\mathfrak{l})$ with itself is an index times a Petersson product (second note), and $\Xi(\mathfrak{l})$ is, up to the inverse of an index, the algebraic form of the theta lift of the constant function 1 on $\mathrm{U}(\mathsf{l}_1)$ with the datum $\mathfrak{v}(\mathfrak{a}) \otimes \Upsilon(\mathfrak{a})$, hence with the datum of G (Lemma 8.3; the constant function is automorphic and, l_1 being definite, rapidly decreasing). The doubling form of the inner product formula (Input 6; the

rank condition is $2 < |\mathbf{n}|$) expresses the Petersson product as $\mathrm{vol}[\mathrm{U}(\mathfrak{l}_{\mathbf{n}})]$ times the integral over $\mathrm{U}(\mathfrak{l}_1)(\mathbb{A})$ of the matrix coefficient of the datum.

The places off $\mathfrak{l}$. The datum is the vacuum off the places of $\mathfrak{l}$ and their conjugates. Integrating out one more place of E^+ off these, that is, the set $\{\mathfrak{p}, \bar{\mathfrak{p}}\}$ (one place of E if $\mathfrak{p} = \bar{\mathfrak{p}}$, two if not), multiplies the partial integral by the local doubling sum

$$\prod_{\mathfrak{u} \in \{\mathfrak{p}, \bar{\mathfrak{p}}\}} \frac{1 - \mathrm{sg}(\mathfrak{u})\ \mathrm{N}\mathfrak{u}^{-e}}{1 - \omega_{\mathfrak{l}}^{\mathrm{bc}}(\varpi_{\mathfrak{u}})^{-1}\ \mathrm{N}\mathfrak{u}^{-e}}$$

(Input 6; here $e > 1$ and $\omega_{\mathfrak{l}}$ is unramified at these places, and $\omega_{\mathfrak{l}}^{\mathrm{bc}}$ is the base change of $\omega_{\mathfrak{l}}$); for $\mathfrak{p} = \bar{\mathfrak{p}}$ it is the single factor $(1 - \mathrm{N}\mathfrak{p}^{-e})/(1 - \omega_{\mathfrak{l}}^{\mathrm{bc}}(\varpi_{\mathfrak{p}})^{-1}\,\mathrm{N}\mathfrak{p}^{-e})$, the sign being 1. By induction on finite sets of places and passage to the limit, the partial integrals converging and the Euler products converging for $e > 1$ (Input 7), the integral is

$$L^{\Sigma(\mathfrak{l}) \cup \overline{\Sigma(\mathfrak{l})}}(e, (\omega_{\mathfrak{l}}^{-1})^{\mathrm{bc}}) \cdot \prod_{\mathfrak{u} \notin \Sigma(\mathfrak{l}) \cup \overline{\Sigma(\mathfrak{l})}} (1 - \mathrm{sg}(\mathfrak{u})\ \mathrm{N}\mathfrak{u}^{-e})$$

times the partial integral at the places of $\mathfrak{l}$, at their conjugates and at infinity.

The places of $\mathfrak{l}$ *and infinity.* The datum of G is sharp (Input 6, by Lemma 8.4), so the remaining partial integral is its constant of sharpness times the archimedean integral, which over the compact group is $\mathrm{vol}[\mathrm{U}(\mathfrak{l}_1)]$ times the Fock pairing of the highest weight vector with itself; the polynomial of the highest vector is a renaming of the harmonic polynomial of $\mathfrak{a}$ (second note), the pairing is invariant under renaming, and its value is the hook number.

Collecting the factors. The local factors that enter are the following. At infinity: $\mathrm{vol}[\mathrm{U}(\mathfrak{l}_1)] \cdot \mathrm{hook}(\mathfrak{l})$; and the passage from the Petersson product to the inner product of algebraic forms contributes $\mathrm{weyl}(\mathfrak{l})$ and an index (second note). At a place $\mathfrak{p}$ of $\mathfrak{l}$, with residue field of $\mathrm{N}\mathfrak{p}$ elements: the stabilizer of a line in $\mathrm{GL}_{\mathbf{n}}$ of the residue field has index $\sum_{n<|\mathbf{n}|} \mathrm{N}\mathfrak{p}^n$, and the stabilizer of a non-zero vector has index $\mathrm{N}\mathfrak{p}^{|\mathbf{n}|} - 1$ (units 44, 103); the number of all vectors is $\mathrm{N}\mathfrak{p}^{|\mathbf{n}|}$, and the number of units of the residue field is $\mathrm{N}\mathfrak{p} - 1$. At the twist places, moreover, $\sum |G|^2 = \mathrm{t}(\mathfrak{a})^2\,(\prod_{\mathfrak{q}} \mathrm{N}\mathfrak{q})^{|\mathbf{n}|-1}$ (Lemma 8.4 with $|\mathbf{r}| = 1$), where $\mathrm{t}(\mathfrak{a}) = \prod_{\mathfrak{q}} (\mathrm{N}\mathfrak{q} - 1)$. These numbers combine as follows. A place $\mathfrak{p}$ of $\mathfrak{l}$ is not its own conjugate, the sign takes both values on the pair $\{\mathfrak{p}, \bar{\mathfrak{p}}\}$, and the product of the two sign factors of the pair is

$$(1 - \mathrm{sg}(\mathfrak{p})\,\mathrm{N}\mathfrak{p}^{-e})(1 - \mathrm{sg}(\bar{\mathfrak{p}})\,\mathrm{N}\mathfrak{p}^{-e}) = 1 - \mathrm{N}\mathfrak{p}^{-|\mathbf{n}|} = \frac{\mathrm{N}\mathfrak{p}^{|\mathbf{n}|} - 1}{\mathrm{N}\mathfrak{p}^{|\mathbf{n}|}},$$

the index of the stabilizer of a vector divided by the number of vectors; with these factors the product of the sign factors off $\mathfrak{l}$ becomes the product over all finite places, the inverse of the Euler product in the denominator of C. The quotient of the index of the stabilizer of a vector by that of a line is $\mathrm{N}\mathfrak{p} - 1$, which leaves $\prod(\mathrm{N}\mathfrak{p} - 1)$ over the chosen places; at the twist places only the power $(\prod \mathrm{N}\mathfrak{q})^{|\mathbf{n}|-1}$ of Lemma 8.4 remains in the result. The formal proof makes this collection in one long step of simplification, which the readers of the dossier mark as coarse; it is reported here by its local factors and its result and was not recomputed line by line. Finally $\mathrm{C} \neq 0$ because the volumes and the Euler product are non-zero. □

9 Notes on the formal text

1. *Half ranks.* Where the formal text divides a number of elements by 2 it uses the division of natural numbers; in Theorems 1 and 3 and in Propositions 4.2 and 5.5 the number is assumed even, and in Definition 5.6 the floor is written. The argument of $\Pi(\mathfrak{c}, \cdot)$ in Definition 5.6 (b) is an integer, possibly negative.

2. *Local sections depend on choices.* The functional $\mathcal{L}_0$ and the representatives of Lemma 3.1 are fixed by the axiom of choice; Definition 3.2 and Definition 3.3 depend on them, and the results of §3 hold for every choice.

3. *Hypotheses that are not used, or only formally.* In Lemma 3.6 the identification e is an argument of the formal statement although the conclusion does not mention it. In Proposition 5.2 the hypothesis on the index i is the existence of a positive sign of Y. Lemma 8.4 is stated in the formal text without Weil data, the surrounding statements with them.

4. *Statements that begin with a local definition.* The formal statements of Proposition 3.13, Proposition 5.3, Lemma 8.4, Lemma 3.8, Lemma 7.3 and Theorem 5 begin with a local definition (of S_{z}, of the datum and the point, of G, of the blocks, of the Fock vector, of C). The reader's tool takes such a definition for the beginning of the proof; the dossier restores the statements, and they were read here in the Lean text.

5. *Arguments that only Lean can settle.* For some declarations of this note the dossier gives the list of arguments by Lean's rules, applied by hand, because the text alone does not determine it; no compiler was run. They are: Lemma 3.1, Lemma 3.6, Lemma 3.8, Proposition 3.13, Proposition 5.3, Lemma 8.4, Theorems 3, 4 and 5, Lemma 8.11, Definition 5.6 (b) and Definition 7.1 (c). The doubt concerns implicit and instance arguments only (which index sets, signs and instances a declaration takes); the explicit arguments and the hypotheses are those written in the Lean statements. For instance arguments of definitions the tool marks some as taken only if the defining term uses them.

6. *A reading in the dossier that the types contradict.* For Lemma 3.10 the dossier gives the identification as $\mathbf{b} \sqcup (\mathbf{b} \sqcup \mathbf{n}) \simeq \mathbf{m}$. The frame of the formal statement has its rows indexed by $\mathbf{n}$, so the identification that the types allow is $\mathbf{b} \sqcup (\mathbf{b} \sqcup \mathbf{m}) \simeq \mathbf{n}$, as stated here; in the application (Theorem 3) the two index sets appear with exchanged names. This was inferred from the types in the Lean text, not checked by a compiler.

7. *The second reading.* A second reader compared the statements of the unit with the Lean text. Three of her points concern this note and are followed here: the constant C of Theorem 5 depends only on the field and the rank, and its L-function omits the Euler factors at the places of the datum and at their conjugates (the dossier allowed a dependence on the chosen places and named the places only); in the period condition of Theorem 3 the character in the denominator is the base change of $\omega_{\mathfrak{c}}$, evaluated at an idele; Proposition 7.4 asserts preimages and no more; and in the proof of Proposition 3.7 the pieces are cut out by the primitivity of $\mathsf{x}^{\mathsf{T}}\mathsf{r}^{-1}$, as written here from the Lean text.

8. *What is taken for granted.* Everything listed as Inputs. In particular the two forms of the inner product formula, Hecke transfer, the matrix Gauss sum, Freitag's

lemma, Landherr's theorem, the tower property and the constancy criterion are results of the background library. The Inputs give, for each, the relation to the literature that the second reader of the unit found by comparing the formal statement with the sources. For the two forms of the inner product formula, for the matrix Gauss sum and for Hecke transfer that relation is *unresolved*: the sources named there contain an analogue or a part, and not the formal statement. None of them is called a special case of a published theorem here.

9. *Weil data.* In the formal text, Weil data include three conditions that an earlier edition of this note left out: E contains a primitive root of unity of order 5040; $(1-\jmath^{-1})/2$ is integral; and υ is a Hecke character of E with the value -1 at the uniformisers of the places $\mathfrak{p}=\bar{\mathfrak{p}}$. The first is used in this note: the diagonalisation of unimodular hermitian matrices over $\hat{\mathcal{O}}_E$ (Input 2), in the proof of Theorem 2, is proved in the formal text under it.

10. *Two local characters.* The partners in all statements are partners for $\chi_{\mathfrak{c}}$, the local character of rank 0. In the proof of Proposition 4.2 the large side is invariant for the local character $\chi_{\mathfrak{c}}^{(|\mathbf{n}|)}$ of rank $|\mathbf{n}|$, and the formal proof factors through the matched pairs of that character; the two agree on units, which is all that partners over the integers see.

11. *Where the results go.* Theorem 1, Proposition 4.2, Lemma 3.12, Lemma 3.8, Lemma 6.1, Lemma 5.4, Proposition 5.5, Proposition 7.4 and Theorem 5 are used in the second note of the unit; Theorem 4 and Lemmas 8.8, 8.10 and 8.6 are used by other units of the argument.

How this note was written. From the dossier of the unit, which three readers made in three parts; this note follows the first part. The program check of the whole dossier covered 879 leaf steps of 122 proofs and found every statement at its lines. I read all statements of this part in the dossier, all structured proofs in their summaries, and step by step those of Theorems 2 and 3, of Theorem 1, of Proposition 5.2, Proposition 3.7, Lemma 3.12, Proposition 3.13, Proposition 4.2, Theorems 4 and 5 and Proposition 7.4. In the Lean text I read the statements of all declarations of the note, the proof of Proposition 3.7, and the definitions of the background library that the statements rest on (partners, standard data, integral frames, tame characters, split tame characters, local data, theta integrals on the large side, equivariant lifts, the products of L-values, representedness). The proofs here are shorter than the formal ones: bookkeeping with characters, measures and matrix identities is merged, and the proofs of Lemma 3.9 and Lemma 3.12 are written as the computations that the formal steps amount to. The readers of the dossier mark the long automation steps of Theorems 2, 3, 5 and of Propositions 5.2 and 7.4 as coarse; these proofs are given here in outline with every input named, and they have not been checked independently of the dossier. Nothing was compiled.

Revision. This edition answers a referee's report on the three notes of the unit. Two proofs were wrong and are corrected from the Lean text, which I read for both: in the proof of Proposition 3.13 the Gauss sum is formed with the shears $S_{\varpi^{-1}\mathfrak{z}}$ and the left inverse is the block $\mathfrak{A}_3'$; in the proof of Proposition 4.2 the invariance of the large side holds for the local character of rank $|\mathbf{n}|$. The definition of Weil data was completed, the relations of the Inputs to the literature follow the second reader's

verdicts, the quoted criteria of Input 5 are stated with their hypotheses, and letters were changed to agree with the handout (the partner on the large side is $\check{\Psi}$, the lift index is i).

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Lemma 3.1	`exists_common_coset_reps`	BlockPartners.lean	20–29
Definition 3.2	`localSection`	BlockPartners.lean	33–34
Definition 3.3	`matchingPairs`	BlockPartners.lean	39–41
Lemma 3.4	`mul_firstCols`	BlockPartners.lean	44–44
Lemma 3.4	`matchingPairs.translate_mem`	BlockPartners.lean	46–51
Lemma 3.5	`localSection_of_primitive`	BlockPartners.lean	55–58
Lemma 3.6	`TameLocalChar.isClopen_primitive`	BlockPartners.lean	60–65
Proposition 3.7	`matchingPairs.exists_linear_factor`	BlockPartners.lean	70–89
Lemma 3.8	`partner_block`	BlockPartners.lean	94–108
Lemma 8.2	`AuxPair.adjoint_theta_eq_formLift`	CharacterLiftNorm.lean	36–44
Lemma 8.3	`AuxPair.lift_auxDatum_eq_lift_modDatum`	CharacterLiftNorm.lean	49–56
Lemma 8.4	`AuxPair.levelIndicator_support_and_normSq`	CharacterLiftNorm.lean	62–75
Definition 8.5	`AuxPair.gramFactor`	CharacterLiftNorm.lean	79–80
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In this note	Lean declaration	File	Lines
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Three statements as they are written in the artifact.

```
lemma exists_stdExchange_top {t t'} {W : Space E a t} (D₁ :
  W₁.Splitting W Unit) [IsBaseAdapted D₁] (V' : Space E b t') σ' (hn : 2 ∣ Fintype.card n) μ {v v₁ :
  PairFock ..} (hv : IsRapidDatum v) (r₁ : IsRapidDatum v₁) : ∃ x x', realSlice D.symm V' 0 x' = v ∧
  StdExchange X (V'.sum W₁) V' μ σ' x x' (equalRankThetaRatio W₁ · v₁)
```

```
def StdExchange {t' r} (X : Space E m r) (V : Space E n s) (V' : Space E b t') μ σ v y (ℓ :
  _ → ℝ) := IsRapidDatum v ∧ IsRapidDatum y ∧ ∃ C, ∀ c : SplitTameChar E μ, ∃ c : ℂ, C * ‖c‖ ^ 2 =
  lValueProdInt c (card n - card m / 2) * ℓ c ∧ ∀ z, IsIntMatrix 0 z →
  thetaLarge X V c v (stdRowDatum σ c.x z) = c • thetaSmall X V' c y (stdColDatum c.x z)
```

```
lemma Data.auxLift_injective_and_hasSum_inner : ∃ A ≠ 0, ∀ θ, Injective (D.auxLift θ j) ∧ ∀ x y,
  HasSum (A * (D.auxPair θ j).gramFactor * D.rallisTerm θ j _ x y ·)
  (inner ℂ (D.auxLift θ j x) (D.auxLift θ j y))
```

Legend: in these texts `X`, `W₁`, `W`, `V'` are the spaces X, W_1, W, V'; `D`, `D₁` the splittings $\mathfrak{S}$, $\mathfrak{S}_1$; `σ'` is e'; `v`, `v₁`, `x`, `x'` are the real data $\mathfrak{v}$, $\mathfrak{v}_1$, $\mathfrak{x}$, $\mathfrak{x}'$; `c` is $\mathfrak{c}$ and `c.x` is $\chi_{\mathfrak{c}}$; `ℓ` is $\mathcal{N}$; `thetaLarge`, `stdRowDatum`, `stdColDatum`, `lValueProdInt` are the theta integral on the large side, the standard data and $\Pi(\mathfrak{c}, e)$. In the third, `D` is the data D, `auxLift` the adjoint lift and `auxPair` the auxiliary pair.

Notation

E, E^+, $\mathbb{A}_E$, $\mathcal{O}_E$, $\hat{\mathcal{O}}_E$ a CM field, its maximal totally real subfield, its adeles, its integers, its integral finite adeles

τ; $\mathfrak{p}$, $\bar{\mathfrak{p}}$, $\mathfrak{q}$, $\mathfrak{u}$; $\varpi_{\mathfrak{p}}$; $\mathrm{N}\,\mathfrak{p}$ an infinite place of E; finite places, the conjugate place; a uniformiser at $\mathfrak{p}$; the norm of a place

$\jmath$, υ, sc_E, $\mathrm{e}_{\mathbb{A}}$ Weil data: the purely imaginary element, the splitting character (a Hecke character of E, and its restriction to $\mathrm{U}(1)(\mathbb{A})$), the scale idele, the additive character of the adeles

P, D, m, ϑ, i parameters, admissible data, the number of lifts, a tame character of D, a lift index ($1 \le i \le m$) in §8

ω^{bc}, $\omega^{\mathrm{bc}}_{\mathfrak{c}}$, $\omega^{\mathrm{bc}}_{\mathfrak{l}}$; ξ the base change $\mathsf{x} \mapsto \omega(\mathsf{x}/\bar{\mathsf{x}})$ of a character of $\mathrm{U}(1)(\mathbb{A})$, a Hecke character of E; a Hecke character

$L^{\Sigma\cup\bar{\Sigma}}(s, \omega^{\mathrm{bc}})$; $\bar{\Sigma}$; s the Hecke L-function of the base change without the Euler factors at Σ and at the conjugate places $\bar{\Sigma}$; the set of conjugates; a real number ≥ 1

$\mathbf{a}, \mathbf{b}, \mathbf{m}, \mathbf{n}, \mathbf{r}, \mathbf{n}', \mathbf{r}'$; $|\mathbf{n}|$ finite index sets; the number of elements

F, O, ϖ; R, $\bar{\mathsf{R}}$ a semilocal field, its compact open subring, its topologically nilpotent unit; a commutative ring; the residue ring of a tame character

$\mathsf{F}_{\mathfrak{c}}$, F_{Σ} the product of the completions of E at the places of $\mathfrak{c}$, of Σ

$\mathrm{M}_{\mathbf{a}\times\mathbf{n}}(\mathsf{R})$, $\mathrm{GL}_{\mathbf{n}}(\mathsf{R})$; x^{T}, x^*, tr matrices with rows $\mathbf{a}$ and columns $\mathbf{n}$, invertible matrices; transpose, conjugate transpose, trace

$\mathsf{g}, \mathsf{h}, \mathsf{k}, \mathsf{x}, \mathsf{y}, \mathsf{z}, \mathsf{s}, \mathsf{q}$; $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{f}, \mathsf{j}, \mathsf{l}, \mathsf{n}, \mathsf{p}, \mathsf{r}, \mathsf{t}, \mathsf{u}, \mathsf{v}, \mathsf{w}, \mathsf{z}$ matrices, vectors and points of groups: k in a general linear or unitary group, z an integral matrix or an element of F, s a left inverse, q a Levi coordinate, v a rational frame, c, w the coordinates of a radical element, t signs, p a point or a family of eigenvalues, r a coset representative

$\mathsf{A}, \mathsf{B}, \mathsf{I}, \mathsf{K}, \mathsf{L}, \mathsf{M}, \mathsf{P}, \mathsf{U}$, J_0, S_0, P_+ matrices of §6 and §7: a local inverse, the form of a split frame, blocks of a mixed model, a matrix of positive coordinates, a frame matrix, a point, a combination of blocks, a skew matrix, the projection to positive coordinates

e, e' identifications of index sets, $\mathbf{b} \sqcup \mathbf{a} \simeq \mathbf{n}$

$\mathrm{col}_{\mathsf{e}}(\mathsf{k})$, $\mathrm{row}_{\mathsf{e}}(\mathsf{k})$ the columns of k indexed by $\mathbf{b}$; the rows of k^{-1} indexed by $\mathbf{a}$

χ, χ', $\chi^\flat$, $\bar{\chi}$; e_{F}, $\bar{\mathrm{e}}$ multiplicative characters, the inverse on integral units, the primitive character of the residue ring; an additive character and its reduction

$\chi_{\mathfrak{c}}$, $\chi^{(e)}_{\mathfrak{c}}$, χ° the local character of a split tame character in rank 0, in rank e, and in rank $|\mathbf{n}|$ (proof of Proposition 4.2)

$\check{\Psi}$, Ψ, Ψ_0, Ψ' functions on the large side ($\mathbf{a}\times\mathbf{n}$ matrices) and on the small side ($\mathbf{n}\times\mathbf{b}$

matrices); partners (the formal text writes Φ, Ψ)

$\Psi^{\mathrm{st}}_{\chi,\mathfrak{s}}$, $\check{\Psi}^{\mathrm{st}}_{\mathfrak{e},\chi,\mathfrak{s}}$ the standard column datum and the standard row datum

$\mathcal{S}(\mathcal{U})$, $\mathcal{S}^{\mathrm{inj}}$, $\mathcal{S}^{\mathrm{sur}}$ Schwartz–Bruhat functions supported in a set $\mathcal{U}$, on matrices with a left inverse, on matrices whose transpose has one

$\mathcal{L}_0$, $\mathcal{R}$, $\mathcal{R}_0$ the functional and the coset representatives of Lemma 3.1; a finite set of them

$\mathrm{sec}_{\chi}(\mathsf{y})$; $\mathrm{Match}(\chi,\mathsf{e})$ the local section (Definition 3.2); the matched pairs (Definition 3.3)

$\mathcal{L}$, $\mathcal{L}'$, $\mathcal{L}_1$; $\mathcal{X}$ linear maps and functionals; a set

$\mathfrak{A}$, $(\mathfrak{A}_1 \mid \mathfrak{A}_2 \mid \mathfrak{A}_3)$, $\mathfrak{A}'_1$, $\mathfrak{A}'_2$, $\mathfrak{A}'_3$; $\mathfrak{A}^{\mathrm{op}}$; $\mathrm{st}_{\mathfrak{A}}(G)$ an integral frame, its column blocks, the row blocks of its inverse; the adapted frame of the opposite splitting; the function G read in the frame

$\mathcal{Q}$, $\mathcal{M}$; S_{z}, Γ_0, $\mathcal{Z}$ a stable space of functions, the domain of a lift; the shears of a standard datum, their Gauss sum, a set of integral representatives

G, G', G_0; n, K_0; i, n functions; a counting function, a positive constant; an index or an exponent (in §§3–5; in §8 the lift index), a natural number

$\mathsf{W},\mathsf{V},\mathsf{V}',\mathsf{X},\mathsf{W}_1,\mathsf{Y},\mathsf{Z}$; $\mathsf{I}_{\mathbf{n}}$ hermitian spaces; the standard positive definite space

J_{W}, x_{W}, $\mathrm{neg}(\mathsf{W})$, $\mathrm{diag}(\mathsf{W})$ the Gram matrix, the adelic frame, the number of negative signs, the diagonal form of the signs

$\mathrm{U}(\mathsf{W})$, $[\mathrm{U}(\mathsf{W})]$, $\mathrm{vol}[\mathrm{U}(\mathsf{W})]$, $\mathcal{K}_f$, $d\mathsf{k}$ the unitary group, its adelic quotient and the mass of the quotient (normalised in §2), the stabilizer of the integral lattice, integration for a Haar measure

ω, ω', ω'', δ; χ'; μ characters of $\mathrm{U}(1)(\mathbb{A})$; a character of $\mathsf{F}_{\mathfrak{c}}$; weights, integers at the infinite places

$\mathfrak{c}$, $\omega_{\mathfrak{c}}$, $\Sigma(\mathfrak{c})$; Σ a split tame character, its character, its set of places; a set of finite places

$\mathfrak{v},\mathfrak{v}',\mathfrak{v}_1,\mathfrak{x},\mathfrak{x}',\mathfrak{y}$; F, F', $F_{\mathfrak{c}}$, F_{y}; f real data; finite data (also an algebraic form in §8); a datum

θ, $\mathrm{I}_{\mathsf{W},\mathsf{V}}(\omega,\omega';f)$ the theta functional; the theta integral over $[\mathrm{U}(\mathsf{V})]$

$\theta^{\mathrm{lg}}_{\mathsf{W},\mathsf{V}}(\mathfrak{c};\mathfrak{v},\check{\Psi})$, $\theta^{\mathrm{sm}}_{\mathsf{W},\mathsf{V}}(\mathfrak{c};\mathfrak{v},\Psi)$ the theta integrals on the large and on the small side

$\theta^{\infty}_{\mathsf{W}_1,\mathsf{V}}(\omega,\omega';F,\mathsf{x},\mathsf{p})$ the theta functional on real data (Definition 5.1)

$\Pi(\mathfrak{c},e)$, $\Pi(\Sigma,\omega,e)$; e, e_0 products of partial Hecke L-values; a natural number or an integer

$\mathfrak{S}$, $\mathfrak{S}_1$; $\mathrm{coef}_{\mathfrak{S}}(T;f)$; T, T', Y splittings; the centre coefficient; matrix indices ($T' = \jmath T$ in the proof of Theorem 2), an adelic matrix

$\mathrm{rad}_{\mathfrak{S}}(\mathfrak{c},\mathsf{w})$, $\mathrm{lev}_{\mathfrak{S}}(\mathfrak{q},\mathfrak{g})$; $\mathrm{sl}_{\mathfrak{S}}(\mathsf{x};\mathfrak{v})$; $\mathrm{vac}_{\mathfrak{S}}(\mathsf{V})$ the radical and Levi elements of the parabolic subgroup of a splitting; the real slice; the vacuum factor of the mixed model

$\mathrm{heis}(\mathsf{x})$, ω_{∞}, $\det_{\infty} B$ the Heisenberg operator of a vector; the real Weil representation; the idele that is $\det B$ at the infinite places and 1 at the finite ones

$(\mathcal{M},\mathcal{D})$ an equivariant lift

$\varrho(\mathsf{V};\mathfrak{c},\mathfrak{v})$ the equal-rank ratio (Definition 5.6)

$\mathrm{Exch}(\mathsf{X},\mathsf{V},\mathsf{V}';\mu,\mathsf{e};\mathfrak{v},\mathfrak{y};\mathcal{N})$ the standard exchange identity (Definition 5.6)

$\mathcal{N}$; c, C, R, k, b, z; l a real function on split tame characters; constants (R is the constant of unfolding in stages; it is not the right translation $R(\mathfrak{g})$, the commutative ring R, or the remainder $R_{d,M}$ of the handout); a complex function on split tame characters

Ω, Ω_0, Ω_1; B weight and bump functions; an invertible adelic $\mathbf{b}\times\mathbf{b}$ matrix

$\varkappa$, $\mathcal{C}$ a point of the unit circle; a compact group (proof of Proposition 5.2)

$\mathrm{pos}_{\mathrm{sgn}_{\mathsf{W}},\mathrm{sgn}_{\mathsf{V}}}(F,\mathsf{u})$ the matrix of positive coordinates (Definition 7.1)

$\mathrm{dv}^{e,\mu}_{\mathrm{sgn}_{\mathsf{W}},\mathrm{sgn}_{\mathsf{V}}}(\mathrm{c},F;\mathsf{u})$, $\mathrm{Det}(e,\mu)$ a determinant vector, the set of determinant vectors

$\Delta_{\tau,\pm}$ determinants of matrices of linear forms (proof of Proposition 7.4)

$\mathfrak{a}$, $\mathsf{W}(\mathfrak{a})$, $\mathcal{A}(\mathfrak{a})$, $\theta_{\mathfrak{a}}$, $\Upsilon(\mathfrak{a})$, $\mathfrak{v}(\mathfrak{a})$, $\mathbb{1}_{\mathfrak{a}}$, $\mathsf{R}(\mathfrak{a})$, $\mathrm{t}(\mathfrak{a})$, $\mathrm{i}(\mathfrak{a})$ an auxiliary pair and its objects (Setting 8.1); the upright $\mathrm{t}(\mathfrak{a})$ is an order of a finite group, not the step $t(D)$ of the handout

$\mathrm{gram}(\mathfrak{a})$, $\mathrm{ral}_{D,\vartheta,i}(\mathsf{V};y',y;\bar{\mathfrak{g}})$ the Gram factor and the terms of the inner product (Definition 8.5)

$\mathfrak{l}$, $\omega_{\mathfrak{l}}$, $\Sigma(\mathfrak{l})$, $\mathfrak{a}(\mathfrak{l})$, $\Xi(\mathfrak{l})$ L-value data, their character, the set of their places, their auxiliary pair, the lift of the trivial character

$\mathrm{hook}(\mathfrak{l})$, $\mathrm{weyl}(\mathfrak{l})$, $\mathrm{sg}(\mathfrak{p})$ the hook number, the number of elements of the Weyl index set, the sign of a finite place (Setting 8.1)

A, A_0; c_F, $\mathrm{ct}(\bar{\mathfrak{g}})$ the constants of the inner product formula; the constant of a sharp datum, the coset term (Input 6)

y, y', Ξ; $\langle y,y'\rangle_{\mathrm{Pet}}$; $R(\mathfrak{g})$; l forms; the Petersson product; right translation; a parameter of a discrete series type

$\mathsf{W}_i(D)$, $\mathsf{Y}_0(D)$, $\mathfrak{a}_D(\vartheta,i)$, $\theta^*_{D,\vartheta,i}$, $\Theta^{\circ}_{D,\vartheta}$, $\mathfrak{m}_D(\vartheta,e)$ the objects attached to the data (Setting 8.1); the modulus of the chosen places to the power e times the places of ϑ

READBACK NOTE 17

The exchange theorem, one square of lifts, and the triangle of lifts and descents

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The exchange theorem (Theorem 1) says: for hermitian spaces V, V' whose ranks add up to the rank of W, the theta integrals over $[\mathrm{U}(\mathsf{V})]$ and $[\mathrm{U}(\mathsf{V}')]$ of data that are partners modulo a modulus are proportional, and the square of the absolute value of the constant is a fixed number times a product of partial Hecke L-values. It is obtained from the identity at the top of a tower (first note of the unit) by removing hyperbolic planes. Applied to a square of spaces it shows that a theta lift followed by a descent is proportional to a descent followed by a lift (Theorem 2). The squares fit into a triangle of lifts and descents attached to the data (Theorem 3), from which the composite of the m lifts is injective (Theorem 4). The norm of the lift of the trivial character from $\mathrm{U}(1)$ is a Hecke L-value, interpolated by one integral measure (Theorems 5 and 6).

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, attributed by its authors to L. Alpöge with the AI system Claude, whose final theorems are the Riemann hypothesis and its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles and that Lean's kernel accepted the proofs; no mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note is the second of three on one unit of the artifact, the unit of the exchange identity, which belongs to the proof of Theorem C of the handout. It treats 129 of the unit's 305 declarations; the notes *Partner data, vanishing on standard data, and the exchange identity at the top of the tower* (the first note) and *The grid of lifts and descents, and the crude rows* (the third note) treat the others. The proofs are written from the structured proofs of the unit's dossier, which a program compared with the Lean text; what they take from other units and from the two other notes is listed as Inputs in §2 and is not proved here. The writer compared with their Lean statements: Theorems 1, 2, 3, 5 and 6; Lemma 4.1, Lemma 9.2, Lemma 7.2, Lemma 8.9 (c), Lemma 3.5 and Propositions 4.4, 4.5 and 4.6; and the definitions of the square law, of triangles and weighted chains, of the forms of the triangle, of tameness for a modulus, of costs, of pairs, of the algebraic part, of integral measures and of marks. The other statements, Theorem 4 among them, are written from the dossier's readings; §11 says so again. A second reader compared all statements of the unit and a passage of every structured proof with the Lean text after the dossier was made; where her reading differs from the dossier's, this note follows hers (§11). The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 129 declarations.

1 The statements

The notions are recalled in §2. In short: a hermitian space W over a CM field E has an index set and signs; partners $\check{\Psi}$, Ψ for a character χ are functions on $\mathbf{a}\times\mathbf{n}$ and on $\mathbf{n}\times\mathbf{b}$ matrices that correspond under the duality between kernels and images; $\mathfrak{x}\otimes[\check{\Psi}]_{\mathfrak{m}}$ is the datum with real part $\mathfrak{x}$ and finite part given by the function $\check{\Psi}$ modulo a modulus $\mathfrak{m}$; and $\Pi(\Sigma,\omega,e)=(\prod_{\mathfrak{p}\in\Sigma}\mathrm{N}\,\mathfrak{p})^{e^2}\prod_{i=1}^{e}|L^{\Sigma\cup\bar{\Sigma}}(i,\omega^{\mathrm{bc}})|^2$, where $L^{\Sigma\cup\bar{\Sigma}}(s,\omega^{\mathrm{bc}})$ is, as in the first note, the Hecke L-function of the base change $\omega^{\mathrm{bc}}(\mathsf{x})=\omega(\mathsf{x}/\bar{\mathsf{x}})$ of ω to E without the Euler factors at the places of Σ and at their conjugates $\bar{\Sigma}$ (for real $s>1$ the Euler product, at $s=1$ its limit from the right); for $e=0$ both products are 1. This is the only L-function of the note and it is always written so.

Theorem 1 (the exchange theorem). *Let E be a CM field with a CM type and Weil data, with splitting character υ. Let W, V, V' be hermitian spaces over E with index sets $\mathbf{n}$, $\mathbf{a}$, $\mathbf{b}$, and $\mathsf{e}\colon\mathbf{b}\sqcup\mathbf{a}\simeq\mathbf{n}$ an identification. Suppose that $\mathrm{neg}(\mathsf{W})=\mathrm{neg}(\mathsf{V})+\mathrm{neg}(\mathsf{V}')$, that $|\mathbf{a}|=|\mathbf{b}|+2e$ for a natural number e, and that $2\,\mathrm{neg}(\mathsf{W})+2\le|\mathbf{n}|$. Let μ be integer weights with $|\mathbf{n}|\le 2\mu(\tau)+2\,\mathrm{neg}(\mathsf{W})$ for every infinite place τ, and $\mathfrak{v}$ a rapid real datum for the pair (V',W). Then there are a real number C and a rapid real datum $\mathfrak{x}$ for (W,V) such that, for every modulus $\mathfrak{m}$, every character ω of $\mathrm{U}(1)(\mathbb{A})$ and every multiplicative character χ of $\mathcal{O}_E/\mathfrak{m}$ for which ω is tame for $\mathfrak{m}$ with weights μ and character χ, writing $\omega^{\circ}=\omega^{-1}\upsilon^{\lfloor|\mathbf{n}|/2\rfloor}$, there is a complex number $\mathrm{c}\neq 0$ with*

$$|\mathrm{c}|^2=\mathrm{C}\cdot\Pi(\Sigma(\mathfrak{m}),\omega,e)=\mathrm{C}\Big(\prod_{\mathfrak{p}\in\Sigma(\mathfrak{m})}\mathrm{N}\,\mathfrak{p}\Big)^{e^2}\prod_{i=1}^{e}\big|L^{\Sigma(\mathfrak{m})\cup\overline{\Sigma(\mathfrak{m})}}(i,\omega^{\mathrm{bc}})\big|^2,$$

$\Sigma(\mathfrak{m})$ the set of places of $\mathfrak{m}$ and $\overline{\Sigma(\mathfrak{m})}$ the set of their conjugates, such that for all partners $\check{\Psi}$, Ψ for χ over $\mathcal{O}_E/\mathfrak{m}$ with respect to e and every point g of $\mathrm{U}(\mathsf{W})(\mathbb{A})$

$$\mathrm{I}_{\mathsf{W},\mathsf{V}}(\omega^{\circ},1;\mathfrak{x}\otimes[\check{\Psi}]_{\mathfrak{m}})(\mathsf{g})=\mathrm{c}\int_{[\mathrm{U}(\mathsf{V}')]}\theta_{\mathsf{V}',\mathsf{W}}(\omega^{\circ},\omega^{\circ};\mathfrak{v}\otimes[\Psi]_{\mathfrak{m}})(\mathsf{h},\mathsf{g})\,d\mathsf{h}.$$

LEAN `exchange_theorem`

Theorem 2 (one square). *Let E be a CM field with a CM type and Weil data. Let A', A, U_1, U_2 be hermitian spaces over E with index sets $\mathbf{n}$, $\mathbf{n}'$, $\mathbf{n}_1$, $\mathbf{n}_2$. Let $\mathcal{I}$ be an index set with families of characters ω_i^{a}, ω_i^{b} (each a character for every rank), characters ω_i, moduli $\mathfrak{m}_i$ and multiplicative characters χ_i of $\mathcal{O}_E/\mathfrak{m}_i$. Suppose given four edge families: for ω^{b} from the type t_2 to the type t in a direction d; for ω^{a} from t_1 to t in the opposite direction; for ω^{a} from t' to t_2 in the opposite direction; and for ω^{b} from t' to t_1 in the direction d. Let e be an identification $\mathbf{n} \sqcup \mathbf{n}' \simeq \mathbf{n}_1 \sqcup \mathbf{n}_2$, and assume: $\operatorname{neg}(\mathsf{U}_1 \oplus \mathsf{U}_2) = \operatorname{neg}(\mathsf{A}) + \operatorname{neg}(\mathsf{A}')$; $|\mathbf{n}'| = |\mathbf{n}| + 2e$ for a natural number e; $2\operatorname{neg}(\mathsf{U}_1 \oplus \mathsf{U}_2) + 2 \le |\mathbf{n}_1| + |\mathbf{n}_2|$; μ are integer weights with $|\mathbf{n}_1| + |\mathbf{n}_2| \le 2\mu(\tau) + 2\operatorname{neg}(\mathsf{U}_1 \oplus \mathsf{U}_2)$ for every infinite place τ; every ω_i is tame for $\mathfrak{m}_i$ with weights μ and character χ_i; and $\omega_i^{\mathrm{a}} \cdot \omega_i^{\mathrm{b}} = \omega_i^{-1} \upsilon^{\lfloor (|\mathbf{n}_1|+|\mathbf{n}_2|)/2 \rfloor}$ for every i, the two characters being taken in ranks that add up to $|\mathbf{n}_1| + |\mathbf{n}_2|$. For a matrix x with column blocks x_1, x_2 and a matrix y with row blocks y_1, y_2 let*

$$\mathsf{u}_i(\mathsf{x}) = (\text{descent from } \mathsf{A} \text{ to } \mathsf{U}_2 \text{ at } \mathsf{x}_2) \circ (\text{lift from } \mathsf{U}_1 \text{ to } \mathsf{A} \text{ at } \mathsf{x}_1),$$
$$\mathsf{v}_i(\mathsf{y}) = (\text{lift from } \mathsf{A}' \text{ to } \mathsf{U}_2 \text{ at } \mathsf{y}_2) \circ (\text{descent from } \mathsf{U}_1 \text{ to } \mathsf{A}' \text{ at } \mathsf{y}_1)$$

be the composites of the theta lifts and descents of the four edges, with the characters of the index i and the finite data that are the indicators of the blocks modulo $\mathfrak{m}_i$. Then there are a real number C and a complex number $\varkappa$ such that every $i \in \mathcal{I}$ has a complex number $\mathrm{c} \neq 0$ with $|\mathrm{c}|^2 = \mathrm{C} \cdot \Pi(\Sigma(\mathfrak{m}_i), \omega_i, e)$ and, for every form F on U_1 and every linear functional l on the forms on U_2,

$$\mathrm{SqLaw}(\chi_i, \mathsf{e}; \varkappa/\mathrm{c}; \mathsf{x} \mapsto \mathsf{l}(\mathsf{u}_i(\mathsf{x})F), \mathsf{y} \mapsto \mathsf{l}(\mathsf{v}_i(\mathsf{y})F)).$$

That is, the two functions have the determinant covariances of Definition 3.1, and for all partners $\check{\Psi}$, Ψ for χ_i over $\mathcal{O}_E/\mathfrak{m}_i$ with respect to e: $\sum_{\mathsf{y}} \Psi(\mathsf{y})\, \mathsf{l}(\mathsf{v}_i(\mathsf{y})F) = (\varkappa/\mathrm{c}) \sum_{\mathsf{x}} \check{\Psi}(\mathsf{x})\, \mathsf{l}(\mathsf{u}_i(\mathsf{x})F)$. The number $\varkappa$ is the same for all i and may be zero.

LEAN `one_square`

Theorem 3 (the triangle of the data). *Let P be parameters, D admissible data for P, and o a flag (false or true) that names one of the two grids. Then there are descents d, linear maps $\mathsf{d}_{\vartheta,i,n} \colon \mathrm{TF}_D(o; i, n+1) \to \mathrm{TF}_D(o; i+1, n)$ for all m-fold twists ϑ of D and natural numbers i, n, such that*

$$\mathrm{Tri}(m; \mathrm{ord}_{D,o}, \mathrm{Cost}_D(\cdot;\cdot); \mathrm{tl}_{D,o,\cdot}(\cdot,\cdot)(\cdot), \mathsf{d}; i \mapsto \mathrm{tn}_{D,o}(i)|_{\mathrm{TB}_D(o)}; o = \mathrm{false})$$

holds: indexed by the m-fold twists ϑ of D, the spaces $\mathrm{TF}_D(o; i, n)$ with the lifts $\mathrm{tl}_{D,o,\vartheta}(i, n)(\cdot)$, the descents d, the weights given by the triangle orders $\mathrm{ord}_{D,o}(\vartheta, n)$ of Definition 9.1 (f), the grading "costs n squares" on the complex functions of m-fold twists, and the embeddings of $\mathrm{TB}_D(o)$ form a triangle of size m in the sense of Definition 6.1, whose additional condition is that the flag is false.

LEAN `Data.exists_isTriangle`

Theorem 4 (the chain of lifts is injective). *Let P be parameters, D admissible data for P and ϑ an m-fold twist of D. Then the composite $\Theta^{\mathrm{lev}}_{D,\vartheta}$ of the m theta lifts from $\mathrm{U}(\mathsf{Y}_0(D))$ to the definite group $\mathrm{U}(\mathsf{Y}_m(D))$ is injective on the forms of principal level on $\mathrm{U}(\mathsf{Y}_0(D))$.*

LEAN `Data.iotaAtLevel_injective`

Theorem 5 (the norm of the lift from U(1) is an L-value). *Let P be parameters, D admissible data, p a pair of D with character ω_{p} and point e_{p}, and $\mathbf{n}$ a finite index set such that p has lift rank $\mathbf{n}$. Then there is a complex number $A \neq 0$ such that for every tame character ϑ of D*

$$\mathrm{alg}(\mathsf{p},\mathbf{n};\vartheta) = A \cdot L^{\Sigma\cup\bar{\Sigma}}(e_{\mathsf{p}}, ((\omega_{\mathsf{p}}\vartheta)^{-1})^{\mathrm{bc}}), \qquad \Sigma = \Sigma_D \cup Q(\vartheta),$$

the value at e_{p} of the Hecke L-function of the base change of $(\omega_{\mathsf{p}}\vartheta)^{-1}$ without the Euler factors at the chosen places Σ_D, at the places of $Q(\vartheta)$ and at their conjugates.

Theorem 6 (one measure for all twists of a modulus). *Let P be parameters, D admissible data, p a pair of D and $\mathbf{n}$ a finite index set such that p has lift rank $\mathbf{n}$, and let ϑ be a tame character of D. Then* $\mathrm{IntMeas}(\vartheta;\vartheta' \mapsto \mathrm{alg}(\mathsf{p},\mathbf{n};\vartheta'))$ *holds: the function $\vartheta' \mapsto \mathrm{alg}(\mathsf{p},\mathbf{n};\vartheta')$ is an integral measure for ϑ (Definition 8.3).*

LEAN `Data.Pair.intAlgPart_eq_mul_L` · `Data.Pair.isIntegralMeasure_intAlgPart`

2 The setting

Conventions. The natural numbers are $0, 1, 2, \ldots$. The letters of the handout keep their meaning: P are parameters, D admissible data, m the number of lifts, ϑ a tame character or a twist of D, τ an infinite place, σ an automorphism of $\mathbb{C}$, ℓ a prime, ι an isomorphism of $\mathbb{C}$ with an algebraic closure of the ℓ-adic numbers and j a ring homomorphism from $\mathbb{C}$ to the completion of that closure. Objects local to the theory of theta series are set in other alphabets, as in the first note: finite index sets are bold ($\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$, $\mathbf{N}$, $\mathbf{T}$, $\mathbf{p}$, $\mathbf{r}$, …) and $|\mathbf{n}|$ is the number of elements; sans-serif letters are a commutative ring R, hermitian spaces, matrices, points of groups and linear maps; fraktur letters are real data, places, moduli ($\mathfrak{m}$), splittings, auxiliary pairs and L-value data; upright letters are constants. The implicit arguments of the declarations of the unit are of the following kinds, and each statement introduces the ones it uses: a finite commutative ring; finite index sets, one of them empty or non-empty where a statement says so; a CM field E with a CM type and, where a statement says so, Weil data; signs and hermitian spaces; splittings, which are base-adapted where a statement says so; weights, characters, moduli and level places; real data, finite data and functions; types of forms, forms and kernels; matrices and points; natural numbers and integers; constants; a prime with an isomorphism or an embedding of $\mathbb{C}$; inner product spaces, with non-zero weights where a statement says so; parameters and admissible data with their tame characters, twists and pairs.

What is recalled from the first note. Hermitian spaces W over a CM field E with index set $\mathbf{n}$, signs, Gram matrix and adelic frame; $\mathrm{neg}(\mathsf{W})$, the number of negative signs; the Weil representation of a pair, real data (rapid when all weighted norms are finite), finite data, the theta kernel $\theta_{\mathsf{W},\mathsf{V}}(\omega,\omega';f)$ and the theta integral $\mathrm{I}_{\mathsf{W},\mathsf{V}}(\omega,\omega';f)$ over $[\mathrm{U}(\mathsf{V})]$, for the measure fixed in the first note (the Haar measure that gives the stabilizer of the integral lattice the mass 1 at the finite places, and the measure of the formal text at the infinite places); Weil data, which include the hypothesis that E contains a primitive root of unity of order 5040; split tame characters $\mathfrak{c}$ with character $\omega_{\mathfrak{c}}$, places $\Sigma(\mathfrak{c})$ and local character $\chi_{\mathfrak{c}}$; the theta integrals $\theta^{\mathrm{lg}}_{\mathsf{W},\mathsf{V}}(\mathfrak{c};\mathfrak{v},\check{\Psi})$ and $\theta^{\mathrm{sm}}_{\mathsf{W},\mathsf{V}'}(\mathfrak{c};\mathfrak{v},\Psi)$ on the large and on the small side; for an identification $\mathsf{e}\colon \mathbf{b}\sqcup\mathbf{a}\simeq\mathbf{n}$ and $\mathsf{k}\in\mathrm{GL}_{\mathbf{n}}(\mathsf{R})$, the matrices $\mathrm{col}_{\mathsf{e}}(\mathsf{k})$ (columns of k indexed by $\mathbf{b}$) and $\mathrm{row}_{\mathsf{e}}(\mathsf{k})$ (rows of k^{-1} indexed by $\mathbf{a}$); partners for χ over a subring; the standard data $\check{\Psi}^{\mathrm{st}}_{\mathsf{e},\chi,\mathsf{z}}$, $\Psi^{\mathrm{st}}_{\chi,\mathsf{z}}$; splittings and real slices; the products $\Pi(\mathfrak{c},e)$; the equal-rank ratio $\varrho(\mathsf{V};\mathfrak{c},\mathfrak{v})$; the standard exchange identity $\mathrm{Exch}(\mathsf{X},\mathsf{V},\mathsf{V}';\mu,\mathsf{e};\mathfrak{v},\mathfrak{y};\mathcal{N})$; determinant vectors; and L-value data with the lift of the trivial character.

Moduli, lifts and descents. A *modulus* $\mathfrak{m}$ of E is a product of powers of finite places of which none is the conjugate of another; $\Sigma(\mathfrak{m})$ is its set of places and $\mathcal{O}_E/\mathfrak{m}$ its residue ring. A character ω of $\mathrm{U}(1)(\mathbb{A})$ is *tame for* $\mathfrak{m}$ *with weights* μ *and character* χ if $\mathfrak{m}$ has a place, ω has weights μ, its tame part is non-trivial exactly at the places of $\mathfrak{m}$, and χ is the character of $\mathcal{O}_E/\mathfrak{m}$ made from the local tame parts. For a function $\check{\Psi}$ on matrices over $\mathcal{O}_E/\mathfrak{m}$ and a real datum $\mathfrak{x}$, $\mathfrak{x}\otimes[\check{\Psi}]_{\mathfrak{m}}$ is the datum with real part $\mathfrak{x}$, the function $\check{\Psi}$ of the reduction modulo $\mathfrak{m}$ at the places of $\mathfrak{m}$, and the vacuum elsewhere. A *type* of forms on a hermitian space is a discrete series type at the infinite places; the *theta lift* and the *descent* attached to a pair of spaces, characters, a real datum and a function modulo $\mathfrak{m}$ are the integrals of a form against the theta kernel of the pair, from the first space to the second and back. An *edge family*, for a family of characters indexed by $\mathcal{I}$, two types and a direction, is a real datum that lifts forms of one type to the other in both directions, uniformly in the family; a real datum that does so for one choice of characters is an *edge*, and with a finite datum it gives an *arrow*.

The data. For parameters P and admissible data D: $E(D)$ is the CM field, Σ_D the set of the chosen places (above $g(D)$), $f(D)$ the number of infinite places. A *tame character* ϑ of D has a finite set $Q(\vartheta)$ of places, its modulus; $\mathrm{N}\,Q(\vartheta)$ is the product of their norms; ϑ is a *twist* if its tame part is non-trivial at every place of $Q(\vartheta)$ and an m*-fold twist* if this holds for its first m powers. $\mathfrak{m}_D(\vartheta,e)$ is the modulus that is the product of the chosen places to the power e times $Q(\vartheta)$. A *pair* p of D consists of a character ω_{p} of $\mathrm{U}(1)(\mathbb{A})$ of weights $-\mu_{\mathsf{p}}(\tau)$, where the $\mu_{\mathsf{p}}(\tau)$ are natural numbers, whose tame part is trivial off the chosen places, and of a natural number e_{p}, the *point*, with $1\le e_{\mathsf{p}}\le t(D)$ and $e_{\mathsf{p}}\le\mu_{\mathsf{p}}(\tau)\le 4b$ for every infinite place τ (b the third scale of the data). Families are indexed by the pairs (ℓ,ι) of a prime ℓ and an isomorphism ι of $\mathbb{C}$ with an algebraic closure of the ℓ-adic numbers, all primes at once: a family $\theta=(\theta_{\ell,\iota})$ of m-fold twists may be *fit* (Chapter VIII), and a family of complex numbers $(y_{\ell,\iota})$ may have *height at most* a real number (Chapter IV; the bound couples the primes, so the height of a family indexed by the ι of one prime is not defined). The spaces $\mathsf{Y}_i(D)$, $0\le i\le m$, are those of the chain of lifts, $\mathsf{Y}_m(D)$

being definite; the two *grids* of spaces, named by a flag o, with their index sets, parameters, characters, harmonics and moduli, are defined in the third note.

Inputs. The proofs below take the following for granted. Numbers are the units of the reading; references are those of the handout's catalogue, and none of their places is confirmed.

Input 1 (the first note of this unit). The exchange identity at the top of the tower (its Theorem 1); the equivariant lift of the difference of the two theta integrals; a stable space that contains a multiple of a standard datum contains every locally constant partner; partners from a function of a block; a datum with prescribed slices; products of data; the constancy of the equal-rank theta integral; the cone integral is onto the determinant vectors; the norm of the lift of the trivial character (its Theorem 5).

Input 2 (the third note of this unit). The parameters of the grid correspond along its edges; a character of the grid is tame for the modulus of the triangle; the index sets, signs, parameters and harmonics of the grids.

Input 3 (units 131 and 98, the tower and Landherr's theorem). The *tower property*: the constant term of a theta integral on the large side along an isotropic line is a multiple $\mathrm{c}_{\mathrm{tow}}$ of the theta integral of the smaller pair with the cone integral of the real datum; the complex number $\mathrm{c}_{\mathrm{tow}} \neq 0$ depends on the split tame character $\mathfrak{c}$, and $|\mathrm{c}_{\mathrm{tow}}|^2\,\Pi(\mathfrak{c}, e) = \mathrm{C}_{\mathrm{tow}}\,\Pi(\mathfrak{c}, e+1)$ with a real number $\mathrm{C}_{\mathrm{tow}}$ that does not depend on $\mathfrak{c}$ (after S. Rallis, Compositio Math. 51 (1984); an analogue); *Landherr's theorem*: two spaces with the same signs differ by a rational change of coordinates and a move of the adelic point (W. Landherr, Abh. Math. Sem. Hamburg 11 (1936); M. Kneser, TIFR lectures 47 (1969), §5.5; a special case); $\Pi(\mathfrak{c}, e) > 0$.

Inputs (continued).

Input 4 (units 26, 27 and 25, inner products of lifts). The inner product formula for theta lifts (the *Rallis inner product formula*) in its Euler form, for an edge and a sharp finite datum, with injectivity of the lift, as stated in Input 6 of the first note (relation to the literature unresolved: S. Rallis, Compositio Math. 52 (1984), §1 and Thm 1.6, treats another dual pair and is an analogue only; the unitary inner product in this rank range is J.-S. Li, J. reine angew. Math. 428 (1992), §2, Thm 2.1, eq. (26), p. 184; the constant uniform in the characters and the sharp data, the evaluation as a sum of coset terms and the injectivity go beyond both); the Petersson product of two descents; an edge gives arrows; the adjoint of a lift along an arrow is the conjugate of a descent; the theta kernel of the opposite space is a multiple of the conjugate kernel; the decay of the theta kernel in the height.

Input 5 (unit 103, Gauss sums). The *matrix Gauss sum* for primitive characters of a finite ring, as stated in Input 1 of the first note (relation to the literature unresolved: T. Kondo, J. Math. Soc. Japan 15 (1963), pp. 244–245, has the law over a finite field at invertible frequencies; the statement over a finite ring and at all frequencies is wider than the source); Jacobi's identity for complementary minors (unit 112). A multiplicative character χ of a finite commutative ring R is *primitive* if for every $\mathsf{a} \neq 0$ in R there is c with $\chi(1+\mathsf{ca}) \notin \{0,1\}$, and an additive character is *primitive* if it is non-trivial on every non-zero ideal (first note).

Input 6 (units 96 and 104, Hecke characters). Partial Hecke L-values of characters of the weights that occur do not vanish at the points that occur: for a character ω of $\mathrm{U}(1)(\mathbb{A})$ with non-zero weights and a finite set Σ, $L^{\Sigma\cup\tilde{\Sigma}}(s,\omega^{\mathrm{bc}}) \neq 0$ for real $s \geq 1$ (D. Ramakrishnan and R. Valenza, *Fourier Analysis on Number Fields*, 1999, Thm 7-28; J. Tate, 1950, in Cassels–Fröhlich, ch. XV, §4.4; a special case: the real axis, with finitely many Euler factors removed); a character of given weights is tame for a modulus that contains the places where its tame part is non-trivial; the residue ring of $\mathfrak{m}_D(\vartheta, e)$.

Input 7 (units 134 Forms, 138 Selmer, 141 Seesaw, 150 Curve, 133 Discrete series, 152 Data). Auxiliary pairs and their data; standard data and admissible combinations of translates of the integral vacuum, with the divisibility of their jets; the conjugation of jets by an automorphism of $\mathbb{C}$; the Harish-Chandra parameter of a weight by steps; fit families of twists and the height of their ratios of L-values (Chapter VIII); the served primes of the data (unit 150); the column and row index sets of the data.

Input 8 (unit 135 Heights, Chapter IV; unit 25, lifts). Heights of families indexed by the pairs (ℓ, ι): heights add in products, and a bound for the height may be enlarged. *Edge families from corresponding parameters*: when the parameters of two spaces correspond, the joint harmonic is an edge, uniformly in a family of characters; this is a result of the background library and not of the argument (relation to the literature unresolved: A. Ichino, arXiv:2002.09148, Thm 4.1 and Lemma 7.8, gives the criterion for the joint types; the formal statement selects witnesses for a two-sided analytic statement that goes beyond it).

3 The square law over a finite ring

Definition 3.1 (the square law and its data)**.** Let R be a finite commutative ring and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ finite index sets.

(a) Let χ be a multiplicative character of R, $\mathsf{e}\colon \mathbf{b} \sqcup \mathbf{a} \simeq \mathbf{n}$ an identification, $c \in \mathbb{C}$, and $\mathcal{L}$, $\mathcal{L}'$ complex functions on $\mathrm{M}_{\mathbf{a}\times\mathbf{n}}(\mathsf{R})$ and on $\mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{R})$. The statement $\mathrm{SqLaw}(\chi, \mathsf{e}; c; \mathcal{L}, \mathcal{L}')$ holds if $\mathcal{L}(\mathsf{kx}) = \chi(\det \mathsf{k})\mathcal{L}(\mathsf{x})$ for $\mathsf{k} \in \mathrm{GL}_{\mathbf{a}}(\mathsf{R})$, $\mathcal{L}'(\mathsf{yk}) = \chi(\det \mathsf{k})\mathcal{L}'(\mathsf{y})$ for $\mathsf{k} \in \mathrm{GL}_{\mathbf{b}}(\mathsf{R})$, and

$$\sum_{\mathsf{y}} \Psi(\mathsf{y})\, \mathcal{L}'(\mathsf{y}) = c \sum_{\mathsf{x}} \check{\Psi}(\mathsf{x})\, \mathcal{L}(\mathsf{x})$$

for all partners $\check{\Psi}$, Ψ for χ over R with respect to e.

(b) For a commutative ring R and an $(\mathbf{a} \sqcup \mathbf{b}) \times \mathbf{a}$ matrix x over it, $\mathbb{1}_{\mathsf{R}}^{\mathrm{incl}}(\mathsf{x})$ is 1 if x is the inclusion matrix $\binom{1}{0}$ and 0 otherwise.

(c) For a complex function G on $\mathrm{M}_{\mathbf{n}\times\mathbf{a}}(\mathsf{R})$ and an $(\mathbf{n} \sqcup \mathbf{b}) \times (\mathbf{a} \sqcup \mathbf{b})$ matrix y, $\mathrm{next}(G)(\mathsf{y})$ is $G(-\mathsf{y}_{11})$ divided by the number of $\mathbf{n} \times \mathbf{b}$ matrices over R if y has the block form $\left(\begin{smallmatrix} \mathsf{y}_{11} & * \\ 0 & 1 \end{smallmatrix}\right)$, and 0 otherwise.

(d) The *swap* is the bijection $\mathbf{a} \sqcup (\mathbf{n} \sqcup \mathbf{b}) \simeq \mathbf{n} \sqcup (\mathbf{a} \sqcup \mathbf{b})$ that exchanges the first two summands.

(e) For $\mathsf{e}\colon \mathbf{b} \sqcup \mathbf{a} \simeq \mathbf{n}$, bl_{e} is the homomorphism $\mathsf{g} \mapsto \mathrm{diag}(\mathsf{g}, 1)$ from $\mathrm{GL}_{\mathbf{b}}(\mathsf{R})$ to $\mathrm{GL}_{\mathbf{n}}(\mathsf{R})$.

(f) For χ, e as in (a) and matrices $\mathsf{x} \in \mathrm{M}_{\mathbf{a}\times\mathbf{n}}(\mathsf{R})$, $\mathsf{y} \in \mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{R})$, the *graph kernel* $\mathrm{gr}_{\chi,\mathsf{e}}(\mathsf{x}, \mathsf{y})$ is $\chi(\det \mathsf{k})/\#\,\mathrm{GL}_{\mathbf{b}}(\mathsf{R})$ if $(\mathsf{x}, \mathsf{y}) = (\mathrm{row}_{\mathsf{e}}(\mathsf{k}), \mathrm{col}_{\mathsf{e}}(\mathsf{k}))$ for some $\mathsf{k} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{R})$, and 0 otherwise.

LEAN SquareLaw · inclusionDatum · nextDatum · swapFirstTwo · blockInlGL · graphKernel

Lemma 3.2 (elementary facts)**.** *Let R be a finite commutative ring (for* (c), (d) *any commutative ring; for* (a), (b) *any finite set) and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{N}$, $\mathbf{n}$ finite index sets.*

(a) *For a complex function G on $(\mathbf{N} \sqcup \mathbf{n}) \times \mathbf{a}$ matrices, the sum of G over all matrices is the double sum over the two row blocks.*

(b) *For a complex function G on $\mathbf{a} \times (\mathbf{N} \sqcup \mathbf{n})$ matrices, the sum of G over all matrices is the double sum over the two column blocks.*

(c) *For $\mathsf{f} \in \mathrm{GL}_{\mathbf{a}}(\mathsf{R})$ and $\mathsf{g} \in \mathrm{GL}_{\mathbf{b}}(\mathsf{R})$ the map $\mathsf{x} \mapsto \mathsf{f}^{-1}\mathsf{x}\mathsf{g}$ on $\mathbf{a} \times \mathbf{b}$ matrices is injective.*

(d) *The inclusion matrix of $\mathbf{a}$ into $\mathbf{a} \sqcup \mathbf{b}$ is the block matrix $\binom{1}{0}$.*

(e) *For a multiplicative character χ of R, an identification $\mathsf{e}\colon \mathbf{b} \sqcup \mathbf{a} \simeq \mathbf{n}$ and $\mathsf{k} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{R})$: $\mathrm{gr}_{\chi,\mathsf{e}}(\mathrm{row}_{\mathsf{e}}(\mathsf{k}), \mathrm{col}_{\mathsf{e}}(\mathsf{k})) = \chi(\det \mathsf{k})/\#\,\mathrm{GL}_{\mathbf{b}}(\mathsf{R})$; in particular the value in Definition 3.1*(f) *does not depend on the choice of k.*

(f) *For χ, e as in* (e) *and matrices x, y with $\mathrm{gr}_{\chi,\mathsf{e}}(\mathsf{x}, \mathsf{y}) \neq 0$ there is $\mathsf{k} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{R})$ with $(\mathsf{x}, \mathsf{y}) = (\mathrm{row}_{\mathsf{e}}(\mathsf{k}), \mathrm{col}_{\mathsf{e}}(\mathsf{k}))$.*

(g) *For χ, e as in* (e)*, matrices x, y and $\mathsf{g} \in \mathrm{GL}_{\mathbf{b}}(\mathsf{R})$, the graph kernel satisfies $\mathrm{gr}_{\chi,\mathsf{e}}(\mathsf{x}, \mathsf{yg}) = \chi(\det \mathsf{g})\ \mathrm{gr}_{\chi,\mathsf{e}}(\mathsf{x}, \mathsf{y})$.*

Proof. (a)–(d) are reindexings and definitions. (e): two elements of $\mathrm{GL}_{\mathbf{n}}(\mathsf{R})$ with the same blocks $\mathrm{row}_{\mathsf{e}}(\mathsf{k})$, $\mathrm{col}_{\mathsf{e}}(\mathsf{k})$ have the same determinant, by Jacobi's identity (Input 5). (f) is the definition. (g): if (x, y) are the blocks of k, then $(\mathsf{x}, \mathsf{yg})$ are the blocks of $\mathsf{k} \cdot \mathrm{bl}_{\mathsf{e}}(\mathsf{g})$, of determinant $\det \mathsf{k} \det \mathsf{g}$. □

LEAN `sum_eq_sum_fromRows` · `sum_eq_sum_fromCols` · `injective_gl_inv_mul_mul` · `inclMatrix_inl` · `graphKernel_invLastRows_firstCols` · `exists_eq_of_graphKernel_ne_zero` · `graphKernel_mul_right`

Lemma 3.3 (the square law at the inclusion)**.** *Let* R *be a finite commutative ring,* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{n}$ *finite index sets,* χ *a multiplicative character of* R*,* $\mathrm{c} \in \mathbb{C}$*, and* $\mathcal{L}$*,* $\mathcal{L}'$ *functions with* $\mathrm{SqLaw}(\chi, \mathrm{swap}; \mathrm{c}; \mathcal{L}, \mathcal{L}')$ *for the swap* $\mathbf{a} \sqcup (\mathbf{n} \sqcup \mathbf{b}) \simeq \mathbf{n} \sqcup (\mathbf{a} \sqcup \mathbf{b})$*. Then for every complex function* G *on* $\mathrm{M}_{\mathbf{n}\times\mathbf{a}}(\mathsf{R})$*, with* x_0 *the inclusion matrix* $\binom{1}{0}$*,*

$$\#\,\mathrm{GL}_{\mathbf{a}}(\mathsf{R}) \sum_{\mathsf{y}} G(\mathsf{y})\, \mathcal{L}'\tbinom{\mathsf{y}}{\mathsf{x}_0} = \mathrm{c} \cdot \#\,\mathrm{GL}_{\mathbf{n}\sqcup\mathbf{b}}(\mathsf{R}) \sum_{\mathsf{x}} \mathrm{next}(G)(\mathsf{x})\, \mathcal{L}(\mathsf{x}_0 \mid \mathsf{x}).$$

Proof. Apply the partner identity of the square law to the block partners of the first note (Input 1) made from the point function of $\binom{\mathsf{x}}{0}$, and move the blocks to the standard ones with the two equivariances of $\mathcal{L}$ and $\mathcal{L}'$; the orbits are counted by the orders of the two groups, Lemma 3.2 (a)–(d) reindexing the sums. □

LEAN `SquareLaw.sum_eq_nextDatum`

Lemma 3.4 (equivariance of the graph kernel)**.** *Let* R *be a finite commutative ring,* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{n}$ *finite index sets,* χ *a multiplicative character of* R *and* $\mathsf{e} \colon \mathbf{b} \sqcup \mathbf{a} \simeq \mathbf{n}$*.*

(a) *For* $\mathsf{g} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{R})$*,* $\mathsf{f} \in \mathrm{GL}_{\mathbf{a}}(\mathsf{R})$*,* $\mathsf{q} \in \mathrm{GL}_{\mathbf{b}}(\mathsf{R})$ *and matrices* x*,* y*:* $\mathrm{gr}_{\chi,\mathsf{e}}(\mathsf{x},\mathsf{y}) = \chi(\det \mathsf{g})\chi(\det \mathsf{f})^{-1}\chi(\det \mathsf{q})^{-1}\, \mathrm{gr}_{\chi,\mathsf{e}}(\mathsf{f}^{-1}\mathsf{x}\mathsf{g}, \mathsf{g}^{-1}\mathsf{y}\mathsf{q})$.

(b) *Let* Ψ *be a complex function on* $\mathrm{M}_{\mathbf{n}\times\mathbf{b}}(\mathsf{R})$ *with* $\Psi(\mathsf{y}\mathsf{g}) = \chi^{-1}(\det \mathsf{g})\Psi(\mathsf{y})$ *for all* y *and* $\mathsf{g} \in \mathrm{GL}_{\mathbf{b}}(\mathsf{R})$*. Then for every* $\mathsf{k} \in \mathrm{GL}_{\mathbf{n}}(\mathsf{R})$*:* $\sum_{\mathsf{y}} \mathrm{gr}_{\chi,\mathsf{e}}(\mathrm{row}_{\mathsf{e}}(\mathsf{k}), \mathsf{y})\, \Psi(\mathsf{y}) = \chi(\det \mathsf{k})\, \Psi(\mathrm{col}_{\mathsf{e}}(\mathsf{k}))$.

Proof. (a) We may assume the kernel is non-zero at one of the two pairs; if (x,y) are the blocks of k, then $(\mathsf{f}^{-1}\mathsf{x}\mathsf{g}, \mathsf{g}^{-1}\mathsf{y}\mathsf{q})$ are the blocks of $\mathsf{g}^{-1}\mathsf{k}\,\mathrm{diag}(\mathsf{q},\mathsf{f})$. (b) The non-zero terms are the $\mathsf{y} = \mathrm{col}_{\mathsf{e}}(\mathsf{k})\mathsf{g}$, $\mathsf{g} \in \mathrm{GL}_{\mathbf{b}}(\mathsf{R})$, and each contributes $\chi(\det \mathsf{k})\Psi(\mathrm{col}_{\mathsf{e}}(\mathsf{k}))/\#\,\mathrm{GL}_{\mathbf{b}}(\mathsf{R})$ by Lemma 3.2 (g). □

LEAN `graphKernel_equivariant` · `sum_graphKernel_mul`

Lemma 3.5 (the graph kernel reproduces the small side)**.** *Let* R *be a finite commutative ring,* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{N}$*,* $\mathbf{n}$ *finite index sets,* χ *a multiplicative character of* R*,* $\mathsf{e} \colon \mathbf{b} \sqcup \mathbf{a} \simeq \mathbf{N} \sqcup \mathbf{n}$ *an identification, and suppose that* (R,R) *has the elementary divisor property (is a valuation pair). There is a function* d_0 *on* $(\mathbf{N} \sqcup \mathbf{n}) \times \mathbf{b}$ *matrices such that:*

(1) *if* $\mathrm{d}_0(\mathsf{y})\, \mathrm{gr}_{\chi,\mathsf{e}}(\mathsf{x},\mathsf{y}) \neq 0$*, then this number is a root of unity, a left inverse of the first row block of* y *gives one of the second column block of* x*, and a left inverse of the second row block of* y *gives one of the first column block of* x*;*

(2) *the average over* $\mathsf{k} \in \mathrm{GL}_{\mathbf{b}}(\mathsf{R})$ *of* $\mathrm{d}_0(\mathsf{y}\mathsf{k})\, \mathrm{gr}_{\chi,\mathsf{e}}(\mathsf{x},\mathsf{y})$ *is* $\mathrm{gr}_{\chi,\mathsf{e}}(\mathsf{x},\mathsf{y})$*;*

(3) *for all* c*,* $\mathcal{L}$*,* $\mathcal{L}'$ *with* $\mathrm{SqLaw}(\chi, \mathsf{e}; \mathrm{c}; \mathcal{L}, \mathcal{L}')$ *and every* y *one of whose row blocks has a left inverse:* $\mathcal{L}'(\mathsf{y}) = \mathrm{c} \sum_{\mathsf{x}} \mathrm{gr}_{\chi,\mathsf{e}}(\mathsf{x},\mathsf{y})\, \mathcal{L}(\mathsf{x})$*.*

Proof. Let d_0 be $\#\,\mathrm{GL}_{\mathbf{b}}(\mathsf{R})$ at one chosen point of each orbit $\mathsf{y}\,\mathrm{GL}_{\mathbf{b}}(\mathsf{R})$ and 0 elsewhere; (2) follows. The values of the kernel are values of χ divided by the order of the group, whence the roots of unity in (1), and the statements on left inverses are read off the blocks of an invertible matrix. For (3), write $\mathsf{y} = \mathrm{col}_{\mathsf{e}}(\mathsf{j})$ with j invertible (this uses the left inverse and the elementary divisor property), and apply the partner identity of the square law to the partners $(\check{\Psi}, \Psi)$ with

$$\Psi = \mathrm{gr}_{\chi^{-1},\mathsf{e}}(\mathrm{row}_{\mathsf{e}}(\mathsf{j}), \cdot), \qquad \check{\Psi}(\mathsf{x}) = \sum_{\mathsf{y}'} \mathrm{gr}_{\chi,\mathsf{e}}(\mathsf{x},\mathsf{y}')\, \Psi(\mathsf{y}');$$

they are partners, and the two sums are evaluated by Lemma 3.4 and Lemma 3.2. □

LEAN `exists_partner_kernel`

Lemma 3.6 (the square law after a Fourier transform)**.** *Let* R *be a finite commutative ring,* $\mathbf{N}$, $\mathbf{T}$ *finite index sets,* χ *a multiplicative character of* R *that is primitive and* e_{R} *an additive character of* R *that is primitive. There is a complex number* c_{G} *with* $|\mathrm{c}_{\mathrm{G}}|^2 \cdot (\#\mathsf{R})^{|\mathbf{T}|(|\mathbf{T}|+2|\mathbf{N}|)} = 1$ *such that for all* c, $\mathcal{L}$, $\mathcal{L}'$ *and every function* G *on* $\mathrm{M}_{\mathbf{N}\times\mathbf{T}}(\mathsf{R})$*: if* $\mathrm{SqLaw}(\chi, \mathrm{swap}; \mathrm{c}; \mathcal{L}, \mathcal{L}')$ *holds for the swap with* $\mathbf{a} = \mathbf{N} \sqcup \mathbf{T}$, $\mathbf{b} = \mathbf{T}$, $\mathbf{n} = \mathbf{N}$*, then, with* x_0 *the inclusion matrix and* $\widehat{\ }$ *the Fourier transform for* e_{R}*,*

$$\sum_{\mathsf{y}} \widehat{\overline{G}}(\mathsf{y})\, \mathcal{L}'\tbinom{\mathsf{y}}{\mathsf{x}_0} = \mathrm{c}_{\mathrm{G}}\, \mathrm{c} \sum_{\mathsf{x}} \widehat{\overline{\mathrm{next}(G)(\cdot)}}(\mathsf{x})\, \mathcal{L}(\mathsf{x}_0 \mid \mathsf{x}).$$

Proof. Apply Lemma 3.3 to the Fourier transform of $\overline{G}$ and compute the Fourier transform of the next datum with the matrix Gauss sum (Input 5); c_{G} is the conjugate Gauss sum divided by the number of matrices. □

LEAN `SquareLaw.sum_fourier_eq_nextDatum`

4 From the top of the tower to the exchange theorem

Lemma 4.1 (the opposite space)**.** *Let* E *be a CM field with a CM type,* $\mathbf{n}$ *a finite index set,* W *a hermitian space with index set* $\mathbf{n}$ *and* S *a hermitian space with the same index set and the opposite signs, and suppose that the identification* $\mathsf{g} \mapsto \mathsf{g}^-$ *of the adelic unitary groups of opposite signs carries the rational points of* W *onto those of* S*:* g^- *is rational for* S *if and only if* g *is rational for* W*. Then:* (1) *for every parameter* l *and every function* F *on* $\mathrm{U}(\mathsf{S})(\mathbb{A})$*, the function* $\mathsf{g} \mapsto F(\mathsf{g}^-)$ *is a form on* W *of the type of* l *if and only if* F *is a form on* S *of the type of* l *for the opposite signs;* (2) *for every complex function* F *on* $\mathrm{U}(\mathsf{S})(\mathbb{A})$*, the integral of* F *over* $[\mathrm{U}(\mathsf{S})]$ *equals the integral of* $\mathsf{g} \mapsto F(\mathsf{g}^-)$ *over* $[\mathrm{U}(\mathsf{W})]$*.*

Proof. The identification is an isomorphism of topological groups that preserves Haar measures and maximal compact subgroups; unfolding the definitions of forms and of quotient measures, both statements are transport along it. □

LEAN `isForm_neg_iff_and_integral_neg`

Lemma 4.2 (moving the spaces)**.** *Let* E *be a CM field with a CM type and Weil data; let* W, W_1 *be hermitian spaces with index set* $\mathbf{n}$ *and the same signs, and* V, V_1 *hermitian spaces with index set* $\mathbf{a}$ *and the same signs. Let* $\mathsf{c} \in \mathrm{GL}_{\mathbf{n}}(E)$ *and let* k *be an adelic point of the unitary group of the signs with* $\mathrm{x}_{\mathsf{W}} = \mathsf{c}^{-1}\, \mathrm{x}_{\mathsf{W}_1} \mathsf{k}$ *for the adelic frames, and let* $\mathfrak{c}$ *be a split tame character. Then there is a complex number* u *of absolute value* 1 *such that for all real data* $\mathfrak{v}$*, functions* $\check{\Psi}$ *and points* g*:* $\theta^{\mathrm{lg}}_{\mathsf{W},\mathsf{V}}(\mathfrak{c}; \mathfrak{v}, \check{\Psi})(\mathsf{g}) = \mathrm{u} \cdot \theta^{\mathrm{lg}}_{\mathsf{W}_1,\mathsf{V}_1}(\mathfrak{c}; \mathfrak{v}, \check{\Psi})(\mathsf{k}\mathsf{g})$*.*

Proof. By Landherr's theorem (Input 3) V_1 is moved to V by a rational change of coordinates and an adelic point. The two theta integrals are then integrals over the same orbit; the theta functionals are compared on the vacuum and on products of theta functions of lines, and the constant is the value of a character at a quotient of determinants. □

LEAN `thetaLarge_move`

Lemma 4.3 (adding hyperbolic planes; slices at zero)**.** *Let E be a CM field with a CM type and Weil data.*

(a) *Let $\mathbf{n}$, $\mathbf{r}$, $\mathbf{p}$ be finite index sets, W a hermitian space with index set $\mathbf{n}$ and $\mathtt{t}$ signs on $\mathbf{r}$ with $|\mathbf{r}| = |\mathbf{n}| + 2|\mathbf{p}|$ and with $|\mathbf{p}| + \mathrm{neg}(\mathsf{W})$ negative signs. Then there are a hermitian space W' with index set $\mathbf{r}$ and signs $\mathtt{t}$ and a base-adapted splitting of W' over W along $\mathbf{p}$.*

(b) *Let W, W', U be hermitian spaces, $\mathfrak{S}$ a base-adapted splitting of W' over W along $\mathbf{p}$, and $\mathfrak{S}'$ a base-adapted splitting of W' over W along $\mathbf{r}$ whose isotropic frame is that of $\mathfrak{S}$ reindexed by a bijection and whose embedding of W is that of $\mathfrak{S}$. Then the real slices at 0 for the target U agree: $\mathrm{sl}_{\mathfrak{S}}(0;\cdot) = \mathrm{sl}_{\mathfrak{S}'}(0;\cdot)$.*

(c) *Let W, W', U, V' be hermitian spaces, $\mathfrak{S}$ a base-adapted splitting of W' over W along $\mathbf{p}$ and $\mathfrak{S}''$ a base-adapted splitting of W over V' along $\mathbf{r}$. For every real datum $\mathfrak{y}$ of the pair $(\mathsf{W}', \mathsf{U})$: the slice at 0 for the composite splitting is $\mathrm{sl}_{\mathfrak{S}''}(0; \mathrm{sl}_{\mathfrak{S}}(0;\mathfrak{y}))$.*

(d) *Let $\mathbf{n}$ be an empty finite index set, W a hermitian space with index set $\mathbf{n}$, μ weights, $\mathfrak{c}$ a split tame character of weight μ and $\breve{\Psi}$ a complex function on the (empty) matrices. Then the vector named by $\breve{\Psi}$ at the places of $\mathfrak{c}$ is $\breve{\Psi}(0)$ times the vacuum.*

Proof. (a) Reindex the sum of W and $|\mathbf{p}|$ hyperbolic planes by a bijection that matches the signs; its standard splitting is base-adapted. (b) holds by definition. (c) Compare integrated zero terms: the zero term of a composite splitting is the composite of the zero terms. (d) On empty matrices $\breve{\Psi}$ is $\breve{\Psi}(0)$ times the indicator of the integral box, and naming by that indicator is a projection that fixes the vacuum. □

LEAN `exists_isBaseAdapted` · `slice_zero_congr` · `slice_zero_comp` · `name_vacElem_of_isEmpty`

Proposition 4.4 (descent: removing hyperbolic planes)**.** *Let E be a CM field with a CM type and Weil data; let $\mathbf{n}$, $\mathbf{n}'$, $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{p}$, $\mathbf{b}$ be finite index sets with $\mathbf{p}$ non-empty; let W (index set $\mathbf{n}$), W' (index set $\mathbf{n}'$), U (index set $\mathbf{b}$), V' (index set $\mathbf{r}'$) be hermitian spaces and $\mathfrak{S}$ a base-adapted splitting of W' over W along $\mathbf{p}$. Let μ be weights, $\mathcal{N}$ a real function on the split tame characters of weight μ, e a natural number and $i(\tau)$ natural numbers at the infinite places τ, V a hermitian space with index set $\mathbf{r}$ such that $\mathrm{neg}(\mathsf{V}') = |\mathbf{p}| + \mathrm{neg}(\mathsf{V})$, and e, e' identifications for the standard row data of (W, V) and $(\mathsf{W}', \mathsf{V}')$; suppose that $|\mathbf{n}|$ is even. Then there is a map f from the real data of $(\mathsf{W}', \mathsf{V}')$ to those of (W, V) such that*

(1) *for all real data $\mathfrak{x}$, $\mathfrak{y}$: if*

$$\mathrm{Exch}(\mathsf{W}', \mathsf{V}', \mathsf{U}; \mu, \mathsf{e}'; \mathfrak{x}, \mathfrak{y}; \mathcal{N})$$

holds, then so does

$$\mathrm{Exch}(\mathsf{W}, \mathsf{V}, \mathsf{U}; \mu, \mathsf{e}; \mathsf{f}(\mathfrak{x}), \mathrm{sl}_{\mathfrak{S}}(0;\mathfrak{y}); \mathcal{N});$$

(2) *if $|\mathbf{n}| = 2(e + \mathrm{neg}(\mathsf{W}))$, $|\mathbf{n}'| = 2(e + \mathrm{neg}(\mathsf{W}'))$ and $\mu(\tau) = e + i(\tau)$ for all τ, then every determinant vector of (W, V) with the parameters e and $(i(\tau))_\tau$ (first note: the natural number and the family of natural numbers at the infinite places) is the image under f of a determinant vector of $(\mathsf{W}', \mathsf{V}')$ with the same parameters.*

Proof. Strong induction on $|\mathbf{p}|$. *One plane.* Let f be the cone integral of the tower property (Input 3). The constant term along the isotropic line of the large theta integral of $(\mathsf{W}', \mathsf{V}')$ is $\mathrm{c}_{\mathrm{tow}}$ times the large theta integral of (W, V) with the datum $\mathsf{f}(\mathfrak{x})$, and $|\mathrm{c}_{\mathrm{tow}}|^2 \Pi(\mathfrak{c}, e') = \mathrm{C}_{\mathrm{tow}} \Pi(\mathfrak{c}, e' + 1)$ for $e' = |\mathbf{r}| - |\mathbf{n}|/2$: this shifts the argument of the product of L-values by one, which is the shift between the two pairs in the definition of the standard exchange identity. On the small side the constant term of the theta integral of $(\mathsf{W}', \mathsf{U})$ along the line is the theta integral of (W, U) with the slice at 0, only the zero term surviving, times the vacuum factor $\mathrm{vac}_{\mathfrak{S}}(\mathsf{U})$ of the mixed model and a character value. Dividing: if c and C are the constants of the identity for $(\mathsf{W}', \mathsf{V}', \mathsf{U})$, the constant for $(\mathsf{W}, \mathsf{V}, \mathsf{U})$ is $\mathrm{c} \cdot \mathrm{vac}_{\mathfrak{S}}(\mathsf{U}) \cdot \mathrm{u}/\mathrm{c}_{\mathrm{tow}}$ with $|\mathrm{u}| = 1$ (Lemma 4.2). It depends on $\mathfrak{c}$ through $\mathrm{c}_{\mathrm{tow}}$; but its squared absolute value is governed by $\mathrm{C}_{\mathrm{tow}}$, which is free of $\mathfrak{c}$, and the real constant $\mathrm{C}\,\mathrm{C}_{\mathrm{tow}}/|\,\mathrm{vac}_{\mathfrak{S}}(\mathsf{U})|^2$ serves for $(\mathsf{W}, \mathsf{V}, \mathsf{U})$. The standard data of an integral matrix restrict to standard data. Assertion (2) for one plane is the surjectivity of the cone integral on determinant vectors (Input 1). *Several planes.* Remove one plane, then the others, and compose: Lemma 4.3 (b), (c) identify the slices at 0 of the composite, and Lemma 4.3 (a) provides the intermediate space. □

LEAN `exists_stdExchange_removing_planes`

Proposition 4.5 (the exchange identity with an implicit constant)**.** *Let E be a CM field with a CM type and Weil data; let $\mathbf{n}$, $\mathbf{m}$, $\mathbf{r}$, $\mathbf{p}$, $\mathbf{b}$ be finite index sets with $\mathbf{p}$ non-empty; let W (index set $\mathbf{n}$), X (index set $\mathbf{m}$), U (index set $\mathbf{b}$) be hermitian spaces; let μ be weights, X^+ the sum of X and one hyperbolic plane, $\mathfrak{S}$ a base-adapted splitting of X^+ over W along $\mathbf{p}$, V a hermitian space with index set $\mathbf{r}$, and $\mathsf{e} \colon \mathbf{b} \sqcup \mathbf{r} \simeq \mathbf{n}$ an identification. Let $\mathfrak{z}$ and $\mathfrak{y}$ be rapid real data, for $(\mathsf{X}^+, \mathsf{X}^+)$ and for (U, W). Suppose $1 + \mathrm{neg}(\mathsf{X}) = |\mathbf{p}| + \mathrm{neg}(\mathsf{V}) + \mathrm{neg}(\mathsf{U})$ and that $|\mathbf{n}|$ is even. Then there are a real number C and a rapid real datum $\mathfrak{v}$ for (W, V) such that every split tame character $\mathfrak{c}$ of weight μ, with $\omega^\circ = \omega_{\mathfrak{c}}^{-1} \upsilon^{|\mathbf{n}|/2}$, has a complex number c with*

$$\mathrm{C}\,|\mathrm{c}|^2 = \Pi(\mathfrak{c}, |\mathbf{r}| - |\mathbf{n}|/2) \cdot \varrho(\mathsf{X}^+; \mathfrak{c}, \mathfrak{z})$$

and, for all partners $\check{\Psi}$, Ψ for $\chi_{\mathfrak{c}}$ over the integers of $\mathsf{F}_{\mathfrak{c}}$ with respect to e with Ψ locally constant, and every point g:

$$\mathrm{I}_{\mathsf{W},\mathsf{V}}(\omega^\circ, 1; \mathfrak{v} \otimes [\check{\Psi}])(\mathsf{g}) = \mathrm{c} \int_{[\mathrm{U}(\mathsf{U})]} \theta_{\mathsf{U},\mathsf{W}}(\omega^\circ, \omega^\circ; \mathfrak{y} \otimes [\Psi])(\mathsf{h}, \mathsf{g})\, d\mathsf{h},$$

where $\mathsf{F}_{\mathfrak{c}}$ is the product of the completions of E at the places of $\mathfrak{c}$ and $[\,\cdot\,]$ the local datum at these places.

Proof. Exchange the two members of the pair (U, W): by Lemma 4.1 the right-hand side is a theta integral on the small side for W and the space opposite to U. Build a space at the top of the tower over W with Lemma 4.3 (a), and matched real data there: a datum with the prescribed slice $\mathfrak{y}$ at 0 (Input 1), and then the exchange identity at the top (Input 1), whose normalising function is the equal-rank ratio of the datum $\mathfrak{z}$. Descend with Proposition 4.4, removing the planes of $\mathfrak{S}$: this gives the standard exchange identity for $(\mathsf{W}, \mathsf{V}, \mathsf{U})$, that is, the identity for the standard data of all integral matrices, with $\mathrm{C}|\mathrm{c}|^2$ as stated. To pass to all partners, use the equivariant lift of the difference of the two integrals (Input 1): its kernel is a stable space of Schwartz–Bruhat functions that contains the standard data, so it contains every locally constant partner (Input 1). □

LEAN exchange_of_partners

Proposition 4.6 (the equal-rank constant does not depend on the character)**.** *Let E be a CM field with a CM type and Weil data, $\mathbf{n}$ a finite index set, W a hermitian space with index set $\mathbf{n}$, μ integer weights and e a natural number with $|\mathbf{n}| = 2e + 2\,\mathrm{neg}(\mathsf{W})$ and $e \le \mu(\tau)$ for every infinite place τ. Let W^+ be the sum of W and one hyperbolic plane. Then there are a rapid real datum $\mathfrak{v}$ for $(\mathsf{W}^+, \mathsf{W}^+)$ and a real number $\mathrm{C} \neq 0$ such that $\varrho(\mathsf{W}^+; \mathfrak{c}, \mathfrak{v}) = \mathrm{C}$ for every split tame character $\mathfrak{c}$ of weight μ.*

Proof. Let J be the standard definite space of rank $2e$, and X_1 a space with the signs of W that has a base-adapted splitting over J (Lemma 4.3 (a)): it is J plus $\mathrm{neg}(\mathsf{W})$ hyperbolic planes. Its sum X_1^+ with one more plane has the signs of W^+, so Lemma 4.2 applies, and it suffices to treat X_1^+, with a determinant vector as real datum. Apply Proposition 4.5 twice, with $\mathsf{X} = \mathsf{X}_1$ and $\mathsf{W} = \mathsf{J}$, to J and the zero space in the two orders: the product of the two identities, evaluated at the standard datum with Lemma 4.3 (d), shows that the square of the ratio is a product $B_1\,\mathrm{C}_1\,|\langle \text{vacuum at } 0\rangle|^2$ of numbers that do not depend on $\mathfrak{c}$, the L-values cancelling because the two exchange constants involve $\Pi(\mathfrak{c}, e')$ and $\Pi(\mathfrak{c}, -e')$. If the ratio vanished for all $\mathfrak{c}$, the descended identity at the standard datum would give $0 = 1$ at a point where the determinant vector does not vanish (Proposition 4.4 (2) keeps determinant vectors). So the constant is non-zero. □

LEAN exists_equalRankThetaRatio_value

Proof of Theorem 1. Put $e_1 = |\mathbf{n}|/2 - \mathrm{neg}(\mathsf{W})$; by the hypotheses $e_1 \ge 1$ is a natural number and $e_1 \le \mu(\tau)$. Proposition 4.6 gives a rapid $\mathfrak{z}$ and $\mathrm{C}_0 \neq 0$ with $\varrho(\mathsf{W}^+; \mathfrak{c}, \mathfrak{z}) = \mathrm{C}_0$ for all $\mathfrak{c}$. Proposition 4.5, for $(\mathsf{V}', \mathsf{W}, \mathsf{V})$, the splitting of W^+ over W that removes its plane, and the data $\mathfrak{z}$, $\mathfrak{v}$, gives C_1 and a rapid $\mathfrak{x}$; its sign condition is $\mathrm{neg}(\mathsf{W}) = \mathrm{neg}(\mathsf{V}) + \mathrm{neg}(\mathsf{V}')$. Take $\mathrm{C} = \mathrm{C}_0/\mathrm{C}_1$ and $\mathfrak{x}$ (if $\mathrm{C}_1 = 0$ the quotient is read as 0; this happens only when there is no tame triple at all, see below). Let ω be tame for $\mathfrak{m}$ with weights μ and character χ. The weights are positive by the bound on μ, the set $\Sigma(\mathfrak{m})$ is non-empty and contains no conjugate of its members, and the tame part of ω is non-trivial exactly on it: this is a split tame character $\mathfrak{c}$ with $\Sigma(\mathfrak{c}) = \Sigma(\mathfrak{m})$. For it Proposition 4.5 gives c with $\mathrm{C}_1|\mathrm{c}|^2 = \Pi(\mathfrak{c}, |\mathbf{a}| - |\mathbf{n}|/2)\,\mathrm{C}_0$; since $|\mathbf{n}| = |\mathbf{a}| + |\mathbf{b}| = 2|\mathbf{b}| + 2e$ the argument is $e \ge 0$ and $\Pi(\mathfrak{c}, e) = \Pi(\Sigma(\mathfrak{m}), \omega, e) > 0$. The right-hand side is not zero; in particular $\mathrm{C}_1 \neq 0$ and $\mathrm{c} \neq 0$, and dividing by C_1 gives $|\mathrm{c}|^2 = \mathrm{C}\,\Pi(\Sigma(\mathfrak{m}), \omega, e)$. Finally let π_0 be the surjective ring homomorphism from the integers of $\mathsf{F}_{\mathfrak{c}}$ onto $\mathcal{O}_E/\mathfrak{m}$ that agrees with the reduction of adelic coordinates. A function on matrices over $\mathcal{O}_E/\mathfrak{m}$ pulls back along π_0, entry by entry, to a function on the integral matrices over $\mathsf{F}_{\mathfrak{c}}$, and is extended by zero off the integral matrices. Partners pull back to locally constant partners over the integers of $\mathsf{F}_{\mathfrak{c}}$, and the data modulo $\mathfrak{m}$ are the local data of the pulled-back functions. The identity of Proposition 4.5 becomes the asserted one. □

5 One square

Lemma 5.1 (three tools)**.** *Let E be a CM field with a CM type; let A, U_1, U_2 be hermitian spaces with index sets $\mathbf{n}'$, $\mathbf{n}_1$, $\mathbf{n}_2$.*

(a) *Let c_i, G_i be complex numbers and F_i functions on $([\mathrm{U}(\mathsf{U}_1)] \times [\mathrm{U}(\mathsf{U}_2)]) \times [\mathrm{U}(\mathsf{A})]$, indexed by a finite set. If every F_i with $\mathrm{c}_i \neq 0$ is integrable with integral G_i, then the iterated integral of $\sum_i \mathrm{c}_i F_i$ is $\sum_i \mathrm{c}_i G_i$.*

(b) *Let y be a form on A, F a form on U_1, F' a form on U_2, let u, d, l be linear maps and c, c', C complex numbers. If $y = \mathrm{c}\,\mathsf{u}(F)$, $F' = \mathrm{c}'\,\mathsf{d}(y)$ and $\mathrm{c}'\mathrm{c} = \mathrm{C}$, then $\mathsf{l}(F') = \mathrm{C}\,\mathsf{l}(\mathsf{d}(\mathsf{u}(F)))$. (The formal statement writes F and F' as their right translates by the unit element.)*

(c) *Suppose Weil data are given. Let y and y' be functions on $\mathrm{U}(\mathsf{U}_1)(\mathbb{A})$ and $\mathrm{U}(\mathsf{U}_2)(\mathbb{A})$ that are automorphic and rapidly decreasing, ω_1, ω_2 characters of $\mathrm{U}(1)(\mathbb{A})$, and $\mathcal{K}_0(\mathfrak{v}, \mathsf{g}, \mathsf{z})$ a function of a vector $\mathfrak{v}$ of the Fock model, a point g of $\mathrm{U}(\mathsf{U}_1 \oplus \mathsf{U}_2)$ and a point z of $\mathrm{U}(\mathsf{A})$, such that $|\mathcal{K}_0(\mathfrak{v}, \mathsf{g}, \mathsf{z})|$ is at most a constant times a weighted norm of $\mathfrak{v}$ times a power of the height of g whenever that weighted norm is finite, and such that each $\mathcal{K}_0(\mathfrak{v}, \cdot, \cdot)$ is continuous. Then there are a weighted norm and a constant such that for every $\mathfrak{v}$ of finite norm the function $y(\mathsf{q}_1)\, y'(\mathsf{q}_2)\, (\omega_1(\det \mathsf{q}_1)\omega_2(\det \mathsf{q}_2))^{-1} \mathcal{K}_0(\mathfrak{v}, \mathsf{q}_1 \oplus \mathsf{q}_2, \mathsf{z})$ is integrable over $[\mathrm{U}(\mathsf{U}_1)] \times [\mathrm{U}(\mathsf{U}_2)] \times [\mathrm{U}(\mathsf{A})]$ and bounded by the constant times the norm of $\mathfrak{v}$.*

Proof. (a) Fubini and linearity. (b) Substitute. (c) The rapid decay of y, y' absorbs the polynomial growth of $\mathcal{K}_0$ in the height; the function is then bounded and continuous on a space of finite measure. □

LEAN `integral_integral_sum_eq` · `apply_eq_mul_of_formRep_one` · `exists_integrable_kernel_bound`

Proof of Theorem 2. Apply Theorem 1 to $\mathsf{W} = \mathsf{U}_1 \oplus \mathsf{U}_2$, $\mathsf{V} = \mathsf{A}$, $\mathsf{V}' = \mathsf{A}'$; its hypotheses are the numerical hypotheses of Theorem 2. It gives C and a rapid datum, and for each i a constant $\mathrm{c} \neq 0$ with $|\mathrm{c}|^2 = \mathrm{C}\,\Pi(\Sigma(\mathfrak{m}_i), \omega_i, e)$. The first two conditions of the square law, the equivariance of $\mathsf{x} \mapsto \mathsf{l}(\mathsf{u}_i(\mathsf{x})F)$ and $\mathsf{y} \mapsto \mathsf{l}(\mathsf{v}_i(\mathsf{y})F)$ under the general linear groups of the blocks, come from the action of the Weil representation on the indicator data. For the partner identity write both composites as iterated integrals of F against theta kernels: the composite through A is the integral of F against the theta integral over $[\mathrm{U}(\mathsf{A})]$ of the pair $(\mathsf{U}_1 \oplus \mathsf{U}_2, \mathsf{A})$, restricted to $\mathrm{U}(\mathsf{U}_1) \times \mathrm{U}(\mathsf{U}_2)$, by the adjointness of lift and descent (Input 4); the composite through A' likewise for the pair $(\mathsf{A}', \mathsf{U}_1 \oplus \mathsf{U}_2)$. The integrals converge and may be interchanged with the finite sums over matrices by Lemma 5.1, the kernel bound coming from the decay of the theta kernel (Input 4). Summing against partners $\check{\Psi}$, Ψ turns the indicator data into the data of $\check{\Psi}$ and Ψ, and the exchange identity of Theorem 1 identifies the two sides up to c. The real data of the edges are polynomial vectors, while Theorem 1 provides only some rapid datum on one side: the integrals for polynomial data lie in a fixed line, which is closed, and approximating the datum of Theorem 1 by polynomial vectors gives the number $\varkappa$ that compares the two; it is not shown to be non-zero. Hence the constant $\varkappa/\mathrm{c}$. □

6 Triangles

Definition 6.1 (triangles and weighted chains)**.** Let $\mathcal{I}$ be an index set, $\mathcal{F}_{i,n}$ $(i,n\ge 0)$ complex inner product spaces and $\mathcal{V}$ a complex inner product space. The elements of $\mathcal{I}$ are written ϑ, because in the only application (Theorem 3) $\mathcal{I}$ is the set of the m-fold twists of the data; in this definition and in Lemma 6.2 $\mathcal{I}$ is an arbitrary set. Let m be a natural number; w a family of natural numbers $\mathrm{w}_\vartheta(n)$ $(\vartheta\in\mathcal{I})$, the *weights*; $\mathcal{M}$ a relation between natural numbers n and families $\mathrm{c}=(\mathrm{c}_\vartheta)_{\vartheta\in\mathcal{I}}$ of complex numbers, read "c has grade at most n"; a and d families of linear maps $\mathsf{a}_{\vartheta,i,n}\colon\mathcal{F}_{i,n}\to\mathcal{F}_{i,n+1}$ and $\mathsf{d}_{\vartheta,i,n}\colon\mathcal{F}_{i,n+1}\to\mathcal{F}_{i+1,n}$; $\mathsf{j}_i\colon\mathcal{V}\to\mathcal{F}_{i,0}$ linear isometries; and G a proposition.

(a) $\mathrm{Tri}(m;\mathrm{w},\mathcal{M};\mathsf{a},\mathsf{d};\mathsf{j};\mathrm{G})$ holds if: (1) the constant family 1 has grade at most 0; (2) if c has grade at most n, c' grade at most n' and $n+n'\le n''$, then $\mathrm{c}\mathrm{c}'$ has grade at most n''; (3) for $i+(n+1)<m$ there is a family b with $\mathrm{w}_\vartheta(n)\langle y'',\mathsf{a}_{\vartheta,i+1,n}\mathsf{d}_{\vartheta,i,n}y\rangle=\overline{\mathrm{b}_\vartheta}^{-1}\mathrm{w}_\vartheta(n+2)\langle y'',\mathsf{d}_{\vartheta,i,n+1}\mathsf{a}_{\vartheta,i,n+1}y\rangle$ for all ϑ, y, y'', and such that, if G holds, b has grade at most 1 and there is a family k of grade at most 0 such that for all ϑ and y'' some y' satisfies $\langle\mathsf{a}_{\vartheta,i,n+1}y,y''\rangle=\langle y,y'\rangle$ for all y and $\langle\mathsf{d}_{\vartheta,i,n}y',y\rangle=\mathrm{k}_\vartheta\langle\mathsf{d}_{\vartheta,i,n+1}y'',\mathsf{a}_{\vartheta,i+1,n}y\rangle$ for all y; (4) for $i<m$, all ϑ and $y\in\mathcal{V}$: $\langle y'',\mathsf{a}_{\vartheta,i,0}\mathsf{j}_iy\rangle=\langle\mathsf{d}_{\vartheta,i,0}y'',\mathsf{j}_{i+1}y\rangle$ for all y'', and $\mathsf{d}_{\vartheta,i,0}\mathsf{a}_{\vartheta,i,0}\mathsf{j}_iy=\mathsf{j}_{i+1}y'$ for some $y'\in\mathcal{V}$ that vanishes only if y does.

(b) $\mathrm{wc}(\mathrm{w},\mathsf{a};\vartheta,i)$ is the family, indexed by n, of the composites $\mathcal{F}_{i,0}\to\mathcal{F}_{i,n}$ of the first n maps $\mathsf{a}_{\vartheta,i,n'}$, the one of index n' multiplied by $\mathrm{w}_\vartheta(n')$; its n-th member is written $\mathrm{wc}(\mathrm{w},\mathsf{a};\vartheta,i)_n$.

LEAN IsTriangle · weightedChain

Lemma 6.2 (weighted chains in a triangle)**.** *Let $\mathcal{I}$, $\mathcal{F}_{i,n}$, $\mathcal{V}$ be as in Definition 6.1, and suppose that* $\mathrm{Tri}(m;\mathrm{w},\mathcal{M};\mathsf{a},\mathsf{d};\mathsf{j};\mathrm{G})$ *holds and that all weights are non-zero.*

(a) *Let i, n be natural numbers with $i+n<m$. There are families* b, k *such that for all ϑ and y*

$$\mathrm{wc}(\mathrm{w},\mathsf{a};\vartheta,i+1)_n(\mathsf{d}_{\vartheta,i,0}\mathsf{a}_{\vartheta,i,0}y)=\frac{\mathrm{w}_\vartheta(n+1)}{\mathrm{w}_\vartheta(1)\mathrm{w}_\vartheta(0)\overline{\mathrm{b}_\vartheta}}\,\mathsf{d}_{\vartheta,i,n}(\mathrm{wc}(\mathrm{w},\mathsf{a};\vartheta,i)_{n+1}y),$$

and such that, if G *holds,* kb *has grade at most n and*

$$\langle y'',\mathsf{a}_{\vartheta,i,n}\,\mathrm{wc}(\mathrm{w},\mathsf{a};\vartheta,i)_n\mathsf{j}_iy\rangle=\mathrm{k}_\vartheta\langle\mathsf{d}_{\vartheta,i,n}y'',\mathrm{wc}(\mathrm{w},\mathsf{a};\vartheta,i+1)_n\mathsf{j}_{i+1}y\rangle$$

for all ϑ, y, y''.

(b) *For natural numbers n, i with $i+n\le m$, every ϑ and every $y\in\mathcal{V}$: if* $\mathrm{wc}(\mathrm{w},\mathsf{a};\vartheta,i)_n(\mathsf{j}_iy)=0$ *then $y=0$.*

(c) *Suppose* G *holds. Let $\mathcal{V}'$ be a complex vector space,* $\mathsf{s}\colon\mathcal{V}'\to\mathcal{V}$ *a linear map and* $\mathsf{S}_{\vartheta,i}$ *endomorphisms of $\mathcal{V}'$ with* $\mathrm{w}_\vartheta(0)\langle\mathsf{a}_{\vartheta,i,0}\mathsf{j}_iy',\mathsf{a}_{\vartheta,i,0}\mathsf{j}_i\mathsf{s}y\rangle=\langle y',\mathsf{s}\,\mathsf{S}_{\vartheta,i}y\rangle$ *for $i<m$ and all ϑ, y', y. Then for all natural numbers n, i with $i+n=m$ there is a family* k *of grade at most $n\cdot n$ such that every ϑ has an endomorphism* T *of $\mathcal{V}'$ with* $\det\mathsf{T}=\prod_{i\le i'<m}\det\mathsf{S}_{\vartheta,i'}$ *and, for all y', y,*

$$\langle\mathrm{wc}(\mathrm{w},\mathsf{a};\vartheta,i)_n\mathsf{j}_iy',\mathrm{wc}(\mathrm{w},\mathsf{a};\vartheta,i)_n\mathsf{j}_i\mathsf{s}y\rangle=\mathrm{k}_\vartheta\,\frac{\mathrm{w}_\vartheta(0)\,\mathrm{w}_\vartheta(1)^n}{\mathrm{w}_\vartheta(n)}\,\langle y',\mathsf{s}\,\mathsf{T}y\rangle.$$

Proof. (a) Induction on n. For $n = 0$ the identity is trivial and the adjoint law is condition (4) of a triangle. The step moves one more square past the chain by condition (3): the commutation with the scalar $\overline{\mathrm{b}}^{-1}\mathrm{w}(n+2)/\mathrm{w}(n)$ gives the first identity, the weights telescoping, and the adjoint law of (3) under G gives the second, the grades adding by (2). (b) Induction on n: by (a), $\mathsf{d}_{\vartheta,i,n-1}$ of the chain of length n of row i is a non-zero multiple of the chain of length $n-1$ of row $i+1$ applied to $\mathsf{j}_{i+1}y'$, with $y' = 0$ only if $y = 0$ by (4). (c) Induction on n: move the first arrow of row i down by (a), use the formula for row $i+1$, and compose the endomorphisms; the grade $n \cdot n$ is the sum of the grades collected. □

LEAN `IsTriangle.weightedChain_descent` · `IsTriangle.eq_zero_of_weightedChain` · `IsTriangle.inner_weightedChain_eq`

7 Costs in squares

Definition 7.1 (costs)**.** Let P be parameters and D admissible data, with first scale x and $t(D)$ as in the handout. For a natural number n and a complex function c on the m-fold twists of D, the statement $\mathrm{Cost}_D(n;\mathrm{c})$ ("c costs n squares") holds if for every fit family $\theta = (\theta_{\ell,\iota})$ of m-fold twists of D, one twist for each prime ℓ and each isomorphism ι of $\mathbb{C}$ with an algebraic closure of the ℓ-adic numbers, the family $(\ell,\iota) \mapsto |\mathrm{c}(\theta_{\ell,\iota})/\mathrm{c}(1)|^{-2}$ has height at most $n \cdot (2\,t(D)\,x^{999})$; here, as in the formal text, the inverse of 0 is 0.

LEAN `Data.CostsSquares`

Lemma 7.2 (rules for costs)**.** *Let P be parameters and D admissible data.*

(a) *If* $\mathrm{Cost}_D(n;\mathrm{c})$ *and* $\mathrm{Cost}_D(n';\mathrm{c}')$ *hold and* $n + n' \le n''$*, then* $\mathrm{Cost}_D(n'';\mathrm{cc}')$ *holds.*

(b) *For a natural number e and a tame character ϑ: the residue ring of $\mathfrak{m}_D(\vartheta,e)$ has $(\prod_{\mathfrak{p}\in\Sigma_D} \mathrm{N}\,\mathfrak{p})^e \cdot \mathrm{N}\,Q(\vartheta)$ elements.*

(c) *Let e be a natural number, C a real number and i_0 an integer. If $|\mathrm{c}(\vartheta)|^2 = \mathrm{C} \cdot \#(\mathcal{O}_E/\mathfrak{m}_D(\vartheta,e))^{i_0}$ for every m-fold twist ϑ, then* $\mathrm{Cost}_D(0;\mathrm{c})$ *holds.*

(d) $\mathrm{Cost}_D(0;1)$ *holds.*

(e) *Let ϑ be a tame character, ω a character of $\mathrm{U}(1)(\mathbb{A})$, μ weights and χ a multiplicative character of $\mathcal{O}_E/\mathfrak{m}_D(\vartheta,1)$. If ω is tame for $\mathfrak{m}_D(\vartheta,1)$ with weights μ and character χ, then χ is primitive.*

(f) *Let $\mathfrak{p}$ be a pair of D that is ramified exactly at the chosen places, $e \neq 0$ a natural number and n a lift index, $1 \le n \le m$. Then for every m-fold twist ϑ the character $(\omega_{\mathfrak{p}}\vartheta^n)^{-1}$ is tame for $\mathfrak{m}_D(\vartheta,e)$, with the weights $\mu_{\mathfrak{p}}$ (the character $\omega_{\mathfrak{p}}$ itself has the weights $-\mu_{\mathfrak{p}}$) and the character of the tame parts; and every* c *with $|\mathrm{c}(\vartheta)|^2 = \mathrm{C} \cdot \Pi(\Sigma(\mathfrak{m}_D(\vartheta,e)),(\omega_{\mathfrak{p}}\vartheta^n)^{-1},e_{\mathfrak{p}})$ for all ϑ and some real C satisfies* $\mathrm{Cost}_D(1;\mathrm{c})$.

Proof. (a) Heights add in products, and a bound for a height may be enlarged (Input 8). (b) is a count (Input 6). (c) For a fit family, $|\mathrm{c}(\theta_{\ell,\iota})/\mathrm{c}(1)|^{-2}$ is a power of $\mathrm{N}\,Q(\theta_{\ell,\iota})$ by (b); the height of the constant family is 0, and a family that differs from one of height at most h_0 by powers of the norms of the moduli of a fit family has height at most any $h_1 \ge h_0$ (Lemma 8.9 (c)). (d) is (c) with exponent 0. (e) A non-zero element a of the residue ring has a non-zero residue at some place of the

modulus; the tame part there is non-trivial, so it takes a value $\neq 1$ at some unit, and lifting gives c with $\chi(1+\mathsf{ca}) \notin \{0,1\}$. (f) Tameness follows from the weights (Input 2, Input 6). By Lemma 8.10 (b) the product of L-values is $\mathrm{N}\,Q(\vartheta)^{e_\mathsf{p}^2}$ times its value at the trivial twist times the squares of the ratios of twisted to untwisted L-values at the points $1, \dots, e_\mathsf{p}$; for a fit family the inverse square of each of these e_p ratios has height at most $2x^{999}$ (Input 7), so together they have height at most $2e_\mathsf{p}x^{999} \le 2\,t(D)\,x^{999}$, since $e_\mathsf{p} \le t(D)$ for a pair; and the power of the norm is harmless as in (c). □

LEAN `Data.CostsSquares.mul` · `Data.card_m` · `Data.CostsSquares.of_norm_sq_eq_mul_card_zpow` · `Data.CostsSquares.one` · `Data.isPrimitiveMulChar_of_isTameFor` · `Data.exists_isTameFor_costsSquares_one`

8 The lift of the trivial character from U(1)

Definition 8.1 (L-value data and the lift of the trivial character)**.** Let E be a CM field with a CM type and $\mathbf{n}$ a finite index set.

(a) If $\mathbf{n}$ is non-empty, one index of $\mathbf{n}$ is chosen once and for all (an embedding of the one-point set).

(b) $\mathrm{LD}(E, \mathbf{n})$ is the set of *L-value data*: such a datum $\mathfrak{l}$ consists of level places of E (chosen places and twist places), jet degrees $\mathrm{dg}_\mathfrak{l}(\tau) \in \mathbb{N}$ at the infinite places, and a character $\omega_\mathfrak{l}$ of $\mathrm{U}(1)(\mathbb{A})$ of weights $\tau \mapsto -(\mathrm{dg}_\mathfrak{l}(\tau) + \lfloor |\mathbf{n}|/2 \rfloor)$ whose tame part is trivial off the places; moreover $|\mathbf{n}| > 1$.

(c) $\mathrm{aux}(\mathfrak{l})$ is the auxiliary pair with the places of $\mathfrak{l}$, the definite line l_1, the chosen index of $\mathbf{n}$, the weight with one row of length $\mathrm{dg}_\mathfrak{l}(\tau)$ at each τ, and the character $\omega_\mathfrak{l}$.

(d) Suppose Weil data are given. For a scale c, a datum f and finite adelic points g of $\mathrm{U}(\mathsf{l}_1)$ and h of $\mathrm{U}(\mathsf{l}_\mathbf{n})$, $\mathrm{kv}_\mathfrak{l}(\mathsf{c}; f)(\mathsf{g}, \mathsf{h})$ is the family of jets, for the one-row weight of $\mathfrak{l}$ and with derivatives scaled by c, of the datum f moved by (g, h) under the Weil representation of the pair of $\mathrm{aux}(\mathfrak{l})$.

(e) $\mathrm{lk}(\mathfrak{l})$ is the theta kernel from l_1 to $\mathsf{l}_\mathbf{n}$ for the weight and the characters of $\mathrm{aux}(\mathfrak{l})$, with scale 1 and the full level on $\mathrm{U}(\mathsf{l}_1)$.

(f) $\mathrm{kf}_\mathfrak{l}(\mathsf{g})$ is the algebraic form $\mathsf{h} \mapsto \mathrm{kv}_\mathfrak{l}(1; \Upsilon)(\mathsf{g}, \mathsf{h})$ on $\mathsf{l}_\mathbf{n}$, where Υ is the finite datum of $\mathrm{aux}(\mathfrak{l})$; it has the level, character and weight of $\mathrm{aux}(\mathfrak{l})$.

(g) $\mathrm{chl}(\mathfrak{l})$ is the principal level on the finite adelic points of $\mathrm{U}(\mathsf{l}_1)$ at the chosen places of $\mathfrak{l}$.

(h) $\Xi(\mathfrak{l})$, the *lift of the trivial character*, is the sum of the forms $\mathrm{kf}_\mathfrak{l}(\mathsf{g})$ over the classes g of $\mathrm{U}(\mathsf{l}_1)$ at the level $\mathrm{chl}(\mathfrak{l})$.

LEAN `chosenIndexEmb` · `LValueData` · `LValueData.auxPair` · `LValueData.kernelValue` · `LValueData.liftKernel` · `LValueData.kernelForm` · `LValueData.chosenLevel` · `LValueData.trivialLift` · the characters of $\mathrm{aux}(\mathfrak{l})$ have the expected weights: `LValueData.a_hasWeight`, `LValueData.b_hasWeight`

Definition 8.2 (pairs of the data and the algebraic part)**.** Let P be parameters, D admissible data, p a pair of D and $\mathbf{n}$ a finite index set.

(a) $\mathrm{LiftRank}(\mathsf{p}, \mathbf{n})$ ("p has lift rank $\mathbf{n}$") holds if $|\mathbf{n}| = 2e_\mathsf{p}$ and $e_\mathsf{p} \ge 2$.

(b) $\mathrm{Flat}(\mathsf{p}, \mathbf{n})$ ("p is flat") holds if $\mu_\mathsf{p}(\tau) = e_\mathsf{p}$ at every infinite place τ and p has lift rank $\mathbf{n}$.

(c) $\mathrm{den}(\mathsf{p}) = g(D)^{2f(D)\cdot 2e_\mathsf{p}} \cdot (g(D)\,d)^{\sum_\tau (\mu_\mathsf{p}(\tau) - e_\mathsf{p})}$, a natural number; d is the prime of the parameters.

(d) Suppose $\mathfrak{p}$ has lift rank $\mathbf{n}$, and let ϑ be a tame character. The formal definitions that follow take the proof of $\mathrm{LiftRank}(\mathfrak{p},\mathbf{n})$ as their first argument; we write the pair and the index set in its place. $\mathrm{ld}(\mathfrak{p},\mathbf{n};\vartheta)$ is the L-value datum for $E(D)$ and $\mathbf{n}$ with the level places of ϑ (chosen places Σ_D, twist places $Q(\vartheta)$), the jet degrees $\mu_{\mathfrak{p}}(\tau)-e_{\mathfrak{p}}$ and the character $\omega_{\mathfrak{p}}\vartheta$.

(e) $\Xi^{\mathrm{sc}}(\mathfrak{p},\mathbf{n};\vartheta)=\mathrm{vs}^{|\mathbf{n}|}\,(\bar{\pi}_g\,\mathrm{per})^{\sum_\tau(\mu_{\mathfrak{p}}(\tau)-e_{\mathfrak{p}})}\cdot\Xi(\mathrm{ld}(\mathfrak{p},\mathbf{n};\vartheta))$, the scaled lift of the trivial character; vs is the vacuum scale and per the jet scale of the data.

(f) $\mathrm{alg}(\mathfrak{p},\mathbf{n};\vartheta)=\mathrm{M}_D\cdot\mathrm{den}(\mathfrak{p})^2\cdot\|\Xi^{\mathrm{sc}}(\mathfrak{p},\mathbf{n};\vartheta)\|^2\big/\,\mathrm{N}\,Q(\vartheta)^{|\mathbf{n}|-1}$, the *algebraic part*; here $\mathrm{M}_D=\#\prod_{\mathfrak{p}\in\Sigma_D}\mathrm{GL}_{r\cdot t(D)}(\mathcal{O}_E/\mathfrak{p})$ is the order of the product, over the chosen places, of the general linear groups of rank $r\cdot t(D)$ of the residue fields.

(g) $\mathrm{pd}(\mathfrak{p},\mathbf{n};\vartheta)$ is $\mathrm{vs}^{|\mathbf{n}|}$ times the datum that is the level indicator of $\mathrm{aux}(\mathrm{ld}(\mathfrak{p},\mathbf{n};\vartheta))$ modulo its places, on the vacuum, and

$$\mathrm{pk}(\mathfrak{p},\mathbf{n};\vartheta)=\mathrm{kv}_{\mathrm{ld}(\mathfrak{p},\mathbf{n};\vartheta)}(\bar{\pi}_g\,\mathrm{per};\mathrm{pd}(\mathfrak{p},\mathbf{n};\vartheta))(\cdot,\cdot),$$

a function of finite points of $\mathrm{U}(\mathsf{l}_1)$ and $\mathrm{U}(\mathsf{l}_{\mathbf{n}})$.

(h) For an integer i, $\mathrm{low}(\mathfrak{p},i)$ is the pair with the character of $\mathfrak{p}$ and the point $\min(\max(1,i),e_{\mathfrak{p}})$.

LEAN `Data.Pair.HasLiftRank` · `Data.Pair.IsFlat` · `Data.Pair.denomBound` · `Data.Pair.lValueData` · `Data.Pair.trivialLift` · `Data.Pair.intAlgPart` · `Data.Pair.pointDatum` · `Data.Pair.pointKernel` · `Data.Pair.lowerPoint`

Definition 8.3 (integral measures)**.** Let P be parameters, D admissible data, ϑ a tame character of D and G a complex function on the tame characters of D. $\mathrm{IntMeas}(\vartheta;G)$ holds if there are finitely many points u_i of $\mathrm{U}(1)(\mathbb{A})$ and complex numbers c_i such that $|j(\mathrm{c}_i)|\le 1$ for every prime ℓ that does not divide $\mathrm{N}\,Q(\vartheta)$ and every ring homomorphism j from $\mathbb{C}$ to the completed algebraic closure $\mathbb{C}_\ell$ of the ℓ-adic numbers, and such that $G(\vartheta')=\sum_i \mathrm{c}_i\,\vartheta'(\mathsf{u}_i)$ for every tame character ϑ' with $Q(\vartheta')=Q(\vartheta)$.

LEAN `Data.TameChar.IsIntegralMeasure`

Small facts used in the proofs. They are not stated as lemmas; each is a single step in the formal text. For data D: the rank of $\mathsf{Y}_i(D)$ is at most that of $\mathsf{Y}_m(D)$ for $i\le m$; the index set of the spaces V of the grid has $r+t(D)$ elements; the number of row indices whose number lies in an interval $[i,i')$ with $i'\le r$ is $i'-i$; the parameter of the rows is strictly decreasing at each infinite place, and the doubled parameter of $\mathsf{W}_i(D)$ is ordered against the numbering of the basis; exactly $m-i$ basis vectors of $\mathsf{W}_i(D)$ are negative. For L-value data $\mathfrak{l}$: the weight of $\mathrm{aux}(\mathfrak{l})$ is a lift weight from the line; the Weyl representation of a one-row weight is trivial on the line; the class-sum lift of a form y at $(1,\mathsf{g}')$ is $\langle\mathrm{kf}_{\mathfrak{l}}(\mathsf{g}'),y\rangle$, and $\langle\Xi(\mathfrak{l}),y\rangle$ is the class sum of $\mathsf{u}\mapsto\theta_{\mathrm{aux}(\mathfrak{l})}(y)(1,\mathsf{ug})$. For a pair of lift rank $\mathbf{n}$: the ratio of Lemma 8.10 is $\mathrm{alg}(\mathfrak{p},\mathbf{n};\vartheta)/\,\mathrm{alg}(\mathfrak{p},\mathbf{n};1)$; $\mathrm{alg}(\mathfrak{p},\mathbf{n};\vartheta)\neq 0$; and $\Xi^{\mathrm{sc}}(\mathfrak{p},\mathbf{n};\vartheta)\neq 0$ (all three from Theorem 5 and Input 6). The passages to the opposite space of Definition 9.1 commute with right translation. A modulus $\mathfrak{m}$ of $E(D)$ whose ideal contains a natural number $e_2>0$ has a residue ring with at most $e_2^{2f(D)}$ elements. There are a number and an integer exponent such that for every m-fold twist the index of the last level of the chain is that number times a power of $\#(\mathcal{O}_E/\mathfrak{m}_D(\vartheta,0))$ times the m-th power of the Gram factor of the first auxiliary

pair times $\mathrm{ord}_{D,\mathrm{false}}(\vartheta,m)/(\mathrm{ord}_{D,\mathrm{false}}(\vartheta,0)\,\mathrm{ord}_{D,\mathrm{false}}(\vartheta,1)^m)$, with the triangle orders of Definition 9.1 (f).

LEAN `Data.card_Y_le` · `Data.card_V` · `Data.card_filter_basisNumber` · `Data.varsigma_strictAnti` · `Data.paramW_lt_iff` · `Data.card_neg_signsW` · `LValueData.isLiftWeight` · `LValueData.conjWeylRep_eq_one` · `LValueData.theta_eq_inner_kernelForm` · `LValueData.inner_trivialLift` · `Data.Pair.ratio_eq_intAlgPart_div` · `Data.Pair.intAlgPart_ne_zero` · `Data.Pair.trivialLift_ne_zero` · `Data.triNegEquiv_formRep` · `Modulus.card_resRing_le` · `Data.stdLevelIndex_lastResLevel_eq`

Lemma 8.4 (parameters of the chain)**.** *Let P be parameters and D admissible data.*

(a) *At each infinite place the doubled parameter of $\mathsf{Y}_m(D)$ has distinct entries and is twice the Harish-Chandra parameter of the weight of the forms on $\mathsf{Y}_m(D)$ with its last entry.*

(b) *For a tame character ϑ and a lift index i, $1 \le i \le m$: the doubled parameter of $\mathsf{W}_i(D)$ fits the i-th auxiliary pair of (D,ϑ).*

Proof. Both are verifications entry by entry. The Harish-Chandra parameter of a weight is computed by steps along its columns (Input 7); the indices of $\mathsf{Y}_m(D)$, the block of the rows and m blocks of size $t(D)$, are matched with these steps. For (b) the same formula gives the parameter of the definite space of the auxiliary pair, and the four conditions of fitting are checked with the small facts above. □

LEAN `Data.isHCParam_Y` · `Data.fitsParams_auxPair`

Lemma 8.5 (inner products of algebraic forms)**.** *Let E be a CM field with a CM type, $\mathbf{n}$ a finite index set and V a hermitian space with index set $\mathbf{n}$ that is a standard space. Let a weight and its parameter l be given such that l is the Harish-Chandra parameter of the weight, and let F, F' be algebraic forms on V of that weight, of a level and a character at given level places. Then $\langle F,F'\rangle$ is the index of the level in the stabilizer of the integral lattice, times the number of elements of the Weyl index set of the weight, times the Petersson product of the functions on $\mathrm{U}(\mathsf{V})(\mathbb{A})$ attached to F and F'.*

Proof. Unfold the Petersson product and integrate the class function over the classes of the level. □

LEAN `LevelPlaces.LevelForm.inner_eq_petersson`

Lemma 8.6 (the kernel forms)**.** *Let E be a CM field with a CM type and Weil data, $\mathbf{n}$ a finite index set and $\mathfrak{l} \in \mathrm{LD}(E,\mathbf{n})$.*

(a) *The map $\mathsf{g} \mapsto \mathrm{kf}_{\mathfrak{l}}(\mathsf{g})$ is invariant on the left under the rational points of $\mathrm{U}(\mathsf{l}_1)$ and on the right under $\mathrm{chl}(\mathfrak{l})$.*

(b) *For every finite point g of $\mathrm{U}(\mathsf{l}_1)$: $\Xi(\mathfrak{l})$ is the sum of the forms $\mathrm{kf}_{\mathfrak{l}}(\mathsf{ug})$ over the classes u.*

(c) *For a real point x of $\mathrm{U}(\mathsf{l}_1)$, a vector w of the weight, a real point y of $\mathrm{U}(\mathsf{l}_{\mathbf{n}})$ and finite points g, h: the theta functional of the Fock datum of w moved by $((\mathsf{x},\mathsf{g}),(\mathsf{y},\mathsf{h}))$ is the Gram form of $\bar{\mathsf{w}}$ with the image of $\mathrm{kf}_{\mathfrak{l}}(\mathsf{g})(\mathsf{h})$ under the conjugate Weyl representation of y^{-1}.*

(d) *Suppose that all jet degrees of $\mathfrak{l}$ vanish. Then for finite points g, h and every coordinate i: $\mathrm{kf}_{\mathfrak{l}}(\mathsf{g})(\mathsf{h})_i$ is the value at $((1,\mathsf{g}),(1,\mathsf{h}))$ of the theta kernel of $(\mathsf{l}_1,\mathsf{l}_{\mathbf{n}})$ with the characters 1 and that of $\mathrm{aux}(\mathfrak{l})$, the polynomial 1 and the finite datum of the determinant modulo the square of the chosen places times the twist places.*

Proof. (a) On the left this is the automorphy of the theta kernel; on the right, the finite datum Υ is fixed by the elements of the maximal compact subgroup that reduce to 1 at the chosen places. (b) A class sum of a function invariant on both sides is unchanged by a translation of its argument, $\mathrm{U}(\mathsf{l}_1)$ being commutative. (c) Separate the Weil representation into its real and finite parts; the theta functional of the Fock datum is a Gram form with the Schur jets. (d) Without derivatives the jets are the theta values, and the coefficient polynomial is the constant 1. □

LEAN `LValueData.kernelForm_invariant` · `LValueData.trivialLift_eq_stdClassSum_translate` · `LValueData.theta_pairWeil_fockDatum` · `LValueData.kernelForm_of_jetDeg_eq_zero`

Lemma 8.7 (the polynomials of a one-row weight)**.** *Let E be a CM field with a CM type, $\mathbf{n}$ a finite index set and $\mathfrak{l} \in \mathrm{LD}(E, \mathbf{n})$.*

(a) *For a coordinate vector w of the weight, the polynomial of w is the harmonic polynomial of $\mathrm{aux}(\mathfrak{l})$ at w with its variables renamed injectively (each variable to the one with the same row index and column 0).*

(b) *For such w, the harmonic polynomial of $\mathrm{aux}(\mathfrak{l})$ at w is the Schur coefficient polynomial of w times the Weyl Gram matrix.*

(c) *The hook number of the weight of $\mathrm{aux}(\mathfrak{l})$ in rank $|\mathbf{n}|$ is $\prod_\tau \mathrm{dg}_{\mathfrak{l}}(\tau)!$.*

Proof. (a) Both sides are the same intertwining map on Schur modules, checked on the Weyl evaluations. (b) Evaluate both polynomials at every point, using (a). (c) The highest polynomial of a one-row weight is the monomial with the exponents $\mathrm{dg}_{\mathfrak{l}}(\tau)$, whose Fock pairing with itself is the product of their factorials. □

LEAN `LValueData.toPoly_eq_rename_harmonicPoly` · `LValueData.harmonicPoly_eq_schurCoeffPoly` · `LValueData.hookNumber_eq_prod_factorial`

Lemma 8.8 (the lift as a sum of point kernels; integrality)**.** *Let P be parameters, D admissible data, p a pair of D that has lift rank $\mathbf{n}$, and ϑ a tame character.*

(a) *For every finite point h of $\mathrm{U}(\mathsf{l}_{\mathbf{n}})$: $\Xi^{\mathrm{sc}}(\mathsf{p}, \mathbf{n}; \vartheta)(\mathsf{h}) = \sum_{(\mathsf{u},\mathsf{A})} \mathrm{pk}(\mathsf{p}, \mathbf{n}; \vartheta)(\mathsf{u}\,\tilde{\mathsf{A}}, \mathsf{h})$, the sum over representatives u of the classes of $\mathrm{U}(\mathsf{l}_1)$ at the principal level at the chosen places and over the elements A of the product of the unit groups of the residue fields at $Q(\vartheta)$, with $\tilde{\mathsf{A}}$ a fixed lift.*

(b) *Let ℓ be a prime with a homomorphism j from $\mathbb{C}$ to the completed algebraic closure of the ℓ-adic numbers, and g, h finite points. If ℓ is prime to $\mathrm{N}\,Q(\vartheta)$, then $\mathrm{den}(\mathsf{p}) \cdot \mathrm{pk}(\mathsf{p}, \mathbf{n}; \vartheta)(\mathsf{g}, \mathsf{h})$ is j-integral: its gauge at h is at most 1.*

Proof. (a) Unfold the lift as a class sum of kernel forms and write the finite datum Υ as the sum, over A, of the translates of the datum of the level indicator (Input 7). (b) The point datum, scaled, lies in the standard data at ℓ (Lemma 10.2), jets of standard data are divisible (Input 7), and the jet scale and the boxes of the weight are cleared by $\mathrm{den}(\mathsf{p})$. □

LEAN `Data.Pair.trivialLift_apply` · `Data.Pair.weylGauge_pointKernel_le_one`

Lemma 8.9 (integral measures and heights)**.** *Let P be parameters and D admissible data.*

(a) *For a tame character ϑ and a function G with $\mathrm{IntMeas}(\vartheta; G)$: $\mathrm{IntMeas}(\vartheta; \vartheta' \mapsto |G(\vartheta')|^2)$ holds.*

(b) *For a prime ℓ, an isomorphism ι of $\mathbb{C}$ with an algebraic closure of the ℓ-adic numbers and a function G with* $\mathrm{IntMeas}(1; G)$*:* $|\iota(G(1))| \le 1$.

(c) *Let $\theta = (\theta_{\ell,\iota})$ be a fit family of m-fold twists, y and y' families of complex numbers indexed by the pairs (ℓ, ι), and $h_0 \le h_1$ real numbers. If y has height at most h_0 and every non-zero $y'_{\ell,\iota}$ equals $\mathrm{N}\, Q(\theta_{\ell,\iota})^{i_0}\, y_{\ell,\iota}$ for an integer i_0, which may depend on (ℓ, ι), then y' has height at most h_1.*

Proof. (a) $|G|^2 = G\bar{G}$ is given by the points $\mathsf{u}_i \mathsf{u}_{i'}^{-1}$ and the coefficients $\mathrm{c}_i \bar{\mathrm{c}}_{i'}$. (b) $G(1)$ is the sum of the coefficients, and the trivial character has empty modulus. (c) For a fit family the prime ℓ is prime to $\mathrm{N}\, Q(\theta_{\ell,\iota})$, so the factor is a unit at ℓ: the witnesses of the height of y serve for y' at every pair (ℓ, ι), and the bound may be enlarged from h_0 to h_1 (Input 8). □

LEAN `Data.TameChar.IsIntegralMeasure.norm_sq` · `Data.TameChar.IsIntegralMeasure.norm_apply_one_le` · `Data.IsFitFamily.heightLe_of_eq_zpow_mul`

Proof of Theorems 5 and 6. Theorem 5. By the first note (Input 1) the squared norm of $\Xi(\mathrm{ld}(\mathsf{p}, \mathbf{n}; \vartheta))$ is a non-zero constant, depending only on the field and the rank, times explicit counts at the chosen and the twist places, the hook number, the Weyl index, the value $L^{\Sigma \cup \bar{\Sigma}}(e_{\mathsf{p}}, ((\omega_{\mathsf{p}} \vartheta)^{-1})^{\mathrm{bc}})$ with $\Sigma = \Sigma_D \cup Q(\vartheta)$, and $\mathrm{N}\, Q(\vartheta)^{|\mathbf{n}|-1}$; here $2e_{\mathsf{p}} = |\mathbf{n}|$ and $e_{\mathsf{p}} > 1$ by the lift rank. The scaling factors of Definition 8.2 (e), (f) do not depend on ϑ, the hook number is given by Lemma 8.7 (c), and the power of $\mathrm{N}\, Q(\vartheta)$ cancels against the normalisation of the algebraic part. What remains is a constant $A \neq 0$.

Theorem 6. By Lemma 8.8 (a) the lift is a finite sum of point kernels at the points $\mathsf{u}\tilde{\mathsf{A}}$. Its squared norm expands into a double sum over pairs of these points; the dependence on ϑ' is through the values of ϑ' at the quotients of the determinants of the points, and the coefficients are $\mathrm{M}_D\, \mathrm{N}\, Q(\vartheta)^{-(|\mathbf{n}|-1)}$ times class sums of Gram forms of the point kernels multiplied by the denominator bound. Let ℓ be a prime that does not divide $\mathrm{N}\, Q(\vartheta)$ and $j \colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism. The Gram forms are j-integral by Lemma 8.8 (b). The inner product of algebraic forms on $\mathsf{l}_{\mathbf{n}}$ is a sum over the classes of the level in which each class is weighted by the inverse of the order of its stabilizer; these orders divide M_D, so M_D times the class sum is j-integral, and this is why M_D stands in the algebraic part. Finally $\mathrm{N}\, Q(\vartheta)^{-(|\mathbf{n}|-1)}$ is a unit at ℓ. So $|j(\mathrm{c}_i)| \le 1$ for every coefficient. The points and the coefficients do not depend on ϑ' as long as its modulus is that of ϑ. □

Lemma 8.10 (ratios of L-values)**.** *Let P be parameters, D admissible data, p a pair of D, ϑ a tame character and $\pm$ a sign. Write, with $\Sigma = \Sigma_D \cup Q(\vartheta)$,*

$$\mathrm{rat}(\mathsf{p}, \vartheta, \pm) = \frac{\left| L^{\Sigma \cup \bar{\Sigma}}\left(e_{\mathsf{p}}, ((\omega_{\mathsf{p}} \vartheta^{\pm 1})^{-1})^{\mathrm{bc}}\right) \right|}{\left| L^{\Sigma_D \cup \bar{\Sigma}_D}\left(e_{\mathsf{p}}, (\omega_{\mathsf{p}}^{-1})^{\mathrm{bc}}\right) \right|}$$

for the ratio of the third note: the Euler factors are removed at the chosen places and their conjugates in both values, and at $Q(\vartheta)$ and its conjugates in the numerator.

(a) $\mathrm{rat}(\mathsf{p}, \vartheta, \pm)$ *is non-zero and real.*

(b) *The products of L-values satisfy*

$$\Pi(\Sigma_D \cup Q(\vartheta), (\omega_{\mathsf{p}} \vartheta^{\pm 1})^{-1}, e_{\mathsf{p}}) = \mathrm{N}\, Q(\vartheta)^{e_{\mathsf{p}}^2} \cdot \Pi(\Sigma_D, \omega_{\mathsf{p}}^{-1}, e_{\mathsf{p}}) \cdot \prod_{1 \le i \le e_{\mathsf{p}}} \mathrm{rat}(\mathrm{low}(\mathsf{p}, i), \vartheta, \pm)^2.$$

Proof. (a) Both partial L-values are non-zero (Input 6), and a quotient of absolute values is real. (b) Unfold the product over the union of the two disjoint sets of places and write each $|L|^2$ of the twisted character as the square of the ratio times $|L|^2$ of the untwisted one. □

LEAN `Data.Pair.ratio_ne_zero_and_real` · `Data.Pair.lValueProd_eq`

Proposition 8.11 (conjugates of the lift)**.** (a) *Let P be parameters, i_1 a natural number (a level of the tower of fields, with field E_{i_1}), $\mathbf{n}$ a finite index set, Σ_1 level places of E_{i_1} without twist places, and σ an automorphism of $\mathbb{C}$ over the quadratic field of the parameters. There are a complex number t_1, whose absolute value is the inverse square root of the period of the parameters at that level, a complex number $\mathrm{l}_1 \neq 0$ and a point u_1 such that for all L-value data $\mathfrak{l}$ on $\mathbf{n}$ with the places Σ_1 and every L-value datum $\mathfrak{l}'$ with the same places whose jet degrees are those of $\mathfrak{l}$ permuted by the action of σ on the infinite places and whose character is the σ-conjugate of $\omega_{\mathfrak{l}}$ on the points with trivial infinite component: if $|\mathbf{n}|$ is even, then for every finite point h and every coordinate, the σ-conjugate of the coordinate of $\mathrm{per}^{\sum_\tau \mathrm{dg}_{\mathfrak{l}}(\tau)}\Xi(\mathfrak{l})(\mathsf{h})$ is $\mathrm{t}_1^{\sum_\tau \mathrm{dg}_{\mathfrak{l}}(\tau)}$ times l_1 times a character value at u_1 times a multiplier ratio of σ at h times the permuted coordinate of $\Xi(\mathfrak{l}')(\mathsf{h})$.*

(b) *Let P be parameters, D admissible data, $\mathbf{n}$ a finite index set and σ an automorphism of $\mathbb{C}$ over the quadratic field of the parameters. There is a real number c such that for every pair p of D that has lift rank $\mathbf{n}$ there is a real number r_0 with $-4f(D) \le \mathrm{r}_0$, with $\mathrm{r}_0 \le 4f(D)$ if p is flat, and with*

$$\log\left|\sigma(\|\Xi^{\mathrm{sc}}(\mathsf{p},\mathbf{n};1)\|^2)\right| = \mathrm{c} + \mathrm{r}_0 + \sum_\tau \mathrm{jet}_D(\mu_{\mathsf{p}}(\tau) - e_{\mathsf{p}}),$$

where $\mathrm{jet}_D(i) = \log i! - i\log(\mathit{period}/g(D))$ is the jet size of the data.

Proof. (a) A σ-semilinear conjugation of data conjugates the jets of the kernel data place by place (Input 7); apply it term by term to the class sums that define the two lifts, the second translated by a scalar. (b) Conjugate the lift by (a): the σ-conjugate of the squared norm is a power of $|\mathrm{t}_1|^2$ times $|\mathrm{l}_1|^2$ times the squared norm of the lift for the conjugate character. Insert the formula of the first note for that norm (Input 1), apply (a) to σ and to σ conjugated by the complex conjugation, so that a pair of conjugate multipliers cancels, and collect the error r_0: it is the logarithm of the number of elements of the Weyl index set plus the logarithm of the absolute value of the partial L-value; it is at least $-4f(D)$ always, and at most $4f(D)$ when the pair is flat, where the Weyl index set has one element (second reading). Then collect the factorials of the hook number (Lemma 8.7 (c)) and the scales into the jet size. □

LEAN `LValueData.map_trivialLift` · `Data.Pair.log_norm_map_norm_sq_trivialLift`

9 The triangle of the data

Definition 9.1 (forms of the triangle)**.** (a) Let E be a CM field with a CM type and Weil data, S a hermitian space with index set $\mathbf{N}$ and l a parameter. $\mathrm{opp}_{\mathsf{S},l}$ is the linear isometry from the forms on S of the type of l to the forms of the type of l on the opposite space of S, given by the identification of the two unitary groups (Lemma 4.1).

(b) Let P be parameters, D admissible data, o a flag and i, n natural numbers. $\mathrm{TF}_D(o;i,n)$ is the space of forms on the space of the grid o at the node that the triangle places at (i,n), of the type with the doubled parameter of the grid at that node.

(c) For an m-fold twist ϑ of D and $y \in \mathrm{TF}_D(o;i,n)$, $\mathrm{tl}_{D,o,\vartheta}(i,n)(y) \in \mathrm{TF}_D(o;i,n+1)$ is the theta lift of y with the characters of the grid of index $n+i$, the joint harmonic of the two nodes, and the indicator $\mathbb{1}^{\mathrm{incl}}_{\mathcal{O}_E/\mathfrak{m}_o^{\triangle}(\vartheta)}(\cdot)$ of the standard inclusion modulo the modulus $\mathfrak{m}_o^{\triangle}(\vartheta)$ of the triangle. It is linear in y.

(d) $\mathrm{tn}_{D,o}(i)$ is the linear isometric isomorphism from $\mathrm{TF}_D(o;0,0)$ to $\mathrm{TF}_D(o;i,0)$ composed of i passages to the opposite space, as in (a).

(e) $\mathrm{TB}_D(o)$ is the subspace of $\mathrm{TF}_D(o;0,0)$ of the forms fixed by the level: the level of the chosen places if the flag is false, the principal level of the generator of the grid if it is true.

(f) For an m-fold twist ϑ and a natural number n, the *triangle order* $\mathrm{ord}_{D,o}(\vartheta,n)$ is the order of the group of invertible matrices, in the rank of the node $(0,n)$ of the grid o, over the residue ring of $\mathfrak{m}_o^{\triangle}(\vartheta)$. (The formal definition belongs to another unit and is recalled here.)

LEAN formNegEquiv · Data.TriForm · Data.triLift · Data.triNegEquiv · Data.triBase

Lemma 9.2 (the first arrows)**.** (a) *Let E be a CM field with a CM type and Weil data, S and T hermitian spaces with index sets $\mathbf{N}$ and $\mathbf{N}'$, $\mathcal{I}$ an index set and ω_i^{a} a family of characters. Suppose given an edge family for ω^{a} from the type of a parameter l on S to a type on T, in the direction false; let $i \in \mathcal{I}$ and let Ψ be a complex function on the matrices modulo a modulus $\mathfrak{m}$. Put $\omega_i^{\mathrm{b}} = (\omega_i^{\mathrm{a}})^{-1}\upsilon^{\mathrm{rank}}$. If there is an edge family for ω^{b} from the type of l on the opposite space to the same type on T, in the direction true, then there is a function Ψ' on the matrices modulo $\mathfrak{m}$ such that for all forms y on S and F on T the inner product of F with the lift of y (for ω_i^{a} and Ψ) equals the inner product of the descent of F to the opposite space (for ω_i^{b} and Ψ') with $\mathrm{opp}_{\mathsf{S},l}(y)$.*

(b) *Let P be parameters, D admissible data, o a flag and i a natural number with $i < m$. There is a complex number C such that for every m-fold twist ϑ and every $y \in \mathrm{TB}_D(o)$, writing $\mathsf{a}(y') = \mathrm{tl}_{D,o,\vartheta}(i,0)(\mathrm{tn}_{D,o}(i)(y'))$ for all y' in the ambient space $\mathrm{TF}_D(o;0,0)$: (1) every y'' in the ambient space with $\langle y'', y'\rangle = \langle \mathsf{a}(y), \mathsf{a}(y')\rangle$ for all y' in the ambient space lies in $\mathrm{TB}_D(o)$, and $y'' = 0$ only if $y = 0$; (2) for every y' in the ambient space $\mathrm{TF}_D(o;0,0)$, $\mathrm{ord}_{D,o}(\vartheta,0)\cdot\langle \mathsf{a}(y'), \mathsf{a}(y)\rangle$ is the sum of an unconditionally convergent series of C times the coset terms of the inner product formula for the character of the grid, off the places of $\mathfrak{m}_o^{\triangle}(\vartheta)$ and their conjugates, divided by the number of matrices modulo $\mathfrak{m}_o^{\triangle}(\vartheta)$ of the size of the two nodes.*

Proof. (a) The theta kernel of the opposite space is a constant multiple of the conjugate kernel, and the Petersson product of a descent with a form is an integral against the kernel (Input 4); Ψ' is the corresponding multiple of the conjugate datum. (b) The parameters of the two nodes correspond (Input 2), so the joint harmonic is an edge; the indicator of the inclusion is a sharp datum; the Euler form of the inner product formula (Input 4) gives the series and the injectivity, and the invariance under the level is seen by translating with lifts of the elements of the level. □

LEAN `IsEdgeFamily.exists_adjoint_first_arrow · Data.hasSum_inner_triLift`

Proof of Theorem 3. Conditions (1) and (2) of a triangle are Lemma 7.2 (d) and (a). The descents are theta descents whose point data are, for the first column, the adjoint data of the first arrows given by Lemma 9.2 (a) (the edge families come from the correspondence of the parameters of the grid, Input 2), and then, column by column, the next data of Definition 3.1 (c). *Condition (4)* is Lemma 9.2: part (a) is the adjointness on the images of the base, and part (b) (1) shows that the descent of the lift of a base form is the image of a base form that vanishes only with the given one. *Condition (3).* For a square of the triangle, Theorem 2 applies to the four spaces of its corners, with the family of all m-fold twists, the characters of the grid and the moduli of the triangle; its hypotheses on signs, ranks and weights are those of the grid, and the tameness is Input 2. It gives a square law over the residue ring of the modulus with constant $\varkappa/\mathrm{c}$, and Lemma 3.3, applied with the inclusion datum of the lift and the next datum of the descent, turns it into the commutation of the lift and the descent up to the scalar; the weights of the triangle, the triangle orders of Definition 9.1 (f), are the orders of the groups that appear in Lemma 3.3. When the flag is false, the family c costs one square by Lemma 7.2 (f), since $|\mathrm{c}|^2$ is a constant times a product of L-values of the twisted character; and the adjoint law follows from the Fourier form of the square law (Lemma 3.6, the character being primitive by Lemma 7.2 (e)) and from the Petersson products of descents (Input 4), with a family of grade 0 by Lemma 7.2 (c). □

Lemma 9.3 (the norm of the chain)**.** *Let P be parameters, D admissible data, ϑ an m-fold twist and y', y forms of principal level on $\mathrm{U}(\mathsf{Y}_0(D))$. Let u be the m-th member of $\mathrm{wc}(\mathrm{ord}_{D,\mathrm{false}}, \mathrm{tl}_{D,\mathrm{false},\cdot}(\cdot,\cdot)(\cdot); \vartheta, 0)$, the chain of the m weighted lifts of the triangle from $\mathsf{Y}_0(D)$ to $\mathsf{Y}_m(D)$. Then $\langle \Theta^{\mathrm{lev}}_{D,\vartheta} y', \Theta^{\mathrm{lev}}_{D,\vartheta} y\rangle$ is the index of the last level times the number of elements of the Weyl index set of the weight on $\mathsf{Y}_m(D)$ times $\langle \mathsf{u}(y'), \mathsf{u}(y)\rangle$.*

Proof. Lemma 8.5 for the standard space $\mathsf{Y}_m(D)$, with Lemma 8.4 (a); the functions attached to the algebraic forms $\Theta^{\mathrm{lev}}_{D,\vartheta} y$ are the chain of lifts. □

LEAN `Data.inner_iotaAtLevel`

Proof of Theorem 4. If $\Theta^{\mathrm{lev}}_{D,\vartheta} y = 0$ then $\langle \mathsf{u}(y), \mathsf{u}(y)\rangle = 0$ by Lemma 9.3, so the weighted chain of the triangle of Theorem 3 (flag false) vanishes at y; by Lemma 6.2 (b), with the non-zero triangle orders, $y = 0$. □

10 Data for the crude rows

The third note and the unit of the rows use the following vocabulary and three facts on standard data.

Definition 10.1 (chains of column lifts; marks)**.** Let P be parameters and D admissible data, with the index sets $\mathsf{X}_i(D)$ of the columns and the weight of the columns.

(a) For a natural number i, $\hom^X(i)$ is the hermitian space of homomorphisms between the standard definite spaces on $\mathsf{X}_i(D)$ and $\mathsf{X}_{i+1}(D)$.

(b) For a tame character ϑ and i, $\mathrm{lk}^{\mathrm{col}}(\vartheta, i)$ is the theta kernel of the lift number $i+1$, from forms on $\mathsf{X}_i(D)$ to forms on $\mathsf{X}_{i+1}(D)$ at the levels of the columns, for the character of that lift twisted by ϑ^{i+1}, with scale $\bar{\pi}_g$ per.

(c) A *translation tuple* is a tuple (x_i) of finite adelic matrices with rows $\mathsf{X}_{i+1}(D)$ and columns $\mathsf{X}_i(D)$. For such a tuple x and i, $\mathrm{td}(\mathsf{x}, i)$ is the normalised vacuum of $\hom^X(i)$ translated by the finite Heisenberg operator of x_i.

(d) For a tame character ϑ, a finitely supported complex function Ψ on translation tuples, a family $\breve{\Psi}$ of data and i: $\mathrm{cc}(\vartheta; \Psi, \breve{\Psi}; i)$ is the sum, over the translation tuples x, of $\Psi(\mathsf{x})$ times the chain of the first i lifts with the kernels $\mathrm{lk}^{\mathrm{col}}(\vartheta, \cdot)$ and the data $\breve{\Psi}(\mathsf{x}, \cdot)$.

(e) $\mathrm{B}_\vartheta(\Psi)$ is a map from the columns to algebraic forms on $\mathsf{Y}_m(D)$: restrict the map $\mathrm{cc}(\vartheta; \Psi, \mathrm{td}(\cdot, \cdot); m)$ to the columns and compose it with the restriction to $\mathsf{Y}_m(D)$.

(f) Let ℓ be a prime with a homomorphism j from $\mathbb{C}$ to the completed algebraic closure of the ℓ-adic numbers. For an infinite place σ_0 of $E(D)$, $\mathrm{ind}(\sigma_0)$ is the prime of $E(D)$ above ℓ induced by j composed with the embedding of σ_0. For a finite set Σ_0 of finite places: $\mathrm{Marked}(\Sigma_0)$ holds if Σ_0 contains the conjugate of none of its members, ℓ is served if Σ_0 is non-empty, every member of Σ_0 is $\mathrm{ind}(\tau)$ for a top τ, and $\ell < x_1$ is not split in K. Here ℓ is *served* in the sense of unit 150 (Input 7): $\chi_K(\ell) = -1$, $\ell \not\equiv \pm 1$ modulo m, and the order of the class modulo N_0 of the infinite place named by $-\ell$ is odd and at most the least natural number n with $m \le \ell^n$. Further, $\mathrm{FullMark}(\Sigma_0)$ holds if $\mathrm{Marked}(\Sigma_0)$ holds and for each top τ, Σ_0 contains $\mathrm{ind}(\tau)$ or its conjugate; $\mathrm{ev}(\Sigma_0)$ is the set of the infinite places σ_0 with $\mathrm{ind}(\sigma_0) \in \Sigma_0$; and for a finite place $\mathfrak{p}$, $\mathrm{mexp}(\mathfrak{p}) = a(D) \cdot \#\{\tau \text{ a top} : \mathrm{ind}(\tau) = \mathfrak{p}\}$.

(g) The *integral columns* are the columns whose values have gauge at most 1 at every point, a module over the ring of j-integral numbers.

(h) For i and an infinite place τ, $\mathrm{lc}(i, \tau)$ is the number of columns of the weight at τ of height less than $|\mathsf{X}_{i+1}(D)|$ and different from 1, and $\mathrm{lb}(i, \tau)$ the number of boxes in these columns.

LEAN `Data.homX` · `Data.liftKernel` · `Data.Translations` · `Data.translateDatum` · `Data.chainComb` · `Data.Bμ` · `Data.inducedPlace` · `Data.IsMarked` · `Data.IsFullMark` · `Data.evenPlaces` · `Data.markExponent` · `Data.integralColumns` · `Data.lowColumns` · `Data.lowBoxes`

Lemma 10.2 (three facts)**.** (a) *Let P be parameters, D admissible data and i a natural number. Then the weight of the columns goes up from $\mathsf{X}_i(D)$ to $\mathsf{X}_{i+1}(D)$: $|\mathsf{X}_i(D)| < |\mathsf{X}_{i+1}(D)|$, the weight has no column of height strictly between the two, and the last entry at τ is $\lfloor |\mathsf{X}_{i+1}(D)|/2 \rfloor$ minus the nominal weight of index $i+1$ plus the number of columns of height $|\mathsf{X}_i(D)|$.*

(b) *Let P be parameters and D admissible data. The ideal of $\mathcal{O}_E$ generated by $\bar{\pi}_g$ is the product of the two chosen primes that contain it.*

(c) *Let P be parameters, i_1 a level of the tower of fields with field E_{i_1}, $\mathbf{n}$, $\mathbf{n}'$ finite index sets, $\mathfrak{m}$ a modulus of E_{i_1} and Ψ a function on $\mathbf{n}' \times \mathbf{n}$ matrices over the residue ring of $\mathfrak{m}$.* (i) *Let N_1, q_1 be natural numbers and ℓ a prime with a homomorphism j. If $\mathrm{q}_1\mathrm{N}_1$ lies in the ideal of $\mathfrak{m}$, every value of Ψ is integral away from q_1, and ℓ is prime to q_1, then $\mathrm{N}_1^{2f_1|\mathbf{n}'\times\mathbf{n}|}\,\mathrm{vs}^{|\mathbf{n}'\times\mathbf{n}|}$ times the datum of Ψ on the vacuum is a standard datum at ℓ of denominator N_1 (f_1 the number of infinite places of E_{i_1}).*

(ii) *Let* M_1 *be a real number with* $|\sigma(\Psi(\mathsf{x}))| \le \mathrm{M}_1$ *for every automorphism* σ *of* $\mathbb{C}$ *and every* x. *Then* $\mathrm{vs}^{|\mathbf{n}'\times\mathbf{n}|}$ *times the datum of* Ψ *on the vacuum is an admissible combination of the finite Heisenberg translates of the integral vacuum, the sum of the absolute values of its coefficients being at most the number of matrices times* M_1 *under every automorphism.*

Proof. (a) Arithmetic on the heights of the columns of the weight. (b) Compare the two ideals through the valuations at the chosen places. (c) The datum of Ψ on the vacuum is the sum of the translates of the vacuum with coefficients the values of Ψ (Input 7); after the scaling the coefficients are ℓ-integral, which is (i), and they are bounded by M_1 under every automorphism, which is (ii). □

LEAN `Data.isLiftWeight_kappa` · `span_piGBar` · `Parameters.smul_vacuumDatum_mem_standardData` · `Parameters.isAdmissibleComb_vacuumDatum`

11 Notes on the formal text

1. *The exchange theorem.* The formal statement writes the two sides with the data modulo $\mathfrak{m}$; the integral on the right is over $[\mathrm{U}(\mathsf{V}')]$ of the theta kernel of the pair $(\mathsf{V}', \mathsf{W})$, so the roles of the two members of the pair are exchanged with respect to the first note, and Lemma 4.1 is what relates the two. The real datum $\mathfrak{v}$ on the small side is given and the datum $\mathfrak{x}$ on the large side is found. The statement does not say that $\mathrm{C} > 0$, but it forces it as soon as one tame triple $(\mathfrak{m}, \omega, \chi)$ exists, because $|\mathrm{c}|^2 = \mathrm{C} \cdot \Pi(\Sigma(\mathfrak{m}), \omega, e)$ with $\mathrm{c} \neq 0$ and a positive product; the proof gives $\mathrm{C} = \mathrm{C}_0/\mathrm{C}_1$.
2. *One square.* The number $\varkappa$ is not shown to be non-zero, as the authors' comment says. The hypothesis on the products of characters is stated for all pairs of ranks that add up to $|\mathbf{n}_1| + |\mathbf{n}_2|$.
3. *Three census entries that are not arguments.* For Lemma 4.1 the tool that reads the statements takes the arrow inside the quantification over functions F in the conclusion for a premise, and for the lift of Definition 9.1 (c) the arrow of the type "linear map" likewise. In the Lean text Lemma 4.1 has three explicit arguments (the two spaces and the hypothesis on rational points, which has a default proof), and the lift takes D, the flag, ϑ, i, n and returns a linear map; here the form to which the map is applied fills the sixth slot of the notation. In the structured proof of Lemma 9.2 (b) one step of kind "case" covers a placeholder of the Lean syntax and carries no mathematical content.
4. *Arguments that only Lean can settle.* For a number of declarations of this note the dossier lists the arguments by Lean's rules applied by hand; no compiler was run. Among the statements above they are Theorems 1, 2, 3 and 5, Lemma 3.2 (a)–(c), (e)–(g), Lemma 3.3, Lemma 3.4, Lemma 3.5, Lemma 4.3 (a), Lemma 7.2 (e), (f), Lemma 8.9 (a), (c), Lemma 8.10 (a), Lemma 9.2, Lemma 9.3, Lemma 10.2 (c) (ii), the definitions of costs, of integral measures, of the point datum and the point kernel, of the lift kernel of the columns and of the marks, and two of the small facts. The doubt concerns implicit and instance arguments: which index sets, signs, primes with their embeddings and instances a declaration takes. In particular, for the definitions of Definition 10.1 (f) that use the embedding of a prime it is

not certain from the text whether Lean keeps the primality of ℓ as a separate argument.

5. *Hypotheses with default proofs.* The sign condition and the evenness in Proposition 4.5, and the hypothesis on rational points in Lemma 4.1, are arguments with a default tactic; they are hypotheses all the same and are stated as such.

6. *Statements that begin with a local definition.* Theorems 1 and 2 and the small fact on the index of the last level begin, after their hypotheses, with local definitions (of ω°, of the two composites, of the weights); the dossier restores them, and they were read in the Lean text for Theorems 1 and 2.

7. *Small facts.* The one-step results are listed after Definition 8.3 without a statement of their own; the build of this note records them as absorbed. Their formal statements are in the concordance.

8. *The second reading.* A second reader compared the statements of the unit with the Lean text. Her points on this note are followed here: the L-function in Theorem 1 is that of the base change, without the Euler factors at the places of the modulus and at their conjugates, while the norms run over the places of the modulus only; in the proof of Theorem 1 the ring homomorphism goes from the integers of the local product onto the residue ring of the modulus, and functions are pulled back along it (the dossier had the opposite direction); in Lemma 9.2 (b) the second form ranges over the ambient space, not over the base; and in Proposition 8.11 (b) the error is the logarithm of a Weyl count plus that of an L-value.

9. *What is taken for granted.* Everything listed as Inputs; in particular the results of the first note, the tower property, Landherr's theorem, the inner product formula in its Euler form and the height bounds for fit families of Chapter VIII. For the inner product formula, the matrix Gauss sum and the edge families the second reader found the relation to the cited literature unresolved, and the Inputs say so; none of them is called a special case of a published theorem here. Proposition 8.11 is used by the unit of the rows and not in this note.

10. *Families over all primes.* In the formal text a fit family is one map from the pairs (ℓ, ι) to the m-fold twists, and a height is a condition on a family indexed by all these pairs: its bound adds the contributions of the primes. An earlier edition of this note quantified Definition 7.1 over one prime at a time, which is not what the formal text says and is not well formed.

11. *Twists, not tame characters, in the triangle.* The triangle orders, the characters of the grid and the moduli of the triangle are defined in the formal text for m-fold twists only, and Theorem 3 is a statement about families indexed by them. Theorems 5 and 6 and the definitions of §8 are stated for all tame characters, as in the formal text.

How this note was written. From the dossier of the unit (its second part). The program check of the whole dossier covered 879 leaf steps of 122 proofs and found every statement at its lines. I read all statements of this part in the dossier and the summaries of all its structured proofs, and step by step the proof of Theorem 1. In the Lean text I read, for the first edition, the statements of Theorems 1, 2 and 3,

of Lemma 4.1, of Lemma 9.2 (b), and the definitions of the square law, of triangles and weighted chains, of the forms of the triangle and of tameness for a modulus; and for this revision, after a referee's report found errors in statements that had been written from the dossier alone, the statements of Theorems 5 and 6, of Lemma 9.2 (a), Lemma 7.2, Lemma 8.9 (c), Lemma 3.5 and of Propositions 4.4, 4.5 and 4.6, the proofs of Proposition 4.4 (its constants), of Proposition 4.6 (its spaces) and of Theorem 6 (its integrality), and the definitions of costs, of heights and fit families, of pairs, of the algebraic part, of integral measures, of the triangle orders and of marks. The revision corrected: Definition 7.1 and Lemma 8.9 (c) (families over all pairs (ℓ, ι)), Theorem 3 with Definition 9.1 and Lemma 9.2 (m-fold twists), Proposition 4.4 (a family $i(\tau)$, and the constant), the definition of a pair, and the definition of the triangle orders, which was missing. The other statements are written from the dossier's readings and were not compared with the Lean text by me; the proofs are outlines that name every input, shorter than the formal ones, and those of Theorem 2, Theorem 3, Proposition 4.4, Proposition 4.5, Proposition 4.6 and Proposition 8.11 follow the summaries of structured proofs whose long automation steps the readers mark as coarse. In Proposition 8.11 (b) the source of the two-sided bound for the logarithm of the L-value is taken from the dossier's summary and was not traced. Nothing was compiled.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Proposition 8.11	`LValueData.map_trivialLift`	AlgebraicPartConjugates.lean	13–47
Proposition 8.11	`Data.Pair.log_norm_map_norm_sq_trivialLift`	AlgebraicPartConjugates.lean	52–81
Definition 8.3	`Data.card_Y_le`	AlgebraicPartDefs.lean	20–21
Lemma 8.4	`Data.isHCParam_Y`	AlgebraicPartDefs.lean	25–38
Definition 8.3	`Data.card_V`	AlgebraicPartDefs.lean	40–40
Definition 8.3	`Data.card_filter_basisNumber`	AlgebraicPartDefs.lean	42–44
Definition 8.3	`Data.varsigma_strictAnti`	AlgebraicPartDefs.lean	46–47
Definition 8.3	`Data.paramW_lt_iff`	AlgebraicPartDefs.lean	49–50
Definition 8.3	`Data.card_neg_signsW`	AlgebraicPartDefs.lean	53–54
Lemma 8.4	`Data.fitsParams_auxPair`	AlgebraicPartDefs.lean	58–71
Lemma 8.5	`LevelPlaces.LevelForm.inner_eq_petersson`	AlgebraicPartDefs.lean	76–82
Definition 8.1	`chosenIndexEmb`	AlgebraicPartDefs.lean	86–86
Definition 8.1	`LValueData`	AlgebraicPartDefs.lean	91–98
Definition 8.1	`LValueData.auxPair`	AlgebraicPartDefs.lean	101–103
Definition 8.1	`LValueData.kernelValue`	AlgebraicPartDefs.lean	110–111
Definition 8.3	`LValueData.isLiftWeight`	AlgebraicPartDefs.lean	117–118
Definition 8.1	`LValueData.a_hasWeight`	AlgebraicPartDefs.lean	120–121
Definition 8.1	`LValueData.b_hasWeight`	AlgebraicPartDefs.lean	123–124
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Definition 8.3	`LValueData.conjWeylRep_eq_one`	AlgebraicPartDefs.lean	132–133
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Definition 8.1	`LValueData.chosenLevel`	AlgebraicPartDefs.lean	146–146
Definition 8.1	`LValueData.trivialLift`	AlgebraicPartDefs.lean	150–150
Lemma 8.6	`LValueData.kernelForm_invariant`	AlgebraicPartDefs.lean	155–159
Lemma 8.6	`LValueData.trivialLift_eq_stdClassSum_translate`	AlgebraicPartDefs.lean	161–164
Lemma 8.7	`LValueData.toPoly_eq_rename_harmonicPoly`	AlgebraicPartDefs.lean	170–175
Lemma 8.7	`LValueData.harmonicPoly_eq_schurCoeffPoly`	AlgebraicPartDefs.lean	178–183
Lemma 8.6	`LValueData.theta_pairWeil_fockDatum`	AlgebraicPartDefs.lean	185–190
Lemma 8.6	`LValueData.kernelForm_of_jetDeg_eq_zero`	AlgebraicPartDefs.lean	195–202

In this note	Lean declaration	File	Lines
Definition 8.3	`LValueData.theta_eq_inner_kernelForm`	AlgebraicPartDefs.lean	204–206
Definition 8.3	`LValueData.inner_trivialLift`	AlgebraicPartDefs.lean	211–213
Lemma 8.7	`LValueData.hookNumber_eq_prod_factorial`	AlgebraicPartDefs.lean	218–222
Definition 8.2	`Data.Pair.HasLiftRank`	AlgebraicPartDefs.lean	227–227
Definition 8.2	`Data.Pair.IsFlat`	AlgebraicPartDefs.lean	231–231
Definition 8.2	`Data.Pair.denomBound`	AlgebraicPartDefs.lean	235–236
Definition 8.2	`Data.Pair.lValueData`	AlgebraicPartDefs.lean	242–244
Definition 8.2	`Data.Pair.trivialLift`	AlgebraicPartDefs.lean	249–250
Definition 8.2	`Data.Pair.intAlgPart`	AlgebraicPartDefs.lean	254–255
Definition 8.3	`Data.stdLevelIndex_lastResLevel_eq`	ChainNorms.lean	19–37
Lemma 9.3	`Data.inner_iotaAtLevel`	ChainNorms.lean	42–47
Theorem 4	`Data.iotaAtLevel_injective`	ChainNorms.lean	53–55
Definition 7.1	`Data.CostsSquares`	CostOfSquares.lean	17–18
Lemma 7.2	`Data.CostsSquares.mul`	CostOfSquares.lean	22–24
Lemma 7.2	`Data.card_m`	CostOfSquares.lean	28–29
Lemma 7.2	`Data.CostsSquares.of_norm_sq_eq_mul_card_zpow`	CostOfSquares.lean	33–36
Lemma 7.2	`Data.CostsSquares.one`	CostOfSquares.lean	38–38
Lemma 7.2	`Data.isPrimitiveMulChar_of_isTameFor`	CostOfSquares.lean	42–49
Lemma 7.2	`Data.exists_isTameFor_costsSquares_one`	CostOfSquares.lean	54–63
Lemma 10.2	`Data.isLiftWeight_kappa`	CrudeModularData.lean	28–31
Definition 10.1	`Data.homX`	CrudeModularData.lean	35–35
Definition 10.1	`Data.liftKernel`	CrudeModularData.lean	40–44
Definition 10.1	`Data.Translations`	CrudeModularData.lean	48–48
Definition 10.1	`Data.translateDatum`	CrudeModularData.lean	52–53
Definition 10.1	`Data.chainComb`	CrudeModularData.lean	57–58
Definition 10.1	`Data.Bμ`	CrudeModularData.lean	62–62
Lemma 10.2	`span_piGBar`	CrudeModularData.lean	66–70
Definition 10.1	`Data.inducedPlace`	CrudeModularData.lean	73–73
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Three statements as they are written in the artifact.

```
lemma exchange_theorem σ (hs : negCount s = negCount t + negCount t') {e} (he : card a =
  card b + 2 * e) (hn : 2 * negCount s + 2 ≤ card n) {μ : _ → ℤ} (hμ : ∀ v,
  card n ≤ 2 * μ v + 2 * negCount s) {v} (hv : IsRapidDatum v) : ∃ C x, IsRapidDatum x ∧ ∀ m v x,
  IsTameFor m v μ x → let γ := rankTwist v⁻¹ n; ∃ c ≠ 0, ‖c‖ ^ 2 = C * lValueProd m.places v e
    ∧ ∀ Φ Ψ, IsPartner τ σ x Φ Ψ → ∀ g, thetaIntegral W V γ 1 (modDatum W V m Φ x) g =
    c * ∫ h, pairTheta V' W γ γ (modDatum _ W m Ψ v) (h, g) ∂V'.quotMeasure
```

```
structure SquareLaw σ c (L : Matrix a n R → ℂ) (L' : Matrix n b R → ℂ) where
  left_mul (k : GL a R) x : L (k.1 * x) = x k.1.det * L x
  right_mul (k : GL b R) y : L' (y * k.1) = x k.1.det * L' y
  partners Φ Ψ : IsPartner ⊤ σ x Φ Ψ → ∑ y, Ψ y * L' y = c * ∑ x, Φ x * L x

theorem exists_isTriangle : ∃ d, IsTriangle D.m (D.triangleOrder z) D.CostsSquares (D.triLift z) d
    (fun i ↦ (D.triNegEquiv z i).toLinearIsometry.comp (D.triBase z).subtypeᵢ) (z = false)
```

Legend: `W`, `V`, `V'` are W, V, V'; `σ` is e; `v` is ω; `γ` is ω°; `m` is $\mathfrak{m}$; `modDatum` is the datum $\mathfrak{x} \otimes [\check{\Psi}]_{\mathfrak{m}}$; `lValueProd` is $\Pi(\Sigma, \omega, e)$; `IsTameFor` is tameness for a modulus; `z` is the flag o; `triangleOrder` are the weights $\mathrm{ord}_{D,o}$ (Definition 9.1 (f)); the formal text writes `Φ`, `Ψ` for the partners $\check{\Psi}$, Ψ.

Notation

E, $\mathcal{O}_E$, $\mathbb{A}$, v; $E(D)$, K a CM field, its integers, its adeles, the splitting character of Weil data; the field of the data and its quadratic subfield

P, D, m, r, $t(D)$, $a(D)$, $g(D)$, $f(D)$, d, x, x_1 parameters, admissible data, the number of lifts, the rank of the rows, the step of the ranks, the order of derivatives, the second level, the number of infinite places, the prime of the parameters, the first two scales

ϑ, ϑ', $Q(\vartheta)$, $\mathrm{N}\,Q(\vartheta)$; θ, $\theta_{\ell,\iota}$ tame characters and twists of D, their set of places and its norm; a family of twists indexed by the pairs (ℓ, ι)

τ; σ; ℓ, ι, j, $\mathbb{C}_\ell$; π_g an infinite place; an automorphism of $\mathbb{C}$; a prime, an isomorphism of $\mathbb{C}$ with an algebraic closure of the ℓ-adic numbers, a ring homomorphism from $\mathbb{C}$ to the completion $\mathbb{C}_\ell$ of that closure; the integer of norm $g(D)$

χ_K, N_0, b; f_1; $y_{\ell,\iota}$ the quadratic character of K, the first parameter, the third scale (handout); the number of infinite places of E_{i_1}; a family of complex numbers

$\mathbf{a}, \mathbf{b}, \mathbf{n}, \mathbf{n}', \mathbf{n}_1, \mathbf{n}_2, \mathbf{N}, \mathbf{N}', \mathbf{T}, \mathbf{p}, \mathbf{r}, \mathbf{r}', \mathbf{m}$; $|\mathbf{n}|$ finite index sets; the number of elements

R; χ, e_{R} a finite commutative ring; a multiplicative character, an additive character

e, e'; $\mathrm{col}_{\mathsf{e}}(\mathsf{k})$, $\mathrm{row}_{\mathsf{e}}(\mathsf{k})$ identifications of index sets; the columns of k indexed by $\mathbf{b}$, the rows of k^{-1} indexed by $\mathbf{a}$

$\check{\Psi}$, Ψ, Ψ'; $\check{\Psi}^{\mathrm{st}}_{\mathsf{e},\chi,\mathsf{z}}$, $\Psi^{\mathrm{st}}_{\chi,\mathsf{z}}$ functions on the large and on the small side, partners (the formal text writes Φ, Ψ); the standard data

$\mathcal{L}$, $\mathcal{L}'$; G; c, c', C, C_0, C_1, $\varkappa$, A, B_1 functions on matrices; a function; constants

$\mathrm{SqLaw}(\chi, \mathsf{e}; \mathrm{c}; \mathcal{L}, \mathcal{L}')$ the square law (Definition 3.1)

$\mathbb{1}^{\mathrm{incl}}_{\mathsf{R}}(\mathsf{x})$, $\mathrm{next}(G)(\mathsf{y})$, bl_{e}, $\mathrm{gr}_{\chi,\mathsf{e}}(\mathsf{x}, \mathsf{y})$ the inclusion datum, the next datum, the block embedding, the graph kernel (Definition 3.1)

$\mathsf{g}, \mathsf{h}, \mathsf{k}, \mathsf{x}, \mathsf{y}, \mathsf{z}, \mathsf{f}, \mathsf{q}, \mathsf{c}, \mathsf{j}, \mathsf{u}, \mathsf{v}, \mathsf{l}, \mathsf{s}, \mathsf{w}, \mathsf{x}_0, \mathsf{A}$ matrices, points of groups, vectors and linear maps: x_0 the inclusion matrix, l a linear functional, w a vector of a weight, u a class or a composite map

$\mathsf{W}, \mathsf{W}', \mathsf{W}^+, \mathsf{V}, \mathsf{V}', \mathsf{X}, \mathsf{X}^+, \mathsf{X}_1, \mathsf{X}_1^+, \mathsf{U}, \mathsf{U}_1, \mathsf{U}_2, \mathsf{A}, \mathsf{A}', \mathsf{S}, \mathsf{T}, \mathsf{J}; \mathsf{I}_{\mathbf{n}}$ hermitian spaces ($^+$: with one more hyperbolic plane); the standard definite space

$\mathrm{neg}(\mathsf{W})$, x_{W}, $\mathrm{U}(\mathsf{W})$, $[\mathrm{U}(\mathsf{W})]$, $d\mathsf{h}$ the number of negative signs, the adelic frame, the unitary group, its adelic quotient, a Haar measure

ω, ω', ω°, ω_i, ω_i^{a}, ω_i^{b}; μ characters of $\mathrm{U}(1)(\mathbb{A})$; weights

$\mathfrak{m}$, $\mathfrak{m}_i$, $\Sigma(\mathfrak{m})$; $\mathfrak{m}_D(\vartheta, e)$, $\mathfrak{m}_o^{\triangle}(\vartheta)$; Σ_D, Σ_0, Σ_1 moduli and their places; the moduli of the data and of the triangle; the chosen places, sets of places

$\mathfrak{c}$, $\omega_{\mathfrak{c}}$, $\chi_{\mathfrak{c}}$, $\Sigma(\mathfrak{c})$, $\mathsf{F}_{\mathfrak{c}}$ a split tame character with its character, local character, places and local product

$\mathfrak{v}, \mathfrak{x}, \mathfrak{y}, \mathfrak{z}$; f; $\mathfrak{x} \otimes [\check{\Psi}]_{\mathfrak{m}}$ real data; a datum; the datum with real part $\mathfrak{x}$ and finite part $\check{\Psi}$ modulo $\mathfrak{m}$

$\theta_{\mathsf{W},\mathsf{V}}$, $\mathrm{I}_{\mathsf{W},\mathsf{V}}(\omega, \omega'; f)$, $\theta^{\mathrm{lg}}_{\mathsf{W},\mathsf{V}}(\mathfrak{c}; \mathfrak{v}, \check{\Psi})$, $\theta^{\mathrm{sm}}_{\mathsf{W},\mathsf{V}}(\mathfrak{c}; \mathfrak{v}, \Psi)$ the theta kernel, the theta integral, the theta integrals on the large and small side

$\Pi(\Sigma, \omega, e)$, $\Pi(\mathfrak{c}, e)$; e, e_1, e_2, e' products of partial Hecke L-values; natural numbers or integers

$L^{\Sigma\cup\bar{\Sigma}}(s, \omega^{\mathrm{bc}})$; ω^{bc}; $\bar{\Sigma}$; s the Hecke L-function of the base change without the Euler factors at Σ and at the conjugate places; the base change $\mathsf{x} \mapsto \omega(\mathsf{x}/\bar{\mathsf{x}})$; the set of the conjugates of the places of Σ; a real number ≥ 1

$\varrho(\mathsf{V}; \mathfrak{c}, \mathfrak{v})$, $\mathrm{Exch}(\mathsf{X}, \mathsf{V}, \mathsf{V}'; \mu, \mathsf{e}; \mathfrak{v}, \mathfrak{y}; \mathcal{N})$ the equal-rank ratio and the standard exchange identity of the first note

$\mathfrak{S}$, $\mathfrak{S}'$, $\mathfrak{S}''$; $\mathrm{sl}_{\mathfrak{S}}(0; \mathfrak{y})$; f; $\mathrm{c}_{\mathrm{tow}}$, $\mathrm{C}_{\mathrm{tow}}$, c_{G}; $\mathrm{vac}_{\mathfrak{S}}(\mathsf{U})$, u splittings; the real slice at 0; the descent map on real data; the complex and the real constant of the tower property, the constant of the Gauss sum; the vacuum factor of the mixed model, a complex number of absolute value 1

$\mathcal{I}$; t, t_1, t_2, t', d; $\mathsf{u}_i(\mathsf{x})$, $\mathsf{v}_i(\mathsf{y})$ an index set; types of forms and a direction; the two composites of one square

F, F', y, y', y''; $\langle y, y' \rangle$, $\langle y, y' \rangle_{\mathrm{Pet}}$; l forms and elements of inner product spaces; inner products; a parameter of a type

$\mathcal{F}_{i,n}$, $\mathcal{V}$, $\mathcal{V}'$; w, $\mathcal{M}$, a, d, j, G; b, k; S, T the spaces, weights, grading, arrows, isometries and condition of a triangle; an index; families of constants; endomorphisms

$\mathrm{Tri}(m; \mathrm{w}, \mathcal{M}; \mathsf{a}, \mathsf{d}; \mathsf{j}; \mathrm{G})$, $\mathrm{wc}(\mathrm{w}, \mathsf{a}; \vartheta, i)$ triangles and weighted chains (Definition 6.1)

$\mathrm{Cost}_D(n; \mathrm{c})$; h_0, h_1; i, i', i_0, i_1, n, n', n'' costs (Definition 7.1); height bounds; natural numbers and integers

$\mathfrak{l}$, $\mathfrak{l}'$, $\mathrm{dg}_{\mathfrak{l}}$, $\omega_{\mathfrak{l}}$; $\mathrm{LD}(E, \mathbf{n})$, $\mathrm{aux}(\mathfrak{l})$, $\mathrm{kv}_{\mathfrak{l}}(\mathrm{c}; f)(\mathsf{g}, \mathsf{h})$, $\mathrm{lk}(\mathfrak{l})$, $\mathrm{kf}_{\mathfrak{l}}(\mathsf{g})$, $\mathrm{chl}(\mathfrak{l})$, $\Xi(\mathfrak{l})$ L-value data and the objects of Definition 8.1

p, ω_{p}, e_{p}, μ_{p}; $\mathrm{LiftRank}(\mathsf{p}, \mathbf{n})$, $\mathrm{Flat}(\mathsf{p}, \mathbf{n})$, $\mathrm{den}(\mathsf{p})$, $\mathrm{ld}(\mathsf{p}, \mathbf{n}; \vartheta)$, $\Xi^{\mathrm{sc}}(\mathsf{p}, \mathbf{n}; \vartheta)$, $\mathrm{alg}(\mathsf{p}, \mathbf{n}; \vartheta)$, $\mathrm{pd}(\mathsf{p}, \mathbf{n}; \vartheta)$, $\mathrm{pk}(\mathsf{p}, \mathbf{n}; \vartheta)$, $\mathrm{low}(\mathsf{p}, i)$ a pair of the data and the objects of Definition 8.2

vs, per, M_D, $\mathrm{ord}_{D,o}(\vartheta, n)$, jet_D, $\mathrm{rat}(\mathsf{p}, \vartheta, \pm)$ the vacuum scale, the jet scale, the order of $\prod_{\mathfrak{p} \in \Sigma_D} \mathrm{GL}_{r \cdot t(D)}(\mathcal{O}_E/\mathfrak{p})$, the triangle order (the order of the invertible matrices of the rank of the node $(0, n)$ of the grid o over the residue ring of $\mathfrak{m}_o^{\triangle}(\vartheta)$, Definition 9.1), the jet size, the ratio of L-values

$\mathrm{IntMeas}(\vartheta; G)$; u_i, c_i integral measures (Definition 8.3); their points and coefficients

o; $\mathrm{opp}_{\mathsf{S},l}$, $\mathrm{TF}_D(o; i, n)$, $\mathrm{tl}_{D,o,\vartheta}(i, n)(y)$, $\mathrm{tn}_{D,o}(i)$, $\mathrm{TB}_D(o)$ the flag of a grid; the objects of Definition 9.1

$\mathsf{Y}_i(D)$, $\mathsf{W}_i(D)$, $\Theta^{\mathrm{lev}}_{D,\vartheta}$; $\mathsf{X}_i(D)$ the spaces of the chain, the composite of the m lifts at principal level; the index sets of the columns

$\hom^{X}(i)$, $\mathrm{lk}^{\mathrm{col}}(\vartheta, i)$, $\mathrm{td}(\mathsf{x}, i)$, $\mathrm{cc}(\vartheta; \Psi, \check{\Psi}; i)$, $\mathrm{B}_{\vartheta}(\Psi)$ chains of column lifts (Definition 10.1)

$\mathrm{ind}(\sigma_0)$, $\mathrm{Marked}(\Sigma_0)$, $\mathrm{FullMark}(\Sigma_0)$, $\mathrm{ev}(\Sigma_0)$, $\mathrm{mexp}(\mathfrak{p})$, $\mathrm{lc}(i, \tau)$, $\mathrm{lb}(i, \tau)$; σ_0, $\mathfrak{p}$ marks and counts of columns (Definition 10.1); an infinite place, a finite place

$\mathcal{K}_0$; d_0; t_1, l_1, u_1, r_0; M_1, N_1, q_1; $\mathcal{N}$ a kernel; a function on matrices; constants of Proposition 8.11; numbers of Lemma 10.2; a function on split tame characters

READBACK NOTE 18

The grid of lifts and descents, and the crude rows

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. Theorem C of the handout completes a family of rows to a basis by *crude rows*: rows whose twisted chains of lifts are integral after multiplication by a constant that costs few squares, and whose untwisted lifts have bounded conjugates. This note proves that such rows span (Theorem 1). The pairing of a chain of lifts with a test form is written as a period over a grid of unitary groups; the exchange theorem, in the form of the square law, moves the period across the grid, each square at the cost of one square of L-values (Theorem 2); at the far end the period is a pairing of two chains of lifts between definite groups, which is integral and small for elementary reasons (Theorem 4); and at the start the periods see every row (Theorem 3).

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, attributed by its authors to L. Alpöge with the AI system Claude, whose final theorems are the Riemann hypothesis and its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles and that Lean's kernel accepted the proofs; no mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note is the third of three on one unit of the artifact, the unit of the exchange identity, which belongs to the proof of Theorem C of the handout. It treats 125 of the unit's 305 declarations; the notes *Partner data, vanishing on standard data, and the exchange identity at the top of the tower* (the first note) and *The exchange theorem, one square of lifts, and the triangle of lifts and descents* (the second note) treat the others. The proofs are written from the structured proofs of the unit's dossier, which a program compared with the Lean text; what they take from other units and from the two other notes is listed as Inputs in §2 and is not proved here. The statements of Theorems 1 to 4 and the definitions of routes, of grid periods and of the maps along a path were compared with their Lean statements by the writer; §9 says what was not. A second reader compared all statements of the unit and a passage of every structured proof with the Lean text after the dossier was made; where her reading differs from the dossier's, this note follows hers (§9). The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 125 declarations.

1 The statements

Let P be parameters and D admissible data, with scales x, x_1, b and number of lifts m. The forms of *principal level* on the first space $\mathsf{Y}_0(D)$ of the chain of lifts are the forms of the type of the rows fixed by the principal level at the chosen places; they contain the rows of the overview. For an m-fold twist ϑ, $\Theta^{\mathrm{lev}}_{D,\vartheta}$ is the composite of the m theta lifts from $\mathsf{Y}_0(D)$ to the definite space $\mathsf{Y}_m(D)$ on these forms, with values in algebraic forms, and $\Theta^{\circ}_{D,1}$ the chain for the trivial twist. The statement $\mathrm{Cost}_D(n;\mathrm{c})$ ("c costs n squares") is defined in the second note: for every fit family $\theta = (\theta_{\ell,\iota})$ of

m-fold twists, indexed by all pairs of a prime ℓ and an isomorphism ι of $\mathbb{C}$ with an algebraic closure of the ℓ-adic numbers, the family $(\ell, \iota) \mapsto |\mathrm{c}(\theta_{\ell,\iota})/\mathrm{c}(1)|^{-2}$ has height at most $n \cdot 2\,t(D)\,x^{999}$. For a prime ℓ, $\mathbb{C}_\ell$ is the completion of an algebraic closure of the ℓ-adic numbers; a ring homomorphism $j\colon \mathbb{C} \to \mathbb{C}_\ell$ is what the formal text calls an ℓ-adic place, and a number, a vector or a form is *j-integral* if its image under j has absolute value, respectively gauge, at most 1 (for a form: at every point).

Theorem 1 (crude rows). *Let P be parameters and D admissible data for P. There is a function* c *from the m-fold twists of D to the non-zero complex numbers such that* $\mathrm{Cost}_D(2m^2; \mathrm{c})$ *holds and such that the space of all forms of principal level on* $\Upsilon_0(D)$ *is spanned by the forms y with the following two properties:*

(1) *for every m-fold twist ϑ, every prime ℓ prime to* $\mathrm{N}\,Q(\vartheta)$ *and every ring homomorphism $j\colon \mathbb{C} \to \mathbb{C}_\ell$ (every ℓ-adic place): the form* $(\mathrm{c}(\vartheta)/\mathrm{c}(1)) \cdot \Theta^{\mathrm{lev}}_{D,\vartheta}(y)$ *is j-integral;*

(2) *for every automorphism σ of $\mathbb{C}$ and every point* g*: the squared Weyl norm (§2) of the vector* $\sigma(\Theta^{\circ}_{D,1}(y)(\mathsf{g}))$ *is at most* $\exp(x_1^2\,b^2)$.

LEAN `Data.crude_rows`

Theorem 2 (the climb). *Let P be parameters, D admissible data and $i \le m$ a natural number. Then* $\mathrm{Route}_D(0, i; 2i \cdot i)$ *holds: there is a function* d *from the m-fold twists to the non-zero complex numbers with* $\mathrm{Cost}_D(2i^2; \mathrm{d})$ *such that for every n and every family $\mathcal{L}$ of data with $\mathcal{L}_\vartheta$ tidy of order n on the path of stage 0 for all ϑ, there is a family $\mathcal{L}'$ with $\mathcal{L}'_\vartheta$ tidy of order $n + 2i^2$ on the path of stage i for all ϑ, with this property: for every complex function T of the triples (ϑ, F, F'), if for every real point* h *the stage-i periods of $\mathcal{L}'$ at* h *are a constant multiple of T, then for every real point* h *the stage-0 periods of $\mathcal{L}$ at* h *are a constant multiple of $T/\mathrm{d}(\vartheta)$.*

Theorem 2′ (one step). *Let P be parameters, D admissible data and $i < m$ a natural number. Then* $\mathrm{Route}_D(i, i+1; 4i+2)$ *holds.*

LEAN `Data.gridPeriod_climb` · `Data.hasRoute_succ`

Theorem 3 (the period at the start sees every row). *Let P be parameters and D admissible data. There are a family $\mathcal{L}$ of data with $\mathrm{Tidy}(4m, \mathrm{path}(0); \mathcal{L}_\vartheta)$ for every m-fold twist ϑ, a real point h, a complex number a, and a linear map r from the forms of principal level π_{gr} on the large definite space $\mathsf{Z}_{0,m}(D)$ to the forms of principal level on $\mathsf{Y}_0(D)$, surjective and independent of ϑ, such that for every m-fold twist ϑ, every form F on $\mathsf{Y}_m(D)$ and every form F' in the domain of r*

$$\mathrm{i}_D(\vartheta) \int_{[\mathrm{U}(\mathsf{Y}_m(D))]} F(\mathsf{x}) \cdot \mathrm{ch}_\vartheta(\mathsf{r}(F'))(\mathsf{x})\, d\mathsf{x} = \mathrm{a} \cdot \mathrm{per}(\mathcal{L}; (\vartheta, F, F'); 0, \mathsf{h}),$$

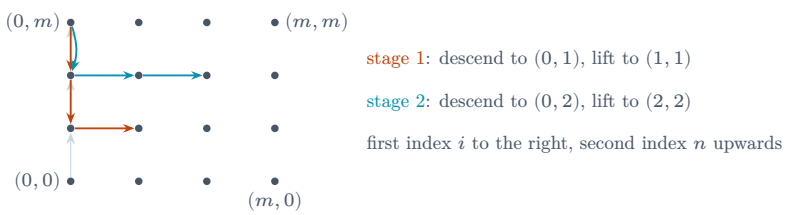
where ch_ϑ is the chain of the m lifts for ϑ, with values in functions on $\mathrm{U}(\mathsf{Y}_m(D))(\mathbb{A})$, and $\mathrm{i}_D(\vartheta)$ is the index of the last level of the chain: the index, in the stabilizer of the integral lattice of $\mathsf{Y}_m(D)$, of the principal level of the modulus $\mathfrak{m}_D(\vartheta, 1)$ (the product of the chosen places and of the places of $Q(\vartheta)$), that is, $\mathrm{i}_D(\vartheta) = \#\,\mathrm{GL}_{\mathrm{GI}_D(\mathrm{false};0,m)}(\mathcal{O}_E/\mathfrak{m}_D(\vartheta, 1)) = \mathrm{ord}_{D,\mathrm{false}}(\vartheta, m)$. It is not the number $\mathrm{i}_D^{\mathrm{gr}}(\vartheta)$ that enters the period (Definition 4.2).

LEAN `Data.exists_isTidy_integral_eq_gridPeriod`

Theorem 4 (the period at the end is integral and small). *Let P be parameters and D admissible data.*
(a) *There is a natural number $\mathrm{d}_{\mathrm{end}} \neq 0$ with $\mathrm{d}_{\mathrm{end}} \leq \exp(x^{99} b^2)$ such that for every m-fold twist ϑ, every n, every datum $\mathcal{L}_0$ with $\mathrm{Tidy}(n, \mathrm{path}(m); \mathcal{L}_0)$, every pair y of forms with $\mathrm{Fix}(\vartheta, y)$, and every prime ℓ with a homomorphism j from $\mathbb{C}$ to the completed algebraic closure of the ℓ-adic numbers such that ℓ is prime to $\mathrm{N}\, Q(\vartheta)$: if the values of the algebraic forms of the two members of y are j-integral, then $\mathrm{d}_{\mathrm{end}} \cdot \mathrm{eper}(\mathcal{L}_0; y)$ is j-integral.*
(b) *Let $\mathcal{L}_0$ be a datum for the trivial twist with $\mathrm{Tidy}(4m + 2m \cdot m, \mathrm{path}(m); \mathcal{L}_0)$, let y be a pair of forms with $\mathrm{Fix}(1, y)$, σ an automorphism of $\mathbb{C}$, and B_0 a real function of the two grids such that the squared Weyl norms of the σ-conjugates of the values of the algebraic form of the member of y in the grid o are at most $B_0(o)$. Then*

$$\left|\sigma(\mathrm{eper}(\mathcal{L}_0; y))\right| \leq \exp(x^{200} b^2) \cdot \sqrt{B_0(\mathrm{false})} \cdot \sqrt{B_0(\mathrm{true})}.$$

LEAN `Data.exists_mul_endPeriod_mem_embValRing` · `Data.endPeriod_le`



One of the two grids, for $m = 3$. A form at the node $(0, m)$ is carried along the path of stage i to the diagonal node (i, i); the period pairs the results in the two grids. At stage 0 the path consists of m descents, at stage m of m lifts along the top row.

2 The setting

Conventions. The natural numbers are $0, 1, 2, \ldots$. The letters of the handout keep their meaning: P are parameters, D admissible data, with scales x, x_1, b, number of lifts m, rank of the rows r, step $t(D)$; ϑ is a tame character or a twist of D, τ an infinite place, σ an automorphism of $\mathbb{C}$, ℓ a prime, ι an isomorphism of $\mathbb{C}$ with an algebraic closure of the ℓ-adic numbers and j a ring homomorphism from $\mathbb{C}$ to the completion of that closure. As in the other notes of the unit, finite index sets are bold, sans-serif letters are rings, hermitian spaces, matrices, tuples of matrices, points of groups and linear maps, fraktur letters are places, moduli and records, and upright letters are constants and functions with constant values. A *flag* o is one of the two values false and true; it names a grid, and $[o]$ is 0 for false and 1 for true. The implicit arguments of the declarations of the unit are of the following kinds, and each statement introduces the ones it uses: parameters and admissible data with their tame characters, twists and pairs; flags and natural numbers; nodes and finite sets of edges; tuples, data and families of matrices at the nodes; forms and linear functionals; finite index sets and bijections between them; characters; a CM field with a CM type and Weil data, hermitian spaces, signs and parameters; a prime with an embedding of $\mathbb{C}$; real numbers and constants.

What is recalled. From the second note: the square law $\mathrm{SqLaw}(\chi, \mathsf{e}; \mathrm{c}; \mathcal{G}, \mathcal{G}')$ for functions $\mathcal{G}$, $\mathcal{G}'$ on matrices over a finite ring, with its partner identity; the graph kernel and the block embedding of general linear groups; costs in squares, their product rule, and the cost of a constant whose squared absolute value is a product of L-values; one square of lifts and descents; triangles, weighted chains and the triangle of the data, with its orders $\mathrm{ord}_{D,o}(\vartheta, n)$; L-value data $\mathfrak{l}$ and their auxiliary pairs; pairs p of the data with character ω_{p} and point e_{p}; standard data and admissible combinations. From other units: the index set of the rows, with r elements, numbered; the weights $w_{D,i}(\tau)$ of the lifts and the parameter of the rows; the chosen places and the integer $\bar{\pi}_g$ of the data, with a non-zero exponent $b_1(D)$; theta lifts lift and descents des between the forms at two neighbouring nodes of a grid, attached to a matrix modulo the modulus of the grid (unit 141); the level of a node and its index; the untwisted and twisted descents from the node $(0, m)$ to the node $(0, n)$ along a grid (unit 141); algebraic forms on definite spaces, their gauges and norms; theta kernels between definite spaces and chains of them (unit 134). The values of an algebraic form are coordinate vectors ξ of the representation of its weight in the basis of products of minors, and the *squared Weyl norm* of such a vector is $\xi^* \mathrm{G}\, \xi$, where G is the Gram matrix of the representation in this basis, a positive definite matrix with integer entries (background library). It is not the sum of the squares of the coordinates. The moduli $\mathfrak{m}_D(\vartheta, e)$ are those of the second note: the product of the chosen places to the power e times the places of $Q(\vartheta)$.

Inputs. The proofs below take the following for granted. Numbers are the units of the reading; references are those of the handout's catalogue, and none of their places is confirmed.

Input 1 (the second note of this unit). One square (its Theorem 2); the triangle of the data (its Theorem 3) and the injectivity of weighted chains; the square law at the inclusion and after a Fourier transform; the graph kernel reproduces the small side of a square law; costs: the product rule, the cost of a product of L-values of a pair ramified at the chosen places; tameness needs only the weights; the standard data of a function modulo a modulus are admissible combinations of translates of the integral vacuum.

Input 2 (unit 141, Seesaw; units 27 and 25 of the background library). (a) The classical *seesaw identity*: for a see-saw pair of unitary groups, the period of a lift against a form is the period of the form against the lift in the other direction (S. Kudla, Seesaw dual reductive pairs, Progr. Math. 46 (1984), 244–268, §1: the adjointness (1.9) and the seesaw identity (1.14)). (b) The *seesaw identity for periods* of the formal text, which is what the proofs use: a proportionality of finite sums of weighted periods is transferred from lifts to descents, under a spanning hypothesis. It is an inference from (a) with additional hypotheses and not a statement of the source (a weaker form). (c) The lifts and descents along the grids and their transformation under the levels; adjointness of a lift and a descent through an edge family.

Input 3 (units 134, 133, 116, 102 and 143, forms). A dual pairing of the weights of two neighbouring definite groups and the branching pairing; delta forms; short algebraic forms span the forms of a level (unit 143); bounds for forms near a cusp and in terms of the first minimum (reduction theory: I. Satake and R. Godement, Sém. H. Cartan 10 (1957/58); the formal statement is specific to this proof); the Laplace method for matrix coefficients of discrete series with steep parameters (reference to be confirmed); Poincaré series as kernels of periods (relation to the literature unresolved: R. Godement, Sém. H. Cartan 10 (1957/58), exp. 10, §4, Thm 5 and 5 bis, and G. Muić, arXiv:1610.05483, Thm 6-4 and Thm 7-6, construct such series under additional hypotheses, and the formal statement allows cases that they exclude).

Input 4 (units 150 Curve and 145 Columns). Jets of admissible combinations of translates of the integral vacuum: their integrality at a prime and the bounds for the norms of their conjugates under all automorphisms of $\mathbb{C}$, uniformly along a chain of kernels.

Input 5 (units 96 and 104). Hecke polynomials and tame characters for a modulus; the places of the moduli of the data.

3 The two grids

Definition 3.1 (index sets, signs, parameters and moduli of the grids)**.** Let P be parameters and D admissible data.

(a) For a flag o and natural numbers i, n, the index set $\mathrm{GI}_D(o;i,n)$ of the node (i,n) of the grid o consists of $r+[o]$ base indices (the index set of the rows and, if o is true, one more point), of i blocks of size $t(D)+[o]$ and of n blocks of size $t(D)-[o]$. The node carries a hermitian space with this index set: $\mathsf{V}_{i,n}(D)$ of rank $r+(i+n)\,t(D)$ in the grid false, and a space $\mathsf{Z}_{i,n}(D)$ of rank $r+1+i(t(D)+1)+n(t(D)-1)$ in the grid true. $\mathrm{num}_D(o)$ is the numbering of the base indices (the given numbering of the rows, and r for the extra point), and gbe_D the bijection that lists the indices of a node as the base indices followed by the blocks.

(b) For an offset n' and an index x of $\mathrm{GI}_D(o;i,n)$, the sign $\mathrm{sg}_D(o,n';i,n)(\mathsf{x})$ is $+1$ on the added blocks and, on the base index number i', the sign -1 or $+1$ according to the parity of an explicit function of n', n and $i'+1-k(D)$.

(c) For an offset n' and an infinite place τ, $\mathrm{par}_D(o,n';i,n)(\tau)$ is the doubled Harish-Chandra parameter at the node (i,n) for these signs: on the base indices twice the parameter of the rows, and on the added blocks arithmetic progressions of step 2 whose position is given by the weights $w_{D,i'}(\tau)$ of the lifts, shifted by n' for the blocks of the second kind and by the parity of $i+n$.

(d) The signs and the doubled parameter of the space $\mathsf{Y}_n(D)=\mathsf{V}_{0,n}(D)$ of the chain are

$$\mathrm{sgY}_D(n)(\mathsf{x})=\mathrm{sg}_D(\mathrm{false},0;0,n)(\mathsf{x}),$$
$$\mathrm{parY}_D(n)(\tau,\mathsf{x})=\mathrm{par}_D(\mathrm{false},0;0,n)(\tau)(\mathsf{x}).$$

(e) For a flag o', a natural number i', a finite index set $\mathbf{n}$ and τ: $\mathrm{gw}_D(o',i';\mathbf{n})(\tau)=\lfloor(|\mathbf{n}|+[\neg o'])/2\rfloor-w_{D,i'+1}(\tau)$, the weight at τ of the character of the grid. The *harmonic of the grid* for two nodes of a grid, an offset, two flags and an index is the product over the infinite places of the chosen joint harmonics for these weights, the parameters of the two nodes (negated or not) and their signs.

(f) $\mathrm{gde}_D(i)$ is a bijection from the index set of $\mathsf{V}_{i,i}(D)$ plus one point to the index set of the diagonal node (i,i) of the grid true that keeps the signs, the extra point having the sign $(-1)^{\min(i,m)}$; it is the identity for $i=0$.

(g) The *generator of the grid* is $\pi_{\mathrm{gr}}=\bar\pi_g^{\,b_1(D)}$. For an m-fold twist ϑ, $\mathfrak{m}^{\mathrm{gr}}(\vartheta)$ is the modulus that is the product of the chosen places to the power $b_1(D)$ times $Q(\vartheta)$; $\mathfrak{m}_o^{\triangle}(\vartheta)$ is the product of the chosen places times $Q(\vartheta)$ if o is false, and the product of the chosen places to the power $b_1(D)$ if o is true; and $\mathrm{red}_{\vartheta,o}$ is the reduction from the residue ring of $\mathfrak{m}^{\mathrm{gr}}(\vartheta)$ to that of $\mathfrak{m}_o^{\triangle}(\vartheta)$.

LEAN `Data.GridIndex` · `Data.gridNumber` · `Data.gridBlockEquiv` · `Data.gridSigns` · `Data.gridParam` · `Data.signsY` · `Data.paramY` · `Data.gridWeight` · `Data.gridHarmonic` · `Data.gridDiagEquiv` · `Data.gridLevelGen` · `Data.gridModulus` · `Data.triangleModulus` · `Data.reduceToTriangle` · two instances record that the powers of $\bar\pi_g$ and $b_1(D)$ are non-zero: `Data.instFactNeRingOfIntegersEHPowGbOfNat`, `Data.instNeZeroNatB₁`

Small facts on the grids. They are single steps or computations in the formal text, and are used inside the proofs below. The sign function of the base indices has a closed form; the index set of a node has $r + [o] + i(t(D) + [o]) + n(t(D) - [o])$ elements, at most twice the rank of $\mathsf{Y}_m(D)$ when $i, n \le m$, and its parity is even in the grid true and that of $i + n + 1$ in the grid false; with offset 0 the node (i, n) has $\lfloor (m - n + [o])/2 \rfloor$ negative signs (a count written first as a sum), none if $m \le n$; the signs of the node $(0, 0)$ of the grid true are those of $\mathsf{Y}_0(D)$ followed by $+1$, and those of the diagonal nodes correspond through $\mathrm{gde}_D(i)$; the weights of the lifts increase by at least the gap of the data from one index to the next, and by twice the gap further on; the parameter of a node has that gap, provided the offset is 0 or $i = 0$; its entries are at most $\lceil 20b \rceil$ in absolute value for $i, n \le m$; and at the node (m, m) the negated parameter of the grid true interlaces the parameter of the grid false. For the weights of $\mathsf{W}_i(D)$: three size comparisons of the data; the order of Definition 3.4 puts the positive vectors first and is compatible with the parameter. For the moduli: the places of the moduli of the data are the chosen places and those of $Q(\vartheta)$; for a lift index i $(1 \le i \le m)$, the i-th power of the basic character times a twist is tame for them; the modulus of the grid for the trivial twist is generated by π_{gr}, which is divisible by $\bar{\pi}_g$; the trivial twist has the trivial character and an empty set of places. For tuples: the product of edge characters is multiplicative in the family; the maps of Definition 4.2 depend only on the matrices on their edges; the number of matrices on an edge between nodes with indices at most m is at most the tuple bound; the end nodes have at most twice the rank of $\mathsf{Y}_m(D)$ indices, and their weights and exponents are bounded in terms of $b_1(D)\, r^4$ and $\lceil 20b \rceil$.

LEAN `Data.entrySign_eq` · `Data.card_gridIndex` · `Data.card_gridIndex_le` · `Data.card_gridIndex_mod_two` · `Data.card_neg_gridSigns` · `Data.card_neg_gridSigns_eq_sum` · `Data.gridSigns_eq_one` · `Data.gridSigns_zero` · `Data.gridDiagEquiv_spec` · `Data.w_add_gap_le` · `Data.hasGap_gridParam` · `Data.abs_gridParam_le` · `Data.gridParam_interlace` · `Data.nStar_bounds` · `Data.rank_bounds_card_bigIndex` · `Data.two_mul_card_detIndex_add_le` · `Data.isPosFirstOrder_orderW` · `Data.paramW_lt_of_orderW_lt` · `Data.places_modulus_eq_union` · `Data.isTameFor_twistedChi` · `Data.gridModulus_one_ideal` · `Data.piGBar_dvd_gridLevelGen` · `Data.twist_one_spec` · `Data.edgeCharProd_mul` · `Data.legDescents_congr` · `Data.gridPathMap_congr` · `Data.card_matrix_le_tupleBound` · `Data.boxes_endWeightData_le`

Lemma 3.2 (the steps of the grids are edges)**.** *Let P be parameters and D admissible data.*

(a) *Let* d *be a flag (a direction), o a flag, n', i, n natural numbers, and let (o', i', i_1, n_1) be $(o, i, i+1, n)$ if* d *is true and $(\neg o, n + n', i, n + 1)$ if it is false. Suppose $i_1 \le m$, $n_1 \le m$, and $n' = 0$ or $i_1 = 0$. Then the parameters of the nodes (i, n) and (i_1, n_1) of the grid o with offset n', both negated if* d *is true, correspond for the weights $\mathrm{gw}_D(o', i'; \mathrm{GI}_D(o; i, n))(\tau)$ and $\mathrm{gw}_D(o', i'; \mathrm{GI}_D(o; i_1, n_1))(\tau)$ and the signs of the two nodes: the step is an edge.*

(b) *Let o be a flag and i, n natural numbers with $i + n < m$. For the complementary weights $|\mathrm{GI}_D(o; 0, n)| - \mathrm{gw}_D(\neg o, i; \mathrm{GI}_D(o; 0, n))(\tau)$, the negated parameters of the nodes $(0, n)$ with offset $i + 1$ and $(0, n + 1)$ with offset i correspond, and so do those of $(0, n + 1)$ with offset i and $(0, n)$ with offset $i + 1$.*

Proof. (a) At each infinite place the two parameters have the gap of the data (a small fact above), and the entries of the two nodes are matched one by one by integer arithmetic. (b) follows from (a) with the offsets: the gaps carry over to the negated parameters, and the matching of the entries is checked again. □

LEAN Data.paramsCorrespond_grid · Data.paramsCorrespond_first_arrow

Definition 3.3 (twists, their powers, and ratios of L-values)**.** Let P be parameters and D admissible data.

(a) For a tame character ϑ and a natural number n, ϑ^n is the tame character with the same set $Q(\vartheta)$ and the n-th power of the tame part.

(b) $\mathrm{Tw}^m(D)$ is the set of the *m-fold twists*: the twists ϑ whose local tame parts satisfy $(\vartheta^{\mathrm{t}}_{\mathfrak{p}})^n \neq 1$ for every $\mathfrak{p} \in Q(\vartheta)$ and every $1 \leq n \leq m$. The trivial twist 1 is one, and an m-fold twist is read as a twist. For a lift index n, $\vartheta^{[n]}$ is the power ϑ^n as a twist.

(c) For a tame character ϑ and a natural number i, $\mathrm{cch}_D(\vartheta, i)$ is the character of $\mathrm{U}(1)(\mathbb{A})$ of the lift number i of a chain, the basic character of that lift times ϑ^i.

(d) For a pair $\mathfrak{p}$ of D, a tame character ϑ and a sign $\pm$,

$$\mathrm{rat}(\mathfrak{p}, \vartheta, \pm) = \frac{\left|L^{\Sigma\cup\bar{\Sigma}}\left(e_{\mathfrak{p}}, ((\omega_{\mathfrak{p}}\vartheta^{\pm 1})^{-1})^{\mathrm{bc}}\right)\right|}{\left|L^{\Sigma_D\cup\bar{\Sigma}_D}\left(e_{\mathfrak{p}}, (\omega_{\mathfrak{p}}^{-1})^{\mathrm{bc}}\right)\right|}, \qquad \Sigma = \Sigma_D \cup Q(\vartheta),$$

where, as in the two other notes, $L^{\Sigma\cup\bar{\Sigma}}(s, \omega^{\mathrm{bc}})$ is the Hecke L-function of the base change ω^{bc} of a character ω to $E(D)$ without the Euler factors at a set Σ of places and at the set $\bar{\Sigma}$ of their conjugates, and Σ_D is the set of the chosen places: the Euler factors at the chosen places and at their conjugates are removed in both values, and those at $Q(\vartheta)$ and its conjugates in the numerator.

LEAN Data.TameChar.pow · Data.MFoldTwist · Data.MFoldTwist.powTwist · Data.chainChar · Data.Pair.ratio · the set of pairs is finite, the trivial twist and the reading of an m-fold twist as a twist are instances: Data.finite_pair, Data.instOneTws, Data.instCoeTwsTwist

Definition 3.4 (weights on the spaces $\mathsf{W}_i(D)$)**.** Let P be parameters and D admissible data. $\mathrm{nst}(D) = f(D) \cdot 2\, t(D)^3$. For a lift index i and an index x of the basis of $\mathsf{W}_i(D)$, $\mathrm{ordW}(i, \mathsf{x})$ is an integer that orders the basis, the positive vectors first and in reversed order. For an infinite place τ, $\mathrm{chw}(i, \tau)$ is the lowest K-type of the Harish-Chandra parameter that is minus the parameter of $\mathsf{W}_i(D)$ at τ, and $\mathrm{kw}(i, \tau)$ is the staircase weight of the kernel of the i-th untwisted auxiliary pair on its small definite space.

LEAN Data.nStar · Data.orderW · Data.conjHolWeight · Data.kernelWeight

Lemma 3.5 (weights and auxiliary pairs)**.** (a) *Let P be parameters, D admissible data, i a lift index and τ an infinite place. Then* $\mathrm{chw}(i, \tau)$ *is the sum of the staircase weight of the big definite space of the i-th auxiliary pair and of* $\mathrm{kw}(i, \tau)$.

(b) *With the same i and τ: the entries of* $\mathrm{chw}(i, \tau)$ *are at most* $5b$ *in absolute value;* $\mathrm{chw}(i, \tau)$ *and* $\mathrm{kw}(i, \tau)$ *are dominant for the signs of* $\mathsf{W}_i(D)$ *and the order* $\mathrm{ordW}(i, \cdot)$*; and both have holomorphic gaps* $300\ \mathrm{nst}(D)$.

(c) *Let E be a CM field with a CM type,* $\mathbf{n}$ *a finite index set and* $\mathfrak{l}$ *an L-value datum for E and* $\mathbf{n}$*. Then the parameter* 0 *fits the auxiliary pair of* $\mathfrak{l}$.

(d) *Let E be a CM field with a CM type and Weil data,* $\mathfrak{a}$ *an auxiliary pair, and* $\mathfrak{p}$ *a finite place that is not its own conjugate and is off the places of* $\mathfrak{a}$ *and their conjugates. If an algebraic form y of* $\mathfrak{a}$ *lifts a family of eigenvalues, then the Hecke polynomial operator of* $\mathfrak{a}$ *at* $\mathfrak{p}$*, evaluated at the test value of* $\mathfrak{a}$ *at* $\mathfrak{p}$*, kills y.*

(e) *Let P be parameters, D admissible data, $e \neq 0$ a natural number, ϑ a twist and ω a character of $\mathrm{U}(1)(\mathbb{A})$ that has weights μ and tame part $\omega^{\mathrm{t}}\vartheta^{\mathrm{t}}$, where ω^{t} is non-trivial exactly at the chosen places. Then ω is tame for the modulus that is the product of the chosen places to the power e times $Q(\vartheta)$, with the weights μ and the character of this tame part.*

Proof. (a) Entry by entry: by the joint harmonic of the fitting auxiliary pair (second note) the lowest K-type is the staircase weight of the pair; split by the sign of the index. (b) The size from the bound for lowest K-types of a parameter with a gap; dominance and gaps by the sign of each index and sums of column lengths, with the small facts. (c) The four conditions of fitting are computed for the one-row weight of $\mathfrak{l}$. (d) The Hecke polynomial acts on y by its value at the test value, where the first factor vanishes because the rank difference is positive. (e) The four conditions of tameness: the modulus has places, the weights are given, the tame part is non-trivial exactly at its places, and the character is that of the tame part. □

LEAN `Data.conjHolWeight_eq_add_kernelWeight` · `Data.conjHolWeight_kernelWeight_bounds` · `LValueData.fitsParams_auxPair` · `AuxPair.heckePolyOnForms_testValue_eq_zero` · `Data.isTameFor_of_hasWeight`

4 Tuples, tidy data and periods

A *node* is a triple (o, i, n) of a flag and two natural numbers; it stands for the node (i, n) of the grid o, with its index set and its space of forms. An *edge* is an ordered pair of nodes.

Definition 4.1 (tuples and data)**.** Let P be parameters, D admissible data and ϑ an m-fold twist; let R_ϑ be the residue ring of $\mathfrak{m}^{\mathrm{gr}}(\vartheta)$.

(a) $\mathrm{Tup}(\vartheta)$ is the set of *tuples*: a tuple S assigns to every edge $(\mathsf{p}, \mathsf{p}')$ a matrix over R_ϑ with columns indexed by the index set of p and rows by that of p'. $\mathrm{Dat}(\vartheta)$ is the space of *data*: the linear functionals on the complex functions of tuples.

(b) For nodes p, q: $\mathrm{MuEdge}(\mathsf{p}, \mathsf{q})$ holds if p lies in the grid true exactly when the first indices of p and q differ; these are the edges whose character carries the twist. For an edge $\mathsf{b} = (\mathsf{p}, \mathsf{p}')$, $\chi^{\mathrm{ed}}_\vartheta(\mathsf{b})$ is the multiplicative character of R_ϑ made from the tame part of the character of the lift along b, twisted if b carries the twist.

(c) For a family g of invertible matrices over R_ϑ, one at each node: $\mathrm{Adm}(\mathsf{g})$ holds if for both grids the value of g at the node $(0, m)$ lies in the level of that node, and for every i the value at the diagonal node (i, i) of the grid true is the block extension by 1, through $\mathrm{gde}_D(i)$, of the value at (i, i) of the grid false. For a tuple S and an edge $\mathsf{b} = (\mathsf{p}, \mathsf{p}')$, $\mathrm{act}(\mathsf{g}, \mathsf{S})(\mathsf{b}) = \mathsf{g}(\mathsf{p}')^{-1}\,\mathsf{S}(\mathsf{b})\,\mathsf{g}(\mathsf{p})$; and for a finite set $\mathcal{E}$ of edges, $\mathrm{ecp}(\mathsf{g}, \mathcal{E}) = \prod_{\mathsf{b}=(\mathsf{p},\mathsf{p}')\in\mathcal{E}} \chi^{\mathrm{ed}}_\vartheta(\mathsf{b})(\det \mathsf{g}(\mathsf{p}')/\det \mathsf{g}(\mathsf{p}))$.

(d) $\mathrm{tb}(\vartheta) = (\#\mathsf{R}_\vartheta)^{(2\,\mathrm{rk}\,\mathsf{Y}_m(D))^2}$. For a tuple S and a finite set $\mathcal{E}$ of edges, $\mathrm{LInv}(\mathsf{S}, \mathcal{E})$ holds if $\mathsf{S}(\mathsf{b})$ has a left inverse for every edge $\mathsf{b} \in \mathcal{E}$ that carries the twist.

(e) For a natural number n, a finite set $\mathcal{E}$ of edges and a datum $\mathcal{L}_0$: $\mathrm{Short}(n, \mathcal{E}; \mathcal{L}_0)$ holds if $\mathcal{L}_0(G) = \sum_{\mathsf{x}\in\mathcal{W}} \mathrm{c}_\mathsf{x}\, G(\mathsf{S}_\mathsf{x})$ for all functions G of tuples, with a finite set $\mathcal{W}$ of at most $\mathrm{tb}(\vartheta)^n$ elements, tuples S_x and coefficients c_x, each of which is 0 or a root of unity with $\mathrm{LInv}(\mathsf{S}_\mathsf{x}, \mathcal{E})$.

(f) For a datum $\mathcal{L}_0$, a finite set $\mathcal{X}$, weights $\mathrm{k}(\mathsf{S}, \mathsf{x}) \in \mathbb{C}$ and tuples $\mathrm{t}(\mathsf{S}, \mathsf{x})$: $\mathrm{bind}(\mathcal{L}_0; \mathrm{k}, \mathrm{t})$ is the datum $G \mapsto \mathcal{L}_0(\mathsf{S} \mapsto \sum_{\mathsf{x}\in\mathcal{X}} \mathrm{k}(\mathsf{S}, \mathsf{x})\, G(\mathrm{t}(\mathsf{S}, \mathsf{x})))$.

(g) $\mathrm{Tidy}(n,\mathcal{E};\mathcal{L}_0)$ ("$\mathcal{L}_0$ is tidy of order n on $\mathcal{E}$") holds if $\mathrm{Short}(n,\mathcal{E};\mathcal{L}_0)$ holds and $\mathcal{L}_0(\mathsf{S}\mapsto G(\mathrm{act}(\mathsf{g},\mathsf{S})(\cdot))) = \mathrm{ecp}(\mathsf{g},\mathcal{E})\cdot\mathcal{L}_0(G)$ for every family g with $\mathrm{Adm}(\mathsf{g})$ and every G. $\mathrm{LTidy}(n,\mathcal{E};\mathcal{L}_0)$ holds if $\mathrm{Short}(n,\mathcal{E};\mathcal{L}_0)$ holds and the same identity holds for every family g whose values at the nodes $(0,i)$, $i\le m$, of both grids lie in the levels of these nodes.

LEAN Data.EdgeTuple · Data.GridDatum · Data.IsMuEdge · Data.edgeChar · Data.IsAdmissibleFamily · Data.tupleAction · Data.edgeCharProd · Data.tupleBound · Data.LeftInvOnMuEdges · Data.IsShortSum · Data.GridDatum.bind · Data.IsTidy · Data.IsLevelTidy

Definition 4.2 (paths, the maps along them, and periods)**.** Let P be parameters, D admissible data and ϑ an m-fold twist.

(a) For a flag o and natural numbers i, n: $\mathrm{ve}(o,i,n)$ is the edge from (o,i,n) to $(o,i,n+1)$ and $\mathrm{re}(o,i,n)$ the edge from (o,i,n) to $(o,i+1,n)$. $\mathrm{leg}(o,n)$ is the set of the edges $\mathrm{ve}(o,0,n')$ with $n\le n'<m$; $\mathrm{gp}(o,i,n)$ is the union of $\mathrm{leg}(o,n)$ and of the edges $\mathrm{re}(o,i',n)$ with $i'<i$; and $\mathrm{path}(n)=\mathrm{gp}(\mathrm{false},n,n)\cup\mathrm{gp}(\mathrm{true},n,n)$ is the *path of stage* n.

(b) For a tuple S, a flag o, a natural number n and a form F at the node $(0,m)$ of the grid o: $\mathrm{ld}_{\mathsf{S}}(o,n)(F)$ is the form at $(0,n)$ obtained from F by the descents along the edges $\mathrm{ve}(o,0,n')$, $n'=m-1,\dots,n$, each with the matrix that S puts on its edge (it is F for $n=m$ and 0 for $n>m$).

(c) For natural numbers i, n: $\mathrm{pm}_{\mathsf{S}}(o;i,n)(F)$ is the form at (i,n) obtained from $\mathrm{ld}_{\mathsf{S}}(o,n)(F)$ by the lifts along the edges $\mathrm{re}(o,i',n)$, $i'=0,\dots,i-1$, with the matrices of S. Both maps are linear in F.

(d) For an m-fold twist ϑ_1, a natural number $i\le m$ and a real point h, $\mathrm{dp}_{\vartheta_1}(i;\mathsf{h})$ is the bilinear form on the forms at the diagonal nodes (i,i) of the two grids

$$\mathrm{dp}_{\vartheta_1}(i;\mathsf{h})(y,y') = \int_{[\mathrm{U}(\mathsf{V}_{i,i}(D))]} y(\mathsf{x})\,y'\big((\mathsf{x}\oplus\mathsf{u}^{\,i})\,\mathsf{h}\big)\,d\mathsf{x},$$

where $\mathsf{x}\oplus\mathsf{u}^{\,i}$ is read in the group of the grid true through $\mathrm{gde}_D(i)$ and u is the scalar of norm one attached to $\mathfrak{m}^{\mathrm{gr}}(\vartheta_1)$. In the formal definition the second argument is the proof of $i\le m$.

(e) For a family $\mathcal{L}$ of data ($\mathcal{L}_\vartheta\in\mathrm{Dat}(\vartheta)$ for every m-fold twist), a triple (ϑ,F,F') of an m-fold twist and forms at the nodes $(0,m)$ of the grids false and true, a natural number $i\le m$ and a real point h, the *period of stage* i is

$$\begin{aligned}&\mathrm{per}(\mathcal{L};(\vartheta,F,F');i,\mathsf{h})\\ &\qquad=\mathcal{L}_\vartheta\Big(\mathsf{S}\mapsto \mathrm{dp}_\vartheta(i;\mathsf{h})\big(\mathrm{pm}_{\mathsf{S}}(\mathrm{false};i,i)(\mathrm{i}^{\mathrm{gr}}_D(\vartheta)F),\ \mathrm{pm}_{\mathsf{S}}(\mathrm{true};i,i)(F')\big)\Big),\end{aligned}$$

where $\mathrm{i}^{\mathrm{gr}}_D(\vartheta)=\mathrm{i}^{\mathrm{gr}}_{D,m}(\vartheta)$ and, for a natural number n,

$$\mathrm{i}^{\mathrm{gr}}_{D,n}(\vartheta)=\#\,\mathrm{GL}_{\mathrm{GI}_D(\mathrm{false};0,n)}\big(\mathcal{O}_E/\mathfrak{m}^{\mathrm{gr}}(\vartheta)\big)$$

is the index of the level of the node $(0,n)$ of the grid false; this level is the trivial subgroup of the invertible matrices over the residue ring of $\mathfrak{m}^{\mathrm{gr}}(\vartheta)$. The modulus $\mathfrak{m}^{\mathrm{gr}}(\vartheta)$ has the exponent $b_1(D)$ at the chosen places, so $\mathrm{i}^{\mathrm{gr}}_D(\vartheta)$ is in general not the number $\mathrm{i}_D(\vartheta)=\mathrm{ord}_{D,\mathrm{false}}(\vartheta,m)$ of Theorem 3, which belongs to the modulus $\mathfrak{m}_D(\vartheta,1)$; the two are related by Lemma 6.1.

LEAN `Data.verticalEdge` · `Data.rowEdge` · `Data.legEdges` · `Data.gridPath` · `Data.path` · `Data.legDescents` · `Data.gridPathMap` · `Data.diagonalPeriodForm` · `Data.gridPeriod`

Definition 4.3 (routes)**.** Let P be parameters, D admissible data, $i \le m$ and $i' \le m$ natural numbers and e a natural number. $\mathrm{Route}_D(i, i'; e)$ holds if there is a function d from the m-fold twists to the non-zero complex numbers with $\mathrm{Cost}_D(e; \mathrm{d})$ such that for every n and every family $\mathcal{L}$ of data with $\mathrm{Tidy}(n, \mathrm{path}(i); \mathcal{L}_\vartheta)$ for all ϑ there is a family $\mathcal{L}'$ with $\mathrm{Tidy}(n+e, \mathrm{path}(i'); \mathcal{L}'_\vartheta)$ for all ϑ and with the following property. For every complex function T of the triples: if for every real point h there is a constant b with $\mathrm{per}(\mathcal{L}'; (\vartheta, F, F'); i', \mathsf{h}) = \mathrm{b} \cdot T(\vartheta, F, F')$ for all triples, then for every real point h there is a constant b with $\mathrm{per}(\mathcal{L}; (\vartheta, F, F'); i, \mathsf{h}) = \mathrm{b} \cdot T(\vartheta, F, F')/\mathrm{d}(\vartheta)$ for all triples. In the formal definition the second and third arguments are the proofs of $i \le m$ and $i' \le m$.

LEAN `Data.HasRoute`

Lemma 4.4 (binding keeps short sums short)**.** *Let P, D and an m-fold twist ϑ be given, n and e natural numbers, $\mathcal{E}$ and $\mathcal{E}'$ finite sets of edges, $\mathcal{X}$ a finite set, k and t weights and tuples as in Definition 4.1(f), and $\mathcal{L}_0$ a datum. Suppose that $\#\mathcal{X} \le \mathrm{tb}(\vartheta)^e$, and that for all S and x with $\mathrm{k}(\mathsf{S}, \mathsf{x}) \neq 0$ and $\mathrm{LInv}(\mathsf{S}, \mathcal{E})$ the number $\mathrm{k}(\mathsf{S}, \mathsf{x})$ is a root of unity and $\mathrm{LInv}(\mathrm{t}(\mathsf{S}, \mathsf{x}), \mathcal{E}')$ holds. If $\mathrm{Short}(n, \mathcal{E}; \mathcal{L}_0)$ holds, then $\mathrm{Short}(n + e, \mathcal{E}'; \mathrm{bind}(\mathcal{L}_0; \mathrm{k}, \mathrm{t}))$ holds.*

Proof. Substitute the short sum of $\mathcal{L}_0$: the new datum is the sum over the pairs of a term of $\mathcal{L}_0$ and an element of $\mathcal{X}$, with coefficients the products, which are 0 or roots of unity, and their number is at most $\mathrm{tb}(\vartheta)^n \cdot \mathrm{tb}(\vartheta)^e$. □

LEAN `Data.IsShortSum.bind`

5 Crossing the grid

Proposition 5.1 (crossing a square)**.** *Let P, D, an m-fold twist ϑ, a flag o, natural numbers i, n', n, a finite set $\mathcal{E}$ of edges and a datum $\mathcal{L}_0$ be given. Let p, v, Q, U, R and r be nodes, $\mathcal{I}_0$ a finite index set and $\mathcal{J}$ a set; let e_1 be a bijection from $\mathrm{GI}_D(o; i, n')$ to the index set of U, e_2 a bijection from the index set of p plus $\mathcal{I}_0$ to that of Q, and e_3 a bijection from the index set of r plus $\mathcal{I}_0$ to that of R; let e_R be an additive character of R_ϑ; and let χ be a multiplicative character of R_ϑ, e an identification, C a complex number and $\mathcal{G}_\mathsf{j}$, $\mathcal{G}'_\mathsf{j}$ ($\mathsf{j} \in \mathcal{J}$) functions with $\mathrm{SqLaw}(\chi, \mathsf{e}; \mathrm{C}; \mathcal{G}_\mathsf{j}, \mathcal{G}'_\mathsf{j})$ for all j. Assume:*

(i) *for every admissible family g, the value $\mathsf{g}(\mathsf{R})$ is the block extension by 1 of $\mathsf{g}(\mathsf{r})$ through e_3, and for every invertible k the family obtained from g by putting k at p and its block extension through e_2 at Q is admissible;*

(ii) *$\mathsf{U} \neq \mathsf{v}$; either (Q, U) carries the twist and (U, R) does not, or (p, v) does and (v, r) does not; the nodes U, R, v, r have indices at most m and are different from p and Q; and every edge of $\mathcal{E}$ has its source outside $\{\mathsf{p}, \mathsf{Q}\}$ and its target outside $\{\mathsf{p}, \mathsf{Q}, \mathsf{R}, \mathsf{r}\}$;*

(iii) *$\mathrm{Tidy}(n, \mathcal{E} \cup \{(\mathsf{p}, \mathsf{v}), (\mathsf{Q}, \mathsf{U})\}; \mathcal{L}_0)$ holds;*

(iv) *$\chi = \left(\chi^{\mathrm{ed}}_\vartheta((\mathsf{p}, \mathsf{v})) \cdot \chi^{\mathrm{ed}}_\vartheta((\mathsf{Q}, \mathsf{U}))\right)^{-1}$, $\chi^{\mathrm{ed}}_\vartheta((\mathsf{U}, \mathsf{R})) = \chi^{\mathrm{ed}}_\vartheta((\mathsf{p}, \mathsf{v}))$ and $\chi^{\mathrm{ed}}_\vartheta((\mathsf{v}, \mathsf{r})) = \chi^{\mathrm{ed}}_\vartheta((\mathsf{Q}, \mathsf{U}))$.*

Then there is a datum $\mathcal{L}_1$ *with* $\mathrm{Tidy}(n+2, \mathcal{E} \cup \{(\mathsf{U},\mathsf{R}),(\mathsf{v},\mathsf{r})\}; \mathcal{L}_1)$ *such that for every function* $\mathsf{j}(\mathsf{S},\mathsf{k})$ *with values in* $\mathcal{J}$ *that depends only on the values of* S *on* $\mathcal{E}$*:*

$$\mathcal{L}_0\Big(\mathsf{S} \mapsto \sum_{\mathsf{k}} \mathrm{e}_{\mathsf{R}}(\mathrm{tr}(\mathsf{k}\,\mathsf{x}_2))\, \mathcal{G}'_{\mathsf{j}(\mathsf{S},\mathsf{k})}\big(\tbinom{\mathsf{S}(\mathsf{p},\mathsf{v})}{\mathsf{x}_1}\big)\Big) = \mathrm{C}\cdot \mathcal{L}_1\Big(\mathsf{S} \mapsto \mathcal{G}_{\mathsf{j}(\mathsf{S},\mathsf{x}'_2)}(\mathsf{S}(\mathsf{v},\mathsf{r}) \mid \mathsf{x}'_1)\Big),$$

where x_1, x_2 *are the column blocks of* $\mathsf{S}(\mathsf{Q},\mathsf{U})$ *reindexed by* e_1, e_2, *and* x'_1, x'_2 *the row blocks of* $\mathsf{S}(\mathsf{U},\mathsf{R})$ *reindexed by* e_3, e_1.

Proof. Let $\mathcal{L}_1$ be the datum obtained from $\mathcal{L}_0$ by binding (Definition 4.1 (f)) with the graph kernel of χ and e as weight, corrected by the function of orbits of the second note (Input 1) and by the Fourier factor e_{R}: a tuple is replaced by the tuples in which the matrices on the two old edges are exchanged for matrices on the two new edges. The kernel is a root of unity or zero, and its non-vanishing transfers left inverses between the blocks (Input 1), so Lemma 4.4 gives a short sum of order $n+2$, the two new matrices being counted by the tuple bound. The equivariance under admissible families follows from the equivariance of the graph kernel, from (i) and (iv), and from the averaging property of the correcting function. The identity is the reproducing property of the graph kernel for a square law, $\mathcal{G}' = \mathrm{C}\sum(\mathrm{kernel})\,\mathcal{G}$ (Input 1), applied term by term in the short sum of $\mathcal{L}_0$; the left inverses that it requires are those of tidiness, by (ii). □

LEAN `Data.IsTidy.exists_square_move`

Lemma 5.2 (the square law of a square of the grid)**.** *Let* P *be parameters,* D *admissible data,* o *a flag and* i, n *natural numbers with* $i < m$ *and* $n < m$*. There are a complex number* $\varkappa$*, an identification* e *and a function* c *from the* m*-fold twists to the non-zero complex numbers with* $\mathrm{Cost}_D(1;\mathrm{c})$ *such that for every* m*-fold twist* ϑ*, every form* y *at the node* $(i,n+1)$ *of the grid* o *and every linear functional* l *on the forms at* $(i+1,n)$*: the functions* $\mathsf{x} \mapsto \mathsf{l}(\mathrm{des}_{\mathsf{x}_2}\,\mathrm{lift}_{\mathsf{x}_1}\,y)$ *(lift, then descent, for the column blocks of* x*) and* $\mathsf{y} \mapsto \mathsf{l}(\mathrm{lift}_{\mathsf{y}_2}\,\mathrm{des}_{\mathsf{y}_1}\,y)$ *(descent, then lift, for the row blocks of* y*) satisfy the square law for the character* $\big(\chi^{\mathrm{ed}}_{\vartheta}(\mathrm{ve}(o,i,n))\cdot\chi^{\mathrm{ed}}_{\vartheta}(\mathrm{re}(o,i,n))\big)^{-1}$*, the identification* e *and the constant* $\varkappa/\mathrm{c}(\vartheta)$*.*

Proof. Apply the theorem on one square (Input 1) to the four spaces at the corners (i,n), $(i+1,n)$, $(i,n+1)$, $(i+1,n+1)$, with the family of all m-fold twists; the edge families are provided by Lemma 3.2, the characters are tame by Lemma 3.5 (e), and the numerical hypotheses hold by the small facts. Its constant has squared absolute value a fixed number times a product of L-values of a pair ramified at the chosen places, so it costs one square (Input 1). □

LEAN `Data.exists_squareLaw_grid`

Proposition 5.3 (a descent moves past the lifts of a path)**.** *Let* P *be parameters,* D *admissible data,* o *a flag and* i, n *natural numbers with* $n < m$ *and* $i \le n$*. There are a complex number* $\varkappa$ *and a function* c *from the* m*-fold twists to the non-zero complex numbers with* $\mathrm{Cost}_D(i;\mathrm{c})$ *such that for every* m*-fold twist* ϑ*, every* n_0*, every finite set* $\mathcal{E}$ *of edges and every datum* $\mathcal{L}_0$*: if every edge of* $\mathcal{E}$ *avoids the grid* o *or is* $\mathrm{re}(o,i',n)$ *with* $i' \ge i$*, and* $\mathrm{Tidy}(n_0, \mathcal{E}\cup\mathrm{gp}(o,i,n); \mathcal{L}_0)$ *holds, then there is a datum* $\mathcal{L}_1$ *with* $\mathrm{Tidy}(n_0+2i, \{\mathrm{ve}(o,i,n)\}\cup\mathrm{gp}(o,i,n+1)\cup\mathcal{E}; \mathcal{L}_1)$ *and*

$$\mathcal{L}_0(\mathsf{S} \mapsto \mathsf{l}_{\mathsf{S}}(\mathrm{pm}_{\mathsf{S}}(o;i,n)(F))) = \frac{\varkappa}{\mathrm{c}(\vartheta)}\,\mathcal{L}_1\Big(\mathsf{S} \mapsto \mathsf{l}_{\mathsf{S}}(\mathrm{des}_{\mathsf{S}(\mathrm{ve}(o,i,n))}\,\mathrm{pm}_{\mathsf{S}}(o;i,n+1)(F))\Big)$$

for every form F at $(0,m)$ and every family of linear functionals l_{S} that depends only on the values of S on $\mathcal{E}$.

Proof. Induction on i. For $i=0$ the two sides agree by the definition of the maps along a path. For $i+1$, apply the induction hypothesis, and then cross the square with corners (i,n), $(i+1,n)$, $(i,n+1)$, $(i+1,n+1)$ by Proposition 5.1, with the square law of Lemma 5.2; each crossing raises the order by 2 and multiplies the constant by one that costs one square, and costs add in products (Input 1). □

LEAN `Data.descent_past_gridPathMap`

Proof of Theorem 2′ and Theorem 2. One step. Let $\mathcal{L}$ be tidy of order n on path(i). In each of the two grids move the last descent of the path, the one from $(0,i+1)$ to $(0,i)$, past the i lifts of the path by Proposition 5.3: each of the two moves has a constant that costs i squares and raises the order by $2i$. The data are now tidy on the path of stage $i+1$ in its first i lifts, followed by a descent from $(i,i+1)$ to (i,i) in each grid. Cross the square at the diagonal by Proposition 5.1, with the Fourier form of the square law (Input 1) for the flipped kernel of the line that the grid true has in addition: its constant costs one square and the order rises by 2. The periods are compared by the seesaw identity for the pair formed by the diagonal space and the subdiagonal one (Input 2), which transfers a proportionality of periods from the new path to the old; the flip contributes a power of the number of residues, which costs no square. So the product of the constants costs $2i+1 \le 4i+2$ squares and the order has risen by $4i+2$; a route records one bound for both numbers. *The climb.* Induction on i: the route of cost 0 from stage 0 to itself is the identity, and the routes compose, the functions d multiplying and the costs adding, with $2i^2+4i+2 = 2(i+1)^2$. □

6 The start: stage zero

At stage 0 the path consists of the m descents of each grid from $(0,m)$ to $(0,0)$. Write $\mathrm{des}^{\vartheta}_{o,n}$ for the descent of unit 141 from the forms at $(0,m)$ to the forms at $(0,n)$ of the grid o for the twist ϑ (untwisted for $\vartheta=1$), and $\mathcal{A}_n^{\pi_{\mathrm{gr}}}(D)$ for the forms of principal level π_{gr} at the node $(0,n)$ of the grid true.

Lemma 6.1 (orders and indices)**.** *Let P be parameters, D admissible data and n a natural number. There is a complex number C such that for every m-fold twist ϑ: $\mathrm{ord}_{D,\mathrm{false}}(\vartheta,n) = \mathrm{C}\cdot \mathrm{i}^{\mathrm{gr}}_{D,n}(\vartheta)$, where $\mathrm{i}^{\mathrm{gr}}_{D,n}(\vartheta)$ is the index of the level of the node $(0,n)$ of the grid false for ϑ (Definition 4.2 (e)).*

Proof. Both numbers count invertible matrices. The index of the level is a mean over the residue ring of $\mathfrak{m}^{\mathrm{gr}}(\vartheta)$, and the reduction $\mathrm{red}_{\vartheta,\mathrm{false}}$, which is surjective, multiplies the counts by a fixed power of the norms of the chosen places. □

LEAN `Data.triangleOrder_eq_mul_levelIndex`

Lemma 6.2 (translations)**.** *Let P be parameters, D admissible data and ϑ an m-fold twist.*

(a) *Let $i \le m$ and h a real point. For every finite point u of $\mathrm{U}(\mathsf{V}_{i,i}(D))$, every form y at the node (i,i) of the grid false and every form y' at the node (i,i) of the grid true: the form $\mathrm{dp}_{\vartheta}(i;\mathsf{h})$ takes the same value at (y,y') and at the translates of y by u and of y' by $\mathsf{u}\oplus 1$.*

(b) *Let* g *be a family of invertible matrices at the nodes with value* 1 *at the nodes* $(0,m)$ *of both grids. Let* o *be a flag,* n *a natural number,* S *a tuple,* F *a form at* $(0,m)$ *of the grid* o*, and* k *an element of the maximal compact subgroup whose reduction modulo* $\mathfrak{m}^{\mathrm{gr}}(\vartheta)$ *is the value of* g *at* $(o,0,n)$*, with* $\mathsf{k}=1$ *if* $n=m$*. Then the translate by* k *of* $\mathrm{ld}_{\mathrm{act}(\mathsf{g},\mathsf{S})(\cdot)}(o,n)(F)$ *is* $\mathrm{ecp}(\mathsf{g},\mathrm{leg}(o,n))\cdot\mathrm{ld}_{\mathsf{S}}(o,n)(F)$.

(c) *Let* g *be as in* (b) *and suppose moreover that its value at* $(\mathrm{true},0,0)$ *is the block extension through* $\mathrm{gde}_{D}(0)$ *of its value at* $(\mathrm{false},0,0)$*. For a real point* h*, forms* F*,* F' *at the nodes* $(0,m)$ *of the grids false and true and a tuple* S*:*

$$\mathrm{dp}_{\vartheta}(0;\mathsf{h})(\mathrm{ld}_{\mathrm{act}(\mathsf{g},\mathsf{S})(\cdot)}(\mathrm{false},0)(F),\mathrm{ld}_{\mathrm{act}(\mathsf{g},\mathsf{S})(\cdot)}(\mathrm{true},0)(F'))$$
$$=\mathrm{ecp}(\mathsf{g},\mathrm{path}(0))\cdot\mathrm{dp}_{\vartheta}(0;\mathsf{h})(\mathrm{ld}_{\mathsf{S}}(\mathrm{false},0)(F),\mathrm{ld}_{\mathsf{S}}(\mathrm{true},0)(F')).$$

Proof. (a) Substitute $\mathsf{x}\mapsto\mathsf{x}\mathsf{u}$ in the integral; the quotient measure is invariant, the point in the larger group moves by $\mathsf{u}\oplus 1$, and the forms are invariant under rational points. (b) Induction along the descents: at each step the descent with the moved matrix is the translate of the descent with the old one, by the transformation law of the datum modulo the modulus, and the character of the edge is collected. (c) Translate both descents by a lift of the value of g at $(0,0)$ and its block extension: (a) leaves the period unchanged and (b) turns each moved descent into the old one times the product of the edge characters over its edges. □

LEAN Data.diagonalPeriodForm_formRep · Data.formRep_legDescents_tupleAction · Data.diagonalPeriodForm_tupleAction

Proposition 6.3 (the descents of unit 141 as tidy data)**.** *Let* P *be parameters and* D *admissible data.*

(a) *Let* ϑ *be an* m*-fold twist and* $n\le m$*. There is a datum* $\mathcal{L}_1\in\mathrm{Dat}(\vartheta)$ *with* $\mathrm{LTidy}(2(m-n),\mathrm{leg}(\mathrm{false},n)\cup\mathrm{leg}(\mathrm{true},n);\mathcal{L}_1)$ *such that for every bilinear form* B_1 *and all forms* F*,* F' *at the nodes* $(0,m)$ *of the two grids:*

$$B_1(\mathrm{des}^{\vartheta}_{\mathrm{false},n}F,\mathrm{des}^{1}_{\mathrm{true},n}F')=\prod_{n\le n'<m}\mathrm{ord}_{D,\mathrm{false}}(\vartheta,n')\,\mathrm{ord}_{D,\mathrm{true}}(\vartheta,n')$$
$$\cdot\ \mathcal{L}_1(\mathsf{S}\mapsto B_1(\mathrm{ld}_{\mathsf{S}}(\mathrm{false},n)(F),\mathrm{ld}_{\mathsf{S}}(\mathrm{true},n)(F'))).$$

(b) *There are a complex number* C *and a family* $\mathcal{L}$ *of data such that for every* m*-fold twist* ϑ*:* $\mathrm{Tidy}(4m,\mathrm{path}(0);\mathcal{L}_\vartheta)$ *holds, and for every real point* h *and all forms* F*,* F' *at the nodes* $(0,m)$ *of the two grids*

$$\mathrm{ord}_{D,\mathrm{false}}(\vartheta,m)\cdot\mathrm{dp}_{\vartheta}(0;\mathsf{h})(\mathrm{des}^{\vartheta}_{\mathrm{false},0}F,\mathrm{des}^{1}_{\mathrm{true},0}F')=\mathrm{C}\cdot\mathrm{per}(\mathcal{L};(\vartheta,F,F');0,\mathsf{h}).$$

Proof. (a) Decreasing induction on n from m. A descent with the point datum of the inclusion is a sum of descents with indicator data over the matrices that reduce to the inclusion; binding the datum with these matrices for the two new edges keeps level-tidiness, with one more edge and order two more per step (Lemma 4.4). (b) Average the datum of (a) for $n=0$ over representatives of the cosets of the levels of the nodes, weighted by their edge characters. This makes it equivariant for all admissible families, hence tidy, of order $4m$. The identity follows from Lemma 6.2 (c) and from the ratios of indices of Lemma 6.1. □

LEAN Data.exists_isLevelTidy_iotaPlus_eq · Data.exists_isTidy_diagonalPeriodForm_eq

Lemma 6.4 (restriction to the rows)**.** *Let P be parameters, D admissible data and h a real point. There is a linear map r_0 from $\mathcal{A}_0^{\pi_{\mathrm{gr}}}(D)$ to the forms on $\mathsf{Y}_0(D)$ of the level of the chosen places such that for every $F' \in \mathcal{A}_0^{\pi_{\mathrm{gr}}}(D)$, every m-fold twist ϑ and every form y on $\mathsf{Y}_0(D)$ of principal level at the level places of ϑ: $\langle y, \mathsf{r}_0(F')\rangle_{\mathrm{Pet}} = \mathrm{dp}_{\vartheta}(0;\mathsf{h})(\bar{y}, F')$.*

Proof. On the finite-dimensional space of forms of the level of the chosen places, r_0 is the adjoint of the period functional for the trivial twist. For a general ϑ the period of a form of the level of ϑ equals that of its average over the level of the chosen places, because the twisted and untwisted periods differ by translations that fix F'. □

LEAN `Data.exists_restrict_bordered`

Lemma 6.5 (the descent is adjoint to the weighted chain)**.** *Let P be parameters and D admissible data.*

(a) *Let o be a flag, ϑ an m-fold twist, $n \le m$ and y a form at the node $(0,0)$ of the grid o; let y_n be the image of y under the chain of the first n lifts of the triangle of the data, each weighted by its triangle order. If $n \neq 0$, then every $\mathrm{des}^{\vartheta}_{o,n-1} F'$ and y_n are fixed by the elements of the maximal compact subgroup that reduce to 1 modulo $\mathfrak{m}^{\triangle}_o(\vartheta)$; and for every form F' at $(0,m)$ the integral of $\mathrm{des}^{\vartheta}_{o,n} F' \cdot y_n$ over the quotient at $(0,n)$ equals the integral of $\mathrm{des}^{\vartheta}_{o,0} F' \cdot y$ over the quotient at $(0,0)$.*

(b) *$\mathcal{A}_0^{\pi_{\mathrm{gr}}}(D)$ is the image of $\mathcal{A}_m^{\pi_{\mathrm{gr}}}(D)$ under the untwisted descent $\mathrm{des}^{1}_{\mathrm{true},0}$.*

Proof. (a) Induction on n: a descent and a lift through an edge family are adjoint (Input 2), and both transform under the elements of the level by the character of the grid, which is trivial on those that reduce to 1. (b) The image lies in the target by the invariance in (a). A form in the target that is orthogonal to the image has, by (a), a weighted chain orthogonal to all forms of level π_{gr} at $(0,m)$, among them the chain itself; so the chain vanishes, and the form vanishes because weighted chains of the triangle are injective (Input 1). □

LEAN `Data.iotaPlus_adjoint_weightedChain` · `Data.gridLevelForms_eq_map_iotaPlus`

Lemma 6.6 (steep parameters)**.** *Let P be parameters, D admissible data and o a flag. There is an ordering of the index set $\mathrm{GI}_D(o;0,0)$ such that at every infinite place τ the parameter $\mathrm{par}_D(o,0;0,0)(\tau)$ is steep along this ordering, with the bound $w_{D,0}(\tau)$ and the signs of the node.*

Proof. Order the indices by the numbering of the base indices, compute the signs and the doubled parameters along the ordering, bound the top weight by $2w_{D,0}(\tau)$ and compute the lowest K-type by induction on the rank; steepness is then a list of inequalities. □

LEAN `Data.exists_isSteep`

Proposition 6.7 (a form orthogonal to all restrictions vanishes)**.** *Let P be parameters, D admissible data, $\mathbf{n}$ a non-empty finite index set, V a hermitian space over $E(D)$ with index set $\mathbf{n}$, and $\mathsf{V} \oplus \langle 1\rangle$ its sum with a positive line. Let A, L, K_0 be natural numbers and B, M real numbers; let l_τ and l'_τ be Harish-Chandra parameters for the signs of V and of $\mathsf{V} \oplus \langle 1\rangle$ at the infinite places τ. Assume: each l_τ has a large gap $4\mathrm{A}$ and each l'_τ a large gap $2\mathrm{A}$; $|l_\tau(i)| \le \mathrm{L}$ for all entries; $8(|\mathbf{n}|+1)^2 < \mathrm{A}$; $\mathrm{K}_0 < \mathrm{A}$; $2f(D)(|\mathbf{n}|+1)+2 \le \mathrm{A}$; $g(D)\,\mathrm{c}_{\mathrm{cusp}}^2 \le \mathrm{B}$ for the cusp constant of the field and $\mathbf{n}$;*

$2000(|\mathbf{n}|+\mathrm{L}+1)^6 \le \mathrm{B}$; $f(D) \le \mathrm{B}$; $\mathrm{B}^{20\mathrm{A}(f(D)+1)} \le \mathrm{M}$; $\mathrm{M}^{3|\mathbf{n}|+2} \le |\pi_{\mathrm{gr}}|_\tau$ *at every* τ*; and for every* τ *there is an element* w_τ *of the real unitary group of* $\mathsf{V} \oplus \langle 1\rangle$ *with* $\|\mathsf{w}_\tau\|^2 \le \mathrm{B}$ *such that the volume of the unit ball of the real group of* V *is at most* $\mathrm{B}^{\mathrm{K}_0}$ *times the absolute value of the integral over that group of the matrix coefficient of the discrete series of* l_τ *against the conjugate of that of* l'_τ *at* $\mathsf{w}_\tau^{-1}(\,\cdot \oplus 1)\mathsf{w}_\tau$*. Then there is a real point* h *of* $\mathrm{U}(\mathsf{V} \oplus \langle 1\rangle)$ *such that every form* y *on* V *of the type of* l *and of the level of the chosen places vanishes if it is orthogonal to* $\mathsf{x} \mapsto F'((\mathsf{x} \oplus 1)\mathsf{h})$ *for every form* F' *on* $\mathsf{V} \oplus \langle 1\rangle$ *of the type of* l' *and of principal level* π_{gr}*.*

Proof. Take for h the point with the components w_τ. Let y be orthogonal to all restrictions. The Poincaré series of the larger group are forms F' of the required kind, and the period of y against the restriction of one of them is, by unfolding, an integral of y against a matrix coefficient near a point (Input 3). The lower bound in the hypothesis for this matrix coefficient shows that $|y|$ at a point is at most a fixed fraction of its supremum on a neighbourhood, wherever the first minimum of the lattice is large enough for the level π_{gr} to separate the terms of the series; near the cusps y is small by the bound for forms in terms of the first minimum (Input 3). The two bounds, with the inequalities between A, B, M and $|\pi_{\mathrm{gr}}|_\tau$, show that the supremum of $|y|$ is at most half of itself; hence $y = 0$. □

LEAN `eq_zero_of_petersson_restrict_eq_zero`

Corollary 6.8 (every row pairs with a restriction)**.** *Let* P *be parameters and* D *admissible data. There is a real point* h *of the group of the node* $(0,0)$ *of the grid true such that every form* y *of principal level on* $\mathsf{Y}_0(D)$ *vanishes if it is orthogonal to* $\mathsf{x} \mapsto F'((\mathsf{x} \oplus 1)\mathsf{h})$ *for every* $F' \in \mathcal{A}_0^{\pi_{\mathrm{gr}}}(D)$*.*

Proof. Proposition 6.7 for $\mathsf{V} = \mathsf{Y}_0(D)$, with $\mathrm{A} = r^4$ and B a power of $g(D)$ with exponent about x; the gaps and sizes of the parameters are small facts of §3, and the lower bound for the matrix coefficient is the Laplace method for the steep parameters of Lemma 6.6 (Input 3). □

LEAN `Data.eq_zero_of_orthogonal_gridLevelForms`

Proof of Theorem 3. Let h be the point of Corollary 6.8, $\mathcal{L}$ and C as in Proposition 6.3 (b), and r_0 the map of Lemma 6.4 for h. Put $\mathsf{r} = \mathsf{r}_0 \circ \mathrm{des}^1_{\mathrm{true},0}$ on $\mathcal{A}_m^{\pi_{\mathrm{gr}}}(D)$ and $\mathrm{a} = \mathrm{C}$. *Surjectivity.* By Lemma 6.5 (b) the image of r is $\mathsf{r}_0(\mathcal{A}_0^{\pi_{\mathrm{gr}}}(D))$. A form y of principal level orthogonal to it satisfies $\mathrm{dp}_1(0;\mathsf{h})(\bar{y}, F') = 0$ for all F' by Lemma 6.4, that is, y is orthogonal to all restrictions, and $y = 0$ by Corollary 6.8. *The identity.* By Lemma 6.4, applied to the conjugate of $\mathrm{des}^{\vartheta}_{\mathrm{false},0} F$, which has the level of ϑ by Lemma 6.5 (a), the Petersson product of that form with $\mathsf{r}(F')$ is $\mathrm{dp}_\vartheta(0;\mathsf{h})(\mathrm{des}^{\vartheta}_{\mathrm{false},0} F, \mathrm{des}^1_{\mathrm{true},0} F')$, which Proposition 6.3 (b) expresses by the period of stage 0. On the other hand, by Lemma 6.5 (a) for $n = m$, the same Petersson product is the integral of F against the weighted chain of $\mathsf{r}(F')$, and the weighted chain is the chain of the m lifts up to the orders and indices that appear on the two sides: the factor $\mathrm{i}_D(\vartheta)$ on the left is $\mathrm{ord}_{D,\mathrm{false}}(\vartheta, m)$, the order of the invertible matrices of the rank of $\mathsf{Y}_m(D)$ modulo $\mathfrak{m}_D(\vartheta, 1)$, while the period carries $\mathrm{i}^{\mathrm{gr}}_D(\vartheta)$ inside. □

7 The end: stage m

Definition 7.1 (the last row of the grids and the end period)**.** Let P be parameters and D admissible data.

(a) For a flag o and a natural number i, $\mathrm{EI}(o,i) = \mathrm{GI}_D(o;i,m)$ is the index set of the *end node* (i,m); the spaces at the end nodes are definite and standard (an instance). $\mathrm{ewd}(o)$ is a chosen weight for the forms at the end nodes of the grid o, with exponents $\mathrm{ede}(o)$, which goes up from each end node to the next and is the Harish-Chandra datum of minus the parameter of each end node.

(b) For an m-fold twist ϑ_1, a flag o and a natural number i: $\mathrm{elev}(\vartheta_1,o,i)$ is the level of the elements whose reduction modulo $\mathfrak{m}^{\mathrm{gr}}(\vartheta_1)$ lies in the level of the node (i,m); $\mathrm{eker}(\vartheta_1,o,i)$ is the theta kernel from the end node (i,m) to $(i+1,m)$ for this weight, the characters of the grid, the jet scale of the data and that level.

(c) The *branching matrix* is a chosen integer matrix that pairs the weights at the nodes (m,m) of the two grids (it exists by the interlacing of their parameters), and the *end pairing* $\langle\ ,\ \rangle_{\mathrm{end}}$ is its bilinear form. For a point h of $\mathrm{U}(\mathsf{V}_{m,m}(D))$, $\mathrm{ebp}(\vartheta_1,\mathsf{h})$ is the point $\mathsf{h}\oplus\mathsf{u}^m$ of the group of the node (m,m) of the grid true, read through $\mathrm{gde}_D(m)$.

(d) alg_o is the isomorphism from the forms at the node $(0,m)$ of the grid o to the algebraic forms of the weight of (a). For a pair $y=(y_{\mathrm{false}},y_{\mathrm{true}})$ of forms at the nodes $(0,m)$ of the two grids, $\mathrm{Fix}(\vartheta_1,y)$ holds if for both o the algebraic form $\mathrm{alg}_o(y_o)$ is fixed by $\mathrm{elev}(\vartheta_1,o,0)$.

(e) For a tuple S (for a twist ϑ), a flag o and i: $\mathrm{edat}(\mathsf{S},o,i)$ is the vacuum scale to the power of the number of entries, times the datum of the indicator of the matrix that S puts on $\mathrm{re}(o,i,m)$; and for a form F, $\mathrm{ech}(\mathsf{S},o,i)(F)$ is the chain of the first i lifts of $\mathrm{alg}_o(F)$ along the end nodes, with the kernels $\mathrm{eker}(\vartheta,o,\cdot)$ and the data $\mathrm{edat}(\mathsf{S},o,\cdot)$.

(f) For a datum $\mathcal{L}_0\in\mathrm{Dat}(\vartheta)$ and a pair y of forms, the *end period* is

$$\mathrm{eper}(\mathcal{L}_0;y)=\mathcal{L}_0\Big(\mathsf{S}\mapsto\sum_{\mathsf{h}}\big\langle\mathrm{ech}(\mathsf{S},\mathrm{false},m)(y_{\mathrm{false}})(\mathsf{h}),\ \mathrm{ech}(\mathsf{S},\mathrm{true},m)(y_{\mathrm{true}})(\mathrm{ebp}(\vartheta,\mathsf{h}))\big\rangle_{\mathrm{end}}\Big),$$

the sum being the class integral over $\mathrm{U}(\mathsf{V}_{m,m}(D))$ at the level of the last end node.

LEAN `Data.EndIndex` · `Data.endWeightData` · `Data.endDetExp` · `Data.endLevel` · `Data.endKernel` · `Data.endBranching` · `Data.endPairing` · `Data.endBlockPoint` · `Data.endToAlgForm` · `Data.FixedByEndLevels` · `Data.endDatum` · `Data.endChain` · `Data.endPeriod` · `Data.isStd_gridSpace`

Lemma 7.2 (the end nodes)**.** *Let P be parameters and D admissible data.*

(a) *For an m-fold twist ϑ_1, a flag o and a natural number i:* $\mathrm{elev}(\vartheta_1,o,i)$ *is contained in the principal level of* π_{gr}.

(b) *For a family $\mathcal{L}$ of data and a real point* h *there is a complex number* b *such that for every triple* (ϑ,F,F')*:* $\mathrm{per}(\mathcal{L};(\vartheta,F,F');m,\mathsf{h})=\mathrm{b}\cdot\mathrm{eper}(\mathcal{L}_\vartheta;(F,F'))$.

Proof. (a) An element of the level reduces into the level of the node, which lies in the kernel of the reduction modulo the chosen places to the power $b_1(D)$. (b) At stage

m the path consists of the m lifts along the end nodes, and these lifts of a form are the chains of theta kernels of Definition 7.1; the diagonal period of two such chains, an integral over a compact quotient, is a class sum, and it is a constant multiple of their branching pairing (Input 3). □

LEAN `Data.endLevel_le_principalLevel` · `Data.gridPeriod_eq_endPeriod`

Proposition 7.3 (the end period as a short sum of pairings)**.** *Let P be parameters, D admissible data, ϑ an m-fold twist, n a natural number, $\mathcal{L}_0$ a datum with* $\mathrm{Tidy}(n, \mathrm{path}(m); \mathcal{L}_0)$ *and y a pair of forms with* $\mathrm{Fix}(\vartheta, y)$*. Let $\mathcal{S}$ be a property of quadruples (flag, index, point, vector), F a normed field, $\sigma_1 : \mathbb{C} \to \mathsf{F}$ a ring homomorphism, M_1 a complex number and R_1 a real number. Assume: $\mathcal{S}$ is preserved by the kernels, that is, for every tuple x, flag o, $i < m$, points g, h and vector v with $\mathcal{S}(o, i, \mathsf{g}, \mathsf{v})$, the property $\mathcal{S}(o, i+1, \mathsf{h}, \cdot)$ holds for the image of v under the jet matrix of* $\ker(\vartheta, o, i)$ *at the datum* $\mathrm{edat}(\mathsf{x}, o, i)$ *moved by* (g, h)*; $\mathcal{S}$ holds at the start,* $\mathcal{S}(o, 0, \mathsf{g}, \mathrm{alg}_o(y_o)(\mathsf{g}))$ *for all o, g; and for vectors v', v with $\mathcal{S}$ at the last end node of the grid false at g and of the grid true at* $\mathrm{ebp}(\vartheta, \mathsf{g})$*: $\|\sigma_1(\mathrm{M}_1 \langle \mathsf{v}', \mathsf{v} \rangle_{\mathrm{end}})\| \le \mathrm{R}_1$. Then there is a natural number $\mathrm{n}_1 \le \mathrm{tb}(\vartheta)^n$ such that $\mathrm{eper}(\mathcal{L}_0; y)$ is a sum of n_1 times the product of the class numbers of the end levels complex numbers a_1, each with $\|\sigma_1(\mathrm{M}_1 \mathrm{a}_1)\| \le \mathrm{R}_1$.*

Proof. Tidiness makes the summand of the end period, as a function of the point, invariant under the levels of the end nodes. Expand $\mathcal{L}_0$ as its short sum, of at most $\mathrm{tb}(\vartheta)^n$ terms with coefficients that are roots of unity; write each class integral as a sum over double cosets and each chain value as an iterated sum over the classes of the end levels of products of jet matrices applied to the values of the form. Each term is a pairing of two vectors that have the property $\mathcal{S}$, by its propagation along the kernels. □

LEAN `Data.endPeriod_mem`

Proof of Theorem 4. (a) Take for $\mathrm{d}_{\mathrm{end}}$ the product, over the infinite places, of powers of $g(D)^{b_1(D)}$ and of the denominators of the branching data, times a power of the denominator of the dual pairing; its size is bounded by the bounds for the weights of the end nodes (small facts). Apply Proposition 7.3 with the completed algebraic closure of the ℓ-adic numbers, $\sigma_1 = j$, $\mathrm{M}_1 = \mathrm{d}_{\mathrm{end}}$, $\mathrm{R}_1 = 1$ and the property "the gauge of the vector is at most 1": it holds at the start by hypothesis, and it is preserved because the data of the end nodes are standard data and jets of standard data are integral at the primes prime to the modulus (Input 1, Input 4). So the end period times $\mathrm{d}_{\mathrm{end}}$ is a sum of j-integral numbers.

(b) Apply Proposition 7.3 with $\sigma_1 = \sigma$ and the property "the norm of the σ-conjugate of the vector at the end node i of the grid o is at most $\exp(i\, x^{99} b^2) \sqrt{B_0(o)}$", which is preserved by the bounds for the conjugates of jets of admissible combinations (Input 4). The number of terms is bounded by the tuple bound to the power of the order and by the class numbers of the end levels, each at most $\exp(x^{99} b^2)$; the sizes of the data give the exponent $x^{200} b^2$. □

8 Proof of Theorem 1

Let $\mathcal{L}$, h, a and r be as in Theorem 3, and $\mathrm{d}_{\mathrm{end}}$ as in Theorem 4 (a). By Theorem 2 for $i = m$ there are a function c with $\mathrm{Cost}_D(2m^2; \mathrm{c})$ and a family $\mathcal{L}'$, tidy of order $4m + 2m^2$ on the path of stage m, such that proportionalities of the periods of stage m of $\mathcal{L}'$ descend to those of stage 0 of $\mathcal{L}$, with the factor $1/\mathrm{c}(\vartheta)$. By Lemma 7.2 (b) the periods of stage m of $\mathcal{L}'$ are a constant multiple of the end periods. So there is a constant b with

$$\mathrm{i}_D(\vartheta) \int F \cdot (\text{chain of lifts of } \mathsf{r}(F')) = \frac{\mathrm{b}}{\mathrm{c}(\vartheta)}\ \mathrm{eper}(\mathcal{L}'_\vartheta; (F, F'))$$

for all triples. The constant b may be zero, and the proof does not divide by it: there is a unit t_0 with $\mathrm{b}\,\mathrm{t}_0 = \mathrm{b}'$, where b' is 1 if $\mathrm{b} \neq 0$ and 0 if $\mathrm{b} = 0$.

From periods to coordinates. A dual pairing of the weights of $\mathsf{Y}_m(D)$ (Input 3; the parameter of $\mathsf{Y}_m(D)$ is a Harish-Chandra parameter by the second note) expresses the coordinates of the value of an algebraic form at a point x through its pairings with delta forms: for a vector v, the delta form at x with value v is an algebraic form of the last level, and the integral of its product with another form is a class sum with one term. Taking for F the form whose algebraic form is this delta form, the identity above shows that the coordinates at x of

$$\frac{\mathrm{c}(\vartheta)}{\mathrm{c}(1)} \cdot \Theta^{\mathrm{lev}}_{D,\vartheta}\left(\mathrm{t}_0\, \mathrm{c}(1)\, \frac{\mathrm{d}_{\mathrm{end}}}{w_0}\, \mathsf{r}(F')\right),$$

with w_0 the number of elements of the Weyl index set and t_0 the unit above, are integer multiples of $\mathrm{b}'\, \mathrm{d}_{\mathrm{end}} \cdot \mathrm{eper}(\mathcal{L}'_\vartheta; (F, F'))$, up to the hook number of the weight.

Crude rows. Call an algebraic form of level π_{gr} on the large definite space *short* if its values are integral and small in every conjugate. The pair (F, F') of a delta form and a short form is fixed by the end levels (Lemma 7.2 (a) and the level of the delta form). By Theorem 4 (a) the numbers $\mathrm{d}_{\mathrm{end}} \cdot \mathrm{eper}(\mathcal{L}'_\vartheta; (F, F'))$ are j-integral for every prime ℓ prime to $\mathrm{N}\,Q(\vartheta)$ and every ring homomorphism $j\colon \mathbb{C} \to \mathbb{C}_\ell$; this is property (1) for the row $y = \mathrm{t}_0\, \mathrm{c}(1)\, \mathrm{d}_{\mathrm{end}}\, w_0^{-1}\, \mathsf{r}(F')$. For property (2), take $\vartheta = 1$ and an automorphism σ; a point at which the Weyl norm of the conjugate vector is attained and the dual pairing reduce the bound to the end period of the conjugated delta form, which Theorem 4 (b) bounds by $\exp(x^{200}b^2)$ times the square roots of the bounds for the delta form and for the short form; with the bounds for the hook number, for $\mathrm{d}_{\mathrm{end}}$ and the comparison of the scales of the data, the squared Weyl norm is at most $\exp(x_1^2 b^2)$.

They span. The short forms span the forms of level π_{gr} on the large definite space (Input 3), and r is surjective and linear. So the rows y above span the forms of principal level on $\mathsf{Y}_0(D)$.

LEAN the formal proof of `Data.crude_rows` uses `Data.exists_isTidy_integral_eq_gridPeriod`, `Data.gridPeriod_climb`, `Data.gridPeriod_eq_endPeriod`, `Data.exists_mul_endPeriod_mem_embValRing` and `Data.endPeriod_le` as here

9 Notes on the formal text

1. *Theorem 1 and the overview.* The overview of the handout says that in the proof of Theorem C "the family is completed to a basis by crude rows". Theorem 1 provides them: rows that span, whose twisted chains of lifts are integral after the factor $\mathrm{c}(\vartheta)/\mathrm{c}(1)$, and whose untwisted lifts are bounded by $\exp(x_1^2 b^2)$ at every conjugate. The cost $2m^2$ of c is what the fit families of twists have to pay for these rows. In the formal statement the forms y are the forms of principal level on $\mathsf{Y}_0(D)$, a space that contains the rows; property (2) is stated for the chain of lifts on all such forms. In property (1) the formal statement quantifies over all ℓ-adic places, that is, over all ring homomorphisms $j\colon \mathbb{C} \to \mathbb{C}_\ell$, as Theorem C of the handout needs; an earlier edition of this note quantified over the isomorphisms of $\mathbb{C}$ with an algebraic closure of the ℓ-adic numbers, which gives only the homomorphisms with image in that closure and is weaker. The norm in property (2) and in Theorem 4 (b) is the Weyl norm of the weight representation.

2. *Routes carry two numbers in one.* In the definition of a route the same number e bounds the cost of the constant and the growth of the order of tidiness. In the step from stage i the constants cost $2i+1$ squares and the order grows by $4i+2$; costs are upper bounds (the accounting is that of the second reading).

3. *The constant* b *in the proof of Theorem 1.* The formal proof replaces the constant b by b/b, which is 1 or 0, and chooses a unit accordingly; the case $\mathrm{b} = 0$ is not excluded, and cancelling b would lose it (second reading).

4. *What Theorem 1 does not say.* It is a spanning statement: the quantifiers over twists, primes, automorphisms and points apply to each generator y, and the statement names no basis and no extension of a given independent family; the completion to a basis is made where the theorem is used. Theorem 4 (b) is proved for the trivial twist only, and the order $4m + 2m^2$ is the order $4m$ of the data of Theorem 3 plus the bound $2m^2$ of the climb.

5. *Two census entries that are not arguments.* For the maps along a path (Definition 4.2 (b), (c)) the tool that reads the statements takes the arrow of the type "linear map" for a premise. In the Lean text these definitions take the tuple, the flag and the indices and return a linear map; here the form to which the map is applied fills the last slot of the notation.

6. *Proofs as arguments.* The formal definitions of the diagonal period, of the period of a stage and of a route take proofs of inequalities $i \le m$ as arguments; in the notation their place is taken by the index.

7. *Arguments that only Lean can settle.* For the declarations that the dossier marks so, the list of arguments was derived by Lean's rules by hand; no compiler was run. Among the statements above they are Theorem 4 (a), Proposition 5.1, Proposition 7.3, Proposition 6.3 (b), Lemma 6.1, Lemma 6.4, Lemma 4.4, Lemma 3.2 (b), Lemma 3.5 (a), (b), (e), the definitions of tidy and level-tidy data, of the tuple bound, of the maps along a path, of the period of a stage and of the end objects, and several small facts. The doubt concerns implicit arguments: whether a declaration takes the data, the twist or an index set explicitly or implicitly, and whether an instance for a prime is a separate argument.

8. *Small facts.* The one-step results and computations listed in §3 have no statement of their own; the build of this note records them as absorbed, together with the definition of the harmonic of the grid, which has more arguments than a notation can carry and is described in words. Their formal statements are in the concordance.

9. *Two indices.* The number $\mathrm{i}_D(\vartheta)$ of Theorem 3 and the number $\mathrm{i}_D^{\mathrm{gr}}(\vartheta)$ inside the period of a stage are different: the first is the order of the invertible matrices of the rank of $\mathsf{Y}_m(D)$ over the residue ring of $\mathfrak{m}_D(\vartheta,1)$, the second the same order over the residue ring of $\mathfrak{m}^{\mathrm{gr}}(\vartheta)$, whose exponent at the chosen places is $b_1(D)$. An earlier edition wrote both with one symbol. They differ by a factor that does not depend on ϑ (Lemma 6.1).

10. *What is taken for granted.* Everything listed as Inputs: the second note (one square, the triangle, costs), the seesaw identity for periods in the weaker form of the background library (an inference from Kudla's identity, not a statement of his paper), the dual and branching pairings, the spanning by short forms, the bounds of reduction theory, the Laplace method, the Poincaré series as kernels of periods (whose relation to the literature the second reader found unresolved), and the integrality and the bounds for jets. The parts (c), (d), (e) of Lemma 3.5 and Definition 3.4 are used by other units and not in this note.

How this note was written. From the dossier of the unit (its third part). The program check of the whole dossier covered 879 leaf steps of 122 proofs and found every statement at its lines. I read all statements of this part in the dossier, the summaries of all its structured proofs, and step by step the proof of Theorem 1. In the Lean text I read the statements of Theorems 1 to 4, the definitions of routes, of the maps along a path, of the diagonal period and of the period of a stage. The other statements are written from the dossier's readings and were not compared with the Lean text by me. The proofs are outlines that name every input, shorter than the formal ones; those of Theorem $2'$, Theorem 4, Proposition 5.1, Proposition 7.3 and Proposition 6.7 follow the summaries of structured proofs whose long steps the readers mark as coarse; the accounting of orders and costs in the proof of Theorem $2'$ is that of the second reader. This edition is a revision after a referee's report on the three notes of the unit: in Theorem 1 (1) the quantifier is now the formal one (all ring homomorphisms $\mathbb{C} \to \mathbb{C}_\ell$; I read the statement and the definition of an ℓ-adic place again in the Lean text), the two indices of Theorem 3 and of Definition 4.2 (e) have two symbols (read in the Lean text: the statements of Theorem 3, of the period of a stage and of Lemma 6.1, and the definitions of the levels of the nodes and of the triangle orders), and the relations of the Inputs to the literature follow the second reader's verdicts. Definition 3.1 (b), (c) describe the signs and the parameters of the grids in words; their formulas are in the dossier and in the concordance. The figure is mine. Nothing was compiled.

Appendix A. Concordance with the Lean artifact

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Theorem 3	`Data.exists_isTidy_integral_eq_gridPeriod`	CrudeRows.lean	23–40
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Theorem 2′	`Data.hasRoute_succ`	CrudeRows.lean	52–104
Theorem 2	`Data.gridPeriod_climb`	CrudeRows.lean	109–116
Theorem 4	`Data.exists_mul_endPeriod_mem_embValRing`	CrudeRows.lean	120–151
Theorem 4	`Data.endPeriod_le`	CrudeRows.lean	156–184
Theorem 1	`Data.crude_rows`	CrudeRows.lean	191–247
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Lemma 6.2	`Data.diagonalPeriodForm_formRep`	GridFirstColumns.lean	35–42
Lemma 6.4	`Data.exists_restrict_bordered`	GridFirstColumns.lean	47–70
Lemma 6.2	`Data.formRep_legDescents_tupleAction`	GridFirstColumns.lean	79–92
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Proposition 6.3	`Data.exists_isTidy_diagonalPeriodForm_eq`	GridFirstColumns.lean	242–272
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Lemma 3.5	`Data.conjHolWeight_eq_add_kernelWeight`	GridModuli.lean	70–76
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Definition 3.1	`Data.instFactNeRingOfIntegersEHPowGbOfNat`	GridModuli.lean	97–97
Definition 3.1	`Data.places_modulus_eq_union`	GridModuli.lean	102–103
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Definition 3.1	`Data.isTameFor_twistedChi`	GridModuli.lean	113–115
Definition 3.1	`Data.instNeZeroNatB₁`	GridModuli.lean	117–117
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Definition 3.3	`Data.TameChar.pow`	LValueRatio.lean	22–23
Definition 3.3	`Data.MFoldTwist`	LValueRatio.lean	27–28
Definition 3.3	`Data.chainChar`	LValueRatio.lean	32–32
Definition 3.3	`Data.instOneTws`	LValueRatio.lean	34–34
Definition 3.3	`Data.instCoeTwsTwist`	LValueRatio.lean	36–36
Definition 3.3	`Data.MFoldTwist.powTwist`	LValueRatio.lean	39–40
Definition 3.1	`Data.twist_one_spec`	LValueRatio.lean	42–43
Definition 3.3	`Data.Pair.ratio`	LValueRatio.lean	47–48
Lemma 5.2	`Data.exists_squareLaw_grid`	RowLadder.lean	16–34
Proposition 5.3	`Data.descent_past_gridPathMap`	RowLadder.lean	40–62
Lemma 6.5	`Data.iotaPlus_adjoint_weightedChain`	RowsReached.lean	21–34
Lemma 6.5	`Data.gridLevelForms_eq_map_iotaPlus`	RowsReached.lean	39–56
Proposition 6.7	`eq_zero_of_petersson_restrict_eq_zero`	RowsReached.lean	67–135
Corollary 6.8	`Data.eq_zero_of_orthogonal_gridLevelForms`	RowsReached.lean	142–162
Lemma 6.6	`Data.exists_isSteep`	SteepParameters.lean	13–29

Three statements as they are written in the artifact.

```
lemma Data.crude_rows : ∃ c : _ → ℂˣ, D.CostsSquares (2 * D.m * D.m) (c ·) ∧ span ℂ {y |
    (∀ θ l [Fact l.Prime] ι, l.Coprime θ.levelNorm →
  (c θ / c 1 : ℂ) • D.iotaAtLevel θ y ∈ LevelPlaces.LevelForm.integralForms (l := l) ι) ∧
  ∀ (σ : ℂ ≃+* ℂ) g, weylNormSq (σ ∘ (D.ι .one y).1 g) ≤ exp (D.x₁ ^ 2 * D.b ^ 2)} = ⊤

def Data.HasRoute {i j} (h : i ≤ D.m) (h' : j ≤ D.m) k := ∃ d : _ → ℂˣ, D.CostsSquares k (d ·) ∧ ∀ n
  Λ, (∀ θ, IsTidy n (D.path i) (Λ θ)) → ∃ Φ, (∀ θ, IsTidy (n + k) (D.path j) (Φ θ)) ∧ ∀ T : _ → ℂ,
  (∀ g, ∃ β, ∀ x, gridPeriod Φ x h' g = β * T x) → ∀ g, ∃ β, ∀ x, gridPeriod Λ x h g =
  β * (T x / d x.1)

@[simp] def gridPeriod (Λ : ∀ θ, D.GridDatum θ) (y : _ × _ × _) h g := Λ y.1 fun S ↦
  diagonalPeriodForm y.1 h g
  (gridPathMap S _ j j (levelIndex y.1 false 0 D.m • y.2.1)) (gridPathMap S _ j j y.2.2)
```

Legend: `D` is the data D; `c` is c; `CostsSquares` is the cost in squares; `iotaAtLevel` is $\Theta^{\mathrm{lev}}_{D,\vartheta}$ and `D.ι .one` is $\Theta^{\circ}_{D,1}$; `θ.levelNorm` is $\mathrm{N}\,Q(\vartheta)$; `Λ`, `Φ` are families of data $\mathcal{L}$, $\mathcal{L}'$; `IsTidy` and `D.path` are tidiness and the path of a stage; `z` is the flag o; `S` is a tuple S; `levelIndex` at the node $(0, m)$ of the grid false is $\mathrm{i}^{\mathrm{gr}}_D(\vartheta)$; the embedding `ι` of the first text is the ring homomorphism $j\colon \mathbb{C} \to \mathbb{C}_\ell$ (an ℓ-adic place).

Notation

- P, D, $E(D)$, m, r, $k(D)$, $t(D)$, $f(D)$, $g(D)$, x, x_1, b, $b_1(D)$ parameters, admissible data and their field, the number of lifts, the ranks of rows and columns, the step of the ranks, the number of infinite places, the second level, the three scales, an exponent of the data
- ϑ, ϑ_1, $Q(\vartheta)$, $\mathrm{N}\,Q(\vartheta)$, θ, $\theta_{\ell,\iota}$ twists, their set of places and its norm; a fit family of twists, indexed by the pairs (ℓ, ι)
- τ; σ; ℓ, ι, j, $\mathbb{C}_\ell$; $\bar{\pi}_g$, π_{gr} an infinite place; an automorphism of $\mathbb{C}$; a prime, an isomorphism of $\mathbb{C}$ with an algebraic closure of the ℓ-adic numbers, a ring homomorphism from $\mathbb{C}$ to the completion $\mathbb{C}_\ell$ of that closure (an ℓ-adic place); the conjugate of the integer of norm $g(D)$, the generator of the grid
- o, o', $[o]$, d flags (false or true), the number 0 or 1 of a flag; a direction or a function of twists
- i, i', i_1, n, n', n_0, n_1, e natural numbers: indices of nodes, stages, offsets, orders of tidiness, costs
- $\mathbf{n}$, $|\mathbf{n}|$; $\mathfrak{p}$; $\mathbb{A}$, $\mathbb{C}$, $\mathcal{O}_E$ a finite index set and its size; a finite place; adeles, complex numbers, integers
- $\mathrm{GI}_D(o; i, n)$, $\mathrm{num}_D(o)$, gbe_D, $\mathrm{gde}_D(i)$ the index set of a node, the numbering of its base, the listing of its indices, the diagonal bijection (Definition 3.1)

$\mathrm{sg}_D(o,n';i,n)(\mathsf{x})$, $\mathrm{par}_D(o,n';i,n)(\tau)$, $\mathrm{sgY}_D(n)(\mathsf{x})$, $\mathrm{parY}_D(n)(\tau,\mathsf{x})$, $\mathrm{gw}_D(o',i';\mathbf{n})(\tau)$ signs, doubled parameters and weights of the grids (Definition 3.1)

$\mathsf{V}_{i,n}(D)$, $\mathsf{Z}_{i,n}(D)$, $\mathsf{Y}_n(D)$, $\mathsf{W}_i(D)$, V, $\mathsf{V}\oplus\langle 1\rangle$; $\mathrm{U}(\mathsf{V})$, $d\mathsf{x}$ the spaces of the two grids, of the chain and of the auxiliary pairs; a hermitian space and its sum with a positive line; the unitary group, integration for an invariant measure

$\mathfrak{m}^{\mathrm{gr}}(\vartheta)$, $\mathfrak{m}_o^{\triangle}(\vartheta)$, $\mathrm{red}_{\vartheta,o}$, R_ϑ; $\mathfrak{m}_D(\vartheta,e)$ the moduli of the grid and of the triangle, the reduction between their residue rings, the residue ring of the grid; the modulus of the chosen places to the power e times $Q(\vartheta)$

ϑ^n, $\mathrm{Tw}^m(D)$, $\vartheta^{[n]}$, $\mathrm{cch}_D(\vartheta,i)$, $\vartheta^{\mathrm{t}}_{\mathfrak{p}}$ powers of tame characters, the m-fold twists, the characters of a chain, a local tame part (Definition 3.3)

p, ω_{p}, e_{p}, $\mathrm{rat}(\mathsf{p},\vartheta,\pm)$; ω, ω^{t}, μ a pair of the data, its character and point, the ratio of L-values; a character of $\mathrm{U}(1)(\mathbb{A})$, its tame part, weights

$L^{\Sigma\cup\bar\Sigma}(s,\omega^{\mathrm{bc}})$; ω^{bc}; Σ, $\bar\Sigma$, Σ_D; s the Hecke L-function of the base change without the Euler factors at Σ and at the conjugate places; the base change of ω; a set of finite places, the set of their conjugates, the chosen places; a real number ≥ 1

$\mathrm{nst}(D)$, $\mathrm{ordW}(i,\mathsf{x})$, $\mathrm{chw}(i,\tau)$, $\mathrm{kw}(i,\tau)$ a size, an order of a basis, a lowest K-type, a kernel weight (Definition 3.4)

$\mathfrak{l}$, $\mathfrak{a}$ an L-value datum, an auxiliary pair

S, S_{x}, $\mathrm{Tup}(\vartheta)$; $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathrm{Dat}(\vartheta)$ tuples of matrices on the edges; data and families of data

p, p', q, v, Q, U, R, r; b, $\mathcal{E}$, $\mathcal{E}'$ nodes; an edge, finite sets of edges

$\mathrm{MuEdge}(\mathsf{p},\mathsf{q})$, $\chi^{\mathrm{ed}}_\vartheta(\mathsf{b})$, $\mathrm{Adm}(\mathsf{g})$, $\mathrm{act}(\mathsf{g},\mathsf{S})(\mathsf{b})$, $\mathrm{ecp}(\mathsf{g},\mathcal{E})$, $\mathrm{tb}(\vartheta)$, $\mathrm{LInv}(\mathsf{S},\mathcal{E})$ edges that carry the twist, edge characters, admissible families, their action, products of edge characters, the tuple bound, left inverses (Definition 4.1)

$\mathrm{Short}(n,\mathcal{E};\mathcal{L}_0)$, $\mathrm{bind}(\mathcal{L}_0;\mathrm{k},\mathrm{t})$, $\mathrm{Tidy}(n,\mathcal{E};\mathcal{L}_0)$, $\mathrm{LTidy}(n,\mathcal{E};\mathcal{L}_0)$; $\mathcal{W}$, $\mathcal{X}$, c_{x}, k, t short sums, binding, tidy and level-tidy data (Definition 4.1); the sets, coefficients, weights and tuples in them

$\mathrm{ve}(o,i,n)$, $\mathrm{re}(o,i,n)$, $\mathrm{leg}(o,n)$, $\mathrm{gp}(o,i,n)$, $\mathrm{path}(n)$ edges and paths (Definition 4.2)

$\mathrm{ld}_{\mathsf{S}}(o,n)(F)$, $\mathrm{pm}_{\mathsf{S}}(o;i,n)(F)$, $\mathrm{dp}_{\vartheta_1}(i;\mathsf{h})$, $\mathrm{per}(\mathcal{L};(\vartheta,F,F');i,\mathsf{h})$ the maps along a path, the diagonal period, the period of a stage (Definition 4.2)

$\mathrm{Route}_D(i,i';e)$; T, b, b′, t_0 routes (Definition 4.3); a function of triples, constants of proportionality, a unit

F, F', y, y', y_n, y_o; $\langle y,y'\rangle_{\mathrm{Pet}}$; l, l_{S}; G, $\mathcal{G}$, $\mathcal{G}'$, $\mathcal{G}_{\mathsf{j}}$; B_0, B_1 forms; the Petersson product; linear functionals; functions of tuples and of matrices; a bound, a bilinear form

g, h, k, x, u, x, y, w_τ; e, e_1, e_2, e_3; j, $\mathcal{J}$, $\mathcal{I}_0$ points of groups, families of matrices, matrices and indices; identifications and bijections of index sets; an index, a set, a finite index set

$\mathrm{SqLaw}(\chi,\mathsf{e};\mathrm{c};\mathcal{G},\mathcal{G}')$, $\mathrm{Cost}_D(n;\mathrm{c})$; χ, e_{R}; c, C, $\varkappa$, a, a_1, $\mathrm{d}_{\mathrm{end}}$, w_0 the square law and costs of the second note; a multiplicative and an additive character of a finite ring; constants

$\Theta^{\mathrm{lev}}_{D,\vartheta}$, $\Theta^{\circ}_{D,1}$, ch_ϑ, $w_{D,i}(\tau)$ the composite of the m lifts at principal level, the chain for the trivial twist, the chain as functions, the weights of the lifts

$\mathrm{i}_D(\vartheta)$; $\mathrm{i}^{\mathrm{gr}}_D(\vartheta)$, $\mathrm{i}^{\mathrm{gr}}_{D,n}(\vartheta)$; $\mathrm{ord}_{D,o}(\vartheta,n)$ the index of the last level of the chain, $\#\,\mathrm{GL}(\mathcal{O}_E/\mathfrak{m}_D(\vartheta,1))$ in the rank of $\mathsf{Y}_m(D)$; the index of the level of the node $(0,m)$, of the node $(0,n)$, of the grid false, $\#\,\mathrm{GL}(\mathcal{O}_E/\mathfrak{m}^{\mathrm{gr}}(\vartheta))$ in the rank of that node; the triangle orders (second note)

ξ, G a coordinate vector of a weight, the Gram matrix of the weight representation (squared Weyl norm $\xi^*\mathrm{G}\xi$)

$\mathrm{lift}_{\mathsf{x}}$, $\mathrm{des}_{\mathsf{x}}$, $\mathrm{des}^{\vartheta}_{o,n}$, $\mathcal{A}^{\pi_{\mathrm{gr}}}_n(D)$, r, r_0 lifts and descents along an edge with a matrix, the descent from $(0,m)$ to $(0,n)$, forms of principal level π_{gr}, the restriction maps

$\mathrm{EI}(o,i)$, $\mathrm{ewd}(o)$, $\mathrm{ede}(o)$, $\mathrm{elev}(\vartheta_1,o,i)$, $\mathrm{eker}(\vartheta_1,o,i)$, $\mathrm{ebp}(\vartheta_1,\mathsf{h})$, alg_o, $\mathrm{Fix}(\vartheta_1,y)$, $\mathrm{edat}(\mathsf{S},o,i)$, $\mathrm{ech}(\mathsf{S},o,i)(F)$, $\mathrm{eper}(\mathcal{L}_0;y)$, $\langle\ ,\ \rangle_{\mathrm{end}}$ the objects of the end nodes and the end period (Definition 7.1)

$\mathcal{S}$, F, σ_1, M_1, R_1, n_1; A, L, K_0, B, M, l_τ, l'_τ, $\mathrm{c}_{\mathrm{cusp}}$ the property, field, homomorphism and numbers of Proposition 7.3; the numbers and parameters of Proposition 6.7

READBACK NOTE 26

Good forms and rows: equal Hecke traces on unitary groups that differ at the real places

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The rows of the proof are cusp forms on an indefinite unitary group, cut out of the forms of principal level at the chosen primes by one idempotent of the group algebra of a finite group: the *good forms*. This note defines these objects: the hermitian spaces $Y_i(D)$ and $W_i(D)$ attached to the data D, the forms of principal level on them, the representation of the finite group, the good forms, the rows and their dimension. Its theorem is a transfer between unitary groups that differ only at the real places: if on two such spaces every Hecke eigensystem meets at most one primitive label at each chosen prime, then every Hecke operator away from the chosen primes has the same trace, hence the same determinant, on the good forms of the two spaces. The hypothesis holds for $Y_0(D)$ and for every $W_i(D)$, so the rows have the dimension of the good forms on each $W_i(D)$. The proof subtracts from the good idempotent its cubic twists, which leaves group elements with an irreducible characteristic polynomial at a chosen prime; for those a comparison of trace formulas is available.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles and that Lean's kernel accepted the proofs; an independent replay is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note treats 37 of the 46 declarations of the unit *Good idempotents, labels and the transfer of traces*; the other declarations, on labels over finite fields, are the subject of the note *Primitive labels*. Proved in full here, from the inputs: the transfer of traces with its three corollaries (Theorems 1 and 2), the reduction of Theorem 3 to its inputs, and the lemmas of §§4–7. Taken for granted and stated as Inputs 1 to 8: a comparison of trace formulas, finiteness of spaces of cusp forms and facts on lattices and on finite general linear groups from the artifact's background library; and results of other units of the argument, among them the uniqueness of primitive labels (unit on Galois representations) and the injectivity and equivariance of theta lifts. Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 70 declarations.

1 The statements

The objects are defined in §2. In brief: D is admissible data (Setting 2.1); $\mathrm{Prin}(D,\mathcal{V})$ is the space of forms of principal level at the chosen primes on a hermitian space $\mathcal{V}$, $\mathrm{Good}(D,\mathcal{V})$ its subspace of good forms, $\mathcal{H}(D)$ the algebra of Hecke operators away from the chosen primes, and $\mathrm{Prin}(D,\mathcal{V})[p]$ the part of $\mathrm{Prin}(D,\mathcal{V})$ with Hecke eigensystem p. Rigidity of labels is Definition 2.12.

Theorem 1 (equal traces). *In Setting 2.1, let $\mathfrak{s}$, $\mathfrak{s}'$, $\mathcal{V}$, $\mathcal{V}'$ be as in Setting 2.2. Suppose that labels are rigid for D on $\mathcal{V}$ and that labels are rigid for D on $\mathcal{V}'$. Then for every $T \in \mathcal{H}(D)$, the trace of T on $\mathrm{Good}(D,\mathcal{V})$ equals the trace of T on $\mathrm{Good}(D,\mathcal{V}')$.*

LEAN `Data.trace_heckeOnGood_eq`

Theorem 2 (determinants, eigensystems, dimensions). *In Setting 2.1, let $\mathfrak{s}$, $\mathfrak{s}'$, $\mathcal{V}$, $\mathcal{V}'$ be as in Setting 2.2, and suppose that labels are rigid for D on $\mathcal{V}$ and that labels are rigid for D on $\mathcal{V}'$.*

(a) *For every $T \in \mathcal{H}(D)$, the determinant of T on $\mathrm{Good}(D,\mathcal{V})$ equals the determinant of T on $\mathrm{Good}(D,\mathcal{V}')$.*

(b) *Let p be an eigensystem. If $\mathrm{Good}(D,\mathcal{V}') \cap \mathrm{Prin}(D,\mathcal{V}')[p] = 0$, then $\mathrm{Good}(D,\mathcal{V}) \cap \mathrm{Prin}(D,\mathcal{V})[p] = 0$.*

Moreover, in Setting 2.1, for every integer i with $1 \le i \le m(D)$:

$$\dim \mathrm{Row}(D) = \dim_{\mathbb{C}} \mathrm{Good}(D, W_i(D)).$$

LEAN `Data.det_heckeOnGood_eq` · `Data.goodForms_inf_eigenPart_eq_bot` · `Data.finrank_rows_eq_finrank_goodForms`

Theorem 3 (rigid labels). *In Setting 2.1: labels are rigid for D on $Y_0(D)$; and for every integer i with $1 \le i \le m(D)$, labels are rigid for D on $W_i(D)$.*

LEAN `Data.hasRigidLabels_spaceY_zero` · `Data.hasRigidLabels_spaceW`

The space $Y_0(D)$ carries the rows, $\mathrm{Row}(D) = \mathrm{Good}(D, Y_0(D))$; it has $m(D)/2$ negative signs at each infinite place. The space $W_i(D)$ is the source of the i-th single theta lift and has $m(D) - i$ negative signs. Both have rank $r(D)$ and the same finite adelic group, so Theorems 1 and 2 compare them. In Chapter VII the equality of determinants turns the volume of the chain of $m(D)$ lifts on the rows into the product of the volumes of the single lifts, and the equality of dimensions lets one count rows on $W_i(D)$.

2 The setting

Conventions. The natural numbers are $0, 1, 2, \ldots$; differences $n - n'$ of natural numbers that the formal text truncates at zero are marked as such. The parameters $P = (N_0, d)$ of the proof are fixed: N_0 is a positive multiple of 18480 and d a prime with $d \equiv N_0/2 - 1$ modulo N_0. For a number field E with ring of integers O_E: a *prime* of E is a non-zero prime ideal, also called a finite place; $\mathbb{F}_v$ is the residue field of a prime v, E_v the completion and O_v its ring of integers; $\mathbb{A}_E$ and $\mathbb{A}_{E,f}$ are the adeles and the finite adeles, and $\hat{O}_E$ is the ring of integral finite adeles. For a CM field E, $z \mapsto \bar{z}$ is complex conjugation, on E, on its adeles and on its primes; a prime v is *split* if $\bar{v} \neq v$. For a finite set I and a commutative ring, GL_I of that ring is the group of the invertible matrices over it with rows and columns indexed by I, and g^* is the conjugate transpose of a matrix g over a ring with conjugation. $\mathbb{C}[G]$ is the complex group algebra of a group G, and a complex representation of G is also a module over it. A *Boolean* is one of the two truth values. Labels, primitive labels and the idempotents $e_{I,c}$ and $e^{\otimes}_{I,\mathbf{c}}$ of $\mathbb{C}[\mathrm{GL}_I(\mathbb{F})]$ and of the group algebra of a product, for a finite field $\mathbb{F}$, are those of the note *Primitive labels*, and its sets of good tuples are written as there. Letters of the handout's overview that are not displayed with the data D as an argument (among them t, Υ, R, f, β, ν and n) have, in this note, only the meaning given in its notation list.

Setting 2.1 (the data)**.** D is admissible data for the parameters P, in the sense of Chapter III of the handout (Setting 1 of its overview). The following objects attached to D are defined in Chapter III and in the unit on the seesaw; they are recalled here and used throughout.

(i) The CM field $E = E(D)$, abelian over $\mathbb{Q}$; it contains a primitive root of unity of order 5040 and has an even number of infinite places. The letter τ denotes an infinite place of $E(D)$.

(ii) Natural numbers $m(D)$, a power of 2, and $r(D)$, a power of 3 greater than 1, with $\sqrt{x(D)} \leq m(D) \leq 2\sqrt{x(D)}$ and $x(D)^5 \leq r(D) \leq 3\,x(D)^5$, where the first scale $x(D)$ of the data is larger than every absolute constant that occurs in this note; the exponent $\mu(D)$ of the data, with $m(D)^{22} = 2^{\mu(D)}$; the numbers $k(D) = r(D) + 1 - m(D) \geq 1$, $t(D) = 5r(D)$, $\varrho(D) = m(D)/2 - 1$, $h_2(D) = r(D)^8$ and $h(D) = 2r(D)^8$; the ranks $n_i(D) = k(D) + i\,(t(D) + 1)$ for $i \geq 0$; and the integer $M_\omega(D) = (N_0 d)^{\mu(D)+6}$.

(iii) For every natural number i and every τ, the *weight* of Chapter III, the natural number

$$w_{D,i}(\tau) = N_1(D)\,\lceil w_D(\tau)/N_1(D)\rceil + i\,h(D) + h_2(D).$$

Here $w_D(\tau)$ is the nominal weight of the overview and $N_1(D)$ the order of the root of unity that generates $E(D)$ over $\mathbb{Q}(\sqrt{-d})$; the first term is the least multiple of $N_1(D)$ that is at least the nominal weight. By the definition, $w_{D,i'}(\tau) - w_{D,i}(\tau) = (i' - i)\,h(D)$; and $w_{D,i}(\tau) \geq h(D)$, which is not read off the formula: it is one of the comparisons of magnitudes of the data (Chapter III).

(iv) The level, a prime number $N(D)$, with an integer $\pi_N(D)$ of $E(D)$ such that $\pi_N(D)\,\overline{\pi_N(D)} = N(D)$.

(v) Two distinct primes $g_0(D)$, $g_1(D)$ and an integer $\pi_g(D)$ of $K = \mathbb{Q}(\sqrt{-d})$ with $\pi_g(D)\overline{\pi_g(D)} = g_0(D)g_1(D)$; all three are components of D. The *chosen primes*

are the primes of $E(D)$ that divide $\overline{\pi_g(D)}$; their set is $\mathrm{Ch}(D)$. No chosen prime is the conjugate of a chosen prime; in particular each is split. A chosen prime v divides exactly one of $g_0(D)$, $g_1(D)$, and $\mathbb{F}_v$ has $g_0(D)$ or $g_1(D)$ elements accordingly. The number $\#\mathbb{F}_v - 1$ is divisible by 9 and by $M_\omega(D)^2$, and $M_\omega(D) + 1 < \#\mathbb{F}_v$.

(vi) The *row index set* $I(D)$, a finite set with $r(D)$ elements, with a bijection num_D from $I(D)$ to $\{0, \dots, r(D)-1\}$; and the finite group

$$\mathsf{G}(D) = \prod_{v \in \mathrm{Ch}(D)} \mathrm{GL}_{I(D)}(\mathbb{F}_v).$$

(vii) For every prime v of $E(D)$, a multiplicative character $\omega_{D,v}$ of $\mathbb{F}_v$: of order $M_\omega(D)$ if v is chosen, trivial otherwise.

(viii) The *row parameter*: for an infinite place τ and a natural number n, the integer

$$\begin{aligned} \varkappa_D(\tau, n) &= \lfloor k(D)/2 \rfloor - 1 - n && \text{if } n + 1 < k(D), \\ \varkappa_D(\tau, n) &= h_2(D) - w_{D,\, n+2-k(D)}(\tau) && \text{otherwise.} \end{aligned}$$

(ix) *Tame characters.* A tame character ϑ of D is a continuous character of the norm-one ideles of $E(D)$, trivial on the norm-one elements of $E(D)$, together with a finite set $Q(\vartheta)$ of primes of $E(D)$ and a family $(\psi_{\vartheta,v})_v$ of multiplicative characters of the residue fields, extended by 0, its *tame part*. The character and the family are tied as in the note on the auxiliary pair: ϑ has weight 0 and tame part $(\psi_{\vartheta,v})_v$, that is, $\vartheta(z) = \prod_v \psi_{\vartheta,v}(z \bmod v)$ for every norm-one idele z whose finite part is integral; in particular ϑ is trivial at the infinite places. It is required that $\psi_{\vartheta,v} = 1$ for $v \notin Q(\vartheta)$, that no conjugate of a member of $Q(\vartheta)$ lies in $Q(\vartheta)$, and that the product of the norms of the members of $Q(\vartheta)$ is prime to $g_0(D)g_1(D)N(D)$. The trivial character, with $Q = \emptyset$, is one.

Hermitian spaces with a point, and forms. These notions belong to the background library; they are recalled as the library defines them. Let E be a CM field, I a finite set, and $\mathfrak{s} \colon I \to \{1, -1\}$ a family of signs. Put

$$\begin{aligned} G_\infty(\mathfrak{s}) &= \prod_\tau \mathrm{U}(\mathfrak{s}), \\ G_f(I) &= \{g \in \mathrm{GL}_I(\mathbb{A}_{E,f}) : g^* g = 1\}, \qquad K_0(I) = G_f(I) \cap \mathrm{GL}_I(\hat{O}_E), \end{aligned}$$

the product over the infinite places of E, where $\mathrm{U}(\mathfrak{s})$ is the group of the complex matrices g with $g^* \,\mathrm{diag}(\mathfrak{s})\, g = \mathrm{diag}(\mathfrak{s})$. For a prime v and $g \in G_f(I)$, $g_v \in \mathrm{GL}_I(E_v)$ is the component at v; an element of $K_0(I)$ has a reduction in $\mathrm{GL}_I(\mathbb{F}_v)$. For a split prime v and $k \in \mathrm{GL}_I(E_v)$, the element $k^{(v)} \in G_f(I)$ has component k at v and 1 at the primes other than v and $\bar{v}$.

A *space over E with index set I and signs $\mathfrak{s}$* is a pair $\mathcal{V} = (J, \mathbf{x})$ of a matrix J over E and an element $\mathbf{x} \in \mathrm{GL}_I(\mathbb{A}_E)$, both indexed by I, such that $\mathbf{x}^* J \mathbf{x}$ is $\mathrm{diag}(\mathfrak{s})$ at every infinite place and 1 at the finite places. So J is a hermitian form whose signature at each infinite place is that of $\mathfrak{s}$, $\mathbf{x}$ identifies its unitary group over the adeles with $G_\infty(\mathfrak{s}) \times G_f(I)$, and the rational points become the discrete subgroup $\Gamma_\mathcal{V} = \mathbf{x}^{-1}\mathrm{U}(J)(E)\,\mathbf{x}$. The space is *standard* if all signs are 1 and $\mathbf{x} = 1$. The *diagonal space* of the signs $\mathfrak{s}$ has $J = \mathrm{diag}(\mathfrak{s})$ and a diagonal point, with entry 1 where the

sign is 1 and, where it is -1, a fixed idele that is 1 at the infinite places and whose product with its conjugate is -1 at the finite places.

Let $\epsilon\colon G_\infty(\mathfrak{s}) \to \mathbb{C}$ be a function (a *type*). A *form of type* ϵ *on* $\mathcal{V}$ is a bounded continuous function $F\colon G_\infty(\mathfrak{s}) \times G_f(I) \to \mathbb{C}$ that is invariant under left translation by $\Gamma_{\mathcal{V}}$ and under right translation by an open subgroup of $G_f(I)$, and that satisfies $\int_{G_\infty(\mathfrak{s})} \epsilon(u)\, F(g \cdot (u,1))\, du = F(g)$ for all g if ϵ is integrable. The group $G_f(I)$ acts on these forms by right translation, $(R(g)F)(g') = F(g'g)$, and they carry the Petersson inner product $\langle F, F' \rangle = \int_{\Gamma_{\mathcal{V}} \backslash G} \overline{F} F'$, for which R is unitary. Note that $G_f(I)$ does not depend on the space: it acts on the forms of every space with index set I.

For an integer vector $l = (l_\iota)_{\iota \in I}$, the library defines a function ϵ_l on $\mathrm{U}(\mathfrak{s})$: the formal degree times the complex conjugate of the canonical matrix coefficient of the discrete series with Harish-Chandra parameter $l/2$. The vector l *has gap* H *for* $\mathfrak{s}$, for a real number H, if its entries are distinct, every $l_\iota + \#I - 1$ is even, and $|l_\iota - l_{\iota'}| \geq 2H$ whenever $\mathfrak{s}_\iota \neq \mathfrak{s}_{\iota'}$. A function ϵ on $G_\infty(\mathfrak{s})$ is a *cusp kernel of order* B if $\|u\|^n \epsilon(u)$ is integrable for $n \leq B+1$, $\|u\|^B \epsilon(u)$ is bounded, and $\alpha * \epsilon = \epsilon$ for some continuous function α of compact support; here $\|u\|$ is the product of the operator norms of the components of u.

In Setting 2.1, for signs $\mathfrak{s}$ on $I(D)$, the *type of the data* is the function on $G_\infty(\mathfrak{s})$

$$\epsilon_{D,\mathfrak{s}}(u) = \prod_\tau \epsilon_{l_D(\tau)}(u_\tau), \qquad l_D(\tau) = \big(2\,\varkappa_D(\tau, \mathrm{num}_D(\iota))\big)_{\iota \in I(D)}.$$

The *principal congruence subgroup at the chosen primes*, $K_g(D) \subset K_0(I(D))$, consists of the elements whose reduction at every chosen prime is 1. For $\gamma \in \mathsf{G}(D)$, $\tilde{\gamma} \in K_0(I(D))$ is a lift that the library chooses: its reductions at the chosen primes are the components of γ, and its components at the primes that are neither chosen nor conjugate to a chosen prime are 1.

Setting 2.2 (two spaces)**.** In Setting 2.1: $\mathfrak{s}$ and $\mathfrak{s}'$ are families of signs on $I(D)$, $\mathcal{V}$ is a space over $E(D)$ with index set $I(D)$ and signs $\mathfrak{s}$, and $\mathcal{V}'$ one with signs $\mathfrak{s}'$; and for every infinite place τ the vector $l_D(\tau)$ has gap $h(D)$ for $\mathfrak{s}$ and has gap $h(D)$ for $\mathfrak{s}'$.

The spaces.

Definition 2.3 (index sets and their bijections)**.** (a) A *shape* is a triple $\Sigma = (\varrho, \mathcal{X}, o)$ of a natural number, a finite set and an element of that set; its index set is the disjoint union $\{\mathsf{b}_0, \dots, \mathsf{b}_{\varrho-1}\} \sqcup \mathcal{X} \sqcup \{z\} \sqcup \{\mathsf{b}^*_0, \dots, \mathsf{b}^*_{\varrho-1}\}$. The bijection bes_Σ from $\mathcal{X} \sqcup \{0, \dots, 2\varrho\}$ to this index set is the identity on $\mathcal{X}$ and sends n to b_n for $n < \varrho$, ϱ to z, and n to $\mathsf{b}^*_{2\varrho-n}$ for $n > \varrho$.

(b) In Setting 2.1, let i be a natural number. $\mathcal{X}_i(D)$ is a set with $n_i(D)$ elements, one of them marked as the last, and $I_i(D) = I(D) \sqcup (\{1, \dots, i\} \times \{0, \dots, t(D)-1\})$; the row index set $I(D) = I_0(D)$ is the index set of the shape $\Sigma(D) = (\varrho(D), \mathcal{X}_0(D), \mathrm{last})$. Suppose $i < m(D)$. Then $\mathrm{ix}_{D,i}^{i<m(D)}$ is a bijection from $\mathcal{X}_i(D) \sqcup \{0, \dots, 2\varrho(D) - i\}$ to $I_i(D)$: for $i = 0$ it is $\mathrm{bes}_{\Sigma(D)}$, and for $i \geq 1$ it is a bijection that the formal text chooses, the two sets having the same number of elements.

LEAN `Shape.besselIndexEquiv` · `Data.indexEquivY`

Lemma 2.4. *In Setting 2.1:* $m(D) = 2\varrho(D) + 2$*; and*

$$\exp(-4)\sqrt{x(D)} \leq \varrho(D) \leq \exp(2)\sqrt{x(D)}.$$

Proof. The number $m(D)$ is even and at least 2, and $\varrho(D) = m(D)/2 - 1$. Now $x(D)$ is large and

$$\sqrt{x(D)} \le m(D) \le 2\sqrt{x(D)},$$

so $m(D)/4 \le \varrho(D) \le m(D)$, which gives the bounds. □

LEAN `Data.m_eq_two_mul_varrho_add_two` · the bounds are an instance without a name, in the form in which the formal text compares magnitudes of the data

Definition 2.5 (ideles)**.** In Setting 2.1:

(a) $\hat{\pi}_g(D)$ is a finite idele of $E(D)$, chosen by the library, that is integral, has component 1 at every prime not dividing $\overline{\pi_g(D)}$, and generates the ideal $\overline{\pi_g(D)}\hat{O}_E$.

(b) $\alpha_g(D) = \alpha(D)\big/\overline{\hat{\pi}_g(D)}$, where $\alpha(D)$ is the finite idele of the Weil data of $E(D)$ (Chapter III).

(c) $\hat{\pi}_N(D)$ is the principal finite idele of the element $\pi_N(D)$ of $E(D)$.

(d) For a natural number i, $\mathrm{sc}_i(D) = \left(\alpha_g(D)\big/\overline{\alpha_g(D)}\right)^{\lfloor (m(D)-i+1)/2 \rfloor}$, with $m(D) - i$ truncated at zero: a finite idele of norm one, equal to 1 for $i \ge m(D)$.

LEAN `Data.chosenIdele` · `Data.ideleA` · `Data.piIdele` · `Data.latticeScale`

Lemma 2.6. *In Setting 2.1: $N(D) = \hat{\pi}_N(D) \cdot \overline{\hat{\pi}_N(D)}$ as finite ideles; for every prime v of $E(D)$ the component of $\hat{\pi}_N(D)$ at v is a uniformizer if v divides $\pi_N(D)$ and a unit otherwise; and if v divides $\pi_N(D)$, then $\bar{v}$ does not.*

Proof. The first assertion is $\pi_N(D)\overline{\pi_N(D)} = N(D)$. The component of $\hat{\pi}_N(D)$ at v is the image of $\pi_N(D)$, so the second and the third assertion are Input 8(e). (They hold because the prime $N(D)$ is unramified in $E(D)$: if v^2 or $v\bar{v}$ divided $\pi_N(D)$, then v^2 would divide $N(D)$.) □

LEAN `Data.piIdele_spec`

Two matrices of the background are needed to state the next lemma. Let β be a bijection from $\mathcal{X} \sqcup \{0, \dots, s-1\}$ to a finite set I', where $\mathcal{X}$ is a finite set with a marked last element and s a natural number. The *graded Gram matrix* $\mathrm{Gr}(\beta, N)$, for N in a commutative ring, is the matrix indexed by I' that is the identity on $\beta(\mathcal{X})$, pairs $\beta(n)$ with $\beta(n')$ to 1 if $n + n' = s - 1$ and $n \ne n'$, to $-N$ if $n = n'$ and $2n = s - 1$, and has all other entries 0. The *graded lattice basis* $\mathrm{Lb}(\beta, a, \pi, u)$, for units a, π and an element u of norm one of a ring with conjugation, is the matrix indexed by I' that for even s is diagonal, with entry 1 on $\beta(\mathcal{X})$, $u\,a^{-1}$ at $\beta(n)$ for $2n < s$ and $u\,\bar{a}$ for $2n \ge s$; for odd s it is diagonal with entry 1 on $\beta(\mathcal{X})$ except $\bar{\pi}^{-1}$ at the marked element, and u on the $\beta(n)$, plus the single entry $u\,(\pi\bar{\pi})^{-1}$ in the row of the middle index $\beta((s-1)/2)$ and the column of the marked element. The signs $\mathfrak{s}^Y_{D,i}$ on $I_i(D)$ are -1 at the $\iota \in I(D)$ for which $\mathrm{num}_D(\iota) - k(D) - i$ is non-negative and even, and 1 elsewhere; the signs $\mathfrak{s}^W_{D,i}$ on $I(D)$ are -1 at the ι with $\mathrm{num}_D(\iota) \ge k(D) + i - 1$ (truncated), and 1 elsewhere.

Lemma 2.7 (existence of the spaces of the chain)**.** *In Setting 2.1, let i be a natural number. There is a space $\mathcal{V}$ over $E(D)$ with index set $I_i(D)$ and signs $\mathfrak{s}^Y_{D,i}$ such that:*

(a) *if $m(D) \le i$, then $\mathcal{V}$ is standard;*

(b) *if* $i < m(D)$*, then the matrix of* $\mathcal{V}$ *is* $\mathrm{Gr}(\mathrm{ix}_{D,i}^{i<m(D)}, N(D))$*, and there is* $k \in \mathrm{GL}_{I_i(D)}(\hat{O}_E)$ *whose reduction at every chosen prime is* 1 *such that the finite part of the point of* $\mathcal{V}$ *is* $\mathrm{Lb}(\mathrm{ix}_{D,i}^{i<m(D)}, \alpha_g(D), \hat{\pi}_N(D), \mathrm{sc}_i(D)) \cdot k$.

LEAN `Data.exists_spaceY`

The proof is given in §4.

Definition 2.8 (the two families of spaces)**.** In Setting 2.1, let i be a natural number. $Y_i(D)$ is a space as in Lemma 2.7, chosen once and for all. $W_i(D)$ is the diagonal space over $E(D)$ of the signs $\mathfrak{s}_{D,i}^W$ on $I(D)$.

LEAN `Data.spaceY` · `Data.spaceW`

So $Y_i(D)$ has rank $r(D) + i\,t(D)$ and $\lfloor (m(D) - i)/2 \rfloor$ negative signs (truncated), and $W_i(D)$ has rank $r(D)$ and $m(D) - i$ negative signs (truncated), at every infinite place.

Lemma 2.9. *In Setting 2.1, the spaces* $Y_{m(D)}(D)$ *and* $W_{m(D)}(D)$ *are standard.*

Proof. The first is Lemma 2.7(a). For the second, $k(D) + m(D) - 1 = r(D)$, so no $\iota \in I(D)$ has $\mathrm{num}_D(\iota) \geq k(D) + m(D) - 1$: all signs are 1, and the diagonal space of these signs is standard. □

LEAN `Data.isStd_spaceY` · for $W_{m(D)}(D)$, an instance without a name

Forms of principal level, good forms, rows.

Definition 2.10 (principal forms and the action of $\mathsf{G}(D)$)**.** In Setting 2.1, let $\mathfrak{s}$ be a family of signs on $I(D)$ and $\mathcal{V}$ a space over $E(D)$ with index set $I(D)$ and signs $\mathfrak{s}$.

(a) $\mathrm{Prin}(D, \mathcal{V})$ is the space of the forms of type $\epsilon_{D,\mathfrak{s}}$ on $\mathcal{V}$ that are fixed by $R(k)$ for all $k \in K_g(D)$.

(b) $\mathrm{rep}_{D,\mathcal{V}}$ is the representation of $\mathsf{G}(D)$ on $\mathrm{Prin}(D, \mathcal{V})$ given by $\mathrm{rep}_{D,\mathcal{V}}(\gamma) = R(\tilde{\gamma})$.

LEAN `Data.principalForms` · `Data.principalRep`

Definition 2.11 (the good idempotent, good forms, rows)**.** In Setting 2.1, the *good idempotent* is the element of $\mathbb{C}[\mathsf{G}(D)]$

$$e_0(D) = \sum_{\mathbf{c}} e^{\otimes}_{I(D),\mathbf{c}},$$

the sum over the families $\mathbf{c} = (c_v)_{v \in \mathrm{Ch}(D)}$ of monic polynomials $c_v \in \mathbb{F}_v[X]$ of degree $r(D)$ such that, for each of the two primes $g_0(D)$ and $g_1(D)$, the family of the c_v with v above that prime belongs to $\mathrm{Good}_{M_\omega(D),r(D)}$. (This definition belongs to the unit on the seesaw.) Let $\mathfrak{s}$ be a family of signs on $I(D)$ and $\mathcal{V}$ a space over $E(D)$ with index set $I(D)$ and signs $\mathfrak{s}$. Then

$$\mathrm{Good}(D, \mathcal{V}) = \mathrm{rep}_{D,\mathcal{V}}(e_0(D))\,\mathrm{Prin}(D, \mathcal{V}),$$

the space of *good forms* on $\mathcal{V}$. The space of *rows* is $\mathrm{Row}(D) = \mathrm{Good}(D, Y_0(D))$, and $\dim \mathrm{Row}(D)$ is its complex dimension.

LEAN `Data.goodForms` · `Data.rows` · `Data.rowDim` · the good idempotent: `Data.e₀`

So a good family has, above each of $g_0(D)$ and $g_1(D)$, a label without a root of order dividing $M_\omega(D)$ at every chosen prime and a primitive label at one chosen prime at least. The rows of the overview of this handout are $\mathrm{Row}(D)$.

Hecke operators and labels (definitions of the unit on fit twists). In Setting 2.1, $\mathcal{H}(D)$ is the algebra of the families $T = (T_\mathcal{V})$, indexed by all spaces $\mathcal{V}$ over $E(D)$ with index set $I(D)$ and any signs, of endomorphisms $T_\mathcal{V}$ of $\mathrm{Prin}(D,\mathcal{V})$ for which there is one element $\sum_g c_g\, g$ of the group algebra of the subgroup of the $g \in G_f(I(D))$ with $g_v = 1$ at every chosen prime v, such that $T_\mathcal{V} = \sum_g c_g\, R(g)$ on $\mathrm{Prin}(D,\mathcal{V})$ for every $\mathcal{V}$. Every $T_\mathcal{V}$ commutes with $\mathrm{rep}_{D,\mathcal{V}}$, so it preserves $\mathrm{Good}(D,\mathcal{V})$; the trace and the determinant of T on $\mathrm{Good}(D,\mathcal{V})$ are those of this restriction. For a prime v of $E(D)$ that is split, with neither v nor $\bar{v}$ chosen, and a natural number n, the Hecke operator $\mathbb{T}^{v,n} \in \mathcal{H}(D)$ is the operator of the double coset of $\mathrm{diag}(\varpi,\dots,\varpi,1,\dots,1)$ with n entries ϖ, a uniformizer, in $\mathrm{GL}_{I(D)}(E_v)$ acting through $k \mapsto k^{(v)}$ (it is zero if n exceeds the rank); for the other pairs (v,n) put $\mathbb{T}^{v,n} = 0$. An *eigensystem* is a complex function p on the pairs (v,n) of a prime of $E(D)$ and a natural number, and

$$\mathrm{Prin}(D,\mathcal{V})[p] = \{y \in \mathrm{Prin}(D,\mathcal{V}) :\ \mathbb{T}^{v,n}_\mathcal{V}\, y = p(v,n)\, y \text{ for all } (v,n)\}.$$

For an eigensystem p and a chosen prime v, $\mathrm{Occ}(D,\mathcal{V},p,v)$ is the set of the polynomials $c \in \mathbb{F}_v[X]$ that are primitive labels of degree $r(D)$ and for which $e_{I(D),c}$, acting through $\mathrm{rep}_{D,\mathcal{V}}$ restricted to the factor $\mathrm{GL}_{I(D)}(\mathbb{F}_v)$ of $\mathsf{G}(D)$, does not kill $\mathrm{Prin}(D,\mathcal{V})[p]$.

Definition 2.12 (rigid labels)**.** In Setting 2.1, let $\mathfrak{s}$ be a family of signs on $I(D)$ and $\mathcal{V}$ a space over $E(D)$ with index set $I(D)$ and signs $\mathfrak{s}$. Then labels are rigid for D on $\mathcal{V}$ if for every eigensystem p and every chosen prime v the set $\mathrm{Occ}(D,\mathcal{V},p,v)$ has at most one element.

LEAN `Data.HasRigidLabels` · the notions it rests on: `Data.heckeAlgebra` · `Data.heckeOperator` · `Data.eigenPart` · `Data.occurringLabels` · `Data.heckeOnGood`

3 Inputs

Inputs from the background library. In Inputs 1 to 3, E is a CM field that contains a primitive root of unity of order 5040, I is a non-empty finite set, and spaces, forms and types are those recalled in §2.

Input 1 (comparison of traces on unitary groups that differ at the real places). *Suppose that E has an even number of infinite places. Let $\mathbf{K}$ be an open subgroup of $K_0(I)$ and $g \in G_f(I)$ an element such that every element of the coset $g\mathbf{K}$ has, at some prime v of E, a component in $\mathrm{GL}_I(E_v)$ whose characteristic polynomial is irreducible. Let B be a natural number with $B \geq 16\,[E:\mathbb{Q}]\,(\#I+1)^3$, let $l = (l(\tau))_\tau$ be a family of integer vectors indexed by I, one for each infinite place, and let H be a real number with $H \geq 500\,(\#I)^3 + 10\,(\#I)^5$. Then there is a complex number t with the following property. For every family of signs $\mathfrak{s}$ on I, every space $\mathcal{V}$ over E with index set I and signs $\mathfrak{s}$, and every finite orthonormal basis $(b_n)_n$ of the space of the forms of type $\epsilon = \prod_\tau \epsilon_{l(\tau)}$ on $\mathcal{V}$ that are fixed by $\mathbf{K}$: if ϵ is a cusp kernel of order B and every $l(\tau)$ has gap H for $\mathfrak{s}$, then $\sum_n \langle b_n, R(g)\,b_n\rangle = t$.*

Input 2 (cusp kernels; finiteness). (a) *Let $\mathfrak{s}$ be signs on I, B a natural number and $(l(\tau))_\tau$ integer vectors such that every $l(\tau)$ has gap $500\,(\#I)^3 + B$ for $\mathfrak{s}$. Then $\prod_\tau \epsilon_{l(\tau)}$ is a cusp kernel of order B.* (b) *If ϵ is a cusp kernel of some order on $G_\infty(\mathfrak{s})$, $\mathcal{V}$ a space with signs $\mathfrak{s}$ and $\mathbf{K}$ an open subgroup of $G_f(I)$, then the space of the forms of type ϵ on $\mathcal{V}$ fixed by $\mathbf{K}$ has finite dimension.*

Inputs from the background library, continued.

Input 3 (frames and lattices). *Let β be a bijection from $\mathcal{X} \sqcup \{0,\dots,s-1\}$ to a finite set I', as before Lemma 2.7.* (a) *Let $N > 0$ be an integer and $\mathfrak{s}$ a family of signs on I' with exactly $\lfloor (s+1)/2 \rfloor$ signs -1. Then there is a complex matrix y with $y^*\,\mathrm{Gr}(\beta,N)\,y = \mathrm{diag}(\mathfrak{s})$.* (b) *Let a, π be finite ideles of E, u a finite idele of norm one, and N an integer with $N = \pi\bar{\pi}$. Then for $B = \mathrm{Lb}(\beta,a,\pi,u)$ the matrix $B^*\,\mathrm{Gr}(\beta,N)\,B$ has entries in $\hat{O}_E$ and an inverse with entries in $\hat{O}_E$.* (c) *Every hermitian matrix b over $\hat{O}_E$ with an inverse over $\hat{O}_E$ has the form $k^*bk = 1$ for some $k \in \mathrm{GL}_{I'}(\hat{O}_E)$.*

Input 4 (finite general linear groups). *Let $(\mathbb{F}_v)_v$ be a finite family of finite fields and I a finite set.* (a) *For every finite set $\mathcal{C}$ of families $\mathbf{c} = (c_v)_v$ of polynomials, $\sum_{\mathbf{c}\in\mathcal{C}} e^{\otimes}_{I,\mathbf{c}}$ is an idempotent of the group algebra of $\prod_v \mathrm{GL}_I(\mathbb{F}_v)$ whose coefficients are constant on conjugacy classes.* (b) *Let χ be a multiplicative character of a finite field $\mathbb{F}$, $x_\chi \in \mathbb{F}^\times$ the unit that names it, $c \in \mathbb{F}[X]$ a polynomial of which x_χ is not a root, and $l \in \mathbb{F}^I$ a non-zero vector. Then $(\sum_h \chi(\det h)^{-1} h)\cdot e_{I,c} = 0$, the sum over the $h \in \mathrm{GL}_I(\mathbb{F})$ of the form $h = 1 + u\,l^{\mathsf{T}}$ with $u \in \mathbb{F}^I$.* (c) *If ρ is a representation of $\prod_v \mathrm{GL}_I(\mathbb{F}_v)$, $y_v \in \mathbb{C}[\mathrm{GL}_I(\mathbb{F}_v)]$ for all v, and for one index v_0 the element y_{v_0}, acting through the factor v_0, kills a vector, then $\bigotimes_v y_v$ kills that vector.* (d) *Over an algebraically closed field of characteristic 0, two square matrices S, S', possibly of different sizes, with $\mathrm{tr}(S^n) = \mathrm{tr}(S'^n)$ for all $n \geq 0$ have the same characteristic polynomial.*

LEAN `Space.exists_trace_eq` · `isCuspKernel_typeKernel` · `Space.finite_levelSpace` · `exists_frame_gradedGram` · `isUnimodular_gradedLatticeBasis_gram` · `exists_orthonormal_Ô` · `isCentralProj_sum_tupleIdem` · `reflectionSum_mul_labelIdem` · `piTensorElem_apply_eq_zero` · `charpoly_eq_mul_of_trace_pow_add`

On Input 1. It is the one deep input of this note, and its relation to the literature needs care. The library's file names as sources J.-P. Labesse, *Changement de base CM et séries discrètes* (2011), Théorème 5.4 and its proof, and J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula* (1989), Chapter 1, Lemmas 2.4 to 2.6; the catalogue of this handout first classed Input 1 as a special case of these. The second reading of this write-up compared the statements and left the identification open: Labesse's Théorème 5.1 is an identity of stabilized trace formulas and his Théorème 5.4 gives weak cuspidal base change and multiplicity one, with pseudo-coefficients as test functions; the lemmas of Arthur and Clozel are simple trace formulas for general linear groups. None of them states the equality of this trace for a fixed coset $g\mathbf{K}$, for integrable kernels with an explicit gap, and for all spaces at once. According to the headers of the library's two files on this comparison (I read the statement of Input 1 and not its proof), the statement is proved there by a comparison of simple trace formulas in which the test function is elliptic at a finite place, because of the irreducible characteristic polynomial, and is at the real places the integrable idempotent kernel of a very regular discrete series in place of a pseudo-coefficient; the rational elliptic classes of the two groups, which differ at the real places and are therefore not the same sets, are matched by changing their real signs with a global element that is locally a norm; and convergence is obtained by packing, without truncation and without endoscopy. So Input 1 is a statement of the formal library in the direction of known results, not yet a quotation of a published theorem. Input 2(b) is the finiteness of spaces of cusp forms in a special case (Harish-Chandra; the catalogue gives A. Borel's Park City lectures, 8.3 and 8.4, as the place, at second hand); Input 3(c) is a special case of the classification of unimodular hermitian lattices, with R. Jacobowitz, *Hermitian forms over local fields*, Amer. J. Math. 84 (1962), named by the catalogue and the place to be confirmed; Input 4(a) is a weaker consequence of Green's theorems, as in the note *Primitive labels*; Inputs 2(a), 3(a), 3(b), 4(b), 4(c), 4(d) are proved in the library without a named source.

Inputs from other units of the argument. In Setting 2.1 throughout.

Input 5 (the Hecke algebra; unit on fit twists). *Let $\mathcal{V}$ be a space over $E(D)$ with index set $I(D)$.* (a) *Every $T_{\mathcal{V}}$, $T \in \mathcal{H}(D)$, commutes with $\mathrm{rep}_{D,\mathcal{V}}(y)$ for all $y \in \mathbb{C}[\mathsf{G}(D)]$; so $\mathrm{rep}_{D,\mathcal{V}}$ preserves every $\mathrm{Prin}(D,\mathcal{V})[p]$.* (b) *If $\mathrm{Prin}(D,\mathcal{V})$ has finite dimension, it is the sum of the subspaces $\mathrm{Prin}(D,\mathcal{V})[p]$, and only finitely many of them are non-zero.* (c) *For every eigensystem p and every finite set $\mathcal{F}$ of eigensystems there is $\Pi \in \mathcal{H}(D)$ such that, for every space $\mathcal{V}$ and every $p' \in \mathcal{F} \cup \{p\}$, $\Pi_{\mathcal{V}}$ is the identity on $\mathrm{Prin}(D,\mathcal{V})[p']$ if $p' = p$ and zero on it if $p' \neq p$.*

Input 6 (at most one primitive label; unit on Galois representations). *Let $\mathfrak{s}$ be signs on $I(D)$ and $\mathcal{V}$ a space over $E(D)$ with index set $I(D)$ and signs $\mathfrak{s}$. Let I' be a non-empty finite set, $\nu\colon I(D) \to I'$ an injection, and L a linear map from $\mathrm{Prin}(D,\mathcal{V})$ to a space of algebraic modular forms, of some weight, level and level character, on the standard space with index set I'. Let p be an eigensystem, $(\Upsilon_w)_w$ a family of functions $\mathbb{C} \to \mathbb{C}$ indexed by the primes of $E(D)$, v a chosen prime, n_1 a natural number, $\beta\colon I' \to \{0,\dots,n_1\}$ a map, $\chi_1,\dots,\chi_{n_1}$ and ψ multiplicative characters of $\mathbb{F}_v$. Suppose:*

(i) *L is injective;*

(ii) *for every $y \in \mathrm{Prin}(D,\mathcal{V})[p]$ and every split prime w with neither w nor $\bar{w}$ chosen, $L(y)$ is fixed by the elements of $K_0(I')$ with component 1 off w and $\bar{w}$, and for every $X \in \mathbb{C}$ the operator $\sum_n (-X)^n (\#\mathbb{F}_w)^{n(n-1)/2}\, \mathbb{T}^{w,n}$ sends $L(y)$ to $\Upsilon_w(X)\, L(y)$; here $\mathbb{T}^{w,n}$ is the Hecke operator of the double coset of $\mathrm{diag}(\varpi,\dots,\varpi,1,\dots,1)$, with n entries ϖ, in $\mathrm{GL}_{I'}(E_w)$, acting by right translation on the functions on $G_f(I')$ (note* At most one primitive label for a Hecke eigensystem*, §2(a));*

(iii) *$\beta^{-1}(0) = \nu(I(D))$, and for every $y \in \mathrm{Prin}(D,\mathcal{V})$, every $x \in \mathrm{GL}_{I(D)}(\mathbb{F}_v)$ and every $k \in \mathrm{GL}_{I'}(O_v)$ whose reduction $\bar{k}$ is block upper triangular for β with block 0 equal to x: $R(k^{(v)})\, L(y) = \prod_{n=1}^{n_1} \chi_n(\det \bar{k}_{nn}) \cdot \psi(\det x) \cdot L\big(\mathrm{rep}_{D,\mathcal{V}}(x_{(v)})\, y\big)$, where $\bar{k}_{nn}$ is the n-th diagonal block of $\bar{k}$ and $x_{(v)} \in \mathsf{G}(D)$ has component x at v and 1 elsewhere;*

(iv) *the characters $\chi_1,\dots,\chi_{n_1}$ are distinct, and $\psi^{M_\omega(D)} = 1$ and $\chi_n^{M_\omega(D)} = 1$ for all n.*

Then $\mathrm{Occ}(D,\mathcal{V},p,v)$ has at most one element.

Inputs from other units of the argument, continued. In Setting 2.1 throughout.

Input 7 (theta lifts; units on the seesaw, on the exchange and on the Siegel–Weil formula). *Write* 1 *for the trivial tame character.* (a) *For* $1 \le i \le m(D)$ *there is a linear map* Λ_i*, a theta lift, from* $\mathrm{Prin}(D, W_i(D))$ *to the algebraic forms on the standard space with index set* $I(D) \sqcup \{0, \dots, t(D)-1\}$ *that have the weight, level and level character of the pair* $\mathrm{Aux}_i(D, 1)$*. It is injective; it satisfies Input 6*(ii) *for every eigensystem* p*, with functions* Υ_w *that depend on* p *only; and* $\Lambda_i(\mathrm{rep}_{D,W_i(D)}(\gamma)\, y) = \rho_i(\gamma)\, \Lambda_i(y)$ *for* $\gamma \in \mathsf{G}(D)$*, where* ρ_i *is the action of* $\mathsf{G}(D)$ *on vectors of that level described in Lemma 8.1.* (b) *There is a linear map* Λ_{ch}*, the composite of the* $m(D)$ *theta lifts for the trivial twist, from* $\mathrm{Prin}(D, Y_0(D))$ *to algebraic forms on the standard space* $Y_{m(D)}(D)$*. It is injective; it satisfies Input 6*(ii) *for every* p*, with some functions* Υ_w*; and for every* $k \in K_0(I_{m(D)}(D))$ *whose reduction* $\bar{k}_v$ *at every chosen prime* v *is block upper triangular for the blocks* $I(D)$*,* $\{1\} \times \{0, \dots, t(D)-1\}$*, …,* $\{m(D)\} \times \{0, \dots, t(D)-1\}$ *of* $I_{m(D)}(D)$*, with block* 0 *equal to* γ_v*,* $R(k)\, \Lambda_{\mathrm{ch}}(y) = \prod_{v \in \mathrm{Ch}(D)} \prod_{n=1}^{m(D)} \omega_{D,v}^{n}(\det \bar{k}_{v,nn}) \cdot \Lambda_{\mathrm{ch}}(\mathrm{rep}_{D,Y_0(D)}(\gamma)\, y)$*.* (c) *Let* O *be an auxiliary pair,* π *a representation of* $G_f(I')$*,* y *a vector of the level and level character of* O*, and* $k \in K_0(I')$ *with reductions* $\bar{k}_w$ *at the places* w *of* O*. Suppose that at each chosen place* w *of* O *the columns of* $\bar{k}_w$ *indexed by the image of the embedding of* O *vanish outside the rows of that image, where they form a matrix* s_w*, and that at each twist place the same holds for the transpose. Then* $\pi(k)\, y = \prod_w \psi_{O,w}(\det s_w)^{-1} \cdot \rho_O((s_w)_{w\ \text{chosen}})\, y$*, with* ψ_O *the tame part of the character of* O *and* ρ_O *as in Lemma 8.1.*

Input 8 (the data; Chapter III and the unit on the exchange). (a) *For every infinite place* τ*, the vector* $l_D(\tau)$ *has gap* $h(D)$ *for the signs* $\mathfrak{s}^Y_{D,0}$*.* (b) *If* $m(D) \le i$*, all signs* $\mathfrak{s}^Y_{D,i}$ *are* 1*; if* $i < m(D)$*, exactly* $\lfloor (m(D) - i)/2 \rfloor$ *of them are* -1*.* (c) $500\, r(D)^3 + 10\, r(D)^5 \le h(D)$*,* $500\, r(D)^3 \le h_2(D)$ *and* $16\, [E(D) : \mathbb{Q}]\, (r(D)+1)^3 \le h_2(D)$*.* (d) *For a chosen prime* v *and an integer* n *with* $0 < |n| \le 8m(D)$*:* $\omega_{D,v}^{n} \ne 1$*.* (e) *The set of the primes dividing* $\pi_N(D)$ *contains no conjugate of a member, and* $\pi_N(D)$ *has valuation* 1 *at each of them.*

LEAN `Data.commute_heckeAlgebra_principalRep` · `Data.iSup_eigenPart` · `Data.finite_eigenPart` · `Data.exists_hecke_proj_eigenPart` · `Data.occurringLabels_subsingleton` · `Data.auxLift` · `Data.auxLift_injective` · `Data.lifts_auxLift` · `Data.auxLift_principalRep` · `Data.iotaAtLevel` · `Data.iotaAtLevel_injective` · `Data.exists_isSphericalWith_iota` · `Data.algFormRep_iota` · `AuxPair.smul_chosenGLRep_eq` · `Data.hasGap_gridParam` · `Data.card_neg_gridSigns` · `Data.omega_zpow_ne_one`

Input 6 is the theorem of the note *At most one primitive label for a Hecke eigensystem*, where it is proved from two results on Galois representations. Further, the theorem on the cubic twist of the note *Primitive labels*, the cubic twist, is used in §7; it belongs to this unit.

Proof of Lemma 2.7. If $m(D) \le i$, the signs $\mathfrak{s}^Y_{D,i}$ are all 1 by Input 8(b), and the diagonal space of these signs is standard. Let $i < m(D)$, and put $\beta = \mathrm{ix}^{i<m(D)}_{D,i}$, $J = \mathrm{Gr}(\beta, N(D))$ and $s = 2\varrho(D) + 1 - i = m(D) - 1 - i$, the length of the string. By Input 8(b) the signs have $\lfloor (s+1)/2 \rfloor$ entries -1, so by Input 3(a) there is, for every infinite place τ, a complex matrix y_τ with $y_\tau^* \tau(J)\, y_\tau = \mathrm{diag}(\mathfrak{s}^Y_{D,i})$; it is invertible. Let $B = \mathrm{Lb}(\beta, \alpha_g(D), \hat{\pi}_N(D), \mathrm{sc}_i(D))$. By Lemma 2.6 and Input 3(b) the matrix $B^* J B$ is hermitian with entries and inverse over $\hat{O}_E$, so by Input 3(c) there is $k_1 \in \mathrm{GL}_{I_i(D)}(\hat{O}_E)$ with $(Bk_1)^* J (Bk_1) = 1$. Let $k_2 \in K_0(I_i(D))$ be the library's lift of the family of the inverses of the reductions of k_1 at the chosen primes, and $k = k_1 k_2$. Then k has reduction 1 at every chosen prime, and $(Bk)^* J (Bk) = k_2^* k_2 = 1$. The pair of J and the adelic matrix with components y_τ and finite part Bk is the space required. □

Lemma 4.1 (the weights are far from the row parameter)**.** *In Setting 2.1, let i and n be natural numbers and τ an infinite place of $E(D)$. Then*

$$\begin{aligned} &\varkappa_D(\tau, n) + w_{D,i}(\tau) \ge h_2(D) && \text{if } n < k(D) + i - 1, \\ &\varkappa_D(\tau, n) + w_{D,i}(\tau) \le -h_2(D) && \text{otherwise.} \end{aligned}$$

Proof. If $n + 1 < k(D)$, then $n < k(D) + i - 1$, and, since $k(D) \le r(D) \le r(D)^8$,

$$\varkappa_D(\tau, n) = \lfloor k(D)/2 \rfloor - 1 - n \;\ge\; -k(D) \;\ge\; -h_2(D);$$

as $w_{D,i}(\tau) \ge h(D) = 2h_2(D)$, the sum is at least $h_2(D)$.

Let now $n + 1 \ge k(D)$, and put $i' = n + 2 - k(D)$, so that $i' \ge 1$. Then

$$\varkappa_D(\tau, n) + w_{D,i}(\tau) = h_2(D) + w_{D,i}(\tau) - w_{D,i'}(\tau) = h_2(D) + (i - i')\, h(D).$$

The condition $n < k(D) + i - 1$ means $i' \le i$, and then the sum is at least $h_2(D)$; otherwise $i' \ge i + 1$ and it is at most $h_2(D) - h(D) = -h_2(D)$. □

LEAN `Data.varsigma_add_w_gap`

Lemma 4.2 (gaps)**.** *In Setting 2.1, let τ be an infinite place of $E(D)$. Then the vector $l_D(\tau)$ has gap $h(D)$ for the signs $\mathfrak{s}^Y_{D,0}$ of $Y_0(D)$; and for every natural number i it has gap $h(D)$ for the signs $\mathfrak{s}^W_{D,i}$ of $W_i(D)$.*

Proof. The first assertion is Input 8(a). That the entries of $l_D(\tau)$ are distinct and of the right parity does not involve the signs. Let $\iota, \iota' \in I(D)$ have different signs in $\mathfrak{s}^W_{D,i}$, say $\mathrm{num}_D(\iota) < k(D) + i - 1 \le \mathrm{num}_D(\iota')$. By Lemma 4.1 the two values of $\varkappa_D(\tau, \cdot) + w_{D,i}(\tau)$ are $\ge h_2(D)$ and $\le -h_2(D)$, so the two entries of $l_D(\tau)$ differ by at least $4h_2(D) = 2h(D)$. □

LEAN `Data.hasGap_signsY_paramW` · `Data.hasGap_paramW`

Lemma 4.3 (cusp kernel, finite dimension)**.** *In Setting 2.1, let $\mathfrak{s}$ be a family of signs on $I(D)$ such that for every infinite place τ the vector $l_D(\tau)$ has gap $h(D)$ for $\mathfrak{s}$. Then $\epsilon_{D,\mathfrak{s}}$ is a cusp kernel of order $h_2(D)$, and for every space $\mathcal{V}$ over $E(D)$ with index set $I(D)$ and signs $\mathfrak{s}$ the space $\mathrm{Prin}(D, \mathcal{V})$ has finite dimension.*

Proof. By Input 8(c), $500 r(D)^3 + h_2(D) \le 2h_2(D) = h(D)$, and a vector with gap $h(D)$ has every smaller gap; so Input 2(a) applies with $B = h_2(D)$. The group $K_g(D)$ is open, so Input 2(b) gives the finite dimension. □

LEAN `Data.isCuspKernel_paramW` · the finite dimension is an instance without a name

By Lemmas 4.2 and 4.3, the spaces $Y_0(D)$ and $W_i(D)$ satisfy the hypothesis on gaps of Setting 2.2, and their spaces of principal forms have finite dimension.

5 Levels, moduli and auxiliary pairs

Lemma 5.1. *In Setting 2.1, let ϑ be a tame character of D and v a prime of $E(D)$. If v or $\bar{v}$ belongs to $Q(\vartheta)$, then the product of the norms of the members of $Q(\vartheta)$ lies in v, and neither $\overline{\pi_g(D)}$ nor $N(D)$ lies in v.*

Proof. The norm of a prime lies in it and in its conjugate, which gives the first assertion. So the product of the norms lies in v, and it is prime to $g_0(D)g_1(D)N(D)$. Hence $g_0(D)g_1(D)N(D)$ does not lie in v: otherwise $v \cap \mathbb{Z}$ would contain two coprime integers, hence 1. As $\overline{\pi_g(D)}\pi_g(D) = g_0(D)g_1(D)$, neither $\overline{\pi_g(D)}$ nor $N(D)$ lies in v. □

LEAN `Data.levelNorm_mem_and_notMem_of_mem_Q`

Definition 5.2 (places and moduli of a tame character)**.** In Setting 2.1, let ϑ be a tame character of D.

(a) $\mathrm{Lv}(D, \vartheta)$ is the system of places with the chosen primes $\mathrm{Ch}(D)$ as its chosen places and the members of $Q(\vartheta)$ as its twist places. (It is a system of places in the library's sense: these primes are distinct, and none is the conjugate of one of them, by Lemma 5.1.)

(b) For a natural number n and a Boolean o, $\mathfrak{m}_{D,\vartheta}^{n,o}$ is the ideal of $O_{E(D)}$

$$\Big(\prod_{v \in \mathrm{Ch}(D)} v\Big)^n \cdot \prod_{v \in Q(\vartheta)} v \quad \text{if } o \text{ is true}, \qquad \Big(\prod_{v \in \mathrm{Ch}(D)} v\Big)^n \quad \text{if } o \text{ is false}.$$

The formal text takes o true when it is not given.

LEAN `Data.levelPlaces` · `Data.𝔪`

Lemma 5.3. *In Setting 2.1, let ϑ be a tame character of D, n a natural number with $n \geq 1$, and o a Boolean. Then some prime of $E(D)$ divides $\mathfrak{m}_{D,\vartheta}^{n,o}$.*

Proof. There is a chosen prime v above $g_0(D)$. The ideal is contained in $(\prod_{v' \in \mathrm{Ch}(D)} v')^n$, which, as $n \geq 1$, is contained in the product of the chosen primes and so in v. □

LEAN `Data.nonempty_modulus_places`

Definition 5.4 (the auxiliary pairs)**.** An *auxiliary pair* over a CM field, in the sense of the unit on the rows, is a record of: a system of places; a space with an index set I_1 and signs; an embedding of I_1 into a larger index set I'; a numbering $I_1 \to \mathbb{N}$; two families of natural numbers $\kappa^+(\tau, n)$, $\kappa^-(\tau, n)$ (numbers of columns of height n, which give the weight of forms on the standard space with index set I'); and a character of the norm-one ideles with a weight and with a tame part that is trivial off the places of the system.

In Setting 2.1, let ϑ be a tame character of D and i a natural number. The auxiliary pair $\mathrm{Aux}_i(D, \vartheta)$ has: the system of places $\mathrm{Lv}(D, \vartheta)$; the space $W_i(D)$ with its signs $\mathfrak{s}_{D,i}^W$; the inclusion of $I(D)$ into $I(D) \sqcup \{0, \ldots, t(D) - 1\}$; the numbering $\iota \mapsto \mathrm{num}_D(\iota)$

where the sign is 1 and $\iota \mapsto r(D) - 1 - \mathrm{num}_D(\iota)$ where it is -1; the column numbers, written $\kappa_O^+(\tau, n)$ and $\kappa_O^-(\tau, n)$ with $O = \mathrm{Aux}_i(D, \vartheta)$ as in the note on the auxiliary pair,

$$\kappa_O^+(\tau, n) = \Big(\varkappa_D(\tau, n-1) - \max(\varkappa_D(\tau, n), \lfloor t(D)/2 \rfloor - w_{D,i}(\tau)) - 1 \Big)_+ ,$$

$$\kappa_O^-(\tau, n) = \Big(\min(\varkappa_D(\tau, r(D) - n - 1), -w_{D,i}(\tau) - \lfloor t(D)/2 \rfloor) - \varkappa_D(\tau, r(D) - n) - 1 \Big)_+ ,$$

where $(\cdot)_+$ is the positive part and the differences $n-1$, $r(D)-n-1$, $r(D)-n$ of natural numbers are truncated at zero; and the character $\phi_{D,i}\,\vartheta$, where $\phi_{D,i}$ is the character of the data of weight $\tau \mapsto -w_{D,i}(\tau)$ with tame part $(\omega_{D,v}^i)_v$, so that the tame part of the product is $(\omega_{D,v}^i\,\psi_{\vartheta,v})_v$.

LEAN `Data.auxPair`

6 The good idempotent

Lemma 6.1. *In Setting 2.1, the element $e_0(D)$ is an idempotent of $\mathbb{C}[\mathsf{G}(D)]$ whose coefficients are constant on conjugacy classes.*

Proof. It is a sum of elements $e^{\otimes}_{I(D),\mathbf{c}}$ over a finite set of families; apply Input 4(a). □

LEAN `Data.isCentralProj_e₀`

So $e_0(D)$ is central, and $\mathrm{rep}_{D,\mathcal{V}}(e_0(D))$ is a projector of $\mathrm{Prin}(D, \mathcal{V})$ onto $\mathrm{Good}(D, \mathcal{V})$ that commutes with $\mathrm{rep}_{D,\mathcal{V}}$.

Lemma 6.2 (good forms see no reflection characters)**.** *In Setting 2.1, let $(y_v)_{v \in \mathrm{Ch}(D)}$ be a family of elements $y_v \in \mathbb{C}[\mathrm{GL}_{I(D)}(\mathbb{F}_v)]$, let v_0 be a chosen prime, χ a multiplicative character of $\mathbb{F}_{v_0}$ and $l \in \mathbb{F}_{v_0}^{I(D)}$ a vector. Suppose that $\chi^{M_\omega(D)} = 1$, that $l \neq 0$, and that $y_{v_0} = \sum_h \chi(\det h)^{-1} h$, the sum over the $h \in \mathrm{GL}_{I(D)}(\mathbb{F}_{v_0})$ of the form $1 + u\,l^{\mathsf{T}}$. Then $\big(\bigotimes_v y_v\big) \cdot e_0(D) = 0$.*

Proof. It suffices to show $\big(\bigotimes_v y_v\big) \cdot e^{\otimes}_{I(D),\mathbf{c}} = 0$ for every family $\mathbf{c}$ in the sum that defines $e_0(D)$. This product is the tensor product of the $y_v\, e_{I(D),c_v}$. By Definition 2.11, no $x \in \mathbb{F}_{v_0}$ with $x^{M_\omega(D)} = 1$ is a root of c_{v_0}; and $x_\chi^{M_\omega(D)} = 1$, because $\chi \mapsto x_\chi$ is a homomorphism. So x_χ is not a root of c_{v_0}, and the factor at v_0 vanishes by Input 4(b). □

LEAN `Data.reflectionSum_mul_e₀`

7 Hecke operators: one operator for two pairings; equal traces

For a tame character ϑ of D, a natural number i and a space $\mathcal{V}$ over $E(D)$ with index set $I(D)$, the unit on the exchange defines the *Rallis terms*. Call a prime of $E(D)$ *bad* if it or its conjugate divides $\mathfrak{m}_{D,\vartheta}^{1,\mathrm{true}}$; the chosen primes are bad. Let $L_{D,\vartheta}$ be the subgroup of the $g \in G_f(I(D))$ whose components at the primes that are not bad are integral. The Rallis terms are the numbers, for $z, y \in \mathrm{Prin}(D, \mathcal{V})$ and $\gamma \in G_f(I(D))/L_{D,\vartheta}$,

$$\mathrm{Ral}_{D,\vartheta,i}(\mathcal{V}; z, y; \gamma) = c_{D,\vartheta,i}(\gamma)\,\langle z, R(g_\gamma)\, y \rangle .$$

Here $g_\gamma \in G_f(I(D))$ is a representative of γ, chosen by the library, whose components at the bad primes are 1, and $c_{D,\vartheta,i}(\gamma)$ is a complex number that does not depend on the space: a matrix coefficient of the vacuum vector of a Weil representation at g_γ, for the character of $\mathrm{Aux}_i(D,\vartheta)$, divided by a power of the norm of that modulus.

Lemma 7.1 (one Hecke operator for two pairings)**.** *In Setting 2.1, let $\mathfrak{s}$, $\mathfrak{s}'$, $\mathcal{V}$, $\mathcal{V}'$ be as in Setting 2.2. Let ϑ be a tame character of D, i a natural number, A and A' complex numbers, B a complex function on $\mathrm{Prin}(D,\mathcal{V}) \times \mathrm{Prin}(D,\mathcal{V})$ and B' one on $\mathrm{Prin}(D,\mathcal{V}') \times \mathrm{Prin}(D,\mathcal{V}')$. Suppose that for all $z, y \in \mathrm{Prin}(D,\mathcal{V})$ the family $\left(A \cdot \mathrm{Ral}_{D,\vartheta,i}(\mathcal{V}; z, y; \gamma)\right)_\gamma$ is summable with sum $B(z,y)$, and that for all $z, y \in \mathrm{Prin}(D,\mathcal{V}')$ the family $\left(A' \cdot \mathrm{Ral}_{D,\vartheta,i}(\mathcal{V}'; z, y; \gamma)\right)_\gamma$ is summable with sum $B'(z,y)$. Then there is $T \in \mathcal{H}(D)$ such that*

$$\begin{aligned} B(z,y) &= A\,\langle z, T_{\mathcal{V}}\, y\rangle && \text{for all } z, y \in \mathrm{Prin}(D,\mathcal{V}),\\ B'(z,y) &= A'\,\langle z, T_{\mathcal{V}'}\, y\rangle && \text{for all } z, y \in \mathrm{Prin}(D,\mathcal{V}'). \end{aligned}$$

Proof. For $T \in \mathcal{H}(D)$ let $\Psi(T)$ be the pair of the function $(z,y) \mapsto A\langle z, T_{\mathcal{V}} y\rangle$ on pairs of forms on $\mathcal{V}$ and of the function $(z,y) \mapsto A'\langle z, T_{\mathcal{V}'} y\rangle$ on pairs of forms on $\mathcal{V}'$. The map Ψ is linear, and it factors through the pair $(T_{\mathcal{V}}, T_{\mathcal{V}'})$ of endomorphisms of two spaces of finite dimension (Lemma 4.3); so its image $\mathcal{Z}$ is a subspace of finite dimension of the space of pairs of functions, hence closed for pointwise convergence. The pair (B, B') is the limit of the finite partial sums of the pairs of terms. So it suffices to show that for every coset γ the pair of the two terms lies in $\mathcal{Z}$.

Fix γ, write $g = g_\gamma$, and let H be the subgroup of the $k \in K_g(D)$ with $g^{-1}kg \in K_g(D)$. It is open in the compact group $K_g(D)$, so of finite index. Put

$$T^\gamma = \frac{c_{D,\vartheta,i}(\gamma)}{[K_g(D) : H]} \sum_{k \in K_g(D)/H} R(kgk^{-1}).$$

The elements kgk^{-1} have component 1 at every chosen prime, because g has: the chosen primes are bad. For a form y fixed by $K_g(D)$, $R(kgk^{-1})y = R(k)R(g)y$; this depends only on the class of k modulo H, and the sum over the classes is again fixed by $K_g(D)$. So T^γ maps $\mathrm{Prin}(D,\mathcal{V})$ to itself for every space $\mathcal{V}$, and $T^\gamma \in \mathcal{H}(D)$. As R is unitary and z is fixed by $K_g(D)$,

$$\langle z, T^\gamma_{\mathcal{V}}\, y\rangle = \frac{c_{D,\vartheta,i}(\gamma)}{[K_g(D) : H]} \sum_k \langle R(k)^{-1} z, R(g) y\rangle = \mathrm{Ral}_{D,\vartheta,i}(\mathcal{V}; z, y; \gamma),$$

and the same holds for $\mathcal{V}'$ with the same T^γ. So the pair of terms is $\Psi(T^\gamma)$. □

LEAN `Data.exists_common_hecke_rallis`

Proof of Theorem 1. By Lemma 4.3, $\mathrm{Prin}(D,\mathcal{V})$ and $\mathrm{Prin}(D,\mathcal{V}')$ have finite dimension. Let $\mathcal{F}$ be the set of the eigensystems p with $\mathrm{Prin}(D,\mathcal{V})[p] \neq 0$ or $\mathrm{Prin}(D,\mathcal{V}')[p] \neq 0$; it is finite, and each of the two spaces is the sum of its parts $[p]$, $p \in \mathcal{F}$ (Input 5(b)). For $p \in \mathcal{F}$ let $\Pi^p \in \mathcal{H}(D)$ be as in Input 5(c), for p and $\mathcal{F}$.

Twisting the labels. Fix $p \in \mathcal{F}$ and a chosen prime v. The sets $\mathrm{Occ}(D,\mathcal{V},p,v)$ and $\mathrm{Occ}(D,\mathcal{V}',p,v)$ have at most one element each, by the two hypotheses of rigidity; $I(D)$ has $r(D)$ elements and $r(D)$ is a power of 3 greater than 1; and 9 divides $\#\mathbb{F}_v - 1$. So the theorem on the cubic twist of the note *Primitive labels* gives, for

every polynomial $c \in \mathbb{F}_v[X]$, a polynomial $c^{p,v} \in \mathbb{F}_v[X]$ such that: if c is a primitive label of degree $r(D)$, then so is $c^{p,v}$, and $c^{p,v}$ lies in neither of the two sets; and $e_{I(D),c}(\gamma) = e_{I(D),c^{p,v}}(\gamma)$ for every $\gamma \in \mathrm{GL}_{I(D)}(\mathbb{F}_v)$ whose characteristic polynomial is not irreducible. Put

$$\tilde{e}_p = \sum_{\mathbf{c}} e^{\otimes}_{I(D),(c_v^{p,v})_v} \ \in \mathbb{C}[\mathsf{G}(D)],$$

the sum over the same families $\mathbf{c} = (c_v)_v$ as in the definition of $e_0(D)$.

The twisted idempotent kills the part $[p]$. Let $\mathcal{V}''$ be $\mathcal{V}$ or $\mathcal{V}'$, and $\mathbf{c}$ a family in that sum. Above $g_0(D)$ there is a chosen prime v at which c_v is a primitive label. Then $c_v^{p,v}$ is a primitive label of degree $r(D)$ outside $\mathrm{Occ}(D, \mathcal{V}'', p, v)$, which by the definition of that set means that $e_{I(D),c_v^{p,v}}$, acting through the factor v, kills $\mathrm{Prin}(D, \mathcal{V}'')[p]$. By Input 4(c) the tensor product kills it too. Hence $\mathrm{rep}_{D,\mathcal{V}''}(\tilde{e}_p)$ is zero on $\mathrm{Prin}(D, \mathcal{V}'')[p]$.

A formula for the trace. On $\mathrm{Prin}(D, \mathcal{V}'')$,

$$T_{\mathcal{V}''} \circ \mathrm{rep}_{D,\mathcal{V}''}(e_0(D)) = \sum_{p \in \mathcal{F}} (T\Pi^p)_{\mathcal{V}''} \circ \mathrm{rep}_{D,\mathcal{V}''}(e_0(D) - \tilde{e}_p). \tag{1}$$

It suffices to check this on each part $\mathrm{Prin}(D, \mathcal{V}'')[p']$, $p' \in \mathcal{F}$. The representation $\mathrm{rep}_{D,\mathcal{V}''}$ preserves that part (Input 5(a)); on it Π^p is the identity for $p = p'$ and zero for the other $p \in \mathcal{F}$; and $\mathrm{rep}_{D,\mathcal{V}''}(\tilde{e}_{p'})$ vanishes there. So both sides of (1) are $T_{\mathcal{V}''} \circ \mathrm{rep}_{D,\mathcal{V}''}(e_0(D))$ on that part. Now $\mathrm{rep}_{D,\mathcal{V}''}(e_0(D))$ is a projector onto $\mathrm{Good}(D, \mathcal{V}'')$ that commutes with $T_{\mathcal{V}''}$ (Lemma 6.1, Input 5(a)), so the trace of the left side of (1) is the trace of T on $\mathrm{Good}(D, \mathcal{V}'')$.

For $p \in \mathcal{F}$ write $(T\Pi^p)_{\mathcal{V}''} = \sum_g c_g^p \, R(g)$, a finite sum over elements $g \in G_f(I(D))$ with component 1 at the chosen primes, the same for $\mathcal{V}$ and $\mathcal{V}'$. Let $(b_n)_n$ be an orthonormal basis of $\mathrm{Prin}(D, \mathcal{V}'')$. The trace of an endomorphism S is $\sum_n \langle b_n, S b_n \rangle$, so by (1) and the definition of $\mathrm{rep}_{D,\mathcal{V}''}$

$$\mathrm{tr}(T \mid \mathrm{Good}(D, \mathcal{V}'')) = \sum_{p \in \mathcal{F}} \sum_{\gamma \in \mathsf{G}(D)} \sum_{g} (e_0(D) - \tilde{e}_p)(\gamma) \; c_g^p \; \sum_n \langle b_n, R(g\tilde{\gamma})\, b_n \rangle. \tag{2}$$

Only elliptic elements remain. Let $\gamma = (\gamma_v)_v \in \mathsf{G}(D)$ be such that no γ_v has an irreducible characteristic polynomial. Then for every family $\mathbf{c}$ and every v, $e_{I(D),c_v}(\gamma_v) = e_{I(D),c_v^{p,v}}(\gamma_v)$; taking products over v and summing over $\mathbf{c}$ gives $(e_0(D) - \tilde{e}_p)(\gamma) = 0$. So in (2) only those γ contribute for which some γ_{v_0} has an irreducible characteristic polynomial. For such a γ, for g with component 1 at the chosen primes and for $k \in K_g(D)$, the component at v_0 of $g\tilde{\gamma}k$ is that of $\tilde{\gamma}k$: a matrix over O_{v_0} with reduction γ_{v_0}. Its characteristic polynomial is monic over O_{v_0} with irreducible reduction, hence irreducible over O_{v_0} and, by Gauss's lemma, over E_{v_0}.

Comparison. So Input 1 applies to $\mathbf{K} = K_g(D)$ and the element $g\tilde{\gamma}$, with $l = l_D$, $B = h_2(D)$ and $H = h(D)$: the field $E(D)$ contains a primitive root of unity of order 5040 and has an even number of infinite places, and the two inequalities hold by Input 8(c). For both $\mathcal{V}$ and $\mathcal{V}'$ the type is a cusp kernel of order $h_2(D)$ (Lemma 4.3) and $l_D(\tau)$ has gap $h(D)$ (Setting 2.2). Hence the inner sum $\sum_n \langle b_n, R(g\tilde{\gamma}) b_n \rangle$ of (2) has the same value for $\mathcal{V}$ and for $\mathcal{V}'$. All other factors of (2) do not depend on the space, and the theorem follows. □

Proof of Theorem 2. (a) For $n \geq 0$, $T^n \in \mathcal{H}(D)$, and the action on good forms respects products. By Theorem 1 the n-th powers of the two restrictions of T have equal traces, for all $n \geq 0$. By Input 4(d) they have the same characteristic polynomial, hence the same determinant.

(b) Let $\mathcal{F}'$ be the finite set of the eigensystems p' with $\mathrm{Prin}(D, \mathcal{V}')[p'] \neq 0$, and Π as in Input 5(c) for p and $\mathcal{F}'$. On $\mathrm{Prin}(D, \mathcal{V}')$, which is the sum of its parts $[p']$, $p' \in \mathcal{F}'$, the image of $\Pi_{\mathcal{V}'}$ lies in $\mathrm{Prin}(D, \mathcal{V}')[p]$. Since $\Pi_{\mathcal{V}'}$ preserves the good forms, it maps $\mathrm{Good}(D, \mathcal{V}')$ into $\mathrm{Good}(D, \mathcal{V}') \cap \mathrm{Prin}(D, \mathcal{V}')[p] = 0$. So $1 - \Pi$ is the identity on $\mathrm{Good}(D, \mathcal{V}')$, with determinant 1; by (a) its determinant on $\mathrm{Good}(D, \mathcal{V})$ is 1, and $1 - \Pi_{\mathcal{V}}$ is injective there. But it kills $\mathrm{Good}(D, \mathcal{V}) \cap \mathrm{Prin}(D, \mathcal{V})[p]$, on which $\Pi_{\mathcal{V}}$ is the identity. So this intersection is zero.

For the last assertion apply Theorem 1 to $\mathcal{V} = Y_0(D)$ and $\mathcal{V}' = W_i(D)$, which satisfy Setting 2.2 by Lemma 4.2 and have rigid labels by Theorem 3, and to $T = 1$: the trace of the identity is the dimension. □

8 Rigid labels: proof of Theorem 3

Lemma 8.1 (the parahoric type of a vector of the level of an auxiliary pair)**.** *In Setting 2.1, let O be an auxiliary pair over $E(D)$ whose space has the index set I_1 and whose embedding is $\nu_O \colon I_1 \to I'$, with I_1 and I' finite sets. Its* level group K_O *consists of the $k \in K_0(I')$ whose reduction fixes the basis vectors indexed by $\nu_O(I_1)$ at the chosen places of O and preserves the span of the other basis vectors at its twist places; its* level character *is $k \mapsto \prod_w \psi_{O,w}^{-1}(\det s_w(k))$, the product over the twist places, with $s_w(k)$ the block of the reduction of k at w indexed by $\nu_O(I_1)$ and $(\psi_{O,w})_w$ the tame part of the character of O. Let $\mathcal{S}$ be a complex vector space of functions on $G_f(I')$ with values in the functions on a finite set, and π a representation of $G_f(I')$ on $\mathcal{S}$. On the subspace of the $u \in \mathcal{S}$ on which K_O acts through the level character, the group $\prod_w \mathrm{GL}_{I_1}(\mathbb{F}_w)$, w the chosen places of O, acts by $\rho_O(\gamma) = \prod_w \psi_{O,w}(\det \gamma_w) \cdot \pi(\tilde{\gamma})$, with $\tilde{\gamma}$ a lift of the block diagonal matrices with blocks γ_w and 1. Let u be a vector of that subspace, v a chosen place of O, which is split, and f a function on $G_f(I')$ with values in the functions on that finite set. Suppose that*

$$\pi(k^{(v)})\, u = R(k^{(v)})\, f \qquad \text{for every } k \in \mathrm{GL}_{I'}(E_v).$$

Then for every $x \in \mathrm{GL}_{I_1}(\mathbb{F}_v)$ and every $k \in \mathrm{GL}_{I'}(O_v)$ whose reduction $\bar{k}$ is block upper triangular for the two blocks $\nu_O(I_1)$ and its complement, with first block equal to x:

$$R(k^{(v)})\, f = \psi_{O,v}(\det x)^{-1} \cdot \rho_O(x_{(v)})\, u,$$

where $x_{(v)}$ has component x at v and 1 at the other chosen places of O. In the language of Input 6: f satisfies its condition (iii) *for the map β that is 0 on $\nu_O(I_1)$ and 1 elsewhere, $n_1 = 1$, $\chi_1 = 1$, $\psi = \psi_{O,v}^{-1}$, and $x \mapsto \rho_O(x_{(v)})\, u$ in the place of $x \mapsto L(\mathrm{rep}_{D,\mathcal{V}}(x_{(v)})y)$.*

Proof. Let x and k be as stated. By hypothesis $R(k^{(v)})f = \pi(k^{(v)})u$. The reductions of $k^{(v)}$ at the places of O are $\bar{k}$ at v and 1 elsewhere. Since $\bar{k}$ is block upper triangular with first block x, its columns indexed by $\nu_O(I_1)$ vanish outside the rows of $\nu_O(I_1)$ and form the matrix x there; at the other places the condition of Input 7(c) holds with the identity matrix. So Input 7(c) gives $\pi(k^{(v)})u = \psi_{O,v}(\det x)^{-1}\rho_O(x_{(v)})u$. □

Proof of Theorem 3. Let p be an eigensystem and v a chosen prime. In both cases Input 6 is applied; the gaps that it does not ask for are not needed.

The space $W_i(D)$, $1 \le i \le m(D)$. Take $I' = I(D) \sqcup \{0, \dots, t(D)-1\}$, ν the inclusion, and $L = \Lambda_i$ of Input 7(a); (i) and (ii) are part of that input. For (iii), Lemma 8.1 is applied to $O = \mathrm{Aux}_i(D, 1)$, to the representation of right translations on the functions on $G_f(I')$, to $u = f = \Lambda_i(y)$, for which its hypothesis is trivially true, and to v: with the equivariance of Λ_i in Input 7(a) it gives (iii) for $n_1 = 1$, $\chi_1 = 1$ and $\psi = \psi_{O,v}^{-1}$. The tame part of the character of this pair at v is $\omega_{D,v}^{i}$, because the trivial tame character has trivial tame part; so $\psi = \omega_{D,v}^{-i}$ and $\psi^{M_\omega(D)} = 1$. A single character is a family of distinct characters, and $1^{M_\omega(D)} = 1$; this is (iv).

The space $Y_0(D)$. Take $I' = I_{m(D)}(D)$, ν the inclusion of $I(D)$, and $L = \Lambda_{\mathrm{ch}}$ of Input 7(b), which gives (i) and (ii). Let β be the map that is 0 on $I(D)$ and n on $\{n\} \times \{0, \dots, t(D)-1\}$, $n_1 = m(D)$, $\chi_n = \omega_{D,v}^{n}$ and $\psi = 1$. For $k \in \mathrm{GL}_{I'}(O_v)$ with reduction block upper triangular for β and block 0 equal to x, the element $k^{(v)}$ lies in $K_0(I')$ and has reductions $\bar{k}$ at v and 1 at the other chosen primes; so the formula of Input 7(b), in which all factors at the other chosen primes are 1, is (iii). The characters $\omega_{D,v}^{n}$, $1 \le n \le m(D)$, are distinct: a quotient of two of them is $\omega_{D,v}^{n-n'}$ with $0 < |n - n'| \le 8m(D)$, which is not trivial by Input 8(d). They have order dividing $M_\omega(D)$, the order of $\omega_{D,v}$. This is (iv). □

9 Where the results are used

The definitions of this note are used throughout Chapters V to VIII: $\mathrm{Row}(D)$ is the space of rows of the three main theorems, and $Y_i(D)$, $W_i(D)$ are the spaces of the chain of theta lifts and of the single lifts. Theorem 2 enters the proof of Theorem C in three places. Its part (a), with Lemma 7.1, is the step that writes the volume of the chain of lifts on the rows as a constant times the product of the volumes of the single lifts (unit on the rows): by Lemma 7.1 the two Rallis inner products are given by one Hecke operator, and by part (a) its determinants on $\mathrm{Row}(D)$ and on $\mathrm{Good}(D, W_i(D))$ agree. The equality of dimensions is used there too, and in Chapter VI to bound $\dim \mathrm{Row}(D)$ from below. Part (b) is used in the construction of the fit family of twists (Chapter VIII), to pass from eigensystems of good forms on $W_i(D)$ to those of rows. Lemma 6.2 is used in the comparison of forms on the rows, and Lemma 8.1 again in Chapter VIII. Lemmas 2.4, 2.6, 5.1 and 5.3 are used in the units on the seesaw and on the exchange.

10 Notes on the formal text

1. *Letters.* The formal text writes `V`, `s` for a space and its signs, `j` for the index i, `w` and `x` for primes and chosen primes, `p` for a Hecke element in the file on traces and for an eigensystem in the file on rigidity, `q` and `d` for eigensystems, `a`, `a'` for the constants A, A', `c`, `b` for the exponent n and the Boolean o of a modulus, `ς` for the row parameter and `paramW` for its double l_D. The spaces $Y_i(D)$, $W_i(D)$ are the authors' `Y_j` and `W_j`.

2. *The hypothesis on gaps* of Setting 2.2 is an instance argument in the formal text: it is found by Lean, not passed by hand. Its only role in Lemma 7.1 is to make the two spaces of forms finite-dimensional.

3. *Lemma 2.6.* The formal text states the valuations as values $\exp(-1)$ and 1 of the valuation at v, written multiplicatively in the value group; this is the order 1 and 0. The proof there quotes Input 8(e).

4. *Lemma 4.1.* The formal proof uses only the inequality $w_{D,i}(\tau) + h(D) \le w_{D,i'}(\tau)$ for $i < i'$, an input from the unit on the exchange, and closes the cases by an automatic tactic; I used the definition of the weights, which gives the difference exactly. After the lemma the formal text registers it as a pattern for that tactic; this line is not part of a proof.

5. *Lemma 2.7.* In (b) the formal statement is made for every proof of $i < m(D)$. The formal proof takes for the infinite components of the point the inverses of the matrices $y_\tau \operatorname{diag}(\mathfrak{s}^{Y}_{D,i})$-type frames that Input 3(a) provides, arranged so that the defining identity of a space holds; I did not trace this arrangement term by term, and the dossier of the unit marks the same place as uncertain.

6. *Theorem 1.* The formal proof first shows identity (1) for an arbitrary space whose occurring labels are among those of $\mathcal{V}$ and $\mathcal{V}'$ and whose eigensystems lie in $\mathcal{F}$, and then applies it twice. It uses of a good family only that it has a primitive label above $g_0(D)$. The Hecke elements Π^p are chosen for all eigensystems, and the theorem on the cubic twist of the note *Primitive labels* is applied to all pairs (p, v), not only for $p \in \mathcal{F}$.

7. *Theorem 2.* In (a) the formal proof writes the determinants as products of the roots of the characteristic polynomials. The traces for $n = 0$ give the equality of the dimensions, which the comparison of characteristic polynomials of matrices of different sizes needs.

8. *Theorem 3.* The applications of Input 6 pass most of its arguments implicitly; in the case of $Y_0(D)$ two side conditions on the places other than v are closed by automatic tactics, and I read them as the statement that the reductions of $k^{(v)}$ at those places are 1. The dossier marks the same lines.

9. *Arguments.* For Lemma 2.7, Lemma 4.3 with its instance, and Lemma 6.2, the dossier records the arguments as derived by hand from the text of the signatures, not as printed by Lean; the second reading of this write-up asks for Lean's own output before these records are called confirmed. The statements above agree with the derived lists: Lemma 2.7 takes D and i only; Lemma 4.3 takes the hypothesis on gaps as an instance argument; Lemma 6.2 takes the family, the character and the vector implicitly and the chosen prime and the three hypotheses explicitly.

10. *What the final theorems do not use.* The plan of the write-up, which follows names in the text of the proofs, records most but not all of the 46 declarations of the unit as reached from the final theorems. Instances are used by Lean without being named, so which declarations are really unused is for the kernel export to say.

How this note was written. From the dossier of the unit: its statements and the structured proofs of Lemmas 2.6, 2.7, 4.1, 5.3, 6.2, 7.1, 8.1 and of Theorems 1 to 3, which a program compared line by line with the source (99 leaf steps in the 16 proofs of the unit), and the second reading of the unit by the other agent of this write-up. I read the four Lean files of the unit completely; in the other units the definitions of the Hecke algebra, of the eigenparts and of the occurring labels, of the good idempotent, of the weights and of the row parameter, of auxiliary pairs and of the Rallis terms; and in the library the definitions of spaces, forms, levels and types, and the statement of Input 1. The statements of the other inputs I took from the dossier's quotations, which the program compared with the source. Every statement of a declaration of this note was compared with its Lean statement argument by argument, and the order of the quantifiers is that of the formal text. The proofs follow the formal ones; the proof of Lemma 4.1 is shorter, and the proof of Theorem 1 is arranged as a computation with an explicit formula for the trace. The descriptions of the objects of Setting 2.1 are recollections, and the notes of their units are the authority for them. The references are those of the library's files, of the catalogue of this handout and of its second reading; I did not open the cited works.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Lemma 7.1	`Data.exists_common_hecke_rallis`	GoodFormTransfer.lean	27–47
Theorem 1	`Data.trace_heckeOnGood_eq`	GoodFormTransfer.lean	56–89
Theorem 2	`Data.det_heckeOnGood_eq`	GoodFormTransfer.lean	94–98
Theorem 2	`Data.goodForms_inf_eigenPart_eq_bot`	GoodFormTransfer.lean	103–110
Theorem 2	`Data.finrank_rows_eq_finrank_goodForms`	GoodFormTransfer.lean	116–118
Definition 2.3	`Shape.besselIndexEquiv`	GoodIdempotent.lean	16–19
Lemma 2.4	`Data.m_eq_two_mul_varrho_add_two`	GoodIdempotent.lean	27–27
Lemma 5.1	`Data.levelNorm_mem_and_notMem_of_mem_Q`	GoodIdempotent.lean	34–36
Definition 5.2	`Data.levelPlaces`	GoodIdempotent.lean	40–42
Definition 5.2	`Data.m`	GoodIdempotent.lean	46–46
Lemma 5.3	`Data.nonempty_modulus_places`	GoodIdempotent.lean	48–50
Definition 2.5	`Data.chosenIdele`	GoodIdempotent.lean	54–55
Definition 2.5	`Data.ideleA`	GoodIdempotent.lean	59–59
Definition 2.5	`Data.piIdele`	GoodIdempotent.lean	62–62
Lemma 2.6	`Data.piIdele_spec`	GoodIdempotent.lean	64–66
Definition 2.5	`Data.latticeScale`	GoodIdempotent.lean	70–70
Definition 2.3	`Data.indexEquivY`	GoodIdempotent.lean	74–75
Lemma 2.7	`Data.exists_spaceY`	GoodIdempotent.lean	80–91
Definition 2.8	`Data.spaceY`	GoodIdempotent.lean	95–95
Lemma 2.9	`Data.isStd_spaceY`	GoodIdempotent.lean	97–97
Definition 2.8	`Data.spaceW`	GoodIdempotent.lean	101–101
Definition 2.10	`Data.principalForms`	GoodIdempotent.lean	105–106
Definition 2.10	`Data.principalRep`	GoodIdempotent.lean	111–111
Definition 2.11	`Data.goodForms`	GoodIdempotent.lean	115–115
Definition 2.11	`Data.rows`	GoodIdempotent.lean	118–118
Definition 2.11	`Data.rowDim`	GoodIdempotent.lean	121–121
Definition 5.4	`Data.auxPair`	GoodIdempotent.lean	125–130
Lemma 4.1	`Data.varsigma_add_w_gap`	GoodIdempotent.lean	138–143
Lemma 4.2	`Data.hasGap_signsY_paramW`	GoodIdempotent.lean	145–146
Lemma 4.2	`Data.hasGap_paramW`	GoodIdempotent.lean	148–149
Lemma 4.3	`Data.isCuspKernel_paramW`	GoodIdempotent.lean	154–155

In this note	Lean declaration	File	Lines
Lemma 6.1	`Data.isCentralProj_e₀`	GoodIdempotent.lean	166–166
Lemma 6.2	`Data.reflectionSum_mul_e₀`	GoodIdempotent.lean	171–174
Lemma 8.1	`Data.hasParahoricType_chosenGLRep`	LabelRigidity.lean	15–25
Definition 2.12	`Data.HasRigidLabels`	LabelRigidity.lean	29–30
Theorem 3	`Data.hasRigidLabels_spaceW`	LabelRigidity.lean	34–38
Theorem 3	`Data.hasRigidLabels_spaceY_zero`	LabelRigidity.lean	42–53
Definition 2.11	`Data.e₀`	IndexTypes.lean	314–314
Definition 2.12	`Data.heckeAlgebra`	FitTwists.lean	122–127
Definition 2.12	`Data.heckeOperator`	FitTwists.lean	131–136
Definition 2.12	`Data.eigenPart`	FitTwists.lean	140–140
Definition 2.12	`Data.occurringLabels`	FitTwists.lean	152–153
Definition 2.12	`Data.heckeOnGood`	FitTwists.lean	177–179
Input 1	`Space.exists_trace_eq`	TraceComparison.lean	195–283
Input 2	`isCuspKernel_typeKernel`	RealConvolution.lean	84–92
Input 2	`Space.finite_levelSpace`	ReproducingKernel.lean	54–64
Input 3	`exists_frame_gradedGram`	FramesForY.lean	29–36
Input 3	`isUnimodular_gradedLatticeBasis_gram`	FramesForY.lean	15–24
Input 3	`exists_orthonormal_Ô`	SpaceOperations.lean	83–87
Input 4	`isCentralProj_sum_tupleIdem`	LabelIdempotents.lean	124–125
Input 4	`reflectionSum_mul_labelIdem`	LabelIdempotents.lean	77–101
Input 4	`piTensorElem_apply_eq_zero`	CentralProjections.lean	37–39
Input 4	`charpoly_eq_mul_of_trace_pow_add`	FibreSum.lean	70–76
Input 5	`Data.commute_heckeAlgebra_principalRep`	FitTwists.lean	169–173
Input 5	`Data.iSup_eigenPart`	FitTwists.lean	192–197
Input 5	`Data.finite_eigenPart`	FitTwists.lean	187–188
Input 5	`Data.exists_hecke_proj_eigenPart`	FitTwists.lean	162–167
Input 6	`Data.occurringLabels_subsingleton`	LabelUniqueness.lean	23–59
Input 7	`Data.auxLift`	FitTwists.lean	226–227
Input 7	`Data.auxLift_injective`	CharacterLiftNorm.lean	147–148
Input 7	`Data.lifts_auxLift`	CharacterLiftNorm.lean	137–143
Input 7	`Data.auxLift_principalRep`	CharacterLiftNorm.lean	152–158
Input 7	`Data.iotaAtLevel`	GradingsOfY.lean	141–145
Input 7	`Data.iotaAtLevel_injective`	ChainNorms.lean	53–55
Input 7	`Data.exists_isSphericalWith_iota`	CharacterLiftNorm.lean	91–101
Input 7	`Data.algFormRep_iota`	GradingsOfY.lean	111–131
Input 7	`AuxPair.smul_chosenGLRep_eq`	AuxPairVanishing.lean	37–49
Input 8	`Data.hasGap_gridParam`	GridParameters.lean	121–124
Input 8	`Data.card_neg_gridSigns`	GridParameters.lean	81–83
Input 8	`Data.omega_zpow_ne_one`	ChosenPlaceLevels.lean	86–90

The main statements as they are written in the artifact.

```
lemma trace_heckeOnGood_eq p : trace ℂ _ (D.heckeOnGood V p) = trace ℂ _ (D.heckeOnGood V' p)

lemma det_heckeOnGood_eq p : (D.heckeOnGood V p).det = (D.heckeOnGood V' p).det

lemma finrank_rows_eq_finrank_goodForms (j : D.LiftIndex) : D.rowDim =
    finrank ℂ (D.goodForms (D.spaceW j))

lemma exists_common_hecke_rallis θ j {a a'} {B B' : _ → _ → ℂ} (h : ∀ z y,
  HasSum (a * D.rallisTerm θ j V z y ·) (B z y))
  (h' : ∀ z y, HasSum (a' * D.rallisTerm θ j _ z y ·) (B' z y)) : ∃ p : D.heckeAlgebra,
  (∀ z y, B z y = a * inner ℂ z (p.1 V y)) ∧ ∀ z y, B' z y = a' * inner ℂ z (p.1 V' y)

theorem exists_trace_eq [Nonempty n] [StablyUnram E]
    l H (j : gapThreshold n (10 * card n ^ 5) ≤ H) : ∃ t, ∀ {s} (V : Space E n
  s) {ι} [Fintype ι] (b : OrthonormalBasis ι ℂ (levelSpace (V.formRep (typeKernel l s)) K)),
  IsCuspKernel B (typeKernel l s) →
```

```
(∀ w, (toHCParam s (l w)).HasGap H) → ∑ i, ⟪(b i).1, V.formRep _ h (b i)⟫_ℂ = t
```

The first three stand after a line that declares the variables `D`, `s`, `s'`, `V`, `V'`, the two instance arguments on gaps and the two hypotheses of rigidity `o`, `o'`. In these texts `principalForms`, `goodForms`, `heckeAlgebra`, `heckeOnGood`, `rallisTerm` are $\mathrm{Prin}(D,\mathcal{V})$, $\mathrm{Good}(D,\mathcal{V})$, $\mathcal{H}(D)$, the action of $\mathcal{H}(D)$ on good forms and the Rallis terms; `spaceW`, `rowDim` are $W_i(D)$ and $\dim \mathrm{Row}(D)$; `typeKernel l s` is $\prod_\tau \epsilon_{l(\tau)}$, `formRep` is R and `levelSpace` takes the vectors fixed by a subgroup.

Notation

$P=(N_0,d)$, D the parameters; admissible data (Setting 2.1)

E, $E(D)$, K, O_E a CM field; the field of the data; $\mathbb{Q}(\sqrt{-d})$; the ring of integers

$\mathbf{K}$ an open subgroup of $K_0(I)$ or of $G_f(I)$ (Inputs 1 and 2)

v, w, $\bar v$, $\mathbb{F}_v$, E_v, O_v primes of a number field, the conjugate prime, residue field, completion, its integers

$\mathbb{A}_E$, $\mathbb{A}_{E,f}$, $\hat O_E$ adeles, finite adeles, integral finite adeles

τ an infinite place of $E(D)$

$m(D)$, $r(D)$, $x(D)$, $\mu(D)$ components of D: the number of lifts, the rank, the first scale, the exponent

$k(D)$, $t(D)$, $\varrho(D)$, $n_i(D)$ $r(D)+1-m(D)$, $5r(D)$, $m(D)/2-1$, $k(D)+i(t(D)+1)$

$h_2(D)$, $h(D)$, $M_\omega(D)$ $r(D)^8$, $2r(D)^8$, $(N_0 d)^{\mu(D)+6}$

$w_{D,i}(\tau)$, $w_D(\tau)$, $N_1(D)$ the weights; the nominal weight of the overview; the order of the root of unity that generates $E(D)$ over K

$\varkappa_D(\tau,n)$, $l_D(\tau)$ the row parameter, its double as a vector on $I(D)$

$N(D)$, $\pi_N(D)$, $g_0(D)$, $g_1(D)$, $\pi_g(D)$ the level and an element of norm $N(D)$; two primes and an element of norm $g_0(D)g_1(D)$

$\mathrm{Ch}(D)$, $\mathsf{G}(D)$ the chosen primes; $\prod_{v\in\mathrm{Ch}(D)} \mathrm{GL}_{I(D)}(\mathbb{F}_v)$

$I(D)$, num_D, $I_i(D)$, $\mathcal{X}_i(D)$ the row index set and its numbering; the index sets of $Y_i(D)$ and of its first summand

$\omega_{D,v}$ the character of order $M_\omega(D)$ of $\mathbb{F}_v^\times$ at a chosen prime

ϑ, $Q(\vartheta)$, $\psi_{\vartheta,v}$ a tame character, its set of primes, its tame part

I, I', I_1, GL_I, g^* finite index sets; invertible matrices indexed by I; conjugate transpose

$\mathfrak{s}$, $\mathfrak{s}'$, $\mathfrak{s}^Y_{D,i}$, $\mathfrak{s}^W_{D,i}$ families of signs; those of $Y_i(D)$ and $W_i(D)$

$G_\infty(\mathfrak{s})$, $\mathrm{U}(\mathfrak{s})$, $G_f(I)$, $K_0(I)$, $K_g(D)$ the unitary groups at the infinite places and over the finite adeles; the integral points; the principal congruence subgroup at the chosen primes

g_v, $k^{(v)}$, $\bar k$, $\tilde\gamma$, $x_{(v)}$ component at v; the element with component k at a split v; reduction; the chosen lift of $\gamma\in\mathsf{G}(D)$; the element of a product with component x at v

$\mathcal{V}=(J,\mathbf{x})$, $\mathcal{V}'$, $\mathcal{V}''$, $\Gamma_{\mathcal{V}}$ spaces with a point; the rational points in the adelic group

F, F', $R(g)$, $\langle\ ,\ \rangle$ forms; right translation; the Petersson product

ϵ, ϵ_l, $\epsilon_{D,\mathfrak{s}}$, u, α types; the kernel of a discrete series; the type of the data; an element of $G_\infty(\mathfrak{s})$ or a vector; a function of compact support, and in $\alpha(D)$ the idele of the Weil data

l, $l(\tau)$, H, B integer vectors (in Lemma 6.2, a vector over a finite field); a gap; an order of a cusp kernel (in Lemma 7.1, with B', a function on pairs of forms; in Lemma 2.7, a lattice basis)

$\Sigma=(\varrho,\mathcal{X},o)$, b_n, b^*_n, z a shape and its index set; o also a Boolean, z also a form or an idele

bes_Σ, $\mathrm{ix}^{i<m(D)}_{D,i}$ bijections of index sets (Definition 2.3)

$\hat\pi_g(D)$, $\alpha_g(D)$, $\hat\pi_N(D)$, $\mathrm{sc}_i(D)$ the ideles of Definition 2.5

$\mathrm{Gr}(\beta,N)$, $\mathrm{Lb}(\beta,a,\pi,u)$, s graded Gram matrix and lattice basis; the length of the string

$Y_i(D)$, $W_i(D)$ the spaces of Definition 2.8

$\mathrm{Prin}(D,\mathcal{V})$, $\mathrm{rep}_{D,\mathcal{V}}$ principal forms and the action of $\mathsf{G}(D)$ (Definition 2.10)

$e_0(D)$, $\mathrm{Good}(D,\mathcal{V})$, $\mathrm{Row}(D)$, $\dim\mathrm{Row}(D)$ the good idempotent, good forms, rows, their dimension

$e_{I,c}$, $e^{\otimes}_{I,\mathbf{c}}$, $\mathrm{Good}_{M_\omega(D),r(D)}$, x_χ idempotents of labels and of families, good tuples, the unit naming a character (note *Primitive labels*)

c, $\mathbf{c}=(c_v)_v$, $c^{p,v}$, $\tilde e_p$, X, $\mathcal{C}$ polynomials, families, cubic twists, the twisted idempotent; the variable; a set of families

$\mathcal{H}(D)$, T, $T_{\mathcal{V}}$, T^γ, Π, Π^p, $\mathbb{T}^{v,n}$ the Hecke algebra away from the chosen primes, its elements and their operators; the Hecke operators at a prime

p, p', $\mathcal{F}$, $\mathcal{F}'$, $\mathrm{Prin}(D,\mathcal{V})[p]$ eigensystems, finite sets of them, eigenparts

$\mathrm{Occ}(D,\mathcal{V},p,v)$, labels are rigid for D on $\mathcal{V}$ occurring primitive labels; Definition 2.12

$\mathrm{Lv}(D,\vartheta)$, $\mathfrak{m}_{D,\vartheta}^{n,o}$ places and moduli of a tame character (Definition 5.2)

$\mathrm{Aux}_i(D,\vartheta)$, κ_O^+, κ_O^-, $\phi_{D,i}$, O, ν_O, K_O, $\psi_{O,w}$, ρ_O, ρ_i, s_w auxiliary pairs and their parts (Definition 5.4, Lemma 8.1)

$\mathrm{Ral}_{D,\vartheta,i}$, $c_{D,\vartheta,i}(\gamma)$, g_γ, $L_{D,\vartheta}$, A, A', Ψ, $\mathcal{Z}$ Rallis terms and the objects of Lemma 7.1

L, Λ_i, Λ_{ch}, Υ_w, ν, β, n_1, χ, χ_n, ψ lifts and the data of Input 6; β also a bijection of index sets, χ a multiplicative character

$\mathcal{S}$, π, f, y, y_v, ρ, S, S' a space of functions, a representation, a function, vectors or elements of group algebras, a representation, matrices

b_n, t, c_g, c_g^p an orthonormal basis; the common trace of Input 1; coefficients of Hecke elements

i, i', n, n', a, k, k_1, k_2, h, γ, ι, ι', ϖ indices and natural numbers; an idele; integral matrices; a matrix over a finite field; elements of $\mathsf{G}(D)$ or cosets; elements of index sets; a uniformizer

y_τ, T frames at the infinite places; transpose

References

All references are given as the library's files, the catalogue of this handout and its second reading record them; I did not open the works.

J. Arthur, L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, 1989; Chapter 1, §2, Lemmas 2.4–2.6.

A. Borel, lectures on automorphic forms (Park City), 8.3–8.4, for the finiteness of spaces of cusp forms; at second hand.

R. Jacobowitz, *Hermitian forms over local fields*, Amer. J. Math. 84 (1962); place to be confirmed.

J.-P. Labesse, *Changement de base CM et séries discrètes*, 2011; Théorèmes 5.1 and 5.4.

READBACK NOTE 19

Primitive labels of general linear groups over finite fields: the share of bad types and the cubic twist

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The rows of the proof are cut out of a space of automorphic forms by an idempotent of the group algebra of a product of finite general linear groups. This note treats the part of that construction which lives over finite fields. A *label* is a squarefree monic polynomial; it names an irreducible character of GL_r of a finite field, by Green's construction. A label is *primitive* when it is irreducible with a root that generates the multiplicative group of the extension of degree r. Two results are proved. Theorem 1: in the regular representation of a product of f groups $\mathrm{GL}_r(\mathbb{F}_v)$, the characters named by tuples of labels that avoid finitely many roots and are primitive at one factor at least have all but a share $f\varepsilon + \exp(-L)$. Theorem 2: when r is a power of 3, a primitive label has a cubic twist that avoids two given polynomials and whose character agrees with the original one at every element with a reducible characteristic polynomial.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles and that Lean's kernel accepted the proofs; an independent replay is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note treats one file of the unit *Good idempotents, labels and the transfer of traces*: 6 of the unit's 46 declarations (the other declarations are the subject of the note *Good forms and rows*). The two theorems and the two lemmas are proved in full. Taken for granted, and stated as Inputs 1 to 3, are facts about Green's characters and about Euler's function that the artifact proves in its background library. Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 15 declarations.

1 The statements

The notions are defined in §2: labels and primitive labels in Definitions 2.1 and 2.2, the idempotents $e_{I,c}$ and $e^{\otimes}_{I,\mathbf{c}}$ in Definitions 2.3 and 2.4, the set $\mathrm{Good}_{M,r}$ of good tuples in Definition 2.5. For an element y of the complex group algebra of a finite group and an element g of the group, $y(g)$ is the coefficient of y at g.

Theorem 1 (bad types have a small share). *Let I and A be finite sets, let $(\mathbb{F}_v)_{v\in A}$ be a family of finite fields, let q, M, r, f be natural numbers with $M \geq 1$, and let ε and L be real numbers. Suppose that every $\mathbb{F}_v$ has q elements, that I has r elements and A has f elements, and that*

$$1 < r, \qquad r < q, \qquad \varepsilon \leq 1, \qquad \frac{1 + M + 2r}{q} \leq \varepsilon, \qquad 2r\,(2r \log q + 1)\,L \leq f.$$

Then the coefficient at the identity of the element $1 - \sum_{\mathbf{c}\in \mathrm{Good}_{M,r}} e^{\otimes}_{I,\mathbf{c}}$ of the complex group algebra of $\prod_{v\in A} \mathrm{GL}_I(\mathbb{F}_v)$ has real part at most

$$f\varepsilon + \exp(-L).$$

LEAN `bad_share_le`

Theorem 2 (the cubic twist). *Let $\mathbb{F}$ be a finite field, I a finite set and a a natural number. Suppose that I has 3^{a+1} elements and that 9 divides $\#\mathbb{F} - 1$. Let $c \in \mathbb{F}[X]$ be a polynomial, and let $\mathcal{S}_1$ and $\mathcal{S}_2$ be sets of polynomials over $\mathbb{F}$ with at most one element each. Then there is a polynomial $\tilde{c} \in \mathbb{F}[X]$ with the following two properties.*

(1) *If c is a primitive label of degree $\#I$, then $\tilde{c}$ is a primitive label of degree $\#I$, and $\tilde{c} \notin \mathcal{S}_1 \cup \mathcal{S}_2$.*

(2) *For every $g \in \mathrm{GL}_I(\mathbb{F})$ whose characteristic polynomial is not irreducible, $e_{I,c}(g) = e_{I,\tilde{c}}(g)$.*

LEAN `exists_labelIdem_eq_off_irreducible`

In Theorem 1 the element $\sum_{\mathbf{c}} e^{\otimes}_{I,\mathbf{c}}$ is a central idempotent (Input 1), and the coefficient at the identity of a central idempotent is the share of its image in the regular representation: the dimension of the image divided by the order of the group. So Theorem 1 says that the representations whose type is not a good tuple fill at most the share $f\varepsilon + \exp(-L)$ of the group algebra. In the proof of the main theorems the fields are the residue fields at the chosen primes above one of two auxiliary primes, M is a fixed integer, and f is the number of infinite places of a large CM field; the theorem is used in Chapter VI. Theorem 2 is the device that makes a comparison of trace formulas possible in the note *Good forms and rows*: subtracting from an idempotent its cubic twist leaves an element of the group algebra supported on matrices with irreducible characteristic polynomial.

2 The setting

Conventions. The natural numbers are $0, 1, 2, \ldots$. Throughout, $\mathbb{F}$ is a finite field, $\bar{\mathbb{F}}$ an algebraic closure of $\mathbb{F}$, and $\#\mathbb{F}$ the number of elements of $\mathbb{F}$; in a family $(\mathbb{F}_v)_{v\in A}$ of finite fields the index set A is a finite set. A finite set I serves as index set of matrices: $\mathrm{GL}_I(\mathbb{F})$ is the group of invertible matrices over $\mathbb{F}$ with rows and columns indexed by I, and $\#I$ is the number of elements of I. The letters r, M, q, f, a denote natural numbers and c a polynomial in $\mathbb{F}[X]$. For a finite group G, $\mathbb{C}[G]$ is its complex group algebra, $y(g)$ is the coefficient of $y \in \mathbb{C}[G]$ at $g \in G$, and a complex

representation of G is also a module over $\mathbb{C}[G]$. A multiplicative character of $\mathbb{F}$ is a homomorphism $\chi\colon \mathbb{F}^\times \to \mathbb{C}^\times$. For a unit $u \in \mathbb{F}^\times$ and a polynomial $c \in \mathbb{F}[X]$ of degree δ, $c^{[u]} = u^\delta\, c(X/u)$ is the polynomial of the same degree whose roots in $\bar{\mathbb{F}}$ are u times the roots of c, with the same multiplicities; $c \mapsto c^{[u]}$ is a bijection of $\mathbb{F}[X]$ that respects products, with inverse $c \mapsto c^{[u^{-1}]}$. Letters of the handout's overview that are not displayed with the data as an argument (among them ε, δ, f, q, n and V) have, in this note, only the meaning given in its notation list.

Definition 2.1 (labels)**.** Let r be a natural number and $c \in \mathbb{F}[X]$. Then c is a label of degree r if c is monic and squarefree, has degree r, and has $c(0) \neq 0$.

LEAN `IsLabel` · a definition of the background library

Definition 2.2 (primitive labels)**.** Let r be a natural number and $c \in \mathbb{F}[X]$. Then c is a primitive label of degree r if c is a label of degree r, if c is irreducible, and if some root $\xi \in \bar{\mathbb{F}}$ of c is a primitive root of unity of order $(\#\mathbb{F})^r - 1$.

LEAN `IsPrimitiveLabel`

So a primitive label of degree r is the minimal polynomial over $\mathbb{F}$ of a generator of the cyclic group $\mathbb{F}_{(r)}^\times$, where $\mathbb{F}_{(r)} \subset \bar{\mathbb{F}}$ is the subfield with $(\#\mathbb{F})^r$ elements. All roots of a primitive label are such generators, because they are conjugate over $\mathbb{F}$.

Definition 2.3 (Green's characters and their idempotents)**.** Fix an injective multiplicative character $T\colon \bar{\mathbb{F}}^\times \to \mathbb{C}^\times$ (the library chooses one for each field). For an integer $\delta \geq 1$ and $x \in \mathbb{F}_{(\delta)}^\times$, let θ_x be the character T^n of $\mathbb{F}_{(\delta)}^\times$, where n is an integer with $T(x) = \exp(2\pi i\, n/((\#\mathbb{F})^\delta - 1))$; the map $x \mapsto \theta_x$ is an isomorphism from $\mathbb{F}_{(\delta)}^\times$ to its group of characters. For $\delta = 1$ it gives, for each multiplicative character χ of $\mathbb{F}$, the unit $x_\chi \in \mathbb{F}^\times$ with $\theta_{x_\chi} = \chi$, and $\chi \mapsto x_\chi$ is an isomorphism of groups.

The complex functions on the similarity classes of square matrices over $\mathbb{F}$ of all sizes form a ring under the product of parabolic induction,

$$(F * H)(g) = \sum_W F(g|_W)\, H(g|_{V/W}),$$

the sum over the g-stable subspaces W of the space V on which g acts. For a monic irreducible $P \in \mathbb{F}[X]$ of degree δ with a root $x \in \bar{\mathbb{F}}^\times$, Green's class function τ_P is $(-1)^\delta$ times the part in size δ of $(\beta_x \cdot \mathbb{1}^{-1}) * \mathbb{1}$; here $\mathbb{1}$ is the constant function 1, $\mathbb{1}^{-1}$ is its inverse in this ring, the dot is the pointwise product, and $\beta_x(g)$ is the sum of θ_x over the eigenvalues of g. For a polynomial $c \neq 0$ over $\mathbb{F}$,

$$\Xi_c = \tau_{P_1} * \cdots * \tau_{P_s}, \qquad P_1, \ldots, P_s \text{ the monic irreducible factors of } c \text{ with multiplicity.}$$

For a finite set I and $c \in \mathbb{F}[X]$, the element $e_{I,c}$ of $\mathbb{C}[\mathrm{GL}_I(\mathbb{F})]$ is

$$e_{I,c} = \sum_{g \in \mathrm{GL}_I(\mathbb{F})} \frac{\Xi_c(1)}{\#\mathrm{GL}_I(\mathbb{F})}\, \overline{\Xi_c(g)}\; g$$

if c is a label of degree $\#I$, and $e_{I,c} = 0$ otherwise.

LEAN `labelIdem` · a definition of the background library, with the class functions of its files on Green's characters

For a label c of degree $\#I$ the restriction of Ξ_c to $\mathrm{GL}_I(\mathbb{F})$ is an irreducible character, by Green's theory as the library proves it; then $e_{I,c}$ is the central idempotent of

$\mathbb{C}[\mathrm{GL}_I(\mathbb{F})]$ that projects every representation to its part of type Ξ_c. If c is irreducible, $\Xi_c = \tau_c$ is a cuspidal character. This note uses nothing of the construction beyond Inputs 1 to 3.

Definition 2.4 (idempotents of tuples)**.** Let $(\mathbb{F}_v)_{v\in A}$ be a family of finite fields, I a finite set, and $\mathbf{c} = (c_v)_{v\in A}$ a family of polynomials $c_v \in \mathbb{F}_v[X]$. Then $e^{\otimes}_{I,\mathbf{c}}$ is the element $\bigotimes_{v\in A} e_{I,c_v}$ of the group algebra of $\prod_{v\in A} \mathrm{GL}_I(\mathbb{F}_v)$: its coefficient at $(g_v)_{v\in A}$ is $\prod_{v\in A} e_{I,c_v}(g_v)$.

LEAN `tupleIdem` · a definition of the background library

Definition 2.5 (good tuples)**.** Let $(\mathbb{F}_v)_{v\in A}$ be a family of finite fields, and let M and r be natural numbers. The set $\mathrm{Good}_{M,r}$ consists of the families $\mathbf{c} = (c_v)_{v\in A}$ of monic polynomials $c_v \in \mathbb{F}_v[X]$ of degree r such that

(i) for every $v \in A$: c_v is a label of degree r, and no $x \in \mathbb{F}_v$ with $x^M = 1$ is a root of c_v;
(ii) for at least one $v \in A$: c_v is a primitive label of degree r.

It is a finite set.

LEAN `goodTuples`

In terms of characters, (i) says that the type Ξ_{c_v} is induced from cuspidal characters among which there is no character $\chi \circ \det$ of GL_1 with $\chi^M = 1$, and (ii) asks for one factor with a cuspidal type that comes from a faithful character of $\mathbb{F}^{\times}_{v,(r)}$.

Inputs from the background library. The statements below are proved in the artifact's library and are taken for granted here. For Inputs 1 and 2 the library's files name J. A. Green, *The characters of the finite general linear groups*, Trans. Amer. Math. Soc. 80 (1955), as their source. The second reading of this write-up compared Input 1 with that paper: both parts are weaker consequences of its §8, Theorems 13 and 14 (pp. 441–444, with the degree formula on pp. 443–444). For Input 2 the place in the paper is to be confirmed; the catalogue of quoted results classes Input 2(b) as folklore.

Input 1 (orthogonality and degree). *Let I be a finite set and $(\mathbb{F}_v)_{v\in A}$ a family of finite fields.* (a) *For every finite set $\mathcal{C}$ of families $\mathbf{c} = (c_v)_{v\in A}$ of polynomials $c_v \in \mathbb{F}_v[X]$, the element $\sum_{\mathbf{c}\in\mathcal{C}} e^{\otimes}_{I,\mathbf{c}}$ is an idempotent whose coefficients are constant on conjugacy classes.* (b) *If $\mathbb{F}$ has q elements, I has r elements and c is a label of degree r, then $e_{I,c}(1)$ is a real number and $e_{I,c}(1) \geq (1 - r/q)\, q^{-r}$.*

Input 2 (twisting; cuspidal support). (a) *Let r be a natural number and suppose that c is a label of degree r; let χ be a multiplicative character of $\mathbb{F}$ and I a finite set. Then $\Xi_c(g)\,\chi(\det g) = \Xi_{c^{[x_\chi]}}(g)$ for all $g \in \mathrm{GL}_I(\mathbb{F})$.* (b) *Let $P \in \mathbb{F}[X]$ be monic and irreducible, and let g be a square matrix over $\mathbb{F}$ with $\tau_P(g) \neq 0$. Then the characteristic polynomial of g is a power of an irreducible polynomial.*

Input 3 (Euler's function). *For every integer $n \geq 1$: $n \leq \varphi(n)\,(2\log n + 1)$, where φ is Euler's function.*

LEAN `isCentralProj_sum_tupleIdem` · `labelChar_degree_bounds` · `labelChar_mul_det` · `isCuspidal_greenTau` · `Hall.IsCuspidal.exists_charpoly_eq_pow` · `le_totient_mul_log`

Input 1(b) is a weak form of Green's formula for the degree of Ξ_c: only this lower bound is proved, and only for labels. (As the second reading records it: for a label whose irreducible factors have degrees $d_1, \dots, d_s$, Green's degree is $\prod_{n=1}^{r}(q^n-1)/\prod_{i=1}^{s}(q^{d_i}-1)$ up to sign, and the bound follows from $\prod_i (q^{d_i}-1) \le q^r$.) Input 2(b) is the vanishing of cuspidal characters off the primary classes; the library proves it for every class function that is cuspidal in the sense that its averages over block triangular matrices with two diagonal blocks of positive size vanish, and proves that τ_P is cuspidal in this sense.

3 Primitive labels

Two facts about finite fields are used without further mention. If $P \in \mathbb{F}[X]$ is irreducible of degree δ and $x \in \bar{\mathbb{F}}$ is a root of P, then the roots of P are $x, x^q, \dots, x^{q^{\delta-1}}$ with $q = \#\mathbb{F}$, each a simple root; and $x^{q^n} = x$ holds exactly when δ divides n.

Lemma 3.1 (scaling the roots)**.** *Let $c \in \mathbb{F}[X]$, let u be a unit of $\mathbb{F}$, and let r and M be natural numbers. Suppose that $u^M = 1$ and that M^2 divides $\#\mathbb{F} - 1$. If c is a primitive label of degree r, then $c^{[u]}$ is a primitive label of degree r.*

Proof. Put $q = \#\mathbb{F}$. The polynomial $c^{[u]}$ is monic of degree r, its roots are u times those of c, hence distinct, and $c^{[u]}(0) = u^r c(0) \neq 0$; so $c^{[u]}$ is a label of degree r. It is irreducible, because $c \mapsto c^{[u]}$ is a bijection of $\mathbb{F}[X]$ that respects products and degrees. Let ξ be a root of c of order $n = q^r - 1$; then $u\xi$ is a root of $c^{[u]}$, and it remains to see that $u\xi$ has order n. Let o be the order of u. It divides M, so o^2 divides $q-1$, which divides n. Since $(u\xi)^n = 1$, it suffices to show $(u\xi)^{n/p} \neq 1$ for every prime p dividing n. If p divides o, then po divides o^2 and so n; hence $u^{n/p} = 1$ and $(u\xi)^{n/p} = \xi^{n/p} \neq 1$. If p does not divide o and $(u\xi)^{n/p} = 1$, then $\xi^{n/p} = u^{-n/p}$; the left side has order p and the order of the right side divides o, a contradiction. □

LEAN `IsPrimitiveLabel.scaleRoots`

Lemma 3.2 (the norm of a generator is a generator)**.** *Let $c \in \mathbb{F}[X]$ and let r be a natural number. Suppose c is a primitive label of degree r. Then the product of the roots of c in $\bar{\mathbb{F}}$, counted with multiplicity, is a primitive root of unity of order $\#\mathbb{F} - 1$; and there is a natural number n such that ξ^n equals this product for every root ξ of c in $\bar{\mathbb{F}}$.*

Proof. Put $q = \#\mathbb{F}$ and $n = 1 + q + \cdots + q^{r-1}$. Let ξ be any root of c. As c is irreducible of degree r, its roots are ξ^{q^i}, $0 \le i < r$, each once, and their product is ξ^n. For the root ξ_0 of order $q^r - 1 = n\,(q-1)$ the power ξ_0^n has order $q-1$. □

LEAN `IsPrimitiveLabel.isPrimitiveRoot_prod_roots`

4 The cubic twist: proof of Theorem 2

Proof of Theorem 2. Put $q = \#\mathbb{F}$ and $r = \#I = 3^{a+1}$. If c is not a primitive label of degree r, take $\tilde{c} = c$: then (1) is empty and (2) is trivial. So assume that c is a primitive label of degree r, and let ξ be a root of c of order $q^r - 1$.

The element ξ^3 has degree r over $\mathbb{F}$. Its degree δ divides r, since $\xi^3 \in \mathbb{F}_{(r)}$. If $\delta \neq r$, then δ divides $r/3$, so ξ^3 lies in the field with $z = q^{r/3}$ elements and its order $(q^r-1)/3$ divides $z-1$. Then $z^3 - 1 \le 3(z-1)$, that is $(z-1)^2(z+2) \le 0$, which is false for $z \ge 2$. (The order of ξ^3 is $(q^r-1)/3$ because 3 divides $q-1$.)

The three cubic twists of c are distinct. As 3 divides $q-1$, the group U_3 of the $u \in \mathbb{F}$ with $u^3 = 1$ has three elements. Let $u, u' \in U_3$ with $c^{[u]} = c^{[u']}$. This polynomial is irreducible and has the roots $u\xi$ and $u'\xi$, so $u'\xi = (u\xi)^{q^l} = u\,\xi^{q^l}$ for some l with $0 \le l < r$. Taking cubes, $\xi^3 = (\xi^3)^{q^l}$; since ξ^3 has degree r, r divides l, so $l = 0$ and $u' = u$.

Choice of $\tilde{c}$. Of the three distinct polynomials $c^{[u]}$, $u \in U_3$, at most two lie in $\mathcal{S}_1 \cup \mathcal{S}_2$. Choose $u \in U_3$ with $c^{[u]} \notin \mathcal{S}_1 \cup \mathcal{S}_2$, and put $\tilde{c} = c^{[u]}$. By Lemma 3.1 with $M = 3$, which applies because $u^3 = 1$ and 9 divides $q-1$, $\tilde{c}$ is a primitive label of degree r. This is (1).

The coefficients. Let χ be the multiplicative character of $\mathbb{F}$ with $x_\chi = u$; then $\chi^3 = 1$, because $\chi \mapsto x_\chi$ is an isomorphism of groups. By Input 2(a), $\Xi_{\tilde{c}}(g) = \Xi_c(g)\,\chi(\det g)$ for $g \in \mathrm{GL}_I(\mathbb{F})$, in particular $\Xi_{\tilde{c}}(1) = \Xi_c(1)$. Both c and $\tilde{c}$ are labels of degree r, so by Definition 2.3

$$e_{I,\tilde{c}}(g) = e_{I,c}(g) \cdot \overline{\chi(\det g)} \qquad (g \in \mathrm{GL}_I(\mathbb{F})).$$

Let g have a characteristic polynomial that is not irreducible. If $\Xi_c(g) = 0$, both coefficients vanish. Otherwise, c being irreducible, $\Xi_c = \tau_c$, and by Input 2(b) the characteristic polynomial of g is P^n with P irreducible and $n \ge 1$. Comparing degrees, n divides 3^{a+1}; and $n \ne 1$, because the characteristic polynomial is not irreducible. So 3 divides n, and, r being odd,

$$\det g = (-1)^r\,P(0)^n = \bigl(-P(0)^{n/3}\bigr)^3$$

is a cube in $\mathbb{F}^\times$. Hence $\chi(\det g) = 1$, and the two coefficients agree. This is (2). □

5 The share of bad types: proof of Theorem 1

Proof of Theorem 1. If $f = 0$, then A is empty, the only family is the empty one, and it does not satisfy (ii) of Definition 2.5; so the element in question is 1, with coefficient 1 at the identity, and $L \le 0$ by the last hypothesis, so $\exp(-L) \ge 1$. Let $f \ge 1$. Put

$$\varepsilon_0 = \Bigl(1 - \frac{r}{q}\Bigr) q^{-r}, \qquad \text{so that} \quad 0 \le \varepsilon_0 \le q^{-r}.$$

For $v \in A$ let $\mathcal{P}_v$ be the set of the monic polynomials of degree r over $\mathbb{F}_v$; it has q^r elements. Let $\mathcal{B}_v \subseteq \mathcal{P}_v$ be the set of those that are not primitive labels of degree r, and let $\mathcal{A}_v \subseteq \mathcal{P}_v$ be the set of those that are neither of the form $(X - x)\,h$ with $x \in \mathbb{F}_v$, $x = 0$ or $x^M = 1$, and h monic of degree $r-1$, nor of the form $u^2 h$ with u monic of positive degree and h monic.

Step 1: $\#\mathcal{A}_v \cdot \varepsilon_0 \ge 1 - \varepsilon$. There are at most $1 + M$ elements x and q^{r-1} polynomials h, hence at most $(1+M)\,q^{r-1}$ polynomials of the first form. A polynomial of degree r of the second form is given by the degree $d < r$ of h, with $r - d$ even, by h and by u; for each d there are at most $q^{d+(r-d)/2} \le q^{r-1}$ of them, hence at most $r\,q^{r-1}$ in all. So $\#\mathcal{A}_v \ge q^r(1 - (1+M+r)/q)$, and, as $\varepsilon_0 \ge 0$,

$$\#\mathcal{A}_v \cdot \varepsilon_0 \;\ge\; \Bigl(1 - \frac{1+M+r}{q}\Bigr)\Bigl(1 - \frac{r}{q}\Bigr) \;\ge\; 1 - \frac{1+M+2r}{q} \;\ge\; 1 - \varepsilon.$$

Step 2: $\#\mathcal{B}_v \cdot \varepsilon_0 \le 1 - L/f$. The field $\bar{\mathbb{F}}_v$ contains exactly $\varphi(q^r - 1)$ primitive roots of unity of order $q^r - 1$. Let ξ be one of them and δ the degree of its minimal

polynomial P over $\mathbb{F}_v$. Then $\xi^{q^\delta-1}=1$, so q^r-1 divides $q^\delta-1$; and $\xi^{q^r}=\xi$, so δ divides r and $q^\delta-1$ divides q^r-1. Hence $\delta=r$, and P is monic, irreducible, of degree r, with $P(0)\neq 0$ and with the root ξ: a primitive label of degree r. Each such P has r roots, so there are at least $\varphi(q^r-1)/r$ primitive labels of degree r over $\mathbb{F}_v$. By Input 3, and because $q^r\geq 2$,

$$q^r \leq 2\,(q^r-1) \leq 2\,\varphi(q^r-1)\big(2\log(q^r-1)+1\big) \leq 2r\,(2r\log q+1)\cdot\frac{\varphi(q^r-1)}{r}.$$

With the hypothesis on L this gives $\#(\mathcal{P}_v\setminus\mathcal{B}_v)\geq q^r/(2r(2r\log q+1))\geq q^r L/f$. Therefore $\#\mathcal{B}_v\leq q^r(1-L/f)$, and $\#\mathcal{B}_v\cdot\varepsilon_0\leq \#\mathcal{B}_v\,q^{-r}\leq 1-L/f$.

Step 3: a family $\mathbf{c}\in\prod_v\mathcal{A}_v$ that does not lie in $\prod_v\mathcal{B}_v$ belongs to $\mathrm{Good}_{M,r}$. Let $c_v\in\mathcal{A}_v$. It is monic of degree r. It has no root $x\in\mathbb{F}_v$ with $x=0$ or $x^M=1$, since a root x gives $c_v=(X-x)h$ with h monic of degree $r-1$. It is squarefree: otherwise the square of a monic irreducible polynomial u divides it, and $c_v=u^2h$ with h monic. So c_v is a label of degree r, and (i) holds; (ii) holds because some c_v is not in $\mathcal{B}_v$.

Step 4: conclusion. Let

$$y=\sum_{\mathbf{c}\in\mathrm{Good}_{M,r}} e^{\otimes}_{I,\mathbf{c}},\qquad \text{where}\quad e^{\otimes}_{I,\mathbf{c}}(1)=\prod_{v\in A} e_{I,c_v}(1)$$

by Definition 2.4. For $\mathbf{c}\in\mathrm{Good}_{M,r}$ every factor is a real number $\geq\varepsilon_0\geq 0$ by Input 1(b). So $y(1)$ is real, and with Step 3, then Steps 1 and 2 (all factors being non-negative, and $1-\varepsilon\geq 0$),

$$y(1)\;\geq\;\#\mathrm{Good}_{M,r}\cdot\varepsilon_0^f\;\geq\;\prod_{v\in A}(\#\mathcal{A}_v\,\varepsilon_0)-\prod_{v\in A}(\#\mathcal{B}_v\,\varepsilon_0)\;\geq\;(1-\varepsilon)^f-\Big(1-\frac{L}{f}\Big)^f.$$

By Bernoulli's inequality, $(1-\varepsilon)^f\geq 1-f\varepsilon$. Moreover $0\leq 1-L/f\leq\exp(-L/f)$, so that $(1-L/f)^f\leq\exp(-L)$. Hence $1-y(1)\leq f\varepsilon+\exp(-L)$. □

6 Where the results are used

Theorem 1 is used once, in the proof of Theorem B of the handout (enough forms, Chapter VI): there the fields are the residue fields at the chosen primes above one of the primes $g_0(D)$, $g_1(D)$ of the data D, with q that prime, the integer M is the integer $M_\omega(D)$ of the data, and $L=9\log b(D)$; the bound says that the good idempotent discards a negligible share of the Poincaré series. The set $\mathrm{Good}_{M,r}$ enters the definition of the good idempotent of the data (note *Good forms and rows*). Theorem 2 is used in the same note, in the comparison of the traces of Hecke operators on good forms of two unitary groups. Lemmas 3.1 and 3.2 are used in the note *At most one primitive label for a Hecke eigensystem*: the first to twist a primitive label by a character of order dividing $M_\omega(D)$, the second to see that the roots of a primitive label cannot all be roots of unity of order dividing $M_\omega(D)$.

7 Notes on the formal text

1. *Letters.* The formal text writes `k` for the field, `n` and `ι` for the index sets I and A, `a` for the unit u of Lemma 3.1 and, as here, for the exponent of Theorem 2, `S` and `T` for the sets $\mathcal{S}_1$ and $\mathcal{S}_2$, `P` for the polynomial $\tilde{c}$, and `m` for the number ε_0. Of these, k, n and ι are kept in this write-up for objects of the data, which is why the field and the index sets are written otherwise here. The letters P, m and a are letters of the data as well (the parameters, the number of lifts, and an integer that is always written with its argument, $a(D)$); none of these objects occurs in this note, so the irreducible polynomial P and the exponent a here are local.

2. *The polynomial $c^{[u]}$.* It is the library's operation of scaling the roots, $\sum_i c_i\, u^{\delta-i} X^i$ for $c = \sum_i c_i X^i$ of degree δ.

3. *Theorem 2 for a polynomial that is not a primitive label.* The statement is made for every polynomial c; for such a c the formal proof takes $\tilde{c} = c$, as here.

4. *The distinctness of the cubic twists.* The formal proof quotes a lemma of the library: for a prime l, an exponent n and ξ of order $q^{3^{a+1}} - 1$ with $9 \mid q-1$, the conjugates of ξ^{l^n} over $\mathbb{F}$ are distinct. It is used for $l = 3$, $n = 1$. The two lines given here for that case are mine; they do not use that 9 divides $q-1$, only that 3 does. The hypothesis $9 \mid \#\mathbb{F} - 1$ of Theorem 2 is needed for Lemma 3.1. The formal proof finds the twist outside $\mathcal{S}_1 \cup \mathcal{S}_2$ by contradiction: otherwise $u \mapsto [c^{[u]} \in \mathcal{S}_1]$ would embed U_3 into a set of two elements.

5. *Lemma 3.1.* The formal proof quotes from Mathlib that the order of a product of two commuting elements u, ξ equals the order of ξ when $p \cdot \mathrm{ord}(u)$ divides $\mathrm{ord}(\xi)$ for every prime p dividing $\mathrm{ord}(u)$; the proof above includes an argument for it. The case $M = 0$ cannot occur, since 0 does not divide $\#\mathbb{F} - 1$.

6. *Theorem 1.* The formal statement asks $M \neq 0$ through an instance argument. The sets $\mathcal{A}_v$ are written there as differences of finite sets: the monic polynomials of degree r, less the products $(X - x)h$ and less the products $u^2 h$ with h monic of degree $d < r$ and u monic of degree $\lfloor (r-d)/2 \rfloor$; products of the wrong degree are removed with them, which changes nothing. The case $f = 0$ needs no separate treatment there, because division by zero gives zero in Lean. The inequalities of Steps 1, 2 and 4 are established in the formal text by chains of rewriting under the order relation, closed by automatic tactics; I derived each of them by hand, as written above, and compared the terms of the chains with it. What the automatic tactics prove in between was not replayed.

7. *What Theorem 1 does not use.* Of Green's theory it uses only the lower bound of Input 1(b) for the coefficient at the identity. It does not use that the elements $e^{\otimes}_{I,\mathbf{c}}$ are orthogonal idempotents; that is needed for the interpretation as a share.

8. *Arguments.* In Definition 2.2 the formal text does not ask that $\mathbb{F}$ be finite; the number $\#\mathbb{F}$ is 0 for an infinite field there, and the notion is used for finite fields only. The dossier of the unit records the arguments of the declarations of Definition 2.5 and of Theorem 1 as derived by hand from the text of their signatures, not as printed by Lean; they agree with what is stated here.

How this note was written. From the dossier of the unit: its statements and the structured proofs of the two lemmas and the two theorems, which a program compared line by line with the source (99 leaf steps in the 16 proofs of the unit). I read the whole Lean file of these six declarations, and in the library the definitions of labels, of Green's class functions and of the idempotents, and the statements of the six results quoted as Inputs. The statements of this note were compared with the Lean statements argument by argument, and the order of the quantifiers is that of the formal text. The proofs follow the formal ones, with the differences listed in notes 4 to 6. The interpretations after the statements and after Definition 2.5 are mine. The reference and its place are those that the library's files, the catalogue of this handout and its second reading give; I did not check them in the literature.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Definition 2.2	`IsPrimitiveLabel`	BadShare.lean	29–30
Lemma 3.1	`IsPrimitiveLabel.scaleRoots`	BadShare.lean	35–41
Lemma 3.2	`IsPrimitiveLabel.isPrimitiveRoot_prod_roots`	BadShare.lean	45–51
Definition 2.5	`goodTuples`	BadShare.lean	55–57
Theorem 2	`exists_labelIdem_eq_off_irreducible`	BadShare.lean	64–86
Theorem 1	`bad_share_le`	BadShare.lean	92–136
Definition 2.1	`IsLabel`	FrobeniusOrbits.lean	48–48
Definition 2.3	`labelIdem`	FrobeniusOrbits.lean	61–63
Definition 2.4	`tupleIdem`	LabelIdempotents.lean	108–108
Input 1	`isCentralProj_sum_tupleIdem`	LabelIdempotents.lean	124–125
Input 1	`labelChar_degree_bounds`	LabelDegree.lean	35–57
Input 2	`labelChar_mul_det`	LabelIdempotents.lean	39–57
Input 2	`isCuspidal_greenTau`	GreenTau.lean	26–27
Input 2	`Hall.IsCuspidal.exists_charpoly_eq_pow`	CuspidalSupport.lean	26–45
Input 3	`le_totient_mul_log`	OstrowskiTaussky.lean	12–19

The two theorems as they are written in the artifact.

```
lemma bad_share_le {n ι} [Fintype n] [Fintype ι] {k : ι → _}
    [∀ w, Field (k w)] [∀ w, Finite (k w)] {q M r f : ℕ} [NeZero M] (ε L : ℝ) (hq : ∀ w, .card (k w)
    = q) (hr : card n = r) (hf : card ι = f) (hn : 1 < r) (h2 : r < q) (hε : ε ≤ 1)
    (hs : (1 + M + 2 * r) / q ≤ ε) (hL : 2 * r * (2 * (r * .log q) + 1) * L ≤ f) :
    ((1 - ∑ c ∈ goodTuples M r, tupleIdem (k := k) n c).coeff 1).re ≤ f * ε + exp (-L)

lemma exists_labelIdem_eq_off_irreducible {k n a} [Field k] [Finite k] [Fintype n] (hr :
  Fintype.card n = 3 ^ (a + 1)) (hq : 9 ∣ .card k - 1) c {S T : Set k[X]} (hS : S.Subsingleton)
  (hT : T.Subsingleton) : ∃ P, (IsPrimitiveLabel (Fintype.card n) c →
  IsPrimitiveLabel (Fintype.card n) P ∧ P ∉ S ∪ T) ∧ ∀ g, ¬Irreducible g.1.charpoly →
  (labelIdem n c).coeff g = (labelIdem n P).coeff g
```

In these texts `k` is the field or the family of fields, `n` and `ι` are the index sets I and A, `labelIdem n c` is $e_{I,c}$, `tupleIdem n c` is $e^{\otimes}_{I,\mathbf{c}}$, `goodTuples M r` is $\mathrm{Good}_{M,r}$, `.card` is the number of elements, and `coeff` takes the coefficient of an element of a group algebra.

Notation

$\mathbb{F}$, $\bar{\mathbb{F}}$, $\#\mathbb{F}$ a finite field, an algebraic closure, the number of its elements

$\mathbb{F}_{(\delta)}$ the subfield of $\bar{\mathbb{F}}$ with $(\#\mathbb{F})^{\delta}$ elements

q the number of elements of the finite field or fields at hand

A, v, $\mathbb{F}_v$, f a finite index set, an index, the field with index v; $f = \#A$

I, $\mathrm{GL}_I(\mathbb{F})$ a finite set; the invertible matrices over $\mathbb{F}$ indexed by I

r a degree; in the theorems $r = \#I$

c, $\tilde{c}$, P, h, u polynomials over a finite field; u also a unit of $\mathbb{F}$

X the variable of polynomials

$c^{[u]}$ the polynomial whose roots are u times those of c

$\mathbf{c} = (c_v)_{v \in A}$ a family of polynomials, $c_v \in \mathbb{F}_v[X]$

M a natural number: roots x with $x^M = 1$ are excluded

ε, L the real numbers of Theorem 1

a a natural number: $\#I = 3^{a+1}$ in Theorem 2 (not the integer $a(D)$ of the data)

$\mathcal{S}_1$, $\mathcal{S}_2$ sets of polynomials with at most one element

$\mathbb{C}[G]$, $y(g)$ the complex group algebra of a finite group G; the coefficient of y at g

T, θ_x, x_χ an injective character of $\bar{\mathbb{F}}^{\times}$; the character named by x; the unit that names χ

χ a multiplicative character of $\mathbb{F}$

τ_P, Ξ_c Green's class functions (Definition 2.3)

$F * H$, $\mathbb{1}$, β_x, W, V the induction product, the constant function 1, the sum of θ_x over eigenvalues; a stable subspace of a space

$e_{I,c}$ the idempotent of the label c in $\mathbb{C}[\mathrm{GL}_I(\mathbb{F})]$ (Definition 2.3)

$e^{\otimes}_{I,\mathbf{c}}$ the tensor product of the e_{I,c_v} (Definition 2.4)

$\mathrm{Good}_{M,r}$ the set of good tuples (Definition 2.5)

$\mathcal{P}_v$, $\mathcal{A}_v$, $\mathcal{B}_v$, $\mathcal{C}$ sets of monic polynomials in the proof of Theorem 1; a finite set of families

ε_0 the number $(1 - r/q)\, q^{-r}$

ξ, ξ_0 roots of a polynomial in $\bar{\mathbb{F}}$

U_3 the group of cube roots of unity in $\mathbb{F}$

φ Euler's function

π, i in $2\pi i$ the circle constant and the imaginary unit

n, l, o, p, d, i, s, z, δ auxiliary integers: exponents, orders, a prime, degrees, indices

g, x, y a matrix or group element; an element of a field; an element of a group algebra

D, $g_0(D)$, $g_1(D)$, $M_\omega(D)$, $b(D)$ the data of the proof and objects attached to it, named in §6 only: two primes, an integer, a scale

References

J. A. Green, *The characters of the finite general linear groups*, Trans. Amer. Math. Soc. 80 (1955). Named by the library's files as the source of Inputs 1 and 2. Place for Input 1, as located by the second reading of this write-up: §8, Theorems 13–14, pp. 441–444. I did not open the paper.

READBACK NOTE 9

The test datum of an auxiliary pair, joint harmonics with staircase weights, and the theta lift as a conjugate descent

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. An *auxiliary pair* is a dual pair of unitary groups over a CM field, an indefinite space W of rank r and the standard definite space of rank $n > r$, with a level at two families of split places and a Hecke character; it carries one test datum for the Weil representation, the tensor product of the Fock vector of a harmonic polynomial and a finite datum Υ, and a map from algebraic forms on the definite group to functions on the indefinite one, defined as a finite sum over classes. This unit proves the three facts about this datum that the rest of Theorem C's proof uses. The finite datum is a sum of translates of an indicator function and transforms under the level by an explicit character. The polynomial at the highest vector is a product, over the infinite places, of products of minors (staircases), which are joint harmonics for matched discrete series parameters. And the class sum is a fixed multiple of the complex conjugate of a theta descent.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, whose final theorems are the Riemann hypothesis and its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note treats one unit of the artifact: 15 of its 16 declarations, all that have a name (the remaining one is the remark that a finite product of finite groups is finite). The proofs are given in full, relative to two kinds of inputs, which are stated and not proved: the definition of an auxiliary pair and of its objects, which belongs to the unit on the rows (Setting 2.3); and results of the background library on the Weil representation, on joint harmonics and on algebraic forms (Setting 2.1 and the facts (Q1) to (Q11)). Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 15 declarations. This is the second of three notes on the theta correspondence behind Theorem C; it uses the Fock space of the note *Decay of Heisenberg matrix coefficients of homogeneous Fock vectors, uniform in the degree.*

1 The statements

The objects are defined in §2. In brief: for an auxiliary pair O, $W(O)$ is its indefinite space, with signs ε_O on the index set $[r]$, and $\mathrm{I}_n(E)$ the definite one; $\mathrm{Form}(O)$ is a space of algebraic modular forms on the definite unitary group, with values in a representation ρ_{κ_O,e_O} on a space $\mathbb{V}_{n,\kappa_O}$; $\mathrm{Th}_O(\Xi)$ is the class-sum lift of such a form, a function on the adelic group $G(\varepsilon_O)$ of $W(O)$; $\mathrm{J}_O(\xi) = v_{O,\mathrm{Har}_O(\xi)} \otimes \Upsilon(O)$ is the test datum, linear in $\xi \in \mathbb{V}_{n,\kappa_O}$; and $\mathrm{Desc}^{W(O),\mathrm{I}_n(E)}_{\chi_O,\chi'_O}$ is the theta descent from the definite to the indefinite group.

Theorem 1 (the class-sum lift is a conjugate descent). *In Setting 2.1, let O be an auxiliary pair as in Setting 2.3 and let $\Xi \in \mathrm{Form}(O)$. Then, as functions on $G(\varepsilon_O)$,*

$$\mathrm{Th}_O(\Xi) \;=\; \dim \mathbb{V}_{n,\kappa_O} \cdot [\mathbf{K}_0(n) : \mathbf{K}(O)] \cdot \overline{\mathrm{Desc}^{W(O),\mathrm{I}_n(E)}_{\chi_O,\chi'_O}\Big(\mathrm{J}_O(v^+_{\kappa_O}(l^V_O(\,\cdot\,,\,\cdot\,))),\ \overline{F_{l^V_O(\,\cdot\,,\,\cdot\,),1_n}(\Xi)}\Big)},$$

where $l^V_O(\,\cdot\,,\,\cdot\,)$ is the function $(\tau,i) \mapsto l^V_O(\tau,i)$, $v^+_{\kappa_O}(l^V_O(\,\cdot\,,\,\cdot\,))$ is the highest vector for it, $F_{l^V_O(\,\cdot\,,\,\cdot\,),1_n}(\Xi)$ is the scalar function $h \mapsto \langle v^+_{\kappa_O}(l^V_O(\,\cdot\,,\,\cdot\,)), \rho_{\kappa_O,e_O}(h^{-1}_\infty)\,\Xi(h_{\mathrm{f}})\rangle_{\kappa_O}$ on the adelic group of $\mathrm{I}_n(E)$, and the bars are complex conjugation of functions. The measure of the descent is normalised as in Setting 2.1.

LEAN `AuxPair.theta_eq_smul_star_lift_swap`

Theorem 2 (staircases are joint harmonics). *Let O be an auxiliary pair as in Setting 2.3 and let l be a family of integers $l(\tau,c)$, indexed by the infinite places τ of E and $c \in [r]$, such that l fits O* (Definition 2.7). *Then there is a family $\Phi = (\Phi_\tau)$ of complex polynomials in the variables X_{ic}, $i \in [n]$, $c \in [r]$, one for each infinite place τ, with these three properties.*

(i) *For every τ, Φ_τ is a joint harmonic* (Definition 2.6) *for the signs $(1_n, \varepsilon_O)$, the weights $(r_-(O), -(\lfloor n/2 \rfloor + \mathrm{wt}_O(\tau)))$ and the parameters $(l^V_O(\tau,\,\cdot\,), -l(\tau,\,\cdot\,))$.*

(ii) *$\mathrm{Har}_O(v^+_{\kappa_O}(l^V_O(\,\cdot\,,\,\cdot\,))) = \prod_\tau \Phi_\tau((X_{\tau,i,c})_{i,c})$: the polynomial of the highest vector is the product of the Φ_τ, each written in the variables of its place.*

(iii) *For every τ and every $c \in [r]$, the lowest K-type of the parameter $-l(\tau,\,\cdot\,)$ for the signs ε_O is a staircase weight:*

$$\Lambda_{\varepsilon_O}(-l(\tau,\,\cdot\,))(c) = \mathrm{sw}_n(\varepsilon_O, \mathrm{ord}_O, \kappa^\pm_O(\tau), -(\lfloor n/2 \rfloor + \mathrm{wt}_O(\tau)))(c),$$

where $\kappa^\pm_O(\tau)$ is the function with $(1,i) \mapsto \kappa^+_O(\tau,i)$ and $(-1,i) \mapsto \kappa^-_O(\tau,i)$.

LEAN `AuxPair.FitsParams.exists_isJointHarmonic`

Theorem 1 turns a statement about a finite sum over classes of a definite group into a statement about an integral of a theta kernel, to which the tools of the theta correspondence apply: the vanishing results of the third note of this sequence, and the adjunction between Th_O and the theta lift from $W(O)$ (§6). Theorem 2 supplies the archimedean input of that correspondence (Corollary 4.4): with the datum

$\mathrm{J}_O(v^+_{\kappa_O}(l^V_O(\,\cdot\,,\,\cdot\,)))$, theta lifts carry forms of the discrete series type of l on $W(O)$ to forms of the type of $l^V_O(\,\cdot\,,\,\cdot\,)$ on the definite group, and complex conjugates of forms of the type of $l^V_O(\,\cdot\,,\,\cdot\,)$ back to complex conjugates of forms of the type of l (Definition 2.8); accordingly Theorem 1 descends the conjugate $\overline{F_{l^V_O(\,\cdot\,,\,\cdot\,),1_n}(\Xi)}$. The two propositions of §3 describe the finite part $\Upsilon(O)$ of the datum.

2 The setting

Conventions. Throughout, E is a CM field with a CM type: a number field that is a totally imaginary quadratic extension of a totally real field, together with the choice, for every infinite place τ of E, of a complex embedding that induces τ. Complex conjugation of E and of its completions and adele rings is written $z \mapsto \overline{z}$; for a matrix X, X^* is the conjugate transpose and X^{T} the transpose. Finite places are written w; $\overline{w}$ is the conjugate place, E_w the completion, $\mathfrak{o}_w$ its ring of integers and $\mathbb{F}_w$ its residue field. The finite adeles of E are $\mathbb{A}_{\mathrm{f}}$, and $\hat{\mathfrak{o}} = \prod_w \mathfrak{o}_w$. The index sets $[r]$, $[n]$ and $[r']$ are finite sets with r, n and r' elements; an $n \times r$ matrix has its rows indexed by $[n]$ and its columns by $[r]$. Where a statement says so, E carries Weil data, described in Setting 2.1. Signs are elements of $\{1,-1\}$, and 1_n is the family of signs on $[n]$ that is constantly 1. The Fock space $\mathcal{F}(\varepsilon)$ of a family of signs ε, the vectors $v_{\varepsilon,p}$ of polynomials p and the group $U(\varepsilon)$ are those of the note on Fock vectors. Letters of the handout's overview that are not displayed with the data as an argument (among them Υ, R, f, ε, β, n and V) have, in this note, only the meaning given in its notation list.

Setting 2.1 (adelic unitary groups, the Weil representation, theta descents; quoted)**.** The following objects are those of the background library; they are recalled, not constructed.

Weil data. The field E contains a primitive root of unity of order 5040 and is equipped with: an element $\jmath \in E$ with $\overline{\jmath} = -\jmath$ whose image under the chosen embedding at every infinite place has positive imaginary part; a finite idele that normalises the self-duality of $\hat{\mathfrak{o}}$; and a *splitting character* υ, a unitary Hecke character of E with $\upsilon(\overline{z}) = \upsilon(z)^{-1}$, equal to $\prod_\tau z_\tau/|z_\tau|$ on the ideles whose finite part is a unit of $\hat{\mathfrak{o}}$. Its restriction to the ideles of norm one is written υ_1.

Groups. For a family of signs ε on $[r]$, $G_\infty(\varepsilon)$ is the product of one copy of $U(\varepsilon)$ for each infinite place of E; $G_{\mathrm{f}}(r)$ is the group of matrices $u \in \mathrm{GL}_r(\mathbb{A}_{\mathrm{f}})$ with $u^*u = 1$; and $G(\varepsilon) = G_\infty(\varepsilon) \times G_{\mathrm{f}}(r)$, with elements $u = (u_\infty, u_{\mathrm{f}})$. The subgroup $\mathbf{K}_0(r) = G_{\mathrm{f}}(r) \cap \mathrm{GL}_r(\hat{\mathfrak{o}})$ is open and compact; a *level* is an open subgroup of it. For a finite place $w \neq \overline{w}$, reduction modulo w is a homomorphism $\mathrm{red}_w \colon \mathbf{K}_0(r) \to \mathrm{GL}_r(\mathbb{F}_w)$. A *hermitian space with a point*, of signs ε, is a pair of a matrix J over E and an element ϕ of GL_r of the adeles with $\phi^* J \phi$ equal to $\mathrm{diag}(\varepsilon)$ at the infinite places and to 1 at the finite ones; its rational points, the $\gamma \in \mathrm{GL}_r(E)$ with $\gamma^* J \gamma = J$, map to $G(\varepsilon)$ by $\gamma \mapsto \phi^{-1}\gamma\phi$. The *standard definite space* $\mathrm{I}_n(E)$ has $J = 1$ and $\phi = 1$, and signs 1_n; its rational points are the unitary matrices over E, and $\gamma \mapsto \gamma_{\mathrm{f}}$ denotes their embedding in $G_{\mathrm{f}}(n)$. An *automorphic character* is a continuous unitary character of the adeles of norm one of E that is trivial on the elements of E of norm one. It has *weight* $(e_\tau)_\tau$, a family of integers, and *tame part* $(\psi_w)_w$, a family of multiplicative characters of the residue fields $\mathbb{F}_w$, extended by 0 to $\mathbb{F}_w$ and almost all trivial, if its

value at every z with integral finite part is $\prod_\tau z_\tau^{e_\tau} \prod_w \psi_w(z \bmod w)$. The character υ_1 has weight 1 at every place and trivial tame part.

The Weil representation. For hermitian spaces with points W, of signs ε on $[r]$, and V, of signs ε' on $[n]$, and automorphic characters χ, χ', the library provides a Banach space $\mathcal{S}(W,V)$ of *data* on $\mathrm{Hom}(W,V)$; a continuous representation $\omega^{W,V}_{\chi,\chi'}$ of $G(\varepsilon)\times G(\varepsilon')$ on it, in which (u,h) acts by $\chi(\det h)\,\chi'(\det u)^{-1}$ times the Weil representation of the unitary group of $\mathrm{Hom}(W,V)$ at the map $X\mapsto hXu^{-1}$; and a bounded linear functional $\theta_{W,V}$ on $\mathcal{S}(W,V)$, the *theta functional*, with

$$\theta_{W,V}(\omega^{W,V}_{\chi,\chi'}(\gamma u,\gamma' h)\varphi) = \theta_{W,V}(\omega^{W,V}_{\chi,\chi'}(u,h)\varphi) \tag{1}$$

for rational points γ, γ' of W and V. The *theta descent* of a function Y on $G(\varepsilon')$ with datum $\varphi\in\mathcal{S}(W,V)$ is the function on $G(\varepsilon)$

$$\mathrm{Desc}^{W,V}_{\chi,\chi'}(\varphi,Y)(u) = \int_{[V]} \theta_{W,V}(\omega^{W,V}_{\chi,\chi'}(u,h)\varphi)\,Y(h)\,\mathrm{d}h,$$

the integral over the quotient of $G(\varepsilon')$ by the rational points of V, with the invariant measure of the library; for $V=\mathrm{I}_n(E)$ it is normalised so that $G_\infty(1_n)\times\mathbf{K}_0(n)$ has volume 1, the rational points carrying the counting measure.

Data. Let $\mathcal{F}(\varepsilon,\varepsilon')$ be the Fock space of the signs $\varepsilon'_i\varepsilon_c$ on the index set of all triples (τ,i,c) of an infinite place, an element of $[n]$ and an element of $[r]$, and $v_p^{\varepsilon,\varepsilon'}$ the vector in it of a polynomial p in the variables $X_{\tau,i,c}$. For $v\in\mathcal{F}(\varepsilon,\varepsilon')$ and $f\in\mathcal{S}(W,V)$ there is a datum $v\otimes f\in\mathcal{S}(W,V)$, bilinear and bounded in (v,f). A *modulus* is a non-zero ideal $\mathfrak{m}$ of the integers of E that is divisible by no pair $w,\overline{w}$. For a complex function Ψ on the $n\times r$ matrices over the integers modulo $\mathfrak{m}$ there is a *finite datum* $[\Psi]^{W,V}_{\mathfrak{m}}$: at the places dividing $\mathfrak{m}$ it is Ψ of the reduction of an integral matrix, and elsewhere it is the vacuum, the characteristic function of the integral matrices. With the Gaussian $\varphi_0\in\mathcal{F}(\varepsilon,\varepsilon')$, the datum $\varphi_0\otimes[\Psi]^{W,V}_{\mathfrak{m}}$ is $[\Psi]^{W,V}_{\mathfrak{m}}$ itself. Finally $\Omega_{\gamma,\gamma'}$, for families of integers γ, γ' indexed by the infinite places, is the representation of $G_\infty(\varepsilon)\times G_\infty(\varepsilon')$ on $\mathcal{F}(\varepsilon,\varepsilon')$ in which (u,h) acts by $\prod_\tau \det(h_\tau)^{\gamma_\tau}\det(u_\tau)^{-\gamma'_\tau}$ times the oscillator representation at $X\mapsto hXu^{-1}$.

LEAN in the background library: `CMType`, `WeilData`, `WeilData.upsilonU1`, `GInf`, `Gf`, `G`, `Kf`, `Kf.reduceAt`, `Space`, `stdSpace`, `stdEmbFin`, `AutChar`, `AutChar.HasWeight`, `pairWeil`, `pairThetaKer`, `Space.lift`, `Space.quotMeasure`, `Space.integrated`, `Modulus`, `vacuumDatum`, `modDatum`, `realPairWeil`

Definition 2.2 (level systems; the twist group and the twist lift)**.** A *level system* Σ over E consists of two finite families of finite places of E, the *chosen places* M_Σ and the *twist places* Q_Σ, such that all these places are distinct and none is the conjugate of another or of itself. For a level system Σ and an index set $[n]$ put

$$\mathrm{GL}_n^{\mathrm{tw}}(\Sigma)=\prod_{w\in Q_\Sigma}\mathrm{GL}_n(\mathbb{F}_w),\qquad \mathrm{GL}_n^{\mathrm{ch}}(\Sigma)=\prod_{w\in M_\Sigma}\mathrm{GL}_n(\mathbb{F}_w),$$

finite groups. For $A=(A_w)\in\mathrm{GL}_n^{\mathrm{tw}}(\Sigma)$, the *twist lift* $\mathrm{lift}_\Sigma(A)$ is the element of $\mathbf{K}_0(n)$ that the library chooses with $\mathrm{red}_w=1$ for $w\in M_\Sigma$ and $\mathrm{red}_w=A_w^{-1}$ for $w\in Q_\Sigma$; its components at the places outside $M_\Sigma\cup Q_\Sigma$ and their conjugates are 1.

LEAN `LevelPlaces.TwistGL` · `LevelPlaces.twistLift` · in the background library: `LevelPlaces`, `LevelPlaces.ChosenGL` (in the unit on the rows), `liftResidue`

Setting 2.3 (auxiliary pairs; Input from the unit on the rows)**.** Let E carry Weil data (Setting 2.1). An *auxiliary pair* O over E, for the index sets $[r]$ and $[n]$, consists of: a level system Σ_O, with chosen places M_O and twist places Q_O; a family of signs ε_O on $[r]$ and a hermitian space with a point $W(O)$ of these signs; an injective map $\mathrm{inc}_O\colon [r] \to [n]$; a function $c \mapsto \mathrm{ord}_O(c)$ from $[r]$ to the natural numbers; natural numbers $\kappa_O^+(\tau, i)$ and $\kappa_O^-(\tau, i)$ for every infinite place τ and every natural number i; and an automorphic character μ_O of weight $(\mathrm{wt}_O(\tau))_\tau$ and tame part $(\psi_{O,w})_w$, with $\psi_{O,w} = 1$ for $w \notin M_O \cup Q_O$. It is required that $r < n$. The following objects are attached to O.

(a) $r_-(O)$ and $r_+(O)$ are the numbers of $c \in [r]$ with $\varepsilon_O(c) = -1$ and with $\varepsilon_O(c) = 1$. Over any ring, Inc_O is the $n \times r$ matrix with entry 1 at the places $(\mathrm{inc}_O(c), c)$ and 0 elsewhere; the inc_O*-rows* of an $n \times r$ matrix form the $r \times r$ matrix of its rows with indices in the image of inc_O, and the inc_O*-block* of an $n \times n$ matrix is the $r \times r$ matrix of its entries with both indices in that image.

(b) *Weights.* For an infinite place τ and a natural number i,

$$\kappa_O(\tau, i) = \kappa_O^+(\tau, i) + \kappa_O^-(\tau, n-i) \qquad (\text{with } n - i = 0 \text{ for } i > n),$$
$$e_O(\tau) = -\sum_{1 \le i < n} \kappa_O^-(\tau, i).$$

For a family κ of natural numbers $\kappa(\tau, i)$, the space $\mathbb{V}_{n,\kappa}$ is the space of coordinate vectors, in a basis fixed by the library, of the Weyl module of $\prod_\tau \mathrm{GL}_n(\mathbb{C})$ given at each place τ by the diagram with $\kappa(\tau, i)$ columns of length i, $0 < i < n$; the matrix of a family $y = (y_\tau)$ of invertible matrices on it is written $S_\kappa(y)$. The space has a hermitian form $\langle \cdot, \cdot \rangle_\kappa$ with an integral Gram matrix, and a complex conjugation $\xi \mapsto \overline{\xi}$ of coordinates. The representation ρ_{κ_O, e_O} of $G_\infty(1_n)$ on $\mathbb{V}_{n,\kappa_O}$ lets $h = (h_\tau)$ act by $\prod_\tau \det(\overline{h_\tau})^{e_O(\tau)}\, S_{\kappa_O}(\overline{h})$, with $\overline{h}$ the family of the complex conjugate matrices; rational points of $\mathrm{I}_n(E)$ act through their images at the infinite places.

(c) *Level.* $\mathbf{L}(O)$ is the group of families (j_w) in $\prod_{w \in M_O \cup Q_O} \mathrm{GL}_n(\mathbb{F}_w)$ such that $j_w \,\mathrm{Inc}_O = \mathrm{Inc}_O$ for $w \in M_O$ and, for $w \in Q_O$, the entries of j_w in the rows of the image of inc_O and the columns outside it vanish. Its character η_O is

$$\eta_O(j) = \prod_{w \in Q_O} \psi_{O,w}(\det(\text{the } \mathrm{inc}_O\text{-block of } j_w))^{-1}.$$

The level $\mathbf{K}(O)$ is the group of $g' \in \mathbf{K}_0(n)$ whose reductions $(\mathrm{red}_w\, g')_w$ lie in $\mathbf{L}(O)$, and $\tilde{\eta}_O(g') = \eta_O((\mathrm{red}_w\, g')_w)$ is its character.

(d) *Forms.* $\mathrm{Form}(O)$ is the space of functions $\Xi\colon G_{\mathrm{f}}(n) \to \mathbb{V}_{n,\kappa_O}$ with

$$\Xi(\gamma_{\mathrm{f}}\, h\, g') = \tilde{\eta}_O(g')\, \rho_{\kappa_O, e_O}(\gamma)\, \Xi(h)$$

for all rational points γ of $\mathrm{I}_n(E)$, all $h \in G_{\mathrm{f}}(n)$ and all $g' \in \mathbf{K}(O)$.

(e) *Characters and the Weil representation.* $\chi_O = \upsilon_1^{r_-(O)}$ and $\chi'_O = \mu_O\, \upsilon_1^{\lfloor n/2 \rfloor}$. We write $\mathcal{S}(O)$, ω_O and θ_O for $\mathcal{S}(W(O), \mathrm{I}_n(E))$, $\omega_{\chi_O, \chi'_O}^{W(O), \mathrm{I}_n(E)}$ and $\theta_{W(O), \mathrm{I}_n(E)}$, and $\mathcal{F}(O)$ and $v_{O,p}$ for the Fock space $\mathcal{F}(\varepsilon_O, 1_n)$ and its vectors $v_p^{\varepsilon_O, 1_n}$.

(f) *The finite datum.* For a family $X = (X_w)$ of $n \times r$ matrices over the fields $\mathbb{F}_w$, $w \in M_O \cup Q_O$, put

$$\Psi_O(X) = \prod_{w \in Q_O} \psi_{O,w}(\det(\text{the } \mathrm{inc}_O\text{-rows of } X_w))$$

if $X_w = \mathrm{Inc}_O$ for all $w \in M_O$, and $\Psi_O(X) = 0$ otherwise. With the moduli $\mathfrak{m}_1(\Sigma_O) = \prod_{w \in M_O} w \prod_{w \in Q_O} w$ and $\mathfrak{m}_2(\Sigma_O) = \prod_{w \in M_O} w^2 \prod_{w \in Q_O} w$, the finite datum of O is

$$\Upsilon(O) = [\Psi_O \circ \mathrm{red}]_{\mathfrak{m}_2(\Sigma_O)}^{W(O),\mathrm{I}_n(E)},$$

where red sends a matrix modulo $\mathfrak{m}_2(\Sigma_O)$ to the family of its reductions at the places of $M_O \cup Q_O$.

(g) *The archimedean datum.* Let $Y_{\tau,i,a}$ be variables indexed by an infinite place, an element of $[n]$ and a natural number a. For signs s on $[r]$, a function $o\colon [r] \to \{0, 1, 2, \dots\}$ and a sign σ, the *letter substitution* $\mathrm{Let}_{s,o,\sigma}$ is the homomorphism of algebras from the polynomials in the $Y_{\tau,i,a}$ to the polynomials in the $X_{\tau,i,c}$ that sends $Y_{\tau,i,a}$ to the sum of the $X_{\tau,i,c}$ over the $c \in [r]$ with $s(c) = \sigma$ and $o(c) = a$. The library provides, for every κ, a linear map Pol_κ from $\mathbb{V}_{n,\kappa}$ to the polynomials in the $Y_{\tau,i,a}$, and an integral matrix $\mathrm{C}_O\colon \mathbb{V}_{n,\kappa_O} \to \mathbb{V}_{n,\kappa_O^+} \otimes \mathbb{V}_{n,\kappa_O^-}$, the *mixed Cartan embedding* (Q5). The linear map $\xi \mapsto \mathrm{Har}_O(\xi)$, from $\mathbb{V}_{n,\kappa_O}$ to the polynomials in the $X_{\tau,i,c}$, is the composite of C_O with the linear map that sends a tensor $\alpha \otimes \beta$ to $\mathrm{Let}_{\varepsilon_O,\mathrm{ord}_O,1}(\mathrm{Pol}_{\kappa_O^+}\,\alpha) \cdot \mathrm{Let}_{\varepsilon_O,\mathrm{ord}_O,-1}(\mathrm{Pol}_{\kappa_O^-}\,\beta)$. The *test datum* is $\mathrm{J}_O(\xi) = v_{O,\mathrm{Har}_O(\xi)} \otimes \Upsilon(O) \in \mathcal{S}(O)$.

(h) *The kernel and the class-sum lift.* For $u \in G(\underline{\varepsilon_O})$ and $h \in G_{\mathrm{f}}(n)$, $\mathcal{C}_O(u,h)$ is the linear form on $\mathbb{V}_{n,\kappa_O}$ with $\mathcal{C}_O(u,h)(\xi) = \theta_O(\omega_O(u,(1,h))\ \mathrm{J}_O(\bar{\xi}))$. For $\Xi \in \mathrm{Form}(O)$,

$$\mathrm{Th}_O(\Xi)(u) = \sum_{[h]} \frac{1}{\#\Gamma_h}\, \mathcal{C}_O(u,h)(\Xi(h)),$$

the sum over the double classes of $G_{\mathrm{f}}(n)$ modulo the rational points of $\mathrm{I}_n(E)$ on the left and $\mathbf{K}(O)$ on the right, with Γ_h the finite group of rational points in $h\,\mathbf{K}(O)\,h^{-1}$.

LEAN in the unit on the rows: `AuxPair` with `AuxPair.κ`, `AuxPair.detExp`, `auxLevel`, `AuxPair.auxLevelChar`, `AuxPair.levelGroup`, `AuxPair.levelChar`, `AuxPair.FormV`, `AuxPair.a`, `AuxPair.b`, `AuxPair.pairThetaKernel`, `AuxPair.resDetDatum`, `AuxPair.auxDatum`, `AuxPair.harmonicPoly`, `AuxPair.fockDatum`, `AuxPair.conjKernel`, `AuxPair.θ` · in the background library: `conjWeylRep`, `mixedCartan`, `letterPoly`, `LevelPlaces.levelModulus`, `stdClassSum`

In the terminology of the theta correspondence: $(U(W(O)), U(\mathrm{I}_n(E)))$ is a dual pair of unitary groups of ranks $r < n$ over the totally real subfield, split at the places of the level; χ_O and χ_O' are the two characters that split the metaplectic cover over the two groups; the datum is a pure tensor whose archimedean component is $\mathrm{Har}_O(\xi)$ times the Gaussian in the Fock model, whose components at the places of the level are given by Ψ_O and whose other components are unramified. The forms of $\mathrm{Form}(O)$ are algebraic modular forms on a definite unitary group in the sense of Gross.

Definition 2.4 (the level indicator)**.** Let O be an auxiliary pair (Setting 2.3). For an $n \times r$ matrix X over the integers of E modulo $\mathfrak{m}_1(\Sigma_O)$, with reductions X_w at the places $w \in M_O \cup Q_O$, put $\mathbf{1}_O(X) = 1$ if $X_w = \mathrm{Inc}_O$ for every $w \in M_O$ and the inc_O-rows of X_w form the identity matrix for every $w \in Q_O$; and $\mathbf{1}_O(X) = 0$ otherwise.

LEAN `AuxPair.levelIndicator`

Definition 2.5 (the parameter on the definite side)**.** Let O be an auxiliary pair (Setting 2.3), τ an infinite place of E and $i \in [n]$. Fix once and for all a numbering of $[n]$ by $0, \dots, n-1$, and let $\pi(i)$ be the number of i. Then

$$l_O^V(\tau, i) = 2\Big(e_O(\tau) + \sum_{\pi(i)<a'<n} \kappa_O(\tau, a')\Big) + n - 1 - 2\pi(i).$$

LEAN `AuxPair.paramV` · in the background library: `hcParameter`

For each τ, the numbers $l_O^V(\tau, i)$, $i \in [n]$, are the entries of twice the Harish-Chandra parameter (highest weight plus half the sum of the positive roots) of the factor at τ of the representation $y \mapsto \prod_\tau \det(y_\tau)^{e_O(\tau)}\, S_{\kappa_O}(y)$ of $G_\infty(1_n)$, a product of compact unitary groups of rank n. This representation is the complex conjugate of ρ_{κ_O, e_O}, which acts through the conjugate matrices (Setting 2.3(b)); so the doubled parameter of ρ_{κ_O, e_O} itself has the entries $-l_O^V(\tau, i)$. The conjugate is the representation through which the scalar functions $F_{l_O^V(\cdot,\cdot),1_n}(\Xi)$ of Theorem 1 transform under right translation by $G_\infty(1_n)$; hence the type $\mathbf{e}_{l_O^V(\cdot,\cdot),1_n}$ in Corollary 4.4. All parameters in this note are doubled in this way, which keeps them integral.

Definition 2.6 (matched parameters, joint harmonics, staircase weights; quoted)**.** Let s' be a family of signs on $[n]$, s a family of signs on $[r]$, and γ, γ' integers. Consider the polynomials in the variables X_{ic}, $i \in [n]$, $c \in [r]$; the variable X_{ic} has the sign $s'_i s_c$. For a variable X of sign 1 let A_X be the multiplication by X and $B_X = \partial/\partial X$; for a variable of sign -1 let $A_X = -\partial/\partial X$ and let B_X be the multiplication by X. The *row* and *column operators* are, with Kronecker's symbol $\delta_{ii'}$,

$$R_{ii'} = \sum_{c\in[r]} A_{X_{ic}} B_{X_{i'c}} + \gamma\,\delta_{ii'} \qquad (i, i' \in [n]),$$
$$C_{cc'} = -\sum_{i\in[n]} B_{X_{ic}} A_{X_{ic'}} + (\gamma' + n)\,\delta_{cc'} \qquad (c, c' \in [r]).$$

They give commuting actions of the Lie algebras of GL_n and GL_r: the derivative of the oscillator representation of the pair, twisted by powers of the determinants.

Parameters. A *parameter* for the signs s' is a family of integers $l_1(i)$, $i \in [n]$; it stands for twice a Harish-Chandra parameter of the group $U(s')$. Its *lowest K-type* is the family

$$\Lambda_{s'}(l_1)(i) = \tfrac{1}{2}(l_1(i) + \#\{j : l_1(j) < l_1(i)\} - \#\{j : l_1(j) > l_1(i)\}) \\ - \#\{j : s'_j = s'_i,\, l_1(j) < l_1(i)\} + \#\{j : s'_j = s'_i,\, l_1(j) > l_1(i)\},$$

the Harish-Chandra parameter plus the half sum of the positive roots minus the sum of the compact positive roots. The parameter is *spread*, relative to a second index set $[r]$, if it is injective, all $l_1(i) + n - 1$ are even, and $|l_1(i) - l_1(j)| \geq 1000\,(n + r)^3$ whenever $s'_i \neq s'_j$. The same words apply to a parameter l_2 for the signs s on $[r]$, with n and r exchanged.

Matching. Put $\xi(i) = l_1(i) - 2\gamma + r$ for $i \in [n]$ and $\xi'(c) = l_2(c) - 2\gamma' - n$ for $c \in [r]$. A pair (i, c) is *matched* if $\xi(i) = -\xi'(c)$ and $|n - r| < s'_i s_c\, \xi(i)$. The parameters (l_1, l_2) *match* for the signs (s', s) and the weights (γ, γ') if both are spread, every $i \in [n]$ lies in a matched pair or has $|\xi(i)| < n - r$, and every $c \in [r]$ lies in a matched pair or has $|\xi'(c)| < r - n$.

Joint harmonics. A polynomial Φ is *jointly highest* of weights (ν, ν') for the parameters (l_1, l_2) if $\Phi \neq 0$ and

(a) $R_{ii}\Phi = \nu(i)\Phi$ for all i, and $R_{ii'}\Phi = 0$ whenever $s'_i = s'_{i'}$ and $l_1(i') < l_1(i)$;
(b) $C_{cc}\Phi = \nu'(c)\Phi$ for all c, and $C_{cc'}\Phi = 0$ whenever $s_c = s_{c'}$ and $l_2(c') < l_2(c)$;
(c) $\partial^2\Phi/\partial X_{ic}\partial X_{i'c} = 0$ for all c whenever $s'_i \neq s'_{i'}$, and $\partial^2\Phi/\partial X_{ic}\partial X_{ic'} = 0$ for all i whenever $s_c \neq s_{c'}$.

It is a *joint harmonic* for the signs (s', s), the weights (γ, γ') and the parameters (l_1, l_2) if these parameters match and Φ is jointly highest of weights $(\Lambda_{s'}(l_1), \Lambda_s(l_2))$.

Staircase weights. For signs s on $[r]$, a function $o\colon [r] \to \{0, 1, 2, \dots\}$, natural numbers $k(\sigma, i)$ for signs σ and natural numbers i, an integer γ' and $c \in [r]$,

$$\mathrm{sw}_n(s, o, k, \gamma')(c) = \gamma' + n \cdot [s_c = -1] - s_c \sum_{o(c)<i<n} k(s_c, i),$$

where $[s_c = -1]$ is 1 if $s_c = -1$ and 0 otherwise.

LEAN in the background library: `FockDualPair.std`, `oscRows`, `HCParam.lowestKType`, `HCParam.HasGap`, `IsMatch`, `FockDualPair.IsJointHighest`, `IsJointHarmonic`, `staircaseWeight`

Joint harmonics are the polynomials of lowest degree in which a K-type of one group of the pair occurs together with a K-type of the other, in Howe's description of the correspondence in the Fock model; the library uses them, for parameters that are far apart, to match discrete series representations of the two groups by their lowest K-types. In this note s' is always 1_n, so that conditions on pairs with $s'_i \neq s'_{i'}$ are empty.

Definition 2.7 (parameters that fit an auxiliary pair)**.** Let O be an auxiliary pair (Setting 2.3) and l a family of integers $l(\tau, c)$, indexed by the infinite places τ of E and $c \in [r]$. We say that l fits O if the following four conditions hold.

(F1) For every τ, the parameters $(l^V_O(\tau, \cdot), -l(\tau, \cdot))$ match for the signs $(1_n, \varepsilon_O)$ and the weights $(r_-(O), -(\lfloor n/2 \rfloor + \mathrm{wt}_O(\tau)))$ (Definition 2.6).
(F2) For every τ and every $c \in [r]$, $\mathrm{ord}_O(c)$ is the number of $c' \in [r]$ with $\varepsilon_O(c') = \varepsilon_O(c)$ and $\varepsilon_O(c)\, l(\tau, c) < \varepsilon_O(c)\, l(\tau, c')$.
(F3) $\kappa^+_O(\tau, i) = 0$ for every τ and every $i > r_+(O)$.
(F4) $\kappa^-_O(\tau, i) = 0$ for every τ and every $i > r_-(O)$.

LEAN `AuxPair.FitsParams` · in the background library: `ParamsCorrespond`

Condition (F2) says that ord_O numbers the indices of each sign: those of sign 1 by decreasing $l(\tau, c)$, those of sign -1 by increasing $l(\tau, c)$, and that this numbering is the same at all infinite places. Conditions (F3) and (F4) bound the lengths of the columns of the diagrams κ^+_O and κ^-_O by the numbers of positive and negative signs of $W(O)$.

Definition 2.8 (forms of a type; edges; quoted)**.** Let s and s' be families of signs on $[r]$ and $[n]$. For a function $\mathbf{e}$ on $G_\infty(s)$, a *form of type* $\mathbf{e}$ on a hermitian space with a point W of signs s is a bounded continuous function F on $G(s)$, left invariant under the rational points of W and right invariant under an open subgroup of $G_{\mathrm{f}}(r)$, with $\int_{G_\infty(s)} \mathbf{e}(y) F(uy)\, \mathrm{d}y = F(u)$ for all u whenever $\mathbf{e}$ is integrable. For a family l of parameters $l(\tau, \cdot)$ for the signs s, $\mathbf{e}_{l,s}$ is the product over the infinite places of the kernels that the library attaches to the discrete series parameters $l(\tau, \cdot)$: the formal degree times the complex conjugate of the normalised matrix coefficient. Let Ω be a

map from $G_\infty(s) \times G_\infty(s')$ to the bounded operators of $\mathcal{F}(s,s')$, $\mathbf{e}$ and $\mathbf{e}'$ functions on $G_\infty(s)$ and $G_\infty(s')$, and $v \in \mathcal{F}(s,s')$. Then v is a *half edge* from $\mathbf{e}$ to $\mathbf{e}'$ for Ω if

(i) $\mathbf{e}$ is a cusp kernel of order 0 in the sense of the library: an idempotent for convolution, bounded, integrable against the first power of the norm, and fixed by convolution with some continuous function of compact support;

(ii) $\int_{G_\infty(s)} \langle v, \Omega(y,1)v\rangle\, \overline{\mathbf{e}(y)}\, \mathbf{e}(1)^{-1}\, \mathrm{d}y \neq 0$, with the inner product of the Fock space;

(iii) for every polynomial vector v' of $\mathcal{F}(s,s')$ there is a continuous function α of compact support on $G_\infty(s')$ such that, for all hermitian spaces with points W, V of signs s, s', all characters, all data f and all forms Y of type $\mathbf{e}$ on W, if the real points act on the data $v'' \otimes f$ through Ω on v'', then the theta lift F of Y to V with datum $v \otimes f$ is a form of type $\mathbf{e}'$ on V as soon as it is right invariant under some open subgroup, and the theta lift of Y with datum $v' \otimes f$ is $h \mapsto \int_{G_\infty(s')} \alpha(y) F(hy)\, \mathrm{d}y$.

It is an *edge* from $\mathbf{e}$ to $\mathbf{e}'$ for Ω if in addition it is a half edge from $\overline{\mathbf{e}'}$ to $\overline{\mathbf{e}}$ for Ω with its two arguments exchanged.

LEAN in the background library: `Space.IsForm`, `typeKernel`, `IsCuspKernel`, `IsHalfEdge`, `IsEdge`

Quoted from the background library. These facts are taken for granted; references and their state are in §7, note 7. In all of them W, V, ε, ε', χ, χ' are as in Setting 2.1, and ω stands for $\omega_{\chi,\chi'}^{W,V}$.

(Q1) *Integral elements move the finite datum.* For $g \in \mathbf{K}_0(r)$, $g' \in \mathbf{K}_0(n)$, a modulus $\mathfrak{m}$, a function Ψ on the $n \times r$ matrices modulo $\mathfrak{m}$ and $v \in \mathcal{F}(\varepsilon,\varepsilon')$: $\omega((1,g),(1,g'))(v \otimes [\Psi]_{\mathfrak{m}}^{W,V}) = \chi(\det g')\,\chi'(\det g)^{-1} \cdot v \otimes [\Psi']_{\mathfrak{m}}^{W,V}$, where $\Psi'(X) = \Psi(\overline{g}'^{-1} X \overline{g})$ and $\overline{g}$, $\overline{g}'$ are the reductions of g, g' modulo $\mathfrak{m}$. A function of the reductions at the places dividing $\mathfrak{m}$ gives the same finite datum for $\mathfrak{m}$ and for every modulus with the same prime divisors.

(Q2) *Characters at integral elements.* If an automorphic character μ has a weight and tame part (ψ_w), trivial outside a finite set S of places as in Definition 2.2, then $\mu(\det g) = \prod_{w \in S} \psi_w(\det \mathrm{red}_w\, g)$ for every $g \in \mathbf{K}_0(r)$. If μ has weight e and tame part ψ, then $\mu\, v_1^a$ has weight $(e_\tau + a)_\tau$ and tame part ψ, for every natural number a.

(Q3) *Real and finite parts act separately.* If χ has weight γ and χ' has weight γ', then for $u = (u_\infty, u_\mathrm{f})$, $h = (h_\infty, h_\mathrm{f})$, $v \in \mathcal{F}(\varepsilon,\varepsilon')$ and a datum f: $\omega(u,h)(v \otimes f) = (\Omega_{\gamma,\gamma'}(u_\infty,h_\infty)v) \otimes \omega((1,u_\mathrm{f}),(1,h_\mathrm{f}))f$.

(Q4) *The definite group on polynomial vectors.* Let $\varepsilon' = 1_n$, let γ be the constant family with value the number of negative signs of ε, and γ' any family of integers. For $y \in G_\infty(1_n)$ and a polynomial p in the $X_{\tau,i,c}$: $\Omega_{\gamma,\gamma'}(1,y)\, v_p^{\varepsilon,1_n} = v_{y\cdot p}^{\varepsilon,1_n}$, where $p \mapsto y \cdot p$ is the automorphism of the polynomial algebra that replaces the column $(X_{\tau,i,c})_i$ by $y_\tau^{\mathsf{T}}(X_{\tau,i,c})_i$ if $\varepsilon_c = 1$ and by $y_\tau^{-1}(X_{\tau,i,c})_i$ if $\varepsilon_c = -1$. The map $p \mapsto v_p^{\varepsilon,1_n}$ is injective.

LEAN in the background library: (Q1) `pairWeil_modDatum_Kf`, `liftFn_eq_residueDatum` · (Q2) `AutChar.HasWeight.apply_det_eq_prod`, `AutChar.HasWeight.hasWeight_rankTwist` · (Q3) `pairWeil_integrated_all` · (Q4) `realPairWeil_polyVector`, `polyVector_injective`

Quoted from the background library, continued.

(Q5) *Weyl modules.* For a family $y = (y_\tau)$ of invertible complex $n \times n$ matrices and every κ: $\mathrm{Pol}_\kappa(S_\kappa(y)\alpha)$ is obtained from $\mathrm{Pol}_\kappa(\alpha)$ by replacing each column $(Y_{\tau,i,a})_i$ by $y_\tau^{\mathsf{T}}(Y_{\tau,i,a})_i$. For an auxiliary pair O the mixed Cartan embedding satisfies

$$\mathrm{C}_O \prod_\tau \det(y_\tau)^{e_O(\tau)}\, S_{\kappa_O}(y) = (S_{\kappa_O^+}(y) \otimes S_{\kappa_O^-}(y^{-\mathsf{T}}))\, \mathrm{C}_O,$$

with $y^{-\mathsf{T}}$ the family of the inverse transposes, and it sends the vector $v^+_{\kappa_O}(l_1)$ of a parameter l_1 as in (Q9) to a pure tensor of extremal vectors, whose two polynomials are products of minors of the matrices $(Y_{\tau,i,a})$. The matrices S_κ have polynomial entries with integral coefficients, so $\overline{S_\kappa(y)\xi} = S_\kappa(\overline{y})\overline{\xi}$.

(Q6) *Staircases* (Kashiwara–Vergne). Let s be signs on $[r]$, l_1 an injective family of integers on $[n]$, l_2 an injective family on $[r]$, and for $c \in [r]$ let $o(c)$ be the number of c' with $s_{c'} = s_c$ and $s_c l_2(c') < s_c l_2(c)$. Let $k(\sigma, a)$ be natural numbers with $k(\sigma, a) = 0$ whenever $\#\{c : s_c = \sigma\} < a < n$, and such that $k(1, a) = 0$ or $k(-1, a') = 0$ whenever $a + a' > n$. For a sign σ and $a < n$ let $D_{\sigma,a}$ be the minor of the matrix (X_{ic}) with the a rows i for which $\sigma\, l_1(i)$ is largest and the columns c with $s_c = \sigma$ and $o(c) < a$, in these orders. Then, for all integers γ, γ', the polynomial $\prod_\sigma \prod_{a<n} D_{\sigma,a}^{k(\sigma,a)}$ is jointly highest for the signs $(1_n, s)$, the weights (γ, γ') and the parameters (l_1, l_2), of weights (ν, ν') with $\nu'(c) = \mathrm{sw}_n(s, o, k, \gamma')(c)$ and $\nu(i) = \gamma - \#\{c : s_c = -1\} + \sum_\sigma \sigma \sum_a k(\sigma, a)$, the inner sum over the a with $\#\{j : \sigma l_1(j) > \sigma l_1(i)\} < a < n$.

(Q7) *Existence of joint harmonics.* If (l_1, l_2) match for the signs (s', s) and the weights (γ, γ'), a joint harmonic for them exists.

(Q8) *Uniqueness.* If (l_1, l_2) match, Φ_0 is a joint harmonic for them and Φ is jointly highest of weights $(\Lambda_{s'}(l_1), \nu')$ for some ν', then Φ_0 is a complex multiple of Φ.

LEAN in the background library: (Q5) `PolyRep.MixedCartanEmbedding`, `mixedCartan`, `weylRep_map` · (Q6) `staircase_harmonic_highest` · (Q7) `IsMatch.exists_isJointHarmonic` · (Q8) `IsMatch.unique`

Quoted from the background library, continued.

(Q9) *Parameters of the definite group.* Call a family $l_1(\tau, i)$ a *doubled Harish-Chandra parameter of* (κ, e) if every $l_1(\tau, \cdot)$ is injective on $[n]$ and each of its values is $2(e(\tau) + \sum_{a<a'<n} \kappa(\tau, a')) + n - 1 - 2a$ for some $a < n$. Then $\Lambda_{1_n}(l_1(\tau, \cdot))(i) = e(\tau) + \sum_{a<a'<n} \kappa(\tau, a')$ with a the number of j with $l_1(\tau, j) > l_1(\tau, i)$. The *highest vector* $v^+_\kappa(l_1) \in \mathbb{V}_{n,\kappa}$ is the vector the library attaches to the orders of $[n]$ by decreasing $l_1(\tau, \cdot)$; it has real coordinates, $\overline{v^+_\kappa(l_1)} = v^+_\kappa(l_1)$, and it is a unit vector: $\langle v^+_\kappa(l_1), v^+_\kappa(l_1)\rangle_\kappa = 1$.

(Q10) *From joint harmonics to edges.* Let $[r]$ and $[n]$ be non-empty, s, s' signs on them, γ, γ' families of integers and l, l' families of parameters, all indexed by the infinite places. If for every τ the polynomial Φ_τ is a joint harmonic for the signs (s', s), the weights $(\gamma_\tau, -\gamma'_\tau)$ and the parameters $(l'(\tau, \cdot), -l(\tau, \cdot))$, then the vector $v_p^{s,s'}$ of $p = \prod_\tau \Phi_\tau((X_{\tau,i,c})_{i,c})$ is an edge from $\mathbf{e}_{l,s}$ to $\mathbf{e}_{l',s'}$ for $\Omega_{\gamma,\gamma'}$.

(Q11) *Automorphy and Schur orthogonality.* (a) Every finite datum is fixed by $\omega((1, g), (1, g'))$ for g, g' in suitable levels; if a datum φ is so fixed, then $h \mapsto \theta_{W,V}(\omega(u, h)\varphi)$ is continuous, left invariant under the rational points of V and right invariant under a level. (b) Let $V = \mathrm{I}_n(E)$, let l_1 be a doubled Harish-Chandra parameter of (κ, e), $\mathbf{K}$ a level in $\mathbf{K}_0(n)$, Ξ an algebraic form of weight (κ, e) as in Setting 2.3(d), and Φ a function on $G(1_n)$ with the invariance of (a) such that $\Phi(y, h) = L_h(\rho_{\kappa,e}(y)v^+_\kappa(l_1))$ for linear forms L_h on $\mathbb{V}_{n,\kappa}$. If $h \mapsto L_h(\Xi(h))$ is right invariant under $\mathbf{K}$, then

$$\int_{[\mathrm{I}_n(E)]} \Phi(h)\, F_{l_1,1_n}(\Xi)(h)\,\mathrm{d}h = \frac{1}{\dim \mathbb{V}_{n,\kappa} \cdot [\mathbf{K}_0(n) : \mathbf{K}]} \sum_{[h]} \frac{1}{\#\Gamma_h} L_h(\Xi(h)),$$

the sum over the double classes modulo the rational points and $\mathbf{K}$, as in Setting 2.3(h), and the measure normalised as in Setting 2.1.

LEAN in the background library: (Q9) `IsHCParam`, `IsHCParam.lowestKType_eq`, `highestVector`, `star_highestVector`, `gramForm_highestVector_self` · (Q10) `isEdge_of_isJointHarmonic` · (Q11) `exists_level_pairWeil`, `ThetaKernel.isAutomorphic`, `Space.integral_mul_eq_classIntegral`, `classIntegral`, `classIntegral_eq_classSum`; behind (b): `Space.integral_eq_classIntegral_of_isLevel`, `schur_orthogonality`

3 The finite datum

Proposition 3.1 (the finite datum is a sum of translates of the level indicator)**.** *In Setting 2.1, let O be an auxiliary pair (Setting 2.3). Then*

$$\Upsilon(O) = \sum_{A \in \mathrm{GL}_r^{\mathrm{tw}}(\Sigma_O)} \omega_O((1, \mathrm{lift}_{\Sigma_O}(A)), 1)\Big(\varphi_0 \otimes [\mathbf{1}_O(\,\cdot\,)]^{W(O),\mathrm{I}_n(E)}_{\mathfrak{m}_1(\Sigma_O)}\Big),$$

where $(1, \mathrm{lift}_{\Sigma_O}(A))$ is the element of $G(\varepsilon_O)$ with trivial component at the infinite places.

Proof. By (Q1) both sides are finite data for the modulus $\mathfrak{m}_1(\Sigma_O)$, the left one of the function $X \mapsto \Psi_O((X_w)_w)$; so it suffices to compare functions of an $n \times r$ matrix X modulo $\mathfrak{m}_1(\Sigma_O)$. Fix $A \in \mathrm{GL}_r^{\mathrm{tw}}(\Sigma_O)$ and put $g = \mathrm{lift}_{\Sigma_O}(A)$; its reduction is 1 at the places of M_O and A_w^{-1} at $w \in Q_O$. By (Q1) the term of A is the finite datum

of $X \mapsto \chi'_O(\det g)^{-1}\,\mathbf{1}_O(X\overline{g})$. By (Q2), applied to $\chi'_O = \mu_O \upsilon_1^{\lfloor n/2 \rfloor}$ with the places $M_O \cup Q_O$, we have $\chi'_O(\det g)^{-1} = \prod_{w \in Q_O} \psi_{O,w}(\det A_w)$. And $\mathbf{1}_O(X\overline{g}) = 1$ exactly when $X_w = \mathrm{Inc}_O$ for all $w \in M_O$ and the inc_O-rows of $X_w A_w^{-1}$ form the identity, that is, the inc_O-rows of X_w are A_w, for all $w \in Q_O$. So the right side is the finite datum of

$$X \;\longmapsto\; \sum_A \prod_{w \in Q_O} \psi_{O,w}(\det A_w) \\ \cdot \,[X_w = \mathrm{Inc}_O \ (w \in M_O),\ \ \mathrm{inc}_O\text{-rows of } X_w = A_w \ (w \in Q_O)],$$

where the bracket is 1 if the conditions in it hold and 0 otherwise. If the inc_O-rows of X_w are invertible for every $w \in Q_O$, exactly one A contributes and the value is $\Psi_O((X_w)_w)$. If for some $w \in Q_O$ they are not, no A contributes, and $\Psi_O((X_w)_w) = 0$ because $\psi_{O,w}$ vanishes at 0. □

LEAN `AuxPair.upsilon_eq_sum`

Proposition 3.2 (the transformation law of the finite datum)**.** *In Setting 2.1, let O be an auxiliary pair (Setting 2.3), let $g \in \mathbf{K}_0(r)$, $g' \in \mathbf{K}_0(n)$ and $j \in \mathbf{L}(O)$, and suppose that $\mathrm{red}_w g' = \mathrm{diag}(\mathrm{red}_w g, 1)\, j_w$ for every $w \in M_O$ and $\mathrm{red}_w g' = j_w$ for every $w \in Q_O$, where $\mathrm{diag}(g,1)$ is the matrix in $\mathrm{GL}_n(\mathbb{F}_w)$ that acts by g on the basis vectors with indices in the image of inc_O and fixes the others. Then*

$$\omega_O((1,g),(1,g'))\,\Upsilon(O) = \Big(\prod_{w \in M_O} \psi_{O,w}(\det \mathrm{red}_w g)\Big)^{-1} \eta_O(j) \cdot \Upsilon(O).$$

Proof. By (Q1) the left side is the finite datum of the function

$$X \;\longmapsto\; \chi_O(\det g')\,\chi'_O(\det g)^{-1}\,\Psi_O(((\mathrm{red}_w g')^{-1} X_w\ \mathrm{red}_w g)_w).$$

By (Q2), $\chi_O = \upsilon_1^{r_-(O)}$ has trivial tame part, so $\chi_O(\det g') = 1$, and

$$\chi'_O(\det g) = \prod_{w \in M_O \cup Q_O} \psi_{O,w}(\det \mathrm{red}_w g).$$

At $w \in M_O$: since j_w fixes the basis vectors with indices in the image of inc_O,

$$(\mathrm{red}_w g')\ \mathrm{Inc}_O = \mathrm{diag}(\mathrm{red}_w g, 1)\ \mathrm{Inc}_O = \mathrm{Inc}_O\ \mathrm{red}_w g.$$

Hence $(\mathrm{red}_w g')^{-1} X_w\ \mathrm{red}_w g = \mathrm{Inc}_O$ if and only if $X_w = \mathrm{Inc}_O$: the condition at M_O is unchanged.

At $w \in Q_O$: $\mathrm{red}_w g' = j_w$, and j_w^{-1}, like j_w, has vanishing entries in the rows of the image of inc_O and the columns outside it. So the inc_O-rows of $j_w^{-1} X_w\ \mathrm{red}_w g$ are the inc_O-block of j_w^{-1} times the inc_O-rows of X_w times $\mathrm{red}_w g$, and their determinant is $\det(\mathrm{inc}_O\text{-block of } j_w)^{-1} \cdot \det(\mathrm{inc}_O\text{-rows of } X_w) \cdot \det \mathrm{red}_w g$.

Together: the translated function is $\eta_O(j) \prod_{w \in Q_O} \psi_{O,w}(\det \mathrm{red}_w g)$ times Ψ_O, and the factors at Q_O cancel against those of $\chi'_O(\det g)^{-1}$. □

LEAN `AuxPair.pairWeil_upsilon` · in the unit on the rows: `firstBlockHom`

For $g = 1$ the hypothesis says that $g' \in \mathbf{K}(O)$ with reductions j, and the law reads $\omega_O(1,(1,g'))\Upsilon(O) = \tilde{\eta}_O(g')\Upsilon(O)$: the finite datum is an eigenvector of the level with the character of the forms. Both families of places contribute inverse character values: at M_O through $\det \mathrm{red}_w g$, at Q_O through the block of j_w, which enters η_O with an inverse.

4 The archimedean datum

Lemma 4.1 (the parameter of the definite side)**.** *Let O be an auxiliary pair (Setting 2.3). Then $(\tau, i) \mapsto l_O^V(\tau, i)$ is a doubled Harish-Chandra parameter of (κ_O, e_O) in the sense of* (Q9)*.*

Proof. By Definition 2.5, the value at (τ, i) is the number of (Q9) for $a = \pi(i)$, and $a \mapsto 2(e_O(\tau) + \sum_{a<a'<n} \kappa_O(\tau, a')) + n - 1 - 2a$ is strictly decreasing, hence injective. □

LEAN `AuxPair.isHCParam_paramV` · in the background library: `isHCParam_hcParameter`

Lemma 4.2 (letter substitutions)**.** *Let s be a family of signs on $[r]$, let $o\colon [r] \to \{0, 1, 2, \dots\}$, let γ' be a family of integers indexed by the infinite places, $y \in G_\infty(1_n)$, σ a sign and F a polynomial in the variables $Y_{\tau,i,a}$. Let γ be the constant family whose value is the number of $c \in [r]$ with $s(c) = -1$. Then*

$$\Omega_{\gamma,\gamma'}(1, y)\ v^{s,1_n}_{\mathrm{Let}_{s,o,\sigma}(F)} = v^{s,1_n}_{\mathrm{Let}_{s,o,\sigma}(F^{y,\sigma})},$$

where $F^{y,\sigma}$ is obtained from F by replacing each column $(Y_{\tau,i,a})_i$ by $y_\tau^{\mathsf{T}}(Y_{\tau,i,a})_i$ if $\sigma = 1$, and by $y_\tau^{-1}(Y_{\tau,i,a})_i$ if $\sigma = -1$.

Proof. By (Q4) the left side is the vector of $y \cdot \mathrm{Let}_{s,o,\sigma}(F)$. Both $F \mapsto y \cdot \mathrm{Let}_{s,o,\sigma}(F)$ and $F \mapsto \mathrm{Let}_{s,o,\sigma}(F^{y,\sigma})$ are homomorphisms of algebras, so it suffices to compare them on a variable $Y_{\tau,i,a}$. Let z_τ be y_τ^{T} or y_τ^{-1} according to σ. The columns c that occur in $\mathrm{Let}_{s,o,\sigma}(Y_{\tau,i,a})$ have $s(c) = \sigma$, so the first map gives $\sum_c \sum_{i'} z_\tau(i, i') X_{\tau,i',c}$, the sums over $i' \in [n]$ and over the c with $s(c) = \sigma$ and $o(c) = a$; the second gives the same double sum in the other order. □

LEAN `realPairWeil_fockLetters`

Lemma 4.3 (the definite group acts on the harmonic polynomials through the weight)**.** *Let O be an auxiliary pair (Setting 2.3), γ' a family of integers indexed by the infinite places, $y \in G_\infty(1_n)$ and $\xi \in \mathbb{V}_{n,\kappa_O}$. Let γ be the constant family with value $r_-(O)$. Then*

$$\Omega_{\gamma,\gamma'}(1, y)\ v_{O,\mathrm{Har}_O(\bar\xi)} = v_{O,\mathrm{Har}_O(\overline{\rho_{\kappa_O,e_O}(y)\,\xi})}.$$

Proof. By Setting 2.3(b) and (Q5),

$$\overline{\rho_{\kappa_O,e_O}(y)\,\xi} = \prod_\tau \det(y_\tau)^{e_O(\tau)}\, S_{\kappa_O}(y)\,\bar\xi,$$

and the mixed Cartan embedding sends this vector to $(S_{\kappa_O^+}(y) \otimes S_{\kappa_O^-}(y^{-\mathsf{T}}))\mathrm{C}_O\bar\xi$. Write $\mathrm{C}_O\bar\xi$ as a sum of pure tensors $\alpha \otimes \beta$. By (Q5), in the notation of Lemma 4.2,

$$\mathrm{Pol}_{\kappa_O^+}(S_{\kappa_O^+}(y)\alpha) = (\mathrm{Pol}_{\kappa_O^+}\alpha)^{y,1}, \qquad \mathrm{Pol}_{\kappa_O^-}(S_{\kappa_O^-}(y^{-\mathsf{T}})\beta) = (\mathrm{Pol}_{\kappa_O^-}\beta)^{y,-1}.$$

So $\mathrm{Har}_O(\overline{\rho_{\kappa_O,e_O}(y)\xi})$ is the sum of the products

$$\mathrm{Let}_{\varepsilon_O,\mathrm{ord}_O,1}((\mathrm{Pol}_{\kappa_O^+}\alpha)^{y,1}) \cdot \mathrm{Let}_{\varepsilon_O,\mathrm{ord}_O,-1}((\mathrm{Pol}_{\kappa_O^-}\beta)^{y,-1}).$$

By Lemma 4.2, (Q4) and the injectivity in (Q4), each factor is $y \cdot$ the corresponding factor of $\mathrm{Har}_O(\bar\xi)$; as $p \mapsto y \cdot p$ is a homomorphism of algebras, the sum is $y \cdot \mathrm{Har}_O(\bar\xi)$. By (Q4) once more, its vector is the left side. □

LEAN AuxPair.realPairWeil_harmonicPoly

Proof of Theorem 2. Fix an infinite place τ. Put $l_1 = l_O^V(\tau, \cdot)$ and $l_2 = -l(\tau, \cdot)$, and let $k(1,i) = \kappa_O^+(\tau,i)$, $k(-1,i) = \kappa_O^-(\tau,i)$, $\gamma = r_-(O)$, $\gamma' = -(\lfloor n/2 \rfloor + \mathrm{wt}_O(\tau))$.

Step 1: the staircase. The hypotheses of (Q6) hold for $s = \varepsilon_O$: l_1 is injective by Lemma 4.1, and l_2 is injective because it is spread, by (F1). The function o of (Q6) is ord_O, by (F2): the inequality $s_c l_2(c') < s_c l_2(c)$ is $s_c\, l(\tau,c) < s_c\, l(\tau,c')$. The vanishing of $k(\sigma,a)$ for $\#\{c : s_c = \sigma\} < a$ is (F3) and (F4). If $a + a' > n$, then $a + a' > r = r_+(O) + r_-(O)$ because $r < n$, so $a > r_+(O)$ or $a' > r_-(O)$, and $k(1,a) = 0$ or $k(-1,a') = 0$. Let Φ_τ be the polynomial of (Q6). It is jointly highest of weights (ν, ν') with $\nu'(c) = \mathrm{sw}_n(\varepsilon_O, \mathrm{ord}_O, k, \gamma')(c)$.

Step 2: its weight on the definite side is the lowest K-type. Let $i \in [n]$ and let a be the number of j with $l_1(j) > l_1(i)$; then $n - 1 - a$ is the number of j with $-l_1(j) > -l_1(i)$. As $\gamma = r_-(O)$, (Q6) gives $\nu(i) = \sum_{a<a'<n} \kappa_O^+(\tau,a') - \sum_{n-1-a<a'<n} \kappa_O^-(\tau,a')$. On the other hand, by Lemma 4.1 and (Q9),

$$\begin{aligned}\Lambda_{1_n}(l_1)(i) &= e_O(\tau) + \sum_{a<a'<n} \kappa_O(\tau,a') \\ &= -\sum_{1\le a'<n} \kappa_O^-(\tau,a') + \sum_{a<a'<n} \kappa_O^+(\tau,a') + \sum_{1\le a'<n-a} \kappa_O^-(\tau,a'),\end{aligned}$$

which is the same number. So $\nu = \Lambda_{1_n}(l_1)$.

Step 3: its weight on the indefinite side. By (F1) the parameters (l_1, l_2) match, so by (Q7) there is a joint harmonic Φ_0 for them, and by (Q8) and Step 2, Φ_0 is a complex multiple of Φ_τ. For $c \in [r]$ the operator C_{cc} multiplies Φ_0 by $\Lambda_{\varepsilon_O}(l_2)(c)$ and every multiple of Φ_τ by $\nu'(c)$; since $\Phi_0 \neq 0$, the two numbers agree. This is (iii). It also shows that Φ_τ is jointly highest of weights $(\Lambda_{1_n}(l_1), \Lambda_{\varepsilon_O}(l_2))$, hence a joint harmonic: this is (i).

Step 4: the polynomial of the highest vector. By (Q5) the mixed Cartan embedding sends $v^+_{\kappa_O}(l_O^V(\cdot,\cdot))$ to a pure tensor of extremal vectors, and their polynomials are products of minors of the matrices $(Y_{\tau,i,a})$: for the diagram κ_O^+, the product over τ and $a < n$ of the $\kappa_O^+(\tau,a)$-th power of the minor with the first a columns and the a rows of largest l_1; for κ_O^-, the same with $\kappa_O^-(\tau,a)$ and the a rows of smallest l_1. The letter substitutions $\mathrm{Let}_{\varepsilon_O,\mathrm{ord}_O,\pm 1}$ replace the column of index a' by the column $X_{\tau,\cdot,c}$ with $\varepsilon_O(c) = \pm 1$ and $\mathrm{ord}_O(c) = a'$, which is unique when it exists, by (F2). This turns the two products into $\prod_\tau \prod_a D_{1,a}^{k(1,a)}$ and $\prod_\tau \prod_a D_{-1,a}^{k(-1,a)}$ in the variables of each place, and their product is $\prod_\tau \Phi_\tau$. This is (ii). □

LEAN AuxPair.FitsParams.exists_isJointHarmonic

Corollary 4.4 (the vector of the highest polynomial is an edge)**.** *In Setting 2.1, let O be an auxiliary pair (Setting 2.3) with $[r]$ not empty, and let l be a family of integers $l(\tau,c)$ with l fits O. Let γ be the constant family with value $r_-(O)$ and $\gamma'(\tau) = \lfloor n/2 \rfloor + \mathrm{wt}_O(\tau)$. Then the vector $v_{O,\mathrm{Har}_O(v^+_{\kappa_O}(l_O^V(\cdot,\cdot)))}$ is an edge* (Definition 2.8) *from $\mathbf{e}_{l,\varepsilon_O}$ to $\mathbf{e}_{l_O^V(\cdot,\cdot),1_n}$ for the representation $\Omega_{\gamma,\gamma'}$, where $l_O^V(\cdot,\cdot)$ is the family $(\tau,i) \mapsto l_O^V(\tau,i)$.*

Proof. Since $[r]$ is not empty and inc_O maps it into $[n]$, the set $[n]$ is not empty. Theorem 2(i) and (ii) give the hypothesis of (Q10) for $s' = 1_n$, $s = \varepsilon_O$ and $l' = l_O^V(\cdot,\cdot)$, and its conclusion is the assertion. □

LEAN `AuxPair.FitsParams.isEdge`

The weights of $\Omega_{\gamma,\gamma'}$ in the corollary are those of the characters χ_O and χ'_O, by (Q2). So, by (Q3), the real points act on the data $v \otimes f$ of the Weil representation ω_O through $\Omega_{\gamma,\gamma'}$, and the corollary applies to the theta lifts between $W(O)$ and $\mathrm{I}_n(E)$ with the test datum.

5 The class-sum lift as a descent

Lemma 5.1 (the transformation law of the test datum)**.** *In Setting 2.1, let O be an auxiliary pair (Setting 2.3), $y \in G_\infty(1_n)$, $g' \in \mathbf{K}(O)$ and $\xi \in \mathbb{V}_{n,\kappa_O}$. Then*

$$\omega_O(1,(y,g'))\ \mathrm{J}_O(\bar\xi) = \tilde\eta_O(g') \cdot \mathrm{J}_O(\overline{\rho_{\kappa_O,e_O}(y)\,\xi}).$$

Proof. By (Q2), the weight of χ_O is $r_-(O)$ at every place, and the weight of χ'_O at τ is $\lfloor n/2 \rfloor + \mathrm{wt}_O(\tau)$. By (Q3) the left side is the tensor product of $\Omega_{\gamma,\gamma'}(1,y)v_{O,\mathrm{Har}_O(\bar\xi)}$, with these weights, and of $\omega_O(1,(1,g'))\Upsilon(O)$. The first is $v_{O,\mathrm{Har}_O(\overline{\rho_{\kappa_O,e_O}(y)\xi})}$ by Lemma 4.3; the second is $\tilde\eta_O(g')\Upsilon(O)$ by Proposition 3.2 with $g = 1$ and j the family of the reductions of g'. □

LEAN `AuxPair.pairWeil_fockDatum`

Lemma 5.2 (the kernel is equivariant)**.** *In Setting 2.1, let O be an auxiliary pair (Setting 2.3), γ a rational point of $\mathrm{I}_n(E)$, $g' \in \mathbf{K}(O)$, $v \in \mathbb{V}_{n,\kappa_O}$, $u \in G(\varepsilon_O)$ and $h \in G_{\mathrm{f}}(n)$. Then*

$$\mathcal{C}_O(u,\gamma_{\mathrm{f}}\,h\,g')(\tilde\eta_O(g')\,\rho_{\kappa_O,e_O}(\gamma)\,v) = \mathcal{C}_O(u,h)(v).$$

Proof. Put $\xi = \rho_{\kappa_O,e_O}(\gamma)v$. As $\xi \mapsto \mathrm{J}_O(\bar\xi)$ is conjugate-linear, the left side is $\tilde\eta_O(g')$ times the complex conjugate of $\theta_O(\omega_O(u,(1,\gamma_{\mathrm{f}}hg'))\,\mathrm{J}_O(\bar\xi))$. By Lemma 5.1 with $y = 1$,

$$\omega_O(1,(1,g'))\ \mathrm{J}_O(\bar\xi) = \tilde\eta_O(g')\ \mathrm{J}_O(\bar\xi);$$

the values of $\tilde\eta_O$ are roots of unity, so the two factors $\tilde\eta_O(g')$ and $\overline{\tilde\eta_O(g')}$ cancel. It remains to show that

$$\theta_O(\omega_O(u,(1,\gamma_{\mathrm{f}}h))\ \mathrm{J}_O(\bar\xi)) = \theta_O(\omega_O(u,(1,h))\ \mathrm{J}_O(\overline{v})).$$

The rational point γ is $(\gamma_\infty,\gamma_{\mathrm{f}})$ in $G(1_n)$, and $(1,\gamma_{\mathrm{f}}h) = \gamma\cdot(\gamma_\infty^{-1},h)$. By (1) the left side is

$$\theta_O\Big(\omega_O(u,(1,h))\,\omega_O(1,(\gamma_\infty^{-1},1))\ \mathrm{J}_O(\bar\xi)\Big),$$

and by Lemma 5.1 with $g' = 1$ the inner datum is $\mathrm{J}_O(\overline{\rho_{\kappa_O,e_O}(\gamma_\infty^{-1})\xi}) = \mathrm{J}_O(\overline{v})$. □

LEAN `AuxPair.conjKernel_equivariant`

Proof of Theorem 1. Write

$$v^+ = v^+_{\kappa_O}(l^V_O(\,\cdot\,,\,\cdot\,)), \qquad F_\Xi = F_{l^V_O(\,\cdot\,,\,\cdot\,),1_n}(\Xi).$$

Fix $u \in G(\varepsilon_O)$ and put $\Phi(h) = \overline{\theta_O(\omega_O(u,h)\,\mathrm{J}_O(v^+))}$ for $h \in G(1_n)$. The complex conjugate of the descent at u is $\int_{[\mathrm{I}_n(E)]}\Phi(h)F_\Xi(h)\,\mathrm{d}h$: the datum is conjugated with the value of the theta functional, and the two conjugations of F_Ξ cancel. We

apply (Q11)(b) with $\mathbf{K} = \mathbf{K}(O)$, $\kappa = \kappa_O$, $e = e_O$, $l_1 = l^V_O(\,\cdot\,,\,\cdot\,)$ (Lemma 4.1) and $L_h = \mathcal{C}_O(u,h)$.

The function Φ has the invariance of (Q11)(a), because $\mathrm{J}_O(v^+)$ is a polynomial vector tensor the finite datum $\Upsilon(O)$. For $h = (y, h_{\mathrm{f}})$, Lemma 5.1 with $g' = 1$, and $v^+ = \overline{v^+}$ from (Q9), give

$$\Phi(y,h_{\mathrm{f}}) = \overline{\theta_O(\omega_O(u,(1,h_{\mathrm{f}}))\ \mathrm{J}_O(\overline{\rho_{\kappa_O,e_O}(y)v^+}))} = \mathcal{C}_O(u,h_{\mathrm{f}})(\rho_{\kappa_O,e_O}(y)v^+).$$

The function $h \mapsto \mathcal{C}_O(u,h)(\Xi(h))$ on $G_{\mathrm{f}}(n)$ is right invariant under $\mathbf{K}(O)$: by Setting 2.3(d), $\Xi(hg') = \tilde{\eta}_O(g')\Xi(h)$, and Lemma 5.2 with $\gamma = 1$ gives

$$\mathcal{C}_O(u,hg')(\tilde{\eta}_O(g')\,\Xi(h)) = \mathcal{C}_O(u,h)(\Xi(h)).$$

So (Q11)(b) applies:

$$\int_{[\mathrm{I}_n(E)]} \Phi(h)\,F_\Xi(h)\,\mathrm{d}h = \frac{1}{\dim \mathbb{V}_{n,\kappa_O}\cdot[\mathbf{K}_0(n):\mathbf{K}(O)]}\sum_{[h]}\frac{1}{\#\Gamma_h}\,\mathcal{C}_O(u,h)(\Xi(h)),$$

and the sum is $\mathrm{Th}_O(\Xi)(u)$ by Setting 2.3(h). □

LEAN `AuxPair.theta_eq_smul_star_lift_swap`

Lemma 5.2 with general γ shows that the summand of the class sum does not depend on the representative of the double class, so that $\mathrm{Th}_O(\Xi)$ is well defined by Setting 2.3(h); the proof of Theorem 1 uses the lemma only for $\gamma = 1$.

6 Where the results are used

In the unit on the rows. The auxiliary pairs of the proof are the pairs (W_j, V_j) of the j-th theta lift of the chain, $1 \le j \le m$, and a parameter that fits them is provided by the unit on the exchange identity. For a form Ξ on the definite group with $\mathrm{Th}_O(\Xi) = 0$, Theorem 1 shows that the theta descent of $\overline{F_{l^V_O(\,\cdot\,,\,\cdot\,),1_n}(\Xi)}$ to $W(O)$ with the datum $\mathrm{J}_O(v^+_{\kappa_O}(l^V_O(\,\cdot\,,\,\cdot\,)))$ vanishes. Corollary 4.4 extends this to the data with any polynomial vector at the infinite places. The third note of this sequence then extends it to all data that are unramified outside the level, and to the spaces $W(O)$ plus hyperbolic planes; this is how forms that are not lifts are shown to have vanishing theta periods.

In the unit on the exchange identity. If l fits O and $W(O)$ is not zero, the map Th_O is adjoint, for the Petersson products, to the theta lift from $W(O)$ to the definite group with the datum $\mathrm{J}_O(v^+_{\kappa_O}(l^V_O(\,\cdot\,,\,\cdot\,)))$: Theorem 1 and Corollary 4.4 are the two inputs of this adjunction, and Lemma 5.1 gives the level and the weight of the lifted form. Proposition 3.1 is used where the datum $\Upsilon(O)$ is compared with data of full level.

In the unit on the twists. The image of Th_O is identified with a space of theta lifts by the same two results.

7 Notes on the formal text

1. *What is a definition and what is a theorem.* The class sum and the kernel of Setting 2.3(h) are the definition of Th_O in the unit on the rows; the integral with its scalar is the theorem of this unit. In Theorem 1 the basic descent is bilinear in the value of the theta functional and in its input function; the input is the complex conjugate of F_Ξ and the output is conjugated again. The formal statement writes the scalar as the number of elements of the index set of the basis of $\mathbb{V}_{n,\kappa_O}$ times the index $[\mathbf{K}_0(n) : \mathbf{K}(O) \cap \mathbf{K}_0(n)]$; since $\mathbf{K}(O) \subseteq \mathbf{K}_0(n)$ this is the index of Theorem 1.

2. *Arguments read from the text, not printed by Lean.* The file declares its variables twice: the field, the index sets and the pair are introduced at the top and again before the definition of $l_O^V(\tau, i)$, with the letters of the index sets changed. The program that reads the arguments of a declaration from the text cannot tell the two declarations of E apart. For the eight lemmas after the second declaration the dossier gives the list of arguments that Lean's rule for variables predicts, by hand, and marks it as not printed by Lean, since nothing was built. I read the two variable lines and the eight statements, and the lists agree with what I find: each statement takes the field with its two structures, the two index sets, the pair where it occurs, the Weil data from the lemma on edges onwards, and its own arguments, in this order. The statements of this note account for exactly these arguments. A print by Lean of the eight signatures is still to be made.

3. *Index sets.* In the formal text $[r]$ and $[n]$ are arbitrary finite types, r and n their numbers of elements, and the numbering π of Definition 2.5 is a bijection chosen by the library. In the first part of the file the two index sets are called by other letters; the note uses $[r]$ for the indefinite and $[n]$ for the definite side throughout. The twist group of Definition 2.2 is defined for any index set; it is used for $[r]$.

4. *The modulus.* The finite datum is defined with the modulus $\mathfrak{m}_2(\Sigma_O)$, in which the chosen places occur squared, and Proposition 3.1 is stated with $\mathfrak{m}_1(\Sigma_O)$. The function Ψ_O depends only on the reductions at the places, so the two data agree; the formal proof begins with this change of modulus.

5. *Steps carried out by automation.* The reader of the unit marked the central steps of Propositions 3.1 and 3.2 as descriptions of what the simplifier computes, not replayed goal by goal. I read these lines. The lemmas they name are those used in the proofs above: the action of integral elements on finite data (Q1), the values of the characters (Q2), the reductions of the twist lift, the two rules for products with matrices in the level (rows of a product with a matrix of block triangular shape, and $\mathrm{diag}(g, 1)\,\mathrm{Inc}_O = \mathrm{Inc}_O\, g$), and the case distinction on the invertibility of the tested blocks, which is explicit in the Lean text. The proofs above are complete as mathematics; that the simplifier's steps are these was not replayed. In Lemma 4.3 the formal proof uses the compatibility of Pol_κ with the matrices S_κ through rules of the library that are not named in the text of the proof; the form in which (Q5) states it, with a transpose, is the one that makes Lemmas 4.2 and 4.3 agree, and I did not read its proof in the library. Likewise Step 4 of the proof of Theorem 2 describes a computation that the formal proof leaves to the simplifier, after unfolding the highest vector, the letter substitutions and the polynomials of extremal vectors.

6. *A weaker hypothesis would do, and a stronger conclusion holds.* Theorem 2 uses of (F1) only that the parameters match; of the requirement $r < n$ only that $i + i' > n$ forces $i > r_+(O)$ or $i' > r_-(O)$. The formal proof of Theorem 2 does not use Weil data, and the theorem is stated without them; Corollary 4.4 has them because edges are defined with theta lifts.

7. *References for the quoted facts.* The catalogue of the handout and the second reader's comparison with the literature give the following; where no place is named, the *reference is to be confirmed.* (Q6): M. Kashiwara, M. Vergne, *On the Segal–Shale–Weil representations and harmonic polynomials*, Invent. Math. 44 (1978), Chapter III, Proposition 6.1 with formula (6.2) and Theorem 6.3, and M. G. Davidson, *The harmonic representation of U(p,q) and its connection with the generalized unit disk*, Pacific J. Math. 129 (1987), §2, Theorem 2.9; the formal statement is a *special case* (the explicit products of minors under the condition on the sizes, in the orders given by the parameters, with the shifts of the weights by the vacuum and the determinant characters). (Q7) and (Q8): R. Howe, *Transcending classical invariant theory*, J. Amer. Math. Soc. 2 (1989); J.-S. Li, Duke Math. J. 61 (1990); A. Paul, J. Funct. Anal. 159 (1998); *special cases* (existence and uniqueness up to a scalar only for matched parameters with large gaps, in the Fock model); places not confirmed. (Q10): A. Ichino, *Theta lifting for discrete series representations of real unitary groups*, §7.2, Proposition 7.2, and §7.4, Lemma 7.8; Howe, loc. cit., Theorem 1; the relation is *unresolved*: the published results give the correspondence of the compact types, and the two-sided analytic conclusions of an edge need further argument; the catalogue describes the formal statement as a *weaker form* of the correspondence of discrete series, without Howe duality and without non-vanishing of the lift. The Weil representation of the pair and its theta functional, (Q1) to (Q4): the library's file names A. Weil, *Sur certains groupes d'opérateurs unitaires*, which the artifact's list of references marks "work found, no place given"; reference to be confirmed. (Q5), (Q9) and (Q11): no reference recorded; (Q11)(b) is Schur's orthogonality for the compact group $G_\infty(1_n)$ together with the definition of the measure: its factor is the product of $1/\dim \mathbb{V}_{n,\kappa}$, from the orthogonality relation for the unit vector $v^+_\kappa(l_1)$ on a group of volume 1, and of $1/[\mathbf{K}_0(n) : \mathbf{K}]$, the volume of $\mathbf{K}$ when $\mathbf{K}_0(n)$ has volume 1. The two normalisations are those of the library: the header of its lemma behind (Q11)(b) says "vol $\mathbf{K}_0 = 1$", and (Q9) has the unit vector.

How this note was written. From the dossier of the unit: sixteen members and seven structured proofs. The program check of the unit's dossier covered 29 leaf steps of 7 proofs, found the statements at their lines and compared the quotations of the imported declarations with the source. I read the whole Lean file of the unit and compared every statement with the statements above, argument by argument. In the unit on the rows I read the definition of an auxiliary pair and of all objects of Setting 2.3. In the background library I read the definitions behind Setting 2.1 and Definitions 2.6 and 2.8 (groups, spaces with a point, characters with weight and tame part, the Weil representation of a pair and the theta functional as they are packaged, the lift, finite data, the row and column operators, lowest K-types, gaps, matching, joint harmonics, staircase weights, cusp kernels, forms of a type, half edges and edges)

and the statements of the lemmas behind (Q1) to (Q11); I read none of their proofs, and I did not read the construction of the space of data, of the theta functional, of the quotient measure or of the Weyl modules. The normalisation of the measure in Setting 2.1 is read from the statements and headers of the library's lemmas behind (Q11)(b) and of its Schur orthogonality, which holds as stated only for total mass 1 of $G_\infty(1_n)$. The second reader of the artifact read the unit independently; her remarks on the two conjugations of Theorem 1, on the inverse in the character η_O and on the signed ordering in (F2) agree with the text above. The proofs follow the formal ones; those of the two propositions of §3 and of Lemma 5.2 are written out from the definitions and are longer than the structured proofs, and the remarks after the statements and §6 are mine. For §6 I read the statements of the consuming lemmas.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Definition 2.2	`LevelPlaces.TwistGL`	AuxiliaryPair.lean	15–16
Definition 2.2	`LevelPlaces.twistLift`	AuxiliaryPair.lean	24–25
Definition 2.4	`AuxPair.levelIndicator`	AuxiliaryPair.lean	29–32
Proposition 3.1	`AuxPair.upsilon_eq_sum`	AuxiliaryPair.lean	39–50
Proposition 3.2	`AuxPair.pairWeil_upsilon`	AuxiliaryPair.lean	55–67
Definition 2.5	`AuxPair.paramV`	AuxiliaryPair.lean	74–74
Lemma 4.1	`AuxPair.isHCParam_paramV`	AuxiliaryPair.lean	79–79
Definition 2.7	`AuxPair.FitsParams`	AuxiliaryPair.lean	85–90
Lemma 4.2	`realPairWeil_fockLetters`	AuxiliaryPair.lean	97–101
Lemma 4.3	`AuxPair.realPairWeil_harmonicPoly`	AuxiliaryPair.lean	106–115
Theorem 2	`AuxPair.FitsParams.exists_isJointHarmonic`	AuxiliaryPair.lean	120–138
Corollary 4.4	`AuxPair.FitsParams.isEdge`	AuxiliaryPair.lean	145–148
Lemma 5.1	`AuxPair.pairWeil_fockDatum`	AuxiliaryPair.lean	153–155
Lemma 5.2	`AuxPair.conjKernel_equivariant`	AuxiliaryPair.lean	160–164
Theorem 1	`AuxPair.theta_eq_smul_star_lift_swap`	AuxiliaryPair.lean	169–179

The two theorems as they are written in the artifact.

```
lemma AuxPair.theta_eq_smul_star_lift_swap Ξ : O.θ Ξ = (card (WeylIndex n O.κ) * stdLevelIndex
  O.levelGroup : ℂ) • star (Space.lift O.pairThetaKernel.swap
  (O.fockDatum (highestVector _ O.paramV)) (star (ofAlgForm O.paramV 1 Ξ.1)))

lemma AuxPair.FitsParams.exists_isJointHarmonic {l} (h : O.FitsParams l) : ∃ Φ : _ → _,
    (∀ w, IsJointHarmonic (.std 1 O.signs (negCount O.signs) (-((card n / 2 : ℕ) + O.weight w)))
  (O.paramV w) (-l w) (Φ w)) ∧ O.harmonicPoly (highestVector _ O.paramV) = ∏ w,
  (Φ w).rename (w, ·) ∧ ∀ w c, (toHCParam O.signs (-l w)).lowestKType c = staircaseWeight n O.signs
  O.ord (if · = 1 then O.colsPlus w else O.colsMinus w) (-((card n / 2 : ℕ) + O.weight w)) c
```

Legend. In the formal text the auxiliary pair is `O`; `O.signs`, `O.W`, `O.incl`, `O.ord`, `O.colsPlus`, `O.colsMinus`, `O.μ`, `O.weight`, `O.ψ` are ε_O, $W(O)$, inc_O, ord_O, κ_O^+, κ_O^-, μ_O, wt_O, ψ_O; `O.κ` and `O.detExp` are κ_O and e_O; `O.a` and `O.b` are χ_O and χ'_O; `negCount O.signs` is $r_-(O)$; `O.auxDatum` is $\Upsilon(O)$, `O.harmonicPoly` is Har_O, `O.fockDatum` is J_O, `O.conjKernel` is $\mathcal{C}_O$ and `O.θ` is Th_O; `O.levelGroup`, `O.levelChar` and `O.auxLevelChar` are $\mathbf{K}(O)$, $\tilde{\eta}_O$ and η_O; `Kf` is $\mathbf{K}_0$; the letters `k`, `k'` for its elements are g, g' here, the letter `g` for a point of the adelic group of $W(O)$ is u, the sign `c` of the lemma on letters is σ, the signs `t` of the library's definitions on $[n]$ are s' (the letter t is kept in this write-up for the step of the ranks), and the infinite places, written `w` in part of the file, are τ. `Space.lift` with the swapped kernel

is the descent, `star` on functions is complex conjugation, `ofAlgForm` gives $F_{l_1,1_n}(\Xi)$, and `realPairWeil γ γ'` is $\Omega_{\gamma,\gamma'}$.

Notation

E, τ, w, $\overline{w}$ a CM field with a CM type; an infinite place; a finite place and its conjugate

E_w, $\mathfrak{o}_w$, $\mathbb{F}_w$, $\mathbb{A}_{\mathrm{f}}$, $\hat{\mathfrak{o}}$ completion, its integers, residue field; finite adeles; their maximal compact subring

$[r]$, $[n]$, $[r']$, r, n finite index sets; their numbers of elements (ranks of the indefinite and the definite space)

i, i', j; c, c'; a, a' indices in $[n]$ (j is also an element of $\mathbf{L}(O)$); indices in $[r]$; natural numbers (column indices of diagrams, sizes of minors, exponents)

X^*, X^{T}, $\overline{z}$ conjugate transpose, transpose, complex conjugate

$\jmath$, υ, υ_1 the Weil data: a purely imaginary element, the splitting character, its restriction to norm one

ε, ε', s, s', σ, 1_n families of signs on index sets; a sign; the constant family 1 on $[n]$

$U(\varepsilon)$, $G_\infty(\varepsilon)$, $G_{\mathrm{f}}(r)$, $G(\varepsilon)$ the unitary group of the signs; its product over the infinite places; the finite adelic unitary group; their product

u, h, y points of the adelic group of the indefinite, resp. definite, space, with components u_∞, u_{f}, h_∞, h_{f}; y a point of a real group, mostly of $G_\infty(1_n)$ (in (Q5), y is any family of invertible complex matrices)

$\mathbf{K}_0(n)$, g, g', red_w, $\mathbf{K}$ the integral points of $G_{\mathrm{f}}(n)$; elements of such groups; reduction modulo w; a level

W, V, J, ϕ, $\mathrm{I}_n(E)$ hermitian spaces with a point; the matrix of the form and the point; the standard definite space

γ, γ_∞, γ_{f} a rational point and its components (in §5); otherwise γ, γ' are weights: integers or families of integers

χ, χ', μ, e, ψ_w automorphic characters; a weight; a tame part

$\mathcal{S}(W,V)$, $\omega^{W,V}_{\chi,\chi'}$, $\theta_{W,V}$ data, the Weil representation of the pair, the theta functional

φ, f, $v\otimes f$, φ_0 data; the datum with archimedean part v; the Gaussian

$\mathrm{Desc}^{W,V}_{\chi,\chi'}(\varphi,Y)$, Y, $\mathrm{d}h$ the theta descent of a function Y from V to W; the invariant measure

$\mathcal{F}(\varepsilon,\varepsilon')$, $v_p^{\varepsilon,\varepsilon'}$, p the Fock space of the pair; the vector of a polynomial p

$X_{\tau,i,c}$, X_{ic}, $Y_{\tau,i,a}$, X the variables of the Fock model; those of one place; the matrix variables of Weyl modules; X also a matrix over a residue ring, with reductions X_w

$\mathfrak{m}$, $[\Psi]^{W,V}_{\mathfrak{m}}$, Ψ, Ψ' a modulus; the finite datum of a function on matrices modulo $\mathfrak{m}$

$\Omega_{\gamma,\gamma'}$, Ω the representation of the real points on the Fock space with weights γ, γ'; a map to operators of the Fock space

Σ, M_Σ, Q_Σ, S a level system, its chosen and twist places; a finite set of places

$\mathrm{GL}^{\mathrm{tw}}_n(\Sigma)$, $\mathrm{GL}^{\mathrm{ch}}_n(\Sigma)$, A the products of $\mathrm{GL}_n(\mathbb{F}_w)$ over the twist, resp. chosen, places; an element of the first

$\mathrm{lift}_\Sigma(A)$ the twist lift of A

O an auxiliary pair; Σ_O, M_O, Q_O its level system and places

$W(O)$, ε_O, $r_+(O)$, $r_-(O)$ its indefinite space, signs, numbers of positive and negative signs

inc_O, Inc_O, $\mathrm{ord}_O(c)$ its embedding of index sets, the matrix of the embedding, its numbering

$\kappa^+_O(\tau,i)$, $\kappa^-_O(\tau,i)$, κ_O, e_O column numbers; the weight on the definite side

μ_O, $\mathrm{wt}_O(\tau)$, $\psi_{O,w}$ its character, with weight and tame part

$\mathbb{V}_{n,\kappa}$, S_κ, $\langle\xi,\eta\rangle_\kappa$, Pol_κ coordinate vectors of the Weyl module, its matrices, its hermitian form, its polynomial realisation

$\rho_{\kappa,e}$, ξ, η, α, β, v the weight representation; vectors of Weyl modules (α also a function of compact support, v also a Fock vector)

$\mathbf{L}(O)$, η_O, $\mathbf{K}(O)$, $\tilde{\eta}_O$ the level in the residue groups, its character; the level, its character

$\mathrm{Form}(O)$, Ξ algebraic forms on the definite group; one of them

χ_O, χ'_O, $\mathcal{S}(O)$, ω_O, θ_O, $\mathcal{F}(O)$, $v_{O,p}$ the characters, data, Weil representation, theta functional, Fock space and polynomial vectors of O

$\Psi_O(X)$, $\mathfrak{m}_1(\Sigma)$, $\mathfrak{m}_2(\Sigma)$, $\Upsilon(O)$ the function of residue matrices, the two moduli, the finite datum

$\mathbf{1}_O(X)$ the level indicator

$\mathrm{Let}_{s,o,\sigma}$, o, C_O, F the letter substitution; a numbering; the mixed Cartan embedding; a polynomial in the $Y_{\tau,i,a}$ (in Definition 2.8 and §5 also a form)

$\mathrm{Har}_O(\xi)$, $\mathrm{J}_O(\xi)$ the harmonic polynomial and the test datum of ξ

$\mathcal{C}_O(u,h)$, $\mathrm{Th}_O(\Xi)$, Γ_h the kernel, the class-sum lift, the stabiliser of a class

$l^V_O(\tau,i)$, $l^V_O(\cdot,\cdot)$, π the parameter on the definite side; the numbering of $[n]$

l, l_1, l_2, l' parameters (doubled Harish-Chandra parameters)

l fits O Definition 2.7

A_X, B_X, $R_{ii'}$, $C_{cc'}$, $\delta_{ii'}$ raising and lowering operators; row and column operators; Kronecker's symbol

$\Lambda_{s'}(l_1)$, $\xi(i)$, $\xi'(c)$ the lowest K-type; the centred parameters

Φ, Φ_τ, Φ_0, ν, ν' polynomials (joint harmonics); weights (Φ is also an automorphic function in (Q11))

$\mathrm{sw}_n(s, o, k, \gamma')(c)$, k, $D_{\sigma,a}$ the staircase weight; column numbers; the minors of (Q6)

$\mathbf{e}_{l,s}$, $\mathbf{e}$, $\mathbf{e}'$ the discrete series kernel of l; functions on the real groups

$v_\kappa^+(l_1)$, v^+, $F_{l_1,1_n}(\Xi)$, F_Ξ, L_h the highest vector; the scalar function of a form; linear forms

$F^{y,\sigma}$, $y \cdot p$, z_τ substitutions in polynomials; a matrix

z an idele or adele (Setting 2.1)

READBACK NOTE 10

Vanishing of theta descents: reflection sums at the level, a maximum principle for Hecke polynomials, and adding hyperbolic planes

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. A function Y on a definite unitary group has *vanishing descents* to a space W if its theta lifts to the unitary group of W are zero for all data that are unramified at a given set of places. The unit proves two transfers of this property. First, for the auxiliary pair of the second note: if the descents vanish for the single test datum at the level, with all polynomial vectors at the infinite places, then they vanish for all data unramified outside the level, provided the form is killed by the reflection sums at the chosen places and the character of the level is non-trivial at the twist places. Second, vanishing climbs the theta tower: from W to W plus ν hyperbolic planes, as long as $2r + 2\nu + 3 < n$ and Y is killed by a Hecke polynomial at one split place at a point that is too small to be a Hecke root of a lift. The second transfer rests on a maximum principle: a bounded function killed by a Hecke polynomial with a small argument is zero.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, whose final theorems are the Riemann hypothesis and its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note treats one unit of the artifact: all 8 of its 8 declarations. The proofs are given in full relative to inputs that are stated and not proved: the auxiliary pair and its objects (Setting 2.3, from the unit on the rows), and results of the background library on the Weil representation, on Hecke operators at split places and on the decay of theta kernels (the facts (Q1) to (Q11)). One step, the bound for the theta kernel near the cusps in the proof of Theorem 2, is described through the library's estimate and was not traced in its constants (§8, note 3). Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 8 declarations. This is the third of three notes on the theta correspondence behind Theorem C; it uses the notation of the note *The test datum of an auxiliary pair, joint harmonics with staircase weights, and the theta lift as a conjugate descent.*

1 The statements

The objects are defined in §2. In brief: $\mathrm{Desc}^{W,V}_{\chi,1}(f,Y)$ is the theta descent of a function Y on the adelic group of a definite space V to the adelic group of W, with datum f, for the Weil representation with characters χ and 1; $\mathrm{DV}(W,V;\chi;S;Y)$ says that it vanishes for all data unramified at the set of places S; for an auxiliary pair O, $\mathrm{pl}(O)$ is the set of its chosen and twist places, $\Upsilon(O)$ its finite datum, and Ξ_L the scalar function of a form $\Xi \in \mathrm{Form}(O)$ and a linear form L; $\mathrm{Refl}(\psi,y)$ is a reflection sum in the group algebra of a general linear group over a finite field; and $H^{\pi}_{w}(T)$ is the Hecke polynomial at a split place w, evaluated at a complex number T.

Theorem 1 (from the test datum to all data unramified outside the level). *In Setting 2.1, let O be an auxiliary pair as in Setting 2.3, $\Xi \in \mathrm{Form}(O)$ and L a linear form on $\mathbb{V}_{n,\kappa_O}$. Suppose:*

(a) *$\psi_{O,w} \neq 1$ for every twist place $w \in Q_O$;*

(b) *for every chosen place $w_0 \in M_O$ and every family $y = (y_w)_{w\in M_O}$ of vectors $y_w \in \mathbb{F}_w^r$ with $y_{w_0} \neq 0$, the element $\bigotimes_{w\in M_O} \mathrm{Refl}(\psi_{O,w}, y_w)$ of the group algebra of $\mathrm{GL}_r^{\mathrm{ch}}(\Sigma_O)$ annihilates Ξ in the action of Setting 2.3(c);*

(c) *for every polynomial p in the variables $X_{\tau,i,c}$, $\mathrm{Desc}^{W(O),\mathrm{I}_n(E)}_{\chi_O,1}(v_{O,p} \otimes \Upsilon(O), \overline{\Xi_L}) = 0$.*

Then $\mathrm{DV}(W(O),\mathrm{I}_n(E);\chi_O;S;\overline{\Xi_L})$ holds for the set S of all finite places of E that are neither in $\mathrm{pl}(O)$ nor conjugate to a place of $\mathrm{pl}(O)$.

LEAN `AuxPair.descentsVanish_of_reflectionSum`

Theorem 2 (climbing the theta tower). *In Setting 2.1, let W, W' and V be hermitian spaces with points, of signs ε on $[r]$, ε' on $[r']$ and 1_n on $[n]$, let χ be an automorphic character, S a set of finite places of E and Y a complex function on $G(1_n)$, and suppose $\mathrm{DV}(W,V;\chi;S;Y)$. Let Π be an adapted splitting of W' over W with planes indexed by a finite set $[\nu]$ of ν elements* (Definition 2.6). *Let w be a finite place of E and T a complex number. Suppose that Y is automorphic on V; that w is split, $w \neq \overline{w}$; that the base change of χ is unramified at $\overline{w}$; that $\overline{w} \in S$; that Y is spherical at w; that $|T| = \left(\sqrt{q_w}^{\,r} q_w^{n-r-1}\right)^{-1}$; that $H^{\mathrm{R}_{1_n}}_{w}(T)\, Y = 0$; and that $2r + 2\nu + 3 < n$. Then $\mathrm{DV}(W',V;\chi\upsilon_1^{\nu};S;Y)$.*

LEAN `descentsVanish_add_planes`

The space W' is W plus ν hyperbolic planes, and the spaces $W \oplus \mathbb{H}^{\nu}$, $\nu = 0, 1, 2, \dots$, form the Witt tower of W; the character changes by υ_1 with every plane. The number $q_w^{-(n-1-r/2)}$ is the absolute value of the smallest of the points at which the Hecke transfer (Q9) makes the Hecke polynomial vanish on theta lifts from a space of rank r to V; for the spaces of rank $r + 2\nu$, $\nu \geq 1$, the corresponding absolute value is larger by the factor q_w^{ν}. So the theorem says that a function with this Hecke root and without descent to W has no descent to the higher spaces of the tower either, in the range $2r + 2\nu + 3 < n$. In the application (§7) Theorems 1 and 2 are applied one after the other, to $Y = \overline{\Xi_L}$.

2 The setting

Conventions. The conventions of the second note of this sequence hold: E is a CM field with a CM type; τ denotes its infinite places and w its finite places, with conjugate $\overline{w}$, completion E_w, integers $\mathfrak{o}_w$, residue field $\mathbb{F}_w$ with q_w elements, a fixed uniformiser ϖ_w and the absolute value $|\cdot|_w$ with $|\varpi_w|_w < 1$; $\mathbb{A}_{\mathrm{f}}$ is the ring of finite adeles and $\hat{\mathfrak{o}}$ the product of the $\mathfrak{o}_w$. The index sets $[r]$, $[n]$, $[r']$, $[\nu]$ are finite sets with r, n, r', ν elements; an $n \times r$ matrix has rows indexed by $[n]$ and columns by $[r]$. For a matrix X, X^* is the conjugate transpose and X^{T} the transpose. A complex vector space is written $\mathcal{V}$ when it carries a representation. Where a statement says so, E carries Weil data (Setting 2.1). Letters of the handout's overview that are not displayed with the data as an argument (among them Υ, f, ε, β, δ, n and V) have, in this note, only the meaning given in its notation list.

Setting 2.1 (groups, the Weil representation, descents; quoted, as in the second note)**.** The field E carries Weil data; of these we use the splitting character and its restriction υ_1 to the ideles of norm one. For a family of signs ε on $[r]$, $G(\varepsilon) = G_\infty(\varepsilon) \times G_{\mathrm{f}}(r)$ is the adelic unitary group: $G_\infty(\varepsilon)$ is the product over the infinite places of the unitary groups of $\mathrm{diag}(\varepsilon)$, and $G_{\mathrm{f}}(r)$ the group of $u \in \mathrm{GL}_r(\mathbb{A}_{\mathrm{f}})$ with $u^*u = 1$. The group $\mathbf{K}_0(r) = G_{\mathrm{f}}(r) \cap \mathrm{GL}_r(\hat{\mathfrak{o}})$ is open and compact, and for $w \neq \overline{w}$, $\mathrm{red}_w \colon \mathbf{K}_0(r) \to \mathrm{GL}_r(\mathbb{F}_w)$ is the reduction. For a set S of finite places, an element of $G_{\mathrm{f}}(r)$ is *supported on* S if its components at all places outside S and not conjugate to a place of S are 1. A hermitian space with a point, of signs ε, has a matrix J over E, a group of rational points embedded in $G(\varepsilon)$, and a lattice at every point of $G(\varepsilon)$; the standard definite space $\mathrm{I}_n(E)$ has signs 1_n. A function on $G(\varepsilon)$ is *automorphic* on such a space if it is continuous, left invariant under the rational points and right invariant under an open subgroup of $G_{\mathrm{f}}(r)$. An automorphic character is a continuous unitary character of the adeles of norm one of E, trivial on the rational ones; its *base change* is the Hecke character $z \mapsto \chi(z/\overline{z})$ of E, and a Hecke character is unramified at w if it is trivial on the units of $\mathfrak{o}_w$.

For spaces W, V of signs ε on $[r]$ and ε' on $[n]$ and automorphic characters χ, χ', the library provides the Banach space $\mathcal{S}(W,V)$ of data on $\mathrm{Hom}(W,V)$, with the Heisenberg operators $\varrho(z)$ of adelic vectors z of $\mathrm{Hom}(W,V)$, the Weil representation $\omega^{W,V}_{\chi,\chi'}$ of $G(\varepsilon) \times G(\varepsilon')$ on it, and the theta functional $\theta_{W,V}$, invariant under the rational points of both groups. The vacuum $\varphi^\circ \in \mathcal{S}(W,V)$ is the Gaussian at the infinite places times the characteristic function of the integral matrices at the finite places. For a Fock vector v of the pair and a datum f there is the datum $v \otimes f$; $v_p^{\varepsilon,\varepsilon'}$ is the Fock vector of a polynomial p in the variables $X_{\tau,i,c}$. The theta descent of a function Y on $G(\varepsilon')$ with datum f is the function

$$\mathrm{Desc}^{W,V}_{\chi,\chi'}(f,Y)(u) = \int_{[V]} \theta_{W,V}(\omega^{W,V}_{\chi,\chi'}(u,h)f)\, Y(h)\,\mathrm{d}h \qquad (u \in G(\varepsilon)),$$

the integral over the quotient of $G(\varepsilon')$ by the rational points of V, with the measure of the second note. In this note the second character is always the trivial character 1.

LEAN in the background library: `WeilData`, `WeilData.upsilonU1`, `G`, `Gf`, `Kf`, `Gf.supportedOn`, `Space`, `stdSpace`, `Space.IsAutomorphic`, `AutChar`, `AutChar.baseChange`, `CharUnramifiedAt`, `pairWeil`, `pairThetaKer`, `Space.lift`, `Space.integrated`

Definition 2.2 (data unramified at a set of places; vanishing descents; quoted)**.** Let W, V be as in Setting 2.1 and S a set of finite places of E. The operator P^S on $\mathcal{S}(W,V)$ is the mean of the Heisenberg operators $\varrho(z)$ over the integral finite vectors z whose components vanish at the places that are neither in S nor conjugate to a place of S; it is a projector, and a datum f with $\mathrm{P}^S f = f$ is called *unramified at* S: at the places of S and their conjugates it behaves as the vacuum. For an automorphic character χ and a function Y on $G(\varepsilon')$, the statement $\mathrm{DV}(W,V;\chi;S;Y)$ is:

$$\mathrm{Desc}^{W,V}_{\chi,1}(f,Y) = 0 \qquad \text{for every } f \in \mathcal{S}(W,V) \text{ with } \mathrm{P}^S f = f.$$

LEAN in the background library: `Space.projOn`, `DescentsVanish`

Setting 2.3 (auxiliary pairs; Input from the unit on the rows)**.** Let E carry Weil data. An auxiliary pair O over E for the index sets $[r]$ and $[n]$, $r < n$, is as in the second note. We use: its level system Σ_O with the chosen places M_O and the twist places Q_O, and $\mathrm{pl}(O) = M_O \cup Q_O$, a finite set of places none of which is conjugate to another or to itself; its space $W(O)$ of signs ε_O on $[r]$; the injection $\mathrm{inc}_O\colon [r] \to [n]$ and the $n \times r$ matrix Inc_O with entries 1 at the places $(\mathrm{inc}_O(c), c)$ and 0 elsewhere; the multiplicative characters $\psi_{O,w}$ of the fields $\mathbb{F}_w$, extended by 0, trivial for $w \notin \mathrm{pl}(O)$; the weight (κ_O, e_O), the space $\mathbb{V}_{n,\kappa_O}$ and the representation ρ_{κ_O,e_O} of $G_\infty(1_n)$ on it; the character χ_O, a power of v_1; the Fock vectors $v_{O,p} = v_p^{\varepsilon_O,1_n}$; and the following.

(a) *Level.* $\mathbf{L}(O)$ is the group of families $(j_w)_{w\in\mathrm{pl}(O)}$, $j_w \in \mathrm{GL}_n(\mathbb{F}_w)$, with $j_w\,\mathrm{Inc}_O = \mathrm{Inc}_O$ for $w \in M_O$ and, for $w \in Q_O$, with vanishing entries in the rows of the image of inc_O and the columns outside it; $\eta_O(j)$ is the product over $w \in Q_O$ of $\psi_{O,w}(\det(\mathrm{inc}_O\text{-block of } j_w))^{-1}$; $\mathbf{K}(O)$ is the group of $g' \in \mathbf{K}_0(n)$ with $(\mathrm{red}_w\, g')_w \in \mathbf{L}(O)$, and $\tilde\eta_O(g') = \eta_O((\mathrm{red}_w\, g')_w)$.

(b) *Forms.* An *algebraic form of weight* (κ_O, e_O) is a function $\Xi\colon G_{\mathrm{f}}(n) \to \mathbb{V}_{n,\kappa_O}$ with $\Xi(\gamma_{\mathrm{f}} h) = \rho_{\kappa_O,e_O}(\gamma)\Xi(h)$ for rational points γ of $\mathrm{I}_n(E)$, whose stabiliser under right translation is open; $G_{\mathrm{f}}(n)$ acts on these forms by right translation, written $\mathrm{R}(g')\Xi = \Xi(\cdot\, g')$. The space $\mathrm{Form}(O)$ consists of the algebraic forms with $\Xi(hg') = \tilde\eta_O(g')\Xi(h)$ for $g' \in \mathbf{K}(O)$. For a linear form L on $\mathbb{V}_{n,\kappa_O}$, Ξ_L is the function $h \mapsto L(\rho_{\kappa_O,e_O}(h_\infty^{-1})\Xi(h_{\mathrm{f}}))$ on $G(1_n)$.

(c) *The action of the chosen places.* Let π be a representation of $G_{\mathrm{f}}(n)$ on a complex vector space $\mathcal{V}$, and let $\mathcal{V}^O$ be the subspace of the ϕ with $\pi(g')\phi = \tilde\eta_O(g')\phi$ for all $g' \in \mathbf{K}(O)$. The group $\mathrm{GL}_r^{\mathrm{ch}}(\Sigma_O) = \prod_{w\in M_O} \mathrm{GL}_r(\mathbb{F}_w)$ acts on $\mathcal{V}^O$: a family $A = (A_w)$ acts by

$$[A]_{O,\pi}\,\phi = \prod_{w\in M_O} \psi_{O,w}(\det A_w)\cdot \pi(g_A)\,\phi.$$

Here $g_A \in \mathbf{K}_0(n)$ is the element that the library chooses with

$$\mathrm{red}_w\, g_A = \mathrm{diag}(A_w, 1) \quad (w \in M_O), \qquad \mathrm{red}_w\, g_A = 1 \quad (w \in Q_O),$$

and $\mathrm{diag}(A_w,1)$ acts by A_w on the basis vectors with indices in the image of inc_O and fixes the others. For $\mathcal{V}$ the space of algebraic forms of (b) with $\pi = \mathrm{R}$, we have $\mathcal{V}^O = \mathrm{Form}(O)$, and $[A]_{O,\mathrm{R}}$ is the action on forms.

(d) *The finite datum.* $\Psi_O(X)$, for a family $X = (X_w)_{w\in\mathrm{pl}(O)}$ of $n \times r$ matrices over the fields $\mathbb{F}_w$, is $\prod_{w\in Q_O} \psi_{O,w}(\det(\mathrm{inc}_O\text{-rows of } X_w))$ if $X_w = \mathrm{Inc}_O$ for all

$w \in M_O$, and 0 otherwise. The finite datum $\Upsilon(O) \in \mathcal{S}(W(O), \mathrm{I}_n(E))$ has, at the places of $\mathrm{pl}(O)$, the component $X \mapsto \Psi_O((X \bmod w)_w)$ on the integral matrices and 0 off them, and is the vacuum at all other places.

LEAN in the unit on the rows: `AuxPair`, `auxLevel`, `AuxPair.auxLevelChar`, `AuxPair.levelGroup`, `AuxPair.levelChar`, `AuxPair.FormV`, `AuxPair.chosenGLRep`, `AuxPair.formAction`, `AuxPair.resDetDatum`, `AuxPair.auxDatum`, `AuxPair.a` · in the background library: `algFormCoeff`, `levelRep`, `liftResidue`

Definition 2.4 (reflection sums; quoted)**.** Let $\mathbb{F}$ be a finite field, ψ a multiplicative character of $\mathbb{F}$ extended by 0, and $y \in \mathbb{F}^r$. The *reflection sum* is the element of the complex group algebra of $\mathrm{GL}_r(\mathbb{F})$

$$\mathrm{Refl}(\psi, y) = \sum_A \psi(\det A)^{-1}\, A,$$

the sum over the invertible matrices of the form $A = 1 + \xi\, y^{\mathsf{T}}$ with a column vector $\xi \in \mathbb{F}^r$. For finite groups G_w and elements a_w of their group algebras, $\bigotimes_w a_w$ is the element of the group algebra of $\prod_w G_w$ whose coefficient at $(A_w)_w$ is the product of the coefficients of the a_w at the A_w.

LEAN in the background library: `reflectionSum`, `piTensorElem`

For $y \neq 0$ the invertible matrices $1 + \xi y^{\mathsf{T}}$ are the elements of $\mathrm{GL}_r(\mathbb{F})$ that fix every vector of the hyperplane on which the linear form y^{T} vanishes: the transvections and the pseudo-reflections along it; for $y = 0$ the sum is the identity. Hypothesis (b) of Theorem 1 is a property that the consumers of the theorem establish for the forms to which they apply it.

Definition 2.5 (Hecke polynomials at a split place; quoted)**.** Let w be a finite place of E with $w \neq \overline{w}$, and π a representation of $G_{\mathrm{f}}(r')$ on a complex vector space $\mathcal{V}$. Since w is split, $\mathrm{GL}_{r'}(E_w)$ embeds in $G_{\mathrm{f}}(r')$: a matrix g goes to the element with component g at w, $(g^*)^{-1}$ at $\overline{w}$, and 1 elsewhere. A vector is *spherical at* w if it is fixed by the elements of $\mathbf{K}_0(r')$ supported on $\{w\}$. On such vectors the Hecke operators $T_{w,a}$, $0 \le a \le r'$, are those of the double cosets of $\mathrm{diag}(\varpi_w 1_a, 1_{r'-a})$ in $\mathrm{GL}_{r'}(E_w)$, and the *Hecke polynomial* at a complex number T is the operator

$$H_w^{\pi}(T) = \sum_{a=0}^{r'} (-T)^a\, q_w^{a(a-1)/2}\, T_{w,a}.$$

For signs ε' on $[r']$, $\mathrm{R}_{\varepsilon'}$ is the representation of $G_{\mathrm{f}}(r')$ on the complex functions on $G(\varepsilon')$ by right translation.

LEAN in the background library: `FinPlace.IsSplit`, `atSplit`, `sphericalAt`, `heckeSplit`, `heckePolyOp`, `finRightRep`

Up to the normalisation of the parameters, $H_w^{\pi}(T)$ acts on the spherical vector of an unramified principal series of $\mathrm{GL}_{r'}(E_w)$ by the product of the factors $1 - \alpha_i T$ over its Satake parameters α_i: it is the reciprocal of the standard local Euler factor in the variable T. For the trivial representation the factors are $1 - q_w^a T$, $0 \le a < r'$, which is the second part of (Q7) below.

Definition 2.6 (splittings and zero terms; quoted)**.** Let W and W' be hermitian spaces with points, of signs ε on $[r]$ and ε' on $[r']$, with matrices J and J'. A *splitting* Π of W' over W with planes indexed by $[\nu]$ consists of matrices k_Π, l_Π of size $r' \times \nu$ and i_Π of size $r' \times r$ over E with $k_\Pi^* J' k_\Pi = 0$, $l_\Pi^* J' l_\Pi = 0$, $l_\Pi^* J' k_\Pi = 1$,

$i_\Pi^* J' i_\Pi = J$, $i_\Pi^* J' k_\Pi = 0$, $i_\Pi^* J' l_\Pi = 0$, and it is required that $r' = r + 2\nu$. So W' is the orthogonal sum of W and of ν hyperbolic planes with isotropic bases given by the columns of k_Π and l_Π; we write $W' \cong W \oplus \mathbb{H}^\nu$. The splitting is *adapted* if, in the coordinates of the points of W and W', the frame (l_Π, k_Π, i_Π) has orthonormal columns at the infinite places and integral entries at the finite places. For a space V of signs 1_n, the *zero term* of a datum along Π is given by the bounded linear map

$$\mathrm{Z}_{\Pi,V}[0] \colon \mathcal{S}(W', V) \to \mathcal{S}(W, V),$$

which in a Schrödinger model takes the mean of a function over the adelic translations along the maps that vanish on the orthogonal complement of the columns of k_Π, modulo the rational ones, and restricts to $\mathrm{Hom}(W, V)$.

LEAN in the background library: `Space.Splitting`, `IsBaseAdapted`, `Space.Splitting.homSource`, `Space.Splitting.zeroTerm`, `Space.Splitting.sourceTerm` (at the argument 0)

Quoted from the background library, I: data and descents. These facts are taken for granted; references and their state are in §8, note 6. W, V, ε, ε', χ are as in Setting 2.1, with $\varepsilon' = 1_n$, and ω stands for $\omega_{\chi,1}^{W,V}$.

(Q1) *The descent as a functional.* Let Y be automorphic on V. Then Y is bounded, the quotient $[V]$ has finite measure, and for every $u \in G(\varepsilon)$ the map $f \mapsto \mathrm{Desc}_{\chi,1}^{W,V}(f,Y)(u)$ is a continuous linear functional on $\mathcal{S}(W,V)$. For $u' \in G(\varepsilon)$ and $h' \in G(1_n)$, $\mathrm{Desc}_{\chi,1}^{W,V}(f,Y)(uu') = \mathrm{Desc}_{\chi,1}^{W,V}(\omega(u',h')f,\, Y(\cdot\, h'))(u)$, and $\mathrm{Desc}_{\chi,1}^{W,V}(f,Y)(\gamma u) = \mathrm{Desc}_{\chi,1}^{W,V}(f,Y)(u)$ for rational points γ of W.

(Q2) *Density.* Let S be a set of finite places and f a datum with $\mathrm{P}^S f = f$. If a continuous linear functional on $\mathcal{S}(W,V)$ vanishes on all data $v_p^{\varepsilon,1_n} \otimes \varrho(z)\varphi^\circ$, with p a polynomial and z a finite vector whose components at the places of S and at their conjugates are zero, then it vanishes on f. These data are unramified at S.

(Q3) *Functions at a finite set of places.* Let P be a finite set of finite places, none conjugate to another or to itself, and $E_P = \prod_{w\in P} E_w$. A Schwartz–Bruhat function Ψ on the $n \times r$ matrices over E_P defines a datum $\Psi \cdot \varphi^\circ$ that has the component Ψ at P and is the vacuum elsewhere; it is linear in Ψ. Let Box be the set of matrices with entries in $\prod_{w\in P} \mathfrak{o}_w$; for an idele $\mathfrak{t}$ with integral components and a matrix z, $\mathbf{1}_{z+\mathfrak{t}\,\mathrm{Box}}$ is the characteristic function of $z + \mathfrak{t}\,\mathrm{Box}$. (a) For a finite vector z supported on P and its conjugates, $\varrho(z)\varphi^\circ$ is a complex multiple of $\Psi \cdot \varphi^\circ$ for a Schwartz–Bruhat function Ψ at P. (b) A linear space of functions that contains all $\mathbf{1}_{z+\mathfrak{t}\,\mathrm{Box}}$ contains every Schwartz–Bruhat function at P. (c) If such a function Ψ is invariant under the translations by $\mathfrak{t}\,\mathrm{Box}$, then $\Psi = \sum_{z'\in\mathcal{Z}} \Psi(z')\,\mathbf{1}_{z'+\mathfrak{t}\,\mathrm{Box}}$ for a finite set $\mathcal{Z}$ of matrices with $\Psi(z') \neq 0$. (d) A function of the residues of an integral matrix at the places of P, extended by 0, is Schwartz–Bruhat, and takes the same value at two matrices whose difference has entries of absolute value < 1 at all $w \in P$.

(Q4) *Finite elements acting on such data.* Let χ be a power of v_1, v a Fock vector and Ψ as in (Q3). (a) For $g' \in \mathbf{K}_0(n)$ supported on P: $\omega(1,(1,g'))(v \otimes \Psi \cdot \varphi^\circ) = v \otimes \Psi' \cdot \varphi^\circ$ with $\Psi'(X) = \Psi(g'^{-1}X)$. (b) For $\alpha \in G_{\mathrm{f}}(r)$ supported on P: $\omega((1,\alpha),1)(v \otimes \Psi \cdot \varphi^\circ)$ is a non-zero complex multiple of $v \otimes \Psi'' \cdot \varphi^\circ$ with $\Psi''(X) = \Psi(X\alpha)$.

(Q5) *Scalar functions of forms.* For an auxiliary pair O, an algebraic form Ξ of weight (κ_O, e_O) and a linear form L on $\mathbb{V}_{n,\kappa_O}$: Ξ_L is automorphic on $\mathrm{I}_n(E)$, and $(\mathrm{R}(g')\Xi)_L(h) = \Xi_L(hg')$ for $g' \in G_{\mathrm{f}}(n)$.

(Q6) *Primitive matrices.* Let w be a finite place. An $n \times r$ matrix over E_w with a left inverse becomes, after multiplication on the right by a suitable element of $\mathrm{GL}_r(E_w)$, a matrix with entries in $\mathfrak{o}_w$ that has a left inverse with entries in $\mathfrak{o}_w$. A square matrix over $\mathfrak{o}_w$ whose reduction is invertible is invertible over $\mathfrak{o}_w$.

LEAN in the background library: (Q1) `ThetaKernel.exists_clm_lift`, `ThetaKernel.lift_equivariant`, `lift_mul_right`, `ThetaKernel.fixesLeft_lift` · (Q2) `Space.ext_integrated` · (Q3) `Space.name`, `cosetIndicator`, `Space.name_schrodingerAct_boxIndicator`, `IsSchwartzBruhat.mem_of_cosets`, `IsSchwartzBruhat.eq_sum_cosetIndicator`, `isSchwartzBruhat_residueDatum`, `residueDatum_congr` · (Q4) `pairWeil_name_of_mem`, `pairWeil_eq_smul_of_projOn_eq`, `pairWeil_name_on`, `pairWeil_integrated_fin` · (Q5) `Space.isAutomorphic_algFormCoeff`, `Space.algFormCoeff_algFormRep` · (Q6)

IsValuationPair.exists_isPrimitiveMatrix_mul, IsPrimitiveMatrix.exists_map_subtype_eq, isUnit_of_mapMatrix

Quoted from the background library, II: Hecke operators. Here W' has signs ε' on $[r']$, V has signs 1_n, χ_1 is an automorphic character and ω' stands for $\omega_{\chi_1,1}^{W',V}$.

(Q7) *Hecke polynomials as sums over subspaces.* Let $w \neq \overline{w}$. There are elements $g_U \in G_{\mathrm{f}}(r')$, one for each linear subspace U of $\mathbb{F}_w^{r'}$, supported on $\{w\}$, such that for every representation π of $G_{\mathrm{f}}(r')$, every vector ϕ spherical at w and every complex T: $H_w^\pi(T)\phi = \sum_U (-T)^{\dim U} q_w^{\dim U(\dim U-1)/2}\, \pi(g_U)\phi$. Moreover (the q-binomial theorem) $\sum_U (-T)^{\dim U} q_w^{\dim U(\dim U-1)/2} = \prod_{0\le a<r'}(1-q_w^a T)$.

(Q8) *Adjoints.* Let $w \neq \overline{w}$, let π, π' be representations of $G_{\mathrm{f}}(r')$ on $\mathcal{V}$, $\mathcal{V}'$ and β a bilinear map on $\mathcal{V}\times\mathcal{V}'$ with $\beta(\pi(g)\phi,\pi'(g)\phi') = \beta(\phi,\phi')$. For vectors spherical at w: $\beta(H_w^\pi(T)\phi,\phi') = \beta(\phi, H_{\overline{w}}^{\pi'}(T)\phi')$. A linear map that commutes with the elements supported on $\{w\}$ commutes with the Hecke polynomials at w. Vectors spherical at w are spherical at $\overline{w}$.

(Q9) *Hecke transfer.* Let $w \neq \overline{w}$ and $r' \le n$. Suppose that the base change of χ_1 is unramified at w and that f is unramified at $\{w\}$. Let Λ be a linear map from $\mathcal{S}(W',V)$ to a complex vector space with commuting representations π of $G_{\mathrm{f}}(r')$ and ρ of $G_{\mathrm{f}}(n)$, such that $\Lambda(\omega'((1,g),(1,h))f) = \pi(g)\rho(h)\Lambda f$ for all $g \in G_{\mathrm{f}}(r')$, $h \in G_{\mathrm{f}}(n)$. Then Λf is spherical at w for π and for ρ, and for every complex T

$$H_w^\rho(T)\,\Lambda f = \prod_{0\le a<n-r'} \left(1 - q_w^a\, A\, T\right) \cdot H_{\overline{w}}^{\pi}(q_w^{n-r'} A B^{-1} T)\,\Lambda f,$$

with complex numbers A, B that depend on χ_1, w, r', n and have $|A| = q_w^{r'/2}$, $|B| = q_w^{n/2}$. The character υ_1 has unramified base change at every split place.

Quoted from the background library, III: zero terms, growth. The conventions are those of II.

(Q10) *Zero terms.* Let Π be an adapted splitting of W' over W with planes indexed by $[\nu]$, and $\chi_1 = \chi v_1^{\nu}$. (a) For every $u \in G(\varepsilon')$ there are an element g of the standard maximal compact subgroup of $G(\varepsilon')$, with finite component in $\mathbf{K}_0(r')$, an element $u_0 \in G(\varepsilon)$ and a complex number a such that for all $h \in G(1_n)$ and all data f: $\mathrm{Z}_{\Pi,V}[0](\omega'(u,h)f) = a \cdot \omega_{\chi,1}^{W,V}(u_0,h)\ \mathrm{Z}_{\Pi,V}[0](\omega'(g^{-1},1)f)$. (b) If $\mathrm{P}^S f = f$, then $\mathrm{P}^S f_1 = f_1$ for $f_1 = \mathrm{Z}_{\Pi,V}[0](\omega'(g^{-1},1)f)$. (c) If $\nu = 0$, then $\theta_{W,V} \circ \mathrm{Z}_{\Pi,V}[0] = \theta_{W',V}$. (d) If $\nu \geq 1$, there are a space W_1 with an adapted splitting over W with $\nu - 1$ planes, and an adapted splitting Π' of W' over W_1 with one plane.

(Q11) *Growth and decay.* Every point u of an adelic unitary group has a height $\mathrm{ht}(u) \geq 1$, invariant under rational points on the left; on a definite space it is 1. (a) There are constants C, a with $|\theta_{W',V}(\omega'(u,h)\varphi)| \leq C\|\varphi\|\,\mathrm{ht}(u)^a$ for all u, h, φ. (b) If $\mathrm{ht}(u) > 2$, the lattice of W' at u contains a non-zero isotropic vector of length at most $2/\mathrm{ht}(u)$; rational frames with the same Gram matrix are conjugate under the rational points (Witt); and the lattice of V at every h has a basis of bounded length. (c) Let Π' be an adapted splitting of W' over W_1 with one plane and $\theta^{\Pi'} = \theta_{W',V} - \theta_{W_1,V} \circ \mathrm{Z}_{\Pi',V}[0]$. Then $\theta^{\Pi'}$ is invariant under the Heisenberg operators of the adelic translations along the isotropic vector $k_{\Pi'}$, and $|\theta^{\Pi'}(\varphi)|$ is at most the supremum of $|\theta_{W',V}(\varphi) - \theta_{W',V}(\varrho(z)\varphi)|$ over these translations z. (d) *The decay estimate, in the case used.* For data $f = v_p^{\varepsilon',1_n} \otimes \varrho(z)\varphi^{\circ}$ there is a constant C_0 with $|\theta^{\Pi'}(\omega'(u,h)f)| \leq C_0$ for all h and all u such that either $\mathrm{ht}(u) \leq 2$, or the isotropic vector $k_{\Pi'}$ is, in the lattice of W' at u, of length at most $2/\mathrm{ht}(u)$.

LEAN in the background library: (Q7) `heckePolyOp_eq_sum`, `q_binomial_submodule` · (Q8) `heckePolyOp_adjoint`, `map_heckePolyOp`, `Gf.supportedOn_star` · (Q9) `hecke_transfer_of_equivariant`, `WeilData.charUnramifiedAt_upsilon` · (Q10) `Space.Splitting.iwasawa`, `Space.Splitting.sourceTerm_radicalElt`, `Space.Splitting.sourceTerm_levi`, `projOn_pairWeil`, `Space.Splitting.IsAdapted.omega_projOn_of_isMixedLaw`, `exists_isBaseAdapted_stage`, `removePlanes` · (Q11) `ThetaKernel.moderate_growth`, `Space.exists_short_isotropic`, `Space.witt`, `Space.exists_short_basis_matrix`, `Space.Splitting.zeroTerm_rho_kAdelic`, `norm_quotMean_le`, `exists_norm_name_le`

3 Three lemmas on the level

Lemma 3.1 (prescribed components at the places of the pair)**.** *Let O be an auxiliary pair (Setting 2.3), $[r']$ a finite index set, and $g = (g_w)_{w \in \mathrm{pl}(O)}$ a family of matrices $g_w \in \mathrm{GL}_{r'}(E_w)$. Then there is an element of $G_{\mathrm{f}}(r')$, supported on $\mathrm{pl}(O)$, whose component at every $w \in \mathrm{pl}(O)$ is g_w.*

Proof. For $w \in \mathrm{pl}(O)$ we have $w \neq \overline{w}$, so $\mathrm{GL}_{r'}(E_w)$ embeds in $G_{\mathrm{f}}(r')$ (Definition 2.5): the image of g_w has the component g_w at w and is supported on $\{w\}$. No place of $\mathrm{pl}(O)$ is conjugate to another, so the image of g_w has the component 1 at the other places of $\mathrm{pl}(O)$, and the product of these images over $w \in \mathrm{pl}(O)$ is the element sought. □

LEAN `AuxPair.exists_atPlace_eq`

Lemma 3.2 (an annihilator, or a congruence to the inclusion matrix)**.** *Let $\iota\colon [r] \to [n]$ be an injective map, w a finite place of E, and X an $n \times r$ matrix over E_w that has a left inverse over E_w. Let Inc_ι be the $n \times r$ matrix with entries 1 at the places $(\iota(c), c)$ and 0 elsewhere. Then there is an $n \times n$ matrix ℓ with entries in $\mathfrak{o}_w$ such that $\ell X = 0$ and such that the following holds: if all entries $\ell_{i,\iota(c)}$, $i \in [n]$, $c \in [r]$, have absolute value < 1, then there is $\alpha \in \mathrm{GL}_r(E_w)$ such that every entry of $X\alpha - \mathrm{Inc}_\iota$ has absolute value < 1.*

Proof. By (Q6) there is $\beta \in \mathrm{GL}_r(E_w)$ such that $Z = X\beta$ has entries in $\mathfrak{o}_w$ and a left inverse Z' with entries in $\mathfrak{o}_w$: $Z'Z = 1$. Put $\ell = ZZ' - 1$. Then $\ell X = (ZZ' - 1)Z\beta^{-1} = 0$. Suppose that all $\ell_{i,\iota(c)}$ have absolute value < 1, and let β' be the $r \times r$ matrix with the entries $Z'_{c',\iota(c)}$. The entry of $Z\beta'$ at (i, c) is $(ZZ')_{i,\iota(c)}$, which is congruent modulo the maximal ideal to the entry of the identity matrix at $(i, \iota(c))$, that is, to the entry of Inc_ι at (i, c). In particular the product of the ι-rows of Z with β' reduces to the identity, so β' is invertible modulo the maximal ideal, hence over $\mathfrak{o}_w$ by (Q6). Then $\alpha = \beta\beta'$ has $X\alpha = Z\beta'$, with the congruence just shown. □

LEAN `AuxPair.exists_annihilator_or_congr_inclMatrix`

Lemma 3.3 (an integral element with block reductions acts through the chosen places)**.** *Let O be an auxiliary pair (Setting 2.3), π a representation of $G_{\mathrm{f}}(n)$ on a complex vector space $\mathcal{V}$, and $\phi \in \mathcal{V}^O$. Let $g \in \mathbf{K}_0(n)$, with reductions $\overline{g}_w = \mathrm{red}_w\, g$ for $w \in \mathrm{pl}(O)$, and let $s = (s_w)_{w \in \mathrm{pl}(O)}$ be a family of matrices $s_w \in \mathrm{GL}_r(\mathbb{F}_w)$. Suppose that $\overline{g}_w\ \mathrm{Inc}_O = \mathrm{Inc}_O\ s_w$ for every $w \in M_O$ and $\overline{g}_w^{\mathsf{T}}\ \mathrm{Inc}_O = \mathrm{Inc}_O\ s_w^{\mathsf{T}}$ for every $w \in Q_O$. Then*

$$\pi(g)\,\phi = \Big(\prod_{w \in \mathrm{pl}(O)} \psi_{O,w}(\det s_w)\Big)^{-1} [(s_w)_{w \in M_O}]_{O,\pi}\,\phi.$$

Proof. Let $A = (s_w)_{w \in M_O}$ and let $g_A \in \mathbf{K}_0(n)$ be as in Setting 2.3(c). The reductions of $g_A^{-1}g$ are $j_w = \mathrm{diag}(s_w, 1)^{-1}\overline{g}_w$ at $w \in M_O$ and $j_w = \overline{g}_w$ at $w \in Q_O$. At $w \in M_O$:

$$j_w\ \mathrm{Inc}_O = \mathrm{diag}(s_w, 1)^{-1}\ \mathrm{Inc}_O\ s_w = \mathrm{Inc}_O\ s_w^{-1} s_w = \mathrm{Inc}_O\,.$$

At $w \in Q_O$: the hypothesis says that the transpose of the rows of $\overline{g}_w$ with indices in the image of inc_O is $\mathrm{Inc}_O\, s_w^{\mathsf{T}}$; so these rows vanish in the columns outside the image, and the inc_O-block of $\overline{g}_w$ is s_w. Hence $j = (j_w) \in \mathbf{L}(O)$ with $\eta_O(j) = \prod_{w \in Q_O} \psi_{O,w}(\det s_w)^{-1}$, so $g_A^{-1}g \in \mathbf{K}(O)$ and

$$\pi(g)\,\phi = \pi(g_A)\,\pi(g_A^{-1}g)\,\phi = \eta_O(j)\,\pi(g_A)\,\phi,$$
$$\pi(g_A)\,\phi = \prod_{w \in M_O} \psi_{O,w}(\det s_w)^{-1}\,[A]_{O,\pi}\,\phi,$$

the second equation by Setting 2.3(c). □

LEAN `AuxPair.smul_chosenGLRep_eq`

Lemma 3.4 (the conjugate scalar function of a form is automorphic)**.** *Let O be an auxiliary pair (Setting 2.3), L a linear form on $\mathbb{V}_{n,\kappa_O}$ and $\Xi \in \mathrm{Form}(O)$. Then the function $\overline{\Xi_L}$ is automorphic on $\mathrm{I}_n(E)$.*

Proof. By (Q5), Ξ_L is automorphic, and the three defining conditions are stable under complex conjugation. □

LEAN `AuxPair.isAutomorphic_star_algFormCoeff`

4 Proof of Theorem 1

Throughout this section O, Ξ, L are as in Theorem 1, $W = W(O)$, $V = \mathrm{I}_n(E)$, $\chi = \chi_O$, $\omega = \omega_{\chi,1}^{W,V}$, and $P = \mathrm{pl}(O)$. Functions at P are functions on the $n \times r$ matrices over E_P, as in (Q3).

Step 0: reductions. Let f be a datum with $\mathrm{P}^S f = f$ and $u \in G(\varepsilon_O)$. By Lemma 3.4 and (Q1), the descent of $\overline{\Xi_L}$ at u is a continuous linear functional of the datum. By (Q2) it suffices to show that it vanishes on $v_{O,p} \otimes \varrho(z)\varphi^\circ$ for every polynomial p and every finite vector z supported on P and its conjugates; by (Q3)(a), on $v_{O,p} \otimes \Psi \cdot \varphi^\circ$ for every Schwartz–Bruhat function Ψ at P; and by (Q3)(b), since the descent is linear in Ψ, for the functions $\Psi = \mathbf{1}_{z+\mathfrak{t}\,\mathrm{Box}}$. So fix p and an idele $\mathfrak{t}$ with integral components, and put, for a matrix z over E_P and $\Xi' \in \mathrm{Form}(O)$,

$$C_z(\Xi') = \mathrm{Desc}_{\chi,1}^{W,V}(v_{O,p} \otimes \mathbf{1}_{z+\mathfrak{t}\,\mathrm{Box}} \cdot \varphi^\circ,\ \overline{\Xi'_L})(u).$$

It is conjugate-linear in Ξ'. We have to show $C_z(\Xi) = 0$ for every z.

Step 1: transport of the centre.

Lemma 4.1. *For $w \in P$ let A_w, B_w be $n \times r$ matrices over $\mathfrak{o}_w$ and $s_w \in \mathrm{GL}_r(\mathbb{F}_w)$, and let z, z' be matrices over E_P. Suppose that for every $w \in P$:* (i) *s_w is the reduction of $1 + B_w^{\mathsf{T}} A_w$;* (ii) *the reduction σ_w of the $n \times n$ matrix $1 + A_w B_w^{\mathsf{T}}$ satisfies $\sigma_w \,\mathrm{Inc}_O = \mathrm{Inc}_O\, s_w$ if $w \in M_O$ and $\sigma_w^{\mathsf{T}} \,\mathrm{Inc}_O = \mathrm{Inc}_O\, s_w^{\mathsf{T}}$ if $w \in Q_O$;* (iii) *the component at w of $\mathfrak{t}^{-1}(z' - (1 + A_w B_w^{\mathsf{T}})z)$ has entries in $\mathfrak{o}_w$. Then*

$$\prod_{w \in P} \psi_{O,w}(\det s_w) \cdot C_{z'}\big([(s_w)_{w \in M_O}]_{O,\mathrm{R}}\, \Xi\big) = C_z(\Xi),$$

where $[\,\cdot\,]_{O,\mathrm{R}}$ is the action of Setting 2.3(c) *on* $\mathrm{Form}(O)$.

Proof. The determinant of $1 + A_w B_w^{\mathsf{T}}$ equals that of $1 + B_w^{\mathsf{T}} A_w$, whose reduction is $\det s_w \neq 0$; so $1 + A_w B_w^{\mathsf{T}} \in \mathrm{GL}_n(\mathfrak{o}_w)$ by (Q6). By Lemma 3.1 there is $g \in G_{\mathrm{f}}(n)$, supported on P, with these components; it lies in $\mathbf{K}_0(n)$, and $\mathrm{red}_w\, g = \sigma_w$. By (Q1), with $u' = 1$ and $h' = (1, g)$,

$$C_z(\Xi) = \mathrm{Desc}_{\chi,1}^{W,V}\Big(\omega(1,(1,g))(v_{O,p} \otimes \mathbf{1}_{z+\mathfrak{t}\,\mathrm{Box}} \cdot \varphi^\circ),\ \overline{\Xi_L}(\,\cdot\, g)\Big)(u).$$

By (Q4)(a) the datum is $v_{O,p} \otimes \Psi' \cdot \varphi^\circ$ with $\Psi'(X) = \mathbf{1}_{z+\mathfrak{t}\,\mathrm{Box}}(g^{-1}X)$, the characteristic function of $gz + \mathfrak{t}\,\mathrm{Box}$, since g preserves Box; by (iii) this set is $z' + \mathfrak{t}\,\mathrm{Box}$. By (Q5), $\Xi_L(hg) = (\mathrm{R}(g)\Xi)_L(h)$, and by Lemma 3.3 and (ii),

$$\mathrm{R}(g)\,\Xi = \prod_{w \in P} \psi_{O,w}(\det s_w)^{-1} \cdot [(s_w)_{w \in M_O}]_{O,\mathrm{R}}\, \Xi.$$

As $C_{z'}$ is conjugate-linear and the values $\psi_{O,w}(\det s_w)$ have absolute value 1, the scalar comes out as $\prod_w \psi_{O,w}(\det s_w)$. □

Step 2: annihilators. Fix z. For $w \in P$ let δ_w be the diagonal $n \times n$ matrix with the entry ϖ_w at the indices outside the image of inc_O if $w \in Q_O$, and all other entries 1; so $\delta_w = 1$ for $w \in M_O$. Apply Lemma 3.2 to the matrix with $n + r$ rows

$$X_w = \begin{pmatrix} \delta_w z_w \\ \mathfrak{t}_w 1_r \end{pmatrix},$$

with the injection of $[r]$ into the first n rows by inc_O; it has the left inverse $(0,\ \mathfrak{t}_w^{-1}1_r)$. We get a square matrix ℓ_w of size $n+r$ over $\mathfrak{o}_w$ with $\ell_w X_w = 0$. For a row index i_0 of ℓ_w let $\overline{\ell}_w(i_0) \in \mathbb{F}_w^r$ be the vector of the residues of the entries $(\ell_w)_{i_0,\mathrm{inc}_O(c)}$, $c \in [r]$. There are two cases.

Step 3: all residues vanish. Suppose $\overline{\ell}_w(i_0) = 0$ for all w and i_0. Lemma 3.2 gives $\alpha_w \in \mathrm{GL}_r(E_w)$ such that $\delta_w z_w \alpha_w - \mathrm{Inc}_O$ and $\mathfrak{t}_w \alpha_w$ have entries of absolute value < 1. By Lemma 3.1 there is $\alpha \in G_{\mathrm{f}}(r)$, supported on P, with the components α_w. Let Ψ be the function at P with $\Psi(X) = \Psi_O(((X\alpha)_w \bmod w)_w)$ if $X\alpha$ has entries in $\prod_{w\in P} \mathfrak{o}_w$, and $\Psi(X) = 0$ otherwise. We collect four facts.

(1) $\Psi(z) = 1$. At $w \in M_O$, $z_w\alpha_w$ is congruent to Inc_O. At $w \in Q_O$, the rows of $z_w\alpha_w$ in the image of inc_O are congruent to those of Inc_O, and the other rows, multiplied by ϖ_w, have absolute value < 1, so they are integral. Hence $z\alpha$ is integral, reduces to Inc_O at M_O, and its inc_O-rows reduce to the identity at Q_O.

(2) Ψ is Schwartz–Bruhat and invariant under the translations by $\mathfrak{t}\,\mathrm{Box}$: for X' in this set, $X'\alpha$ has entries of absolute value < 1, and (Q3)(d) applies. By (Q3)(c), $\Psi = \sum_{z'\in\mathcal{Z}} \Psi(z')\mathbf{1}_{z'+\mathfrak{t}\,\mathrm{Box}}$ with a finite set $\mathcal{Z}$ and all $\Psi(z') \neq 0$; by (1), $\Psi \neq 0$, so $\mathcal{Z}$ is not empty.

(3) For every z' with $\Psi(z') \neq 0$: $\Psi(z')\, C_{z'}(\Xi) = C_{\mathrm{Inc}_O\,\alpha^{-1}}(\Xi)$. Indeed, $X = z'\alpha$ is integral, congruent to Inc_O at M_O, and its inc_O-rows have invertible reductions s_w at $w \in Q_O$; put $s_w = 1$ for $w \in M_O$. Apply Lemma 4.1 with $A_w = X_w - \mathrm{Inc}_O$, $B_w = \mathrm{Inc}_O$, the centres $\mathrm{Inc}_O\,\alpha^{-1}$ and z'. Since $\mathrm{Inc}_O^{\mathsf{T}}\,\mathrm{Inc}_O = 1$: $1 + B_w^{\mathsf{T}}A_w$ is the matrix of the inc_O-rows of X_w, which gives (i); $(1 + A_wB_w^{\mathsf{T}})\,\mathrm{Inc}_O = X_w$, whose reduction is Inc_O at M_O, and the rows of $1 + A_wB_w^{\mathsf{T}}$ in the image of inc_O are those of $X_w\,\mathrm{Inc}_O^{\mathsf{T}}$, which gives (ii); and $(1 + A_wB_w^{\mathsf{T}})\,\mathrm{Inc}_O\,\alpha_w^{-1} = X_w\alpha_w^{-1} = z'_w$, which gives (iii). The lemma yields $\prod_{w\in Q_O} \psi_{O,w}(\det s_w)\, C_{z'}(\Xi) = C_{\mathrm{Inc}_O\,\alpha^{-1}}(\Xi)$, and the product is $\Psi(z')$.

(4) By hypothesis (c) of the theorem at the point $u\alpha$, by (Q1) with $u' = (1,\alpha)$, $h' = 1$, and by (Q4)(b) applied to the component $X \mapsto \Psi_O((X \bmod w)_w)$ of $\Upsilon(O)$:

$$\begin{aligned} 0 &= \mathrm{Desc}_{\chi,1}^{W,V}(v_{O,p} \otimes \Upsilon(O),\, \overline{\Xi_L})(u\alpha) \\ &= a \cdot \mathrm{Desc}_{\chi,1}^{W,V}(v_{O,p} \otimes \Psi\cdot\varphi^\circ,\, \overline{\Xi_L})(u) = a \sum_{z'\in\mathcal{Z}} \Psi(z')\, C_{z'}(\Xi) \end{aligned}$$

with a complex number $a \neq 0$. By (3), applied to every $z' \in \mathcal{Z}$ and to z, each term is $C_{\mathrm{Inc}_O\,\alpha^{-1}}(\Xi) = \Psi(z)C_z(\Xi) = C_z(\Xi)$. So $\#\mathcal{Z}\cdot C_z(\Xi) = 0$, and $C_z(\Xi) = 0$.

Step 4: some residue does not vanish. Suppose $\overline{\ell}_{w_0}(i_0) \neq 0$ for some $w_0 \in P$ and some row index i_0. First, for every family $s = (s_w)_{w\in P}$ of invertible matrices of the form $s_w = 1 + \xi_w\overline{\ell}_w(i_0)^{\mathsf{T}}$, $\xi_w \in \mathbb{F}_w^r$:

$$\prod_{w\in P} \psi_{O,w}(\det s_w)\cdot C_z([(s_w)_{w\in M_O}]_{O,\mathrm{R}}\,\Xi) = C_z(\Xi). \tag{1}$$

This is Lemma 4.1 with $z' = z$, $A_w = \mathrm{Inc}_O$ and $B_w = \mu_w\tilde{\xi}_w^{\mathsf{T}}$, where $\tilde{\xi}_w$ is a lift of ξ_w to $\mathfrak{o}_w^r$ and $\mu_w \in \mathfrak{o}_w^n$ is the vector with the entries $(\ell_w)_{i_0,i}(\delta_w)_{ii}$, $i \in [n]$. Indeed the matrix

$$1 + B_w^{\mathsf{T}}A_w = 1 + \tilde{\xi}_w\,(\mathrm{Inc}_O^{\mathsf{T}}\,\mu_w)^{\mathsf{T}}$$

reduces to s_w, because δ_w is 1 at the indices in the image of inc_O; this is (i). The matrix $1 + A_w B_w^{\mathsf{T}} = 1 + \mathrm{Inc}_O\, \tilde{\xi}_w \mu_w^{\mathsf{T}}$ differs from 1 only in the rows of the image of inc_O; its inc_O-block reduces to s_w, and its entries in these rows and in the columns outside the image are multiples of ϖ_w when $w \in Q_O$; this gives (ii) at both kinds of places. Finally $\mu_w^{\mathsf{T}} z_w$ is the part of the row i_0 of ℓ_w in the first n columns times $\delta_w z_w$, which by $\ell_w X_w = 0$ equals $-\mathfrak{t}_w$ times the part of that row in the last r columns; so $\mathfrak{t}_w^{-1}(z_w - (1 + A_w B_w^{\mathsf{T}}) z_w) = -\mathfrak{t}_w^{-1}\, \mathrm{Inc}_O\, \tilde{\xi}_w \mu_w^{\mathsf{T}} z_w$ is integral; this is (iii).

If $w_0 \in M_O$. Sum (1) over the families s with $s_w = 1$ for $w \in Q_O$ and s_w, $w \in M_O$, running over all invertible matrices of the form $1 + \xi\, \overline{\ell}_w(i_0)^{\mathsf{T}}$. By Definition 2.4 and conjugate-linearity, the left sides add up to the value of C_z at $\bigotimes_{w \in M_O} \mathrm{Refl}(\psi_{O,w}, \overline{\ell}_w(i_0))$ applied to Ξ, which is 0 by hypothesis (b), the vector at w_0 being non-zero. The right sides add up to the number of families, which is at least 1, times $C_z(\Xi)$. So $C_z(\Xi) = 0$.

If $w_0 \in Q_O$. By hypothesis (a) there is $a \in \mathbb{F}_{w_0}$ with $a \neq 0$ and $\psi_{O,w_0}(a) \neq 1$. Choose $c_0 \in [r]$ with the entry $\overline{\ell}_{w_0}(i_0)_{c_0} \neq 0$, and let ξ_{w_0} be $(a-1)/\overline{\ell}_{w_0}(i_0)_{c_0}$ times the basis vector of index c_0; then $s_{w_0} = 1 + \xi_{w_0} \overline{\ell}_{w_0}(i_0)^{\mathsf{T}}$ has determinant a. With $s_w = 1$ for $w \neq w_0$, (1) reads $\psi_{O,w_0}(a)\, C_z(\Xi) = C_z(\Xi)$. So $C_z(\Xi) = 0$.

This proves Theorem 1.

LEAN `AuxPair.descentsVanish_of_reflectionSum`

5 A maximum principle

Proposition 5.1 (bounded functions killed by a Hecke polynomial at a small argument)**.** *Let ε' be a family of signs on $[r']$, F a complex function on $G(\varepsilon')$, w a finite place of E and T a complex number. Suppose that w is split, $w \neq \overline{w}$; that F is spherical at w for $\mathrm{R}_{\varepsilon'}$; that $H_w^{\mathrm{R}_{\varepsilon'}}(T)\, F = 0$; that $q_w^{r'+1}\, |T| \leq 1$; and that $|F|$ is bounded. Then $F = 0$.*

Proof. By (Q7), for every $u \in G(\varepsilon')$, $0 = \sum_U (-T)^{\dim U} q_w^{\dim U(\dim U - 1)/2} F(u\, g_U)$. Replace u by $u\, g_0^{-1}$, with g_0 the element of the zero subspace: the term of $U = 0$ is $F(u)$, so $|F(u)| \leq \sum_{U \neq 0} |T|^{\dim U} q_w^{\dim U(\dim U-1)/2} \sup |F|$. By the second part of (Q7) at $-|T|$, the sum of the coefficients over all U is $\prod_{0 \leq a < r'} (1 + q_w^a |T|) \leq \exp(|T| \sum_{a<r'} q_w^a)$. As $q_w \geq 2$, $\sum_{a<r'} q_w^a < q_w^{r'}$, so the exponent is less than $q_w^{r'} |T| \leq q_w^{-1} \leq \frac{1}{2}$, which is less than $\log 2$. Hence the sum of the coefficients over $U \neq 0$ is less than 1, and the inequality forces $\sup |F| = 0$. □

LEAN `eq_zero_of_heckePolyOp_eq_zero` · in the background library: `maximum_principle`

6 Proof of Theorem 2

Lemma 6.1 (the zero term has vanishing integral)**.** *In Setting 2.1, let W, W' and V be hermitian spaces with points, of signs ε on $[r]$, ε' on $[r']$ and 1_n on $[n]$, χ an automorphic character, S a set of finite places and Y a complex function on $G(1_n)$ with $\mathrm{DV}(W, V; \chi; S; Y)$. Let Π be an adapted splitting of W' over W with planes indexed by a finite set $[\nu]$. Let $f \in \mathcal{S}(W', V)$ with $\mathrm{P}^S f = f$ and $u \in G(\varepsilon')$. Then*

$$\int_{[V]} \theta_{W,V}\Big(\mathrm{Z}_{\Pi,V}[0](\omega^{W',V}_{\chi v_1^\nu, 1}(u,h) f)\Big)\, Y(h)\, \mathrm{d}h = 0.$$

Proof. With g, u_0, a as in (Q10)(a) and $f_1 = \mathrm{Z}_{\Pi,V}[0](\omega^{W',V}_{\chi\upsilon_1^{\nu},1}(g^{-1},1)f)$, the integral is $a\int_{[V]}\theta_{W,V}(\omega^{W,V}_{\chi,1}(u_0,h)f_1)Y(h)\,\mathrm{d}h = a\cdot\mathrm{Desc}^{W,V}_{\chi,1}(f_1,Y)(u_0)$. By (Q10)(b), $\mathrm{P}^S f_1 = f_1$, so this vanishes by $\mathrm{DV}(W,V;\chi;S;Y)$. □

LEAN `DescentsVanish.integral_sourceTerm_eq_zero`

Proof of Theorem 2. By induction on ν, for all W' and Π at once. Write $\chi_1 = \chi\upsilon_1^{\nu}$ and $\omega' = \omega^{W',V}_{\chi_1,1}$; let f be a datum with $\mathrm{P}^S f = f$ and $F = \mathrm{Desc}^{W',V}_{\chi_1,1}(f,Y)$. We show $F = 0$.

The case $\nu = 0$. By (Q10)(c), $F(u)$ is the integral of Lemma 6.1, which vanishes.

The case $\nu \geq 1$. By (Q1) and (Q2) we may assume $f = v_p^{\varepsilon',1_n}\otimes\varrho(z)\varphi^{\circ}$ with a polynomial p and a finite vector z whose components at S and its conjugates vanish; such a datum is unramified at S, in particular at $\{\overline{w}\}$.

Step 1: a Hecke relation for F. Let Λ be the map that sends a datum f' to the linear map $Y' \mapsto \mathrm{Desc}^{W',V}_{\chi_1,1}(f',Y')$ from the automorphic functions on V to the functions on $G(\varepsilon')$. On such linear maps Λ' let $g \in G_{\mathrm{f}}(r')$ act by right translation of the values, and $h \in G_{\mathrm{f}}(n)$ by the inverse translation of the argument:

$$(\pi(g)\Lambda')(Y') = \Lambda'(Y')(\,\cdot\, g), \qquad (\rho(h)\Lambda')(Y') = \Lambda'(Y'(\,\cdot\, h^{-1})).$$

These are commuting representations. By (Q1), applied with $u' = (1,g)$, $h' = (1,h)$ and the function $Y'(\,\cdot\, h^{-1})$, we have $\Lambda(\omega'((1,g),(1,h))f') = \pi(g)\rho(h)\Lambda f'$ for every datum f', which is the hypothesis of (Q9) on Λ; and the pairing $(Y',\Lambda') \mapsto \Lambda'(Y')$ is invariant under the pairs $(\mathrm{R}_{1_n}(h),\rho(h))$, as (Q8) requires. Moreover $r' = r+2\nu$ and $2r+2\nu+3 < n$, so $r' < n$: the condition $r' \leq n$ of (Q9) holds. The base change of χ_1 is unramified at $\overline{w}$, by hypothesis for χ and by (Q9) for υ_1. So (Q9), at the split place $\overline{w}$, shows that Λf is spherical for π and ρ and gives, for ρ on the left and π on the right,

$$(\Lambda f)(H_w^{\mathrm{R}_{1_n}}(T)\,Y) = \prod_{0\leq a<n-r'}(1-q_w^a A\,T)\cdot H_w^{\mathrm{R}_{\varepsilon'}}(T')\,F, \qquad T' = q_w^{n-r'}AB^{-1}T;$$

here the left side has been rewritten by the adjunction (Q8) for the invariant pairing above, which moves the Hecke polynomial of ρ at $\overline{w}$ on linear maps to the Hecke polynomial of right translation at w on Y; on the right, evaluation at Y commutes with π, and $F = (\Lambda f)(Y)$ is spherical at w. The left side is 0 by hypothesis. On the right, $|q_w^a AT| \leq q_w^{n-r'-1+r'/2}\,|T| = q_w^{(r-r')/2} = q_w^{-\nu} < 1$ for $a < n-r'$, because $|T| = q_w^{-(n-1-r/2)}$ and $r' = r+2\nu$; so no factor of the product vanishes, and $H_w^{\mathrm{R}_{\varepsilon'}}(T')F = 0$.

Step 2: the size of T'. We have $|T'| = q_w^{(n-r')/2}|T|$, so

$$q_w^{r'+1}\,|T'| = q_w^{r'+1+(n-r')/2-(n-1-r/2)} = q_w^{(2r+2\nu+4-n)/2} \leq 1,$$

because $2r+2\nu+3 < n$.

Step 3: F is bounded. By (Q10)(d) there are a space W_1 with an adapted splitting over W with $\nu-1$ planes and an adapted splitting Π' of W' over W_1 with one plane. The induction hypothesis, applied to W_1, gives $\mathrm{DV}(W_1,V;\chi\upsilon_1^{\nu-1};S;Y)$: its hypotheses are those of the theorem, with $2r+2(\nu-1)+3 < n$. By Lemma 6.1 for Π' and the character $\chi\upsilon_1^{\nu-1}$,

$$\int_{[V]}\theta_{W_1,V}\Big(\mathrm{Z}_{\Pi',V}[0](\omega'(u,h)f)\Big)\,Y(h)\,\mathrm{d}h = 0 \qquad \text{for every } u.$$

Hence, with $\theta^{\Pi'} = \theta_{W',V} - \theta_{W_1,V} \circ \mathrm{Z}_{\Pi',V}[0]$ as in (Q11)(c),

$$F(u) = \int_{[V]} \theta^{\Pi'}(\omega'(u,h)f)\, Y(h)\, \mathrm{d}h \qquad \text{for every } u \in G(\varepsilon').$$

Fix u. If $\mathrm{ht}(u) > 2$, (Q11)(b) gives a non-zero isotropic vector of the lattice at u of length at most $2/\mathrm{ht}(u)$ and a rational point γ of W' that carries it to $k_{\Pi'}$; then $k_{\Pi'}$ has this length in the lattice at γu. If $\mathrm{ht}(u) \leq 2$, put $\gamma = 1$. In both cases (Q11)(d) gives $|\theta^{\Pi'}(\omega'(\gamma u, h)f)| \leq C_0$ for all h, with a constant C_0 that does not depend on u. By (Q1), $F(u) = F(\gamma u)$, so $|F(u)| \leq C_0 \sup|Y|$ times the measure of $[V]$.

Step 4. By Steps 1 to 3 and Proposition 5.1, applied to F, w and T', we get $F = 0$. □

LEAN `descentsVanish_add_planes`

7 Where the results are used

In the unit on the rows. Let O be an auxiliary pair with a parameter that fits it, as in the second note, and let $\Xi \in \mathrm{Form}(O)$ be killed by the class-sum lift, by the reflection sums of hypothesis (b), and by the Hecke polynomial at one split place w outside the level at a point T of the absolute value of Theorem 2. Let first L be the hermitian product with the highest vector of the second note, so that Ξ_L is the scalar function of its first theorem. By that theorem the descent of $\overline{\Xi_L}$ with the test datum at the highest vector vanishes; the edge property extends this to all polynomial vectors, and passing from the characters of the pair to the characters $(\chi_O, 1)$ only multiplies descents by a character. This is hypothesis (c) of Theorem 1 for this one linear form. The unit on the rows needs (c) for the hermitian products with all vectors of $\mathbb{V}_{n,\kappa_O}$, and obtains it in two steps. For the product with a translate of the highest vector by an element of $G_\infty(1_n)$, the scalar function is a right translate of the former one, and (Q1) moves the translation to the datum, which remains a polynomial vector tensor $\Upsilon(O)$. And the translates of the highest vector span $\mathbb{V}_{n,\kappa_O}$, while the descent of $\overline{\Xi_L}$ is additive in L. Then Theorem 1 gives $\mathrm{DV}(W(O), \mathrm{I}_n(E); \chi_O; S; \overline{\Xi_L})$ with S the places outside the level, and Theorem 2, applied to the sum of $W(O)$ and of ν hyperbolic planes with its tautological splitting, gives vanishing descents to this sum for the character $\chi_O \upsilon_1^\nu$, provided $2r + 2\nu + 3 < n$. The unit on the rows deduces that such forms have vanishing theta periods along a definite subgroup, which makes the test forms of the proof orthogonal to them: forms that are not lifts but share one Hecke root with the lifts do not interfere.

In the unit on traces. Lemma 3.3 is used on its own, to identify the action of elements of $\mathbf{K}_0(n)$ with block reductions on vectors of the level of O.

8 Notes on the formal text

1. *The characters.* In this unit the Weil representation of the pair is taken with the second character trivial: the statements use $\omega_{\chi,1}^{W,V}$, also in hypothesis (c) of Theorem 1, whereas the second note uses the pair of characters of O. The definition of vanishing descents has only one character as an argument.
2. *Forms and their scalar functions.* In Theorem 1 the function whose descents vanish is the complex conjugate of Ξ_L, and the source of the descent is the

definite group. The formal statement takes Ξ as an implicit argument and L as the first explicit one. Lemma 3.3 is stated in the formal text for an arbitrary representation π of $G_{\mathrm{f}}(n)$ and a vector of the level, with the family of reductions of g as a separate implicit argument tied to g by an equation; here the reductions are written $\overline{g}_{w}$.

3. *Steps read at the level of their role.* In the proof of Theorem 2, Step 3 follows lines 80 to 107 of the formal proof. The formal text applies the library's decay estimate with the exponent of the height equal to 0; its hypothesis at points of height greater than 2 asks for a short isotropic vector in the lattice of the indefinite space, carried to the frame of the one-plane splitting by a rational point (Witt's theorem), for a basis of bounded length of the lattice of the definite space, for the invariance of the functional $\theta^{\Pi\prime}$ along the frame, and for the inequality of (Q11)(c); the height of the definite space is 1, which makes the bound uniform. The reader of the unit described these lines at the level of their role, and so does (Q11)(d), which is a summary of the estimate in the case used and not a quotation of its statement. I read the statement of the estimate and the lines that supply its hypotheses, and did not trace the constants. The second reader points out that the argument which the dossier calls a plane index in these lines is the hypothesis that the height exceeds 2; the text above follows her reading, which agrees with the Lean text.

4. *The reflection argument.* In Step 3 of the proof of Theorem 1 the dossier speaks of "the number of coset terms". In the Lean text the finite set $\mathcal{Z}$ is a set of centres at which Ψ does not vanish, the coefficient of the indicator at z' is $\Psi(z')$, and the identity of fact (3) makes every term equal to $\Psi(z)C_{z}(\Xi)$; the set is not empty because $\Psi \neq 0$. This is the second reader's correction, and it is what I find in lines 129 to 139. The matrix to which Lemma 3.2 is applied in Step 2 has the lower block $\mathfrak{t}_{w}1_{r}$; its left inverse does not depend on the upper block.

5. *Automation.* Several steps of both long proofs are closed by the simplifier with long lists of lemmas: the verification of the conditions (i) to (iii) of Lemma 4.1 in its three uses, fact (1), and the arithmetic of Steps 1 and 2 in the proof of Theorem 2. The computations given above are mine and are complete; that the simplifier performs these computations was not replayed. In Proposition 5.1 the formal proof applies a general maximum principle of the library with the constant 2 to the sum over all subspaces; the proof above is the same argument with the term of the zero subspace taken out.

6. *References for the quoted facts.* From the catalogue of the handout and the second reader's comparison with the literature; where no place is named, the *reference is to be confirmed.* (Q7), the q-binomial theorem in the form of a sum over subspaces: the catalogue attributes it to Gauss and Cauchy, relation "the theorem", place not confirmed. (Q9): the catalogue names S. Rallis, *Langlands' functoriality and the Weil representation*, Amer. J. Math. 104 (1982), R. Howe, *θ-series and invariant theory* (Corvallis 1979), and T. Tamagawa, Ann. of Math. 77 (1963), places not confirmed, relation *special case* (split places, unramified characters, the unramified vacuum; an identity of Hecke polynomials obtained by sums over subspaces of the residue space, without the Satake transform); the second reader locates the correspondence of unramified parameters in A. Mínguez, *Correspondance de*

Howe explicite : paires duales de type II, Theorem 1 and Corollary 6.3, and marks the relation as *unresolved* for the identity of operators and its normalisation. (Q10)(a), the Iwasawa decomposition: J. Arthur, *A trace formula for reductive groups I*, §1, p. 917; relation *special case*; the formulas for the zero term are the standard formulas of the mixed model (the catalogue compares with Kudla 1994 and Harris–Kudla–Sweet 1996, §1, without a confirmed place). (Q11)(b), Witt's theorem: U. A. First, *Witt's extension theorem for quadratic spaces over semiperfect rings*, J. Pure Appl. Algebra 219 (2015), Corollaries 4.9 and 4.11; relation "the theorem". (Q11)(d): the catalogue compares with the decay of theta kernels on Siegel sets in S. Kudla, S. Rallis, *A regularized Siegel–Weil formula: the first term identity*, Ann. of Math. 140 (1994), place not confirmed, and notes that the method of the formal text is different (small shifts along a short isotropic vector). (Q1) to (Q6), (Q8), (Q11)(a) and (c): no reference recorded.

How this note was written. From the dossier of the unit: eight members and seven structured proofs. The program check of the unit's dossier covered 89 leaf steps of 7 proofs, found the statements at their lines and compared the quotations of the imported declarations with the source. I read both Lean files of the unit in full and compared every statement with the statements above, argument by argument. In the background library I read the definitions behind Definitions 2.2 to 2.6 and the statements of the lemmas behind (Q1) to (Q11); I read none of their proofs. The second reader of the artifact read the unit independently; her two corrections to the dossier's words are taken over (notes 3 and 4), and I checked both against the Lean lines. The proofs of the three lemmas of §3, of Proposition 5.1 and of Lemma 6.1 follow the structured proofs step by step. The proof of Theorem 1 follows the formal one in its four steps; the verifications inside Steps 3 and 4 are written out by me. The proof of Theorem 2 follows the formal one; its Step 3 is at the level described in note 3. The remarks after the statements and §7 are mine; for §7 I read the statement and the proof of the consuming lemma in the unit on the rows.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Lemma 3.1	`AuxPair.exists_atPlace_eq`	AuxPairVanishing.lean	12–15
Lemma 3.2	`AuxPair.exists_annihilator_or_congr_inclMatrix`	AuxPairVanishing.lean	20–33
Lemma 3.3	`AuxPair.smul_chosenGLRep_eq`	AuxPairVanishing.lean	37–49
Lemma 3.4	`AuxPair.isAutomorphic_star_algFormCoeff`	AuxPairVanishing.lean	51–52
Theorem 1	`AuxPair.descentsVanish_of_reflectionSum`	AuxPairVanishing.lean	59–158
Proposition 5.1	`eq_zero_of_heckePolyOp_eq_zero`	DescentsVanishPlanes.lean	17–28
Lemma 6.1	`DescentsVanish.integral_sourceTerm_eq_zero`	DescentsVanishPlanes.lean	35–44
Theorem 2	`descentsVanish_add_planes`	DescentsVanishPlanes.lean	49–109

The two theorems as they are written in the artifact.

```
lemma AuxPair.descentsVanish_of_reflectionSum [WeilData E]
    {y} ∪ (hQ : ∀ i, O.ψ (O.twistPlace i) ≠ 1)
    (hM : ∀ i (v : ∀ _, _), v i ≠ 0 → O.formAction.asAlgebraHom (piTensorElem fun i ↦
  reflectionSum (O.ψ _) (v i)) y = 0) (h0 : ∀ P, lift (pairThetaKer _ _ O.a 1).swap (integrated _
  (polyVector _ P) O.auxDatum.1) (star (algFormCoeff 1 ∪ y.1)) = 0) : DescentsVanish O.W (stdSpace E
  n) O.a O.places.conjClosureᶜ (star (algFormCoeff 1 ∪ y.1))
```

```
lemma descentsVanish_add_planes [IsBaseAdapted D] (hY : V.IsAutomorphic Y) {w} (hw :
  w.IsSplit) (ha : CharUnramifiedAt a.baseChange.finPart (star w))
  (hw' : star w ∈ S) (hK : Y ∈ sphericalAt (finRightRep ..) w) {X} (hX : ‖X‖ = (√w.q ^ card r *
  w.q ^ (card n - card r - 1))⁻¹) (hP : heckePolyOp (finRightRep ..) hw X Y = 0) (h : 2 * card
  r + 2 * card p + 3 < card n) : DescentsVanish W' V (a * upsilonU1 ^ card p) S Y
```

The second statement is to be read with the variable line at the head of its file, which introduces W, W', V, the hypothesis of vanishing descents for W and the splitting: `(h0 : DescentsVanish W V a S Y) (D : W'.Splitting W p)`.

Legend. In the formal text `y` is the form Ξ and `u` the linear form L; `algFormCoeff 1 u y.1` is Ξ_L and `star` is complex conjugation; `O.places` is $\mathrm{pl}(O)$ and `O.places.conjClosure^c` the set S of Theorem 1; `O.formAction` is the action of Setting 2.3(c) on forms and `O.chosenGLRep π` the action $[\,\cdot\,]_{O,\pi}$; `reflectionSum` and `piTensorElem` are $\mathrm{Refl}(\psi, y)$ and the tensor product; `lift (pairThetaKer W V a 1).swap f Y` is the descent $\mathrm{Desc}^{W,V}_{\chi,1}(f, Y)$, with `a` the character χ; `projOn _ S` is P^S; `heckePolyOp (finRightRep ..) hw X` is $H^{\mathrm{R}_{\varepsilon'}}_w(T)$, with `X` the number T; `w.q` is q_w; the index types `m` and `p` are $[r']$ and $[\nu]$, the signs `s`, `t` are ε, ε', the splitting `D` is Π, and `upsilonU1` is υ_1; `D.sourceTerm V 0` is $\mathrm{Z}_{\Pi,V}[0]$; the points `g` of the adelic group of the indefinite space are u, and the elements `k` of $\mathbf{K}_0(n)$ are g.

Notation

E, τ, w, $\overline{w}$, w_0 a CM field with a CM type; an infinite place; finite places and conjugates

E_w, $\mathfrak{o}_w$, $\mathbb{F}_w$, q_w, ϖ_w, $|\cdot|_w$ completion, integers, residue field, its number of elements, a uniformiser, the absolute value

$\mathbb{A}_{\mathrm{f}}$, $\hat{\mathfrak{o}}$, E_P, P finite adeles, their integers; the product of the E_w over a finite set P of places

$[r]$, $[n]$, $[r']$, $[\nu]$; r, n, r', ν finite index sets; their numbers of elements (ranks of W, V, W'; number of planes)

i, i_0; c, c', c_0; a row indices; indices in $[r]$; a natural number, and in the proofs also a complex number or an element of a finite field

X^*, X^{T}, $\overline{z}$ conjugate transpose, transpose, conjugation

υ_1, χ, χ', χ_1 the splitting character on norm one; automorphic characters

ε, ε', 1_n families of signs; the constant family 1

$G_\infty(\varepsilon)$, $G_{\mathrm{f}}(r)$, $G(\varepsilon)$, $\mathbf{K}_0(r)$, red_w the adelic unitary groups, the integral points, reduction modulo w

u, u', u_0; h, h'; γ; g, g', g_A, g_U, g_0 points of the adelic groups of the indefinite spaces; of the definite space; rational points; elements of the finite adelic groups, mostly integral

W, W', W_1, V, J, J', $\mathrm{I}_n(E)$, $[V]$ hermitian spaces with points, their matrices, the standard definite space, the automorphic quotient

$\mathcal{S}(W,V)$, $\varrho(z)$, $\omega^{W,V}_{\chi,\chi'}$, ω, ω', $\theta_{W,V}$ data, Heisenberg operators, the Weil representation of the pair, the theta functional

f, f', f_1, φ, φ°, v, $v_p^{\varepsilon,\varepsilon'}$, p data; the vacuum; a Fock vector; the vector of a polynomial p in the variables $X_{\tau,i,c}$

$\mathrm{Desc}^{W,V}_{\chi,\chi'}(f,Y)$, Y, Y', $\mathrm{d}h$ the theta descent of a function on the definite group; the invariant measure

S, P^S, $\mathrm{DV}(W,V;\chi;S;Y)$ a set of finite places; the projector onto data unramified at S; vanishing of descents (Definition 2.2)

O, Σ_O, M_O, Q_O, $\mathrm{pl}(O)$ an auxiliary pair, its level system, chosen places, twist places, all its places

$W(O)$, ε_O, inc_O, Inc_O, $\psi_{O,w}$, χ_O its space, signs, embedding of index sets, the matrix of the embedding, its residue characters, its character

κ_O, e_O, $\mathbb{V}_{n,\kappa_O}$, ρ_{κ_O,e_O} the weight on the definite side, its space and representation

$\mathbf{L}(O)$, j, η_O, $\mathbf{K}(O)$, $\tilde{\eta}_O$ the level in the residue groups, an element, its character; the level and its character

$\mathrm{Form}(O)$, Ξ, Ξ', L, Ξ_L, $\mathrm{R}(g')$ forms; a linear form; the scalar function; right translation

π, π', ρ, $\mathcal{V}$, $\mathcal{V}'$, $\mathcal{V}^O$, ϕ, ϕ' representations of finite adelic groups, their spaces, the vectors of the level of O, vectors

$\mathrm{GL}^{\mathrm{ch}}_r(\Sigma)$, A, $[A]_{O,\pi}$ the product of the $\mathrm{GL}_r(\mathbb{F}_w)$ over the chosen places, an element, its action

$\Psi_O(X)$, $\Upsilon(O)$, $v_{O,p}$ the function of residue matrices, the finite datum, the polynomial vectors of O

$\mathbb{F}$, ψ, y, ξ, $\mathrm{Refl}(\psi, y)$, G_w, a_w a finite field, a multiplicative character, vectors, the reflection sum; finite groups and elements of their group algebras

T, T', $T_{w,a}$, $H_w^{\pi}(T)$, $\mathrm{R}_{\varepsilon'}$ complex numbers; Hecke operators; the Hecke polynomial; right translation of functions

U, F, α_i a subspace of $\mathbb{F}_w^{r'}$; a function on the adelic group of W'; Satake parameters

Π, Π', k_Π, l_Π, i_Π, $W \oplus \mathbb{H}^\nu$, $\mathrm{Z}_{\Pi,V}[0]$ splittings, their frames, the sum with ν hyperbolic planes, the zero term

Ψ, Ψ', Ψ'', Box, $\mathfrak{t}$, $\mathbf{1}_{z+\mathfrak{t}\,\mathrm{Box}}$, $\mathcal{Z}$ functions at a finite set of places, the integral matrices, an idele, the indicator of a coset, a finite set of centres

z, z', X, X', X_w, Z, Z' vectors and matrices over products of local fields (z also an idele in Setting 2.1)

$C_z(\Xi)$ the descent with the indicator of the coset of z (Step 0 of §4)

A_w, B_w, s_w, σ_w, $\overline{g}_w$ integral matrices; elements of $\mathrm{GL}_r(\mathbb{F}_w)$; reductions

ι, Inc_ι, ℓ, ℓ_w, $\overline{\ell}_w(i_0)$ an injection of index sets, its matrix; annihilators; rows of residues

α, α_w, β, β', δ_w, μ_w, ξ_w, $\tilde{\xi}_w$ matrices and vectors in the proofs of Lemma 3.2 and Theorem 1 (β is a bilinear map in (Q8))

Λ, Λ', $\theta^{\Pi\prime}$, A, B linear maps on data; the difference of theta functionals; in (Q9), A and B are complex numbers

$\mathrm{ht}(u)$, C, C_0 the height; constants

READBACK NOTE 8

Decay of Heisenberg matrix coefficients of homogeneous Fock vectors, uniform in the degree

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. In the Fock model of the oscillator representation of a unitary group $U(\varepsilon)$ of any signature, let v be the vector of a homogeneous polynomial. The matrix coefficient of v against its image under a group element followed by a Heisenberg translation by a vector y decays faster than every power of $|y|$. The unit proves this with explicit constants that depend on the degree of the polynomial only through a bound $A \geq \text{degree} + \text{number of variables}$: the decay of order $|y|^{-2\nu}$ costs the factor $(4(A+\nu))^{\nu}$. This is the sense of "uniform in the degree" in the title: no feature of the polynomial other than A enters the constants. The bound itself is not independent of the degree: for fixed ν the factor grows like the ν-th power of A. The proof writes the coefficient as an integral, compares v with a dilate of itself, and uses three elementary inequalities. This is the first of three notes on the theta correspondence behind Theorem C; it is used where a theta kernel is compared with a matrix coefficient.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, whose final theorems are the Riemann hypothesis and its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note treats one unit of the artifact, which consists of a single lemma (1 of 1 declarations). Its proof is given in full. The Fock model itself (the space, the Heisenberg operators, the oscillator representation) belongs to the background library of the artifact and is quoted in §2, with the few properties that the proof uses. The statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 1 declaration.

1 The statement

The objects are defined in §2: $U(\varepsilon)$ is the unitary group of the diagonal hermitian form with entries ε, the space $\mathcal{F}(\varepsilon)$ is the Fock space of signature ε with its inner product $\langle \cdot, \cdot \rangle_{\varepsilon}$, the operators $\varrho_{\varepsilon}(y)$ and $\omega_{\varepsilon}(u)$ are the Heisenberg translation by y and the oscillator representation at u, and $v_{\varepsilon,p}$ is the vector of the polynomial p. Norms of vectors of $\mathbb{C}^I$ are Euclidean, and $\|u\|$ is the operator norm of the matrix u for these norms.

Theorem 1 (decay of matrix coefficients, with explicit dependence on the degree). *Let I be a non-empty finite set, $\varepsilon\colon I \to \{1,-1\}$ a family of signs, $u \in U(\varepsilon)$, p a complex polynomial in the variables X_i, $i \in I$, and v an element of the Fock space $\mathcal{F}(\varepsilon)$ with $v = v_{\varepsilon,p}$. Let* dg*, A and ν be natural numbers and let $y \in \mathbb{C}^I$. Suppose that p is homogeneous of degree* dg*, in the sense that $p(cz) = c^{\mathrm{dg}}p(z)$ for every real number c and every $z \in \mathbb{C}^I$; that $\mathrm{dg} + |I| \le A$; that $A \neq 0$; and that $|y| > 0$. Then*

$$\left|\langle v, \varrho_\varepsilon(y)\,\omega_\varepsilon(u)\,v\rangle_\varepsilon\right| \;\le\; \|u\|^{|I|+2\nu} \cdot \mathrm{Re}\langle v, v\rangle_\varepsilon \cdot \big(4(A+\nu)\big)^{\nu} \cdot \big(|y|^{-1}\big)^{2\nu}.$$

LEAN `norm_realInner_realHeisOp_le`

The number $\langle v, v\rangle_\varepsilon$ is the integral of $|v|^2$, so it is real and its real part is itself. For $\nu = 0$ the bound reads $\|u\|^{|I|}\langle v, v\rangle_\varepsilon$, which is weaker than the bound $\langle v, v\rangle_\varepsilon$ that the Cauchy–Schwarz inequality gives together with the unitarity in (Q6), a fact this note does not use; the content lies in large ν. If $\nu \le |y|^2/(4\mathrm{e}\|u\|^2) - A$, then $4(A+\nu)\|u\|^2|y|^{-2} \le \mathrm{e}^{-1}$ and the right side is at most $\|u\|^{|I|}\langle v, v\rangle_\varepsilon\,\mathrm{e}^{-\nu}$; so the theorem gives a bound of Gaussian type in $|y|$. What matters in the application is that the constant $(4(A+\nu))^\nu$ depends on the polynomial only through the bound A for its degree plus the number of variables. The theorem is a quantitative form, with the dependence on the degree made explicit, of the rapid decay in y of such coefficients.

2 The setting

Conventions. The natural numbers are $0, 1, 2, \ldots$. Throughout, I is a non-empty finite set and $\varepsilon\colon I \to \{1,-1\}$ a family of signs; $|I|$ is the number of elements of I. Vectors of $\mathbb{C}^I$ are columns indexed by I, with the Euclidean norm $|z| = (\sum_{i\in I}|z_i|^2)^{1/2}$, and for $z, w \in \mathbb{C}^I$ we put $w^*\varepsilon z = \sum_{i\in I}\overline{w_i}\,\varepsilon_i\,z_i$. Integrals over $\mathbb{C}^I$ are taken with respect to the Lebesgue measure divided by $\pi^{|I|}$, written $\mathrm{d}z$; so $\int_{\mathbb{C}^I}\exp(-|z|^2)\,\mathrm{d}z = 1$, and the substitution $z \mapsto z/c$, for a real number $c > 0$, multiplies integrals by $c^{2|I|}$. For a square matrix u indexed by I, $\|u\|$ is its operator norm on $\mathbb{C}^I$ with the Euclidean norm. The objects of this section are those of the background library of the artifact; they are recalled here because the statement of Theorem 1 uses them. Letters of the handout's overview that are not displayed with the data as an argument (among them ε, β, ν and j) have, in this note, only the meaning given in its notation list; the degree is written dg, because δ is the bound of Theorem C in the overview.

Definition 2.1 (the Heisenberg bicharacter and the Fock space)**.** For $z, w \in \mathbb{C}^I$ put

$$\beta_\varepsilon(z,w) = \exp\Big(\tfrac{1}{2}(w^*\varepsilon z - z^*\varepsilon w)\Big) = \exp\big(\mathrm{i}\,\mathrm{Im}(w^*\varepsilon z)\big), \qquad \varphi_0(z) = \exp(-\tfrac{1}{2}|z|^2).$$

So β_ε has absolute value 1, is multiplicative in each variable, and

$$\beta_\varepsilon(w,z) = \beta_\varepsilon(z,w)^{-1} = \overline{\beta_\varepsilon(z,w)};$$

the Gaussian φ_0 is real, even, and $0 < \varphi_0 \le 1$. The *Fock space* $\mathcal{F}(\varepsilon)$ is the space of all integrable functions $v\colon \mathbb{C}^I \to \mathbb{C}$ that satisfy the reproducing identity

$$v(w) = \int_{\mathbb{C}^I} v(z)\,\beta_\varepsilon(z,w)\,\varphi_0(w-z)\,\mathrm{d}z \qquad \text{for every } w \in \mathbb{C}^I,$$

with the norm $\|v\|_1 = \int |v(z)|\,\mathrm{d}z$. For $v, w \in \mathcal{F}(\varepsilon)$ the *inner product* is

$$\langle v, w\rangle_\varepsilon = \int_{\mathbb{C}^I} \overline{v(z)}\, w(z)\,\mathrm{d}z;$$

it is conjugate-linear in the first variable. For $y \in \mathbb{C}^I$ the *Heisenberg operator* $\varrho_\varepsilon(y)$ sends $v \in \mathcal{F}(\varepsilon)$ to the function

$$\big(\varrho_\varepsilon(y)v\big)(z) = \beta_\varepsilon(y,z)\, v(z-y).$$

LEAN in the background library: `realHeis`, `realVac`, `realMeasure`, `fockSpace`, `realInner`, `realHeisOp` · the general construction is `Heisenberg.IsVacuum.Model` with `heisOp` and `modelInner`

The elements of $\mathcal{F}(\varepsilon)$ are functions, not classes of functions: the reproducing identity determines the values. It also gives $|v(w)| \le \|v\|_1$ for all w, because $|\beta_\varepsilon| = 1$ and $\varphi_0 \le 1$. So every $v \in \mathcal{F}(\varepsilon)$ is bounded, $|v|^2 \le \|v\|_1\,|v|$ is integrable, and the inner product is defined and bounded: $|\langle v, w\rangle_\varepsilon| \le \|v\|_1 \|w\|_1$. The Heisenberg operators map $\mathcal{F}(\varepsilon)$ to itself, preserve the norm and the inner product, and satisfy

$$\varrho_\varepsilon(y)\,\varrho_\varepsilon(w) = \beta_\varepsilon(y,w)\,\varrho_\varepsilon(y+w) \qquad (y, w \in \mathbb{C}^I). \tag{1}$$

The Gaussian φ_0 belongs to $\mathcal{F}(\varepsilon)$; it is called the *vacuum vector*. For $\varepsilon = 1$ the functions $z \mapsto v(z)/\varphi_0(z)$, $v \in \mathcal{F}(\varepsilon)$, are antiholomorphic, and one has the integrable functions of the classical Fock space; in general the positive coordinates enter through their conjugates and the negative ones directly, as the next definition shows. The background library works with this space of integrable functions and its norm of integrable functions, and not with the Hilbert space completion.

Definition 2.2 (polynomial vectors)**.** For $z \in \mathbb{C}^I$ let $\mathrm{hol}_\varepsilon(z) \in \mathbb{C}^I$ be the vector with coordinates $\overline{z_i}$ for $\varepsilon_i = 1$ and z_i for $\varepsilon_i = -1$. For a complex polynomial p in the variables X_i, $i \in I$, the *Fock vector of* p is the function

$$v_{\varepsilon,p}(z) = p\big(\mathrm{hol}_\varepsilon(z)\big)\,\varphi_0(z) \qquad (z \in \mathbb{C}^I).$$

LEAN in the background library: `holCoord`, `polyVector`

Definition 2.3 (the unitary group, its multiplier, the oscillator representation)**.** The group $U(\varepsilon)$ consists of the invertible complex matrices u indexed by I with $u^*\,\mathrm{diag}(\varepsilon)\,u = \mathrm{diag}(\varepsilon)$, where u^* is the conjugate transpose; it acts on $\mathbb{C}^I$ by $z \mapsto uz$. For $u \in U(\varepsilon)$ let $u__$ be the square block of u whose rows and columns are the indices i with $\varepsilon_i = -1$, and put $\Delta_\varepsilon(u) = \overline{\det u__}$, the *multiplier* (it is 1 if no sign is -1). The *oscillator representation* of $u \in U(\varepsilon)$ is the operator on $\mathcal{F}(\varepsilon)$

$$\omega_\varepsilon(u)\,v = \Delta_\varepsilon(u) \int_{\mathbb{C}^I} v(z)\;\varrho_\varepsilon(uz)\,\varphi_0\,\mathrm{d}z,$$

an integral of a function with values in the Banach space $\mathcal{F}(\varepsilon)$.

LEAN in the background library: `USign`, `realMultiplier`, `realAction`, `realWeil` · the general construction is `Heisenberg.IsVacuum.weilOp`, `weilRep`

Quoted from the background library. These facts about the Fock model are taken for granted. Facts (Q2) to (Q5) have proofs of a line, which are indicated; (Q1) and (Q6) are part of the classical theory of the Fock model of the oscillator representation of unitary groups of any signature (references in §5, note 5).

(Q1) For every complex polynomial p in the variables X_i, $i \in I$, the function $v_{\varepsilon,p}$ belongs to $\mathcal{F}(\varepsilon)$; in particular it is integrable and satisfies the reproducing identity.

(Q2) *Reproducing property.* For $v \in \mathcal{F}(\varepsilon)$ and $w \in \mathbb{C}^I$: $\langle \varrho_\varepsilon(w)\varphi_0, v\rangle_\varepsilon = v(w)$. Indeed the left side is $\int \overline{\beta_\varepsilon(w,z)}\,\varphi_0(z-w)\,v(z)\,\mathrm{d}z$, which is the right side of the reproducing identity because $\overline{\beta_\varepsilon(w,z)} = \beta_\varepsilon(z,w)$ and φ_0 is real and even.

(Q3) For $u \in U(\varepsilon)$: $\beta_\varepsilon(uz,uw) = \beta_\varepsilon(z,w)$ for all $z,w \in \mathbb{C}^I$, and $z \mapsto uz$ preserves the measure $\mathrm{d}z$. The first holds because $(uw)^*\varepsilon(uz) = w^*\varepsilon z$, the second because $|\det u| = 1$.

(Q4) For $u \in U(\varepsilon)$: $\|u\| \geq 1$ and $|\Delta_\varepsilon(u)| \leq \|u\|^{|I|}$. Indeed $u^{-1} = \mathrm{diag}(\varepsilon)\,u^*\,\mathrm{diag}(\varepsilon)$ has the norm of u, so $1 = \|u^{-1}u\| \leq \|u\|^2$; and a principal minor of size j of a matrix u has absolute value at most $\|u\|^j \leq \|u\|^{|I|}$.

(Q5) A bounded linear map T from $\mathcal{F}(\varepsilon)$ to $\mathbb{C}$ commutes with the integral of Definition 2.3: $T(\int v(z)\,\varrho_\varepsilon(uz)\varphi_0\,\mathrm{d}z) = \int v(z)\,T(\varrho_\varepsilon(uz)\varphi_0)\,\mathrm{d}z$.

(Q6) *Not used in this note.* The map $u \mapsto \omega_\varepsilon(u)$ is a homomorphism from $U(\varepsilon)$ to the bounded invertible operators of $\mathcal{F}(\varepsilon)$, it preserves the inner product, and $\omega_\varepsilon(u)\,\varrho_\varepsilon(y) = \varrho_\varepsilon(uy)\,\omega_\varepsilon(u)$. This is the statement that the oscillator representation splits over $U(\varepsilon)$ with the multiplier Δ_ε.

LEAN in the background library: (Q1) is part of `polyVector` · (Q2) `Heisenberg.IsVacuum.modelInner_heisOp_vacElem` · (Q3) is part of `realAction` · (Q4) `U.one_le_norm`, `norm_realMultiplier_le` · (Q5) `Heisenberg.Rep.map_integrated_eq_integral` · (Q6) is part of `realAction`, with `weilRep`, `modelInner_weilRep`, `weilRep_heisOp`

3 Proof of Theorem 1

Three lemmas prepare the proof. In all of them I and ε are as in the conventions.

Lemma 3.1 (the coefficient as an integral)**.** *Let $u \in U(\varepsilon)$, $v \in \mathcal{F}(\varepsilon)$ and $y \in \mathbb{C}^I$. Then*

$$\langle v, \varrho_\varepsilon(y)\,\omega_\varepsilon(u)\,v\rangle_\varepsilon = \Delta_\varepsilon(u)\int_{\mathbb{C}^I} v(z)\,\beta_\varepsilon(y,uz)\,\overline{v(uz+y)}\,\mathrm{d}z, \qquad \textit{and so}$$

$$\big|\langle v, \varrho_\varepsilon(y)\,\omega_\varepsilon(u)\,v\rangle_\varepsilon\big| \leq \|u\|^{|I|}\int_{\mathbb{C}^I} |v(z)|\,|v(uz+y)|\,\mathrm{d}z.$$

Proof. The map $T\colon w \mapsto \langle v, \varrho_\varepsilon(y)w\rangle_\varepsilon$ is linear and bounded on $\mathcal{F}(\varepsilon)$. By Definition 2.3 and (Q5),

$$T(\omega_\varepsilon(u)v) = \Delta_\varepsilon(u)\int v(z)\,\langle v, \varrho_\varepsilon(y)\,\varrho_\varepsilon(uz)\,\varphi_0\rangle_\varepsilon\,\mathrm{d}z.$$

By (1) and by (Q2), for every $w \in \mathbb{C}^I$,

$$\varrho_\varepsilon(y)\,\varrho_\varepsilon(uz)\,\varphi_0 = \beta_\varepsilon(y,uz)\,\varrho_\varepsilon(uz+y)\,\varphi_0, \qquad \langle v, \varrho_\varepsilon(w)\varphi_0\rangle_\varepsilon = \overline{\langle \varrho_\varepsilon(w)\varphi_0, v\rangle_\varepsilon} = \overline{v(w)}.$$

This is the identity. The inequality follows from $|\beta_\varepsilon| = 1$ and (Q4). □

Lemma 3.2 (dilation of the vector of a homogeneous polynomial)**.** *Let p be a complex polynomial in the variables X_i, $i \in I$, and* dg *a natural number with $p(cz) = c^{\mathrm{dg}} p(z)$ for every real c and every $z \in \mathbb{C}^I$. Then for every real $c > 0$ and every $z \in \mathbb{C}^I$*

$$\bigl|v_{\varepsilon,p}(z)\bigr| = c^{\mathrm{dg}}\,\exp\Bigl(-\tfrac{1}{2}(1-c^{-2})|z|^2\Bigr)\,\bigl|v_{\varepsilon,p}(z/c)\bigr|.$$

Proof. The map $z \mapsto \mathrm{hol}_\varepsilon(z)$ is linear over the real numbers, so

$$p(\mathrm{hol}_\varepsilon(z/c)) = c^{-\mathrm{dg}}\,p(\mathrm{hol}_\varepsilon(z)), \qquad \varphi_0(z/c) = \varphi_0(z)\exp(\tfrac{1}{2}(1-c^{-2})|z|^2).$$

Multiply the two. □

Lemma 3.3 (three elementary inequalities)**.** (i) $1 + B \le \mathrm{e}^B$ *for real* B. (ii) $\mathrm{e}^{1-B} \le B^{-1}$ *for real* $B > 0$. (iii) *For $u \in U(\varepsilon)$ and $y, z \in \mathbb{C}^I$:*

$$|y|^2 \le 2\|u\|^2(|z|^2 + |uz+y|^2).$$

Proof. (i) is the convexity of the exponential function, and (ii) is (i) applied to $B-1$: $B \le \mathrm{e}^{B-1}$. For (iii), the triangle inequality and (Q4) give

$$|y| \le |uz+y| + \|u\||z| \le \|u\|(|uz+y| + |z|),$$

and $(\alpha+\gamma)^2 \le 2(\alpha^2+\gamma^2)$ for real α, γ. □

Proof of Theorem 1. Since I is not empty and $\mathrm{dg} + |I| \le A$, we have $A \ge 1$. Put $c = \sqrt{\nu/A + 1}$, so that $c \ge 1$ and $1 - c^{-2} = \nu/(\nu + A)$. By (Q1), $v = v_{\varepsilon,p}$ lies in $\mathcal{F}(\varepsilon)$, so $|v|^2$ is integrable; hence so are $z \mapsto |v(z/c)|^2$ and, by (Q3), $z \mapsto |v((uz+y)/c)|^2$, with

$$\int_{\mathbb{C}^I} |v(z/c)|^2\,\mathrm{d}z = \int_{\mathbb{C}^I} \bigl|v((uz+y)/c)\bigr|^2\,\mathrm{d}z = c^{2|I|}\,\langle v, v\rangle_\varepsilon. \tag{2}$$

Fix $z \in \mathbb{C}^I$ and put $B = (|z|^2 + |uz+y|^2)/(2(\nu+A))$. If B were 0, then $z = 0$ and $uz + y = 0$, so $y = 0$, which is excluded; hence $B > 0$. Lemma 3.2, applied at the points z and $uz+y$, gives

$$|v(z)|\,|v(uz+y)| = c^{2\mathrm{dg}}\,\mathrm{e}^{-\nu B}\,|v(z/c)|\,\bigl|v((uz+y)/c)\bigr|.$$

Now $c^2 = 1 + \nu/A \le \mathrm{e}^{\nu/A}$ by Lemma 3.3(i), and $(\mathrm{dg} + |I|)/A \le 1$, so

$$c^{2\mathrm{dg}} = c^{2(\mathrm{dg}+|I|)}\,c^{-2|I|} \le \mathrm{e}^{\nu}\,c^{-2|I|}.$$

By Lemma 3.3(ii) and (iii),

$$\mathrm{e}^{\nu}\,\mathrm{e}^{-\nu B} = \bigl(\mathrm{e}^{1-B}\bigr)^{\nu} \le B^{-\nu} \le \Bigl(\frac{4(A+\nu)\|u\|^2}{|y|^2}\Bigr)^{\nu}.$$

With $2\alpha\gamma \le \alpha^2 + \gamma^2$ for the last two factors this gives, for every z,

$$|v(z)|\,|v(uz+y)| \;\le\; \frac{1}{2\,c^{2|I|}}\Bigl(\frac{4(A+\nu)\|u\|^2}{|y|^2}\Bigr)^{\nu}\Bigl(|v(z/c)|^2 + \bigl|v((uz+y)/c)\bigr|^2\Bigr).$$

Integrate over z and use (2): the integral of the left side is at most

$$\bigl(4(A+\nu)\bigr)^{\nu}\,\|u\|^{2\nu}\,|y|^{-2\nu}\,\langle v, v\rangle_\varepsilon.$$

Lemma 3.1 multiplies this by $\|u\|^{|I|}$. □

LEAN `norm_realInner_realHeisOp_le` · the formal proof is one chain of the same steps: the constant c, the two integrable functions, the integral formula, the bound of the multiplier, the pointwise inequality

4 Where the result is used

The theorem has one consumer, in the unit on the rows (the forms on the indefinite unitary group). There a theta kernel for a pair of unitary groups, formed with a datum that is a character on the matrices modulo a modulus, is compared with the matrix coefficient $\langle v, \omega_\varepsilon(u)v\rangle_\varepsilon$ of a homogeneous polynomial vector v for real points u of the two groups. The difference is a sum, over the non-zero vectors y of a lattice, of the coefficients $\langle v, \varrho_\varepsilon(y)\omega_\varepsilon(u)v\rangle_\varepsilon$. Theorem 1 is applied to each term, with $A = \mathrm{dg} + |I|$; the vectors y have norm at least a quantity fixed by the first minimum of the lattice, and the sum of $|y|^{-2\nu}$ over the lattice converges when 2ν exceeds its real rank. The outcome is a bound for the difference whose dependence on the degree dg is explicit, through the factor $(4(\mathrm{dg} + |I| + \nu))^\nu$, polynomial in dg for fixed ν; this is what allows degrees that grow with the data. That unit states the result as an Input.

5 Notes on the formal text

1. *A hypothesis that follows from the others.* The formal statement assumes $A \neq 0$. Since I is not empty and $\mathrm{dg} + |I| \leq A$, this follows from the other hypotheses; the formal proof uses it where the proof above uses $A \geq 1$, namely for the identity $1 - c^{-2} = \nu/(\nu + A)$. Theorem 1 keeps the hypothesis because the formal statement has it.

2. *Homogeneity.* The formal statement does not use the notion of a homogeneous polynomial of its library. It asks for $p(cz) = c^{\mathrm{dg}}p(z)$ for all real c, zero and negative values included, and all $z \in \mathbb{C}^I$. For a polynomial this is the same as homogeneity of degree dg; only positive c is used.

3. *The form of the bound.* The right side is written in the formal text as the product $\|u\|^{|I|+2\nu} \cdot \mathrm{Re}\langle v, v\rangle_\varepsilon \cdot (4(A + \nu))^\nu \cdot (|y|^{-1})^{2\nu}$, with the real part and with the inverse of $|y|$ raised to the power 2ν; Theorem 1 reproduces this form. The vector v is an argument of its own, tied to p by the hypothesis $v = v_{\varepsilon,p}$.

4. *Steps carried out by automation.* The reader of the unit marked the last four steps of the structured proof (the dilation identity, the collection of powers, the two chains of inequalities) as claims about the mathematical content, not replayed goal by goal. I read these lines in the Lean text. The dilation identity is stated there explicitly, as $|v(z/c)| = |v(z)| \exp(|z|^2/(\nu + A)/2)^\nu/|c|^{\mathrm{dg}}$, which is Lemma 3.2 for the chosen c. The chains then use, in this order: the square of the square root, $1 + B \leq \mathrm{e}^B$, the hypothesis $\mathrm{dg} + |I| \leq A$, $A/A \leq 1$, $\mathrm{e} \cdot B \leq \mathrm{e}^B$; then the triangle inequality, the bound of $|uz|$ by $\|u\||z|$, $\|u\| \geq 1$, and $(\alpha + \gamma)^2 \leq 2(\alpha^2 + \gamma^2)$. These are the steps of the proof above. What the automation proves between them was not replayed by me either: no Lean was run for this note.

5. *One place where the dossier's words differ from the Lean text.* The structured proof describes the second integrable function as $z \mapsto |v(uz + y)|^2$ and says that it integrates to $\langle v, v\rangle_\varepsilon$. In the Lean text it is the composite of $z \mapsto |v(z/c)|^2$ with the measure-preserving map $z \mapsto uz + y$, that is $z \mapsto |v((uz + y)/c)|^2$, and both functions integrate to $c^{2|I|}\langle v, v\rangle_\varepsilon$, as in (2). With the dilated second function the constants match exactly; with the undilated one the two integrals would have different scales. The quotations of the dossier are exact; only its words are

affected. The second reader of the artifact, who read the unit independently, reports the same reading of these two lines.

6. *References for the quoted facts.* The catalogue of the handout records no confirmed place for the facts (Q1) to (Q6): *reference to be confirmed.* The files of the background library name their sources in their headers, for the Fock model of the oscillator representation of the indefinite unitary groups: M. G. Davidson, *The harmonic representation of U(p,q) and its connection with the generalized unit disk*, Pacific J. Math. 129 (1987), §1 and Prop. 2.2; G. B. Folland, *Harmonic Analysis in Phase Space* (1989), ch. 1 and 4; H. Reiter, *Metaplectic Groups and Segal Algebras*, Lecture Notes in Math. 1382 (1989). The artifact's own list of references marks the first as "work found, no place given" and the second as "place confirmed at second hand". The same headers say how the formal text differs from these sources: the space is the space of integrable functions with the reproducing identity, not the Hilbert space, and no kernel formula for $\omega_\varepsilon(u)$ is proved. For (Q6) the catalogue notes that the formal definition of the action contains the proof that the cocycle is trivial, by a Gaussian integral. The second reader located the classical statement in Davidson's paper, §1, formula (1.9a), Lemma 1.11 and Theorem 1.12, pp. 39–40, where the phases of the Fock intertwiners are fixed and shown to give a continuous representation of the indefinite unitary group; the relation of the two is *unresolved*: the conversion of Davidson's kernel scalar into the multiplier Δ_ε of the formal text has not been verified.

How this note was written. From the dossier of the unit: one structured proof of ten steps. The program check of the unit's dossier covered 10 leaf steps of 1 proof, found the statement at its lines and compared the quotations of the imported declarations with the source. I read the whole Lean file of the unit, statement and proof, and compared the statement with Theorem 1 argument by argument, in Lean's order: the index set with its two instance arguments, the signs, the polynomial, the vector, the translation, the group element, the equation $v = v_{\varepsilon,p}$, the three natural numbers, and the four hypotheses. In the background library I read the definitions quoted in Definitions 2.1 to 2.3 (the bicharacter, the Gaussian, the normalised measure, the space with its reproducing identity, the Heisenberg operators, the inner product, the polynomial vectors, the multiplier, the operators $\omega_\varepsilon(u)$) and the statements of the lemmas behind (Q2), (Q4), (Q5) and (Q6); I did not read the proof that polynomial vectors lie in the space (Q1), nor the proof of the cocycle identity (Q6). The proof above follows the formal one step by step; its arrangement into three lemmas, the remark on the best ν and §4 are mine. For §4 I read the statement of the consuming lemma and the lines where Theorem 1 is applied.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Theorem 1	`norm_realInner_realHeisOp_le`	FockCoeffDecay.lean	13–35

The statement as it is written in the artifact.

```
lemma norm_realInner_realHeisOp_le {ι : Type} [Fintype ι] [Nonempty ι] {ε P v y} (σ : USign (n := ι)
  ε) (hv : v = polyVector ε P) {d A n} (hP : ∀ (c : ℝ) x, eval (c • x) P = c ^ d * eval x P) (hA :
  d + card ι ≤ A) (h0 : A ≠ 0) (hy : 0 < ‖toLp 2 y‖) : ‖realInner v (realHeisOp y (realWeil σ v))‖ ≤
  ‖σ.1.1‖ ^ (card ι + 2 * n) * (realInner v v).re * (4 * (A + n)) ^ n * ‖toLp 2 y‖⁻¹ ^ (2 * n)
```

In this text `ι` is the index set I, `σ` the group element u, `P` the polynomial p, `d` the degree dg, `n` the exponent ν; `USign ε` is $U(\varepsilon)$, `polyVector ε P` is $v_{\varepsilon,p}$, `realInner` is the inner product, `realHeisOp y` is $\varrho_\varepsilon(y)$ and `realWeil σ` is $\omega_\varepsilon(u)$; `‖toLp 2 y‖` is the Euclidean norm $|y|$, and `‖σ.1.1‖` the operator norm of the matrix of u. The letters of the formal text were changed where the write-up reserves them for other objects (the letters d, P and n).

Notation

I, $|I|$, i a non-empty finite set, its number of elements, an element of it

ε a family of signs $\varepsilon_i \in \{1, -1\}$, $i \in I$

$\mathbb{C}^I$, $|z|$ the vectors indexed by I; the Euclidean norm

z, w, y vectors of $\mathbb{C}^I$; y is the translation of Theorem 1; w also denotes an element of the Fock space

$w^*\varepsilon z$ the hermitian form $\sum_i \overline{w_i}\varepsilon_i z_i$

$\mathrm{d}z$ Lebesgue measure on $\mathbb{C}^I$ divided by $\pi^{|I|}$

β_ε, $\beta_\varepsilon(z,w)$ the bicharacter $\exp(\mathrm{i}\,\mathrm{Im}(w^*\varepsilon z))$

φ_0 the Gaussian $\exp(-|z|^2/2)$, the vacuum vector

$\mathcal{F}(\varepsilon)$, $\|v\|_1$ the Fock space of signature ε; its norm

$\langle v, w\rangle_\varepsilon$ the inner product $\int \overline{v}\, w\, \mathrm{d}z$ of $\mathcal{F}(\varepsilon)$

$\varrho_\varepsilon(y)$ the Heisenberg operator of $y \in \mathbb{C}^I$

$U(\varepsilon)$, u, u^* the unitary group of $\mathrm{diag}(\varepsilon)$; an element; its conjugate transpose

u_{--}, Δ_ε, $\Delta_\varepsilon(u)$ the block of u at the negative signs; the multiplier $\overline{\det u_{--}}$

$\|u\|$ the operator norm of a matrix for the Euclidean norm

$\omega_\varepsilon(u)$ the oscillator representation at u

X_i, p the variables; a complex polynomial in them

$\mathrm{hol}_\varepsilon(z)$ z with its coordinates at positive signs conjugated

$v_{\varepsilon,p}$, v the Fock vector of p; an element of the Fock space

dg the degree of the homogeneous polynomial p (a natural number)

δ the bound of Theorem C in the overview; not used for an object of this note

A a natural number with $\mathrm{dg} + |I| \le A$

ν a natural number: the decay is of order $|y|^{-2\nu}$

c a real number; in the proof $c = \sqrt{\nu/A + 1}$

B a real number; in the proof $(|z|^2 + |uz + y|^2)/(2(\nu + A))$

T a bounded linear map on the Fock space

j the size of a minor

α, γ real numbers in elementary inequalities

e, π the base of the natural logarithm; the area of the unit disc

READBACK NOTE 37

At most one primitive label for a Hecke eigensystem, by attached Galois representations

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. At a chosen prime the forms of principal level carry a representation of a finite general linear group, and a primitive label names a cuspidal type of that group. This note proves that a Hecke eigensystem can meet at most one primitive label at a chosen prime, as soon as the forms can be lifted injectively to a definite unitary group in a way that respects the Hecke operators away from the level and the action of a parabolic subgroup at the prime. The proof passes to Galois representations: a lifted form is an eigenform, so a 2-adic Galois representation is attached to its Hecke polynomials; local-global compatibility at tame level reads the label off the action of tame inertia; and two different primitive labels would force the eigenvalues of one inertia element to be roots of unity of small order, against the determinant.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles and that Lean's kernel accepted the proofs; an independent replay is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note treats the unit *At most one primitive label per eigensystem at a chosen place*: all 2 of its 2 declarations, a lemma and a theorem. Both are proved in full from the inputs. Taken for granted, and stated as Inputs 1 to 5, are: the existence of Galois representations attached to eigenforms on definite unitary groups, and local-global compatibility at tame level, both in the weak forms in which the artifact's background library proves them; three further facts of that library; and facts about the data and about primitive labels from other units. Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 10 declarations.

1 The statement

The setting and the notions are those of the note *Good forms and rows*, recalled in Setting 2.1; algebraic forms, Hecke polynomials and block triangular matrices are explained in §2.

Theorem 1 (at most one primitive label). *In Setting 2.1, let $\mathfrak{s}$ be a family of signs on $I(D)$ and $\mathcal{V}$ a space over $E(D)$ with index set $I(D)$ and signs $\mathfrak{s}$. Let I' be a non-empty finite set, $\nu\colon I(D) \to I'$ an injection, and L a linear map from $\mathrm{Prin}(D,\mathcal{V})$ to a space of algebraic modular forms on the standard space with index set I', in the library's sense: such a space is given by a weight in two parts, a system of places of $E(D)$, a subgroup of the product of the groups $\mathrm{GL}_{I'}(\mathbb{F}_w)$ over these places, and a character of that subgroup. Let p be an eigensystem, $(\Upsilon_w)_w$ a family of functions $\mathbb{C} \to \mathbb{C}$ indexed by the primes of $E(D)$, v a chosen prime, n_1 a natural number, $\beta\colon I' \to \{0,\dots,n_1\}$ a map, $\chi = (\chi_1,\dots,\chi_{n_1})$ a family of multiplicative characters of $\mathbb{F}_v$, and ψ a multiplicative character of $\mathbb{F}_v$. Suppose:*

(i) *L is injective;*

(ii) *for every $y \in \mathrm{Prin}(D,\mathcal{V})[p]$ and every split prime w of $E(D)$ such that neither w nor $\bar{w}$ is chosen, the function $L(y)$ is spherical at w with Υ_w: it is fixed by right translation by the elements of $K_0(I')$ with component 1 at the primes other than w and $\bar{w}$, and for every $X \in \mathbb{C}$*

$$\sum_{n=0}^{\#I'} (-X)^n \, (\#\mathbb{F}_w)^{n(n-1)/2} \, \mathbb{T}^{w,n} L(y) = \Upsilon_w(X)\, L(y);$$

(iii) *for every $y \in \mathrm{Prin}(D,\mathcal{V})$, the function $L(y)$ has parahoric type at v for (ν,β,ψ,χ) and the functions $x \mapsto L(\mathrm{rep}_{D,\mathcal{V}}(x_{(v)})\, y)$: that is, $\beta^{-1}(0) = \nu(I(D))$, and for every $x \in \mathrm{GL}_{I(D)}(\mathbb{F}_v)$ and every $k \in \mathrm{GL}_{I'}(O_v)$ whose reduction $\bar{k}$ is block upper triangular for β with block 0 equal to x,*

$$R(k^{(v)})\, L(y) = \prod_{n=1}^{n_1} \chi_n(\det \bar{k}_{nn}) \cdot \psi(\det x) \cdot L(\mathrm{rep}_{D,\mathcal{V}}(x_{(v)})\, y);$$

(iv) *the characters $\chi_1,\dots,\chi_{n_1}$ are pairwise distinct;*

(v) *$\psi^{M_\omega(D)} = 1$;*

(vi) *$\chi_n^{M_\omega(D)} = 1$ for every n.*

Then the set $\mathrm{Occ}(D,\mathcal{V},p,v)$ has at most one element.

LEAN `Data.occurringLabels_subsingleton`

The theorem says nothing about existence: if $\mathrm{Prin}(D,\mathcal{V})[p]$ is zero, no label occurs. It is applied twice in the note *Good forms and rows*: to the space of the rows, with L the chain of $m(D)$ theta lifts to a definite space, and to the source of each single lift, with L that lift. In both cases the lift carries the cuspidal type at v into a type induced from a parabolic subgroup, which is what (iii) expresses.

2 The setting

Conventions. The natural numbers are $0, 1, 2, \dots$. The parameters $P = (N_0, d)$ of the proof are fixed. For a number field E: a prime of E is a non-zero prime ideal of its ring of integers; $\mathbb{F}_w$, E_w, O_w are the residue field, the completion and its ring of integers at a prime w; for a CM field, $z \mapsto \bar{z}$ is complex conjugation, and a prime w is split if $\bar{w} \neq w$. For a finite set I', $\mathrm{GL}_{I'}$ of a ring is the group of invertible matrices over it indexed by I'; $G_f(I')$ is the group of the matrices g over the finite adeles

of E, indexed by I', with $g^*g = 1$, where g^* is the conjugate transpose; $K_0(I')$ is its subgroup of the matrices with integral entries; for a split prime w and $k \in \mathrm{GL}_{I'}(E_w)$, $k^{(w)} \in G_f(I')$ has component k at w and 1 at the primes other than w and $\bar{w}$; and R is right translation of functions on $G_f(I')$, $(R(g)f)(g') = f(g'g)$. $\bar{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$ that contains E. Labels, primitive labels, the polynomial $c^{[u]}$ whose roots are u times those of c, the injective character T of the multiplicative group of an algebraic closure $\bar{\mathbb{F}}$ of a finite field $\mathbb{F}$, the unit x_χ that names a multiplicative character χ, Green's class functions Ξ_c and their induction product $*$, and the idempotents $e_{I,c}$ are those of the note *Primitive labels*. Letters of the handout's overview that are not displayed with the data D as an argument (among them t, λ, Υ, δ, ζ and j) have, in this note, only the meaning given in its notation list.

Setting 2.1 (the data and its forms)**.** D is admissible data for the parameters P (Chapter III of the handout). The following objects are those of the setting and of the definitions of the note *Good forms and rows*: the CM field $E(D)$, which contains a primitive root of unity of order 5040; the chosen primes, none of which is conjugate to a chosen prime; the rank $r(D) > 1$; the row index set $I(D)$ with $r(D)$ elements and the group $\mathsf{G}(D) = \prod_v \mathrm{GL}_{I(D)}(\mathbb{F}_v)$ over the chosen primes; the integer $M_\omega(D)$, for which $M_\omega(D)^2$ divides $\#\mathbb{F}_v - 1$ and $0 < M_\omega(D) < \#\mathbb{F}_v - 1$ at every chosen prime v; the fact that $\#\mathbb{F}_v$ is an odd prime for chosen v; for signs $\mathfrak{s}$ on $I(D)$ and a space $\mathcal{V}$ over $E(D)$ with index set $I(D)$ and signs $\mathfrak{s}$, the space $\mathrm{Prin}(D, \mathcal{V})$ of forms of principal level at the chosen primes and the representation $\mathrm{rep}_{D,\mathcal{V}}$ of $\mathsf{G}(D)$ on it; the Hecke operators $\mathbb{T}^{w,n}$ at the split primes w with neither w nor $\bar{w}$ chosen; eigensystems p and the eigenparts $\mathrm{Prin}(D, \mathcal{V})[p]$; and, for a chosen prime v, the set $\mathrm{Occ}(D, \mathcal{V}, p, v)$ of the primitive labels $c \in \mathbb{F}_v[X]$ of degree $r(D)$ for which $e_{I(D),c}$, acting through the factor v of $\mathsf{G}(D)$, does not kill $\mathrm{Prin}(D, \mathcal{V})[p]$. For $x \in \mathrm{GL}_{I(D)}(\mathbb{F}_v)$, $x_{(v)} \in \mathsf{G}(D)$ has component x at v and 1 elsewhere.

Functions on $G_f(I')$: Hecke polynomials, eigenforms, types, labels. These notions belong to the background library. Let E be a CM field, I' a finite set, $\mathcal{W}$ a set, and f a function on $G_f(I')$ with values in the complex functions on $\mathcal{W}$.

(a) For a split prime w and a natural number n, $\mathbb{T}^{w,n}f$ is the sum of the right translates of f by the $k^{(w)}$, k running over representatives of the cosets modulo $\mathrm{GL}_{I'}(O_w)$ in the double coset of $\mathrm{diag}(\varpi, \dots, \varpi, 1, \dots, 1)$, with n entries equal to a uniformizer ϖ.

(b) *f has Hecke polynomial Q at a split prime w*, for a complex polynomial Q, if f is fixed by right translation by the elements of $K_0(I')$ with component 1 at the primes other than w and $\bar{w}$, and there are complex numbers $t_0, t_1, \dots$ with $\mathbb{T}^{w,n}f = t_n f$ for all n and $Q = \sum_{n=0}^{\#I'} (-1)^n (\#\mathbb{F}_w)^{n(n-1)/2}\, t_n\, X^{\#I'-n}$.

(c) *f is an eigenform for a family $(\mathrm{Hk}_w)_w$ of complex polynomials*, indexed by the primes of E, if $f \neq 0$ and for all but finitely many primes w: if the norm $\#\mathbb{F}_w$ is a prime number, then w is split and f has Hecke polynomial Hk_w at w.

(d) *Algebraic modular forms* on the standard space with index set I', of a weight κ: functions f as above, for a finite set $\mathcal{W}$ fixed by the weight, that are invariant under right translation by an open subgroup and satisfy $f(\gamma g) = \mathrm{alg}_\kappa(\gamma) f(g)$ for all rational points γ of the definite unitary group, with alg_κ the algebraic representation of the weight.

(e) Let $\beta\colon I' \to \{0, \dots, n_1\}$ be a map. A matrix $\bar{k}$ indexed by I' is *block upper triangular for* β if its entry at (ι, ι') vanishes whenever $\beta(\iota') < \beta(\iota)$; its n-th diagonal block $\bar{k}_{nn}$ is its part with rows and columns in $\beta^{-1}(n)$. With an injection ν of a set I onto $\beta^{-1}(0)$, block 0 is read as a matrix indexed by I.

(f) Let w be a split prime, $c \in \mathbb{F}_w[X]$, $\chi_1, \dots, \chi_{n_1}$ multiplicative characters of $\mathbb{F}_w$ and $t_1, \dots, t_{n_1}$ natural numbers. Then f *has the product label* $(c; \chi; t)$ *of degree* δ *at* w if: c is a label of degree δ; the polynomial $c \cdot \prod_n (X - x_{\chi_n})$ is separable; $\delta + \sum_n t_n = \#I'$; f is fixed by $R(k^{(w)})$ for the $k \in \mathrm{GL}_{I'}(O_w)$ that reduce to 1; and $\sum_x \overline{\Xi(x)}\, R(\tilde{x}^{(w)}) f \neq 0$, the sum over $x \in \mathrm{GL}_{I'}(\mathbb{F}_w)$ with lifts $\tilde{x} \in \mathrm{GL}_{I'}(O_w)$, where Ξ is the induction product of Ξ_c and of the functions $\chi_n \circ \det$ in size t_n.

Galois representations. Let E be a CM field inside $\bar{\mathbb{Q}}$, ℓ a prime number and $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ an isomorphism of fields. A *place of* $\bar{\mathbb{Q}}$ *over a prime* w of E is a valuation ring $\mathfrak{P}$ of $\bar{\mathbb{Q}}$ whose maximal ideal meets the integers of E in w; its inertia group consists of the $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/E)$ that preserve $\mathfrak{P}$ and act trivially on its residue field, which is an algebraic closure $\bar{\mathbb{F}}_w$ of $\mathbb{F}_w$.

(g) *Tame exponents.* Let a and $\delta \neq 0$ be natural numbers and $q = (\#\mathbb{F}_w)^\delta$. An element σ of the inertia group of $\mathfrak{P}$ *has tame exponent* a *of level* δ if there are a uniformizer ϖ of E at w and $z \in \bar{\mathbb{Q}}$ with $z^{q-1} = \varpi$ such that the reduction $\zeta \in \bar{\mathbb{F}}_w$ of the root of unity $\sigma(z)/z$ satisfies $T(\zeta) = \exp(2\pi i\, a/(q-1))$.

(h) *Attached representations.* A homomorphism $\rho\colon \mathrm{Gal}(\bar{\mathbb{Q}}/E) \to \mathrm{GL}_{I'}(\bar{\mathbb{Q}}_\ell)$ is *attached to* a family $(\mathrm{Hk}_w)_w$ of complex polynomials if it is continuous, if the trace form $(a, b) \mapsto \mathrm{tr}(ab)$ is non-degenerate on the linear span of its image (so ρ is semisimple), and if for every real $\eta > 0$ the following set of Frobenius data is *large*: the triples $(w, \mathfrak{P}, \sigma)$ of a prime w of prime norm, a place $\mathfrak{P}$ over it and a geometric Frobenius element σ at $\mathfrak{P}$ such that every coefficient of the characteristic polynomial of $\rho(\sigma)$ differs from the corresponding coefficient of $\iota(\mathrm{Hk}_w)$ by at most η in the ℓ-adic absolute value. A set of Frobenius data is large if every non-empty open subset of the Galois group contains a non-empty open subset U such that, for all but finitely many w, all data $(w, \mathfrak{P}, \sigma)$ with $\sigma \in U$ belong to the set.

So "attached" is an approximate notion: it does not say that the characteristic polynomial of a Frobenius element is the Hecke polynomial, only that this holds up to every ℓ-adic precision on a set of Frobenius elements that is dense in a strong sense. By the theorem of Chebotarev (which the library proves in its existence form) and continuity, this still determines the characteristic polynomial of ρ on the whole group, and hence ρ up to isomorphism by the theorem of Brauer and Nesbitt; the proof below needs neither, because Input 2 holds for every attached representation.

3 Inputs

Inputs from the background library. E is a CM field inside $\bar{\mathbb{Q}}$ with Weil data in the library's sense (in particular it contains a primitive root of unity of order 5040); I' is a non-empty finite set; ℓ is a prime and $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ an isomorphism of fields.

Input 1 (attached Galois representations exist). *Let $(\mathrm{Hk}_w)_w$ be a family of complex polynomials and f an algebraic modular form of some weight on the standard space with index set I' that is an eigenform for $(\mathrm{Hk}_w)_w$. Then there is a homomorphism $\rho\colon \mathrm{Gal}(\bar{\mathbb{Q}}/E) \to \mathrm{GL}_{I'}(\bar{\mathbb{Q}}_\ell)$ attached to $(\mathrm{Hk}_w)_w$.*

Input 2 (local-global compatibility at tame level). *Let f be an algebraic modular form of some weight on the standard space with index set I' that is an eigenform for $(\mathrm{Hk}_w)_w$, and let ρ be* any *homomorphism attached to $(\mathrm{Hk}_w)_w$. Let w be a split prime at which f has the product label $(c;\chi;t)$ of degree δ, $\mathfrak{P}$ a place of $\bar{\mathbb{Q}}$ over w, and σ an element of the inertia group of $\mathfrak{P}$ that has tame exponent a of level $\delta!$. Let $Q = \prod_x (X - \iota(T(x)^a))$, the product over the roots x in $\bar{\mathbb{F}}_w$, with multiplicity, of $c \cdot \prod_n (X - x_{\chi_n})^{t_n}$. If ℓ is not the residue characteristic of w, then $Q(\rho(\sigma)) = 0$, and $\det \rho(\sigma)$ is the product of the roots of Q.*

Input 3 (type implies label). *Let w be a split prime, I a finite set, and let $\mathrm{GL}_I(\mathbb{F}_w)$ act linearly on a complex vector space, written $(x,y) \mapsto x \cdot y$; let y be a vector of it, and L an injective linear map from that space to the functions on $G_f(I')$ with values in the functions on a set. Let $c \in \mathbb{F}_w[X]$ with $e_{I,c} \cdot y \neq 0$. Let ν, β, ψ, $\chi_1, \dots, \chi_{n_1}$ be as in Theorem 1*(iii)*, with the χ_n pairwise distinct, and suppose that $L(y)$ has parahoric type at w for (ν,β,ψ,χ) and the functions $x \mapsto L(x \cdot y)$. If no x_{χ_n} is a root of $c^{[x_\psi]}$, then $L(y)$ has the product label $(c^{[x_\psi]};\chi;t)$ of degree $\#I$ at w, with $t_n = \#\beta^{-1}(n)$.*

Input 4 (tame inertia; Hecke polynomials). (a) *For every prime w of E, every place $\mathfrak{P}$ of $\bar{\mathbb{Q}}$ over w and every natural number δ, some element of the inertia group of $\mathfrak{P}$ has tame exponent 1 of level $\delta!$.* (b) *Let w be a split prime and $\Upsilon\colon \mathbb{C} \to \mathbb{C}$ a function. There are complex numbers $t_0, t_1, \dots$ such that $\mathbb{T}^{w,n} f = t_n f$ for all n and every function f on $G_f(I')$ that is spherical at w with Υ in the sense of Theorem 1*(ii)*.* (c) *A prime of E whose norm is a prime number is split.*

LEAN `sphericalCompat_std` · `Attached.isLabelledAt` · `HasParahoricType.hasProductLabel` · `exists_isTameExponent` · `exists_heckeSplit_eq_smul` · `isSplit_of_q_prime`

Input 5 (primitive labels; the two lemmas of the note *Primitive labels*). *Let $\mathbb{F}$ be a finite field, $c \in \mathbb{F}[X]$ a primitive label of degree r.* (a) *If $u \in \mathbb{F}^\times$ and M is a natural number with $u^M = 1$ and M^2 dividing $\#\mathbb{F} - 1$, then $c^{[u]}$ is a primitive label of degree r.* (b) *The product of the roots of c in $\bar{\mathbb{F}}$ is a primitive root of unity of order $\#\mathbb{F} - 1$.*

LEAN `IsPrimitiveLabel.scaleRoots` · `IsPrimitiveLabel.isPrimitiveRoot_prod_roots`

On Inputs 1 and 2. These are the two deep inputs. The references and their relation to the formal statements are those of the catalogue of this handout and of its second reading, which compared the statements with the sources.

Input 1 is a weaker consequence of the construction of Galois representations for automorphic representations of unitary groups: G. Chenevier and M. Harris, *Construction of automorphic Galois representations, II*, Camb. J. Math. 1 (2013), Theorem 3.2.3; T. Barnet-Lamb, T. Gee, D. Geraghty and R. Taylor, *Potential automorphy and change of weight*, Ann. of Math. 179 (2014), Theorem 2.1.1; with base change from the unitary group by J.-P. Labesse, *Changement de base CM et séries discrètes*, Corollaire 5.3. It is weaker in three ways: only the standard definite hermitian space over a CM field with Weil data is treated; "attached" has the approximate meaning of (h) above, in place of the equality of the characteristic polynomial of Frobenius with the Hecke polynomial at almost all places; and the transport between eigenforms on the unitary group, automorphic representations of the general linear group and their weights is part of the library's own argument. The formal statement has a second half, on the local behaviour of attached representations at places where the form is spherical, which is not used in this unit.

Input 2 goes in the direction of local-global compatibility, for which the library's file names M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties* (2001), R. Taylor and T. Yoshida, *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. 20 (2007), Theorem A, and T. Barnet-Lamb, T. Gee, D. Geraghty and R. Taylor, *Local-global compatibility for* $l = p$*, II*, Ann. Sci. ENS 47 (2014), Theorem 1.1. Those theorems identify the Weil–Deligne representation of ρ at w with the local Langlands parameter of the local component. Input 2 retains much less: only split places, only the restriction to tame inertia, only that a polynomial kills $\rho(\sigma)$ together with the value of the determinant, and with the local component described by a finite type (Green's character induced with characters of the determinant) in place of a parameter. The second reading leaves open the identification of this formal statement with a published one. According to the header of the library's file (I read the statement and not its proof), it is reduced there to the spherical case by cyclic base change. In classical terms it says that tame inertia acts through the sum of the characters θ_x, x the roots of the label, composed with the fundamental character of level $\delta!$, as the depth-zero local Langlands correspondence predicts. The formal statement also contains assertions on finite ramification and on Hodge–Tate weights, and a clause for ℓ equal to the residue characteristic, none of which is used here.

Input 3 is Frobenius reciprocity for the parahoric subgroup together with the orthogonality of Green's characters; Input 4(a) is Kummer theory of the tame quotient of inertia; Input 4(b) says that the values of the Hecke polynomial determine the eigenvalues; Input 4(c) holds because a prime fixed by complex conjugation has square norm when E contains a primitive root of unity of order 5040. The library proves these four without a named source.

4 Proofs

Lemma 4.1 (eigenforms from Hecke polynomials off the level)**.** *In Setting 2.1, let I' be a finite set, $\mathcal{W}$ a set, f a function on $G_f(I')$ with values in the complex functions on $\mathcal{W}$, and $(\mathrm{Hk}_w)_w$ a family of complex polynomials indexed by the primes of $E(D)$. Suppose that $f \neq 0$, and that for every split prime w of $E(D)$ such that neither w nor $\bar{w}$ is chosen, f has Hecke polynomial Hk_w at w. Then f is an eigenform for $(\mathrm{Hk}_w)_w$.*

Proof. There are finitely many chosen primes, so all but finitely many primes w are such that neither w nor $\bar{w}$ is chosen. If such a w has prime norm, it is split by Input 4(c), and the hypothesis applies to it. □

LEAN `Data.isEigenform_of_hasHeckePolyAt`

Proof of Theorem 1. Let $c_1, c_2 \in \mathrm{Occ}(D, \mathcal{V}, p, v)$; we show $c_1 = c_2$. Write $q = \#\mathbb{F}_v$ and $r = r(D)$, and let T be the injective character of $\bar{\mathbb{F}}_v^{\times}$. Take $\ell = 2$; an isomorphism of fields $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_2$ exists, because both fields are algebraically closed, of characteristic 0 and of the cardinality of the continuum. Fix a place $\mathfrak{P}$ of $\bar{\mathbb{Q}}$ over v and, by Input 4(a), an element σ of its inertia group with tame exponent 1 of level $r!$. As q is odd, 2 is not the residue characteristic of v.

For $j \in \{1, 2\}$ choose $y_j \in \mathrm{Prin}(D, \mathcal{V})[p]$ such that $e_{I(D),c_j}$, acting through the factor v of $\mathsf{G}(D)$, does not kill y_j; put $f_j = L(y_j)$ and $c_j' = c_j^{[x_\psi]}$.

Step 1: c_j' is a primitive label of degree r. By (v), $x_\psi^{M_\omega(D)} = 1$, and $M_\omega(D)^2$ divides $q - 1$; apply Input 5(a).

Step 2: f_j has the product label $(c_j'; \chi; t)$ of degree r at v, with $t_n = \#\beta^{-1}(n)$. This is Input 3 for the representation $x \mapsto \mathrm{rep}_{D,\mathcal{V}}(x_{(v)})$ of $\mathrm{GL}_{I(D)}(\mathbb{F}_v)$: L is injective by (i), $e_{I(D),c_j}$ does not kill y_j, the parahoric type is (iii), the χ_n are distinct by (iv), and no $x_{\chi_n} \in \mathbb{F}_v$ is a root of c_j', which is irreducible of degree $r > 1$.

Step 3: f_1 and f_2 are eigenforms for one family of polynomials. By Input 4(b) there are, for every split prime w with neither w nor $\bar{w}$ chosen, numbers $t_{w,0}, t_{w,1}, \ldots$ that depend only on Υ_w such that every function spherical at w with Υ_w is an eigenvector of $\mathbb{T}^{w,n}$ with eigenvalue $t_{w,n}$. Let $\mathrm{Hk}_w = \sum_n (-1)^n (\#\mathbb{F}_w)^{n(n-1)/2}\, t_{w,n}\, X^{\#I'-n}$ for these primes, and $\mathrm{Hk}_w = 0$, say, for the others. By (ii), f_1 and f_2 have Hecke polynomial Hk_w at every such w. They are not zero, by the last condition of a product label. So both are eigenforms for $(\mathrm{Hk}_w)_w$, by Lemma 4.1.

Step 4: one Galois representation, two labellings. By Input 1, applied to f_1, there is a homomorphism ρ attached to $(\mathrm{Hk}_w)_w$. Input 2 applies to ρ with f_1 and also with f_2, since it holds for every attached representation. For $j \in \{1, 2\}$ let

$$Q_j = \prod_{x\colon c_j'(x)=0} (X - \iota T(x)) \cdot \prod_{n=1}^{n_1} (X - \iota T(x_{\chi_n}))^{t_n},$$

the first product over the roots of c_j' in $\bar{\mathbb{F}}_v$. Then

$$Q_1(\rho(\sigma)) = 0, \qquad Q_2(\rho(\sigma)) = 0, \qquad \det \rho(\sigma) = \prod_{x\colon c_1'(x)=0} \iota T(x) \cdot \prod_{n=1}^{n_1} \iota T(x_{\chi_n})^{t_n}. \quad (1)$$

Step 5: if $c_1' \neq c_2'$, every eigenvalue of $\rho(\sigma)$ is a root of unity of order dividing $M_\omega(D)$. Let λ be an eigenvalue. By (1) it is a root of Q_1 and of Q_2. Suppose $\lambda = \iota T(x)$ with

x a root of c_1'. As ι and T are injective and λ is a root of Q_2, either x is a root of c_2', or $x = x_{\chi_n}$ for some n. The first is impossible: two different monic irreducible polynomials have no common root. So in every case $\lambda = \iota T(x_{\chi_n})$ for some n, and $\lambda^{M_\omega(D)} = 1$ by (vi), because $\chi \mapsto x_\chi$ is a homomorphism.

Step 6: conclusion. Suppose $c_1' \neq c_2'$. The determinant of $\rho(\sigma)$ is the product of its eigenvalues, so $(\det \rho(\sigma))^{M_\omega(D)} = 1$ by Step 5. In the formula of (1) the second product also has $M_\omega(D)$-th power 1. Hence the product Π of the roots of c_1' satisfies $\iota T(\Pi)^{M_\omega(D)} = 1$, and $\Pi^{M_\omega(D)} = 1$ because ι and T are injective and multiplicative. But by Step 1 and Input 5(b), Π is a primitive root of unity of order $q - 1$, and $0 < M_\omega(D) < q - 1$: a contradiction. So $c_1' = c_2'$, and $c_1 = c_2$, since $c \mapsto c^{[x_\psi]}$ is a bijection. □

5 Where the results are used

Theorem 1 is an input of the note *Good forms and rows*; it gives there that the space of the rows and the sources of the single lifts have rigid labels, which is the hypothesis of the comparison of the traces of Hecke operators on good forms. Lemma 4.1 is used once more, in Chapter VIII, where rows are shown to give eigenforms.

6 Notes on the formal text

1. *Letters.* The formal text writes `x` for the chosen prime v, `v` for the injection ν, `n` for I', `m` and `b` for n_1 and β, `F` for the family Υ, `Q` for the family of Hecke polynomials, `M` for the set $\mathcal{W}$, `r` for the Galois representation ρ, `s` for the inertia element σ, `A` for the place $\mathfrak{P}$, and `Ω` for $\bar{\mathbb{Q}}$.

2. *The functions Υ_w.* In (ii) the value Υ_w is a function of a complex variable, not a list of eigenvalues: the hypothesis prescribes, for every complex X, the scalar by which the Hecke polynomial evaluated at X acts. That the family is common to all forms of the eigenpart is what makes Step 3 work.

3. *The target of L.* The weight, the system of places, the subgroup and the character that determine the space of algebraic forms are implicit arguments; the proof uses only that its elements are algebraic modular forms on the standard space, so that Inputs 1 and 2 apply.

4. *The prime 2.* The formal proof fixes $\ell = 2$ and obtains the isomorphism ι from a lemma of the library on algebraically closed fields of the same cardinality (the catalogue names Steinitz; the second reading derives it from the existence of transcendence bases and the uniqueness of algebraic closures). Any prime other than the residue characteristic of v would do.

5. *The end of the proof.* The formal text closes Steps 5 and 6 with two automatic tactics after a rewriting that turns the claim $\Pi^{M_\omega(D)} = 1$ into the statement that every root of the characteristic polynomial of $\rho(\sigma)$ has $M_\omega(D)$-th power 1; the roots of the characteristic polynomial enter as the multiset whose product is the determinant, and each of them is shown to be a root of Q_1 and of Q_2. I read these lines as Steps 5 and 6 and did not replay the goals. The quantified objects are the eigenvalues of $\rho(\sigma)$: the roots of its characteristic polynomial, in lines 51 and 58 to 59 of the unit's file. The roots of c_1' and c_2' enter twice: as the case excluded

by irreducibility in Step 5, and through their product, by the determinant, in Step 6. An independent referee of this chapter of the handout read the same lines term by term and confirms this reading, and the dossier of the unit, in its corrected state, follows it. The second reader describes the contradiction as a root of c_1' falling among the roots x_{χ_n}; read literally, that route would need every root of Q_1 to be an eigenvalue of $\rho(\sigma)$, which Input 2 does not give. The determinant clause of Input 2 is essential: that Q_1 and Q_2 kill $\rho(\sigma)$ would not suffice.

6. *Lemma 4.1.* In the formal statement the set $\mathcal{W}$ is an arbitrary type and I' need not be non-empty. The family of polynomials is only constrained at the split primes off the level.

7. *What is not used.* The hypotheses (v) and (vi) enter only through the orders of x_ψ and of the x_{χ_n}. The embedding ν and the map β enter only through Input 3.

How this note was written. From the dossier of the unit: its two statements and its two structured proofs, which a program compared line by line with the source (14 leaf steps), and the second reading of the unit by the other agent of this write-up. I read the Lean file of the unit completely, in particular its last thirteen lines term by term, and in the library the definitions of Hecke polynomials, eigenforms, spherical vectors, parahoric types, product labels, tame exponents, attached representations and the predicate that a polynomial kills an element, and the statements of Inputs 1 to 4. The two statements of this note were compared with their Lean statements argument by argument, and the order of the quantifiers is that of the formal text. The proof follows the formal one; its division into six steps is mine. The references are those of the library's files, of the catalogue of this handout and of its second reading; I did not open the cited works.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Lemma 4.1	`Data.isEigenform_of_hasHeckePolyAt`	LabelUniqueness.lean	13–16
Theorem 1	`Data.occurringLabels_subsingleton`	LabelUniqueness.lean	23–59
Input 1	`sphericalCompat_std`	SphericalAllRanks.lean	16–88
Input 2	`Attached.isLabelledAt`	LabelCompatibility.lean	24–84
Input 3	`HasParahoricType.hasProductLabel`	ParahoricTypeLabel.lean	31–56
Input 4	`exists_isTameExponent`	TameExponent.lean	60–79
Input 4	`exists_heckeSplit_eq_smul`	SphericalHecke.lean	255–258
Input 4	`isSplit_of_q_prime`	InertResidue.lean	31–32
Input 5	`IsPrimitiveLabel.scaleRoots`	BadShare.lean	35–41
Input 5	`IsPrimitiveLabel.isPrimitiveRoot_prod_roots`	BadShare.lean	45–51

The two statements, and the two deep inputs, as they are written in the artifact.

```
lemma Data.isEigenform_of_hasHeckePolyAt {M} {f : Gf _ n → M → ℂ} (h0 : f ≠ 0) {Q} (h : ∀ w
    (h : D.IsSplitOffLevel w), HasHeckePolyAt f w h.1 (Q w)) : IsEigenform f Q
```

```
lemma Data.occurringLabels_subsingleton {v : _ ↬ n} {S J x' κ v} {L : _ →ₗ[ℂ] LevelPlaces.LevelForm
  S J x' κ v} [Nonempty n] (hL : (⇑L).Injective) {p} {F : _ → _} (hF : ∀ y ∈ D.eigenPart V p, ∀ w
  (h : D.IsSplitOffLevel w), IsSphericalWith (rightRegular ..) h.1 (F w) (L y).toSmoothForm.1) x {m
  b} x ψ (hk : ∀ y, HasParahoricType (s := m) (L y).toSmoothForm.1 (D.isConjApart_chosen.2 x x) v b
```

```
  ψ x fun g ↦ (L (D.principalRep V (Pi.mulSingle x g) y)).1.1) (hx : x.Injective) (hψ : ψ ^ P.Mω D.s
  = 1) (hM : ∀ l, x l ^ P.Mω D.s = 1) : (D.occurringLabels V p x).Subsingleton

lemma sphericalCompat_std {ℓ E n} [ThePrime ℓ] [PadicIso ℓ] [Field E] [EmbeddedNumberField E]
  [CMType E]
  [Fintype n] [Nonempty n] [WeilData E] w : SphericalCompat E n (card n) w

lemma Attached.isLabelledAt {ℓ} [ThePrime ℓ] [PadicIso ℓ] {E n} [Nonempty n] [Field E] [CMType E]
  [EmbeddedNumberField E] [Fintype n] {κ f P} {r :
  GalRep ℓ E n} {w A hw K} [Field K] [Algebra (ResField w) K] {T} [IsFaithfulMulChar K T] {s
  d c} {x : Fin s → MulChar (ResField w) ℂ} {t} [WeilData E] (hf : f ∈ (stdSpace E n).forms κ) (he :
  IsEigenform f P) (hr : Attached P r) (h : HasProductLabel f hw T d c x t) (hA : IsPlaceOver A w) :
  IsFiniteTypeAt r A ∧ HasJumpsAt r A (jumps κ (Fintype.card n)) ∧
  IsLabelledAt A T d.factorial (fun s Q ↦ PolyKillsAt r A s Q ∧ (A.valuation
  ℓ = 1 → (r s).1.det = Q.roots.prod)) (c * ∏ l, (X - C (charName T (x l)).1) ^ t l)
```

The first two stand after a line that declares the variables `D`, `s`, `V` and the finite type `n`. In these texts `IsSphericalWith`, `HasParahoricType`, `HasProductLabel`, `IsEigenform`, `Attached` are the notions (ii), (iii), (f), (c) and (h) above; `IsLabelledAt` with `PolyKillsAt` is the conclusion of Input 2; `SphericalCompat`, in its first half, is Input 1; `P.Mω D.s` is $M_\omega(D)$.

Notation

$P = (N_0, d)$, D the parameters; admissible data (Setting 2.1)

E, $E(D)$, $\bar{\mathbb{Q}}$ a CM field; the field of the data; an algebraic closure of $\mathbb{Q}$

w, v, $\bar{w}$, $\mathbb{F}_w$, E_w, O_w primes (v a chosen one), the conjugate prime, residue field, completion, its integers

$r(D)$, $m(D)$, $I(D)$, $\mathsf{G}(D)$, $M_\omega(D)$ the rank, the number of lifts, the row index set, the finite group and the integer of the data

$\mathfrak{s}$, $\mathcal{V}$ a family of signs; a space with a point

$\mathrm{Prin}(D, \mathcal{V})$, $\mathrm{rep}_{D,\mathcal{V}}$, $\mathrm{Prin}(D, \mathcal{V})[p]$, $\mathrm{Occ}(D, \mathcal{V}, p, v)$ principal forms, the action of $\mathsf{G}(D)$, eigenparts, occurring primitive labels (note *Good forms and rows*)

p an eigensystem

I, I', $\mathrm{GL}_{I'}$, $G_f(I')$, $K_0(I')$, g^* finite index sets; matrix groups; the unitary group over the finite adeles and its integral points; conjugate transpose

$k^{(w)}$, $\bar{k}$, $\bar{k}_{nn}$, $x_{(v)}$, $\tilde{x}$ the element with component k at w; reduction; its diagonal blocks; the element with component x at v; a lift

$R(g)$, f, f_j, $\mathcal{W}$ right translation; functions on $G_f(I')$ with values in functions on $\mathcal{W}$

$\mathbb{T}^{w,n}$, Hk_w, Q, t_n, $t_{w,n}$ Hecke operators, Hecke polynomials, eigenvalues

L, ν, β, n_1, Υ_w, Υ the data of Theorem 1

ψ, χ, χ_n, x_χ, x_ψ multiplicative characters of $\mathbb{F}_v$ and the units that name them

$\mathbb{F}$, $\bar{\mathbb{F}}$, T, θ_x, Ξ_c, Ξ, $e_{I,c}$, $c^{[u]}$ a finite field, its closure, the injective character, the character named by x, Green's class functions, idempotents, scaled polynomials (note *Primitive labels*)

c, c_j, c'_j, X, $t = (t_n)$, δ, M, u labels and their twists; the variable; sizes of blocks; a degree or level; an exponent; a unit

κ, alg_κ, γ a weight, its representation, a rational point

ℓ, ι, $\bar{\mathbb{Q}}_\ell$ a prime, an isomorphism $\mathbb{C} \to \bar{\mathbb{Q}}_\ell$, an algebraic closure of the ℓ-adic numbers

$\mathfrak{P}$, σ, ϖ, z, ζ, a, q a place of $\bar{\mathbb{Q}}$, a Galois element, a uniformizer, a root of it, a root of unity; a tame exponent; the order of a finite field

ρ, Q_j, λ, Π, η, U a Galois representation, the polynomials of the two labellings, an eigenvalue, the product of the roots of c'_1; a precision; an open set

y, y_j, $x \cdot y$ vectors, forms; the action of a finite group

n, j, i, x, g, k, b, ι', π indices, elements of fields and groups; in $2\pi i$ the circle constant and the imaginary unit

References

All references are given as the library's files, the catalogue of this handout and its second reading record them; I did not open the works.

T. Barnet-Lamb, T. Gee, D. Geraghty, R. Taylor, *Potential automorphy and change of weight*, Ann. of Math. 179 (2014); Theorem 2.1.1.

T. Barnet-Lamb, T. Gee, D. Geraghty, R. Taylor, *Local-global compatibility for $l = p$, II*, Ann. Sci. ENS 47 (2014); Theorem 1.1 and Corollary 1.2.

G. Chenevier, M. Harris, *Construction of automorphic Galois representations, II*, Camb. J. Math. 1 (2013); Theorem 3.2.3.

M. Harris, R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, 2001.

J.-P. Labesse, *Changement de base CM et séries discrètes*, 2011; Corollaire 5.3.

J. S. Milne, *Class Field Theory*, Chapter VIII, Theorem 7.4, for the theorem of Chebotarev.

R. Taylor, T. Yoshida, *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. 20 (2007); Theorem A.

Chapter VIII

The twists: congruences, Selmer groups, lifting

- **The choice of the twists: Galois representations of lifts, Selmer groups made short by one anticyclotomic twist, and congruence bounds by Ribet's method** (unit 138)
  read: note built, structured proofs checked. 205 declarations, 2\,152 proof lines.
- **A Taylor–Wiles prime at a time: the deformed trace at one auxiliary place clears the denominator** (unit 137)
  read: note built, structured proofs checked. 10 declarations, 262 proof lines.
- **Lifting crystalline extension classes in large weight: Kisin modules with exponents, cokernel killers and cup products that vanish up to a power of p** (unit 136)
  read: note built, structured proofs checked. 141 declarations, 870 proof lines.

READBACK NOTE 31

Columns and rows as Galois representations: labels at the chosen places, anticyclotomic characters at new primes, families of L-values and the crystalline loss at one place

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The twists of Chapter VIII are chosen on the Galois side. For admissible data D and a prime ℓ this note sets up that side. It defines the ℓ-adic representations that play the part of columns and rows (conjugate self-dual, with prescribed Hodge–Tate jumps, labelled at the chosen primes by the roots of polynomials over the residue fields), the bound on Selmer modules that a good twist must achieve, the systems of auxiliary primes (anticyclotomic data, pencils, ties, prime triangles), and the loss that the crystalline condition at ℓ costs. Five results are proved for the two notes that follow: an anticyclotomic character of order ℓ^N ramified at one new prime, trivial at a Frobenius element there that lies in a prescribed open set (Theorem A); labelled columns of row type are rows (Theorem B); the forms cut out by an apt idempotent lie in a base level whose eigensystems are labelled columns with a quadratic relation on the inertia at the twist primes (Theorem C); the inverse ratios of twisted L-values are controlled by families of interpolated values (Theorem D); and at one place above ℓ the crystalline cocycles of a block lie, up to an explicit loss, in the cup-orthogonal of a class of lattices (Theorem E).

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles and that Lean's kernel accepted the proofs; an independent replay is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note is the first of three on the unit *The choice of the twists* (the folder of the formal text on Selmer groups): it treats 116 of the unit's 205 declarations, those of the first fifteen files. The other declarations are the subject of the notes *Short Selmer groups after one anticyclotomic twist* and *A fit family of twists exists.* All definitions are written out; the lemmas are proved from the inputs named in the proofs, which are results of the background library or of other units and are quoted with their owners. Where a proof below is an outline of a long formal computation, it says so. Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 116 declarations.

1 The statements

The notions are defined in §2 (background) and §3 (the definitions of the unit); the conventions are at the head of §2. Throughout, ℓ is a prime number and $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ a fixed isomorphism of fields.

Theorem A (an anticyclotomic character at one new prime). *Let E be a CM field as in the conventions, N a natural number, S a finite set of finite places of E and R a finite set of finite places of E. Let $z \in G_E$ lie in the normal subgroup of G_E generated by the inertia $I_{R\setminus(S\cup\bar{S}\cup\Sigma_\ell(E))}$, and let W be a neighbourhood of z in G_E. Then there are a finite place $\mathfrak{q} \notin R \cup \bar{R}$ of E whose norm $\mathrm{N}(\mathfrak{q})$ is a prime number, and a datum $l = (\mathfrak{P}, \gamma, \phi, h) \in \mathrm{Aux}(E, \ell^N)$, such that*

$$\phi \in W, \qquad h(\phi) = 0, \qquad \text{and} \quad \mathsf{Acyc}(T, \mathfrak{q};\, l) \ \text{ for every set } T \subseteq S \cup \Sigma_\ell(E).$$

LEAN `exists_isAnticyclotomicAt`

Theorem B (labelled columns of row type are rows). *Let P be parameters, D admissible data for P, and r, n finite sets. Let O be an auxiliary pair of type (r, n) over $E(D)$, ϑ a twist of D and $c = (c_i)_i$ a family of polynomials over the residue fields of the chosen places of O with $\mathsf{Apt}(O, c;\, \vartheta)$. Let $R\colon G_{E(D)} \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$ be a homomorphism, k a natural number and $\mathcal{H}$ a type. If $\mathsf{LabCol}(O;\, k, \mathcal{H}, \emptyset;\, \mathrm{tl}_O(c, \cdot);\, R)$ and $\mathsf{RowType}_D(\#r, k;\, \mathcal{H})$, then $\mathsf{RowRep}_D(k, \mathcal{H};\, R)$.*

LEAN `IsLabelledColumn.isRow`

Theorem C (a base level for an apt idempotent). *Let P, D be as in Theorem B, r a finite set and n a non-empty finite set, O an auxiliary pair of type (r, n) over $E(D)$, ϑ a twist of D with set of places $Q(\vartheta)$, and c a family of polynomials with $\mathsf{Apt}(O, c;\, \vartheta)$. Suppose $\mathsf{Auxil}(\Sigma_{\mathrm{bad}}(D, \ell);\, Q(\vartheta))$. Then there are a type $\mathcal{H}$ and a base level B of forms of the weight of O on the standard definite hermitian space of rank $\#n$, with bound $\mathrm{ord}_{\mathrm{ch}}(D)$, such that:*

1. *every form in the image of $\mathrm{pr}_O(c)$ is a form of B of level $\emptyset$;*
2. *for every form y of O that is ℓ-integral, $\mathrm{ord}_{\mathrm{ch}}(D) \cdot \mathrm{pr}_O(c)(y)$ is ℓ-integral;*
3. *for every finite set K, every family $u = (u_k)_{k\in K}$ of finite places of $E(D)$ none of which lies in $\Sigma_{\mathrm{bad}}(D, \ell)$, and every eigensystem x of B of level u, the representation $R = \mathrm{nm}_O(\varrho_x)$ satisfies, with the set of places $S_u = Q(\vartheta) \cup Q(\bar{\vartheta}) \cup u(K) \cup u(\bar{K})$ and the polynomials $c'_i = \mathrm{tl}_O(c, i) \cdot (X-1)^{\#n - \#r}$,*

$$\mathsf{LabCol}(O;\, \#n - \#r, \mathcal{H}, S_u;\, (c'_i)_i;\, R)$$

and $(R(g) - \mathrm{av}^-_\vartheta(g))(R(g) - 1) = 0$ for every $g \in I_{Q(\vartheta)\cup Q(\bar{\vartheta})}$.

LEAN `exists_baseLevel`

Theorem D (families of values control ratios of L-values). *Let P, D be as in Theorem B, w a finite place of $E(D)$ that is chosen and does not lie above ℓ, p a pair of D that is ramified, and $s \in \{1,-1\}$ a sign. Then there are a family $\nu = (\nu_i)_{i\in\{0,1\}\times\{1,-1\}}$ of homomorphisms $G_{E(D)} \to \bar{\mathbb{Q}}_\ell^\times$ and a function* val *with* $\mathsf{Fam}(\Sigma_{\mathrm{bad}}(D,\ell);\, \nu, \mathrm{val})$ *such that*

(1) $v_\ell(\nu_i(\tau_w) - 1) \le v_\ell(\mathrm{N}(w) - 1)$ *for every i, where τ_w is the chosen tame generator at the chosen place of $\bar{\mathbb{Q}}$ over w;*

(2) *for every twist τ of D with* $\mathsf{Auxil}(\Sigma_{\mathrm{bad}}(D,\ell);\, Q(\tau))$ *and every real number $a \ge 0$: if $v_\ell(\mathrm{val}(Q(\tau), \mathrm{av}_\tau^-(\cdot))) \le a + v_\ell(\mathrm{val}(\emptyset, 1))$, then $v_\ell(\iota(\varrho_p(\tau, s))) \le a$, where $\varrho_p(\tau, s)$ is the ratio of L-values of p at the twist τ^s.*

LEAN `exists_isFamily`

Theorem E (the loss at one place above ℓ). *Let P, D be as in Theorem B. Let Γ be a topological group with a homomorphism $j\colon \Gamma \to G_{\mathbb{Q}_\ell}$ of open image, and $e\colon E(D) \to \bar{\mathbb{Q}}_\ell$ a ring homomorphism such that j is an open embedding whose image is the group of the elements of $G_{\mathbb{Q}_\ell}$ that fix $e(E(D))$ pointwise. Let r, t be natural numbers and $\mathcal{H}$ a type with* $\mathsf{RowType}_D(r, t;\, \mathcal{H})$. *Then for all large coefficient rings $\mathcal{O}$ the following holds. Let $(L_i)_i$ be a family of Galois lattices over $\mathcal{O}$ for Γ indexed by $i \in \{*\} \sqcup \{0, \dots, t-1\}$, let F be a filter on Γ, K a relation between the elements of Γ and the polynomials over $\mathcal{O}$, and U a function from the $\mathbb{Z}_\ell$-algebra homomorphisms $\mathcal{O} \to \mathbb{C}_\ell$ to the finite sets of integers. Suppose that*

(i) *either $\ell = g_\nu(D)$ for an index ν and, for every $s \in \Gamma$ with $j(s)$ in the inertia group of $G_{\mathbb{Q}_\ell}$, the property $K(s,\cdot)$ is of tame type $(\ell, r(D), (r(D) + t(D))r(D))$; or $F = \top$ and K holds for no pair;*

(ii) *for every homomorphism $g'\colon \mathcal{O} \to \mathbb{C}_\ell$ and every $a \in U(g')$, some L_i has the jump a at g';*

(iii) *for every $g \in G_{\mathbb{Q}_\ell}$, the lattice L_* has the jumps $\mathcal{H}(\mathrm{ce}(e,g))$ at $\mathrm{emb}_{\mathcal{O}}(g)$, and for $c < t$ the lattice L_c has the single jump c at $\mathrm{emb}_{\mathcal{O}}(g)$;*

(iv) $\mathrm{rk}_{\mathcal{O}}\, L_* = r$.

Then for every index c, with $x = 1$ if $c = *$ and $x = t(D)$ otherwise: if L_c belongs to the class $\mathcal{Q}(F, K, U)$, then

$$\mathrm{nlen}_{\mathcal{O}}\Big(Z_{\mathrm{f}}(L_c^\vee(1))\Big/ Z_{\mathrm{f}}(L_c^\vee(1)) \cap (L_c^\vee(1))^{\perp_{\mathcal{Q}(F,K,U)}}\Big) \;\le\; [G_{\mathbb{Q}_\ell} : j(\Gamma)] \cdot \mathrm{rk}_{\mathcal{O}}\, L_c \cdot \frac{\mathrm{loss}_D(\ell)}{x}.$$

LEAN `block_defect_le_at_place`

Theorem A is the source of the tied primes of the pencils in the second note; the Taylor–Wiles primes of the third note and the two further primes of its lemma on reciprocity laws come directly from Chebotarev's theorem. The proof of Theorem A uses Chebotarev's theorem (existence form) and the existence theorem of class field theory. Theorems B and C turn the automorphic input (the representations attached to eigenforms on definite unitary groups, with their local behaviour) into the two predicates on which Ribet's method works in the following notes. Theorem D is used to bound the heights of ratios of L-values in the last clause of a fit family. Theorem E

is the local half of the statement that "crystalline losses above ℓ are affordable"; the global half is in the third note.

2 Conventions and background

Conventions. The natural numbers are $0, 1, 2, \ldots$; differences of natural numbers written in exponents and indices are truncated at 0 where the formal text truncates them, and this is said at each place. Throughout, ℓ is a prime number; $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell$ are the ℓ-adic numbers and integers, $\bar{\mathbb{Q}}_\ell$ is an algebraic closure of $\mathbb{Q}_\ell$ with the absolute value $|\cdot|$ normalised by $|\ell| = 1/\ell$, $\bar{\mathbb{Z}}_\ell$ its valuation ring, $\mathbb{C}_\ell$ its completion, and $G_{\mathbb{Q}_\ell} = \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. Where a statement involves complex numbers and ℓ-adic numbers together, $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ is a fixed isomorphism of fields. $\bar{\mathbb{Q}}$ is a fixed algebraic closure of $\mathbb{Q}$, and c a fixed complex conjugation in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. A *CM field as in the conventions* is a CM number field E (also written E', E_1) with a fixed embedding in $\bar{\mathbb{Q}}$ that contains a primitive root of unity of order 5040; $G_E = \mathrm{Gal}(\bar{\mathbb{Q}}/E)$ with its profinite topology. In some statements E is only a number field with a fixed embedding in $\bar{\mathbb{Q}}$; this is then said. P denotes parameters and D admissible data for P (Chapter III); its field $E(D)$ is a CM field as in the conventions (Lemma 5.1(a)). The letters n, r, m, d, σ, J, X, T, K, when they index matrices or families, denote finite sets (the letter d is the prime of the parameters where it stands with N_0, $N_1(D)$ or under a square root; as an index set and as an exponent, $1 \le d \le \ldots$, it is a bound variable of its statement, as in the formal text; in Definition 3.1 and in Definitions 3.28, 3.34, 3.35 the sets X, J and the index set of the slopes are arbitrary sets); $\#n$ is the number of elements, $M_n(A)$ the ring of matrices over a commutative ring A with rows and columns indexed by n, and $\mathrm{GL}_n(A)$ its group of units. A *coefficient ring* is the ring of integers $\mathcal{O}$ (also $\mathcal{O}'$) of a finite Galois extension of $\mathbb{Q}_\ell$ inside $\bar{\mathbb{Q}}_\ell$; "for all large $\mathcal{O}$" means: there is a coefficient ring $\mathcal{O}_0$ such that the assertion holds for all coefficient rings $\mathcal{O} \supseteq \mathcal{O}_0$. A, Λ denote commutative rings, Γ a group (a topological group where continuity is mentioned), and B a module.

The rest of this section recalls, in words, the notions of the background library and of other units in which the declarations of the unit are stated. They are not declarations of this unit; their formal definitions are in the files named in the concordances of the notes of their units.

2.1. Places, inertia, Frobenius elements. Let E be a number field embedded in $\bar{\mathbb{Q}}$. A *finite place* w (also v, $\mathfrak{q}$) of E is a non-zero prime ideal of its ring of integers; $\mathrm{N}(w)$ is its norm, $\mathbb{F}_w$ its residue field. For a natural number M, $\Sigma_M(E)$ is the set of finite places of E that contain M. When E is a CM field, $\bar{w}$ is the complex conjugate place, $\bar{S} = \{\bar{w} : w \in S\}$ for a set S of places, and w is *split* if $\bar{w} \ne w$; a place of prime norm is split. A *place of* $\bar{\mathbb{Q}}$ is a valuation subring $\mathfrak{P}$ of $\bar{\mathbb{Q}}$; it *lies over* w if an integer of E belongs to the maximal ideal of $\mathfrak{P}$ exactly when it belongs to w. For each w the library fixes one place $\mathfrak{P}_w$ of $\bar{\mathbb{Q}}$ over w. The decomposition group $G_{\mathfrak{P}} \subseteq G_E$ consists of the g with $g(\mathfrak{P}) = \mathfrak{P}$, the inertia group $I_{\mathfrak{P}}$ of those which moreover act trivially on the residue field of $\mathfrak{P}$; for a set S of finite places, I_S is the union of the $I_{\mathfrak{P}}$ over the places $\mathfrak{P}$ of $\bar{\mathbb{Q}}$ lying over a place of S. For a natural number q, an element $\phi \in G_{\mathfrak{P}}$ is a *q-Frobenius at* $\mathfrak{P}$ if $\phi(x) \equiv x^q$ modulo the maximal ideal of $\mathfrak{P}$ for all $x \in \mathfrak{P}$; for $q = \mathrm{N}(w)$ this is an arithmetic Frobenius element. For $\mathfrak{P}$ over w

the library fixes a *tame generator* $\tau_{\mathfrak{P}} \in I_{\mathfrak{P}}$: the local Artin symbol of a unit of the completion E_w whose residue generates $\mathbb{F}_w^\times$; we write $\tau_w = \tau_{\mathfrak{P}_w}$. $\mathfrak{P}^{\mathsf{c}} = \mathsf{c}^{-1}(\mathfrak{P})$ is the conjugate place. For a CM field, $g^{\mathsf{c}} = \mathsf{c}g\mathsf{c}$ for $g \in G_E$, and for a sign b we put $g^{[+]} = g$ and $g^{[-]} = (g^{\mathsf{c}})^{-1}$. $\mathrm{cyc}\colon G_E \to \mathbb{Z}_\ell^\times$ is the ℓ-adic cyclotomic character, read in any $\mathbb{Z}_\ell$-algebra; $\mathrm{cyc}_M\colon G_E \to (\mathbb{Z}/M)^\times$ is the action on the M-th roots of unity. The *Artin map* Art sends the finite ideles of E to the abelianisation of G_E; we write $\mathrm{Art}(x) = g$ when g has the class $\mathrm{Art}(x)$.

2.2. Characters from U(1)**.** Let E be a CM field as in the conventions, S and I sets of finite places of E, and $\chi\colon G_E \to C$ a homomorphism to a commutative monoid. Then χ is *from* U(1) *relative to* S *with modulus* I if χ is locally constant and there is a homomorphism η from the finite ideles of E to C, trivial on the principal ideles and on every idele that is a unit at all places outside $S \cup \bar{S}$ and is congruent to 1 at the places of $I \setminus (S \cup \bar{S})$, such that $\chi(g) = \eta(x/\bar{x})$ whenever $\mathrm{Art}(x) = g$. So such a character is anticyclotomic, trivial on the decomposition groups above $S \cup \bar{S}$, at most tamely ramified at I and unramified elsewhere.

2.3. Relations and Selmer modules. Let Γ be a group (or a set), n a finite set and A a commutative ring. A *relation* is a finitely supported function $\theta\colon \Gamma \to M_n(A)$; a function $c\colon \Gamma \to B^n$ to the n-th power of an A-module *obeys* θ if $\sum_g \theta(g)\, c(g) = 0$. For a set L of relations, $\mathrm{Ob}(L; B)$ is the module of the functions that obey all of them; for a module Z of functions $\Gamma \to A^n$, $\mathrm{Rel}(Z)$ is the set of the relations that all elements of Z obey. For a map $\varrho\colon \Gamma \to M_n(A)$, $\mathrm{Coc}(\varrho)$ is the set of the relations $\delta_{gh} - \delta_g - \varrho(g)\delta_h$, where δ_g is the function with value the identity matrix at g and 0 elsewhere; a function obeys them exactly when it is a cocycle, $c(gh) = c(g) + \varrho(g)c(h)$. For $\Gamma = G_E$, a set S of finite places and a set L of relations, the *Selmer module* $\mathrm{Sel}_S^L(\varrho; B)$ is the module of the locally constant cocycles $G_E \to B^n$ for ϱ that vanish on $I_{S^{\mathsf{c}}}$, the inertia above the places outside S, and obey L. For a place $\mathfrak{P}$ of $\bar{\mathbb{Q}}$, the module of *local cocycles* at $\mathfrak{P}$ consists of the restrictions to $G_{\mathfrak{P}}$ of the continuous cocycles of G_E for ϱ with values in B^n that obey L. For $a \in A$ we write A/a for the quotient by the ideal generated by a. len_A is the length of an A-module; for a coefficient ring $\mathcal{O}$ with uniformiser ϖ, $\mathrm{nlen}_{\mathcal{O}} = v_\ell(\varpi) \cdot \mathrm{len}_{\mathcal{O}}$ is the normalised length, so that $\mathrm{nlen}_{\mathcal{O}}(\mathcal{O}/\ell) = 1$. $v_\ell\colon \bar{\mathbb{Q}}_\ell \to [0, \infty]$ is the valuation $-\log_\ell |x|$, truncated: it is ∞ at 0 and 0 when $|x| \geq 1$. For a coefficient ring $\mathcal{O}$ and $e \in \mathbb{N}$, $\mathcal{O}(-e)$ is the representation $g \mapsto \mathrm{cyc}(g)^{-e}$ of G_E on $\mathcal{O}$, a homomorphism to $M_1(\mathcal{O})$.

2.4. Representations and their local conditions. Let E be a CM field as in the conventions and n a finite set. We consider homomorphisms $R\colon G_E \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ (continuity is said when it is assumed). A homomorphism $u\colon G_E \to M_n(\mathcal{O})$ is a *lattice of* R if $u(g) = Y^{-1}R(g)Y$ for some $Y \in \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ and all g. The *trace form of* R *is non-degenerate* if an element x of the linear span of $R(G_E)$ with $\mathrm{tr}(x\, R(g)) = 0$ for all g is zero. Let $\mathfrak{P}$ be a place of $\bar{\mathbb{Q}}$. An *ℓ-adic chart at* $\mathfrak{P}$ is an embedding $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_\ell$ that induces $\mathfrak{P}$, together with the open embedding $G_{\mathfrak{P}} \to G_{\mathbb{Q}_\ell}$ that it defines; it exists when $\mathfrak{P}$ lies above ℓ. R is *of finite type at* $\mathfrak{P}$ if it is trivial on the elements of $I_{\mathfrak{P}}$ near 1 when ℓ is a unit at $\mathfrak{P}$, and potentially crystalline along every ℓ-adic chart at $\mathfrak{P}$. R is *good at* $\mathfrak{P}$ if it is trivial on $I_{\mathfrak{P}}$ when ℓ is a unit at $\mathfrak{P}$, and crystalline along every chart at $\mathfrak{P}$. A *type* is a function $\mathcal{H}$ from the complex embeddings of E to the finite multisets of integers. R *has the jumps* $\mathcal{H}$ *at* $\mathfrak{P}$ if along every chart at $\mathfrak{P}$, for every embedding

$E \to \bar{\mathbb{Q}}_\ell$ compatible with the chart, read as a complex embedding σ through ι, the Hodge–Tate jumps of R, with multiplicities, form the multiset $\mathcal{H}(\sigma)$. For $s \in G_E$ and a polynomial Q over $\bar{\mathbb{Q}}_\ell$, *Q kills s on R at $\mathfrak{P}$* if $Q(R(s)) = 0$ when ℓ is a unit at $\mathfrak{P}$, and if a corresponding condition on the crystalline Frobenius holds along every chart at $\mathfrak{P}$ when $\mathfrak{P}$ lies above ℓ. Let $\mathfrak{P}$ lie over a finite place w, let T be an injective character $\bar{\mathbb{F}}_w^\times \to \mathbb{C}^\times$ (the library fixes one, the *faithful character* of $\mathbb{F}_w$) and $f \geq 1$. An element $s \in I_{\mathfrak{P}}$ has *tame exponent a* (for T and f) if for some z with $z^{\mathrm{N}(w)^f-1}$ a uniformiser of E at w, T sends the residue of $s(z)/z$ to $\exp(2\pi i\, a/(\mathrm{N}(w)^f - 1))$. For a polynomial c over $\mathbb{F}_w$, R is *labelled by c at $\mathfrak{P}$* (for T and f) if for every s of tame exponent a the polynomial $\prod_x (X - \iota(T(x)^a))$, the product over the roots x of c in $\bar{\mathbb{F}}_w$ with multiplicities, kills s on R at $\mathfrak{P}$. For a multiplicative character χ of $\mathbb{F}_w$, $x_\chi \in \mathbb{F}_w^\times$ is the element that names it: $T^a(x_\chi) = \chi(u)$ when s has tame exponent a and is the local Artin symbol of a unit with residue u; $\chi \mapsto x_\chi$ is an isomorphism of groups.

2.5. Crystalline cocycles and classes of lattices. A *group over $G_{\mathbb{Q}_\ell}$* is a group Γ with a homomorphism $j\colon \Gamma \to G_{\mathbb{Q}_\ell}$ of open image. A *Galois lattice over $\mathcal{O}$ for Γ* is a finite free $\mathcal{O}$-module L with an $\mathcal{O}$-linear action of Γ; $\mathrm{rk}_{\mathcal{O}} L$ is its rank. $Z_{\mathrm{f}}(L)$ is the $\mathcal{O}$-module of the *crystalline cocycles*: the maps $c\colon \Gamma \to L$ for which some y in $B_{\mathrm{cris}} \otimes L$ has $1 \otimes c(g) = g\,y - y$ for all g (Colmez's ring B_{max} serves as period ring). L is *potentially crystalline* if the periods fixed by the g with $j(g)$ near 1 span $B_{\mathrm{cris}} \otimes L$. For a $\mathbb{Z}_\ell$-algebra homomorphism $g'\colon \mathcal{O} \to \mathbb{C}_\ell$ and an integer u, the *multiplicity of the jump u of L at g'* is the dimension of the space of g'-semilinear maps $L \to \mathbb{C}_\ell$ on which Γ acts, near 1, through cyc^{-u}; *L has the jumps $\mathfrak{M}$ at g'*, for a multiset $\mathfrak{M}$ of integers, if these multiplicities are the multiplicities in $\mathfrak{M}$. $L^\vee(1) = \mathrm{Hom}_{\mathcal{O}}(L, \mathcal{O}(1))$ is the twisted dual. For a filter F on Γ, a relation K between elements of Γ and polynomials over $\mathcal{O}$, and a function U from the homomorphisms $\mathcal{O} \to \mathbb{C}_\ell$ to the finite sets of integers, the class $\mathcal{Q}(F, K, U)$ consists of the Galois lattices Y whose periods fixed near 1 and by F-almost all elements span, whose jumps at each g' lie in $U(g')$, and on which Q kills s whenever $K(s, Q)$ holds and $j(s)$ lies in the inertia group. For $F = \top$ and K empty this is the class of the crystalline lattices with jumps in U. A property of polynomials is *of tame type* (ℓ, ϱ, M) if it holds for $X^{M'} - 1$, with M' the least common multiple of the $\ell^i - 1$, $1 \leq i \leq \varrho$, and for some monic polynomial over $\mathbb{Z}_\ell$ of degree at most M. For a class $\mathcal{Q}$ and a lattice L, the *cup-orthogonal* $L^{\perp\mathcal{Q}}$ is the $\mathcal{O}$-module of the maps $c\colon \Gamma \to L$ whose cup product with every crystalline cocycle of every $Y \in \mathcal{Q}$, under every pairing of Y and L modulo a power of ℓ with values in $\mathcal{O}(1)$, is a coboundary. These notions belong to the units on ℓ-adic Hodge theory and on lifting (Chapter VIII, the note on crystalline lifts); only their names enter the proofs below.

2.6. Characters of $\mathrm{U}(1)$, forms and labels. Let E be a CM field with its Weil data (Chapter VII), which include a Hecke character υ of E with $\upsilon(\varpi_w)\upsilon(\varpi_{\bar{w}}) = 1$ at the chosen uniformisers. A *character of* $\mathrm{U}(1)$ is a continuous character μ of the norm-one adeles of E, trivial on the norm-one elements of E; it has *weight* $e = (e_v)_v$ (integers indexed by the infinite places) and *tame part* $\psi = (\psi_w)_w$ (characters of the groups $\mathbb{F}_w^\times$, almost all trivial) if $\mu(u) = \prod_v u_v^{e_v} \prod_w \psi_w(u \bmod w)$ for u with integral finite part. For integers k, j the *algebraic Hecke character* $\mu[k, j]$ is $x \mapsto \mu(x/\bar{x})\, \upsilon(x)^{-k}\, |x|^{j-k/2}$, and its *avatar* $\mathrm{av}(\mu[k, j])\colon G_E \to \bar{\mathbb{Q}}_\ell^\times$ is the continuous

character with $\mathrm{av}(\mu[k,j])(\mathrm{Art}(x)) = \iota(\mu[k,j](x))$ times the usual factor at the places above ℓ. For admissible data D (Chapters III and V): a *tame character* ϑ of D is a character of $\mathrm{U}(1)$ of weight 0 whose tame part ψ_ϑ is trivial off a finite set $Q(\vartheta)$ of finite places, with $Q(\vartheta)$ disjoint from its conjugate and of norms prime to $g(D)N(D)$; a *twist* is a tame character whose tame part is non-trivial at every place of $Q(\vartheta)$. A *pair* $p = (\nu_p, z_p)$ of D consists of an integer $1 \le z_p \le t(D)$ and a character ν_p of $\mathrm{U}(1)$ with tame part supported on the chosen places; it is *ramified* if its tame part is non-trivial exactly at the chosen places, and $\varrho_p(\vartheta, s)$ is its ratio of twisted to untwisted L-values (unit of the exchange identity, Chapter VII). A finite place of $E(D)$ is *chosen* if it contains the conjugate of the integer π_g of norm $g(D)$; $\mathrm{ord}_{\mathrm{ch}}(D)$ is the product of the orders of the groups $\mathrm{GL}_{r(D)t(D)}(\mathbb{F}_w)$ over the chosen places; ω_D is the family of residue characters of the data, of order $M_\omega(P, \mu(D)) = (N_0 d)^{\mu(D)+6}$ at the chosen places (Chapter III). An *auxiliary pair O of type (r, n) over E* (Chapter VII) carries a system of places (chosen places w_i, $i \in I_O$, and twist places), a hermitian space of rank $\#r$ with signs, an embedding $r \hookrightarrow n$, a character μ_O of $\mathrm{U}(1)$ with tame part ψ_O, the character $a_O = \upsilon_{\mathrm{U}(1)}^{r_-}$ (with r_- the number of signs -1), and a space $\mathcal{F}(O)$ of algebraic forms on the standard definite space of rank $\#n$, on which the group algebra of $\prod_i \mathrm{GL}_r(\mathbb{F}_{w_i})$ and the Hecke operators at the split places off the places of O act; $\#n - \#r$ (truncated) is its *rank difference*. For an eigenform, its *Hecke polynomial* at a split place and the representations *attached* to its system of Hecke polynomials are those of Chapter VII: attachment asks continuity, a non-degenerate trace form, and that characteristic polynomials of Frobenius elements are ℓ-adically close to the Hecke polynomials at most Frobenius elements. Precisely, a *Frobenius datum* of a number field $E \subseteq \bar{\mathbb{Q}}$ is a triple $(w, \mathfrak{P}, \phi)$ of a finite place of prime norm, a place of $\bar{\mathbb{Q}}$ over it and an element $\phi \in G_E$ whose inverse is an $\mathrm{N}(w)$-Frobenius at $\mathfrak{P}$. A set W of Frobenius data *contains most Frobenius data* if every non-empty open subset of G_E contains a non-empty open subset U such that, for all but finitely many finite places w, every Frobenius datum $(w, \mathfrak{P}, \phi)$ with $\phi \in U$ belongs to W; it *contains most Frobenius data near g* if it contains the intersection of such a set with the set of the data whose element ϕ lies in some neighbourhood of g; and a statement holds *for most Frobenius data* (near g) if the set of the data for which it holds has this property. A *label of degree δ* over a finite field is a monic squarefree polynomial of degree δ with non-zero constant term, a *primitive label* an irreducible one with a root of order $q^\delta - 1$; $e^{\otimes}_{r,c}$ is the idempotent of a tuple c of labels for the index set r, in the group algebra of the product of the groups GL_r of the residue fields (note *Primitive labels*). A *base level* with bound M (unit of the adelic cusp forms) is a finite set of places with a space of forms of open level there, stable under the operators that commute with the translations at these places, all stabilisers having order dividing M; its *eigensystems of level u* are the isotypic spaces that contain a non-zero form of the base level fixed by the principal level at the places u_k, each with its chosen attached representation ϱ_x.

3 The definitions of the unit

The definitions are given in the order of the formal text. Each names all its arguments; the implicit ones are the ambient objects of the conventions.

3.1. Cocycles, blocks and the crystalline relations.

Definition 3.1 (cocycles that obey the relations among the $u_x(g)-1$)**.** Let A be a commutative ring, Γ a group, X a set, σ and d finite sets, $u=(u_x)_{x\in X}$ a family of homomorphisms $u_x\colon \Gamma\to M_\sigma(A)$, $\varrho\colon \Gamma\to M_d(A)$ a homomorphism and $a\in A$. Then $\mathcal{Z}_u(\varrho;\,a)$ is the A-module of the maps $c\colon \Gamma\to (A/a)^d$, continuous or not, that are cocycles for ϱ and satisfy $\sum_g \theta(g)\,c(g)=0$ for every finitely supported function $\theta\colon \Gamma\to A$ with $\sum_g \theta(g)\,(u_x(g)-1)=0$ in $M_\sigma(A)$ for all $x\in X$.

LEAN `obeyingCocycles`

Definition 3.2 (blocks of a row with its string)**.** Let r be a set, t a natural number, and i an element of $\{*\}\sqcup\{0,\dots,t-1\}$. Then $\mathrm{I}_r(i)$ is the set r if $i=*$, and a set with one element otherwise; it is finite when r is. Let further E be a number field embedded in $\bar{\mathbb{Q}}$, $\mathcal{O}$ a coefficient ring, d a finite set and $T\colon G_E\to M_d(\mathcal{O})$ a homomorphism. For $g\in G_E$, the matrix $\mathrm{blk}_T(i)(g)$, indexed by $\mathrm{I}_d(i)$, is $T(g)$ if $i=*$ and the scalar $\mathrm{cyc}(g)^{-e}$ if $i=e<t$. So $\mathrm{blk}_T(i)(\cdot)$ is T for $i=*$ and $\mathcal{O}(-e)$ for $i=e$.

LEAN `RowBlockIndex` · `blockRep` · the finiteness of $\mathrm{I}_d(i)$ is an unnamed instance

Definition 3.3 (the chart at a place above ℓ)**.** Let E be a CM field as in the conventions and $v\in\Sigma_\ell(E)$. The formal text fixes an ℓ-adic chart at the place $\mathfrak{P}_v$ of $\bar{\mathbb{Q}}$: an embedding $e_v\colon\bar{\mathbb{Q}}\to\bar{\mathbb{Q}}_\ell$ under which $\bar{\mathbb{Z}}_\ell$ pulls back to $\mathfrak{P}_v$, with an open embedding $j_v\colon G_{\mathfrak{P}_v}\to G_{\mathbb{Q}_\ell}$, compatible with e_v, whose image is the group of the elements that fix $e_v(E)$ pointwise. It makes $G_{\mathfrak{P}_v}$ a group over $G_{\mathbb{Q}_\ell}$.

LEAN `localChart` · an instance, chosen from two existence statements of the background library

Definition 3.4 (embeddings read through $G_{\mathbb{Q}_\ell}$)**.** Let $\mathcal{O}'$ be a coefficient ring and $g\in G_{\mathbb{Q}_\ell}$. Then $\mathrm{emb}_{\mathcal{O}'}(g)\colon\mathcal{O}'\to\mathbb{C}_\ell$ is the $\mathbb{Z}_\ell$-algebra homomorphism $x\mapsto g(x)$. Let E' be a field and $e\colon E'\to\bar{\mathbb{Q}}_\ell$ a ring homomorphism. Then $\mathrm{ce}(e,g)=\iota^{-1}\circ g^{-1}\circ e$ is a ring homomorphism $E'\to\mathbb{C}$.

LEAN `coeffEmb` · `complexEmbOf`

Definition 3.5 (the crystalline relations)**.** Let E be a CM field as in the conventions, $\mathcal{O}$ a coefficient ring, n a finite set and $u\colon G_E\to M_n(\mathcal{O})$ a homomorphism. For $v\in\Sigma_\ell(E)$, $Z_{\mathrm{cris}}(v;\,u)$ is the $\mathcal{O}$-module $Z_{\mathrm{f}}(L)$ of the crystalline cocycles of the lattice $L=\mathcal{O}^n$ on which $G_{\mathfrak{P}_v}$ acts through u, the group $G_{\mathfrak{P}_v}$ being a group over $G_{\mathbb{Q}_\ell}$ by Definition 3.3. The set $\mathcal{R}_{\mathrm{cris}}(u)$ of the *crystalline relations of* u consists of the relations on G_E obtained, for some $v\in\Sigma_\ell(E)$, from a relation in $\mathrm{Rel}(Z_{\mathrm{cris}}(v;\,u))$ by extension by zero from $G_{\mathfrak{P}_v}$ to G_E.

LEAN `ZfAt` · `crysRels`

A function on G_E obeys $\mathcal{R}_{\mathrm{cris}}(u)$ exactly when, for each v above ℓ, its restriction to $G_{\mathfrak{P}_v}$ obeys every relation of the crystalline cocycles there (Lemma 4.2(a)). This is the Bloch–Kato finite condition at the places above ℓ, written as linear relations on global cocycles.

Definition 3.6 (the loss of a block)**.** Let E, $\mathcal{O}$, n and $u\colon G_E \to M_n(\mathcal{O})$ be as in Definition 3.5, let X be a set, σ a finite set, $w = (w_x)_{x\in X}$ a family of homomorphisms $w_x\colon G_E \to M_\sigma(\mathcal{O})$, and $M \in \mathbb{N}$. Then

$$\mathrm{bl}(u;\, w, M) = \mathrm{len}_{\mathcal{O}}\Big(\mathcal{Z}_w(u;\, \ell^M)\big/\mathcal{Z}_w(u;\, \ell^M) \cap \mathrm{Ob}(\mathcal{R}_{\mathrm{cris}}(u);\, \mathcal{O}/\ell^M)\Big).$$

LEAN `blockLoss`

Definition 3.7 (the sign of an embedding)**.** Let E be a field with a choice, for each infinite place, of a complex embedding that induces it (for a CM field: a CM type). For a complex embedding σ of E, $\mathrm{sgn}_{\mathrm{CM}}(\sigma)$ is 1 if σ is the chosen embedding of its place, and -1 otherwise.

LEAN `cmSign`

Definition 3.8 (row types)**.** Let D be admissible data, r and t natural numbers and $\mathcal{H}$ a type for $E(D)$. Then $\mathsf{RowType}_D(r, t;\, \mathcal{H})$ holds if

(R1) $1 \le r \le r(D)$ and $t \le t(D)$;

(R2) there is a natural number $j \le m(D)$ such that for every complex embedding σ of $E(D)$ and every $u \in \mathcal{H}(\sigma)$: either $h_2(D) \le 2|u|$ and $|u| \le h_2(D)^2$ (a *low* jump), or $\big|\mathrm{sgn}_{\mathrm{CM}}(\sigma)\, u + w_{D,j}(\sigma)\big| \le t(D)$, where $w_{D,j}(\sigma)$ is the weight of the j-th lift at the infinite place of σ;

(R3) for every σ, $\#\{u \in \mathcal{H}(\sigma) : |u| \le 2h_2(D)^2\} \cdot k(D) \le r \cdot m(D)$;

(R4) for every σ, $\#\{u \in \mathcal{H}(\sigma) : u < 0\} + \#\{u \in \mathcal{H}(\bar\sigma) : u < 0\} \le r$, with $\bar\sigma$ the complex conjugate of σ.

The counts are with multiplicity.

LEAN `IsRowType`

Definition 3.9 (the local loss)**.** For natural numbers h, b, l, $\mathrm{lam}(h, b;\, l)$ is the exponent of l in $(2b)! \cdot \big(\prod_{b-30h^3<k\le b+30h^3} k\big)^{h^3}$, the product over the natural numbers $k \ge 1$ in that range. For admissible data D and a natural number l,

$$\mathrm{loss}_D(l) = 10^6 \cdot \frac{m(D)}{k(D)}\Big(b_1(D)\cdot v_l(g(D)) + \mathrm{lam}(h_2(D), b_1(D);\, l)\Big),$$

a non-negative real number; v_l is the exponent of l (zero when l is not a prime).

LEAN `lambdaLoss` · `loss`

3.2. The field of the data and the avatar of a tame character.

Definition 3.10 (places of the data)**.** Let D be admissible data. Then $G_{E(D)}$ is the group $\mathrm{Gal}(\bar{\mathbb{Q}}/E(D))$, and $\Sigma_g(D)$ is the set of the finite places of $E(D)$ that contain $g_0(D)$ or $g_1(D)$. For a natural number ℓ (a prime, in all uses), $\Sigma_{\mathrm{bad}}(D, \ell)$ is the finite set of the finite places of $E(D)$ that contain $\ell \cdot g(D) \cdot N(D)$.

LEAN `AbsGal` · `placesAboveG` · `badPlaces` · the structure of a CM field as in the conventions on $E(D)$ is an unnamed instance

Definition 3.11 (the avatar of the inverse of a tame character)**.** Let D be admissible data and ϑ a tame character of D. For $g \in G_{E(D)}$, $\mathrm{av}^-_\vartheta(g) = \mathrm{av}(\vartheta^{-1}[0,0])(g) \in \bar{\mathbb{Q}}_\ell$: the value at g of the ℓ-adic avatar, through ι, of the Hecke character $x \mapsto \vartheta^{-1}(x/\bar x)$.

LEAN `Data.TameChar.invAvatar`

3.3. Columns, rows and labels.

Definition 3.12 (conjugate self-dual)**.** Let D be admissible data, n a finite set, t a natural number and $R\colon G_{E(D)} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ a homomorphism. Then $\mathsf{CSD}_t(R)$ holds if R is continuous, the trace form of R is non-degenerate, and $\operatorname{tr} R(g^{\mathsf{c}}) = \mathrm{cyc}(g)^{-(t-1)} \operatorname{tr} R(g^{-1})$ for all $g \in G_{E(D)}$, where $t-1$ is read as 0 for $t = 0$.

LEAN `IsConjSelfDual`

Definition 3.13 (the crude unit)**.** For a natural number ℓ and admissible data D, $\mathrm{cr}_\ell(D) = v_\ell(\mathrm{ord}_{\mathrm{ch}}(D)) + 2^{-\ell}$, a positive real number.

LEAN `crudeUnit`

Definition 3.14 (the Selmer length bound)**.** Let D be admissible data, n a finite set, $\psi\colon G_{E(D)} \to \bar{\mathbb{Q}}_\ell$ a multiplicative homomorphism, $R\colon G_{E(D)} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ a homomorphism, Q a set of finite places of $E(D)$, $m \in \mathbb{N}$, $\gamma \in G_{E(D)}$ and $z \in \bar{\mathbb{Z}}_\ell$. Then $\mathsf{Short}(\psi, R; Q, m; \gamma, z)$ holds if for all large $\mathcal{O}$ there are a homomorphism $u\colon G_{E(D)} \to M_n(\mathcal{O})$, a multiplicative homomorphism $\psi'\colon G_{E(D)} \to \mathcal{O}$ and $y \in \mathcal{O}$ such that $y = z$, u is a lattice of R, $\psi'(g) = \psi(g)$ for all g, and, with $\varrho = \psi' \cdot u$,

$$\varrho(\gamma) = y \cdot 1, \qquad y - 1 \in \mathcal{O}^\times, \qquad \mathrm{cyc}(\gamma) = 1,$$

$$\begin{aligned} &\operatorname{nlen}_{\mathcal{O}}\bigl\{c \in \mathrm{Sel}^{\mathcal{R}_{\mathrm{cris}}(\varrho)}_{\Sigma_\ell(E(D))\cup\Sigma_g(D)\cup Q\cup\bar{Q}}(\varrho; \mathcal{O}/\ell^m) : c(\gamma) = 0\bigr\} \\ &\quad\le\ 10\, f(D) \cdot \#n \cdot \mathrm{cr}_\ell(D) + \mathrm{cr}_\ell(D). \end{aligned}$$

LEAN `SelmerLengthLe`

Definition 3.15 (columns, as Galois representations)**.** Let D be admissible data, n a finite set, t a natural number, $\mathcal{H}$ a type for $E(D)$, S a set of finite places of $E(D)$ and $R\colon G_{E(D)} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ a homomorphism. Then $\mathsf{ColRep}_D(t, \mathcal{H}, S; R)$ (*R is a column of level S*) holds if $\mathsf{CSD}_t(R)$ and, for every finite place v of $E(D)$ that does not lie in $S \setminus (\Sigma_\ell(E(D)) \cup \Sigma_g(D))$ and every place $\mathfrak{P}$ of $\bar{\mathbb{Q}}$ over v: R is of finite type at $\mathfrak{P}$, has the jumps $\mathcal{H}$ at $\mathfrak{P}$, and, if $v \notin \Sigma_g(D)$, is good at $\mathfrak{P}$.

LEAN `IsColumn`

In the chapters on forms a column is an algebraic modular form; here, as in the formal text of this unit, a column is a Galois representation of the kind attached to such forms, and a row is one of the kind that occurs as the irreducible block of the representation of a theta lift.

Definition 3.16 (rows, as Galois representations)**.** Let D, n, t, $\mathcal{H}$ and R be as in Definition 3.15. Then $\mathsf{RowRep}_D(t, \mathcal{H}; R)$ holds if

(W1) $\mathsf{ColRep}_D(t, \mathcal{H}, \emptyset; R)$ and $\mathsf{RowType}_D(\#n, t; \mathcal{H})$;

(W2) for every $v \in \Sigma_g(D) \setminus \Sigma_\ell(E(D))$ and every place $\mathfrak{P}$ of $\bar{\mathbb{Q}}$ over v there are $\gamma \in G_{E(D)}$ and $N \in \mathbb{N}$ such that $\gamma = \tau_{\mathfrak{P}}$ if v is chosen, N is a unit at $\mathfrak{P}$, $R(i^N) = 1$ for all $i \in I_{\mathfrak{P}}$, and $v_\ell(\det(R(\gamma^{[b]}) - 1)) \le \#n \cdot \mathrm{cr}_\ell(D)$ for both signs b;

(W3) for all large $\mathcal{O}$ and every lattice $u\colon G_{E(D)} \to M_n(\mathcal{O})$ of R: the $\mathcal{O}$-span of $u(G_{E(D)})$ is $M_n(\mathcal{O})$, and there are $g \in G_{E(D)}$, an index $i \in \{0,1\}$ and $\alpha \in \mathcal{O}$ with $\mathrm{cyc}(g) = g_i(D)^{\#n}$ and $u(g) = \alpha \cdot 1$;

(W4) there are $\gamma \in I_{\Sigma_{g(D)} \setminus \Sigma_\ell(E(D))}$ and $y \in \bar{\mathbb{Q}}_\ell$ with $R(\gamma) = y \cdot 1$ and $\left|(y^{\ell^a})^j - 1\right| = 1$ for all $a \in \mathbb{N}$ and all $1 \le j \le 4\,\#n + 2$.

LEAN `IsRow`

Definition 3.17 (the normalising character)**.** Let D be admissible data, r, n finite sets and O an auxiliary pair of type (r,n) over $E(D)$. Then $\chi_{1,O}^{-1}$ is the character $\mathrm{av}(a_O^{-1}[-\#r, 0])$ of $G_{E(D)}$ with values in $\bar{\mathbb{Q}}_\ell^\times$, the avatar of the Hecke character $x \mapsto a_O^{-1}(x/\bar{x})\,\upsilon(x)^{\#r}\,|x|^{\#r/2}$. For a homomorphism R from $G_{E(D)}$ to the invertible matrices over $\bar{\mathbb{Q}}_\ell$ indexed by a finite set, $\mathrm{nm}_O(R)$ is the homomorphism $g \mapsto \chi_{1,O}^{-1}(g) \cdot R(g)$.

LEAN `chiOneInv` · `normalisedRep`

Definition 3.18 (twisted labels)**.** Let D, r, n, O be as in Definition 3.17, let $c = (c_i)_{i \in I_O}$ be a family of polynomials, c_i over the residue field of the i-th chosen place w_i of O, and let $i \in I_O$. Then $\mathrm{tl}_O(c,i)$ is the polynomial of the degree of c_i whose roots are the roots of c_i multiplied by $x_{\psi_{O,w_i}^{-1}} \in \mathbb{F}_{w_i}^\times$, the element that names the inverse of the tame part of μ_O at w_i.

LEAN `twistedLabel`

Definition 3.19 (apt labels)**.** Let D, r, n, O, c be as in Definition 3.18 and ϑ a twist of D. Then $\mathsf{Apt}(O, c;\, \vartheta)$ holds if

(A1) the system of places of O is that of ϑ: its chosen places are the chosen places of D and its twist places are the places of $Q(\vartheta)$;

(A2) $\psi_O = \omega_D^j \cdot \psi_\vartheta$ for some $1 \le j \le m(D)$;

(A3) $\#r$ and the rank difference $\#n - \#r$ are odd, $\#r \le r(D)$ and $\#n - \#r \le t(D)$;

(A4) every c_i is a label of degree $\#r$;

(A5) for each $k \in \{0,1\}$ some chosen place w_i lies above $g_k(D)$ with: $\#r = r(D)$ and c_i a primitive label of degree $\#r$, or $\#r = 1$ and $c_i = X - 1$;

(A6) for every i, 1 is not a root of $\mathrm{tl}_O(c,i)$.

LEAN `IsApt`

Definition 3.20 (the projector of a tuple of labels)**.** Let D, r, n, O, c be as in Definition 3.18. Then $\mathrm{pr}_O(c)$ is the endomorphism of the space $\mathcal{F}(O)$ of forms by which the element $e_{r,c}^{\otimes}$ of the group algebra of $\prod_i \mathrm{GL}_r(\mathbb{F}_{w_i})$ acts.

LEAN `tupleProj`

Definition 3.21 (a pair that has a row)**.** Let D, r, n, O be as in Definition 3.17, let z be Hecke data in rank $\#r$ (a polynomial for each finite place) and $R\colon G_{E(D)} \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$ a homomorphism. Then $\mathsf{HasRow}(O, z;\, R)$ holds if

(a) $\mathsf{RowRep}_D(\#n - \#r, \mathcal{H};\, R)$ for some type $\mathcal{H}$; and

(b) for every tame character ϑ of D, every auxiliary pair V of type (r,n) over $E(D)$ with $\mu_V = \mu_O \cdot \vartheta$, every form y of V that lifts z, every system

H of Hecke polynomials for which y is an eigenform and every homomorphism $S\colon G_{E(D)} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ attached to H:

$$\operatorname{tr} \mathrm{nm}_V(S)(g) = \mathrm{av}_\vartheta^-(g) \cdot \operatorname{tr} R(g) + \sum_{e<\#n-\#r} \mathrm{cyc}(g)^{-e} \qquad \text{for all } g \in G_{E(D)}.$$

LEAN `HasRow`

Clause (b) says that the normalised representation of an eigenform lifting z is, on traces, the row twisted by the avatar of ϑ^{-1} plus the string $1, \mathrm{cyc}^{-1}, \dots, \mathrm{cyc}^{-(\#n-\#r-1)}$.

Definition 3.22 (labelled columns)**.** Let D, r, n, O, c be as in Definition 3.18, m a finite set, k a natural number, $\mathcal{H}$ a type, S a set of finite places of $E(D)$ and $R\colon G_{E(D)} \to \mathrm{GL}_m(\bar{\mathbb{Q}}_\ell)$ a homomorphism. Then $\mathsf{LabCol}(O;\, k, \mathcal{H}, S;\, c;\, R)$ holds if $\mathsf{ColRep}_D(k, \mathcal{H}, S;\, R)$ and, for every $i \in I_O$ and every place $\mathfrak{P}$ of $\bar{\mathbb{Q}}$, R is labelled by c_i at $\mathfrak{P}$ for the faithful character of $\mathbb{F}_{w_i}$ and $f = (\#r)!$.

LEAN `IsLabelledColumn`

3.4. Auxiliary primes.

Definition 3.23 (data at an auxiliary prime)**.** Let E be a CM field as in the conventions and q a natural number. $\mathrm{Aux}(E, q)$ is the set of the quadruples $l = (\mathfrak{P}, \gamma, \phi, h)$ of a place $\mathfrak{P}$ of $\bar{\mathbb{Q}}$, elements $\gamma, \phi \in G_E$ and a homomorphism $h\colon G_E \to \mathbb{Z}/q$. (The formal text writes $\mathbb{Z}/q$ multiplicatively.)

LEAN `AuxPrimeData`

Definition 3.24 (anticyclotomic data)**.** Let E be a CM field as in the conventions, T a set of finite places of E, $\mathfrak{q}$ a finite place, q a natural number and $l = (\mathfrak{P}, \gamma, \phi, h) \in \mathrm{Aux}(E, q)$. Then $\mathsf{Acyc}(T, \mathfrak{q};\, l)$ holds if h is from U(1) relative to T with modulus $\{\mathfrak{q}\}$, $\mathrm{N}(\mathfrak{q}) \equiv 1 \bmod q$, $\mathfrak{P}$ lies over $\mathfrak{q}$, ϕ is an $\mathrm{N}(\mathfrak{q})$-Frobenius at $\mathfrak{P}$, $\mathfrak{q} \notin T$, and $\gamma \in I_{\mathfrak{P}}$ with $h(\gamma) = 1$ and $\gamma = \tau_{\mathfrak{P}}$.

LEAN `IsAnticyclotomicAt`

Definition 3.25 (auxiliary sets)**.** Let E' be a CM field as in the conventions and S, I finite sets of finite places of E'. Then $\mathsf{Auxil}(S;\, I)$ holds if every $v \in I$ satisfies: $v \notin S$, $\mathrm{N}(v)$ is a prime number, $\bar{v} \notin I$ and $\bar{v} \notin S$.

LEAN `IsAuxiliary`

Definition 3.26 (families of values)**.** Let E be a CM field as in the conventions, S a finite set of finite places of E, T a set, $\nu = (\nu_i)_{i \in T}$ a family of homomorphisms $\nu_i\colon G_E \to \bar{\mathbb{Q}}_\ell^\times$, and val a function that assigns an element $\mathrm{val}(I, \chi) \in \bar{\mathbb{Q}}_\ell$ to a finite set I of finite places of E and a multiplicative homomorphism $\chi\colon G_E \to \bar{\mathbb{Q}}_\ell$. Then $\mathsf{Fam}(S;\, \nu, \mathrm{val})$ holds if

- (F1) every ν_i is continuous;
- (F2) for every I with $\mathsf{Auxil}(S;\, I)$ there are a finite set α, elements $g_a \in G_E$ and $c_a \in \bar{\mathbb{Z}}_\ell$ $(a \in \alpha)$ with $\mathrm{val}(I, \chi) = \sum_a c_a\, \chi(g_a)$ for all χ;
- (F3) for every I, every finite place $\mathfrak{q} \notin I$ with $\mathsf{Auxil}(S;\, I \cup \{\mathfrak{q}\})$, every χ from U(1) relative to $\Sigma_\ell(E)$ with modulus I, every place $\mathfrak{P}$ of $\bar{\mathbb{Q}}$ over $\mathfrak{q}$, every $\mathrm{N}(\mathfrak{q})$-Frobenius ϕ at $\mathfrak{P}$ and every real $a \geq 0$: if $v_\ell(1 - \nu_i(\phi)\chi(\phi)^{\pm 1}) \leq a$ for all i and both signs, then $v_\ell(\mathrm{val}(I \cup \{\mathfrak{q}\}, \chi)) \leq v_\ell(\mathrm{val}(I, \chi)) + 8a$;

(F4) $\mathrm{val}(\emptyset, 1) \neq 0$.

LEAN IsFamily

Definition 3.27 (even classes)**.** Let E be a CM field as in the conventions, $\mathcal{O}$ a coefficient ring, S a set of finite places of E and ν, e natural numbers. Then $\mathrm{Ev}(\mathcal{O}, S;\, \nu, e)$ is the set of the cocycles $c \in \mathrm{Sel}^{\emptyset}_{\Sigma_\ell(E)\cup S\cup\bar{S}}(\mathcal{O}(-e);\, \mathcal{O}/\ell^\nu)$ with $c(g^{\mathsf{c}}) = c(g)$ for all $g \in G_E$ that vanish at every g which lies in the decomposition group of a place of $\bar{\mathbb{Q}}$ above ℓ and has $\mathrm{cyc}(g) = 1$.

LEAN evenClasses

Definition 3.28 (new places)**.** Let E be a CM field as in the conventions, ν a natural number, J a set, $w = (w_j)_{j\in J}$ a family of finite places of E and $\tau = (\tau_j)_j$, $\phi = (\phi_j)_j$ families in G_E. Then $\mathsf{New}_\nu(w, \tau, \phi)$ holds if for every j: $w_j \notin \Sigma_\ell(E)$, $\mathrm{cyc}_{\ell^\nu}(\phi_j) = 1$, and $\tau_j = \tau_{\mathfrak{P}}$ for some place $\mathfrak{P}$ of $\bar{\mathbb{Q}}$ over w_j. For $k = (j, b)$, a pair of an index and a sign, $\tau^{[k]}$ denotes $\tau_j^{[b]}$, that is τ_j or $(\tau_j^{\mathsf{c}})^{-1}$.

LEAN IsNewPlaces · withConjInv

Definition 3.29 (prime triangles)**.** Let E be a CM field as in the conventions, ν a natural number, J a finite totally ordered set, $w = (w_j)_{j\in J}$ finite places of E, $\tau = (\tau_j)_j$ and $\phi = (\phi_j)_j$ families in G_E, $a \in G_E$, t a natural number, $\mathrm{ty}\colon J \to \{e : 0 < e < t\}$ a monotone map, and h, c natural numbers. Then $\mathsf{Tri}_\nu(w, \tau, \phi;\, a;\, t, \mathrm{ty};\, h, c)$ holds if

(T1) $\mathsf{New}_\nu(w, \tau, \phi)$;

(T2) there is a finite set $s \subseteq G_E$ with $(\#J + \#s)\cdot h \le c$ such that for every coefficient ring $\mathcal{O}$ and every $e \in \mathbb{N}$, an element of $\mathrm{Sel}^{\emptyset}_{\Sigma_\ell(E)}(\mathcal{O}(-e);\, \mathcal{O}/\ell^\nu)$ that vanishes on s is zero;

(T3) for every $e < t$, the product $\prod_{k<t,\, k\neq e}(\mathrm{cyc}(a^{-1})^e - \mathrm{cyc}(a^{-1})^k)$ divides ℓ^h in $\mathbb{Z}_\ell$;

(T4) for every $\mathcal{O}$ and every $j \in J$: if $z \in \mathrm{Ev}(\mathcal{O}, w(J);\, \nu, \mathrm{ty}(j))$ satisfies, for all $i < j$, $z(\tau_i) = 0$, and $z(\phi_i) = 0$ when $\mathrm{ty}(i) = \mathrm{ty}(j)$, then $\ell^h \cdot z(\tau_j) = 0$;

(T5) for every $\mathcal{O}$ and every $0 < e < t$, with $\delta(g) = \mathrm{cyc}(g)^{-e} - 1 \in \mathcal{O}$: $4\,\delta(a)^2$ divides ℓ^h in $\mathcal{O}$, and every $z \in \mathrm{Ev}(\mathcal{O}, \emptyset;\, \nu, e)$ with $z(\phi_j) = 0$ for all j with $\mathrm{ty}(j) = e$ satisfies $\delta(a)\cdot 2z(g) = \delta(g)\cdot 2z(a)$ for all $g \in G_E$.

LEAN IsPrimeTriangle

In a prime triangle the ordered places w_j serve the twists $0 < \mathrm{ty}(j) < t$ and cut down the even classes modulo ℓ^ν one after the other, up to ℓ^h; the element a separates the characters cyc^{-e} up to ℓ^h; and c bounds the total cost.

Definition 3.30 (auxiliary twists)**.** Let E be a CM field as in the conventions, S and Q finite sets of finite places of E, $\psi\colon G_E \to \bar{\mathbb{Q}}_\ell$ a multiplicative homomorphism and $c \in [0, \infty]$. Then $\mathsf{AuxTw}(S, Q;\, \psi, c)$ holds if $\mathsf{Auxil}(S;\, Q)$, ψ is from U(1) relative to $\Sigma_\ell(E)$ with modulus Q, and there is a family $\sigma = (\sigma_{\mathfrak{q}})_{\mathfrak{q}\in Q}$ in G_E with $\mathsf{New}_0((\mathfrak{q})_{\mathfrak{q}\in Q}, \sigma, 1)$ (each $\mathfrak{q} \in Q$ is not above ℓ and $\sigma_{\mathfrak{q}}$ is the tame generator at a place over $\mathfrak{q}$) and $\sum_{\mathfrak{q}\in Q} v_\ell(\psi(\sigma_{\mathfrak{q}}) - 1) \le c$.

LEAN IsAuxTwist

Definition 3.31 (a union with conjugates)**.** For finite sets S, J of finite places of a CM field as in the conventions, the set $S \cup J \cup \overline{J}$ is the union of S, of J and of the set of the conjugates of the places of J.

LEAN `unionWithConj`

Definition 3.32 (twists at a point)**.** Let Γ be a monoid, A and A' commutative rings, n and σ finite sets, $q \geq 1$ an integer, $h = (h_j)_{j\in\sigma}$ a family of homomorphisms $h_j \colon \Gamma \to \mathbb{Z}/q$ and $y = (y_j)_{j\in\sigma}$ a family in A. The *point character* of h and y is defined under the hypothesis that $y_j^q = 1$ for all j: it is the homomorphism $\Gamma \to A$ whose value at $g \in \Gamma$ is $\prod_j y_j^{h_j(g)}$, the exponents being the representatives in $[0, q)$. For a ring homomorphism $e \colon A' \to A$, a function $\varrho \colon \Gamma \to M_n(A')$ and $g \in \Gamma$,

$$\mathrm{tp}_{h,y}(e, \varrho)(g) = \Big(\prod_j y_j^{h_j(g)}\Big) \cdot e(\varrho(g)) \in M_n(A),$$

with e applied to the entries. For natural numbers l, M and a homomorphism $\varrho \colon \Gamma \to M_n(A)$, $\varrho \bmod l^M$ is ϱ followed by the reduction $M_n(A) \to M_n(A/l^M)$.

LEAN `pointChar` · `twistAtPoint` · `reduceModPow`

Definition 3.33 (balanced representations)**.** Let E be a CM field as in the conventions, A the ring of integers of a finite extension of $\mathbb{Q}_\ell$, with valuation v_A normalised by $v_A(\varpi) = 1$ and residue field κ_A, n a finite set, S a finite set of finite places of E, $\varrho \colon G_E \to M_n(A)$ a homomorphism, L a set of relations on G_E with values in $M_n(A)$, $\gamma \in G_E$, β a non-negative extended real number (a bound for lengths) and $M \in \mathbb{N}$. Then $\mathsf{Bal}_\ell(S;\, \varrho, L;\, \gamma, \beta, M)$ holds if

(B1) $\varrho \bmod \ell^M$ is locally constant, and $\varrho(g) = 1$ for all $g \in I_{S^{\mathbb{C}}}$;

(B2) the A-span of $\varrho(G_E)$ is $M_n(A)$, and $4\,\#n + 2 < \#\kappa_A$;

(B3) $\gamma \in I_{S\setminus\Sigma_\ell(E)}$, and there is $s \in A$ with $\varrho(\gamma^{[-]}) = s \cdot 1$, $\varrho(\gamma) = s \cdot 1$ and $s + 1,\ s - 1 \in A^\times$;

(B4) for every $a \in A$ and $b \in (A/a)^n$, the coboundary $g \mapsto \varrho(g)b - b$ obeys L;

(B5) there are $g \in G_E$ and $\alpha, \alpha' \in A$ with $\varrho(g) = \alpha \cdot 1$ and $\varrho(g^{[-]}) = \alpha' \cdot 1$ such that for every a dividing ℓ^M in A:

$$\sum_{v\in S} \min_{\mathfrak{P} \text{ over } v} \lambda_{\mathfrak{P}}(a) + \#n \cdot v_A(\alpha - \mathrm{cyc}(g^{\mathsf{c}})\,\alpha') \ \leq\ (\#S + f_E) \cdot \mathrm{len}_A((A/a)^n) + \beta,$$

where $\lambda_{\mathfrak{P}}(a)$ is the A-length of the module of the local cocycles of ϱ and L at $\mathfrak{P}$ with values in $(A/a)^n$, and f_E is the number of infinite places of E.

LEAN `IsBalanced`

Definition 3.34 (pencils of tied primes)**.** Let E be a CM field as in the conventions, A a commutative ring, $\mathcal{J}$ a set (of rows), S_0 a finite set of finite places of E, $U = (U_i)_{i\in\mathcal{J}}$ a family of properties of elements of G_E, W a subset of G_E, $N \in \mathbb{N}$, $\mathfrak{q}_0$ a finite place, J a finite set of finite places, and x a function that assigns to each finite place v a pair $x(v) = (l_v, s_v)$ of a datum $l_v = (\mathfrak{P}_v, \gamma_v, \phi_v, h_v) \in \mathrm{Aux}(E, \ell^N)$ and a family $s_v = (s_{v,i})_{i\in\mathcal{J}}$ in A (the *slopes*). Then $\mathsf{Pencil}_\ell(S_0;\, U, W;\, N, \mathfrak{q}_0, J;\, x)$ holds if

(P1) $\mathsf{Auxil}(S_0;\, J)$ and $\mathfrak{q}_0 \in J$;

(P2) $\mathsf{Acyc}(\Sigma_\ell(E), v;\, l_v)$ for every $v \in J$;

(P3) for all $v, v' \in J$ with $v \neq \mathfrak{q}_0$: $h_{v'}(\phi_v) = 1$ if $v' = \mathfrak{q}_0$, and $h_{v'}(\phi_v) = 0$ otherwise;
(P4) $\phi_{\mathfrak{q}_0} \in W$;
(P5) for every $v \in J$ and i with $s_{v,i} \neq 0$, the property U_i holds at ϕ_v and at $\phi_v^{[-]}$.

LEAN `IsPencil`

Definition 3.35 (ties)**.** Let E be a CM field as in the conventions, A a commutative ring, n a finite set, $\mathcal{J}$ a set, $q \geq 1$, J a finite set of finite places of E, x a function assigning to each finite place v a pair (l_v, s_v) with $l_v = (\mathfrak{P}_v, \gamma_v, \phi_v, h_v) \in \mathrm{Aux}(E, q)$ and $s_v = (s_{v,i})_{i \in \mathcal{J}}$ in A, and $i \in \mathcal{J}$. Then $\mathrm{Ties}_i(J, x)$ is the set of the relations $\delta_{\gamma_v^{[b]}} - s_{v,i}\, \delta_{\phi_v^{[b]}}$ with values in $M_n(A)$, for $v \in J$ and both signs b. A function c obeys them when $c(\gamma_v) = s_{v,i}\, c(\phi_v)$ and $c(\gamma_v^{[-]}) = s_{v,i}\, c(\phi_v^{[-]})$ for all $v \in J$.

LEAN `tieRelations`

3.5. Two further definitions.

Definition 3.36 (the tame character of an auxiliary set)**.** Let D be admissible data, I a finite set of finite places of $E(D)$ such that $\mathsf{Auxil}(\Sigma_{\mathrm{bad}}(D, \ell);\, I)$, and χ a character of $\mathrm{U}(1)$ of weight 0 whose tame part is trivial off I. The *tame character of D defined by I and χ* is χ with $Q = I$; it is a tame character because I is disjoint from its conjugate and the norms of its places are prime to $g(D)N(D)$ (Lemma 8.2).

LEAN `tameCharOfAuxiliary`

Definition 3.37 (the lattice of a matrix representation)**.** Let Γ be a group, n a finite set, A a commutative $\mathbb{Z}_\ell$-algebra and $\varrho\colon \Gamma \to M_n(A)$ a homomorphism. Then $\mathrm{L}(\varrho)$ is the module A^n with Γ acting through ϱ, as a Galois lattice over A for Γ.

LEAN `matrixLattice`

4 Cocycles, embeddings and a determinant bound

Lemma 4.1 (the graph of a cocycle that obeys the relations)**.** *Let A be a noetherian commutative ring, Γ a group, X a finite set, σ and d finite sets, $u = (u_x)_{x \in X}$ a family of homomorphisms $u_x\colon \Gamma \to M_\sigma(A)$, $\varrho\colon \Gamma \to M_d(A)$ a homomorphism and $a \in A$, and let $c \in \mathcal{Z}_u(\varrho;\, a)$. Let Γ act on $\mathbb{M} = M_\sigma(A)^X$ by $g \cdot (m_x)_x = (u_x(g) m_x)_x$ and on $(A/a)^d$ through ϱ. Then there are $k \in \mathbb{N}$ and a Γ-stable A-submodule Y of $\mathbb{M} \times (A/a)^d$ such that the first projection $Y \to \mathbb{M}$ is injective and*

$$\big((u_x(g) - 1)_x + a^k v,\ c(g)\big) \in Y \qquad \text{for all } g \in \Gamma,\ v \in \mathbb{M}.$$

Proof. Let Λ_0 be the module of the finitely supported functions $\theta\colon \Gamma \to A$ and $\mathcal{L}\colon \Lambda_0 \to \mathbb{M}$, $\theta \mapsto \sum_g \theta(g)(u_x(g) - 1)_x$. By the lemma of Artin and Rees for the ideal (a), the finitely generated module $\mathbb{M}$ and its submodule $\mathcal{L}(\Lambda_0)$, there is k_0 with

$$a^{k_0+1}\mathbb{M} \cap \mathcal{L}(\Lambda_0) = a\,\big(a^{k_0}\mathbb{M} \cap \mathcal{L}(\Lambda_0)\big).$$

Put $k = k_0 + 1$ and let Y be the image of

$$\mathcal{N}\colon \Lambda_0 \times \mathbb{M} \to \mathbb{M} \times (A/a)^d, \qquad \mathcal{N}(\theta, v) = \Big(\mathcal{L}\theta + a^k v,\ \sum_g \theta(g)\, c(g)\Big).$$

Stability. For $g \in \Gamma$ put $\theta^g = \sum_h \theta(h)(\delta_{gh} - \delta_g)$. Then

$$\mathcal{L}(\theta^g) = \sum_h \theta(h)(u(gh) - u(g)) = u(g)\,\mathcal{L}\theta,$$
$$\sum_h \theta(h)(c(gh) - c(g)) = \varrho(g) \sum_h \theta(h)c(h),$$

the second by the cocycle identity; hence $g \cdot \mathcal{N}(\theta, v) = \mathcal{N}(\theta^g, u(g)v)$. *Injectivity.* If $\mathcal{L}\theta + a^k v = 0$, then $\mathcal{L}\theta \in a^k\mathbb{M} \cap \mathcal{L}(\Lambda_0)$, so $\mathcal{L}\theta = -a\,\mathcal{L}\eta$ for some η, that is $\mathcal{L}(a\eta + \theta) = 0$. Since c obeys the relations among the $u_x(g) - 1$, $\sum_g (a\eta + \theta)(g)c(g) = 0$, and a kills $(A/a)^d$; so $\sum_g \theta(g)c(g) = 0$. *The points.* $\mathcal{N}(\delta_g, v) = ((u(g) - 1) + a^k v, c(g))$, with δ_g here the scalar indicator of g. □

LEAN `obeyingCocycles.exists_graph_subrepresentation`

So c is, on the elements $(u(g) - 1) + a^k v$, the value of a Γ-equivariant map defined on the projection of Y, and Y is its graph. The third note uses this to turn a cocycle that obeys the relations of the lifts into an extension.

Lemma 4.2 (four small facts)**.** (a) *Let E, $\mathcal{O}$, n, u be as in Definition 3.5 and $a \in \mathcal{O}$. A function $c\colon G_E \to (\mathcal{O}/a)^n$ obeys $\mathcal{R}_{\mathrm{cris}}(u)$ if and only if, for every $v \in \Sigma_\ell(E)$, its restriction to $G_{\mathfrak{P}_v}$ obeys* $\mathrm{Rel}(Z_{\mathrm{cris}}(v; u))$. (b) *Let E be a CM field as in the conventions, $\mathfrak{P}$ a place of $\bar{\mathbb{Q}}$ with an ℓ-adic chart, of open embedding $j_{\mathfrak{P}}\colon G_{\mathfrak{P}} \to G_{\mathbb{Q}_\ell}$, and $c \in \mathbb{N}$. Then $g \in G_{\mathfrak{P}}$ acts on $\mathcal{O}(-c)$ by the scalar* $\mathrm{cyc}(j_{\mathfrak{P}}(g))^{-c}$, *with* cyc *the cyclotomic character of $G_{\mathbb{Q}_\ell}$.* (c) *Let E be a field, Γ a topological group, $e\colon E \to \bar{\mathbb{Q}}_\ell$ a ring homomorphism and $j\colon \Gamma \to G_{\mathbb{Q}_\ell}$ an open embedding whose image is the group of the elements fixing $e(E)$ pointwise. Then for $g, g' \in G_{\mathbb{Q}_\ell}$:* $\mathrm{ce}(e, g) = \mathrm{ce}(e, g')$ *if and only if $g\,g'^{-1} \in j(\Gamma)$.* (d) *In the situation of* (c)*, the index $[G_{\mathbb{Q}_\ell} : j(\Gamma)]$ is the number of the homomorphisms* $\mathrm{ce}(e, g)$, *$g \in G_{\mathbb{Q}_\ell}$.*

Proof. (a) unfolds Definition 3.5: a relation extended by zero is evaluated on the restriction. (b) is the compatibility of the chart with the cyclotomic characters. (c): $\iota^{-1}g^{-1}e = \iota^{-1}g'^{-1}e$ means that $g'g^{-1}$, hence its inverse, fixes $e(E)$. (d) follows from (c): the fibres of $g \mapsto \mathrm{ce}(e, g)$ are the cosets of $j(\Gamma)$. □

LEAN `obeying_crysRels` · `tateTwist_smul` · `IsCompletionGroup.complexEmbOf_eq_iff` · `IsCompletionGroup.index_eq_card_range`

Lemma 4.3. *Let $\mathcal{O}$ be a coefficient ring. Every $\mathbb{Z}_\ell$-algebra homomorphism $\mathcal{O} \to \mathbb{C}_\ell$ is* $\mathrm{emb}_{\mathcal{O}}(g)$ *for some $g \in G_{\mathbb{Q}_\ell}$.*

Proof. $\mathcal{O}$ is integral over $\mathbb{Z}_\ell$, so the homomorphism extends to a $\mathbb{Z}_\ell$-algebra homomorphism $\bar{\mathbb{Q}}_\ell \to \mathbb{C}_\ell$ (the field $\mathbb{C}_\ell$ is algebraically closed and torsion-free over $\mathcal{O}$). Its image is the algebraic closure of $\mathbb{Q}_\ell$ in $\mathbb{C}_\ell$, which is $\bar{\mathbb{Q}}_\ell$; so it is an element of $G_{\mathbb{Q}_\ell}$ (an input of the unit on Tate–Sen theory). □

LEAN `coeffEmb_surjective`

Lemma 4.4. *Let E be a CM field as in the conventions and a a function from the complex embeddings of E to $\mathbb{N}$. For $v \in \Sigma_\ell(E)$ let $\mathfrak{S}_v$ be the set of the complex embeddings* $\mathrm{ce}(e_v|_E, g)$, *$g \in G_{\mathbb{Q}_\ell}$, with e_v the chart of Definition 3.3. Then $\sum_{v \in \Sigma_\ell(E)} \sum_{\sigma \in \mathfrak{S}_v} a(\sigma) \le \sum_\sigma a(\sigma)$.*

Proof. The sets $\mathfrak{S}_v$ are pairwise disjoint: if $\sigma = \iota^{-1}g^{-1}e_v|_E$, then an integer x of E lies in v exactly when $|\iota\sigma(x)| < 1$, because e_v induces the place $\mathfrak{P}_v$ over v and g is

an isometry; so σ determines v. The left side is thus a sum over a subset of the embeddings. □

LEAN `sum_complexEmbOf_le`

Lemma 4.5 (operators that clear denominators bound a Gram determinant)**.** *Let V and W be complex vector spaces, $\mathcal{A} \subseteq V$ a set whose linear span Z has finite dimension N, $j\colon V \to W$ a linear map, $\eta_1, \dots, \eta_N$ linear forms on V, $x_1, \dots, x_N$ vectors of Z, γ a real number and $m \in \bar{\mathbb{Q}}_\ell$. Suppose that for every non-zero $p \in \mathcal{A}$ there is a linear map $G\colon V \to W$ such that* (i) *for every $v \in \mathcal{A}$ there is $l \in \mathbb{C}$ with $G(v) = l \cdot j(v)$;* (ii) *$e^{-\gamma} \le |\iota(l)\, m|$ for every $l \in \mathbb{C}$ with $G(p) = l \cdot j(p)$;* (iii) *$|m \cdot \iota(\eta_a(w))| \le 1$ for all a, b and every $w \in Z$ with $j(w) = G(x_b)$. Then $\left|\iota(\det(\eta_a(x_b))_{a,b})\right| \le e^{N\gamma}$.*

Proof. Choose a basis $\omega_1, \dots, \omega_N$ of Z inside $\mathcal{A}$, for each p a map G_p as in the hypothesis for ω_p, and scalars $l_{p,k}$ with $G_p(\omega_k) = l_{p,k}\, j(\omega_k)$. For finitely many polynomials over an algebraically closed valued field there is a point with integral coordinates at which each of them has absolute value at least that of each of its coefficients (an input of the unit on local heights). Applied to the linear forms $\sum_p \iota(l_{p,k})\, m\, X_p$ it gives $d_1, \dots, d_N \in \bar{\mathbb{Z}}_\ell$ with $|\lambda_k| \ge |\iota(l_{k,k})\, m| \ge e^{-\gamma}$ for $\lambda_k = \sum_p \iota(l_{p,k})\, m\, d_p$, by (ii). Write $x_b = \sum_k P_{b,k}\omega_k$. The matrix

$$\left(\iota\eta_a(\omega_k)\right)_{a,k} \cdot \operatorname{diag}(\lambda_k) \cdot \left(\iota P_{b,k}\right)_{k,b}$$

has the entries $\sum_p d_p \cdot m\, \iota\eta_a(w_{p,b})$, where $w_{p,b}$ is the element $\sum_k P_{b,k} l_{p,k} \omega_k$ of Z. Moreover $j(w_{p,b}) = G_p(x_b)$, so by (iii) the entries lie in $\bar{\mathbb{Z}}_\ell$. Hence its determinant $\iota(\det(\eta_a(x_b))) \cdot \prod_k \lambda_k$ has absolute value at most 1. □

LEAN `norm_gram_det_le_of_operators`

Lemma 4.5 is the form in which a congruence bound is used in the third note: the maps G are Hecke operators that act on lifts by scalars, m is a power of ℓ, and the determinant is a Gram determinant.

5 The field of the data, avatars and lattices

Lemma 5.1 (small facts on the field of the data)**.** *Let D be admissible data.* (a) *$E(D)$ is a CM field as in the conventions: it is embedded in $\bar{\mathbb{Q}}$ through the cyclotomic field of conductor $N_1(D) \cdot d$, and the image of every element lies in that cyclotomic field.* (b) *A finite place of $E(D)$ in $\Sigma_\ell(E(D)) \cup \Sigma_g(D)$ lies in $\Sigma_{\mathrm{bad}}(D, \ell)$.* (c) *$\bar{v} \in \Sigma_g(D)$ if and only if $v \in \Sigma_g(D)$.* (d) *If $x \ne y$ in $\bar{\mathbb{Q}}_\ell$ and $x^N = y^N = 1$ for a natural number N, then $|N| \le |x - y|$.* (e) *$\mathrm{cyc}^{-1}\colon G_{E(D)} \to \bar{\mathbb{Q}}_\ell$ is continuous.* (f) *If $(w, \mathfrak{P}, s)$ is a Frobenius datum of $E(D)$ (a finite place of prime norm, a place of $\bar{\mathbb{Q}}$ over it and s with s^{-1} an $\mathrm{N}(w)$-Frobenius at $\mathfrak{P}$) and $w \notin \Sigma_\ell(E(D))$, then $\mathrm{cyc}(s)^{-1} = \mathrm{N}(w)$.* (g) *If the system of places of an auxiliary pair is that of a tame character ϑ, then each of its chosen places w_i is chosen, lies in $\Sigma_g(D)$, and $\psi_{\vartheta, w_i} = 1$.*

Proof. (a) is how the formal text equips $E(D)$ with its embedding. (b), (c), (g) unfold the definitions; for (g), the set $Q(\vartheta)$ has norms prime to $g(D)$, so it contains no chosen place. (d): $x - y$ divides $\prod_{\zeta^N = 1, \zeta \ne 1}(1 - \zeta) = N$ up to a unit, since $x^{-1}y$ is an N-th root of unity different from 1. (e) and (f) are the continuity of the cyclotomic character and its value at a Frobenius element. □

LEAN `Data.algebraMap_mem_cycloField` · `placesDividing_union_subset_badPlaces` · `star_mem_placesAboveG` · `norm_natCast_le_norm_sub_of_pow_eq_one` · `continuous_cycInv` · `cycInv_eq_of_isFrob` · `isChosen_chosenPlace`

Lemma 5.2 (conjugate duality of an avatar)**.** *Let D be admissible data, a a character of $\mathrm{U}(1)$ of $E(D)$ with a weight e and a tame part ψ, and k, j integers. Then, for every $g \in G_{E(D)}$,*

$$\mathrm{av}(a[k,j])(g^{\mathsf{c}}) \cdot \mathrm{av}(a[k,j])(g) = \mathrm{cyc}(g)^{2j-k}.$$

Proof. Both sides are continuous in g, and the Frobenius elements of most Frobenius data are dense (an input of the unit on Chebotarev's theorem); so it suffices to prove the identity for $g = s$ with $(w, \mathfrak{P}, s)$ a Frobenius datum, w and $\bar{w}$ outside ℓ and outside the support of ψ. Then s^{c} is the Frobenius element of a datum at $\bar{w}$, and the avatar at such an element is ι of the value of the Hecke character at the uniformiser. The factor $a(x/\bar{x})$ takes inverse values at ϖ_w and $\varpi_{\bar{w}}$; $v(\varpi_w)v(\varpi_{\bar{w}}) = 1$; and $|\varpi_w|^{2(j-k/2)} = \mathrm{N}(w)^{-(2j-k)} = \mathrm{cyc}(s)^{2j-k}$ by Lemma 5.1(f). □

LEAN `avatar_conjGal_mul`

Lemma 5.3 (the label of an avatar)**.** *Let D be admissible data, a a character of $\mathrm{U}(1)$ of $E(D)$ with a weight e and a tame part ψ, and k, j integers. Let w be a finite place of $E(D)$, $\mathfrak{P}$ a place of $\bar{\mathbb{Q}}$ and $f \in \mathbb{N}$. Suppose that $\psi_{\bar{w}} = 1$ and that w is split. Then the one-dimensional representation $\mathrm{av}(a[k,j])$ is labelled at $\mathfrak{P}$, for the faithful character of $\mathbb{F}_w$ and f, by the polynomial $X - x_{\psi_w}$.*

Proof. Let $s \in I_{\mathfrak{P}}$ have tame exponent a'; in particular $\mathfrak{P}$ lies over w. By the local compatibility of avatars (unit of the Galois representations), s is the local Artin symbol of a unit u of E_w and $X - \iota(a[k,j]_w(u))$ kills s. The characters v and $|\cdot|$ are trivial on units, and $a(u/\bar{u}) = \psi_w(u \bmod w)$ because $\psi_{\bar{w}} = 1$ and $\bar{w} \neq w$. By the property that defines x_{ψ_w}, $\psi_w(u \bmod w) = T(x_{\psi_w})^{a'}$. □

LEAN `isLabelledAt_avatar`

Lemma 5.4 (the normalising character at Frobenius elements)**.** *Let D be admissible data, r, n finite sets, O an auxiliary pair of type (r,n) over $E(D)$, w a finite place of $E(D)$ and $\mathfrak{P}$ a place of $\bar{\mathbb{Q}}$. Suppose that $\mathfrak{P}$ lies over w and that w does not contain ℓ. Let $k \in \mathbb{N}$ and let $s \in G_{E(D)}$ be such that s^{-1} is an $\mathrm{N}(w)^k$-Frobenius at $\mathfrak{P}$. Then $\chi_{1,O}^{-1}(s) = \iota(\nu_w^{-1})^k$, where $\nu_w = \chi(\varpi_w)\,\mathrm{N}(w)^{\#r/2}$, with χ the Hecke character $x \mapsto a_O(x/\bar{x})\,v(x)^{-\#r}$, is the* Frobenius value *of a_O in rank $\#r$ at w.*

Proof. s is the local Artin symbol of an element of the completion at w of order k, and the avatar at s is ι of the value there of the Hecke character $a_O^{-1}[-\#r, 0]$. That character is unramified at w, so the value is the k-th power of its value at ϖ_w, which is $\chi(\varpi_w)^{-1}\,\mathrm{N}(w)^{-\#r/2} = \nu_w^{-1}$. □

LEAN `chiOneInv_spec`

Lemma 5.5 (divisibility and counting at the level)**.** *Let D be admissible data.* (a) *For $i \in \{0,1\}$ and an integer d with $1 \le d \le r(D)t(D)$, the number $g_i(D)^d - 1$ divides $\mathrm{ord}_{\mathrm{ch}}(D)$.* (b) *If T is a finite set of finite places of $E(D)$ with $T \subseteq \Sigma_g(D)$, then $\#T \le 4\,f(D)$.* (c) *There are a finite place w of $E(D)$ and an index i such that w is chosen, $w \in \Sigma_g(D) \setminus \Sigma_\ell(E(D))$ and $\mathrm{N}(w) = g_i(D)$.*

Proof. (a) $\mathrm{ord}_{\mathrm{ch}}(D)$ is divisible by the order of $\mathrm{GL}_d(\mathbb{F}_w)$ for a chosen place w above $g_i(D)$ (Chapter III), and that order $\prod_{k<d}(q^d - q^k)$, $q = \mathrm{N}(w) = g_i(D)$, has the factor $q^d - 1$. (b) A place above $g_0(D)$ or $g_1(D)$ is a chosen place or the conjugate of one, and there are $f(D)$ chosen places above each of the two primes. (c) Take i with $g_i(D) \neq \ell$ and a chosen place above $g_i(D)$. □

LEAN `g_pow_sub_one_dvd_chosenGroupOrder` · `card_le_of_subset_placesAboveG` · `exists_chosen_notMem_placesDividing`

Lemma 5.6 (a lattice as a conjugate)**.** *Let D be admissible data, n a finite set, $\mathcal{O}$ a coefficient ring, $R\colon G_{E(D)} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ a homomorphism and $u\colon G_{E(D)} \to M_n(\mathcal{O})$ a lattice of R. Then there is an injective homomorphism of $\mathcal{O}$-algebras $F\colon M_n(\mathcal{O}) \to M_n(\bar{\mathbb{Q}}_\ell)$ with $F(u(g)) = R(g)$ for all g and $\det F(Y) = \det Y$, $\operatorname{tr} F(Y) = \operatorname{tr} Y$ for all $Y \in M_n(\mathcal{O})$.*

Proof. With $u(g) = Y_0^{-1} R(g) Y_0$, take $F(Y) = Y_0\, Y Y_0^{-1}$; conjugation preserves determinant and trace. □

LEAN `IsLatticeIn.exists_algHom`

Lemma 5.7 (small facts used later)**.** *Let D be admissible data.* (a) *For $i \in \{0,1\}$ and $1 \le d \le r(D)t(D)$: $v_\ell(g_i(D)^d - 1) \le \mathrm{cr}_\ell(D)$.* (b) *Let u be a lattice of R as in Lemma 5.6. If R is continuous, so is u; if $R(g) = 1$ then $u(g) = 1$; and $\operatorname{tr} u(g) = \operatorname{tr} R(g)$ for all g.* (c) *Let O be an auxiliary pair of type (r, n) over $E(D)$ and c a family of polynomials as in Definition 3.18. If $\mathsf{Apt}(O, c; \vartheta)$ for a twist ϑ, then $\#n \le 2\,(k(D) + m(D)(t(D) + 1))$. The endomorphism $\mathrm{pr}_O(c)$ is symmetric for the inner product of $\mathcal{F}(O)$, and it fixes its image. Infinitely many finite places of $E(D)$ are split and lie outside the places of O and their conjugates. If a subspace Z of $\mathcal{F}(O)$ is stable under the Hecke operators $H_w(X)$ of all split places w off the places of O and their conjugates and all $X \in \mathbb{C}$, so is its orthogonal complement.* (d) *Let $\mathfrak{P}$ be a place of $\bar{\mathbb{Q}}$ over a finite place $\mathfrak{q}$ of a CM field E as in the conventions, $\tau \in I_{\mathfrak{P}}$, and S a set of finite places with $\mathfrak{q}, \bar{\mathfrak{q}} \in S$. Then $\tau^{[b]} \in I_S$ for both signs b.* (e) *For a tame character τ of D, a finite idele x of $E(D)$ and $g \in G_{E(D)}$ with $\mathrm{Art}(x) = g$: $\mathrm{av}_\tau^-(g) = \iota(\tau(x/\bar{x}))^{-1}$.* (f) *For a complex embedding σ of $E(D)$: $\sigma(\sqrt{-d}) = \mathrm{sgn}_{\mathrm{CM}}(\sigma)\sqrt{-d}$, and the chosen embedding of the place of σ sends the root of unity ς of $E(D)$ to $\sigma(\varsigma)^{\mathrm{sgn}_{\mathrm{CM}}(\sigma)}$.*

Proof. (a) By Lemma 5.5(a), $v_\ell(g_i(D)^d - 1) \le v_\ell(\mathrm{ord}_{\mathrm{ch}}(D))$. (b) follows from Lemma 5.6 and from the fact that conjugation is a homeomorphism onto its image. (c) The bound on $\#n$ is (A3) with the inequalities of the data. The idempotent $e^{\otimes}_{r,c}$ is central and self-adjoint (note *Primitive labels*, Input 1), so its action is a symmetric projector. A place of prime norm is split, and all but finitely many places avoid a finite set. The Hecke operator $H_w(X)$ has an adjoint of the same kind, at $\bar{w}$, so the orthogonal complement of a stable space is stable. (d) $\tau^{[-]} = (\tau^{\mathsf{c}})^{-1}$ lies in the inertia group of the conjugate place, which is over $\bar{\mathfrak{q}}$. (e) is the defining property of the avatar; the weight is 0, so there is no factor at ℓ. (f) is read off from the two possible values of the chosen embedding. □

LEAN `ellVal_le_crudeUnit` · `IsLatticeIn.continuous` · `IsLatticeIn.eq_one` · `IsLatticeIn.trace_eq` · `IsApt.card_le` · `IsApt.isSymmetric_tupleProj` · `IsApt.tupleProj_eq_self_of_mem_range` · `frequently_isSplit_notMem_places` · `orthogonal_mem_invtSubmodule_hecke` · `idOrConjInv_mem_inertiaAbove` · `invAvatar_eq_of_artinMap_eq` · `cmSign_spec`

Lemma 5.8 (the image of the projector is Hecke-stable)**.** *Let D be admissible data, r, n finite sets, O an auxiliary pair of type (r,n) over $E(D)$ and c a family of polynomials as in Definition 3.18. Let w be a finite place of $E(D)$ with $\bar{w} \neq w$ that lies outside the places of O and their conjugates, and $X \in \mathbb{C}$. Then the image of $\mathrm{pr}_O(c)$ is stable under the Hecke operator $H_w(X) = \sum_i (-X)^i \, \mathrm{N}(w)^{i(i-1)/2} \, T_{w,i}$ on $\mathcal{F}(O)$.*

Proof. The image of the action of an element of a group algebra is stable under every operator that commutes with the action of the group. The Hecke operators at w are translations by elements supported at w, which commute with the translations by the lifts of elements of $\prod_i \mathrm{GL}_r(\mathbb{F}_{w_i})$, supported at the chosen places and trivial at w. □

LEAN `IsApt.range_tupleProj_mem_invtSubmodule`

6 Apt labels occur; attached representations are conjugate self-dual

Lemma 6.1 (the auxiliary pairs have apt labels)**.** (a) *Let D be admissible data, ϑ a twist of D, j a lift index $(1 \le j \le m(D))$ and c a family of labels that belongs to the good tuple families of D. Then $\mathsf{Apt}(O_D(\vartheta, j), c; \vartheta)$, where $O_D(\vartheta, j)$ is the j-th auxiliary pair of ϑ.* (b) *Let D be admissible data, ϑ a twist of D, p a pair of D and j such that p is of rank one for j (third note). Let $O_p(\vartheta)$ be the auxiliary pair of the L-value data of p and ϑ. Then $\mathsf{Apt}(O_p(\vartheta), (X-1)_i; \vartheta)$.* (c) *In the situation of* (b)*, the lift $\Xi_p(\vartheta)$ of the trivial character, a form of $O_p(\vartheta)$, lies in the image of $\mathrm{pr}_{O_p(\vartheta)}((X-1)_i)$.*

Proof. (a) (A1) and (A2) hold by the construction of the auxiliary pair (its tame part is $\omega_D^j \psi_\vartheta$). (A3): its type is (r,n) with $\#r = r(D)$ and rank difference $t(D)$, both odd. Each component of a good family is a good tuple: every c_i is a label of degree $r(D)$ none of whose roots is a root of unity of order dividing $M_\omega(P, \mu(D))$, and over each $g_k(D)$ some c_i is primitive; this gives (A4) and (A5). (A6): the roots of the twisted label are the roots of c_i times an element whose $M_\omega(P, \mu(D))$-th power is 1 (the name of a power of ω_D, the twist being trivial at the chosen places by Lemma 5.1(g)); if 1 were among them, a root of c_i would be a root of unity of order dividing $M_\omega(P, \mu(D))$.

(b) Unfold the hypothesis: the hermitian space is a line, $\#r = 1$, the rank difference is $2z_p - 1$, odd and at most $t(D)$, and the tame part of ν_p is ω_D^j. $X - 1$ is a label of degree 1, a chosen place above each $g_k(D)$ carries it, and the twisted label $X - x$ has $x \neq 1$ because ω_D^j is non-trivial at the chosen places for $1 \le j \le m(D)$.

(c) Write $\Xi = \Xi_p(\vartheta)$ and $G = \prod_i \mathrm{GL}_1(\mathbb{F}_{w_i})$. For $s \in G$ and a form y, $\langle \Xi, s \cdot y \rangle = \langle \Xi, y \rangle$: both sides are class sums of the theta descent of y (Chapter VII), and the descent of $s \cdot y$ is the descent of y translated by a lift of s, which the class sum absorbs. The idempotent of the label $X - 1$ in rank one is the average over the group twisted by the trivial character, so $e^{\otimes}_{r, X-1} = \#G^{-1} \sum_{s \in G} s$, r being the one-element index set of the rows. Write e for this element. As $\mathrm{pr}_{O_p(\vartheta)}((X-1)_i)$ is symmetric, $\langle \mathsf{e}\,\Xi, y \rangle = \langle \Xi, \mathsf{e}\,y \rangle = \langle \Xi, y \rangle$ for all y, so $\mathsf{e}\,\Xi = \Xi$. □

LEAN `isApt_auxPair` · `isApt_rank_one` · `trivialLift_mem_range_tupleProj_rank_one`

Lemma 6.2 (attached representations are conjugate self-dual)**.** *Let D be admissible data, n a finite set, f a form in rank $\#n$ over $E(D)$ (a function on the finite adelic points with values in the functions on a set M), H a system of Hecke polynomials and $\varrho\colon G_{E(D)} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ a homomorphism. If f is an eigenform for H and ϱ is attached to H, then $\mathsf{CSD}_{\#n}(\varrho)$.*

Proof. Continuity and the non-degeneracy of the trace form are part of attachment. For the trace identity let $\#n = k+1$ (for $\#n = 0$ there is nothing to prove) and write $c_i(Y)$ for the coefficient of X^i in the characteristic polynomial of a matrix Y. At a split place w of prime norm outside ℓ at which f is spherical at w and $\bar{w}$, the translate by the scalar ϖ_w of the i-th Hecke operator at $\bar{w}$ is the $(\#n - i)$-th Hecke operator at w (duality of the Hecke operators at a split place, an input of the unit on the Weil representation). Applied to f and compared coefficient by coefficient this gives $H_w(0)\, H_{\bar{w}}(k) = H_w(1)\ \mathrm{N}(w)^k$, where $H_w(i)$ denotes the coefficient of X^i. Now let the Frobenius data approach g. By attachment, ιH_w tends coefficientwise to the characteristic polynomial of $\varrho(g)$; the system $w \mapsto H_{\bar{w}}$ has $\varrho \circ (\cdot)^{\mathsf{c}}$ attached, because conjugation is a homeomorphism of $G_{E(D)}$ that carries Frobenius data at w to Frobenius data at $\bar{w}$; and $\mathrm{N}(w) = \mathrm{cyc}(s)^{-1}$ tends to $\mathrm{cyc}(g)^{-1}$ (Lemma 5.1(e),(f)). By the uniqueness of limits,

$$c_0(\varrho(g)) \cdot c_k(\varrho(g^{\mathsf{c}})) = c_1(\varrho(g)) \cdot \mathrm{cyc}(g)^{-k}.$$

Since $\operatorname{tr}\varrho(g^{\mathsf{c}}) = -c_k(\varrho(g^{\mathsf{c}}))$ and $\operatorname{tr}\varrho(g^{-1}) = -c_1(\varrho(g))/c_0(\varrho(g))$, this is the identity of Definition 3.12 with $t = \#n$. □

LEAN `isConjSelfDual_of_attached`

For a weight κ and a natural number N, $\mathcal{H}_{\kappa,N}$ denotes the type of the representations attached to forms of weight κ in rank N (Chapter VII): at the embedding of the CM type the multiset of the $i - 1 - \lambda_i$, at its conjugate that of the $\lambda_i + N - i$, $1 \le i \le N$, with λ the partition of κ.

Lemma 6.3 (local conditions pass to the conjugate place)**.** *Let D be admissible data, r a finite set, j a natural number, $R\colon G_{E(D)} \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$ a homomorphism and $\mathfrak{P}$ a place of $\bar{\mathbb{Q}}$. Suppose $\mathsf{CSD}_j(R)$ and that R is of finite type at the conjugate place $\mathfrak{P}^{\mathsf{c}}$. Then* (1) *$R$ is of finite type at $\mathfrak{P}$;* (2) *for every weight κ: if R has the jumps $\mathcal{H}_{\kappa,j}$ at $\mathfrak{P}^{\mathsf{c}}$, it has them at $\mathfrak{P}$;* (3) *for every $s \in I_{\mathfrak{P}^{\mathsf{c}}}$ and every polynomial Q: if Q kills s on R at $\mathfrak{P}^{\mathsf{c}}$, then Q kills $(s^{-1})^{\mathsf{c}}$ on R at $\mathfrak{P}$.*

Proof. Let $R'(g) = \mathrm{cyc}(g)^{-(j-1)}\, {}^t R(g^{-1})$, the dual of R twisted by cyc^{1-j}. Its trace form is non-degenerate, and by the trace identity of Definition 3.12 it has the traces of $g \mapsto R(g^{\mathsf{c}})$, whose span is semiprime. By the theorem of Brauer and Nesbitt in the form of the background library there is $Y \in \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$ with $R'(g)\, Y = Y R(g^{\mathsf{c}})$ for all g. Conjugation by c carries the place $\mathfrak{P}$ to $\mathfrak{P}^{\mathsf{c}}$, so the local clauses of R' at $\mathfrak{P}^{\mathsf{c}}$ are those of R at $\mathfrak{P}$ (transport along a conjugation, an input of the unit on ℓ-adic characters). It remains to read the clauses of R' at $\mathfrak{P}^{\mathsf{c}}$ from those of R there. The pairing $\langle x, y\rangle = {}^t x\, y$ between R' and R is equivariant up to $\mathrm{cyc}^{-(j-1)}$; so along a chart R' is potentially crystalline when R is, and the jump u of R' has the multiplicity of the jump $(j-1) - u$ of R. On inertia at a place prime to ℓ the cyclotomic character is trivial, so R' is trivial near 1 where R is: this is (1). For (2): the type $\mathcal{H}_{\kappa,j}$ is carried to itself by $u \mapsto (j-1) - u$ together with the exchange of an embedding with its conjugate, which is what the passage from $\mathfrak{P}^{\mathsf{c}}$ to $\mathfrak{P}$ does to the embeddings.

For (3) at a place prime to ℓ: $Q(R(s)) = 0$ gives, after transposition and because $\mathrm{cyc}(s) = 1$, $Q(R'(s^{-1})) = 0$, hence $Q(R((s^{-1})^{c})) = 0$; above ℓ an inertial element in a Weil pair has degree 0, and the clause of the pairing gives the same conclusion. □

LEAN `local_clauses_of_conjPlace`

Lemma 6.4 (a Hecke-stable space is spanned by isotypic vectors)**.** *Let D be admissible data, r, n finite sets, O an auxiliary pair of type (r, n) over $E(D)$, and Z, Z' subspaces of $\mathcal{F}(O)$. Suppose that Z is stable under the Hecke operators $H_w(X)$ of all split finite places w outside the places of O and their conjugates and all $X \in \mathbb{C}$, and that every $v \in Z$ which is non-zero and lies in an isotypic space belongs to Z'. Then $Z \subseteq Z'$.*

Proof. Z is finite-dimensional, fixed by the level that is trivial at the places of O, and stable under all the operators $T_{w,i}$ at the split places off these places (they are the coefficients of the polynomial $H_w(X)$). Such a space is spanned by its Hecke eigenvectors off the places of O, and a non-zero eigenvector of maximal level off these places lies in an isotypic space (two inputs of the units on local–global compatibility and on adelic cusp forms). □

LEAN `le_of_isotypic_mem`

7 Auxiliary primes

Proof of Theorem A. Step 1: units. Choose a finite set T_0 of finite places of E that contains $\bar{S} \cup S \cup \Sigma_\ell(E)$, meets R only inside that set, and is such that every finite idele is a principal idele times an idele that is a unit outside T_0 (finiteness of the class group). By Dirichlet's theorem the group of T_0-units of E has a finite set $\mathfrak{s}$ of generators. For $u \in \mathfrak{s}$ choose $y_u \in \bar{\mathbb{Q}}$ with $y_u^{\ell^N} = u$.

Step 2: the prime. Let $\mathcal{U}$ be the intersection of the kernel of cyc_{ℓ^N}, of the stabilisers of the y_u, and of the interior of W. It is open, and it contains z: the first two groups contain every conjugate of an inertia element at a place v of R outside $S \cup \bar{S} \cup \Sigma_\ell(E)$ (inertia at a place prime to ℓ fixes the ℓ^N-th roots of unity and the ℓ^N-th roots of the units at v, and the T_0-units are units at v because $v \notin T_0$), hence the normal subgroup generated by these inertia groups. By Chebotarev's theorem (Input 1) there are a finite place $\mathfrak{q} \notin T_0 \cup R \cup \bar{R}$ of prime norm, a place $\mathfrak{P}$ of $\bar{\mathbb{Q}}$ over $\mathfrak{q}$ and an $\mathrm{N}(\mathfrak{q})$-Frobenius $\sigma_0 \in \mathcal{U}$ at $\mathfrak{P}$ such that every $\mathrm{N}(\mathfrak{q})$-Frobenius at $\mathfrak{P}$ lies in $\mathcal{U}$. Since σ_0 fixes the ℓ^N-th roots of unity, $\mathrm{N}(\mathfrak{q}) = 1 + \ell^N k$ for some k.

Step 3: a character of the ideles. Let $\gamma = \tau_{\mathfrak{P}}$, the local Artin symbol of a unit u_0 of the completion $E_{\mathfrak{q}}$ whose residue generates $\mathbb{F}_{\mathfrak{q}}^{\times}$. Let $\rho_0 \colon E_{\mathfrak{q}}^{\times} \to \mathbb{F}_{\mathfrak{q}}^{\times}$ send t to the residue of $t\varpi_{\mathfrak{q}}^{-\operatorname{ord} t}$, and let $f_0 \colon E_{\mathfrak{q}}^{\times} \to \mathbb{Z}/\ell^N$ be ρ_0 followed by the homomorphism that sends the residue of u_0 to 1 (it exists because ℓ^N divides $\mathrm{N}(\mathfrak{q}) - 1$). If $\rho_0(t^k) = 1$, then $f_0(t) = 0$. For $u \in \mathfrak{s}$: u is a unit at $\mathfrak{q}$, the root y_u is a unit at $\mathfrak{P}$, and since σ_0 fixes y_u and is a Frobenius element, $y_u \equiv y_u^{\mathrm{N}(\mathfrak{q})}$ modulo $\mathfrak{P}$; so $u^k = y_u^{\mathrm{N}(\mathfrak{q})-1} \equiv 1$ modulo $\mathfrak{q}$ and $f_0(u) = 0$. Hence the homomorphism $a \mapsto f_0(a_{\mathfrak{q}})$ on the ideles that are units outside T_0 is trivial on the T_0-units, and by the choice of T_0 it extends uniquely to a homomorphism η on the finite ideles that is trivial on the principal ideles. It is continuous, with open kernel.

Step 4: the Galois character. By the existence theorem of class field theory (Input 2) there is a continuous homomorphism $h \colon G_E \to \mathbb{Z}/\ell^N$ with $h(g) = \eta(x/\bar{x})$ whenever $\mathrm{Art}(x) = g$. As $\mathrm{N}(\mathfrak{q})$ is prime, $\bar{\mathfrak{q}} \neq \mathfrak{q}$, so the conjugate of the local idele u_0 is a unit

idele supported at $\bar{\mathfrak{q}}$ and $h(\gamma) = f_0(u_0) = 1$. Put $\phi = \sigma_0\gamma^a$ with a the representative of $-h(\sigma_0)$; it is again an $\mathrm{N}(\mathfrak{q})$-Frobenius at $\mathfrak{P}$, so $\phi \in \mathcal{U} \subseteq W$, and $h(\phi) = 0$.

Step 5: the clauses. Let $T \subseteq S \cup \Sigma_\ell(E)$ and $l = (\mathfrak{P}, \gamma, \phi, h)$. The character η is trivial on the ideles that are units outside $T \cup \bar{T} \subseteq T_0$ and congruent to 1 at $\mathfrak{q}$ (they have $f_0(a_\mathfrak{q}) = 0$), and h is locally constant: h is from $\mathrm{U}(1)$ relative to T with modulus $\{\mathfrak{q}\}$. The other clauses of Definition 3.24 were established: $\mathrm{N}(\mathfrak{q}) \equiv 1 \bmod \ell^N$, $\mathfrak{P}$ over $\mathfrak{q}$, ϕ a Frobenius element, $\mathfrak{q} \notin T_0 \supseteq T$, and $\gamma = \tau_\mathfrak{P} \in I_\mathfrak{P}$ with $h(\gamma) = 1$. Finally $\mathfrak{q} \notin R \cup \bar{R}$ and $\mathrm{N}(\mathfrak{q})$ is prime. □

Input 1 (Chebotarev's theorem, existence form; background library, unit of Chebotarev's theorem). Let E be a number field embedded in $\bar{\mathbb{Q}}$, $\mathcal{U}$ a non-empty open subset of G_E and T a finite set of finite places. Then there are a finite place $\mathfrak{q} \notin T$ of prime norm, a place $\mathfrak{P}$ of $\bar{\mathbb{Q}}$ over it and an $\mathrm{N}(\mathfrak{q})$-Frobenius at $\mathfrak{P}$ in $\mathcal{U}$, such that every $\mathrm{N}(\mathfrak{q})$-Frobenius at $\mathfrak{P}$ lies in $\mathcal{U}$. *Reference* (as located by the second reading of this write-up): J. S. Milne, *Class Field Theory*, v4.03, Chapter VIII, §7, Theorem 7.4. *Relation:* an existence consequence of the density theorem for finite Galois extensions, with a chosen place above the prime; no density is used.

Input 2 (existence theorem of class field theory; background library, unit of the global Artin map). Let E be a totally complex number field embedded in $\bar{\mathbb{Q}}$ and θ a continuous homomorphism from the finite ideles of E to a group with a T_1 topology, trivial on the principal ideles. Then there is a continuous homomorphism μ on G_E with $\mu(g) = \theta(x)$ whenever $\mathrm{Art}(x) = g$. *Reference* (second reading): Milne, *Class Field Theory*, v4.03, Chapter V, §5, Theorems 5.3 and 5.5 and Remark 5.7(a), pp. 178–180. *Relation:* a special case (totally complex fields, finite ideles): the factorisation follows from the description of the kernel of the reciprocity map.

Lemma 7.1. *Let E be a CM field as in the conventions and $a \in G_E$. Then 4 divides $\mathrm{cyc}(a) - 1$ in $\mathbb{Z}_\ell$.*

Proof. If $\ell = 2$: E contains a primitive fourth root of unity, which a fixes, so $\mathrm{cyc}(a) \equiv 1 \bmod 4$. If ℓ is odd, 4 is a unit of $\mathbb{Z}_\ell$. □

LEAN `four_dvd_cycChar_sub_one`

Lemma 7.2 (the cyclotomic character at new places)**.** *Let E be a CM field as in the conventions, J a set, $\nu \in \mathbb{N}$, and $w = (w_j)_{j\in J}$, $\tau = (\tau_j)_j$, $\phi = (\phi_j)_j$ with $\mathsf{New}_\nu(w, \tau, \phi)$. Let $\mathcal{O}$ be a coefficient ring and $k = (j, b)$ a pair of an index and a sign. Then $\mathrm{cyc}(\tau^{[k]}) = 1$, and for every $e \in \mathbb{N}$ the element ℓ^ν divides $\mathrm{cyc}(\phi^{[k]})^{-e} - 1$ in $\mathcal{O}$.*

Proof. τ_j and $(\tau_j^{\mathsf{c}})^{-1}$ lie in inertia groups at places prime to ℓ, where cyc is trivial. ϕ_j fixes the ℓ^ν-th roots of unity, and so do its conjugate by c, its inverse and its powers; so cyc of these elements is $\equiv 1 \bmod \ell^\nu$. □

LEAN `IsNewPlaces.cycInv_withConjInv`

Lemma 7.3 (vanishing at the tame generators makes a cocycle unramified)**.** *Let E be a CM field as in the conventions, Λ a commutative ring, $e, \nu \in \mathbb{N}$, J a set*

and $w = (w_j)_{j\in J}$, τ, ϕ *with* $\mathsf{New}_\nu(w,\tau,\phi)$. *Let* n *be a finite set,* $\varrho\colon G_E \to M_n(\Lambda)$ *a homomorphism,* S *and* T *sets of finite places of* E, L *a set of relations and* $c\colon G_E \to (\Lambda/\ell^e)^n$ *a function. Suppose that* $c \in \mathrm{Sel}_T^L(\varrho;\, \Lambda/\ell^e)$, *that* $c(\tau^{[k]}) = 0$ *for every* $k = (j,b)$, *that* $T \subseteq S \cup w(J) \cup \bar{w(J)}$, *and that* $\varrho(g) = 1$ *for every* $g \in I_{S^\mathbb{C}}$ *with* $\mathrm{cyc}(g) = 1$. *Then* $c \in \mathrm{Sel}_S^L(\varrho;\, \Lambda/\ell^e)$.

Proof. Only the vanishing of c on the inertia above the places of $T \setminus S$ is to be shown. Such a place is w_j or $\bar{w_j}$ for some j, it is prime to ℓ, and ϱ is trivial on its inertia. For the tame generator at a place over w_j and a module killed by a power of ℓ, a Selmer cocycle that kills the tame generator on the appropriate side, τ_j or $(\tau_j^{\mathsf{c}})^{-1}$, is unramified at the corresponding place (an input of the unit on tame cocycles: wild inertia is pro-prime-to-ℓ and tame inertia is topologically generated by the tame generator). Both sides vanish by hypothesis. □

LEAN `IsNewPlaces.mem_selmerModule_of_tame_eq_zero`

Lemma 7.4 (Selmer modules are finitely presented, uniformly in the coefficients)**.** *Let* E *be a number field embedded in* $\bar{\mathbb{Q}}$, n *a finite set,* Λ *a commutative ring,* $m \geq 1$ *an integer,* $\varrho\colon G_E \to M_n(\Lambda)$ *a function,* W *a neighbourhood of* 1 *in* G_E, S *a finite set of finite places of* E *and* L *a set of relations. Then there are a finite set* $\mathcal{K}$, *a family* $(g_k, i_k)_{k\in\mathcal{K}}$ *in* $G_E \times n$ *and a* Λ*-submodule* $\mathcal{R}$ *of* $\Lambda^{\mathcal{K}}$ *containing* $m\Lambda^{\mathcal{K}}$ *such that, for every* Λ*-module* B *with* $mB = 0$ *on which* W *acts trivially* ($\varrho(g)b = b$ *for* $g \in W$, $b \in B^n$)*, the map* $c \mapsto (c(g_k)_{i_k})_k$ *is a bijection from* $\mathrm{Sel}_S^L(\varrho;\, B)$ *onto the set of the* $y \in B^{\mathcal{K}}$ *with* $\sum_k \rho_k y_k = 0$ *for all* $\rho \in \mathcal{R}$.

Proof. This is an outline of the formal proof. Choose an open subgroup $U \subseteq W$. The continuous cocycles of G_E that are unramified outside S, with values in the finite module of the $\mathbb{Z}/m$-valued functions on G_E/U, are finite in number (Hermite–Minkowski, in the form of the unit on Selmer groups); so there is an open subgroup V on which all of them vanish. Take $\mathcal{K} = (G_E/V) \times n$, with g_k a representative of the coset, and let $\mathcal{R}$ be generated by $m\Lambda^{\mathcal{K}}$ and by the vectors obtained by writing the relations of $\mathrm{Coc}(\varrho)$, the relations δ_g for $g \in I_{S^\mathbb{C}}$ and the relations of L in the coordinates. A $y \in B^{\mathcal{K}}$ killed by $\mathcal{R}$ is a function on G_E/V, locally constant, that obeys all these relations: a Selmer cocycle. Conversely a Selmer cocycle c with values in B^n is constant on the cosets of V: for a homomorphism $\chi\colon B \to \mathbb{Z}/m$ and an index i, the function $g \mapsto (xU \mapsto \chi((c(x) - c(g^{-1}x))_i))$ is a continuous cocycle of the kind above (because W acts trivially), so it vanishes on V; and the homomorphisms χ separate the elements of B. □

LEAN `exists_corepresenting_coords`

8 Families of interpolated values: proof of Theorem D

For a pair p of D and a tame character ϑ, $y_v(p,\vartheta)$ denotes the value at the uniformiser of a finite place v of the Hecke character $x \mapsto (\nu_p\vartheta)(x/\bar{x})$, and $\mathrm{N}(Q(\vartheta))$ the product of the norms of the places of $Q(\vartheta)$. A function F on the tame characters of D is an *integral measure at the modulus of* ϑ if there are finitely many points u_a of the norm-one adeles and complex coefficients c_a, integral at every prime that is prime to $\mathrm{N}(Q(\vartheta))$, with $F(\vartheta') = \sum_a c_a \vartheta'(u_a)$ for all tame ϑ' with $Q(\vartheta') = Q(\vartheta)$ (unit of the exchange identity).

Lemma 8.1 (an integral measure for a ramified pair)**.** *Let D be admissible data and p a pair of D that is ramified. Then there are a complex function F on the tame characters of D, a set $K \subseteq \{0,1\}$ and $m \in \mathbb{N}$ such that*

(1) *$F(1) \neq 0$, and F is an integral measure at the modulus of every tame character ϑ;*
(2) *for tame characters ϑ, ϑ' and a finite place $v \notin Q(\vartheta)$ with $Q(\vartheta') = Q(\vartheta) \cup \{v\}$, the same character of $\mathrm{U}(1)$ and the same tame part:*

$$F(\vartheta') = F(\vartheta) \cdot \prod_{k \in K} \prod_{\varepsilon = \pm 1} (1 - y_v(p,\vartheta)^{\varepsilon}\ \mathrm{N}(v)^{k - z_p});$$

(3) *for every twist ϑ: $F(\vartheta) = F(1) \cdot \mathrm{N}(Q(\vartheta))^m \cdot \varrho_p(\vartheta, 1)^{m+1}$.*

Proof. *The case $z_p \geq 2$.* Take for F the integral algebraic part of p: the normalised squared norm $\mathrm{ord}_{\mathrm{ch}}(D) \cdot \mathrm{den}^2 \cdot \|\Xi_p(\vartheta)\|^2 / \mathrm{N}(Q(\vartheta))^{\#n-1}$ of the lift of the trivial character, with $K = \{0\}$ and $m = 0$. It is non-zero at 1, an integral measure at each modulus, and $\varrho_p(\vartheta, 1) = F(\vartheta)/F(1)$ (three inputs of the unit of the exchange identity), which is (1) and (3). For (2): $F(\vartheta) = A \cdot L(z_p, (\nu_p \vartheta)^{-1})$ with a constant $A \neq 0$ and the Euler factors removed at the chosen places, at $Q(\vartheta)$ and at their conjugates. Passing from ϑ to ϑ' removes the Euler factors at v and $\bar{v}$; at $z_p > 1$ this multiplies the Euler product by $\prod_{w \in \{v, \bar{v}\}} (1 - a_w\, \mathrm{N}(w)^{-z_p})$, and the coefficients at v and $\bar{v}$ are y_v and y_v^{-1}.

The case $z_p = 1$. Let $\Lambda(\vartheta)$ be the period, at the level of the chosen places and of $Q(\vartheta)$, of the determinant kernel of rank 2 for the character $\nu_p \vartheta\, \upsilon_{\mathrm{U}(1)}$ (unit of the columns), let $N' \neq 0$ be an integer that makes the theta kernels of rank 2 integral at the primes prime to the modulus (unit of the rows; it is not the parameter N_0), and put $F(\vartheta) = |\,\mathrm{ord}_{\mathrm{ch}}(D)\, N'\, \Lambda(\vartheta)|^2$, $K = \{0,1\}$, $m = 1$. $\Lambda(\vartheta)$ is a sum, over the classes and over GL_2 of the residue ring of the modulus, of values of the kernel, each divided by the order of a stabiliser, which divides $\mathrm{ord}_{\mathrm{ch}}(D)$; the values times N' are integral at the primes prime to $\mathrm{N}(Q(\vartheta))$ after the point is moved into the integral points above the prime. So F, the product of such a sum with its conjugate, is an integral measure. Adding v to the modulus multiplies the period by the Euler factor $\prod_{k<2}(1 - \mathrm{N}(v)^k X)$ of rank 2 at v, hence F by the product in (2). By the class-sum identity of the determinant kernel, $|\Lambda(\vartheta)|^2$ is a constant times the product of L-values of $(\nu_p \vartheta)^{-1}$ (here p ramified is used), and that product is $\mathrm{N}(Q(\vartheta))^{z_p^2}$ times the product for ν_p^{-1} times $\varrho_p(\vartheta, 1)^2$; with $z_p = 1$ this is (3), and $F(1) \neq 0$. The formal proof of this case is a computation of some forty lines; the steps above are its outline. □

LEAN `Data.Pair.exists_isIntegralMeasure`

Lemma 8.2. *Let D be admissible data and I a finite set of finite places of $E(D)$ with $\mathsf{Auxil}(\Sigma_{\mathrm{bad}}(D,\ell);\, I)$. Then ℓ is prime to $\prod_{w \in I} \mathrm{N}(w)$, and so is $g(D)\, N(D)$.*

Proof. Each $\mathrm{N}(w)$, $w \in I$, is a prime number, and it does not divide $\ell\, g(D)\, N(D)$: otherwise w would contain that number and lie in $\Sigma_{\mathrm{bad}}(D, \ell)$. □

LEAN `coprime_q_of_isAuxiliary`

Lemma 8.3 (characters from U(1) are avatars of tame characters)**.** *Let D be admissible data, I a finite set of finite places of $E(D)$ with $\mathsf{Auxil}(\Sigma_{\mathrm{bad}}(D,\ell);\, I)$, and $\psi \colon G_{E(D)} \to \bar{\mathbb{Q}}_\ell$ a multiplicative homomorphism that is from $\mathrm{U}(1)$ relative to $\Sigma_\ell(E(D))$ with modulus I. Then there is a tame character τ of D with $Q(\tau) = I$ and $\psi(g) = \mathrm{av}_\tau^-(g)$ for all $g \in G_{E(D)}$.*

Proof. Let η be the idele character of ψ. Then $u \mapsto \iota^{-1}\eta(u_{\mathrm{f}})$, with u_{f} the finite part of a norm-one adele u read as a finite idele, is trivial on the norm-one elements of $E(D)$ and on the integral norm-one adeles that are congruent to 1 at the places of I: a character of the class group of $\mathrm{U}(1)$ of tame level I. Such a character is a character μ of $\mathrm{U}(1)$ of weight 0 with tame part trivial off I (an input of the unit on Hecke characters). Let τ be the tame character defined by I and μ^{-1} (Definition 3.36). By Lemma 5.7(e), $\mathrm{av}^-_\tau(g) = \iota(\mu(x/\bar{x})) = \eta(x/\bar{x}) = \psi(g)$ when $\mathrm{Art}(x) = g$, and the Artin map is surjective. □

LEAN `exists_tameChar_invAvatar_eq`

Proof of Theorem D. Let F, K, m be as in Lemma 8.1. For each I with $\mathsf{Auxil}(\Sigma_{\mathrm{bad}}(D,\ell); I)$ fix the presentation $F(\vartheta') = \sum_a c_{I,a}\vartheta'(u_{I,a})$ at the modulus I, and for each point a finite idele $y_{I,a}$ with $u_{I,a}$ of finite part $y_{I,a}/y^-_{I,a}$; for other I let the sum be empty. Define

$$\mathrm{val}(I,\chi) = \sum_a \iota(c_{I,a})\cdot\chi(g_{I,a})^{-s}, \qquad g_{I,a} \text{ an element with } \mathrm{Art}(y_{I,a}) = g_{I,a},$$

and let $\nu_{(k,\varepsilon)}$ be the avatar of $\nu_p^{\varepsilon}[0,-(z_p-k)]$. By Lemma 5.7(e),

$$\mathrm{val}(I,\mathrm{av}^-_\tau(\cdot)) = \iota(F(\tau^s)) \quad \text{for tame } \tau \text{ with } Q(\tau) = I, \qquad \mathrm{val}(\emptyset,1) = \iota(F(1)). \tag{1}$$

(F1) holds because avatars are continuous; (F2) because ℓ is prime to the norms of I (Lemma 8.2), so the $\iota(c_{I,a})$ are integral; (F4) by (1) and $F(1)\neq 0$.

(F3). Let I, $\mathfrak{q}$, χ, $\mathfrak{P}$, ϕ, a be as there. By Lemma 8.3, $\chi = \mathrm{av}^-_\tau(\cdot)$ with $Q(\tau) = I$, and by (1) and Lemma 8.1(2), $\mathrm{val}(I\cup\{\mathfrak{q}\},\chi) = \mathrm{val}(I,\chi)\cdot\prod_{k\in K}\prod_{\varepsilon}\iota(1-y_{\mathfrak{q}}^{\varepsilon}\,\mathrm{N}(\mathfrak{q})^{k-z_p})$. Each factor is $1-\nu_{(k,-\varepsilon)}(\phi)\chi(\phi)^{s\varepsilon}$, by the value of an avatar at a Frobenius element of a place where the character is unramified; it has valuation at most a. There are at most four factors, and the valuation of a product is at most the sum: the increase is at most $4a \le 8a$.

(1). The tame generator τ_w is the Artin symbol of a unit whose residue γ_0 generates $\mathbb{F}_w^\times$; as w is chosen and $\bar{w}$ is not, $\nu_i(\tau_w) = \iota(\psi_{p,w}(\gamma_0))^{\pm 1}$. This is a root of unity of order dividing $\mathrm{N}(w)-1$, different from 1 because p is ramified. By Lemma 5.1(d), $|\,\mathrm{N}(w)-1| \le |\nu_i(\tau_w)-1|$.

(2). Let τ be a twist with $Q(\tau)$ auxiliary. By (1) and Lemma 8.1(3) (the character τ^s is again a twist), $\mathrm{val}(Q(\tau),\mathrm{av}^-_\tau(\cdot)) = \iota(F(1))\cdot\mathrm{N}(Q(\tau))^m\cdot\iota(\varrho_p(\tau,s))^{m+1}$. The norm is prime to ℓ (Lemma 8.2) and $|\iota F(1)|\le 1$, so the hypothesis gives $\ell^{-a} \le |\iota\varrho_p(\tau,s)|^{m+1}$, that is $(m+1)\,v_\ell(\iota\varrho_p(\tau,s)) \le a$. □

9 Labelled columns, rows and the base level: proofs of Theorems B and C

Lemma 9.1. *Let $x \in \bar{\mathbb{Q}}_\ell$ be a root of unity and a, b natural numbers such that $(x^a)^{\ell^u} \neq (x^b)^{\ell^u}$ for all $u \in \mathbb{N}$. Then $|x^a - x^b| = 1$.*

Proof. Write the order of x as $\ell^u k$ with k prime to ℓ. Then $(x^a)^{\ell^u}$ and $(x^b)^{\ell^u}$ are distinct k-th roots of unity, so their difference has absolute value at least $|k| = 1$ (Lemma 5.1(d)); and it is a multiple of $x^a - x^b$ in $\bar{\mathbb{Z}}_\ell$. □

LEAN `norm_pow_sub_pow_eq_one`

Lemma 9.2 (the eigenvalues of inertia above g)**.** *Let D be admissible data, r, n, m finite sets, O an auxiliary pair of type (r,n) over $E(D)$, ϑ a twist and c a family of polynomials with $\mathsf{Apt}(O,c;\vartheta)$. Let $l\in\{0,1\}$, let v be a finite place of $E(D)$ containing $g_l(D)$, $\mathfrak{P}$ a place of $\bar{\mathbb{Q}}$ over v, and $t\in\mathbb{N}$. Then there is $\gamma\in I_{\mathfrak{P}}$, equal to $\tau_{\mathfrak{P}}$ if v is chosen, such that for every $s\in I_{\mathfrak{P}}$ there is a multiset $\mathfrak{M}$ of elements of $\bar{\mathbb{Q}}_\ell$ with $\#r+t$ elements such that*

(1) *every $y\in\mathfrak{M}$ satisfies $y^{g_l(D)^{(\#r)!}-1}=1$ and $y^{g_l(D)^d-1}=1$ for some $1\le d\le r(D)$, and $y\neq 1$ if $t=0$ and $s=\gamma$;*
(2) *for every homomorphism $R\colon G_{E(D)}\to \mathrm{GL}_m(\bar{\mathbb{Q}}_\ell)$, every $k\in\mathbb{N}$, every type $\mathcal{H}$ and every set S of finite places with $\mathsf{LabCol}(O;\,k,\mathcal{H},S;\,(\mathrm{tl}_O(c,i)\cdot(X-1)^t)_i;\,R)$, the polynomial $\prod_{y\in\mathfrak{M}}(X-y)$ kills s on R at $\mathfrak{P}$ and kills $(s^{-1})^{\mathsf{c}}$ on R at $\mathfrak{P}^{\mathsf{c}}$.*

Proof. If v is not chosen, $\bar v$ is; the statement for $\bar v$ and $\mathfrak{P}^{\mathsf{c}}$ gives the statement for v on applying $s\mapsto(s^{-1})^{\mathsf{c}}$. So let $v=w_i$ be a chosen place of O (A1), with faithful character T. By (A4) the twisted label is a label of degree $\#r$. Take $\gamma=\tau_{\mathfrak{P}}$. Every $s\in I_{\mathfrak{P}}$ has a tame exponent a for $f=(\#r)!$, which for $s=\gamma$ can be taken prime to $\mathrm{N}(v)^f-1$ (an input of the unit on local–global compatibility). Let $\mathfrak{M}$ be the multiset of the $\iota(T(x)^a)$ over the roots x of $\mathrm{tl}_O(c,i)\cdot(X-1)^t$. Then (2) at $\mathfrak{P}$ is the definition of a labelled column, and at $\mathfrak{P}^{\mathsf{c}}$ it follows from Lemma 6.3(3), a column being conjugate self-dual and of finite type at $\mathfrak{P}$. For (1): a root x of a label of degree $\#r$ over $\mathbb{F}_v$ satisfies $x^{q^f-1}=1$ and $x^{q^d-1}=1$ for the degree $d\le\#r\le r(D)$ of its minimal polynomial, $q=\mathrm{N}(v)=g_l(D)$; for $x=1$ take $d=1$. If $t=0$ and $s=\gamma$, then $\iota T(x)^a=1$ would give $x=1$ (a prime to the order, T and ι injective), against (A6). □

LEAN `exists_inertia_eigenvalues`

Proof of Theorem B. (W1) is given. Choose l with $g_l(D)\neq\ell$ and, by (A5), a chosen place w_i above $g_l(D)$ with c_i primitive and $\#r=r(D)$, or $c_i=X-1$ and $\#r=1$. Put $q=\mathrm{N}(w_i)=g_l(D)$, let $\mathfrak{P}$ be a place over w_i (it is prime to ℓ), ϕ a q-Frobenius at $\mathfrak{P}$, and $s\in I_{\mathfrak{P}}$ of tame exponent 1. By the labelled clause, $\prod_\xi(X-\iota T(\xi))$, over the roots ξ of $\mathrm{tl}_O(c,i)$, kills $R(s)$; so the eigenvalues of $R(s)$ are among the $\iota T(\xi)$, and every polynomial vanishing at these numbers kills $R(s)$ (R is of finite type at $\mathfrak{P}$).

The roots. Recall that $M_\omega(P,\mu(D))=(N_0d)^{\mu(D)+6}$, that 9 and $M_\omega(P,\mu(D))^2$ divide $q-1$ (third note; Chapter III), and let j be the index of (A2). In the primitive case the twisted label is again primitive (its roots are those of c_i times a root of unity of order dividing $M_\omega(P,\mu(D))$; note *Primitive labels*, the lemma on the scaling of roots): its roots are ξ^{q^a}, $a<\#r$, for a generator ξ of the multiplicative group of the extension of degree $\#r$ of $\mathbb{F}_{w_i}$. They satisfy $\xi^{q^{\#r}}=\xi$; the $\xi^{q^a\ell^u}$, $a<\#r$, stay distinct for every u ($\#r=r(D)$ is a power of 3 and $9\mid q-1$; an input of the unit on labels); and all have the same power $y=\xi^e$, $e=(q^{\#r}-1)/(q-1)$, their norm, which has order $q-1$. In the rank-one case the only root is the scaling element $y=\xi$, which names the character ω_{D,w_i}^{-j}, and $e=1$. In both cases $y^x=1$ implies that $M_\omega(P,\mu(D))$ divides $j\cdot x$: in the first because $M_\omega(P,\mu(D))^2\mid q-1$, in the second because ω_{D,w_i} has order $M_\omega(P,\mu(D))$ (so that the order of y is that number divided by its greatest common divisor with j).

(W3). Let $\mathcal{O}$ contain the eigenvalues of $R(s)$ and let u be a lattice of R. Tame inertia gives $R(\phi s\phi^{-1})=R(s)^q$ for the arithmetic q-Frobenius ϕ (only the fact that conjugation by ϕ permutes the eigenvalues by the q-th power is used); $R(s)^{q^{\#r}}=R(s)$;

and $R(s)$ has an eigenvalue $\zeta = \iota T(\xi)$ whose conjugates ζ^{q^a}, $a < \#r$, are distinct modulo the maximal ideal, by Lemma 9.1 and the distinctness above. Under these conditions $u(G_{E(D)})$ spans $M_r(\mathcal{O})$ and $u(\phi^{-\#r})$ is a scalar (an input of the background library on residual irreducibility); and $\mathrm{cyc}(\phi^{\#r}) = q^{\#r} = g_l(D)^{\#r}$. (Take $g = \phi^{\#r}$: its inverse acts by a scalar, hence so does g.)

(W4). $\gamma = s^e$ lies in $I_{\Sigma_g(D)\setminus\Sigma_\ell(E(D))}$ and acts by the scalar $\iota T(y)$, since $X^e - \iota T(y)$ vanishes at all $\iota T(\xi^{q^a})$. For $1 \le j \le 4\#r + 2$ and $a \in \mathbb{N}$, $|(\iota T(y)^{\ell^a})^j - 1| = 1$ by Lemma 9.1, provided $y^{j\ell^{a+u}} \neq 1$ for all u. Suppose $y^{j'\ell^{a+u}} = 1$ with $1 \le j' \le 4\#r + 2$. Then $M_\omega(P, \mu(D))$ divides $j\,j'\ell^{a+u}$. Let $p_0 \neq \ell$ be a prime dividing $N_0 d$; one of 2 and 5 will do, as both divide N_0. Then $p_0^{\mu(D)}$ divides $j\,j'$, so that

$$2^{\mu(D)} \le j\,j' \le m(D)\,(4r(D) + 2),$$

which one of the inequalities of the data excludes (Chapter III).

(W2). Let $v \in \Sigma_g(D) \setminus \Sigma_\ell(E(D))$ contain $g_{i'}(D)$ and let $\mathfrak{P}'$ be a place over v. Take γ and the multisets of Lemma 9.2 with $t = 0$, and $N = g_{i'}(D)^{(\#r)!} - 1$, a unit at $\mathfrak{P}'$. Then $R(s'^N) = 1$ for $s' \in I_{\mathfrak{P}'}$, because $X^N - 1$ vanishes at the eigenvalue candidates. The matrix $R(\gamma^{[b]})$ has $\#r$ eigenvalues x, each $\neq 1$ with $x^{g_{i'}(D)^d - 1} = 1$ for some $1 \le d \le r(D)$ (at $\mathfrak{P}'$ or at the conjugate place, according to b); so $v_\ell(x - 1) \le v_\ell(g_{i'}(D)^d - 1) \le \mathrm{cr}_\ell(D)$ by Lemma 5.1(d) and Lemma 5.7(a), and $v_\ell(\det(R(\gamma^{[b]}) - 1)) \le \#r \cdot \mathrm{cr}_\ell(D)$. □

Lemma 9.3 (the quadratic relation on the inertia at the twist primes)**.** *Let D be admissible data, r a finite set, n a non-empty finite set, O an auxiliary pair of type (r, n) over $E(D)$, ϑ a twist and c a family of polynomials with $\mathsf{Apt}(O, c; \vartheta)$; suppose $\mathsf{Auxil}(\Sigma_{\mathrm{bad}}(D, \ell); Q(\vartheta))$. Let f be a form of the weight of O on the standard definite space of rank $\#n$, H a system of Hecke polynomials and $R\colon G_{E(D)} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ a homomorphism. Suppose that f is an eigenform for H, that R is attached to H, and that at each twist place w of O the form f has the parahoric type with characters $(1, \psi_{O,w}^{-1})$. Then for every $g \in I_{Q(\vartheta)\cup Q(\bar{\vartheta})}$,*

$$(\mathrm{nm}_O(R)(g) - \mathrm{av}^-_\vartheta(g))(\mathrm{nm}_O(R)(g) - 1) = 0.$$

Proof. Let $g = s \in I_{\mathfrak{P}}$ with $\mathfrak{P}$ over $w \in Q(\vartheta) \cup Q(\bar{\vartheta})$; w is prime to ℓ. As $\chi_{1,O}^{-1}(s) = 1$ (Lemma 5.4 with $k = 0$), it suffices that R is of finite type at $\mathfrak{P}$ and that $(X - \mathrm{av}^-_\vartheta(s))(X - 1)$ kills s on R at $\mathfrak{P}$. If $w \notin Q(\vartheta)$, apply the case of $\bar{w}$ to $(s^{-1})^{\mathsf{c}}$ and transfer it by Lemmas 6.2 and 6.3, using $\mathrm{av}^-_\vartheta(s^{\mathsf{c}}) \cdot \mathrm{av}^-_\vartheta(s) = 1$ (Lemma 5.2 with $k = j = 0$). So let w be a twist place of O and a a tame exponent of s. A parahoric type gives a product label (Frobenius reciprocity), here with the characters 1 and $\psi_{\vartheta,w}^{-1}$ (the tame part of O at w is that of ϑ, since ω_D is trivial off the chosen places); and a representation attached to an eigenform with a product label is of finite type and labelled accordingly (two inputs of the unit on local–global compatibility): the polynomial with the roots 1 and $\iota T(x_{\psi_{\vartheta,w}^{-1}})^a$ kills s. By Lemma 5.3 the second root is $\mathrm{av}^-_{\bar{\vartheta}}(s)$. □

LEAN `normalisedRep_quadratic_on_inertia`

Lemma 9.4 (twisting a labelled column by an avatar)**.** *Let D be admissible data, r, n, m finite sets, O an auxiliary pair of type (r, n) over $E(D)$, c and c' families of polynomials as in Definition 3.18, ϑ a twist, $k \in \mathbb{N}$, $\mathcal{H}$ a type, S a set of finite places*

and $R\colon G_{E(D)} \to \mathrm{GL}_m(\bar{\mathbb{Q}}_\ell)$ *a homomorphism with* $\mathsf{LabCol}(O;\, k, \mathcal{H}, S;\, c;\, R)$. *Let* a *be a character of* $\mathrm{U}(1)$ *with a weight* e *and a tame part* ψ, *let* k', j *be integers and* b *a natural number with* $(b-1) + (2j - k') = k - 1$ *(both differences* $b-1$ *and* $k-1$ *truncated at* 0*). Suppose* $\mathsf{Apt}(O, c';\, \vartheta)$ *and that* $\psi_v = 1$ *for every finite place* v *that is not chosen. Then*

$$\mathsf{LabCol}(O;\, b, \mathcal{H}', S;\, \tilde{c};\, \mathrm{av}(a[k', j]) \cdot R),$$

where $\mathcal{H}'(\sigma)$ *is* $\mathcal{H}(\sigma)$ *shifted by minus the infinity type of* $a[k', j]$ *at* σ, *and* $\tilde{c}_i$ *is* c_i *with its roots multiplied by* $x_{\psi_{w_i}}$.

Proof. The twist of a continuous representation with non-degenerate trace form by a continuous character is of the same kind, and by Lemma 5.2 the trace identity for $\mathrm{av}(a[k', j])R$ holds with the exponent $(k-1) - (2j - k') = b - 1$. At a place $\mathfrak{P}$ over v as in Definition 3.15, the avatar is of finite type, has its infinity type as jumps, and is good where the character is unramified, that is at the places that are not chosen (the hypothesis on ψ); these clauses and the jumps add under twisting (an input of the unit on ℓ-adic characters). For the labels: if $X - \alpha$ kills s on the avatar and $Q(\alpha X)$ kills s on R, then Q kills s on the twist; by Lemma 5.3, $\alpha = \iota T(x_{\psi_{w_i}})^{a'}$ for s of tame exponent a' (the chosen place w_i is split and its conjugate is not chosen), which scales the roots of c_i by $x_{\psi_{w_i}}$. □

LEAN `IsLabelledColumn.twist_avatar`

Lemma 9.5 (product labels give labelled columns)**.** *Let* D *be admissible data,* r, n *finite sets,* m *a non-empty finite set,* O *an auxiliary pair of type* (r, n) *over* $E(D)$, c *and* c' *families of polynomials as in Definition 3.18,* ϑ *a twist with* $\mathsf{Apt}(O, c';\, \vartheta)$, S *a set of finite places,* κ *a weight,* f *a form of weight* κ *on the standard definite space of rank* $\#m$, H *a system of Hecke polynomials and* $R\colon G_{E(D)} \to \mathrm{GL}_m(\bar{\mathbb{Q}}_\ell)$. *Suppose that* f *is an eigenform for* H *and* R *is attached to* H*; that* f *is spherical at every finite place* v *which lies neither in* $S \setminus (\Sigma_\ell(E(D)) \cup \Sigma_g(D))$ *nor among the chosen places of* O *and their conjugates; and that for a natural number* N *and block sizes* $t_1, \dots, t_N$, *at each chosen place* w_i *the form* f *has the product label of* c_i, *of degree* $\#r$, *with trivial characters on the blocks of sizes* $t_\bullet$. *Then*

$$\mathsf{LabCol}(O;\, \#m, \mathcal{H}_{\kappa, \#m}, S;\, (c_i \cdot (X-1)^{\#m - \#r})_i;\, R).$$

Proof. R is conjugate self-dual by Lemma 6.2. Where f is spherical, R is good with the jumps of the weight (Input 3 below); at a chosen place it is of finite type with these jumps and labelled by the product label (the input used in Lemma 9.3), the trivial characters contributing the factor $(X-1)^{\sum t_\bullet}$ with $\#r + \sum t_\bullet = \#m$; at the conjugates of the chosen places the clauses follow from Lemma 6.3. □

LEAN `isLabelledColumn_of_hasProductLabel`

Input 3 (Galois representations of eigenforms on definite unitary groups; background library, unit of the Galois representations). Every eigenform f on the standard definite unitary group of rank N over a CM field as in the conventions has an attached ℓ-adic representation. Each attached representation is, at a place where f is spherical, unramified (prime to ℓ) or crystalline (above ℓ) with the jumps of the weight of f, and the Hecke polynomial kills the geometric Frobenius element. *References* (second reading): G. Chenevier, M. Harris, *Construction of automorphic Galois representations II*, Theorem 3.2.3; J.-P. Labesse, *Changement de base CM et séries discrètes*, Corollaire 5.3; M. Harris, K.-W. Lan, R. Taylor, J. Thorne, *On the rigid cohomology of certain Shimura varieties*, Theorem A. *Relation:* a weaker form of the published attachment: closeness of characteristic polynomials to Hecke polynomials at most Frobenius elements, in place of equality at all unramified split places.

Proof of Theorem C. This is an outline of the formal proof: it unfolds definitions of the units on forms, and the identities between adelic sums are left to the formal text. Let G be the group $\prod_i \mathrm{GL}_r(\mathbb{F}_{w_i})$ and π the action on forms on the standard space of rank $\#n$ by right translation by the adelic points supported at the places of O. The group algebra of G acts on the forms of the level and character of O (through lifts, twisted by $\psi_O \circ \det$), and for a form y of O the action of $e^{\otimes}_{r,c}$ on its smooth form is the smooth form of $\mathrm{pr}_O(c)(y)$.

The base level. Its places are those of O; its space is the image of $e^{\otimes}_{r,c}$ in the smooth forms of that level. It is stable under the operators that commute with the translations at these places, the level is open, and the stabilisers have order dividing $\mathrm{ord}_{\mathrm{ch}}(D)$ because $\#n \le 2(k(D) + m(D)(t(D)+1))$ (Lemma 5.7(c) and an input of Chapter III).

(1) For y in the image of the projector, $\mathrm{pr}_O(c)(y) = y$, so its smooth form is fixed by $e^{\otimes}_{r,c}$.

(2) $\#G$ divides $\mathrm{ord}_{\mathrm{ch}}(D)$ (Chapter III; $\#r \le r(D)$), and $\mathrm{ord}_{\mathrm{ch}}(D) \cdot e^{\otimes}_{r,c}$ is a sum over $g \in G$ of an integer times $\#G \cdot e^{\otimes}_{r,c}(g)$ times g. The number $\#G \cdot e^{\otimes}_{r,c}(g)$ is a value of a character of G, an algebraic integer; the scalar by which the lift of g is twisted is a root of unity; and a translate of an ℓ-integral form by an element of the integral level is ℓ-integral.

(3) Let x be an eigensystem of level u and f a non-zero form in it; it is the smooth form of a form of O in the image of $e^{\otimes}_{r,c}$, an eigenform, and $\varrho_{\underline{x}}$ is attached to its Hecke system. It is spherical outside $Q(\vartheta) \cup Q(\bar{\vartheta}) \cup u(K) \cup u(\bar{K})$ and the chosen places and their conjugates. At a chosen place w_i its type under $\mathrm{GL}_r(\mathbb{F}_{w_i})$ is cut out by the i-th factor of $e^{\otimes}_{r,c}$, twisted by ψ_{O,w_i}: this is a parahoric type, which by Frobenius reciprocity gives the product label of $\mathrm{tl}_O(c,i)$ with one block of size $\#n - \#r$ and trivial character (here (A6) is used: the label must be prime to $X-1$). By Lemma 9.5, ϱ_x is a labelled column with jumps $\mathcal{H}_{\kappa,\#n}$, exponent $\#n$ and the labels $\mathrm{tl}_O(c,i)(X-1)^{\#n-\#r}$. The character $\chi^{-1}_{1,O}$ is the avatar of $a_O^{-1}[-\#r, 0]$, whose tame part is trivial; so Lemma 9.4, with $k' = -\#r$, $j = 0$ and $b = \#n - \#r$, shows that $R = \mathrm{nm}_O(\varrho_x)$ is a labelled column with exponent $\#n - \#r$, the same labels and the shifted type, which is the type $\mathcal{H}$ of the theorem; it depends on O only. At a twist place w the forms of O transform under the parahoric subgroup by the

character $\psi_{O,w}^{-1}$ of the determinant of the block indexed by r; this is the parahoric type of Lemma 9.3, which gives the quadratic relation. □

10 Strings of roots of Hecke polynomials

Lemma 10.1. *Let F_1, F_1', F_2, F_3 be polynomials over $\bar{\mathbb{Q}}_\ell$. Suppose that F_2 is monic, that all coefficients of F_3 have absolute value at most 1, that all coefficients of $F_1F_2 - F_3$ and of $F_3 - F_1'F_2$ have absolute value less than 1, and that $|F_1'(1)| \geq 1$. Then there is a polynomial e over $\bar{\mathbb{Z}}_\ell$ such that F_1 divides e in $\bar{\mathbb{Q}}_\ell[X]$ and $e(1) = 1$.*

Proof. Division by a monic polynomial does not increase the maximum of the absolute values of the coefficients. So $F_1 - F_1'$, the quotient of $(F_1F_2 - F_3) + (F_3 - F_1'F_2)$ by F_2, has coefficients of absolute value less than 1, and $|F_1(1)| \geq 1$; likewise F_1 has coefficients of absolute value at most 1. Take $e = F_1/F_1(1)$. □

LEAN `exists_integral_multiple_eval_one`

Lemma 10.2 (no eigenvalue 1 near one of the elements δ_i)**.** *Let D be admissible data, n a finite set, $T\colon G_{E(D)} \to M_n(\bar{\mathbb{Q}}_\ell)$ a continuous function, $(\delta_i)_{i\in\mathbb{N}}$ a sequence in $G_{E(D)}$, $y \in \bar{\mathbb{Q}}_\ell$ and $Y \in M_n(\bar{\mathbb{Q}}_\ell)$. Suppose that $T(\delta_i) = y^i\,Y$ for all i and that $|y^j - 1| = 1$ for $1 \leq j \leq 4\#n + 2$. Then there is an i such that $|\det(1 - T(\phi))| \geq 1$ for all ϕ in a neighbourhood of δ_i.*

Proof. Otherwise, by the continuity of T and of the determinant, $|\det(1 - y^iY)| < 1$ for $i = 0, \dots, \#n$. Now $\det(1 - y^iY) = \prod_\alpha (1 - y^i\alpha)$ over the eigenvalues α of Y with multiplicity, so for each such i some eigenvalue α_i has $|1 - y^i\alpha_i| < 1$. There are at most $\#n$ eigenvalues, so $\alpha_i = \alpha_j$ for some $i < j \leq \#n$; call it α. Then $|y^i\alpha| = |y^j\alpha| = 1$, hence $|y| = |\alpha| = 1$, and $|y^{j-i} - 1| = |y^j\alpha - y^i\alpha| < 1$, against the hypothesis for the exponent $j - i \leq \#n$. □

LEAN `exists_eventually_no_eigenvalue_one`

Lemma 10.3 (the string of roots)**.** *Let D be admissible data, r a finite set, n a non-empty finite set, O an auxiliary pair of type (r, n) over $E(D)$, x an isotypic space of forms of the weight of O on the standard definite space of rank $\#n$ and $f \in x$, $f \neq 0$. Suppose that for every $g \in G_{E(D)}$ the matrix $\mathrm{nm}_O(\varrho_x)(g)$ has the eigenvalue 1. Let w be a finite place of $E(D)$ that is split and does not contain ℓ, and S a polynomial such that f has the Hecke polynomial S at w. Then, for every $i < \#n - \#r$, the number $\nu_w\,\mathrm{N}(w)^i$ is a root of S, with ν_w as in Lemma 5.4.*

Proof. Let s be a geometric Frobenius element at $\mathfrak{P}_w$ (the inverse of an $\mathrm{N}(w)$-Frobenius). By Input 3 the polynomial ιS kills $\varrho_x(s)$, and by Lemma 5.4 with $k = 1$, $\chi_{1,O}^{-1}(s) = \iota(\nu_w)^{-1}$. So the eigenvalue 1 of $\mathrm{nm}_O(\varrho_x)(s)$ is an eigenvalue $\iota(\nu_w)$ of $\varrho_x(s)$, and ν_w is a root of S. The local component of f at w generates a unitary unramified representation of $\mathrm{GL}_{\#n}$, and for the Satake parameters of such a representation a root c of the Hecke polynomial with $|c|^2\,\mathrm{N}(w)^k = \mathrm{N}(w)^{\#n}$ brings the roots $c, c\,\mathrm{N}(w), \dots, c\,\mathrm{N}(w)^{k-1}$ (an input of the unit on unramified representations of GL_N); here $k = \#n - \#r$ by the absolute value of ν_w. □

LEAN `isRoot_hecke_string`

Lemma 10.4 (the Hecke polynomial is the string times an integral factor)**.** *Let D be admissible data, r a finite set, n a non-empty finite set, O an auxiliary pair*

of type (r,n) over $E(D)$, x an isotypic space of forms of the weight of O on the standard definite space of rank $\#n$, and f a form with $f \in x$, $f \neq 0$, such that every $\mathrm{nm}_O(\varrho_x)(g)$, $g \in G_{E(D)}$, has the eigenvalue 1. Suppose moreover that all coefficients of the characteristic polynomials of the matrices $\mathrm{nm}_O(\varrho_x)(g)$ have absolute value at most 1. Then for most Frobenius data $(w,\mathfrak{P},\phi)$ the following holds: for every polynomial S over $\bar{\mathbb{Q}}_\ell$ with $|S(1)| \geq 1$ such that the characteristic polynomial of $\mathrm{nm}_O(\varrho_x)(\phi)$ is congruent to $S \cdot \prod_{i<\#n-\#r}(X - \mathrm{cyc}(\phi)^{-i})$ coefficientwise modulo the maximal ideal, there is a complex polynomial e with ℓ-integral coefficients and $e(1) = 1$ such that the Hecke polynomial $H_x(w)$ of x at w satisfies

$$H_x(w)(d_0 X) = \prod_{i<\#n-\#r} (X - \mathrm{N}(w)^i) \cdot (a \text{ divisor of } e), \qquad d_0 = \iota^{-1}(\chi_{1,O}^{-1}(\phi))^{-1}.$$

Proof. For most Frobenius data, w is split and prime to ℓ, f has the Hecke polynomial $H_x(w)$ at w, and the characteristic polynomial of $\varrho_x(\phi)$ is within $1/2$ of $\iota H_x(w)$ (attachment). By Lemma 10.3 and Lemma 5.4, $H_x(w)(d_0X)$ is divisible by the string $\prod_{i<\#n-\#r}(X - \mathrm{N}(w)^i)$; let F_1 be ι of the quotient, suitably scaled. By Lemma 5.1(f) the string is $\prod(X - \mathrm{cyc}(\phi)^{-i})$, monic. Lemma 10.1, with F_2 the string, F_3 the characteristic polynomial of the normalised representation and $F_1' = S$, gives e. □

LEAN `eventually_heckePoly_eq_string_mul`

11 The loss at one place above ℓ: proof of Theorem E

In this section ς is the root of unity of order $N_1(D)$ that generates $E(D)$ over $\mathbb{Q}(\sqrt{-d})$; the infinite places of $E(D)$ form a group, identified with $(\mathbb{Z}/N_1(D))^\times$, and for a divisor m of $N_1(D)$, cl_m is the reduction of a place modulo m (Chapter III); $[\sigma]$ is the infinite place of a complex embedding σ; $\mathrm{thick}(D)$ is the set of thick places.

Lemma 11.1. *Let D be admissible data, $e\colon E(D) \to \bar{\mathbb{Q}}_\ell$ a ring homomorphism and $g \in G_{\mathbb{Q}_\ell}$. If $g(\zeta) = \zeta$ for every $\zeta \in \bar{\mathbb{Q}}_\ell$ with $\zeta^{N_1(D)d} = 1$, then $g(e(x)) = e(x)$ for every $x \in E(D)$.*

Proof. $E(D)$ lies in the cyclotomic field of conductor $N_1(D)d$ (Lemma 5.1(a)), e extends to it, and g fixes the image of its generator. □

LEAN `fixes_emb_of_fixes_rootsOfUnity`

Lemma 11.2 (the class of a ratio of two places)**.** *Let D be admissible data, $e\colon E(D) \to \bar{\mathbb{Q}}_\ell$ a ring homomorphism, $g, g' \in G_{\mathbb{Q}_\ell}$ and $a, n \in \mathbb{N}$. Suppose that $(g'g^{-1})(e(\varsigma)^{\ell^a}) = (e(\varsigma)^{\ell^a})^{\ell^n}$. Let y be an infinite place of $E(D)$ such that*

$$[\mathrm{ce}(e,g')] \cdot y = [\mathrm{ce}(e,g)],$$

and m a natural number that divides $N_1(D)$ and is prime to ℓ^a. Then

$$\mathrm{cl}_m(y) = \mathrm{sgn}_{\mathrm{CM}}(\mathrm{ce}(e,g)) \cdot \mathrm{sgn}_{\mathrm{CM}}(\mathrm{ce}(e,g')) \cdot \ell^n \quad \text{in } (\mathbb{Z}/m)^\times.$$

Proof. Write $\sigma = \mathrm{ce}(e,g)$, $\sigma' = \mathrm{ce}(e,g')$. The chosen embedding of the place $[\sigma']\,y$ sends ς to the $\mathrm{cl}(y)$-th power of its image under the chosen embedding of $[\sigma']$, and the chosen embedding of a place $[\sigma]$ sends ς to $\sigma(\varsigma)^{\mathrm{sgn}_{\mathrm{CM}}(\sigma)}$ (Lemma 5.7(f)). Applying $g'\iota$, raising to the power ℓ^a and inserting the hypothesis, one finds that $e(\varsigma)$ to the power $\ell^a(\mathrm{sgn}_{\mathrm{CM}}(\sigma)\,\mathrm{sgn}_{\mathrm{CM}}(\sigma')\,\mathrm{cl}(y) - \ell^n)$ is 1. As $e(\varsigma)$ has order $N_1(D)$, the number $N_1(D)$ divides that exponent, and m, prime to ℓ^a, divides the second factor. □

LEAN placeClass_ratio

Lemma 11.3 (spacing of thick places)**.** *Let D be admissible data, $e\colon E(D)\to\bar{\mathbb{Q}}_\ell$ a ring homomorphism, $g,g',g_1\in G_{\mathbb{Q}_\ell}$ and $a,i,n\in\mathbb{N}$. Suppose that $[\mathrm{ce}(e,g)]$ and $[\mathrm{ce}(e,g')]$ are thick, that $(g'g^{-1})(e(\varsigma)^{\ell^a})=(e(\varsigma)^{\ell^a})^{\ell^n}$ and $(g'g_1^{-1})(e(\varsigma)^{\ell^a})=(e(\varsigma)^{\ell^a})^{\ell^i}$, and that $\ell\le k(D)t(D)$, $0<i$, $i<n$ and $n\le 27\,\mu(D)$. Then $[\mathrm{ce}(e,g_1)]$ is not thick.*

Proof. Suppose it is. Squaring the identity of Lemma 11.2 removes the signs: the ratios of the three places have the classes ℓ^{2n} and ℓ^{2i} modulo every divisor of $N_1(D)$ that is prime to ℓ^a. If $\ell=2$, put $k_3=\lfloor\log_3\mu(D)\rfloor+5$. The divisor 3^{k_3} of $N_1(D)=N_0\cdot 3^{k_3}\cdot 2^{\mu(D)}$ is prime to 2, and the thick places are $\equiv -1$ modulo 3^{k_3} (Chapter III: they are $\equiv -1$ modulo the divisors of $N_0\cdot 3^{k_3}$, not modulo other powers of 3); so $4^i\equiv 1$ modulo 3^{k_3}. As $v_3(4-1)=1$, the order of 4 modulo 3^{k_3} is $3^{k_3-1}=27\cdot 3^{\lfloor\log_3\mu(D)\rfloor+1}$, which is greater than $27\mu(D)\ge n>i$: a contradiction. If ℓ is odd, take a power of 2 dividing $N_1(D)$: the three-term condition on thick places (Chapter III) gives $\ell^{2i}\equiv 1$ modulo $2^{\mu(D)+1}$, hence $2^{\mu(D)}\le(\ell+1)\cdot 2i$, against $\ell\le k(D)t(D)$, $i<27\mu(D)$ and the inequalities of the data. □

LEAN not_thick_of_short_arc

Lemma 11.4 (thick places have one sign when ℓ divides the conductor)**.** *Let D be admissible data, $e\colon E(D)\to\bar{\mathbb{Q}}_\ell$ a ring homomorphism and $g,g'\in G_{\mathbb{Q}_\ell}$. Suppose that $[\mathrm{ce}(e,g)]$ and $[\mathrm{ce}(e,g')]$ are thick and that ℓ divides $N_1(D)\,d$. Then $\mathrm{sgn}_{\mathrm{CM}}(\mathrm{ce}(e,g))=\mathrm{sgn}_{\mathrm{CM}}(\mathrm{ce}(e,g'))$.*

Proof. Suppose the signs differ. If $\ell\mid N_1(D)$: put $c_0=(1+\sqrt{-d})/2$, so that $c_0(1-c_0)=(d+1)/4$, which ℓ divides (Chapter III). The two embeddings send $\sqrt{-d}$ to opposite values, so $g\,g'^{-1}$ exchanges $e(c_0)$ and $1-e(c_0)$; it is an isometry, so both have the same absolute value, their product has absolute value less than 1, and their sum is 1: a contradiction with the ultrametric inequality. If $\ell\nmid N_1(D)$, then $\ell=d$, and Lemma 11.2 with $a=0$ and $m=N_0$ gives the class $-d^n$ for the ratio of two thick places, whose class modulo N_0 is 1; but no power of d is $\equiv -1$ modulo N_0 (Chapter III). □

LEAN cmSign_eq_of_thick

Lemma 11.5 (the rank of the crystalline cocycles)**.** *Let $\mathcal{O}$ be a coefficient ring, Γ a topological group over $G_{\mathbb{Q}_\ell}$ with homomorphism j, L a Galois lattice over $\mathcal{O}$ for Γ, E a CM field as in the conventions and $e\colon E\to\bar{\mathbb{Q}}_\ell$ a ring homomorphism. Suppose that j is an open embedding whose image is the group of the elements fixing $e(E)$ pointwise, that L is potentially crystalline, and that $\mathcal{H}$ is a type such that for every $g\in G_{\mathbb{Q}_\ell}$ the lattice L has the jumps $\mathcal{H}(\mathrm{ce}(e,g))$ at $\mathrm{emb}_{\mathcal{O}}(g)$. Then*

$$\mathrm{rk}_{\mathcal{O}}\,Z_{\mathrm{f}}(L)=\mathrm{rk}_{\mathcal{O}}\,L+\sum_{\sigma}\#\{u\in\mathcal{H}(\sigma):u<0\},$$

the sum over the complex embeddings σ of the form $\mathrm{ce}(e,g)$, $g\in G_{\mathbb{Q}_\ell}$.

Proof. By the rank formula of Bloch and Kato over $\mathbb{Z}_\ell$ (Input 4),

$$\mathrm{rk}_{\mathbb{Z}_\ell}\,Z_{\mathrm{f}}(L)=\mathrm{rk}_{\mathbb{Z}_\ell}\,L+[G_{\mathbb{Q}_\ell}:j(\Gamma)]\cdot\#\{\text{negative jumps over }\mathbb{Z}_\ell\}.$$

The jumps over $\mathbb{Z}_\ell$ are the union of the jumps at the $[\mathcal{O}:\mathbb{Z}_\ell]$ embeddings of $\mathcal{O}$, each of which is $\mathrm{emb}_{\mathcal{O}}(g)$ for some g (Lemma 4.3); the index is the number of the

embeddings $\mathrm{ce}(e,g)$ (Lemma 4.2(d)); and each such complex embedding is read by the same number of embeddings of $\mathcal{O}$. Dividing by $[\mathcal{O}:\mathbb{Z}_\ell]$ gives the formula. □

LEAN `finrank_crysCocycles_eq_add_neg_jumps`

Input 4 (the rank formula of Bloch and Kato; background library, unit of the Bloch–Kato condition). For a group Γ over $G_{\mathbb{Q}_\ell}$ with homomorphism j and a potentially crystalline lattice T, finite free over $\mathbb{Z}_\ell$, with Hodge–Tate jumps $\mathfrak{M}$ over $\mathbb{Z}_\ell$: $Z_{\mathrm{f}}(T)$ is finitely generated over $\mathbb{Z}_\ell$ of rank $\mathrm{rk}\,T + [G_{\mathbb{Q}_\ell} : j(\Gamma)] \cdot \#\{u \in \mathfrak{M} : u < 0\}$. *References* (second reading): S. Bloch, K. Kato, *L-functions and Tamagawa numbers of motives*, §3.7.2–3 and Corollary 3.8.4, pp. 353–355; P. Colmez, *Théorie d'Iwasawa des représentations de de Rham d'un corps local*, §§III.2 and III.4; J.-M. Fontaine, *Représentations p-adiques semi-stables*, Exposé III, Théorème 3.8(i). *Relation:* the local dimension formula applied to the integral module of cocycles (rank of the coboundaries plus $\dim H^1_{\mathrm{f}}$), with Colmez's ring $B_{\max}$ as period ring.

Lemma 11.6 (a power of ℓ bounds the defect)**.** *Let $\mathcal{O}$ be a coefficient ring, Γ a group over $G_{\mathbb{Q}_\ell}$ with homomorphism j, L a potentially crystalline Galois lattice over $\mathcal{O}$ for Γ, S an $\mathcal{O}$-submodule of the maps $\Gamma \to L$, $k \in \mathbb{N}$ and $y \geq 0$ a real number. Suppose $\mathrm{rk}_{\mathcal{O}} L \cdot 2k \leq y$ and $\ell^k c \in S$ for every $c \in Z_{\mathrm{f}}(L)$. Then $\mathrm{nlen}_{\mathcal{O}}(Z_{\mathrm{f}}(L)/Z_{\mathrm{f}}(L) \cap S) \leq [G_{\mathbb{Q}_\ell} : j(\Gamma)] \cdot y$.*

Proof. The quotient is a quotient of $Z_{\mathrm{f}}(L)/\ell^k Z_{\mathrm{f}}(L)$, which is isomorphic to $(\mathcal{O}/\ell^k)^\rho$ with $\rho = \mathrm{rk}_{\mathcal{O}} Z_{\mathrm{f}}(L)$ and has normalised length $k\rho$. By Input 4 (with a count of jumps at each embedding), $\rho \leq (1 + [G_{\mathbb{Q}_\ell} : j(\Gamma)])\,\mathrm{rk}_{\mathcal{O}} L \leq 2\,[G_{\mathbb{Q}_\ell} : j(\Gamma)]\,\mathrm{rk}_{\mathcal{O}} L$. □

LEAN `normLength_defect_le`

In the next lemma a *Kummer chart* k of Γ (unit on Wach and Kisin modules) has a ramification index e_k and a residue degree f_k, a set of *letters* on which $G_{\mathbb{Q}_\ell}$ acts through the roots of unity of order $\ell^{f_k} - 1$, and may be *unramified*; a *root of the letters* is a $\zeta \in \mathcal{O}$ with $\zeta^{\ell^{f_k}-1} = 1$. A finite set U of integers *lies in the window* $(\eta, \epsilon, \beta, W)$, for natural numbers η, β, W and a sign ϵ, if every $u \in U$ has $|u| \leq \eta$ or $\beta \leq \epsilon u \leq \beta + W$.

Lemma 11.7 (the crystalline case)**.** *Let Γ be a topological group over $G_{\mathbb{Q}_\ell}$ with homomorphism j, $\mathcal{O}$ a coefficient ring, L a Galois lattice over $\mathcal{O}$ for Γ, and $Q, m, h, \varkappa, G, b, \delta, \eta, W$ natural numbers. Let* th *and ϵ be functions on the $\mathbb{Z}_\ell$-algebra homomorphisms $g' : \mathcal{O} \to \mathbb{C}_\ell$, with truth values and with signs as values, and U, U' functions from these homomorphisms to the finite sets of integers; put $\beta_{g'} = b + \delta$ if $\mathrm{th}(g')$ holds and $\beta_{g'} = G + \delta$ otherwise. Let $y \geq 0$ be a real number, k a Kummer chart of Γ, ζ a root of its letters in $\mathcal{O}$, and T_0, s_0 functions on $\mathbb{C}_\ell$ with truth values and signs as values. Suppose:*

(a) $0 < Q$; $99\,(Q^2 + (W+1)Q + Q + W + m) \leq h$; $99h^3 \leq G$; $99\,(Gh^3 + G) \leq b$; $2\log(4b) < m$; $\delta < \eta$; $99\,((\eta + W)Q + \eta + h) \leq h^3$;

(b) *for every g', $U(g')$ lies in the window $(\eta, -\epsilon(g'), \beta_{g'}, W)$ and $U'(g')$ in the window $(\eta, \epsilon(g'), \beta_{g'}, W)$;*

(c) *L belongs to the class $\mathcal{Q}(\top, \emptyset, U')$;*

(d) *either, for every* g'*, every negative* $u \in U(g') \cup U'(g')$ *has* $2u \le -h$ *and the jumps of* L *at* g' *in* $[-\eta,\eta]$ *number at most* $\varkappa$ *with multiplicity; or, for every* g'*,* $U'(g') \subseteq [-\eta, 0]$*;*

(e) $10^5(\mathrm{rk}_{\mathcal{O}} L \cdot \lfloor 2\lambda/Q \rfloor \cdot m + \varkappa\lambda) \le y$*, where* $\lambda = \mathrm{lam}(h, b;\, \ell)$*;*

(f) j *is an open embedding,* $e_k f_k = [G_{\mathbb{Q}_\ell} : j(\Gamma)]$ *and* $e_k \le h$*;*

(g) *for every* g' *with* $\mathrm{th}(g')$ *or* $Q < \ell$*:* $\mathrm{th}(g') = T_0(g'(\zeta))$ *and* $\epsilon(g') = s_0(g'(\zeta))$*;*

(h) *either* k *is unramified and* $7 \le \ell$*; or* $\ell \le Q$ *and* s_0 *is constant;*

(i) *if* $\ell \le Q$*: for every* $z \in \mathbb{C}_\ell$ *and all* $1 \le a, a' \le \lfloor \log_\ell(h^3) \rfloor$*,* $T_0(z)$ *and* $T_0(z^{\ell^a})$ *imply that* $T_0(z^{\ell^{a+a'}})$ *does not hold.*

Then $\mathrm{nlen}_{\mathcal{O}}(Z_{\mathrm{f}}(L)/Z_{\mathrm{f}}(L) \cap L^{\perp \mathcal{Q}(\top,\emptyset,U)}) \le [G_{\mathbb{Q}_\ell} : j(\Gamma)] \cdot y$.

Proof. The lemma assembles three theorems of the note on crystalline lifts; its content lies there. Put $K_0 = \lfloor 2\lambda/Q \rfloor$. By the definition of λ and Legendre's formula, $2b < \ell(\lambda+1)$, and an integer that divides $(2b)! \prod k^{h^3}$ at most to the power λ gives control of the divisibility by ℓ of the integers near b. By (a), $\log_\ell H < m$ for every $H \le 4b$. If, for some $x_0 \le m$, every pairing of L with a lattice of the class $\mathcal{Q}(\top,\emptyset,U)$ satisfies the scaled cup bounds with exponent $10^4 K_0 x_0$, then $\ell^{10^4 K_0 x_0} Z_{\mathrm{f}}(L) \subseteq L^{\perp \mathcal{Q}(\top,\emptyset,U)}$ (first input), and Lemma 11.6 with (e) gives the claim. *Case* $\ell \le Q$. By (g)–(i) the thick letters are sparse (no three at short gaps) and the signs are constant; the theorem on sparse thick letters gives the scaled cup bounds. *Case* $Q < \ell$ *and* $h^3 \le \lambda$. The chart is unramified and $\ell \ge 7$ by (h), and the theorem for two sizes of windows applies, the inequalities following from $2b < \ell(\lambda+1)$ and (a). *Case* $Q < \ell$ *and* $\lambda < h^3$. Then $\ell \ge 9G$ (from the inequalities (a) and the case hypothesis), and ℓ divides no non-zero integer in the windows in play, for two different reasons. In the low window and in the far windows near $\pm G$ the integers are smaller than $9G$ in absolute value, hence smaller than ℓ. In the far windows near $\pm b$ the integers are factors of the product in the definition of λ, which enters it to the power h^3: a multiple of ℓ among them would give $\lambda \ge h^3$. The theorem on few low jumps then shows that the defect is zero unless λ is at least $h/5$, and in that case bounds its length by $11 f_k(\varkappa(2\lambda+3) + (W+1)\,\mathrm{rk}_{\mathcal{O}} L)$ times the length of $\mathcal{O}/\ell$, which is at most $[G_{\mathbb{Q}_\ell} : j(\Gamma)]\, y$ by (e), (f) and (a); here alternative (d) is used. □

LEAN `crystalline_defect_le`

Proof of Theorem E. Choose a Kummer chart k of Γ with $e_k f_k = [G_{\mathbb{Q}_\ell} : j(\Gamma)]$, unramified when $j(\Gamma)$ contains the inertia group (an input of the unit on Wach and Kisin modules), and let $\mathcal{O}$ be large enough to contain a primitive root of unity ζ of order $\ell^{f_k} - 1$. Let j_0 be the index of (R2) for $\mathcal{H}$. By Lemma 4.3 every embedding $g' : \mathcal{O} \to \mathbb{C}_\ell$ is $\mathrm{emb}_{\mathcal{O}}(\rho(g'))$ for some $\rho(g') \in G_{\mathbb{Q}_\ell}$; put $\sigma_{g'} = \mathrm{ce}(e, \rho(g'))$. By (ii), (iii) and (R2), for every g' the set $U(g')$ lies in the window

$$(h_2(D)^2 + 1,\ -\mathrm{sgn}_{\mathrm{CM}}(\sigma_{g'}),\ \beta + j_0 h(D) + h_2(D) - 2t(D),\ 4t(D)),$$

with $\beta = b_1(D)$ if $[\sigma_{g'}] \in \mathrm{thick}(D)$ and $\beta = b_2(D)$ otherwise, because the weight $w_{D,j_0}(\sigma)$ is $b_1(D)$ or $b_2(D)$ plus $j_0 h(D) + h_2(D)$, and the string jumps $0 \le c < t$ are low; the set $\{-u-1 : u \in U(g')\}$, which contains the jumps of the twisted duals, lies in the window with the opposite sign.

The tame case: $\ell = g_\nu(D)$. Then $\ell \equiv 1$ modulo $N_1(D)d$, so every element of $G_{\mathbb{Q}_\ell}$ fixes the roots of unity of that order, hence $e(E(D))$ (Lemma 11.1): j is a homeomorphism onto $G_{\mathbb{Q}_\ell}$. The theorem on tame types of the note on crystalline

lifts gives an explicit exponent k_1 with $\ell^{k_1} Z_{\mathrm{f}}(L_c^\vee(1))$ in the cup-orthogonal, and Lemma 11.6 applies: $2k_1 \operatorname{rk}_{\mathcal{O}} L_c \le \operatorname{rk}_{\mathcal{O}} L_c \cdot \operatorname{loss}_D(\ell)/x$ because $v_\ell(g(D)) \ge 1$, so that $\operatorname{loss}_D(\ell) \ge 10^6\, m(D) b_1(D)/k(D)$, and $x \le t(D)$.

The crystalline case: $F = \top$ and K empty. Apply Lemma 11.7 to $L_c^\vee(1)$ with

$$Q = k(D)t(D), \quad m = m(D), \quad h = h_2(D), \quad \varkappa = \left\lfloor \frac{\operatorname{rk}_{\mathcal{O}} L_c \cdot m(D)}{k(D)\, x} \right\rfloor, \quad G = b_2(D),$$

$b = b_1(D)$, $\delta = j_0 h(D) + h_2(D) - 2t(D)$, $\eta = h_2(D)^2 + 1$, $W = 4t(D)$, $\operatorname{th}(g')$ the statement that $[\sigma_{g'}]$ is thick, and $\epsilon(g') = \operatorname{sgn}_{\mathrm{CM}}(\sigma_{g'})$. Condition (a) consists of inequalities of the data (Chapter III). (b) was shown; (c) is the hypothesis on L_c, dualised. (d): for $c = *$ the first alternative holds, the bound on the low jumps being (R3); for a string index the dual of L_c has the single jump $-c - 1 \in [-\eta, 0]$. (e) is a comparison with Definition 3.9: $10^5(\operatorname{rk} \cdot \frac{2\lambda}{k(D)t(D)} m(D) + \varkappa\lambda) \le 3 \cdot 10^5 \operatorname{rk} \cdot \frac{m(D)}{k(D)} \cdot \frac{\lambda}{x}$. (f): $e_k \le e_k f_k = [G_{\mathbb{Q}_\ell} : j(\Gamma)]$, the number of embeddings $\operatorname{ce}(e, g)$ (Lemma 4.2(d)), at most $[E(D) : \mathbb{Q}] \le h_2(D)$. (g)–(i): thickness and sign of $\sigma_{g'}$ must be functions of $g'(\zeta)$. If ℓ does not divide $N_1(D)d$, the chart is unramified (inertia fixes the roots of unity of order $N_1(D)d$, hence $e(E(D))$), $\ell \ge 7$ because $2 \cdot 3 \cdot 5$ divides N_0, and $g'(\zeta)$ determines the action of $\rho(g')$ on the roots of unity of order prime to ℓ, hence $\sigma_{g'}$ (Lemma 4.2(c)). If ℓ divides $N_1(D)d$, then $\ell \le k(D)t(D) = Q$; thickness is by definition the property "$g'(\zeta) = z$ for some g' with $[\sigma_{g'}]$ thick", the sign is constant on thick places by Lemma 11.4, and (i) is Lemma 11.3: a Frobenius element of the chart raises the roots of unity of order prime to ℓ to the power ℓ^{f_k}, the exponents a, a' are at most $\log_\ell(h_2(D)^3) \le 12\mu(D)$, and $a + a' \le 27\mu(D)$. The formal proof of this theorem has seventy lines, mostly inequalities between the numbers of the data; the paragraph above is its outline. □

12 Where the results are used

All results of this note are used inside the unit, by the two notes that follow. Theorem A is the source of the pencils of tied primes in the note *Short Selmer groups after one anticyclotomic twist*, which also uses Lemma 7.4 (to read Selmer modules in finitely many coordinates), Lemma 4.1 and Lemma 11.5 (for balanced rows). The note *A fit family of twists exists* uses Theorem B and Lemmas 9.4, 6.1 to show that the auxiliary pairs have rows; Theorem C and Lemmas 4.5, 6.4, 10.2, 10.3, 10.4 in the bound for the denominator of the projection to lifts; Theorem D and Lemma 8.3 for the heights of the ratios of L-values; Theorem E, with Lemmas 4.4, 7.2 and 7.3, to show that the crystalline losses above ℓ are affordable. The definitions of §3 are the vocabulary of both notes.

13 Notes on the formal text

1. *Letters.* The formal text writes `𝕆` for $\bar{\mathbb{Q}}$, `ιℓ` for ι, `𝒪` both for a coefficient ring (here $\mathcal{O}$) and for an auxiliary pair (here O), `𝔸` for a place of $\bar{\mathbb{Q}}$ (here $\mathfrak{P}$), `𝔓` for an auxiliary place (here $\mathfrak{q}$) as well as for the parameters, `𝕁` both for the jumps (here $\mathcal{H}$) and for sets of places, `ι` for the index set of the rows of a pencil (here $\mathcal{J}$), `star w` for $\bar{w}$, `conjGal g` for $g^{\mathfrak{c}}$, `idOrConjInv b g` for $g^{[b]}$, `ellVal` for v_ℓ, `Gp ℓ` for $G_{\mathbb{Q}_\ell}$, and `ℓ'`, `E₁` for the prime and the field of Theorem A.

2. *The group $\mathbb{Z}/q$.* The homomorphisms h of Definition 3.23 take values in the multiplicative copy of $\mathbb{Z}/q$; "$h(\gamma) = 1$" above is the generator, "$h(\phi) = 0$" the neutral element.

3. *Truncations.* In Definition 3.12 the exponent is $t - 1$ in natural numbers, so $t = 0$ reads as $t = 1$. The rank difference $\#n - \#r$, the lower end $b - 30h^3$ in Definition 3.9 and the differences in Lemma 9.4 are truncated at 0. The valuation v_ℓ is truncated: it is 0 for $|x| \geq 1$. In Definitions 3.13 and 3.10 the number ℓ may be any natural number.

4. *Places of $\bar{\mathbb{Q}}$.* The formal type of places contains the trivial valuation ring $\bar{\mathbb{Q}}$ itself; no finite place lies under it.

5. *Arguments.* The arguments of the declarations were read from the text of the files; for 35 declarations of this note the dossier gives a list reconstructed by hand (section variables declared twice, variables that occur only as names of fields or bound names, statements that contain a local definition), and no list has been confirmed by Lean's own answer, since the artifact was not compiled here. Three readings of the program differ from the mathematics and were kept as slots or words: the element of the Galois group is counted as an argument of $\mathrm{blk}_T(i)(\cdot)$ and of $\mathrm{av}^-_\vartheta(\cdot)$ (an arrow read as an argument); the point character of Definition 3.32 counts the hypothesis $y_j^q = 1$ and the group element as arguments, and is therefore written out; in Theorem E the quantifier "for all large $\mathcal{O}$" is read as two arguments. Instance arguments of definitions that the definition may not use are not listed.

6. *Statements with local definitions.* Lemmas 5.2, 5.3, 9.2 and Theorems C and E contain a local definition in their formal statement, at which the census of the workspace cuts the statement; they were read in full in the source (the second reading of this write-up lists their true ends).

7. *Hypotheses with defaults, and hypotheses not used.* In Lemma 7.3 the last two hypotheses have default proofs in the formal text (a tactic closes them at the uses). The symmetry of $\mathrm{pr}_O(c)$ and the fact that it fixes its image (Lemma 5.7(c)) are stated in the formal text without the hypothesis $\mathsf{Apt}(O, c; \vartheta)$, in its namespace. In Lemma 9.4 and Lemma 9.5 the apt family c' serves only to identify the chosen places of O with those of D. In Theorem D the index set of the family is the product of a set with two elements and the group of signs.

8. *Theorem A.* The formal statement has two primes in scope; only the one of the conclusion matters. The set R is a set with a proof of finiteness, S a finite set.

9. *Assemblies and outlines.* Lemma 11.7 and the proof of Theorem E apply theorems of the unit on lifting to the numbers of the data; Lemma 8.1 in the case $z_p = 1$ and Theorem C unfold definitions of the units on forms. For these four, and for Lemma 7.4, the proofs above are outlines of the formal proofs, as said at the head or at the end of each; the inequalities and the identities between adelic sums were not re-derived by hand.

10. *Differences from the dossier.* The dossier's reading of "from U(1)" says that the idele character is trivial on the ideles that are units at $S \cup \bar{S}$; the formal definition asks triviality on the ideles that are units *outside* $S \cup \bar{S}$ (and congruent to 1 at

the modulus), which is the reading of §2.2 and of the authors' comment ("trivial at S and ℓ").

11. *What the final theorems use.* All the lemmas of this note are used by declarations of the two notes that follow (§12); whether those are reached by the final theorems is recorded with them.

How this note was written. From the dossier of the unit (its first chunk): the statements in words and the structured proofs, which a program compared line by line with the source (975 leaf steps in the 96 proofs of the whole unit). I read the fifteen Lean files of these declarations completely, and in the background library and in other units the definitions that the statements rest on: Selmer modules, relations and the characters from U(1); Galois lattices, classes, cup-orthogonals, jumps and crystalline cocycles; tame characters, twists, pairs, ramified pairs and the ratio of L-values; the number $M_\omega(P, \mu(D))$ and the divisibilities at the chosen places. The statements of this note were compared with the Lean statements argument by argument, and the order of the quantifiers is that of the formal text. The proofs follow the formal ones, with these differences: the proofs of Lemma 5.1(d) and of Lemma 10.2 are mine (the formal proofs quote a lemma on roots of unity, and a count of the roots of a polynomial); Lemma 7.4, Theorems C and E and Lemmas 8.1, 11.7 are given as outlines. I did not open the proofs of the inputs. The references and their places are those of the second reading of this write-up and of its catalogue; I did not check them in the literature.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Definition 3.12	`IsConjSelfDual`	AptTupleDefs.lean	21–24
Definition 3.13	`crudeUnit`	AptTupleDefs.lean	27–27
Definition 3.14	`SelmerLengthLe`	AptTupleDefs.lean	34–38
Definition 3.15	`IsColumn`	AptTupleDefs.lean	43–45
Definition 3.16	`IsRow`	AptTupleDefs.lean	50–59
Definition 3.17	`chiOneInv`	AptTupleDefs.lean	68–68
Lemma 5.4	`chiOneInv_spec`	AptTupleDefs.lean	73–79
Definition 3.17	`normalisedRep`	AptTupleDefs.lean	82–82
Definition 3.18	`twistedLabel`	AptTupleDefs.lean	86–87
Definition 3.19	`IsApt`	AptTupleDefs.lean	92–100
Definition 3.20	`tupleProj`	AptTupleDefs.lean	104–104
Definition 3.21	`HasRow`	AptTupleDefs.lean	108–110
Definition 3.22	`IsLabelledColumn`	AptTupleDefs.lean	114–116
Lemma 5.5	`g_pow_sub_one_dvd_chosenGroupOrder`	AptTuples.lean	15–20
Lemma 5.7	`ellVal_le_crudeUnit`	AptTuples.lean	24–26
Lemma 5.5	`card_le_of_subset_placesAboveG`	AptTuples.lean	28–31
Lemma 5.5	`exists_chosen_notMem_placesDividing`	AptTuples.lean	33–36
Lemma 5.6	`IsLatticeIn.exists_algHom`	AptTuples.lean	44–48
Lemma 5.7	`IsLatticeIn.continuous`	AptTuples.lean	50–51
Lemma 5.7	`IsLatticeIn.eq_one`	AptTuples.lean	53–54
Lemma 5.7	`IsLatticeIn.trace_eq`	AptTuples.lean	56–57
Lemma 5.7	`IsApt.card_le`	AptTuples.lean	64–65
Lemma 5.7	`IsApt.isSymmetric_tupleProj`	AptTuples.lean	67–68
Lemma 5.7	`IsApt.tupleProj_eq_self_of_mem_range`	AptTuples.lean	70–71
Lemma 5.8	`IsApt.range_tupleProj_mem_invtSubmodule`	AptTuples.lean	73–77

In this note	Lean declaration	File	Lines
Lemma 5.7	`frequently_isSplit_notMem_places`	AptTuples.lean	79–81
Theorem C	`exists_baseLevel`	BaseLevelOfLabel.lean	23–74
Definition 3.1	`obeyingCocycles`	BlocksAndCocycles.lean	16–17
Lemma 4.1	`obeyingCocycles.` `exists_graph_subrepresentation`	BlocksAndCocycles.lean	22–37
Definition 3.2	`RowBlockIndex`	BlocksAndCocycles.lean	41–41
Definition 3.2	`blockRep`	BlocksAndCocycles.lean	50–51
Definition 3.23	`AuxPrimeData`	Corepresentable.lean	38–40
Definition 3.24	`IsAnticyclotomicAt`	Corepresentable.lean	46–51
Definition 3.25	`IsAuxiliary`	Corepresentable.lean	55–55
Definition 3.26	`IsFamily`	Corepresentable.lean	60–68
Theorem A	`exists_isAnticyclotomicAt`	Corepresentable.lean	77–132
Definition 3.27	`evenClasses`	Corepresentable.lean	136–138
Definition 3.28	`IsNewPlaces`	Corepresentable.lean	142–143
Definition 3.28	`withConjInv`	Corepresentable.lean	147–147
Definition 3.29	`IsPrimeTriangle`	Corepresentable.lean	152–162
Definition 3.30	`IsAuxTwist`	Corepresentable.lean	166–167
Lemma 7.1	`four_dvd_cycChar_sub_one`	Corepresentable.lean	169–173
Lemma 7.2	`IsNewPlaces.cycInv_withConjInv`	Corepresentable.lean	178–185
Lemma 7.3	`IsNewPlaces.mem_selmerModule_of_tame_eq_zero`	Corepresentable.lean	190–199
Definition 3.31	`unionWithConj`	Corepresentable.lean	206–206
Definition 3.32	`pointChar`	Corepresentable.lean	210–211
Definition 3.32	`twistAtPoint`	Corepresentable.lean	215–216
Definition 3.32	`reduceModPow`	Corepresentable.lean	219–220
Definition 3.33	`IsBalanced`	Corepresentable.lean	227–238
Definition 3.34	`IsPencil`	Corepresentable.lean	243–250
Definition 3.35	`tieRelations`	Corepresentable.lean	254–256
Lemma 5.7	`idOrConjInv_mem_inertiaAbove`	Corepresentable.lean	258–260
Lemma 7.4	`exists_corepresenting_coords`	Corepresentable.lean	267–286
Lemma 11.7	`crystalline_defect_le`	CrystallineBound.lean	17–74
Definition 3.37	`matrixLattice`	CrystallineRank.lean	22–23
Lemma 11.5	`finrank_crysCocycles_eq_add_neg_jumps`	CrystallineRank.lean	28–42
Lemma 11.6	`normLength_defect_le`	CrystallineRank.lean	47–56
Definition 3.3	`localChart`	DeterminantBound.lean	22–26
Definition 3.4	`coeffEmb`	DeterminantBound.lean	34–35
Definition 3.5	`ZfAt`	DeterminantBound.lean	39–39
Definition 3.5	`crysRels`	DeterminantBound.lean	43–43
Definition 3.6	`blockLoss`	DeterminantBound.lean	47–48
Lemma 4.2	`obeying_crysRels`	DeterminantBound.lean	50–52
Definition 3.7	`cmSign`	DeterminantBound.lean	56–56
Definition 3.4	`complexEmbOf`	DeterminantBound.lean	59–60
Definition 3.8	`IsRowType`	DeterminantBound.lean	64–70
Definition 3.9	`lambdaLoss`	DeterminantBound.lean	74–75
Definition 3.9	`loss`	DeterminantBound.lean	78–79
Lemma 4.3	`coeffEmb_surjective`	DeterminantBound.lean	81–85
Lemma 4.2	`tateTwist_smul`	DeterminantBound.lean	87–89
Lemma 4.2	`IsCompletionGroup.complexEmbOf_eq_iff`	DeterminantBound.lean	91–94
Lemma 4.2	`IsCompletionGroup.index_eq_card_range`	DeterminantBound.lean	98–100
Lemma 4.4	`sum_complexEmbOf_le`	DeterminantBound.lean	102–106
Lemma 4.5	`norm_gram_det_le_of_operators`	DeterminantBound.lean	111–127
Lemma 6.1	`isApt_auxPair`	HeckeStableSubspaces.lean	22–30
Lemma 6.1	`isApt_rank_one`	HeckeStableSubspaces.lean	34–38
Lemma 6.1	`trivialLift_mem_range_tupleProj_rank_one`	HeckeStableSubspaces.lean	42–54
Lemma 6.2	`isConjSelfDual_of_attached`	HeckeStableSubspaces.lean	59–77
Lemma 6.3	`local_clauses_of_conjPlace`	HeckeStableSubspaces.lean	83–110

In this note	Lean declaration	File	Lines
Lemma 5.7	`orthogonal_mem_invtSubmodule_hecke`	HeckeStableSubspaces.lean	112–115
Lemma 6.4	`le_of_isotypic_mem`	HeckeStableSubspaces.lean	119–126
Lemma 10.1	`exists_integral_multiple_eval_one`	HeckeString.lean	15–26
Lemma 10.2	`exists_eventually_no_eigenvalue_one`	HeckeString.lean	30–42
Lemma 10.3	`isRoot_hecke_string`	HeckeString.lean	52–62
Lemma 10.4	`eventually_heckePoly_eq_string_mul`	HeckeString.lean	68–87
Lemma 9.1	`norm_pow_sub_pow_eq_one`	LabelledColumns.lean	25–32
Lemma 9.2	`exists_inertia_eigenvalues`	LabelledColumns.lean	40–62
Theorem B	`IsLabelledColumn.isRow`	LabelledColumns.lean	67–125
Lemma 9.3	`normalisedRep_quadratic_on_inertia`	LabelledColumns.lean	130–155
Lemma 9.4	`IsLabelledColumn.twist_avatar`	LabelledColumns.lean	167–182
Lemma 9.5	`isLabelledColumn_of_hasProductLabel`	LabelledColumns.lean	187–197
Lemma 5.1	`Data.algebraMap_mem_cycloField`	LevelPlacesChosen.lean	20–21
Definition 3.10	`AbsGal`	LevelPlacesChosen.lean	25–25
Definition 3.10	`placesAboveG`	LevelPlacesChosen.lean	28–28
Definition 3.10	`badPlaces`	LevelPlacesChosen.lean	32–32
Lemma 5.1	`placesDividing_union_subset_badPlaces`	LevelPlacesChosen.lean	34–36
Lemma 5.1	`star_mem_placesAboveG`	LevelPlacesChosen.lean	40–41
Lemma 5.1	`norm_natCast_le_norm_sub_of_pow_eq_one`	LevelPlacesChosen.lean	44–46
Lemma 5.1	`continuous_cycInv`	LevelPlacesChosen.lean	48–49
Lemma 5.1	`cycInv_eq_of_isFrob`	LevelPlacesChosen.lean	51–53
Definition 3.11	`Data.TameChar.invAvatar`	LevelPlacesChosen.lean	59–60
Lemma 5.2	`avatar_conjGal_mul`	LevelPlacesChosen.lean	64–73
Lemma 5.3	`isLabelledAt_avatar`	LevelPlacesChosen.lean	78–84
Lemma 5.1	`isChosen_chosenPlace`	LevelPlacesChosen.lean	86–89
Theorem E	`block_defect_le_at_place`	OnePlaceDefect.lean	15–84
Lemma 11.1	`fixes_emb_of_fixes_rootsOfUnity`	ThickSpacing.lean	12–16
Lemma 5.7	`cmSign_spec`	ThickSpacing.lean	18–20
Lemma 11.2	`placeClass_ratio`	ThickSpacing.lean	30–39
Lemma 11.3	`not_thick_of_short_arc`	ThickSpacing.lean	45–57
Lemma 11.4	`cmSign_eq_of_thick`	ThickSpacing.lean	64–75
Lemma 8.1	`Data.Pair.exists_isIntegralMeasure`	ValueFamilies.lean	30–80
Lemma 8.2	`coprime_q_of_isAuxiliary`	ValueFamilies.lean	82–85
Definition 3.36	`tameCharOfAuxiliary`	ValueFamilies.lean	89–90
Lemma 5.7	`invAvatar_eq_of_artinMap_eq`	ValueFamilies.lean	92–95
Lemma 8.3	`exists_tameChar_invAvatar_eq`	ValueFamilies.lean	97–106
Theorem D	`exists_isFamily`	ValueFamilies.lean	111–153

Three statements as they are written in the artifact.

```
lemma exists_isAnticyclotomicAt [ThePrime ℓ'] N (S : Finset (FinPlace E₁)) {R} (h₀ : R.Finite) {z W}
  (h : z ∈ normalClosure (inertiaAbove (R \ (.conjClosure S ∪ placesDividing E₁ ℓ')))) (hW :
  W ∈ nhds z) : ∃ P ∉ R.conjClosure, P.q.Prime ∧ ∃ l : AuxPrimeData E₁ (ℓ' ^ N), l.φ ∈ W ∧
  l.h l.φ = 1 ∧ ∀ T ⊆ ↑S ∪ placesDividing E₁ ℓ', IsAnticyclotomicAt T P l

lemma IsLabelledColumn.isRow (hc : IsApt O c θ) {R k J} (h :
  IsLabelledColumn (m := r) O k J (twistedLabel O c) ∅ R)
  (hJ : IsRowType D (card r) k J) : IsRow D k J R

def SelmerLengthLe Q m γ (z : intQpBar ℓ) := ∀ᶠ O : CoeffRing ℓ in .atTop, ∃ υ ψ' γ, γ.1 = ↑z ∧
  IsLatticeIn R υ ∧ (∀ g, ψ' g = ψ g) ∧ let ρ := twistRep (.id O) ψ' υ; ρ γ = (γ : ↑O) • 1 ∧
    IsUnit (γ - 1) ∧ cycChar Ω ℓ γ = 1 ∧ O.normLength (Module.length O ↥(selmerModule ρ
    (placesDividing _ ℓ ∪ placesAboveG D ∪ .conjClosure Q) (crysRels ρ) (QuotSpan (ℓ ^ m : ↑O)) ⊓
    (LinearMap.proj γ).ker)) ≤ 10 * D.f * Fintype.card n * crudeUnit ℓ D + crudeUnit ℓ D
```

In these texts `Gal(Ω/E₁)` is G_E, `AbsGal D` is $G_{E(D)}$, `placesDividing E ℓ` is $\Sigma_\ell(E)$, `placesAboveG D` is $\Sigma_g(D)$, `.conjClosure S` is $S \cup \bar{S}$, `inertiaAbove` is the inertia above a set of places, `selmerModule`

`p S L B` is $\mathrm{Sel}^L_S(\varrho;\, B)$, `QuotSpan a` is $\mathcal{O}/a$, `crysRels` gives the crystalline relations, `normLength` is $\mathrm{nlen}_{\mathcal{O}}$, and `∀ᶠ O in atTop` is "for all large $\mathcal{O}$".

Notation

ℓ, $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell$, $\bar{\mathbb{Q}}_\ell$, $\bar{\mathbb{Z}}_\ell$, $\mathbb{C}_\ell$, $G_{\mathbb{Q}_\ell}$ a prime; the ℓ-adic numbers, integers, an algebraic closure, its integers, its completion, $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$

ι a fixed isomorphism $\mathbb{C} \to \bar{\mathbb{Q}}_\ell$

$\bar{\mathbb{Q}}$, c, g^{c}, $g^{[b]}$ an algebraic closure of $\mathbb{Q}$; complex conjugation; $\mathsf{c}g\mathsf{c}$; g or $(g^{\mathsf{c}})^{-1}$

E, E', E_1, G_E, f_E CM fields as in the conventions; $\mathrm{Gal}(\bar{\mathbb{Q}}/E)$; the number of infinite places

P, D, N_0, d parameters and admissible data; the two parameters

$E(D)$, $m(D)$, $r(D)$, $k(D)$, $t(D)$, $f(D)$ field, number of lifts, ranks, step, number of infinite places of the data

$g_i(D)$, $g(D)$, $N(D)$, π_g the two auxiliary primes, their product, the level, an integer of norm $g(D)$

$h_2(D)$, $h(D)$, $b_1(D)$, $b_2(D)$, $w_{D,j}(\sigma)$ half-gap, gap, the two rounded quotients, the weight of lift j

$N_1(D)$, $\mu(D)$, ς, $M_\omega(P, \mu(D))$, ω_D order of the root of unity, exponent, the root of unity, the order and the family of residue characters

$\mathrm{ord}_{\mathrm{ch}}(D)$, $\mathrm{thick}(D)$, cl_m the order of the chosen groups; the thick places; the class of a place modulo m

w, v, $\mathfrak{q}$, $\mathfrak{q}_0$, $\bar{w}$, $\mathrm{N}(w)$, $\mathbb{F}_w$, ϖ_w finite places, the conjugate, the norm, the residue field, a uniformiser

$\mathfrak{P}$, $\mathfrak{P}_w$, $\mathfrak{P}^{\mathsf{c}}$, $G_{\mathfrak{P}}$, $I_{\mathfrak{P}}$, $\tau_{\mathfrak{P}}$, τ_w places of $\bar{\mathbb{Q}}$; the chosen one over w; the conjugate; decomposition and inertia groups; tame generators

$\Sigma_M(E)$, I_S, $S^{\mathbb{C}}$ finite places containing M; inertia above S; the complement of S

cyc, cyc_M, Art, υ cyclotomic characters; the Artin map; the Hecke character of the Weil data

$\mathcal{O}$, $\mathcal{O}'$, $\mathcal{O}_0$, ϖ, $\mathcal{O}(-e)$ coefficient rings; a uniformiser; a Tate twist

A, A', Λ, B, Γ, v_A, κ_A commutative rings, a module, a group; valuation and residue field of A

v_ℓ, v_l, len, nlen, rk, tr, $\det$, diag valuations, lengths, rank, trace, determinant, diagonal matrix

$M_n(A)$, $\mathrm{GL}_n(A)$, $\#n$ matrices and invertible matrices indexed by a finite set n; its cardinality

θ, δ_g, $\mathrm{Ob}(L;\, B)$, $\mathrm{Rel}(Z)$, $\mathrm{Coc}(\varrho)$ a relation; the indicator relation; functions obeying L; relations of Z; cocycle relations

$\mathrm{Sel}^L_S(\varrho;\, B)$ the Selmer module

R, u, ϱ, T, ψ, ψ', χ, η representations, lattices, characters, idele characters; T also a faithful character and a set of places

$\mathcal{H}$, $\mathcal{H}'$, $\mathcal{H}_{\kappa,N}$, κ, λ_i types; the type of a weight κ in rank N; its partition

$Z_{\mathrm{f}}(L)$, $L^\vee(1)$, $L^{\perp Q}$, $\mathcal{Q}(F, K, U)$ crystalline cocycles, twisted dual, cup-orthogonal, a class of lattices

$\mathfrak{M}$, $\varkappa$, N', S_u, c'_i, e, k_3 a multiset (of jumps, of eigenvalues); the bound for the number of low jumps in Lemma 11.7; the integer of Lemma 8.1; the set of places and the polynomials of Theorem C(3); the averaging idempotent in Lemma 6.1; the exponent of 3 in $N_1(D)$

μ, ψ_w, e_v, $\mu[k, j]$, av a character of U(1), its tame part and weight, the algebraic Hecke character, the avatar

ϑ, τ, $Q(\vartheta)$, ψ_ϑ tame characters and twists (τ also a family of tame generators); the set of places; the tame part

p, ν_p, z_p, $\varrho_p(\vartheta, s)$, $y_v(p, \vartheta)$ a pair; its ratio of L-values; a value at a uniformiser

O, V, I_O, w_i, μ_O, ψ_O, a_O, $\mathcal{F}(O)$ auxiliary pairs, chosen index set and places, characters, forms

$O_D(\vartheta, j)$, $O_p(\vartheta)$, $\Xi_p(\vartheta)$ the auxiliary pairs of the data and of a pair; the lift of the trivial character

c, c', $\tilde{c}$, c_i, $e^{\otimes}_{r,c}$, x_χ, ξ families of polynomials, the idempotent of a tuple of labels, the name of a character, a root

H, $H_w(X)$, $H_x(w)$, $T_{w,i}$, S systems of Hecke polynomials and Hecke operators; S also a set of places and a polynomial

x, ϱ_x, f, y, z an isotypic space or eigensystem and its representation; forms; Hecke data; also elements

$\mathcal{Z}_u(\varrho;\, a)$, $\mathrm{I}_r(i)$, $\mathrm{blk}_T(i)(g)$ Definitions 3.1, 3.2

$\mathrm{emb}_{\mathcal{O}}(g)$, $\mathrm{ce}(e, g)$, $Z_{\mathrm{cris}}(v;\, u)$, $\mathcal{R}_{\mathrm{cris}}(u)$, $\mathrm{bl}(u;\, w, M)$ Definitions 3.4–3.6

$\mathrm{sgn}_{\mathrm{CM}}(\sigma)$, $\mathsf{RowType}_D(r, t;\, \mathcal{H})$, $\mathrm{lam}(h, b;\, l)$, $\mathrm{loss}_D(l)$ Definitions 3.7–3.9

$G_{E(D)}$, $\Sigma_g(D)$, $\Sigma_{\mathrm{bad}}(D, \ell)$, $\mathrm{av}^-_\vartheta(g)$ Definitions 3.10, 3.11

$\mathsf{CSD}_t(R)$, $\mathrm{cr}_\ell(D)$, $\mathsf{Short}(\psi, R;\, Q, m;\, \gamma, z)$ Definitions 3.12–3.14

$\mathsf{ColRep}_D(t, \mathcal{H}, S;\, R)$, $\mathsf{RowRep}_D(t, \mathcal{H};\, R)$ columns and rows (Definitions 3.15, 3.16)

$\chi^{-1}_{1,O}$, $\mathrm{nm}_O(R)$, $\mathrm{tl}_O(c, i)$, $\mathrm{pr}_O(c)$ Definitions 3.17–3.20

$\mathsf{Apt}(O, c;\, \vartheta)$, $\mathsf{HasRow}(O, z;\, R)$, $\mathsf{LabCol}(O;\, k, \mathcal{H}, S;\, c;\, R)$ Definitions 3.19, 3.21, 3.22

$\mathrm{Aux}(E, q)$, $\mathsf{Acyc}(T, \mathfrak{q};\, l)$, $\mathsf{Auxil}(S;\, I)$, $\mathsf{Fam}(S;\, \nu, \mathrm{val})$ Definitions 3.23–3.26

$\mathrm{Ev}(\mathcal{O}, S;\, \nu, e)$, $\mathsf{New}_\nu(w, \tau, \phi)$, $\tau^{[k]}$ Definitions 3.27, 3.28

$\mathsf{Tri}_\nu(w, \tau, \phi;\, a;\, t, \mathrm{ty};\, h, c)$ prime triangles (Definition 3.29)

$\mathsf{AuxTw}(S, Q;\, \psi, c)$, $S \cup J \cup \overline{J}$ Definitions 3.30, 3.31

$\mathrm{tp}_{h,y}(e, \varrho)(g)$, $\varrho \bmod l^M$ Definition 3.32

$\mathsf{Bal}_\ell(S;\, \varrho, L;\, \gamma, \beta, M)$ balanced representations (Definition 3.33)

$\mathsf{Pencil}_\ell(S_0;\, U, W;\, N, \mathfrak{q}_0, J;\, x)$, $\mathrm{Ties}_i(J, x)$ pencils and ties (Definitions 3.34, 3.35)

$\mathrm{L}(\varrho)$ the lattice of a matrix representation (Definition 3.37)

$l = (\mathfrak{P}, \gamma, \phi, h)$, l_v, s_v, $s_{v,i}$, $\mathcal{J}$ data at an auxiliary prime; the slopes; the index set of the rows

γ, ϕ, σ_0, g, g', g_1, s, a, z elements of Galois groups; s also a sign, a also an exponent or a bound

σ, $\bar{\sigma}$, $[\sigma]$, $\mathfrak{S}_v$, $\sigma_{g'}$ complex embeddings, the place, the CM type, sets of embeddings; σ also a finite set and a family

j, j_v, $j_{\mathfrak{P}}$, e, e_v, $\rho(g')$ homomorphisms to $G_{\mathbb{Q}_\ell}$, embeddings into $\bar{\mathbb{Q}}_\ell$, a preimage; j, e also integers

n, r, m, d, t, k, q, N, M, i, b, h finite sets or natural numbers: index sets, ranks, exponents, moduli, signs

I, J, K, Q, S_0, T_0, W, X, $\mathcal{K}$, $\mathcal{U}$ sets of places, index sets, neighbourhoods; X also the variable of polynomials, W a window width, Q a natural number in §11

ν, val, α, c_a, g_a, F, u_a a family of characters and of values; a presentation; an integral measure (ν also a natural number)

$\mathbb{M}$, $\mathcal{L}$, $\mathcal{N}$, Λ_0, Y, k_0, θ^g objects of the proof of Lemma 4.1; Y also a matrix

V, W, $\mathcal{A}$, Z, Z', η_a, x_b, ω_k, $l_{p,k}$, λ_k, d_p, $P_{b,k}$, $w_{p,b}$, G_p objects of Lemma 4.5; Z, Z' also spaces of forms

T_0, $\mathfrak{s}$, y_u, u_0, ρ_0, f_0, $\mathcal{R}$, ρ_k, U objects of the proofs of §7; U also a subgroup and a function to finite sets

γ_0, $g_{I,a}$, $c_{I,a}$, $u_{I,a}$, $y_{I,a}$, u_{f}, $\Lambda(\vartheta)$, den objects of §8

ζ, p_0, F_1, F_1', F_2, F_3, δ_i, α_i, d_0, $c_i(Y)$, $H_w(i)$ roots of unity, a prime, polynomials, a sequence, eigenvalues, coefficients

G, δ, η, ϵ, th, β, $\beta_{g'}$, λ, s_0, K_0, x_0, k_1, j_0, c_0, e_k, f_k, ρ numbers and functions of §11; G also a finite group or an operator, δ also the function of (T5)

ε, π, C, $\top$, $*$, ty, $\mathrm{U}(1)$ a sign; the constant in $2\pi i$; a monoid; the trivial filter; the index of the row block; the map of a triangle; the unitary group of rank one

References

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READBACK NOTE 32

Short Selmer groups after one anticyclotomic twist, balanced rows, and Ribet's lemma to first order

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. A good twist must make the Selmer modules of the rows short, for m powers at once, and must not make the relevant L-values too divisible. This note proves that such a twist exists (Theorem A): given finitely many *balanced* representations and finitely many families of values, one anticyclotomic character of ℓ-power order, ramified at boundedly many new primes, has all its powers $\psi, \dots, \psi^m$ within the bounds. The primes come from Theorem B: boundedly many primes *tied* to Frobenius elements make every Selmer module short; the proof plays a mirror lemma against a tilting lemma for pencils. The passage from ties to an honest twist is an argument on valuations of polynomials at roots of unity. Theorem C shows that rows are balanced. The second half is Ribet's lemma to first order (Theorem D): from congruences between a block and a string of characters it produces, by Cramer's rule used twice and Fitting ideals, a Hecke-type element whose top coefficient has bounded valuation; Theorem E supplies its hypothesis for the Tate twists from a prime triangle and the odd part of a Tate–Shafarevich group.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles and that Lean's kernel accepted the proofs; an independent replay is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note is the second of three on the unit *The choice of the twists*: it treats the declarations of eight files (roots of unity, the mirror and tilting lemmas, the twist theorem, balanced rows, Ribet's lemma), and recalls from the first note, *Columns and rows as Galois representations*, the definitions it needs; together 58 of the unit's 205 declarations appear here. The statements are complete. The proofs of Theorems B and E and of Lemmas 4.2, 4.3, 5.2, 6.1, 6.2, 6.3, 6.5, 7.1, 7.2, 8.1, 8.3 and 9.1 follow the formal proofs. The proofs of Theorems A, C and D and of Lemmas 4.1, 5.1, 6.4, 7.3 and 8.2 are outlines of the formal proofs: they give the steps and where each hypothesis enters, leave the computations to the formal text, and say so where they stand. Results of the background library and of other units are quoted as Inputs with their owners. Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 58 declarations.

1 The statements

The conventions and the recalled notions are in §2, the definitions of this note in §3. Throughout, ℓ is a prime number.

Theorem A (one anticyclotomic twist makes all Selmer modules short). *Let E be a CM field as in the conventions, $\mathcal{I}$ a finite set and $(n_i)_{i\in\mathcal{I}}$ finite sets, $S \subseteq S_0$ finite sets of finite places of E, $z \in I_{S_0\setminus\Sigma_\ell(E)}$, and $M \in \mathbb{N}$. Let κ and T be finite sets, $\mathcal{O}$ a coefficient ring, and for $i \in \mathcal{I}$ let $\varrho_i \colon G_E \to M_{n_i}(\mathcal{O})$ be a homomorphism, L_i a set of relations, $\gamma_i \in G_E$, and β_i, B_i elements of $[0,\infty]$. Let $\mu = (\mu_{k,t})_{k\in\kappa,t\in T}$ be homomorphisms $G_E \to \bar{\mathbb{Q}}_\ell^\times$ and $(\mathrm{val}_k)_{k\in\kappa}$ functions as in the definition of a family. Let $c \geq 0$ and $\eta > 0$ be real numbers and $m \in \mathbb{N}$. Suppose that*

(i) $\mathsf{Bal}_\ell(S;\, \varrho_i, L_i;\, \gamma_i, \beta_i, M)$ *for every i, and* $\mathsf{Fam}(S_0;\, \mu_k, \mathrm{val}_k)$ *for every k;*

(ii) *$B_i \neq \infty$ and $\mathrm{nlen}_{\mathcal{O}}(\beta_i + \sum_b v_{\mathcal{O}}(\det(\varrho_i(z^{[b]}) - 1))) \leq B_i$ for every i, the sum over the two signs;*

(iii) *$v_\ell(\mu_{k,t}(z) - 1) \leq c$ for all k and t.*

Then there are a finite set I of finite places of E and a multiplicative homomorphism $\psi \colon G_E \to \bar{\mathbb{Q}}_\ell$ such that for every $1 \leq d \leq m$:

(1) *$v_\ell(\mathrm{val}_k(I, \psi^d)) \leq v_\ell(\mathrm{val}_k(\emptyset, 1)) + 8c + \eta$ for every k;*

(2) $\mathsf{AuxTw}(S_0, I;\, \psi^d, \eta)$;

(3) *for every i and all large coefficient rings $\mathcal{O}'$: $\mathcal{O} \subseteq \mathcal{O}'$, and there is a multiplicative homomorphism $\psi' \colon G_E \to \mathcal{O}'$ with $\psi'(g) = \psi^d(g)$ for all g and*

$$\mathrm{nlen}_{\mathcal{O}'}\, \mathrm{Sel}^{L_i\cup\{\delta_{\gamma_i}\}}_{S\cup I\cup\bar{I}}(\psi' \cdot \varrho_i;\, \mathcal{O}'/\ell^M) \;\leq\; B_i + \eta.$$

LEAN `exists_anticyclotomic_twist`

Theorem B (tied conditions at new primes shrink Selmer modules). *Let E, $\mathcal{I}$, $(n_i)_i$, $S \subseteq S_0$, $z \in I_{S_0\setminus\Sigma_\ell(E)}$ and $M \in \mathbb{N}$ be as in Theorem A. Let A be the ring of integers of a finite extension of $\mathbb{Q}_\ell$, and for $i \in \mathcal{I}$ let $\varrho_i \colon G_E \to M_{n_i}(A)$ be a homomorphism, L_i a set of relations, $\gamma_i \in G_E$ and $\beta_i \in [0,\infty]$ with $\mathsf{Bal}_\ell(S;\, \varrho_i, L_i;\, \gamma_i, \beta_i, M)$. Let W be a neighbourhood of z in G_E and $a \in A$ with $a \mid \ell^M$. For $i \in \mathcal{I}$ let U_i be the property $\varrho_i \bmod \ell^M(g) = 1$ of $g \in G_E$. Then there is $J_0 \in \mathbb{N}$ such that for every $N \in \mathbb{N}$ there are a finite place $\mathfrak{q}_0$, a finite set J of finite places with $\#J \leq J_0$, and a function x with $\mathsf{Pencil}_\ell(S_0;\, U, W;\, N, \mathfrak{q}_0, J;\, x)$ and*

$$\mathrm{len}_A\, \mathrm{Sel}^{L_i\cup\mathrm{Ties}_i(J,x)\cup\{\delta_{\gamma_i}\}}_{S\cup J\cup\bar{J}}(\varrho_i;\, A/a) \;\leq\; \beta_i \qquad \text{for all } i \in \mathcal{I}.$$

LEAN `exists_isPencil`

Theorem C (rows are balanced). *Let P be parameters, D admissible data for P, $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ the fixed isomorphism, n a finite set, t a natural number, $\mathcal{H}$ a type and $R\colon G_{E(D)} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ a homomorphism with $\mathsf{RowRep}_D(t, \mathcal{H}; R)$. Let w be a finite place of $E(D)$ that is chosen and lies in $\Sigma_g(D) \setminus \Sigma_\ell(E(D))$, and $M \in \mathbb{N}$. The set $\Sigma = \Sigma_\ell(E(D)) \cup \Sigma_g(D)$ is finite. For all large coefficient rings $\mathcal{O}$ there are a homomorphism $u\colon G_{E(D)} \to M_n(\mathcal{O})$, an element $\gamma \in G_{E(D)}$ and $\beta \in [0, \infty]$ such that u is a lattice of R, $\mathsf{Bal}_\ell(\Sigma; u, \mathcal{R}_{\mathrm{cris}}(u); \gamma, \beta, M)$, and*

$$\mathrm{nlen}_{\mathcal{O}}\Big(\beta + \sum_b v_{\mathcal{O}}(\det(u(\tau_w^{[b]}) - 1))\Big) \;\le\; 10\, f(D) \cdot \#n \cdot \mathrm{cr}_\ell(D).$$

LEAN `IsRow.eventually_isBalanced`

Theorem D (Ribet's lemma to first order). *Let A be a discrete valuation ring with valuation v_A, $\mathbb{L}$ a commutative A-algebra, Y, σ, W and r finite sets, J a finite totally ordered set, Γ a group, $u = (u_x)_{x \in Y}$ a family of homomorphisms $u_x\colon \Gamma \to M_\sigma(A)$, $\mathcal{S}$ an A-subalgebra of the algebra of functions $Y \to \mathbb{L}[X_j : j \in J]$ and $\mathcal{J}$ an $\mathcal{S}$-submodule of that algebra. Let $t \ge 1$, $\epsilon\colon r \sqcup \{0, \dots, t-1\} \to \sigma$ a bijection, $\mathcal{X} \subseteq Y$, $\pi \in \mathcal{X}$, and for $x \in \mathcal{X}$ let $\varrho_x\colon \Gamma \to M_r(A)$ be a homomorphism, $\gamma_x \in \Gamma$ and $y_x \in A$; let $\xi_e\colon \Gamma \to A$, $e < t$, be multiplicative homomorphisms. Let $p, p_0, C \in A$, let $\tau, \phi\colon J \times \{+,-\} \to \Gamma$ be functions, $\varepsilon = (\varepsilon_j)_{j \in J}$ polynomials in $A[X]$, $a \in \Gamma$, $\mathrm{ty}\colon J \to \{0, \dots, t-1\}$ a monotone map, θ a finitely supported function $\Gamma \to \mathbb{N}$, $\beta\colon Y \to \mathbb{L}$ a function, b_0 and $(b_e)_{e<t}$ elements of $\mathbb{N} \cup \{\infty\}$, $(\delta_v)_{v \in W}$ elements of Γ, $(o_v)_{v \in W}$ polynomials in $A[X]$, and s a finite subset of Γ. Suppose:*

(a) *$\xi_0 = 1$; for $x \in \mathcal{X}$ and $g \in \Gamma$, $u_x(g)$ is the block sum of $\varrho_x(g)$ and the diagonal matrix of the $\xi_e(g)$, transported by ϵ; and the A-span of $\varrho_\pi(\Gamma)$ is $M_r(A)$;*

(b) *$\varrho_x(\gamma_x) = y_x \cdot 1$, $y_x - 1 \in A^\times$ and $\xi_e(\gamma_x) = 1$ for all $x \in \mathcal{X}$ and $e < t$;*

(c) *$p \neq 0$; $\mathsf{TrId}(\mathcal{J}; u; p; \tau, \phi, \mathrm{pr}_1, \varepsilon)$, with pr_1 the first projection of $J \times \{+,-\}$; and p divides $\varepsilon_j(\xi_e(\phi_k)) - 1$ for all e and all $k = (j, b)$;*

(d) *for every $e < t$, the product $\prod_{k<t,\, k \neq e}(\xi_e(a) - \xi_k(a))$ divides C;*

(e) *$p_0 \neq 0$; $\beta(x)$ is the image in $\mathbb{L}$ of $\det(\sum_g \mathrm{rn}_{\gamma_\pi, a}(y_\pi, p_0 C, \theta)(g)\,(u_x(g) - 1))$ for every $x \in Y$; $b_0 < v_A(p_0)$; and the A-module of the $c \in \mathcal{Z}_u(\varrho_\pi; p_0)$ that vanish at all τ_k and at γ_π has length at most b_0;*

(f) *$\xi_e(\delta_v) = 1$ for all v, e, and $(o_v \cdot (X-1))(u_x(\delta_v)) = 0$ for all $v \in W$, $x \in Y$;*

(g) *for every $e < t$ and every A-submodule Z of the functions $\Gamma \to A/p$: if every $c \in Z$ lies in $\mathcal{Z}_u(\xi_e; p)$, vanishes at all δ_v, and satisfies "for every $m \in A/p$, if $c(g) = (\xi_e(g) - 1)m$ for all g, then $(\xi_e(a) - 1)\,c(g) = 0$ for all g", then $\mathsf{TriCost}(Z; \tau, \phi; \{j : \mathrm{ty}(j) = e\}; C, b_e)$, and every $c \in Z$ that vanishes on s is zero.*

Then there is $c \in A$ such that $\mathsf{Rib}(\mathcal{J}; \beta; \pi, c)$ (with c read in $\mathbb{L}$) and

$$v_A(c) \;\le\; b_0 + \sum_{e<t} b_e + t\Big(\sum_{v \in W} v_A(o_v(1)) + 2\,\#s \cdot v_A(C)\Big) + (\#J + 1)\, v_A(C).$$

LEAN `ribet_first_order`

Theorem E (the cost of a Tate twist). *Let E be a CM field as in the conventions, J a finite totally ordered set, $\mathcal{O}$ a coefficient ring, ν, t, h, c natural numbers, $w = (w_j)_{j\in J}$ finite places of E, $\tau = (\tau_j)_j$ and $\phi = (\phi_j)_j$ families in G_E, $\gamma \in G_E$, $e < t$, and $\mathrm{ty}\colon J \to \{e' : 0 < e' < t\}$ a monotone map with $\mathsf{Tri}_\nu(w, \tau, \phi; \gamma; t, \mathrm{ty}; h, c)$. Let Z be an $\mathcal{O}$-submodule of the functions $G_E \to \mathcal{O}/\ell^\nu$ and $b \in \mathbb{N} \cup \{\infty\}$. Suppose that*

(i) $Z \subseteq \mathrm{Sel}^{\emptyset}_{\Sigma_\ell(E)\cup w(J)\cup \bar{w}(J)}(\mathcal{O}(-e); \mathcal{O}/\ell^\nu)$;

(ii) *for every $z \in Z$ and $m \in \mathcal{O}/\ell^\nu$: if $z(g) = (\mathrm{cyc}(g)^{-e} - 1)\, m$ for all g, then $(\mathrm{cyc}(\gamma)^{-e} - 1)\, z(g) = 0$ for all g;*

(iii) $\mathrm{len}_{\mathcal{O}}(Z/Z \cap \mathrm{Ob}(\mathcal{R}_{\mathrm{cris}}(\mathcal{O}(-e)); \mathcal{O}/\ell^\nu)) + \mathrm{len}_{\mathcal{O}}\, \mathrm{Sha}^{\mathrm{odd}}(\ell, E, \mathcal{O}, e, \nu) \le b$.

Then $\mathsf{TriCost}(Z; \tau^{[\cdot]}, \phi^{[\cdot]}; \{j : \mathrm{ty}(j) = e\}; \ell^h, b)$.

LEAN `hasTriangleCost_tateTwist`

In Theorem A the representations ϱ_i are, in the third note, lattices of the rows of the auxiliary pairs, balanced by Theorem C with L_i the crystalline relations, and the families are those of Theorem D of the first note; the conclusion (3) is then the Selmer length bound of the first note, and (1) bounds the inverse ratios of L-values. In Theorem D the points of Y are Hecke eigensystems, $\mathcal{X}$ the lifts, the variables X_j the tie parameters at the auxiliary primes, and the element f of $\mathsf{Rib}(\mathcal{J}; \beta; \pi, c)$ a Hecke element that kills a determinant vanishing on the lifts modulo the ties; its coefficient c at $\prod_j X_j$ measures the congruence.

2 Conventions and recalled notions

Conventions. The conventions are those of the first note of the unit. The natural numbers are $0, 1, 2, \ldots$. ℓ is a prime number; $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell$, $\bar{\mathbb{Q}}_\ell$, $\bar{\mathbb{Z}}_\ell$, $\mathbb{C}_\ell$, $G_{\mathbb{Q}_\ell} = \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ are as usual, with $|\ell| = 1/\ell$; $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ is a fixed isomorphism of fields where complex numbers occur. $\bar{\mathbb{Q}}$ is a fixed algebraic closure of $\mathbb{Q}$ with a fixed complex conjugation c. A *CM field as in the conventions* is a CM number field E with a fixed embedding in $\bar{\mathbb{Q}}$ that contains a primitive root of unity of order 5040; $G_E = \mathrm{Gal}(\bar{\mathbb{Q}}/E)$; in Lemmas 5.2 and 6.5 only a number field with a fixed embedding is needed, or no field at all, as said there. P denotes parameters and D admissible data for P. The letters n, r, m, d, σ, κ, T, N, X, Y, W, J, $\mathcal{J}$, when they index matrices or families, denote finite sets (in §8 some of them are arbitrary sets, as said there); $\#n$ is the number of elements, $M_n(A)$ the matrices over a commutative ring indexed by n. A *coefficient ring* is the ring of integers $\mathcal{O}$ (also $\mathcal{O}'$) of a finite Galois extension of $\mathbb{Q}_\ell$ inside $\bar{\mathbb{Q}}_\ell$; "for all large $\mathcal{O}$" means: for all coefficient rings containing a suitable one; "for all large N" is meant in the same way for natural numbers. A, A_0, A', Λ, $\mathbb{L}$ denote commutative rings, Γ a group, B a module. A *discrete valuation ring* is a domain; v_A is its valuation with $v_A(\varpi) = 1$, with values in $\mathbb{N} \cup \{\infty\}$.

2.1. Background. The background notions are those of §2 of the first note: finite places w, v, $\mathfrak{q}$ of E, their conjugates $\bar{w}$, norms $\mathrm{N}(w)$ and the sets $\Sigma_M(E)$; places $\mathfrak{P}$ of $\bar{\mathbb{Q}}$, decomposition and inertia groups $G_{\mathfrak{P}}$, $I_{\mathfrak{P}}$, the inertia I_S above a set of places, q-Frobenius elements, the tame generators $\tau_{\mathfrak{P}}$ and τ_w; $g^{\mathsf{c}} = \mathsf{c}g\mathsf{c}$, $g^{[+]} = g$, $g^{[-]} = (g^{\mathsf{c}})^{-1}$; the cyclotomic character cyc and cyc_M; characters *from* $\mathrm{U}(1)$ *relative*

to S with modulus I; relations θ, the indicator relation δ_g, the module $\mathrm{Ob}(L;\,B)$ of the functions that obey a set L of relations, the relations $\mathrm{Rel}(Z)$ of a module of functions, the cocycle relations, the Selmer module $\mathrm{Sel}_S^L(\varrho;\,B)$ (locally constant cocycles for ϱ with values in B^n, unramified outside S, obeying L) and the local cocycles; A/a; the length len_A; the normalised length $\mathrm{nlen}_{\mathcal{O}} = v_\ell(\varpi)\cdot\mathrm{len}_{\mathcal{O}}$, applied to a module or to an element of $\mathbb{N}\cup\{\infty\}$; the truncated valuation v_ℓ on $\bar{\mathbb{Q}}_\ell$; the Tate twist $\mathcal{O}(-e)$; lattices of a representation $R\colon G_E\to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$; Galois lattices over $\mathcal{O}$ for a group over $G_{\mathbb{Q}_\ell}$, their crystalline cocycles $Z_{\mathrm{f}}(L)$, jumps, twisted duals $L^\vee(1)$, the classes $\mathcal{Q}(F,K,U)$, tame types and cup-orthogonals $L^{\perp_{\mathcal{Q}}}$. For a representation ϱ and a multiplicative homomorphism ψ' with values in the same ring, $\psi'\cdot\varrho$ is $g\mapsto\psi'(g)\varrho(g)$; when ϱ has coefficients in a subring, it is read in the larger ring. For $e,\nu\in\mathbb{N}$, $\mathrm{Sha}^{\mathrm{odd}}(\ell,E,\mathcal{O},e,\nu)$ is the module of the cocycles in $\mathrm{Sel}^{\emptyset}_{\Sigma_\ell(E)}(\mathcal{O}(-e);\,\mathcal{O}/\ell^\nu)$ that vanish on the inertia groups above ℓ when $e=0$, and, when $e\neq 0$, vanish at the elements of the decomposition groups above ℓ with $\mathrm{cyc}=1$ and satisfy $z(g^{\mathsf{c}})=-z(g)$ (background library; the name "odd" is apt for $e\neq 0$ only). For a submodule $\mathcal{N}$ of a free module Λ^s of finite rank, its *Fitting ideal* $\mathrm{Fitt}(\mathcal{N})$ is the ideal generated by the determinants of the square matrices whose rows lie in $\mathcal{N}$ (the zeroth Fitting ideal of the quotient).

Definition 2.1 (notions recalled from the first note)**.** The following are defined in the note *Columns and rows as Galois representations*; the numbers are those of its definitions.

(1) (3.1) For a family $u=(u_x)_x$ of homomorphisms $\Gamma\to M_\sigma(A)$, a homomorphism $\varrho\colon\Gamma\to M_d(A)$ and $a\in A$, $\mathcal{Z}_u(\varrho;\,a)$ is the module of the cocycles $c\colon\Gamma\to(A/a)^d$ for ϱ with $\sum_g\theta(g)c(g)=0$ for every finitely supported $\theta\colon\Gamma\to A$ such that $\sum_g\theta(g)(u_x(g)-1)=0$ for all x.

(2) (3.4, 3.5, 3.37) For $g\in G_{\mathbb{Q}_\ell}$: $\mathrm{emb}_{\mathcal{O}}(g)$ is the map $\mathcal{O}\to\mathbb{C}_\ell$, $x\mapsto g(x)$, and $\mathrm{ce}(e,g)$ is $\iota^{-1}g^{-1}e$, for a ring homomorphism e into $\bar{\mathbb{Q}}_\ell$. For a homomorphism $u\colon G_E\to M_n(\mathcal{O})$ and $v\in\Sigma_\ell(E)$, $Z_{\mathrm{cris}}(v;\,u)$ is the module of the crystalline cocycles of the restriction of u to the decomposition group of the chosen place over v, and $\mathcal{R}_{\mathrm{cris}}(u)$ is the set of the crystalline relations of u: the relations of the modules $Z_{\mathrm{cris}}(v;\,u)$, extended by zero. For a homomorphism $\varrho\colon\Gamma\to M_n(A)$ into matrices over a $\mathbb{Z}_\ell$-algebra, $\mathrm{L}(\varrho)$ is A^n with Γ acting through ϱ.

(3) (3.10, 3.13) For admissible data D: $G_{E(D)}=\mathrm{Gal}(\bar{\mathbb{Q}}/E(D))$; $\Sigma_g(D)$ is the set of the finite places of $E(D)$ above the two primes $g_0(D)$, $g_1(D)$; $\Sigma_{\mathrm{bad}}(D,\ell)$ the set of those containing $\ell g(D)N(D)$; and $\mathrm{cr}_\ell(D)=v_\ell(\mathrm{ord}_{\mathrm{ch}}(D))+2^{-\ell}$.

(4) (3.15, 3.16) $\mathsf{ColRep}_D(t,\mathcal{H},S;\,R)$: R is continuous with non-degenerate trace form, $\mathrm{tr}\,R(g^{\mathsf{c}})=\mathrm{cyc}(g)^{-(t-1)}\,\mathrm{tr}\,R(g^{-1})$, and at the places outside $S\setminus(\Sigma_\ell(E(D))\cup\Sigma_g(D))$ it is of finite type with the jumps $\mathcal{H}$, and good outside $\Sigma_g(D)$. $\mathsf{RowRep}_D(t,\mathcal{H};\,R)$: a column of level $\emptyset$ of row type, with the clauses (W2)–(W4) on the inertia above $\Sigma_g(D)$, on the lattices (they span all matrices and contain a scalar $u(g)$ with $\mathrm{cyc}(g)=g_i(D)^{\#n}$), and on a scalar inertial element of large residual order.

(5) (3.23–3.25) $\mathrm{Aux}(E,q)$ is the set of the data $l=(\mathfrak{P},\gamma,\phi,h)$ at an auxiliary prime, with $h\colon G_E\to\mathbb{Z}/q$. $\mathsf{Acyc}(T,\mathfrak{q};\,l)$: h is from $\mathrm{U}(1)$ relative to T with modulus $\{\mathfrak{q}\}$, $\mathrm{N}(\mathfrak{q})\equiv 1 \bmod q$, $\mathfrak{P}$ lies over $\mathfrak{q}$, ϕ is an $\mathrm{N}(\mathfrak{q})$-Frobenius at $\mathfrak{P}$, $\mathfrak{q}\notin T$, and $\gamma=\tau_{\mathfrak{P}}$ with $h(\gamma)=1$. $\mathsf{Auxil}(S;\,I)$: every $v\in I$ has prime norm, and $v,\bar{v}\notin S$, $\bar{v}\notin I$.

(6) (3.26) $\mathsf{Fam}(S;\,\nu,\mathrm{val})$: the ν_i are continuous; for auxiliary I the value $\mathrm{val}(I,\chi)$ is a sum $\sum_a c_a\chi(g_a)$ with integral c_a; adding a place $\mathfrak{q}$ to I raises $v_\ell(\mathrm{val}(I,\chi))$ by at most $8a$ when, at a Frobenius element ϕ of $\mathfrak{q}$, all $v_\ell(1-\nu_i(\phi)\chi(\phi)^{\pm1})$ are at most a; and $\mathrm{val}(\emptyset,1)\neq 0$.

(7) (3.27, 3.28) $\mathrm{Ev}(\mathcal{O},S;\,\nu,e)$: the cocycles in $\mathrm{Sel}^{\emptyset}_{\Sigma_\ell(E)\cup S\cup\bar S}(\mathcal{O}(-e);\,\mathcal{O}/\ell^\nu)$ with $c(g^{\mathsf c})=c(g)$ that vanish at the elements of the decomposition groups above ℓ with $\mathrm{cyc}=1$. $\mathsf{New}_\nu(w,\tau,\phi)$: no w_j is above ℓ, $\mathrm{cyc}_{\ell^\nu}(\phi_j)=1$, and τ_j is the tame generator at a place over w_j. $\tau^{[k]}=\tau_j^{[b]}$ for $k=(j,b)$.

(8) (3.29) $\mathsf{Tri}_\nu(w,\tau,\phi;\,a;\,t,\mathrm{ty};\,h,c)$: the clauses (T1)–(T5) of the first note; (T1) is $\mathsf{New}_\nu(w,\tau,\phi)$, (T4) the step "$\ell^h z(\tau_j)=0$ for an even class of type $\mathrm{ty}(j)$ that vanishes at the earlier τ_i, and at the earlier ϕ_i of the same type", (T5) the clause "$4\delta(a)^2\mid\ell^h$ and $\delta(a)\cdot 2z(g)=\delta(g)\cdot 2z(a)$ for even classes unramified outside ℓ that vanish at the ϕ_j of type e", with $\delta(g)=\mathrm{cyc}(g)^{-e}-1$.

(9) (3.30, 3.31) $\mathsf{AuxTw}(S,Q;\,\psi,c)$: $\mathsf{Auxil}(S;\,Q)$, ψ is from $\mathrm{U}(1)$ relative to $\Sigma_\ell(E)$ with modulus Q, no place of Q is above ℓ, and $\sum_{\mathfrak{q}\in Q} v_\ell(\psi(\sigma_\mathfrak{q})-1)\le c$ for tame generators $\sigma_\mathfrak{q}$ at places over the $\mathfrak{q}\in Q$. $S\cup J\cup\bar J$ is the union of S, J and the conjugates of J.

(10) (3.32) For homomorphisms $h_j\colon\Gamma\to\mathbb{Z}/q$ and $y_j\in A$ $(j\in\sigma)$ with $y_j^q=1$, the *point character* is $g\mapsto\prod_j y_j^{h_j(g)}$, written y^h; for a ring homomorphism $e\colon A'\to A$ and $\varrho\colon\Gamma\to M_n(A')$, $\mathrm{tp}_{h,y}(e,\varrho)(g)=\prod_j y_j^{h_j(g)}\cdot e(\varrho(g))$; and $\varrho \bmod l^M$ is ϱ modulo l^M.

(11) (3.33) $\mathsf{Bal}_\ell(S;\,\varrho,L;\,\gamma,\beta,M)$: the clauses (B1)–(B5) of the first note: ϱ modulo ℓ^M is locally constant and unramified outside S; it spans all matrices and the residue field has more than $4\#n+2$ elements; $\gamma\in I_{S\setminus\Sigma_\ell(E)}$ and $\gamma^{[-]}$ act by one scalar s with $s\pm1$ units; coboundaries obey L; and for some g with $\varrho(g)=\alpha$, $\varrho(g^{[-]})=\alpha'$ scalars and every $a\mid\ell^M$: the sum over $v\in S$ of the least lengths of the local cocycles modulo a, plus $\#n\cdot v_A(\alpha-\mathrm{cyc}(g^{\mathsf c})\alpha')$, is at most $(\#S+f_E)\,\mathrm{len}_A((A/a)^n)+\beta$.

(12) (3.34, 3.35) $\mathsf{Pencil}_\ell(S_0;\,U,W;\,N,\mathfrak{q}_0,J;\,x)$: x assigns to each place v a datum $l_v=(\mathfrak{P}_v,\gamma_v,\phi_v,h_v)\in\mathrm{Aux}(E,\ell^N)$ and slopes $(s_{v,i})_i$; (P1) $\mathsf{Auxil}(S_0;\,J)$ and $\mathfrak{q}_0\in J$; (P2) $\mathsf{Acyc}(\Sigma_\ell(E),v;\,l_v)$ for $v\in J$; (P3) for $v\neq\mathfrak{q}_0$ in J, $h_{v'}(\phi_v)$ is 1 for $v'=\mathfrak{q}_0$ and 0 for the other $v'\in J$; (P4) $\phi_{\mathfrak{q}_0}\in W$; (P5) U_i holds at $\phi_v^{[\pm]}$ when $s_{v,i}\neq0$. $\mathrm{Ties}_i(J,x)$ is the set of the relations $\delta_{\gamma_v^{[b]}}-s_{v,i}\delta_{\phi_v^{[b]}}$, $v\in J$.

LEAN `obeyingCocycles` · `coeffEmb` · `complexEmbOf` · `ZfAt` · `crysRels` · `matrixLattice` · `AbsGal` · `placesAboveG` · `badPlaces` · `crudeUnit` · `IsColumn` · `IsRow` · `AuxPrimeData` · `IsAnticyclotomicAt` · `IsAuxiliary` · `IsFamily` · `evenClasses` · `IsNewPlaces` · `withConjInv` · `IsPrimeTriangle` · `IsAuxTwist` · `unionWithConj` · `pointChar` · `twistAtPoint` · `reduceModPow` · `IsBalanced` · `IsPencil` · `tieRelations` · all defined in the first note of the unit

3 The definitions of this note

Definition 3.1 (evaluation at a conjugate pair)**.** Let E be a CM field as in the conventions, n a finite set, A a commutative ring, B a commutative A-algebra with the discrete topology, and $g \in G_E$. Then $\mathrm{ev}_g^{\pm}$ is the A-linear map that sends a function $z\colon G_E \to B^n$ to the pair $(z(g^{[+]}), z(g^{[-]})) \in B^n \times B^n$.

LEAN `evalConjPair`

Definition 3.2 (flattening of blocks)**.** Let N and σ be finite sets and $\mathbb{L}$ a commutative algebra over a commutative ring Λ. Then fl_N is the isomorphism of Λ-algebras from the $N \times N$ matrices with entries in $M_\sigma(\mathbb{L})$ to the matrices over $\mathbb{L}$ indexed by $N \times \sigma$.

LEAN `blockFlattenAlgEquiv`

Definition 3.3 (the normaliser)**.** Let Γ be a group, Λ a commutative ring, $\gamma, a \in \Gamma$, $y, p \in \Lambda$ and θ a finitely supported function $\Gamma \to \mathbb{N}$. Then $\mathrm{rn}_{\gamma,a}(y,p,\theta)(\cdot)$ is the finitely supported function $\Gamma \to \Lambda$ which is the sum of the indicator of γa, of $-y$ times the indicator of a, and of $p\,\theta$; its value at $g \in \Gamma$ is $\mathrm{rn}_{\gamma,a}(y,p,\theta)(g)$. Paired with a function $g \mapsto u(g) - 1$ it gives $(u(\gamma a) - 1) - y\,(u(a) - 1) + p \sum_g \theta(g)(u(g) - 1)$.

LEAN `ribetNormaliser`

Definition 3.4 (triangle cost)**.** Let A be a commutative ring, Γ a set, J a totally ordered set, B an A-module, Z a submodule of the functions $\Gamma \to B$, $\tau, \phi\colon J \times \{+,-\} \to \Gamma$ functions, I a finite subset of J, $C \in A$ and $b \in \mathbb{N} \cup \{\infty\}$. Then $\mathsf{TriCost}(Z;\, \tau, \phi;\, I;\, C, b)$ holds if there is a subset $\Gamma_0 \subseteq \Gamma$ such that

(C1) for every $j \in I$ and every $c \in Z$ that vanishes on Γ_0: if for all $i < j$ one has $c(\tau_{(i,+)}) = c(\tau_{(i,-)})$, and also $c(\phi_{(i,+)}) = c(\phi_{(i,-)})$ when $i \in I$, then

$$C\,c(\tau_{(j,+)}) = C\,c(\tau_{(j,-)});$$

(C2) for every submodule Z' of the elements of Z that vanish on Γ_0, if every $c \in Z'$ vanishes at all τ_k and has $c(\phi_{(j,+)}) = c(\phi_{(j,-)})$ for $j \in I$, then

$$\mathrm{len}_A\big(Z / \{c \in Z : c|_{\Gamma_0} = 0\}\big) + \mathrm{len}_A\big(Z' / Z'[C]\big) \le b,$$

with $Z'[C]$ the module of the elements of Z' killed by C.

LEAN `HasTriangleCost`

Definition 3.5 (trace ideals)**.** Let A be a commutative ring, $\mathbb{L}$ a commutative A-algebra, Y, J, K sets (J is totally ordered in the formal text, which the definition does not use), σ a finite set, Γ a group, $\mathcal{S}$ an A-subalgebra of the functions $Y \to \mathbb{L}[X_j : j \in J]$ and $\mathcal{J}$ an $\mathcal{S}$-submodule of these functions. Let A_1 be a commutative A-algebra over which $\mathbb{L}$ is an algebra, $u = (u_x)_{x \in Y}$ a family of functions $u_x\colon \Gamma \to M_\sigma(A_1)$, $p \in A$, $\tau, \phi\colon K \to \Gamma$, $k\colon K \to J$ and $\varepsilon = (\varepsilon_j)_{j \in J}$ polynomials in $A[X]$. Then $\mathsf{TrId}(\mathcal{J};\, u;\, p;\, \tau, \phi, k, \varepsilon)$ holds if

(1) every constant function X_j lies in $\mathcal{S}$, and for all $g \in \Gamma$ and all i the function $x \mapsto$ (the i-th coefficient of the characteristic polynomial of $u_x(g)$) lies in $\mathcal{S}$;

(2) for every $g \in \Gamma$, the function $x \mapsto p\,\mathrm{tr}\,u_x(g)$ lies in $\mathcal{J}$, and for every $i \in K$ so does $x \mapsto \mathrm{tr}\big((u_x(\tau_i) - 1)u_x(g)\big) - X_{k(i)}\ \mathrm{tr}\big(((X-1)\varepsilon_{k(i)})(u_x(\phi_i))\,u_x(g)\big)$.

LEAN IsTraceIdeal

Definition 3.6 (Ribet elements)**.** Let A, $\mathbb{L}$, Y, $\mathcal{S}$, $\mathcal{J}$ be as in Definition 3.5 with J finite (and totally ordered), let $\beta\colon Y \to \mathbb{L}$ be a function, $\mathcal{X} \subseteq Y$, $\pi \in \mathcal{X}$ and $c \in \mathbb{L}$. Then $\mathsf{Rib}(\mathcal{J};\, \beta;\, \pi, c)$ holds if β, as a function with constant polynomials as values, lies in $\mathcal{S}$, and there is $f \in \mathcal{S}$ with $f\beta \in \mathcal{J}$ such that for every $x \in \mathcal{X}$: $\beta(x) = 0$ and the coefficient in $f(x)$ of every monomial that properly divides $\prod_{j\in J} X_j$ is zero; and the coefficient of $\prod_{j\in J} X_j$ in $f(\pi)$ is c.

LEAN IsRibetElement

4 Roots of unity at which polynomials keep low valuations

Lemma 4.1 (one variable)**.** *Let κ be a finite set, $y = (y_k)_{k\in\kappa}$ real numbers and r a real number. Suppose $y_k > 0$ for all k, $0 < r$, $r < 1$ and $|\ell| < r^{\ell-1}$. Then for all large N and, given N, all large $\mathcal{O}$: for every family $(F_k)_{k\in\kappa}$ of polynomials in $\mathcal{O}[X]$ such that for each k some coefficient $a_{k,i}$ of X^i in F_k has $y_k \le |a_{k,i}|\, r^i$, there is $\zeta \in \mathcal{O}$ with $\zeta^{\ell^N} = 1$ and $r\, y_k \le |F_k(\zeta - 1)|$ for all k.*

Proof. For $\rho > 0$ let $\|F\|_\rho = \max_i |a_i|\rho^i$ be the Gauss norm, a multiplicative non-archimedean norm, and put $\rho_j = r^{1/\ell^j}$, an increasing sequence below 1. Choose a with $r^a < |\ell|$, let N be large (how large is fixed at the end), and let $\mathcal{O}$ contain primitive roots of unity η_j of order ℓ^{j+1} for $j < N$. Suppose no ζ works. Fix $j < N$ and $\eta = \eta_j$; for every $i < \ell^{j+1}$ some $k = k(i)$ has $|F_k(\eta^i - 1)| < r\, y_k$. Let $\Pi = \prod_k F_k$ and $\Pi = F_k Q_k$. Differentiating the identity $(1+X)^{\ell^{j+1}} - 1 = \prod_i (X - (\eta^i - 1))$ and inserting, at each i, the remainder of $F_{k(i)}$ on division by $X - (\eta^i - 1)$, one writes $\ell^{j+1}(1+X)^{\ell^{j+1}-1}\,\Pi^{a(j+1)}$ as a sum of terms each divisible by $(1+X)^{\ell^{j+1}} - 1$, plus terms carrying a factor $F_{k(i)}(\eta^i - 1)$. Since $|\eta^i - 1| \le \rho_t$ for $t \ge j$ (the absolute value of $\eta_j - 1$ is $|\ell|^{1/(\ell^j(\ell-1))} < \rho_j$ by the hypothesis on r), the linear factors have Gauss norm ρ_t at the radius ρ_t, and the ultrametric inequality yields an inequality between the Gauss norms of Π at the radii ρ_j and ρ_{j+1}:

$$\log \|\Pi\|_{\rho_{j+1}} - \log \|\Pi\|_{\rho_j} \;\ge\; \frac{-\log r}{a\,(j+1)} \cdot (\text{a positive constant}).$$

Summing over $j < N$, the left side telescopes and is bounded ($\|\Pi\|_\rho \le 1$ and $\|\Pi\|_{\rho_0} \ge \prod_k y_k$), while the right side grows like the harmonic series. This is impossible for large N. The estimate of the Gauss norms is a computation of twenty lines in the formal proof, of which this is the outline. □

LEAN exists_root_of_unity_eval_large

Lemma 4.2 (several variables and all powers up to m)**.** *Let κ be a finite set, $\delta > 0$ a real number, m, L, a, q natural numbers and $B = (B_k)_{k\in\kappa}$ non-negative real numbers. Then for all large N and, given N, all large $\mathcal{O}$ the following holds. Let σ and T be finite sets with $\#\sigma \le q$, $(D_n)_{n\in T}$ monic polynomials in $\mathcal{O}[X]$ with $\sum_n \deg D_n \le L$, $(F_{k,d})_{k\in\kappa,\,1\le d\le m}$ polynomials over $\mathcal{O}$ in the variables indexed by σ, and $(u_{k,d})$ points of $\mathcal{O}^\sigma$ such that ℓ divides $(u_{k,d,j} - 1)^a$ for every coordinate j and $v_\ell(F_{k,d}(u_{k,d})) \le B_k$. Then there is $\zeta \in \mathcal{O}^\sigma$ with*

$$v_\ell(F_{k,d}(\zeta^d)) \le \delta + B_k \;\text{ for all } (k,d), \qquad \zeta_j^{\ell^N} = 1 \;\text{ and }\; v_\ell(D_n(\zeta_j)) \le \delta \;\text{ for all } j,\ n,$$

where ζ^d is the tuple of the d-th powers.

Proof. Choose r with $\max(|\ell|, \ell^{-\delta}) < r^{\,q+L+\ell+ma+1} < 1$. The tuple is built one coordinate at a time: suppose the coordinates in a set $s \subseteq \sigma$ have been replaced by roots of unity so that $|F_{k,d}|$ at the mixed point is at least $\ell^{-B_k} r^{\#s}$. For a further coordinate j, restrict each $F_{k,d}$ to the line where only the j-th coordinate varies, substituting $(1+X)^d$ for its d-th power; the hypothesis $\ell \mid (u_{k,d,j}-1)^a$ shows that the value at the old point is, up to the factor r^{da}, a lower bound for the Gauss norm of radius r of the restriction. Lemma 4.1, applied to these restrictions and to the Taylor shift of $\prod_n D_n$ (a monic polynomial, of Gauss norm at least r^L), gives a root of unity for the j-th coordinate that loses at most one more factor r. After $\#\sigma \le q$ steps all coordinates are roots of unity, and the choice of r turns the accumulated factors into the bound δ. □

LEAN `exists_roots_of_unity_ellVal_le`

Lemma 4.3 (points on a line through a root of unity)**.** *Let A be a commutative ring, $t, u \in A$, and k, M, N natural numbers. Suppose $u^{\ell^k} = 1$ and $k + M \le N$. Then ℓ divides $\left((1+t(u-1))-1\right)^{\ell^k}$, and ℓ^M divides $(1+t(u-1))^{\ell^N} - 1$.*

Proof. Modulo ℓ one has $(x+y)^{\ell^k} \equiv x^{\ell^k} + y^{\ell^k}$. So $(u-1)^{\ell^k} \equiv u^{\ell^k} - 1 = 0$, and then $(1+t(u-1))^{\ell^k} \equiv 1$. If $x \equiv 1 \bmod \ell$ then $x^{\ell^M} \equiv 1 \bmod \ell^M$, and ℓ^N is a multiple of ℓ^{k+M}. □

LEAN `dvd_one_add_mul_pow_sub_one`

5 The mirror lemma and the sharp polynomial

Lemma 5.1 (mirror lemma)**.** *Let A be a commutative ring, Γ a group, n a finite set, $\mathfrak{a}$ an ideal of A with $A/\mathfrak{a}$ finite, $\varrho\colon \Gamma \to M_n(A)$ a homomorphism, $\chi\colon \Gamma \to A$ a multiplicative homomorphism, $\mathsf{c}'\colon \Gamma \to \Gamma$ a homomorphism, H and U subgroups of Γ with H normal, $\gamma, g_0, z, g \in \Gamma$, $s, \alpha, \alpha' \in A$, $y\colon \Gamma \to (A/\mathfrak{a})^n$ a function, i an index, W a set, and $x = (x_w)_{w \in W}$ functions that assign to each $g' \in \Gamma$ an additive map $x_w(g')\colon (A/\mathfrak{a})^n \to \mathbb{Q}/\mathbb{Z}$. Suppose:*

(1) *the A-span of $\varrho(\Gamma)$ is $M_n(A)$, and $\mathsf{c}'(\mathsf{c}'(g')) = g'$ for all g';*
(2) *if $x_w(g') = x_v(g')$ for all $g' \in U$, then $w = v$;*
(3) *$\varrho(g')m = m$ for $g' \in U$ and $m \in (A/\mathfrak{a})^n$; $\mathsf{c}'(H) \subseteq H \subseteq U$; and every commutator $[g', \gamma]$ lies in H;*
(4) *$s-1$ and $s+1$ are units; $\varrho(\mathsf{c}'(\gamma))^{-1} = s$, $\varrho(\gamma) = s$, $\chi(\mathsf{c}'(\gamma)) = \chi(\gamma) = 1$ and $y(\gamma) = 0$;*
(5) *$z \in U$ and $\mathsf{c}'(z) \in U$; y is a cocycle for ϱ; and the coordinate $y(g_0)_i$ is a unit;*
(6) *$x_w(a \cdot g' b g'^{-1})(m) = x_w(a)(m) + x_w(b)(\chi(g')\varrho(g')^{-1}m)$ for all w, $g' \in \Gamma$, $a, b \in U$ and m;*
(7) (isotropy on the coset zH) *$x_w(zh)(t\,y(zh)) = x_w(\mathsf{c}'(zh))(t\,y(\mathsf{c}'(zh)))$ for all w, $h \in H$ and $t \in A$;*
(8) *$\varrho(g) = \alpha$ and $\varrho(\mathsf{c}'(g))^{-1} = \alpha'$ are scalars.*

Then $\#W$ is at most the number of the $m \in (A/\mathfrak{a})^n$ with $(\alpha - \chi(\mathsf{c}'(g))\alpha')\,m = 0$.

Proof. Let $\tau = [g_0, \gamma] \in H$. By the cocycle identity and (4), $y(\tau) = (1-s)\,y(g_0)$ up to the action of elements that act trivially, so $y(\tau)_i$ is a unit. Since ϱ spans all matrices, the vectors $Yy(\tau)$, $Y \in M_n(A)$, fill $(A/\mathfrak{a})^n$. From (6), (7) applied to the

elements zh, $zh \cdot k\tau k^{-1}$ and $zh\gamma$-conjugates, and from the fact that $s^2 - 1$ is a unit, one derives for all $t \in A$, $k \in \Gamma$, $b \in H$:

$$x_w(b)\big(t\,\varrho(k)y(\tau)\big) = x_w(\mathsf{c}'(\tau))\big(\chi(\mathsf{c}'(k))\,t\,\varrho(\mathsf{c}'(k))^{-1}y(\mathsf{c}'(b))\big).$$

Taking $b = \mathsf{c}'(\tau)$ and comparing k with kg shows, by (8), that $x_w(\mathsf{c}'(\tau))$ vanishes on $(\alpha - \chi(\mathsf{c}'(g))\alpha') \cdot (A/\mathfrak{a})^n$; so it factors through the cokernel of that scalar. The map $w \mapsto x_w(\mathsf{c}'(\tau))$ is injective: by (6) with $b = [g', \gamma]$ the values of x_w on U are determined by $x_w(\mathsf{c}'(\tau))$, and then (2) applies. The dual of the cokernel has as many elements as the kernel of the scalar. The formal proof is one identity obtained by a linear combination of six instances of (6) and (7); the display is what it yields. □

LEAN `card_le_torsion_of_isotropic`

Lemma 5.2 (one polynomial bounds all twisted Selmer lengths, sharply at one point)**.** *Let E be a number field with a fixed embedding in $\bar{\mathbb{Q}}$, n and σ finite sets, M, q natural numbers, A_0 a commutative ring, A a discrete valuation ring of characteristic 0, $\varrho\colon G_E \to M_n(A_0)$ a homomorphism, $e_0\colon A_0 \to A$ a ring homomorphism, S a finite set of finite places of E, $h = (h_j)_{j\in\sigma}$ homomorphisms $G_E \to \mathbb{Z}/q$, L a set of relations with values in $M_n(A_0)$, and $y_0 \in A^\sigma$. Suppose that every h_j is locally constant and that $\varrho \bmod \ell^M$ is locally constant. Then there is a polynomial F over A in the variables indexed by σ such that*

$$v_A(F(y_0)) \le \operatorname{len}_A \operatorname{Sel}_S^{e_0(L)}(\operatorname{tp}_{h,y_0}(e_0,\varrho)(\cdot);\, A/\ell^M),$$

and such that for every discrete valuation ring A', ring homomorphism $e\colon A \to A'$ and $y \in A'^\sigma$,

$$\operatorname{len}_{A'} \operatorname{Sel}_S^{(e\circ e_0)(L)}(\operatorname{tp}_{h,y}(e\circ e_0,\varrho)(\cdot);\, A'/\ell^M) \le v_{A'}(e(F)(y)).$$

Proof. Work over the polynomial ring $\Lambda = A[X_j : j \in \sigma]$ with the universal twist $\operatorname{tp}_{h,X}(e_0,\varrho)(\cdot)$. By Lemma 7.4 of the first note (with $m = \ell^M$ and the neighbourhood on which ϱ modulo ℓ^M and all h_j are trivial) there are finitely many coordinates and a module $\mathcal{R}$ of relations over Λ such that, for every specialisation $X \mapsto y$ into a ring A', the Selmer module with values in A'/ℓ^M is the module of the vectors killed by the specialised relations. For a discrete valuation ring, the length of such a module is at most the valuation of the determinant of any square matrix with rows among the relations, and some such matrix attains the length (an input of the unit on Fitting ideals). Choose a square matrix over Λ with rows in $\mathcal{R}$ whose specialisation at y_0 attains the length there, and let F be its determinant. □

LEAN `exists_sharp_fitting_poly`

6 Pencils of tied primes: proofs of Theorems B and A

Lemma 6.1 (Selmer modules cut by relations lie in Selmer groups with local conditions)**.** *Let E be a CM field as in the conventions, n a finite set, A a commutative ring, B a commutative A-algebra with the discrete topology, $\varrho\colon G_E \to M_n(A)$ a homomorphism, S, P', Q' sets of finite places, $K \subseteq L$ sets of relations, and $x \in \operatorname{Sel}_S^L(\varrho;\, B)$. Suppose $S \subseteq P' \cup Q'$. Then x is a continuous cocycle of the discrete module B^n that is unramified outside $P' \cup Q'$ and whose restriction to the decomposition group at any chosen place over a place of P' is a local cocycle of ϱ and K there: an element*

of the Selmer group, in the sense of the unit on the formula of Greenberg and Wiles, with these local conditions at P' and no condition at Q'.

Proof. Unfold both definitions: the cocycle identity, continuity (local constancy), the vanishing on inertia outside $S \subseteq P' \cup Q'$, and the local condition, witnessed by x itself since $K \subseteq L$. A totally complex field has no real places, so there is no condition at infinity. □

LEAN `mem_selmerZ1_of_mem_selmerModule`

Lemma 6.2 (the point character of a pencil)**.** *Let* $\mathsf{Pencil}_\ell(S_0; U, W; N, \mathfrak{q}_0, J; x)$ *hold, with* $x(v) = (l_v, s_v)$ *and* $l_v = (\mathfrak{P}_v, \gamma_v, \phi_v, h_v)$. (a) *For* $v, v' \in J$: $h_{v'}(\gamma_v)$ *is* 1 *if* $v' = v$ *and* 0 *otherwise.* (b) *Let* Λ *be a commutative ring and* $y = (y_v)_{v \in J}$ *elements with* $y_v^{\ell^N} = 1$. *Then the point character* y^h *takes the value* y_v *at* $\gamma_v^{[b]}$ *for both signs* b, *and, for* $v \neq \mathfrak{q}_0$, *the value* $y_{\mathfrak{q}_0}$ *at* $\phi_v^{[b]}$.

Proof. (a) $h_{v'}$ is from $\mathrm{U}(1)$ with modulus $\{v'\}$, hence trivial on the inertia above the other places of J, and $h_v(\gamma_v) = 1$ by (P2). (b) follows from (a), (P3) and the fact that a character from $\mathrm{U}(1)$ takes the same value at g and at $g^{[-]}$. □

LEAN `IsPencil.char_tame_gen` · `IsPencil.pointChar_idOrConjInv`

Lemma 6.3 (dropping ties costs only determinants)**.** *Let* E *be a CM field as in the conventions,* A *a commutative ring and* $\mathcal{I}$ *a set. Suppose that* $\mathsf{Pencil}_\ell(S_0; U, W; N, \mathfrak{q}_0, J; x)$ *holds; write* $x(v) = (l_v, s_v)$ *for the value of* x *at a place* v, *where* $l_v = (\mathfrak{P}_v, \gamma_v, \phi_v, h_v)$. *Let* n *be a finite set,* $\varrho \colon G_E \to M_n(A)$ *a homomorphism,* $i \in \mathcal{I}$, *and* $k, M \in \mathbb{N}$. *Suppose that* $\varrho \bmod \ell^M(g) = 1$ *for all* $g \in I_{S_0^{\complement}}$, *that* $k + M + M \leq N$, *and that* $U_i(g)$ *implies* $\varrho \bmod \ell^M(g) = 1$. *Let* A' *be a discrete valuation ring,* $e \colon A \to A'$ *a ring homomorphism,* $u \in A'$ *with* $u^{\ell^k} = 1$, I *a set of finite places with* $I \subseteq J$, *such that* $u = 1$ *if* $\mathfrak{q}_0 \in I$, *and let* $(w_{v,b})_{v \in I, b}$ *be elements of* G_E *with* $\varrho \bmod \ell^M(\phi_v^{[b]}) = \varrho \bmod \ell^M(w_{v,b})$ *for all* $v \in I$ *and both signs* b. *Let* S *be a set of finite places and* L *a set of relations. Put* $y_{\mathfrak{q}_0} = u$ *and* $y_v = 1 + e(s_{v,i})(u - 1)$ *for* $v \in J \setminus \{\mathfrak{q}_0\}$, *and* $\varrho_y = \mathrm{tp}_{h,y}(e, \varrho)(\cdot)$. *Then*

$$\begin{aligned} \operatorname{len}_{A'} \mathrm{Sel}_S^{e(L)}(\varrho_y;\, A'/\ell^M) \;\leq\; & \operatorname{len}_{A'} \mathrm{Sel}_S^{e(L \cup \mathrm{Ties}_i(I,x))}(\varrho_y;\, A'/\ell^M) \\ & + \sum_{v \in I,\, b} v_{A'}(\det(u \cdot e(\varrho(w_{v,b})) - 1)). \end{aligned}$$

Proof. The module with the ties is cut out of the module without them by the kernels of the maps $c \mapsto c(\gamma_v^{[b]}) - s_{v,i}\, c(\phi_v^{[b]})$, $v \in I$; so it suffices to bound, for each (v, b), the length of the image of this map by $v_{A'}(\det(u\, e(\varrho(w_{v,b})) - 1))$, which holds as soon as the image is killed by the matrix $u\, e(\varrho(w_{v,b})) - 1$ (the length of the kernel of a matrix on $(A'/\ell^M)^n$ is at most the valuation of its determinant). By Lemma 4.3 the y_v are ℓ^N-th roots of unity modulo ℓ^M, so ϱ_y modulo ℓ^M is the twist of ϱ by the point character. Let c be a Selmer cocycle and write $g \cdot m = \varrho_y(g)m + c(g)$ for the affine action on $(A'/\ell^M)^n$ that it defines. At the place $\mathfrak{P}_v$ (and at its conjugate for $b = -$) the action is tame, because ℓ^{k+2M} divides ℓ^N and $\mathrm{N}(v) \equiv 1$ modulo ℓ^N: the Frobenius element and the tame generator commute in the image (an input of the unit on tame cocycles). Evaluating the commutation at 0 gives $(\varrho_y(\phi_v^{[b]}) - 1)\, c(\gamma_v^{[b]}) = (\varrho_y(\gamma_v^{[b]}) - 1)\, c(\phi_v^{[b]})$. Now $\varrho(\gamma_v^{[b]}) \equiv 1$ (inertia outside S_0), $\varrho_y(\gamma_v^{[b]}) = y_v$ and, for $v \neq \mathfrak{q}_0$, $\varrho_y(\phi_v^{[b]}) = u\, \varrho(\phi_v^{[b]})$ by Lemma 6.2; with $y_v - 1 = s_{v,i}(u - 1)$ this gives $(u\varrho(\phi_v^{[b]}) - 1)(c(\gamma_v^{[b]}) - s_{v,i} c(\phi_v^{[b]})) = 0$, using (P5)

when $s_{v,i} \neq 0$ (then $\varrho(\phi_v^{[b]}) \equiv 1$) and the triviality when $s_{v,i} = 0$. For $v = \mathfrak{q}_0$ one has $u = 1$ and the same identity. □

LEAN `IsPencil.length_le_of_drop_ties`

Lemma 6.4 (mirrors: a Selmer module that no new prime shortens is short)**.** *Let E be a CM field as in the conventions, A the ring of integers of a finite extension of $\mathbb{Q}_\ell$, $\mathcal{J}$ a set, and suppose $\mathsf{Pencil}_\ell(S_0; U, W; N, \mathfrak{q}_0, J; x)$. Let n be a finite set, $\varrho\colon G_E \to M_n(A)$ a homomorphism, $i \in \mathcal{J}$, S a finite set of finite places, L a set of relations, $\gamma \in G_E$, $\beta \in [0, \infty]$ and $M \in \mathbb{N}$. Suppose $\mathsf{Bal}_\ell(S;\, \varrho, L;\, \gamma, \beta, M)$, that U_i is the property $\varrho \bmod \ell^M(g) = 1$, that $S \subseteq S_0$, and let $a \in A$ be non-zero with $a \mid \ell^M$. For a finite set T of finite places put*

$$\mathcal{X}(T) = \mathrm{Sel}^{L \cup \mathrm{Ties}_i(J,x) \cup \{\delta_\gamma\}}_{S \cup T \cup \overline{T}}(\varrho;\, A/a).$$

Suppose that some $y \in \mathcal{X}(J)$ has a coordinate $y(g)_e$ that is a unit, and that for every finite place $\mathfrak{q} \notin J$ and every $l \in \mathrm{Aux}(E, \ell^N)$, with elements γ_l, ϕ_l, such that for all $t \in A$ the function x changed at $\mathfrak{q}$ to $(l, t \cdot \mathbf{1}_i)$ satisfies $\mathsf{Pencil}_\ell(S_0; U, W; N, \mathfrak{q}_0, J \cup \{\mathfrak{q}\}; \cdot)$, one has

$$\mathcal{X}(J) \subseteq \left(\mathrm{ev}^{\pm}_{\phi_l}\right)^{-1}\Big(\mathrm{ev}^{\pm}_{\gamma_l}(\mathcal{X}(J \cup \{\mathfrak{q}\}))\Big).$$

Then $\mathrm{len}_A\, \mathcal{X}(J) \leq \beta$.

Proof. Let X be the discrete module $(A/a)^n$ with the action through ϱ, and let U' be the intersection of the kernels of $\varrho \bmod \ell^M$ and of $\mathrm{cyc}_{\ell^{M+1}}$, an open subgroup acting trivially on X. By Lemma 6.1, $\mathcal{X}(J)$ lies in the Selmer group of X with the local conditions of ϱ and $L \cup \mathrm{Ties}_i(J, x)$ at the places of $S \cup J \cup \overline{J}$, and vanishing at γ. The formula of Greenberg and Wiles in the form with relations at split primes (Input 1 below, applied with $K = E$, $m = \ell^{M+1}$, the places $S \cup J \cup \overline{J}$ and $N' = \mathbb{Q}/\mathbb{Z}$; its hypothesis on trivialised cocycles holds because γ is central modulo U' and $s - 1$ is a unit) bounds the order of this Selmer group, by Input 1(a), by the product of the orders of the local conditions, divided by a power of $\#X$, times the number of the *relation families* (n_i). By Input 1(b) each relation family defines, through $x_w(g') = n_{\varpi'(g')}$ for $g' \in U'$, a function on U' with values in the dual of X that satisfies hypothesis (6) of the mirror lemma and the reciprocity relations at the split places outside T'; these functions x_w, w running over the relation families, are the family to which the mirror lemma is applied, and they are separated by their values on U' because $n_{\varpi'(\phi_i)} = n_i$. As γ acts by a scalar s with $s - 1$ a unit and is central modulo U', evaluation at γ maps the Selmer group onto X (coboundaries), which accounts for the condition at γ.

The number of relation families is bounded by the mirror lemma, Lemma 5.1, applied with $\chi = \mathrm{cyc}$, $\mathsf{c}' = (\cdot)^{\mathsf{c}}$, H the intersection with its conjugate of the group of commutators with the normal closure of $\{\gamma, \gamma^{[-]}\}$ inside U', the cocycle y with its unit coordinate, $z = \gamma_{\mathfrak{q}_0}$ times a commutator, and the scalars α, α' of (B5). Its isotropy hypothesis is where the hypothesis on new primes enters. Given $h \in H$, Theorem A of the first note provides a prime $\mathfrak{q}$ of prime norm outside all places in play, with a datum l whose Frobenius element is so close to zh that the pairings in question take the same values, and which is anticyclotomic relative to $\Sigma_\ell(E) \cup J \setminus \{\mathfrak{q}_0\}$; changing x at $\mathfrak{q}$ gives pencils for all slopes. By hypothesis the values of y at $\phi_l^{[\pm]}$ are the values at $\gamma_l^{[\pm]}$ of some $x' \in \mathcal{X}(J \cup \{\mathfrak{q}\})$. The values of x' on the inertia at $\mathfrak{q}$ are multiples of

its value at a tame generator, and the reciprocity relation of the relation family at the pair of split places $\mathfrak{q}, \bar{\mathfrak{q}}$ then reads exactly as the isotropy identity (7).

It remains to convert orders into lengths over A and to sum the local terms. At a place of $J \cup \bar{J}$ the local cocycles that obey the ties are determined by their value at the Frobenius element (the tame generator detects ramification on the appropriate side), so the local term is at most $\operatorname{len}_A((A/a)^n)$; at the places of S one takes the places of $\bar{\mathbb{Q}}$ that minimise the local length; the torsion of the mirror lemma has length at most $\#n \cdot v_A(\alpha - \operatorname{cyc}(g^{\mathrm{c}})\alpha')$. The inequality (B5) at a is precisely what makes the total at most β. The formal proof has a hundred lines; this is its outline, and the bookkeeping of the local terms was not re-derived here. $\square$

LEAN `IsPencil.length_le_of_unseen_class`

Input 1 (relations at split primes; background library, unit of the formula of Greenberg and Wiles). Let K be a number field with a fixed embedding in $\bar{\mathbb{Q}}$ (a CM field or not), $G_K = \mathrm{Gal}(\bar{\mathbb{Q}}/K)$, $m \geq 1$ an integer, X a discrete module of G_K over a commutative ring, killed by m, S a finite set of finite places of K, with a chosen place of $\bar{\mathbb{Q}}$ over every finite place v and a submodule L_v of the functions from the decomposition group at that place to X (the *local condition* at v), N' an abelian group, and U' an open normal subgroup of G_K. Suppose that the inertia groups above the places outside S act trivially on X, that L_v contains the coboundaries of the decomposition group at the chosen place for every $v \in S$, that every complex conjugation in G_K acts trivially on X, that U' acts trivially on X and on the m-th roots of unity, and that U' contains no complex conjugation. For a set Π' of finite places let $Y(\Pi')$ be the *Selmer group relaxed at* Π': the continuous cocycles $G_K \to X$ that vanish on the inertia groups above the places outside $S \cup \Pi'$ and on the complex conjugations, and whose restriction to the decomposition group at the chosen place lies in L_v for every $v \in S$. Then there are an integer $q \geq 1$, a finite set ι', distinct finite places $(v_i)_{i\in\iota'}$ of K not dividing q with arithmetic Frobenius elements $\phi_i \in U'$ at places of $\bar{\mathbb{Q}}$ over them (*split places* in the sense of the library, for the modulus m, the level q and U'), elements $(\gamma_i)_{i\in\iota'}$ of G_K, a locally constant map $\varpi' \colon G_K \to \iota'$ and a finite set $T' \supseteq S$ of finite places, such that every place of S divides q, the ϕ_i detect roots (every element of $\bar{\mathbb{Q}}^\times$ whose m-th power is a unit outside q fixed by U', and which every ϕ_i fixes, is fixed by U'), and the following hold.

(a) *The count.* If X is finite and has at most $(\#X)^{\#\iota'+1}$ trivialised cocycles (pairs of a function f from the places of $\bar{\mathbb{Q}}$ above q and ∞ to X and a continuous function $c \colon G_K \to X$ that vanishes on the inertia groups above the places not dividing q and different from the v_i, with $g\,f(x) = f(gx) + c(g)$), then

$$\#Y(\emptyset) \cdot (\#X)^{\#S+\#\{\text{infinite places of } K\}} \;\leq\; \#\mathcal{R}_{\mathbb{Q}/\mathbb{Z}} \cdot \#X \cdot \prod_{v\in S} \#(L_v \cap Z^1_v),$$

where Z^1_v is the module of the cocycles of the decomposition group at the chosen place over v and, for an abelian group G', $\mathcal{R}_{G'}$ is the set of the *relation families*: the families $(n_i)_{i\in\iota'}$ of homomorphisms $X \to G'$ with $\sum_i n_i(x(\gamma_i)) = 0$ for all $x \in Y(\{v_i : i \in \iota'\})$.

(b) *The relations.* Every $(n_i) \in \mathcal{R}_{N'}$ satisfies $n_{\varpi'(\phi_i)} = n_i$ for all i; $n_{\varpi'(a\,gbg^{-1})}(x) = n_{\varpi'(a)}(x) + n_{\varpi'(b)}(\mathrm{cyc}_m(g)\,g^{-1}x)$ for all $g \in G_K$, $a, b \in U'$ and $x \in X$; and the reciprocity relation $\sum_j n_{\varpi'(\phi'_j)}(x(\delta_j)) = 0$ for every finite family of distinct finite places $v'_j \notin T'$ with places of $\bar{\mathbb{Q}}$ over them, tame generators δ_j for m and Frobenius elements $\phi'_j \in U'$ there, and every $x \in Y(\{v'_j\})$.

The set T' is what the further places must avoid. For a totally complex K the conditions on complex conjugations are empty; the third note applies the statement over a totally real field, where they are not. *Reference:* to be confirmed; the catalogue of this write-up classes the statement as a substantial result of the library, part of the lower bound in the formula of Greenberg and Wiles, without a printed source for this form. The hypothesis of (a) is supplied by a second result of the same unit (few trivialised cocycles when an element central modulo U' acts on X without fixed vectors), for which the second reading located no published derivation either.

Lemma 6.5 (tilting)**.** *Let A be a discrete valuation ring, m a finite set, B an A-module, $\mathfrak{a}$ an ideal of A such that $A/\mathfrak{a}$ is artinian, $\Phi_0, \Phi_1 \colon B \to (A/\mathfrak{a})^m$ A-linear maps and Y a submodule of B. Then there is a list T of at most $\#m$ elements of the residue field of A such that for every $s \in A$ whose residue is not in T,*

$$\operatorname{len}_A(Y \cap \ker(s\Phi_0 - \Phi_1)) \;\le\; \operatorname{len}_A(Y \cap \ker \Phi_0 \cap \Phi_1^{-1}(\Phi_0(Y))).$$

Proof. Let $Z = \sum_s Y \cap \ker(s\Phi_0 - \Phi_1)$, the sum over all $s \in A$. On each summand $\Phi_1 = s\Phi_0$; hence $\Phi_1(Z) \subseteq \Phi_0(Z)$. Let $\mathcal{M} \subseteq A^m$ be the preimage of $\Phi_0(Z)$ under the reduction $A^m \to (A/\mathfrak{a})^m$, $v \mapsto \bar{v}$; it is a free A-module of rank at most $\#m$. As $\mathcal{M}$ is projective and Φ_0 maps Z onto $\Phi_0(Z)$, there is an A-linear map $\sigma' \colon \mathcal{M} \to Z$ with $\Phi_0(\sigma'(v)) = \bar{v}$ for all $v \in \mathcal{M}$; and, as $\Phi_1(Z) \subseteq \Phi_0(Z)$ and $\mathcal{M}$ maps onto $\Phi_0(Z)$, there is an endomorphism ψ of $\mathcal{M}$ with $\overline{\psi(v)} = \Phi_1(\sigma'(v))$ for all v. Let T be the list of the roots, in the residue field, of the reduction of the characteristic polynomial of ψ; it has at most $\#m$ elements. Let the residue of s not be in T. Then $\det(s - \psi)$ is a unit, so $s - \psi$ is an automorphism of $\mathcal{M}$, and the identity $(s\Phi_0 - \Phi_1)(\sigma'(v)) = \overline{(s - \psi)(v)}$ shows that $(s\Phi_0 - \Phi_1)(Z) \supseteq \Phi_0(Z)$. (The map Φ_1 need not vanish on $Z \cap \ker \Phi_0$, so the two kernels on Z may differ; only their lengths are compared.) Both images lie in $(A/\mathfrak{a})^m$, a module of finite length, and in $\mathbb{N} \cup \{\infty\}$

$$\begin{aligned}\operatorname{len}_A(Z \cap \ker(s\Phi_0 - \Phi_1)) + \operatorname{len}_A((s\Phi_0 - \Phi_1)(Z)) &= \operatorname{len}_A Z \\ &= \operatorname{len}_A(Z \cap \ker \Phi_0) + \operatorname{len}_A \Phi_0(Z).\end{aligned}$$

The second terms are finite, and that on the left is at least that on the right. Hence

$$\operatorname{len}_A(Z \cap \ker(s\Phi_0 - \Phi_1)) \;\le\; \operatorname{len}_A(Z \cap \ker \Phi_0).$$

Finally $Y \cap \ker(s\Phi_0 - \Phi_1)$ is contained in Z, so the left side here is the left side of the lemma; and $Z \cap \ker \Phi_0$ is contained in $Y \cap \ker \Phi_0 \cap \Phi_1^{-1}(\Phi_0(Y))$, because $\Phi_1(Z) \subseteq \Phi_0(Z) \subseteq \Phi_0(Y)$. □

LEAN `tilting`

Proof of Theorem B. By induction on the factorisation of a into irreducible elements. If a is a unit, the Selmer modules are zero; take $J_0 = 1$, and for given N one helper prime $\mathfrak{q}_0$ from Theorem A of the first note (applied with $R = S_0$ and the neighbourhood W of z), with slopes 0: this is a pencil. Now suppose the statement holds for a with a bound J_1, let ϖ be a uniformiser with $\varpi a \mid \ell^M$, and fix N. All Selmer modules below are finite (Lemma 7.4 of the first note). Among the pencils (J, x) with the given helper prime whose lengths modulo a are at most β_i and for which $\#J$ plus the sum over i of the lengths modulo ϖa is at most a bound J_0, choose one for which the sum of the lengths modulo ϖa is least. The bound J_0 depends on J_1, the ranks $\#n_i$, $\operatorname{len}_A(A/\varpi a)$ and the (finite) lengths of the Selmer modules of the ϱ_i modulo ϖa that are unramified outside S, but not on N: it is chosen so that the pencil of the induction hypothesis qualifies. For that pencil, the cocycles modulo ϖa that vanish at the tame generators $\gamma_v^{[\pm]}$, $v \in J$, are unramified at J and its conjugates (Lemma 7.3 of the first note), so the length modulo ϖa is at most $2\,\#n_i\,\#J \cdot \operatorname{len}_A(A/\varpi a)$, the length of the target of the evaluation at these generators, plus the length of that Selmer module; and $\#J \le J_1$. We claim that the chosen pencil has all lengths modulo ϖa at most β_i. Fix i. If no element of the module modulo ϖa has a unit coordinate, the module is the image of the module

modulo a under multiplication by ϖ, of length at most β_i. Otherwise, if the inclusion of Lemma 6.4 holds at every admissible new prime, that lemma gives the bound. In the remaining case there are a new prime $\mathfrak{q}$ and a datum l at which the inclusion fails. Apply Lemma 6.5 to $\Phi_0 = \mathrm{ev}^{\pm}_{\gamma_l}$, $\Phi_1 = \mathrm{ev}^{\pm}_{\phi_l}$ on the module for $J \cup \{\mathfrak{q}\}$ with the old ties, modulo a and modulo ϖa: the residue field has more than $4\#n_i + 2$ elements (B2), so there is a unit slope t whose inverse avoids both lists of bad residues. For the pencil $(J \cup \{\mathfrak{q}\}, x')$, with x' equal to x changed at $\mathfrak{q}$ to $(l, t^{-1}\mathbf{1}_i)$, the tied module is contained in $\ker(\Phi_0 - t^{-1}\Phi_1)$, so its length is at most that of $\ker \Phi_0 \cap \Phi_1^{-1}(\Phi_0(\cdot))$, which is contained in the module for J (vanishing at the tame generators of $\mathfrak{q}$ makes a cocycle unramified there) and, modulo ϖa, properly so because the inclusion failed. The modules of the other rows only shrink, the lengths modulo a stay at most β_i, and the count of primes grows by one while the sum of lengths drops by at least one. This contradicts the minimality. $\square$

Proof of Theorem A. This is an outline of the formal proof: its steps and the hypotheses used at each are those of the formal text; the estimates inside the steps are left to it.

Step 1: a pencil. Let W be a neighbourhood of z on which every $\mu_{k,t}$ stays within c of its value (so that $v_\ell(\mu_{k,t}(\phi) - 1) \le c$) and on which the ϱ_i modulo ℓ^M agree with their values at z and at $z^{[-]}$. Let J_0 be the bound of Theorem B for $a = \ell^M$, choose $\delta > 0$ with $\delta\,(8J_0 + 2) < \eta$, and choose k and $N \ge k + 2M$ large as required by Lemma 4.2 in the two uses below. Theorem B gives a pencil $(\mathfrak{q}_0, J, x)$ for N with $\#J \le J_0$ and tied lengths at most β_i.

Step 2: polynomials. For a tuple $y = (y_v)_{v \in J}$ of ℓ^N-th roots of unity the point character y^h is from $\mathrm{U}(1)$ relative to $\Sigma_\ell(E)$ with modulus the set of the v with $y_v \ne 1$; every subset of J is auxiliary for S_0. By (F2), over a large coefficient ring $\mathcal{O}' \supseteq \mathcal{O}$ each value $\mathrm{val}_k(I, y^h)$, $I \subseteq J$, is a polynomial $V_{k,I}(y)$ with integral coefficients. By Lemma 5.2 there are polynomials F_i whose valuations bound the Selmer lengths of the twists $y^h\varrho_i$ with the relations $L_i \cup \{\delta_{\gamma_i}\}$, and polynomials G_i that do the same with the ties at $J \setminus \{\mathfrak{q}_0\}$ added and are sharp at the point $y = 1$.

Step 3: along a line. Consider the line $y_{\mathfrak{q}_0} = X$, $y_v = 1 + s_v(X - 1)$, with s_v the slope of v for the row i at hand; the pencil carries slopes for the rows only, and for a family one takes $s_v = 0$, that is $y_v = 1$ for $v \ne \mathfrak{q}_0$. At $X = 1$ the restricted polynomials have valuations at most B_i (for G_i: sharpness, the bound β_i of the pencil, Lemma 6.3 for $I = \{\mathfrak{q}_0\}$ with $w = z^{[\pm]}$, and hypothesis (ii)) and at most $v_\ell(\mathrm{val}_k(\emptyset, 1)) + 8c$ (for $V_{k,\{\mathfrak{q}_0\}}$: (F3) at the helper prime, whose Frobenius element lies in W). Lemma 4.2 in one variable gives $u \in \mathcal{O}'$ with $u^{\ell^k} = 1$ at which these valuations grow by at most δ and at which the characteristic polynomials of the $\varrho_i(\phi_v^{[b]})^{-1}$ and the polynomials $X - \mu_{k,t}(\phi_v)^{\pm 1}$ have valuation at most δ.

Step 4: the point on the line. Let y be the point of the line at $X = u$; by Lemma 4.3 its coordinates are ℓ^N-th roots of unity modulo the relevant powers, and ℓ divides $(y_v - 1)^{\ell^k}$. By Lemma 6.3 with $I = J \setminus \{\mathfrak{q}_0\}$ and $w = \phi_v^{[b]}$, the Selmer length of $y^h\varrho_i$ is at most the tied length plus at most $2J_0$ determinant terms, each of valuation at most δ by Step 3; so $v_\ell(F_i(y)) \le B_i + \delta + 2J_0\delta$ in normalised terms. By (F3), adding the places of $J \setminus \{\mathfrak{q}_0\}$ one at a time raises $v_\ell(V_{k,\cdot}(y))$ by at most 8δ each: at the Frobenius element of v the point character takes the value $y_{\mathfrak{q}_0} = u$ (Lemma 6.2), and $|1 - \mu u^{\pm 1}|$ was controlled in Step 3.

Step 5: roots of unity, for all powers. Apply Lemma 4.2 with the variables $v \in J$, the powers $1 \le d \le m$, the polynomials F_i and $V_{k,J}$ (the same for every d), the point y of Step 4 for every (k, d), and the monic polynomials $D_d = X^d - 1$, $1 \le d \le m$, of total degree $m(m+1)/2$. It gives $\zeta \in \mathcal{O}'^J$ with $\zeta_v^{\ell^N} = 1$ for all v,

$$v_\ell(\zeta_v^d - 1) \le \delta \qquad \text{for all } v \in J \text{ and all } 1 \le d \le m,$$

and the valuations of the F_i and $V_{k,J}$ at ζ^d within δ of those at y, for every d. The polynomial $X - 1$ alone would not do: for a root of unity ζ_v of ℓ-power order and ℓ dividing d, $v_\ell(\zeta_v^d - 1)$ exceeds $v_\ell(\zeta_v - 1)$, and $\zeta_v^d = 1$ is not excluded. Put $I = J$ and $\psi = \zeta^h$; then $\psi^d = (\zeta^d)^h$, a point character at ℓ^N-th roots of unity. Conclusion (1) holds because the total loss is at most $\delta(8J_0 + 2) < \eta$; (2) because J is auxiliary, ψ^d is from U(1) with modulus J (Step 2), and $\sum_v v_\ell(\psi^d(\gamma_v) - 1) = \sum_v v_\ell(\zeta_v^d - 1) \le \#J \cdot \delta \le \eta$ (Lemma 6.2); (3) by the second property of F_i in Lemma 5.2, for every coefficient ring containing $\mathcal{O}'$. The same I and ψ serve all d and all i. The bound in (2) for every power d, not only for $d = 1$, is what makes the twist m-fold in the third note. □

7 Balanced rows: proof of Theorem C

Lemma 7.1 (classes of polynomials vanishing on roots of unity are tame)**.** *Let $\mathcal{O}$ be a coefficient ring, N and r natural numbers, K a property of polynomials over $\mathcal{O}$, and $\mathfrak{M}$ a finite multiset of elements of $\bar{\mathbb{Z}}_\ell$. Suppose that $\mathfrak{M}$ has at most N elements, that every $y \in \mathfrak{M}$ satisfies $y^{\ell^a - 1} = 1$ for some $1 \le a \le r$, and that K holds for every $Q \in \mathcal{O}[X]$ with $Q(y) = 0$ for all $y \in \mathfrak{M}$. Then K is of tame type (ℓ, r, Nr).*

Proof. A root of unity of order dividing $\ell^a - 1$ lies in the unramified extension of degree a of $\mathbb{Q}_\ell$; its minimal polynomial over $\mathbb{Z}_\ell$ is monic of degree at most $a \le r$. So the product of the minimal polynomials of the elements of $\mathfrak{M}$ is monic over $\mathbb{Z}_\ell$, of degree at most Nr, and vanishes on $\mathfrak{M}$; and $X^{M'} - 1$ vanishes on $\mathfrak{M}$ for the least common multiple M' of the $\ell^a - 1$, $a \le r$. □

LEAN `isTameType_of_roots`

Lemma 7.2 (the local lattices of a column)**.** *Let D be admissible data, $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ the fixed isomorphism, $\mathcal{O}$ a coefficient ring, n a finite set, $\mathcal{H}$ a type, $u\colon G_{E(D)} \to M_n(\mathcal{O})$ a homomorphism, $R\colon G_{E(D)} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ a homomorphism, t a natural number, S a set of finite places, $w \in \Sigma_\ell(E(D))$, and $\mathcal{R}$ a property of homomorphisms $G_{E(D)} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$. Suppose $\mathsf{ColRep}_D(t, \mathcal{H}, S; R)$ and that R has the property $\mathcal{R}$. Let $\mathfrak{P}_w$ be the chosen place over w, with its chart e_w, let $\Lambda_w = \mathrm{L}(u|_{G_{\mathfrak{P}_w}})$, and $\mathcal{H}_w(g) = \mathcal{H}(\mathrm{ce}(e_w|_{E(D)}, g))$ for $g \in G_{\mathbb{Q}_\ell}$. If u is a lattice of R, then for every $g \in G_{\mathbb{Q}_\ell}$ the lattice Λ_w has the jumps $\mathcal{H}_w(g)$ at $\mathrm{emb}_{\mathcal{O}}(g)$, and Λ_w belongs to the class $\mathcal{Q}(F, K, U)$, where: F is the trivial filter (no condition) if $w \in \Sigma_g(D)$ and $F = \top$ otherwise; $K(s, Q)$ holds if $w \in \Sigma_g(D)$ and for every R' with the property $\mathcal{R}$ there is a polynomial that kills s on R' at $\mathfrak{P}_w$ and at all of whose roots Q vanishes; and $U(g')$ is the set of the elements of $\mathcal{H}_w(g)$ for a g with $\mathrm{emb}_{\mathcal{O}}(g) = g'$.*

Proof. Let $u(g) = Y^{-1}R(g)Y$. The map $x \mapsto Yx$ from $\mathcal{O}^n$ to $\bar{\mathbb{Q}}_\ell^n$ intertwines Λ_w with the restriction of R to $G_{\mathfrak{P}_w}$. By the definition of a column, at the place $\mathfrak{P}_w$ above ℓ the representation R is potentially crystalline with the jumps $\mathcal{H}$, and crystalline unless $w \in \Sigma_g(D)$. These properties and the killing polynomials pass to a lattice

along a jointly injective family of equivariant maps, and the jumps are compatible with base change from $\mathcal{O}$ to $\bar{\mathbb{Q}}_\ell$ (inputs of the unit on crystalline representations). □

LEAN `inClass_of_isLatticeIn`

Lemma 7.3 (obeying cocycles and the cup-orthogonal)**.** *Let X, n, d be finite sets, A the ring of integers of a finite extension of $\mathbb{Q}_\ell$, Γ a topological group over $G_{\mathbb{Q}_\ell}$ with homomorphism j, Γ' a group, $\eta\colon \Gamma \to \Gamma'$ a homomorphism, $u = (u_x)_{x\in X}$ homomorphisms $u_x\colon \Gamma' \to M_n(A)$, $\varrho\colon \Gamma' \to M_d(A)$ a homomorphism, (F, K, U) as in the definition of a class, and $m \in \mathbb{N}$. Suppose that j is an open embedding and that $\mathrm{L}(u_x \circ \eta)$ belongs to $\mathcal{Q}(F, K, U)$ for every x. Let $L = \mathrm{L}(\varrho \circ \eta)$. If L belongs to $\mathcal{Q}(F, K, U)$, then*

$$\begin{aligned}\operatorname{len}_A\Big(\mathcal{Z}_u(\varrho;\, \ell^m)\big/\{c : c\circ\eta \in \mathrm{Ob}(\mathrm{Rel}(Z_{\mathrm{f}}(L));\, A/\ell^m)\}\Big)\\ \le\ \operatorname{len}_A\Big(Z_{\mathrm{f}}(L^\vee(1))\big/Z_{\mathrm{f}}(L^\vee(1)) \cap (L^\vee(1))^{\perp_{\mathcal{Q}(F,K,U)}}\Big),\end{aligned}$$

the first quotient being by the intersection with the set shown.

Proof. This is an outline of the formal proof. There is a perfect pairing between L/ℓ^m and the reduction of the twisted dual, so that the cocycles with values in $(A/\ell^m)^d$ which obey the relations of $Z_{\mathrm{f}}(L)$ are those orthogonal to $Z_{\mathrm{f}}(L^\vee(1))$ under the cup product; and for potentially crystalline lattices an index of orthogonal complements on one side divides the corresponding index on the other (inputs of the unit on local duality; this is where j must be an open embedding). It therefore suffices to show that the cup product of an obeying cocycle c, restricted along η, with a class of $Z_{\mathrm{f}}(L^\vee(1))$ that lies in the cup-orthogonal of the class is a coboundary. By Lemma 4.1 of the first note, c is the value of an equivariant map on a subrepresentation Y_0 of $M_n(A)^X \times (A/\ell^m)^d$ that projects injectively to the first factor; Y_0, restricted along η, is a Galois lattice which embeds jointly into the lattices $\mathrm{L}(u_x \circ \eta)$, hence belongs to the class, and it carries a crystalline cocycle whose image is c (the coboundary of a point of the graph, divided by a power of ℓ). Lifting the map $Y_0 \to L/\ell^m$ to L gives a pairing of Y_0 with $L^\vee(1)$ modulo ℓ^m in the sense of the definition of the cup-orthogonal, and the definition gives the coboundary. □

LEAN `relLength_obeyingCocycles_le`

Proof of Theorem C. This is an outline of the formal proof; the local length counts in (B5) rest on the inputs named. The finiteness of Σ is clear. Let $\mathcal{O}$ be large enough for (W3) and such that R has a lattice $u\colon G_{E(D)} \to M_n(\mathcal{O})$ (a continuous representation has one over every large coefficient ring; an input of the background library). Let F' be the algebra map of Lemma 5.6 of the first note. By (W3), $u(G_{E(D)})$ spans $M_n(\mathcal{O})$ and $u(g_1) = \alpha$ is a scalar for some g_1 with $\mathrm{cyc}(g_1) = g_i(D)^{\#n}$. By (W4) there are an inertial g_0 above $\Sigma_g(D) \setminus \Sigma_\ell(E(D))$ and y with $R(g_0) = y$, and then $u(g_0) = y$, $y \in \mathcal{O}$, and $y^j - 1$ is a unit of $\mathcal{O}$ for $1 \le j \le 4\#n + 2$.

A scalar and its conjugate. If $u(g) = y'$ is a scalar, then y' is a unit and $u(g^{[-]}) = \mathrm{cyc}(g)^{t-1}\, y'$ (with $t - 1$ truncated): by the trace identity of a column, for every k the traces of $u(k)\, u(g^{[-]})$ and of $\mathrm{cyc}(g)^{t-1} y'\, u(k)$ agree, and $u(G_{E(D)})$ spans all matrices, on which the trace form is non-degenerate.

The clauses. Take $\gamma = g_0$. (B1): u is continuous (Lemma 5.7(b) of the first note), and it is trivial on the inertia outside Σ because a column of level $\emptyset$ is good outside $\Sigma_g(D)$. (B2): the span was shown; the residue of y has order greater than $4\#n + 2$. (B3):

$u(g_0) = y = u(g_0^{[-]})$, because $\mathrm{cyc}(g_0) = 1$ on inertia prime to ℓ, and $y \pm 1$ are units since $y^2 - 1$ is. (B4): coboundaries are crystalline cocycles, so they obey $\mathcal{R}_{\mathrm{cris}}(u)$. (B5) holds with $g = g_1$, $\alpha' = \mathrm{cyc}(g_1)^{t-1}\alpha$ and

$$\beta = \sum_{v \in \Sigma_g(D) \setminus \Sigma_\ell(E(D))} v_{\mathcal{O}}(\det(u(\gamma_v) - 1)) + \#n \cdot v_{\mathcal{O}}(\alpha - \mathrm{cyc}(g_1^{\mathsf{c}})\,\alpha'),$$

where γ_v is the element that (W2) provides at the chosen place of $\bar{\mathbb{Q}}$ over v. Indeed, let $a \mid \ell^M$. At a place v above ℓ the local cocycles modulo a that obey $\mathcal{R}_{\mathrm{cris}}(u)$ obey the relations of the crystalline cocycles of the local lattice, and their length is at most $\mathrm{rk}_{\mathcal{O}}\, Z_{\mathrm{cris}}(v;\, u) \cdot \mathrm{len}(\mathcal{O}/a)$ (an input of the unit on Fitting ideals); by Lemma 7.2 and Lemma 11.5 of the first note the rank is $\#n$ plus the number of negative jumps at the embeddings read at v. Summing over v, Lemma 4.4 of the first note and (R4) bound the total number of negative jumps by $\#n$ times the number $f(D)$ of infinite places. At a place v of $\Sigma_g(D)$ prime to ℓ, (W2) and an input of the unit on tame cocycles bound the local length by $\mathrm{len}((\mathcal{O}/a)^n)$ plus the length of the kernel of $u(\gamma_v) - 1$, which is at most the valuation of its determinant. Adding up gives (B5).

The size of β. Put $t' = \max(t, 1)$. As $\mathrm{cyc}(g_1^{\mathsf{c}}) = \mathrm{cyc}(g_1)$ and $\alpha' = \mathrm{cyc}(g_1)^{t-1}\alpha$ with $t - 1$ truncated, $\alpha - \mathrm{cyc}(g_1^{\mathsf{c}})\alpha' = \alpha\,(1 - \mathrm{cyc}(g_1)^{t'})$ exactly. Here α is a unit, and $\mathrm{cyc}(g_1)^{t'} - 1 = g_i(D)^{\#n\,t'} - 1$ has valuation at most $\mathrm{cr}_\ell(D)$ by Lemma 5.7(a) of the first note, applied with the exponent $\#n\,t'$, which lies between 1 and $r(D)t(D)$ because $1 \le \#n \le r(D)$ and $t' \le t(D)$ for a row. By (W2) each $v_\ell(\det(u(\gamma_v) - 1))$ is at most $\#n\,\mathrm{cr}_\ell(D)$, and there are at most $4f(D)$ places v (Lemma 5.5(b) of the first note). At the chosen place w, (W2) gives $\gamma_w = \tau_w$ and the same bound for $\det(u(\tau_w^{[b]}) - 1)$, for both signs. In normalised length the left side of the theorem is therefore at most $(4f(D) + 1 + 2)\,\#n\,\mathrm{cr}_\ell(D) \le 10\,f(D)\,\#n\,\mathrm{cr}_\ell(D)$. □

LEAN `IsColumn.eq_one_of_mem_inertiaAbove` · the triviality of a column on the inertia outside $\Sigma_\ell(E(D)) \cup \Sigma_g(D)$ and its level, used for (B1)

8 Ribet's lemma to first order: proof of Theorem D

In this section Λ_0 is a commutative ring, $\mathbb{L}$ a commutative ring, and index sets may be finite sets of any kind.

Lemma 8.1 (Amitsur's formula for block matrices)**.** *Let σ and N be finite sets, $\mathbb{L}$ a commutative algebra over a commutative ring Λ_0, $\mathfrak{A}$ a Λ_0-algebra (not necessarily commutative) and $\varrho\colon \mathfrak{A} \to M_\sigma(\mathbb{L})$ a homomorphism of Λ_0-algebras. Suppose that every coefficient of the characteristic polynomial of every $\varrho(w)$, $w \in \mathfrak{A}$, lies in the image of Λ_0. Let $P' \in M_N(\mathfrak{A})$ and $k \in \mathbb{N}$. Then the k-th coefficient of the characteristic polynomial of the matrix $\mathrm{fl}_N((\varrho(P'_{ij}))_{i,j})$ lies in the image of Λ_0.*

Proof. By Input 2 it suffices to check the property on elements that span $M_N(\mathfrak{A})$: the matrices with one non-zero entry. For such a matrix the block image has, up to a power of X, the characteristic polynomial of the image of one entry of $\mathfrak{A}$ (the trace of a product is insensitive to cyclic rotation), or the polynomial $X^{\#N\#\sigma}$. □

LEAN `amitsur_charpoly_mem_matrix`

Input 2 (Amitsur's formula, integrality consequence; background library, unit of the linear algebra of pseudocharacters). If $\varrho\colon \mathfrak{A} \to M_\sigma(\mathbb{L})$ is a homomorphism of Λ_0-algebras and $\mathfrak{A}$ is spanned over Λ_0 by elements whose images have characteristic polynomials with coefficients in the image of Λ_0, then every $\varrho(w)$ has this property; and then every w has an "adjugate": an element v and F' in the image of Λ_0 with $\det \varrho(w) = F'$ and $\varrho(v)\varrho(w) = \varrho(w)\varrho(v) = F'$. *Reference* (second reading): G. Chenevier, *The p-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings*, §1, equation (1.5). *Relation:* a weaker form: the descent of the coefficients on a spanning family follows from the integral formula; no formula for the coefficients is stated.

Lemma 8.2 (Ribet's construction: Cramer's rule twice)**.** *Let A be a commutative ring, Γ a group, Y a set, σ a non-empty finite set, N a finite set, Λ_0 a commutative ring, $\mathbb{L}$ a commutative ring that is an algebra over A and over Λ_0, and $\mathcal{S}$ a Λ_0-subalgebra of the functions $Y \to \mathbb{L}$. Let $u = (u_x)_{x\in Y}$ be homomorphisms of A-algebras $u_x\colon A[\Gamma] \to M_\sigma(A)$, $\kappa\colon A \to \mathcal{S}$ a ring homomorphism, $\mathcal{J}$ an $\mathcal{S}$-submodule of the functions $Y \to \mathbb{L}$, $w \in A[\Gamma]^N$, $V\colon A[\Gamma]^N \to A[\Gamma]$ a homomorphism of left $A[\Gamma]$-modules, and $\beta\colon Y \to \mathbb{L}$ a function. Suppose that κ followed by the inclusion of $\mathcal{S}$ is the structure map of the A-algebra of functions; that for all $g \in \Gamma$ and k the function $x \mapsto$ (k-th coefficient of the characteristic polynomial of $u_x(g)$, read in $\mathbb{L}$) lies in $\mathcal{S}$; and that $\beta(x)$ is the image of $\det u_x(V(w))$ for all x. Then $\beta \in \mathcal{S}$, and for every $h \in A[\Gamma]$, every $M' \in M_N(A[\Gamma])$ and every left ideal $\mathfrak{a}$ of $A[\Gamma]$ such that each $V(M'_p)$ (the image of the p-th row) lies in the sum of $\mathfrak{a}\cdot V(w)$ and of the left ideal generated by the r with $(x \mapsto \operatorname{tr}(u_x(r)u_x(g))) \in \mathcal{J}$ for all g, there is $f \in \mathcal{S}$ with $f\beta \in \mathcal{J}$ and*

$$f(x) = \det(\mathrm{fl}_N(u_x(M'))) \cdot \operatorname{tr} u_x(h) \qquad \text{for every } x \text{ with } u_x(m\,h) = 0 \text{ for all } m \in \mathfrak{a}.$$

Proof. This is an outline of the formal proof. Let $\rho'\colon \mathcal{S}[\Gamma] \to M_\sigma(Y \to \mathbb{L})$ be the $\mathcal{S}$-algebra homomorphism with $\rho'(g)(x) = u_x(g)$. By hypothesis and Input 2, all characteristic polynomials of ρ' have coefficients in $\mathcal{S}$. The adjugate of $V(w)$ gives $\beta = \det \rho'(V(w)) \in \mathcal{S}$ and an element η' with $\rho'(\eta')\rho'(V(w)) = \beta$. Write $V(M'_p) = r_p + m_p V(w)$ with r_p in the trace ideal and $m_p \in \mathfrak{a}$. By Lemma 8.1 the block matrix of M' has an adjugate W' with entries in $\mathcal{S}[\Gamma]$ and determinant $F'' \in \mathcal{S}$. Put $a' = F'' - \sum_{p,q} w_q\, W'_{qp}\, m_p$ and $f = \operatorname{tr}\rho'(a'h)$, an element of $\mathcal{S}$ (a coefficient of a characteristic polynomial). Where $u_x(mh) = 0$ for $m \in \mathfrak{a}$, the sum does not contribute and $f(x) = F''(x)\operatorname{tr} u_x(h)$. Finally $a'V(w) = \sum_{p,q} w_q W'_{qp}\, r_p$ by Cramer's rule for W' and the linearity of V, so $f\beta = \operatorname{tr}\rho'(\eta'\, h\, a'\, V(w))$-type sums of traces of multiples of the r_p, which lie in $\mathcal{J}$ by the definition of the trace ideal. □

LEAN `exists_ribet_element_cramer`

In the next lemma, for an A-linear map $\Phi\colon A^n \to \Lambda^s$, a function $z\colon \Gamma \to A^n$, a submodule $\mathcal{N}$ of Λ^s and $p \in A$, the *module of forms* $\mathfrak{F}(\Phi, z, \mathcal{N}, p)$ consists of the functions $g \mapsto \Psi(z(g))$ on Γ, for the A-linear maps $\Psi\colon A^n \to A/p$ that vanish on $\Phi^{-1}(\mathcal{N})$ (background library).

Lemma 8.3 (a Fitting ideal contains a multiple of the product of the tie parameters)**.** *Let A be a discrete valuation ring, Γ a set, J a finite totally ordered set, n a finite set, s a finite subset of Γ, $\Lambda = A[X_j : j \in J]$, $\Phi\colon A^n \to \Lambda^s$ an A-linear map, $z\colon \Gamma \to A^n$ a function, $\mathcal{N}$ a Λ-submodule of Λ^s, $p, C \in A$, $\tau, \phi\colon J \times \{+,-\} \to \Gamma$*

functions, I a finite subset of J and $b \in \mathbb{N} \cup \{\infty\}$. Suppose that $p \neq 0$; that $\mathsf{TriCost}(\mathfrak{F}(\Phi, z, \mathcal{N}, p); \tau, \phi; I; C, b)$*; that $\Phi(pv) \in \mathcal{N}$ for all v; that $\Phi(z(g)) - \mathbf{1}_g \in \mathcal{N}$ for all $g \in s$; that $\Phi(z(\tau_k)) - X_j\,\Phi(z(\phi_k)) \in \mathcal{N}$ for all $k = (j, b')$; and that for all $j \in I$ and $i < j$, either $i \in I$ or $X_i \Lambda^s \subseteq \mathcal{N}$. Then there is $G \in \Lambda$ with*

$$v_A(G(0)) \leq b + (\#I + \#s)\, v_A(C) \qquad \text{and} \qquad G \cdot \prod_{j \in I} X_j \in \mathrm{Fitt}(\mathcal{N}).$$

Proof. Let Γ_0 be the set of the triangle cost. Three facts on Fitting ideals are used (background library): *(cost)* for a module $\mathcal{V}$ of functions $\Gamma \to A/p$ there are $\mathcal{K} \supseteq \mathcal{N}$ containing $\Phi(v)$ whenever all forms in $\mathcal{V}$ kill v, and $e \in A$ with $v_A(e) \leq \mathrm{len}(\mathfrak{F}/\mathfrak{F} \cap \mathcal{V})$ and $e\,\mathrm{Fitt}(\mathcal{K}) \subseteq \mathrm{Fitt}(\mathcal{N})$; *(triangle)* if $r_j v_j$ lies in a module plus the span of the earlier v_i, for all $j \in I$, then $\prod_{j \in I} r_j$ multiplies the Fitting ideal of the module with the v_j adjoined into that of the module; *(constants)* $\mathrm{Fitt}(\mathcal{N} + \mathfrak{b}\Lambda^s) \subseteq \mathrm{Fitt}(\mathcal{N}) + \mathfrak{b}$. First apply (cost) to the functions vanishing on Γ_0: this gives $\mathcal{K}$ and e. Then (triangle) with $r_j = CX_j$ and $v_j = \Phi(z(\phi_{(j,+)})) - \Phi(z(\phi_{(j,-)}))$: by the hypothesis on the ties, $X_j v_j$ is congruent modulo $\mathcal{N}$ to the difference of the $\Phi(z(\tau_{(j,\pm)}))$, and (C1), read through the duality between forms and membership (a vector lies in a module containing $p\,\Phi(A^n)$ if every form vanishing on the module kills it), says that C times this difference lies in the module generated by the earlier ones; the hypothesis on $i \notin I$ disposes of the earlier indices outside I. Then (cost) again, for the C-torsion, with a constant f'; by (C2) $v_A(e) + v_A(f') \leq b$. The matrix $C \cdot \mathbf{1}$ of size $\#s$ has its rows in the last module, so $C^{\#s}$ lies in its Fitting ideal, and (constants) for the ideal of polynomials without constant term gives an element d_4 of the Fitting ideal of the middle module with constant term $f' C^{\#s}$. Put $G = e\,C^{\#I} d_4$. □

LEAN `exists_mem_fittingIdeal_block`

Proof of Theorem D. This is an outline. The formal proof has 150 lines; here is its architecture, with the places where each hypothesis enters. Let $\Lambda = A[X_j : j \in J]$ and work in the group algebra $\Lambda[\Gamma]$; let χ_x, ψ_x and η_e be the algebra homomorphisms from $\Lambda[\Gamma]$ to matrices over Λ and to Λ induced by u_x, ϱ_x and ξ_e. On $\mathcal{X}$, χ_x is the block sum of ψ_x and the η_e, by (a).

Two elements of the group algebra. By (b) the element $k_0 = \prod_{x \in \mathcal{X}} (y_x - 1)^{-1}(y_x - \gamma_x)$ has $\eta_e(k_0) = 1$ for all e and $\psi_x(k_0) = 0$ for all $x \in \mathcal{X}$. By (d) there are q_e with $\eta_k(q_e) = C$ for $k = e$ and 0 otherwise (the interpolation polynomial at the values $\xi_k(a)$, evaluated at a).

Coordinates and relations. Choose a finite set $s_0 \subseteq \Gamma$ such that the $u(g) - 1$, $g \in s_0$, span all $u(g) - 1$ over A; index coordinates by $W \sqcup s_0 \sqcup s$ (the elements δ_v, s_0 and s), and write each $u(g) - 1$ as a combination $z(g)$ of the coordinates. Let V be the $\Lambda[\Gamma]$-linear map sending a coordinate vector to the corresponding combination of the $g - 1$, and ω the vector of the normaliser of (e). Let $\mathcal{N}$ be the module of the coordinate vectors whose image under V lies in the sum of the trace ideal of $\mathcal{J}$ and of $\ker(\eta_0) \cdot V(\omega)$. It contains: the cocycle relations $z(gg') - z(g) - g\,z(g')$; the linear relations among the $u(g) - 1$; the ties $z(\tau_k) - X_j \varepsilon_j(\phi_k) z(\phi_k)$ and all multiples of p (by (c)); and $(g - 1)\omega$.

The block. Evaluating the variables at 0 and applying ψ_π, a vector killed by $\mathcal{N}$ becomes an element of $\mathcal{Z}_u(\varrho_\pi; p_0)$ that vanishes at the τ_k and, by the normaliser relation and the unit $y_\pi - 1$, at γ_π. By (e) these have length at most $b_0 < v_A(p_0)$,

and since ϱ_π spans all matrices there is a square matrix B_0 with rows in $\mathcal{N}$ and $v_A(\det \psi_\pi(B_0)|_{X=0}) \le b_0$ (an input on Fitting ideals).

The characters. Fix e. The forms on the image of $\mathcal{N}$ under η_e give cocycles in $\mathcal{Z}_u(\xi_e;\, p)$ that vanish at the δ_v; and the third condition of (g) holds for them: if such a cocycle were of the form $(\xi_e(g)-1)m$ without being killed by $\xi_e(a)-1$, the relation $(g-1)\omega \in \mathcal{N}$ together with (d) and $b_0 < v_A(p_0)$ would make the unit $y_\pi - 1$ a sum of two non-units. So (g) applies: it gives the triangle cost and, by the detection on s, elements of $\mathcal{N}$ that eliminate the coordinates of s_0. Let $\mathfrak{q}$ be the ideal of Λ spanned by the monomials that do not divide $\prod_j X_j$. Lemma 8.3 is applied with $I = \{j : \mathrm{ty}(j) = e\}$ and with the image of $\mathcal{N}$ enlarged by $(\mathfrak{q} : \prod_{\mathrm{ty}(i)<e} X_i)\,\Lambda^s$; the variables of the earlier types are not inverted: for $\mathrm{ty}(i) < e$ the multiples of X_i lie in this colon module, which is the alternative "$X_i\Lambda^s \subseteq \mathcal{N}$" of the lemma, and for $i < j$ with $\mathrm{ty}(j) = e$ the monotonicity of ty gives $\mathrm{ty}(i) = e$ or $\mathrm{ty}(i) < e$. The lemma gives $\Gamma_e \in \Lambda$ with

$$v_A(\Gamma_e(0)) \;\le\; b_e + \big(\#\{j : \mathrm{ty}(j) = e\} + \#s\big)\, v_A(C)$$

and $\Gamma_e \prod_{\mathrm{ty}(j)=e} X_j$ in the Fitting ideal of the enlarged module; it is, up to the colon ideal, a combination of determinants of matrices B_e with rows in the image of $\mathcal{N}$.

Assembly. For each choice of such matrices form the matrix M' with the rows $o_v(\delta_v)$ at the coordinates of W (in $\mathcal{N}$ by (f)), the eliminating rows, and $\sum_e q_e\cdot$(the rows of B_e). On the characters its determinant is $\prod_v o_v(1)\cdot C^{\#s}$ times $\det B_e$. Apply Lemma 8.2 to $h = k_0\cdot(\dots)\cdot q_0$ and $B_0 + k_0(M' - B_0)$, with the ideal $\mathfrak{a} = \ker \eta_0$ (for $x \in \mathcal{X}$ one has $u_x(mh) = 0$ for $m \in \mathfrak{a}$, because $\xi_0 = 1$ is a constituent of u_x and k_0 kills the block): it gives f with $f\beta \in \mathcal{J}$, and summing over the choices, an f whose value at $x \in \mathcal{X}$ is, modulo monomials not dividing $\prod_j X_j$,

$$\det \psi_x(B_0)\cdot C\cdot \Big(\prod_v o_v(1)\, C^{\#s}\Big)^{t}\cdot \prod_e \Gamma_e \cdot \prod_j X_j.$$

So no monomial properly dividing $\prod_j X_j$ occurs, and the coefficient c at π has

$$v_A(c) \;\le\; b_0 + \sum_e v_A(\Gamma_e(0)) + t\,\Big(\sum_v v_A(o_v(1)) + \#s\, v_A(C)\Big) + v_A(C).$$

The numbers $\#\{j : \mathrm{ty}(j) = e\}$ add up to $\#J$, so the bounds for the $\Gamma_e(0)$ add up to $\sum_e b_e + (\#J + t\,\#s)\, v_A(C)$, and the total is exactly the bound of the theorem. That $\beta \in \mathcal{S}$ is part of Lemma 8.2, and $\beta(x) = 0$ on $\mathcal{X}$ because the matrix of the normaliser has a zero column at the constituent $\xi_0 = 1$. □

9 The cost of a Tate twist: proof of Theorem E

Lemma 9.1 (symmetrisation gives even classes)**.** *Let E be a CM field as in the conventions, $\mathcal{O}$ a coefficient ring, e, ν natural numbers, S a set of finite places of E and $c\colon G_E \to \mathcal{O}/\ell^\nu$ a function. Suppose that $e \neq 0$, that $c \in \mathrm{Sel}^{\emptyset}_{\Sigma_\ell(E)\cup S\cup\bar{S}}(\mathcal{O}(-e);\, \mathcal{O}/\ell^\nu)$, and that c vanishes at every element of a decomposition group above ℓ with* $\mathrm{cyc} = 1$*. Then $g \mapsto c(g) + c(g^{\mathsf{c}})$ lies in* $\mathrm{Ev}(\mathcal{O}, S;\, \nu, e)$*.*

Proof. Conjugation by c preserves the cyclotomic character, the set $\Sigma_\ell(E)\cup S\cup\bar{S}$ and the set of the elements of decomposition groups above ℓ with $\mathrm{cyc} = 1$. So $c\circ(\cdot)^{\mathsf{c}}$ is a cocycle of the same kind, and the sum is visibly invariant. □

LEAN `add_conjGal_mem_evenClasses`

Proof of Theorem E. Take for Γ_0 the set of the elements of inertia groups above ℓ if $e = 0$, and of the elements of decomposition groups above ℓ with $\mathrm{cyc} = 1$ otherwise. Write $I = \{j : \mathrm{ty}(j) = e\}$; it is empty for $e = 0$.

(C1). Let $j \in I$ and $z \in Z$ vanish on Γ_0, with $z(\tau_i) = z(\tau_i^{[-]})$ for $i < j$, and $z(\phi_i) = z(\phi_i^{[-]})$ for $i < j$ in I. By Lemma 9.1, $z^+ = z + z \circ (\cdot)^{\mathsf{c}}$ is an even class of type $e = \mathrm{ty}(j)$. For a cocycle of $\mathcal{O}(-e)$ and an element g with $\mathrm{cyc}(g) \equiv 1$, $z(g^{-1}) = -z(g)$; with Lemma 7.2 of the first note this turns the hypotheses into $z^+(\tau_i) = 0$ and $z^+(\phi_i) = 0$. Clause (T4) gives $\ell^h z^+(\tau_j) = 0$, that is $\ell^h z(\tau_j) = \ell^h z(\tau_j^{[-]})$.

(C2), first term. A function of Z that obeys $\mathcal{R}_{\mathrm{cris}}(\mathcal{O}(-e))$ vanishes on Γ_0. Indeed, an element of Γ_0 is conjugate to an element s of the decomposition group of a chosen place above ℓ with $\mathrm{cyc}(s) = 1$, inertial if $e = 0$; for $e \neq 0$ such an s need not be inertial, and on the decomposition group the module $\mathcal{O}(-e)$ is the character cyc^{-e}, not the trivial one. A crystalline cocycle of $\mathcal{O}(-e)$ vanishes at such an s: for $e > 0$ it is a coboundary after inverting ℓ (the finite part of H^1 of a negative Tate twist vanishes), hence it vanishes where $\mathrm{cyc}^{-e} = 1$; for $e = 0$ it is unramified. This is an input of the unit on the Bloch–Kato condition; the second reading relates it to Example 3.9 of Bloch and Kato (a weaker form). The cocycle identity transports the vanishing along the conjugation. Hence $\mathrm{len}(Z/\{z : z|_{\Gamma_0} = 0\}) \leq \mathrm{len}(Z/Z \cap \mathrm{Ob}(\mathcal{R}_{\mathrm{cris}}(\mathcal{O}(-e)); \mathcal{O}/\ell^\nu))$.

(C2), second term. Let Z' be as in (C2). By Lemma 7.3 of the first note its elements, which vanish at all τ_k, are unramified outside ℓ. If $e = 0$ they lie in $\mathrm{Sha}^{\mathrm{odd}}(\ell, E, \mathcal{O}, 0, \nu)$, and $\mathrm{len}(Z'/Z'[\ell^h]) \leq \mathrm{len}\, Z'$. If $e \neq 0$, the map $z \mapsto z \circ (\cdot)^{\mathsf{c}} - z$ sends Z' into $\mathrm{Sha}^{\mathrm{odd}}(\ell, E, \mathcal{O}, e, \nu)$, and its kernel is killed by ℓ^h. Indeed, let z be in the kernel. Then $z^+ = 2z$ is an even class unramified outside ℓ, and it vanishes at the ϕ_j of type e: the hypothesis on the ϕ-pairs gives $z(\phi_j) = z(\phi_j^{[-]}) = -z(\phi_j)$, that is $2z(\phi_j) = 0$ (and not $z(\phi_j) = 0$, which is why (T5) is applied to z^+ and not to z). By (T5) for the even class z^+, $\delta(\gamma) \cdot 4z(g) = \delta(g) \cdot 4z(\gamma)$ for all g: the element $4\delta(\gamma)\, z$ of Z is of the form $g \mapsto (\mathrm{cyc}(g)^{-e} - 1)m$. Hypothesis (ii) then gives $4\delta(\gamma)^2 z = 0$, and $4\delta(\gamma)^2$ divides ℓ^h by (T5). So $\mathrm{len}(Z'/Z'[\ell^h]) \leq \mathrm{len}\, \mathrm{Sha}^{\mathrm{odd}}(\ell, E, \mathcal{O}, e, \nu)$, and (iii) gives the bound b. □

10 Where the results are used

The results are used in the third note of the unit, *A fit family of twists exists*. Theorem A, with Theorem C for its hypothesis, gives there the one twist for each pair of a prime and an isomorphism that makes all Selmer length bounds hold at once. Theorem D is the engine of the bound for the denominator of the projection to theta lifts: it is applied to the Galois representations of a base level, with the lifts as the set $\mathcal{X}$; its hypothesis (g) is supplied by Theorem E for the characters cyc^{-e} of the string, and its hypothesis (e) by Theorem A of this note through the Selmer length bound. Lemmas 7.1, 7.2 and 7.3 enter the estimate of the crystalline losses above ℓ. Theorem B and the lemmas of §§4–6 are used only inside this note.

11 Notes on the formal text

1. *Letters.* Besides the changes of the first note: the formal text writes `0` for the rings called A, A_0, A' here, `L` for the algebra $\mathbb{L}$ and for sets of relations, `S` for the subalgebra $\mathcal{S}$, `J` for $\mathcal{J}$, `X` for $\mathcal{X}$, `ι` for the index set $\mathcal{I}$, `G` for the group Γ, `R` for the rings A of Lemma 7.3 and Λ_0 of Lemma 8.1, `A` for the algebra $\mathfrak{A}$, `c` for the involution c' of Lemma 5.1, `I` for its ideal $\mathfrak{a}$, `n` for the function θ of Theorem D, `L` for the set Γ_0 of Definition 3.4, and `P₀` for the helper prime $\mathfrak{q}_0$.

2. *Arguments.* The arguments were read from the text of the files. For seven declarations (the four lemmas on pencils, Theorem D, Theorem C and Lemma 7.2) the dossier gives a list reconstructed by hand, because the files declare the same variables twice or the statement contains a local definition; none was confirmed by Lean's own answer. The program reads the argument of the function of Definition 3.3 (the group element) as an argument, and so do we; in Theorem C it reads the quantifier "for all large $\mathcal{O}$" as two arguments. The instance "discrete valuation ring" is not used by Definition 3.4, and "group" not by Definition 3.5; they are not listed there.

3. *Statements with local definitions.* Lemmas 6.2(b), 6.3, 6.4, 7.2, 7.3 and Theorem B have a local definition inside their formal statement; they were read to their true end in the source.

4. *Theorem C.* The finiteness of the set Σ is an argument of the formal statement with a default proof; the set enters the conclusion as a finite set made from that proof.

5. *Theorem A.* The hypotheses $S \subseteq S_0$ and $z \in I_{S_0 \setminus \Sigma_\ell(E)}$ stand in a declaration of variables before the statement and are included in it. The same I and ψ serve every d and every i; only the coefficient ring $\mathcal{O}'$ and ψ' depend on them. The bounds B_i are extended non-negative reals, c and η non-negative reals.

6. *Theorem D.* The index set of the characters is $\{0, \dots, t-1\}$ with $t \neq 0$; the map ty is a monotone map whose target the formal text leaves to be inferred. Hypothesis (g) quantifies over all submodules Z; its conclusion is used for the modules of forms of the proof. The formal bound is $b_0 + (\sum_e b_e + t(\sum_v v_A(o_v(1)) + 2\#s\, v_A(C)) + (\#J+1)v_A(C))$; the outline above reproduces it: each character contributes $(\#\{j : \mathrm{ty}(j) = e\} + \#s)\, v_A(C)$ through Lemma 8.3, the assembly $t\,\#s\, v_A(C)$ and one more $v_A(C)$.

7. *Leaves of the structured proofs.* Two leaf steps of the dossier quote nine lines (one long simultaneous choice in Lemma 4.2, one chain of rewritings in Lemma 6.4); the bound of the workspace is eight.

8. *Outlines.* The proofs of Lemmas 4.1, 5.1, 6.4, 7.3, 8.2 and of Theorems A, C and D are outlines of long formal computations, as said at the head or at the end of each. For them I compared the hypotheses used at each step with the formal text but did not re-derive the identities. Two points were re-derived in the revision of this note: in Theorem A the roots of unity are chosen for the polynomials $X^d - 1$, $1 \leq d \leq m$, as in the formal proof; and in Theorem D the multiples of $v_A(C)$ of the outline add up to those of the statement (note 6).

9. *Input 1.* The formal statement of the relations at split primes is long (it has a local definition and was read to its end by the second reading); it is stated

above for a number field, with all its hypotheses, because the third note uses it over a totally real field. Its hypotheses on the local conditions, on the complex conjugations and on the chosen places stand in a declaration of variables before the formal statement and are included in it. The write-up has not yet read its proof, and no printed source for this form is known to the catalogue; the same holds for the count of trivialised cocycles that supplies the hypothesis of its part (a).

10. *The tilting lemma.* The formal proof compares lengths: with the notation of the proof above, it shows $(s\Phi_0 - \Phi_1)(Z) \supseteq \Phi_0(Z)$ and concludes by additivity of the length. The map Φ_1 does not factor through Φ_0 on Z in general (for $A = \mathbb{Z}_\ell$, $\mathfrak{a} = (\ell)$, $\#m = 1$, $B = Y = (\mathbb{Z}_\ell/\ell)^2$, $\Phi_0(x, y) = x$, $\Phi_1(x, y) = y$ one has $Z = B$, and the kernel of $s\Phi_0 - \Phi_1$ is not $\ker \Phi_0$).

How this note was written. From the dossier of the unit (its second chunk): the statements in words and the structured proofs, which a program compared line by line with the source (975 leaf steps in the 96 proofs of the whole unit). I read the eight Lean files of these declarations completely, and for the long proofs the reader's step-by-step rendering beside them. The statements of this note were compared with the Lean statements argument by argument, and the order of the quantifiers is that of the formal text (in Theorem A: first I and ψ, then d, then i, then the coefficient ring; in Theorem B: first J_0, then N). The proofs of Lemmas 4.2, 4.3, 5.2, 6.1, 6.2, 6.3, 6.5, 7.1, 7.2, 8.1, 8.3, 9.1 and of Theorems B and E follow the formal ones; the others are outlines (note 8). I did not open the proofs of the inputs. The reference of Input 2 and its place are those of the second reading of this write-up; I did not check them in the literature.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Definition 2.1	`crudeUnit`	AptTupleDefs.lean	27–27
Definition 2.1	`IsColumn`	AptTupleDefs.lean	43–45
Definition 2.1	`IsRow`	AptTupleDefs.lean	50–59
Definition 2.1	`obeyingCocycles`	BlocksAndCocycles.lean	16–17
Definition 2.1	`AuxPrimeData`	Corepresentable.lean	38–40
Definition 2.1	`IsAnticyclotomicAt`	Corepresentable.lean	46–51
Definition 2.1	`IsAuxiliary`	Corepresentable.lean	55–55
Definition 2.1	`IsFamily`	Corepresentable.lean	60–68
Definition 2.1	`evenClasses`	Corepresentable.lean	136–138
Definition 2.1	`IsNewPlaces`	Corepresentable.lean	142–143
Definition 2.1	`withConjInv`	Corepresentable.lean	147–147
Definition 2.1	`IsPrimeTriangle`	Corepresentable.lean	152–162
Definition 2.1	`IsAuxTwist`	Corepresentable.lean	166–167
Definition 2.1	`unionWithConj`	Corepresentable.lean	206–206
Definition 2.1	`pointChar`	Corepresentable.lean	210–211
Definition 2.1	`twistAtPoint`	Corepresentable.lean	215–216
Definition 2.1	`reduceModPow`	Corepresentable.lean	219–220
Definition 2.1	`IsBalanced`	Corepresentable.lean	227–238
Definition 2.1	`IsPencil`	Corepresentable.lean	243–250
Definition 2.1	`tieRelations`	Corepresentable.lean	254–256
Definition 2.1	`matrixLattice`	CrystallineRank.lean	22–23

In this note	Lean declaration	File	Lines
Definition 2.1	`coeffEmb`	DeterminantBound.lean	34–35
Definition 2.1	`ZfAt`	DeterminantBound.lean	39–39
Definition 2.1	`crysRels`	DeterminantBound.lean	43–43
Definition 2.1	`complexEmbOf`	DeterminantBound.lean	59–60
Definition 2.1	`AbsGal`	LevelPlacesChosen.lean	25–25
Definition 2.1	`placesAboveG`	LevelPlacesChosen.lean	28–28
Definition 2.1	`badPlaces`	LevelPlacesChosen.lean	32–32
Lemma 5.1	`card_le_torsion_of_isotropic`	LinePointPowers.lean	19–53
Lemma 5.2	`exists_sharp_fitting_poly`	LinePointPowers.lean	64–86
Lemma 4.3	`dvd_one_add_mul_pow_sub_one`	LinePointPowers.lean	88–95
Definition 3.2	`blockFlattenAlgEquiv`	RibetCramer.lean	16–18
Lemma 8.1	`amitsur_charpoly_mem_matrix`	RibetCramer.lean	27–35
Lemma 8.2	`exists_ribet_element_cramer`	RibetCramer.lean	40–75
Definition 3.3	`ribetNormaliser`	RibetCramer.lean	79–80
Definition 3.4	`HasTriangleCost`	RibetFirstOrder.lean	20–24
Lemma 8.3	`exists_mem_fittingIdeal_block`	RibetFirstOrder.lean	30–55
Definition 3.5	`IsTraceIdeal`	RibetFirstOrder.lean	67–71
Definition 3.6	`IsRibetElement`	RibetFirstOrder.lean	75–77
Theorem D	`ribet_first_order`	RibetFirstOrder.lean	91–245
Lemma 4.1	`exists_root_of_unity_eval_large`	RootsOfUnityPoints.lean	21–64
Lemma 4.2	`exists_roots_of_unity_ellVal_le`	RootsOfUnityPoints.lean	72–105
Lemma 7.1	`isTameType_of_roots`	RowBalanced.lean	22–32
Theorem C	`IsColumn.eq_one_of_mem_inertiaAbove`	RowBalanced.lean	36–38
Lemma 7.2	`inClass_of_isLatticeIn`	RowBalanced.lean	43–66
Lemma 7.3	`relLength_obeyingCocycles_le`	RowBalanced.lean	73–115
Theorem C	`IsRow.eventually_isBalanced`	RowBalanced.lean	124–175
Lemma 9.1	`add_conjGal_mem_evenClasses`	TateTwistSteps.lean	17–23
Theorem E	`hasTriangleCost_tateTwist`	TateTwistSteps.lean	28–60
Definition 3.1	`evalConjPair`	TiltingLemma.lean	18–19
Lemma 6.1	`mem_selmerZ1_of_mem_selmerModule`	TiltingLemma.lean	21–25
Lemma 6.2	`IsPencil.char_tame_gen`	TiltingLemma.lean	33–35
Lemma 6.2	`IsPencil.pointChar_idOrConjInv`	TiltingLemma.lean	40–44
Lemma 6.3	`IsPencil.length_le_of_drop_ties`	TiltingLemma.lean	50–87
Lemma 6.4	`IsPencil.length_le_of_unseen_class`	TiltingLemma.lean	94–198
Lemma 6.5	`tilting`	TiltingLemma.lean	206–224
Theorem B	`exists_isPencil`	TwistTheorem.lean	31–94
Theorem A	`exists_anticyclotomic_twist`	TwistTheorem.lean	103–195

Two statements as they are written in the artifact.

```
lemma exists_anticyclotomic_twist {κ T : Type} [Finite κ] [Finite T] {O : CoeffRing ℓ} {ρ : ∀ i,
  _ →* Matrix _ (n i) O} {μ : κ → T → _} {L : ∀ _, _} {B : ι → _} (hρ : ∀ i, IsBalanced ℓ S (ρ i)
  (L i) (γ i) (β i) M) {val : κ → _} (hf : ∀ k, IsFamily S₀ (μ k) (val k)) (hB : ∀ i, B i ≠ ⊤) (hβ :
  ∀ i, O.normLength (β i + ∑ b, addVal _ (ρ i (idOrConjInv b z) - 1).det) ≤ B i) {c n : ℝ≥0} (hc :
  ∀ k t, ellVal ((μ k t z).1 - 1) ≤ c) (hn : 0 < n) m : ∃ I ψ, ∀ d ∈ Set.Icc 1 m,
  (∀ k, ellVal (val k I (ψ ^ d)) ≤ ellVal (val k ∅ 1) + 8 * c + n) ∧ IsAuxTwist S₀ I (ψ ^ d) n ∧
  ∀ i, ∀ᶠ O' in atTop, ∃ hO : O ≤ O', ∃ ψ', (∀ g, (ψ' g).1 = (ψ ^ d) g) ∧
  O'.normLength (length O' (selmerModule (twistRep (inclusion hO).1 ψ' (ρ i)) (unionWithConj S I)
  (mapRel (inclusion hO).1 '' (L i ∪ {.single (γ i) 1}))) (QuotSpan (ℓ ^ M : ↑O')))) ≤ B i + n

lemma hasTriangleCost_tateTwist [Fintype J] [LinearOrder J] {v w τ φ t γ h c} {e : Fin t} {ty : J →o
  _} (tri : IsPrimeTriangle v w τ φ γ t ty h c) {Z : Submodule O _} (N : Z ≤ selmerModule (tateTwist
  ℓ E O e) (placesDividing E ℓ ∪ .conjClosure (.range w)) ∅ (QuotSpan (ℓ ^ v : ↑O))) (H : ∀ c ∈ Z,
  ∀ m, (∀ g, c g = (cycInv ℓ E O g ^ e.1 - 1) • m) → ∀ g, (cycInv ℓ E O γ ^ e.1 - 1) • c g = 0) {b}
  (hb : relLength Z (obeying (crysRels (tateTwist ℓ E O e)) _) + length O (shaOdd ℓ E O e v) ≤ b) :
  HasTriangleCost Z (withConjInv τ) (withConjInv φ) {j | (ty j).1 = e}
  (ℓ ^ h) b
```

In these texts `IsBalanced`, `IsFamily`, `IsAuxTwist`, `IsPrimeTriangle` and `HasTriangleCost` are the predicates written above with the macros of the same names; `idOrConjInv b z` is $z^{[b]}$, `unionWithConj S I` is $S \cup I \cup \overline{I}$, `twistRep` forms $\psi' \cdot \varrho_i$, `mapRel` carries relations to the larger ring, `.single (γ i) 1` is δ_{γ_i}, `tateTwist ℓ E 0 e` is $\mathcal{O}(-e)$, `shaOdd` is Ш^{odd}, and `relLength S T` is the length of $S/S \cap T$.

Notation

ℓ, $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell$, $\bar{\mathbb{Q}}_\ell$, $\bar{\mathbb{Z}}_\ell$, $\mathbb{C}_\ell$, $G_{\mathbb{Q}_\ell}$, ι a prime and the ℓ-adic objects; the fixed isomorphism $\mathbb{C} \to \bar{\mathbb{Q}}_\ell$

$\bar{\mathbb{Q}}$, c, g^{c}, $g^{[b]}$ an algebraic closure of $\mathbb{Q}$; complex conjugation; $\mathsf{c}g\mathsf{c}$; g or $(g^{\mathsf{c}})^{-1}$

E, G_E, f_E a CM field as in the conventions; $\mathrm{Gal}(\bar{\mathbb{Q}}/E)$; its number of infinite places

P, D, $E(D)$, $f(D)$, $g_i(D)$, $g(D)$, $N(D)$, $r(D)$, $t(D)$, $\mathrm{ord}_{\mathrm{ch}}(D)$ parameters, data and its numbers, as in the first note

w, v, $\mathfrak{q}$, $\mathfrak{q}_0$, $\bar{w}$, $\mathrm{N}(w)$, $\mathfrak{P}$, $\mathfrak{P}_w$, $G_{\mathfrak{P}}$, $I_{\mathfrak{P}}$, $\tau_{\mathfrak{P}}$, τ_w places, norms, decomposition and inertia groups, tame generators

$\Sigma_M(E)$, I_S, cyc, cyc_M places containing M; inertia above S; cyclotomic characters

$\mathcal{O}$, $\mathcal{O}'$, ϖ, $\mathcal{O}(-e)$, v_ℓ, v_ℓ, nlen, len coefficient rings, a uniformiser, Tate twists, valuations and lengths

A, A_0, A', A_1, Λ, Λ_0, $\mathbb{L}$, $\mathfrak{A}$, B, v_A rings, algebras, a module, the valuation of a discrete valuation ring

Γ, Γ', Γ_0, j, η groups; a subset; the homomorphism to $G_{\mathbb{Q}_\ell}$; a homomorphism (also an idele character and a real number)

$M_n(A)$, GL_n, $\#n$, tr, $\det$, rk matrices, cardinality, trace, determinant, rank

θ, δ_g, $\mathrm{Ob}(L;\, B)$, $\mathrm{Rel}(Z)$, $\mathrm{Sel}_S^L(\varrho;\, B)$ relations, the indicator relation, obeying functions, relations of a module, Selmer modules

$Z_{\mathrm{f}}(L)$, $L^\vee(1)$, $L^{\perp_Q}$, $\mathcal{Q}(F, K, U)$, Ш^{odd} crystalline cocycles, twisted dual, cup-orthogonal, classes, the module of §2.1

$\mathrm{Fitt}(\mathcal{N})$, $\mathfrak{F}(\Phi, z, \mathcal{N}, p)$ Fitting ideal; module of forms

$\mathcal{Z}_u(\varrho;\, a)$, $\mathcal{R}_{\mathrm{cris}}(u)$, $\mathrm{L}(\varrho)$, $\mathrm{emb}_{\mathcal{O}}(g)$, $\mathrm{ce}(e, g)$ Definition 2.1(1),(2)

$G_{E(D)}$, $\Sigma_g(D)$, $\Sigma_{\mathrm{bad}}(D, \ell)$, $\mathrm{cr}_\ell(D)$ Definition 2.1(3)

$\mathsf{ColRep}_D(t, \mathcal{H}, S;\, R)$, $\mathsf{RowRep}_D(t, \mathcal{H};\, R)$ columns and rows (Definition 2.1(4))

$\mathrm{Aux}(E, q)$, $\mathsf{Acyc}(T, \mathfrak{q};\, l)$, $\mathsf{Auxil}(S;\, I)$, $\mathsf{Fam}(S;\, \nu, \mathrm{val})$ Definition 2.1(5),(6)

$\mathrm{Ev}(\mathcal{O}, S;\, \nu, e)$, $\mathsf{New}_\nu(w, \tau, \phi)$, $\tau^{[k]}$ Definition 2.1(7)

$\mathsf{Tri}_\nu(w, \tau, \phi;\, a;\, t, \mathrm{ty};\, h, c)$ prime triangles (Definition 2.1(8))

$\mathsf{AuxTw}(S, Q;\, \psi, c)$, $S \cup J \cup \overline{J}$, $\mathrm{tp}_{h,y}(e, \varrho)(g)$, $\varrho \bmod l^M$, y^h Definition 2.1(9),(10)

$\mathsf{Bal}_\ell(S;\, \varrho, L;\, \gamma, \beta, M)$ balanced representations (Definition 2.1(11))

$\mathsf{Pencil}_\ell(S_0;\, U, W;\, N, \mathfrak{q}_0, J;\, x)$, $\mathsf{Ties}_i(J, x)$ pencils and ties (Definition 2.1(12))

$\mathrm{ev}_g^{\pm}$, fl_N, $\mathrm{rn}_{\gamma,a}(y, p, \theta)(g)$ Definitions 3.1–3.3

$\mathsf{TriCost}(Z;\, \tau, \phi;\, I;\, C, b)$, $\mathsf{Trld}(\mathcal{J};\, u;\, p;\, \tau, \phi, k, \varepsilon)$, $\mathsf{Rib}(\mathcal{J};\, \beta;\, \pi, c)$ Definitions 3.4–3.6

R, R', u, ϱ, ϱ_i, ϱ_x, ϱ_y, ψ, ψ', χ, ξ_e, $\mu_{k,t}$, ν representations, lattices and characters (ν also a natural number)

$\mathcal{H}$, $\mathcal{H}_w$, Λ_w, e_w, $\mathcal{R}$, $\mathfrak{M}$ types; the local lattice and chart at w; a property; a multiset

$l = (\mathfrak{P}, \gamma, \phi, h)$, l_v, h_v, γ_v, ϕ_v, γ_l, ϕ_l, $s_{v,i}$, $\mathbf{1}_i$, $\mathbf{1}_g$ data at auxiliary primes; slopes; indicator vectors

I, J, J_0, J_1, S, S_0, T, W, Q, P', Q', Σ, $\mathcal{X}(T)$ sets of places and index sets; bounds for the number of primes; the Selmer modules of Lemma 6.4; W also a neighbourhood, T a list of residues, Q a polynomial

$\mathcal{J}$, κ, Y, X, $\mathcal{X}$, π, σ, n, n_i, r, d, m, N, M, k, q, t, e, i, b index sets and natural numbers; X also a variable, π a point, κ also a ring homomorphism, e also a ring homomorphism, b also a bound or a sign

β, β_i, B_i, b_0, b_e, c, η, δ, C, p, p_0, a, α, α', s bounds, real numbers and ring elements (a, s also group elements or a finite set, c also a cocycle or a ring element, δ also the function of (T5))

γ, γ_i, γ_x, z, g, g', g_0, g_1, h, τ, ϕ, ζ, u, y, y_v, y_x, y_0 elements of groups, roots of unity and points (u also a lattice, y also a cocycle, h also an element of a group algebra)

F, F_k, $F_{k,d}$, F_i, G, G_i, D_n, $V_{k,I}$, val_k, $a_{k,i}$, ρ_j, $\|F\|_\rho$, Π, Q_k, η_j polynomials and Gauss norms of §4 and of the proof of Theorem A; F also a filter, G also the element of Lemma 8.3

c', H, U, U', U_i, $\mathfrak{a}$, x_w, $\tau = [g_0, \gamma]$, Y objects of the mirror lemma and its use (U also a function to finite sets, Y also a matrix or a submodule)

Φ_0, Φ_1, Φ, Ψ, ψ, σ', $\mathcal{M}$, $\bar{v}$, Z, Z', $Z'[C]$ linear maps and modules of §§6, 8; the free module of the tilting lemma; reduction modulo $\mathfrak{a}$; the C-torsion

K, G_K, X, L_v, N', U', $Y(\Pi')$, Π', ι', v_i, ϕ_i, γ_i, ϖ', T', Z_v^1, $\mathcal{R}_{G'}$, G', n_i, v'_j, ϕ'_j,

δ_j objects of Input 1: the field, the module, local conditions, the relaxed Selmer group, split places, relation families

$\mathfrak{q}$, t', D_d the ideal of the monomials not dividing $\prod_j X_j$; $\max(t,1)$; the polynomial X^d-1

$\mathcal{S}$, $\mathcal{J}$, ε_j, ϵ, pr_1, ty, δ_v, o_v, ω, k_0, q_e, s_0, B_0, B_e, Γ_e, M', P', W', F', F'', a', η', ρ', r_p, m_p, w, V, f objects of §8: subalgebra and submodule, polynomials, the bijection, maps, group-algebra elements, matrices

$\mathcal{N}$, $\mathcal{K}$, $\mathcal{V}$, $\mathcal{R}$, $\mathfrak{b}$, e, f', d_4, r_j, v_j, L, K, Λ^s modules of relations, ideals and constants; L also a set of relations or a lattice, K also a property of polynomials

z^+, $\delta(g)$, χ_x, ψ_x, η_e the symmetrisation; $\mathrm{cyc}(g)^{-e}-1$; the algebra homomorphisms of the proof of Theorem D

$\top$, $\mathbb{Q}/\mathbb{Z}$, $\mathrm{U}(1)$ the trivial filter; the dualising group; the unitary group of rank one

References

The reference is the one named by the second reading of this write-up, with the place given there; I did not open it.

G. Chenevier, *The p-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings.*

READBACK NOTE 33

A fit family of twists exists: Ribet's method with tied primes, triangular Taylor–Wiles primes and the denominator of the projection to theta lifts

One unit of the Lean artifact grh-compact, read back into mathematics

Theorem A (a fit family of twists exists). *Let P be parameters and D admissible data for P. Then there is a family $\theta = (\theta_{\ell,\iota})$ of m-fold twists of D, one for each pair (ℓ, ι) of a prime number ℓ and a field isomorphism $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$, with the following three properties (the family is* fit for *D).*

(F1) *For every (ℓ, ι), the prime ℓ is prime to the product $\mathrm{N}(Q(\theta_{\ell,\iota}))$ of the norms of the places of $Q(\theta_{\ell,\iota})$.*

(F2) *For every lift index $1 \le j \le m(D)$ there are real numbers $(\eta_l)_{l\in\mathbb{N}}$ such that $\sum_{l\in T}\eta_l \le B_0(D)$ for every finite set T of prime numbers, and such that for every (ℓ, ι), with $\vartheta = \theta^j_{\ell,\iota}$:*

$$\mathsf{Den}_\iota(O_D(\vartheta, j), e_0(D), \operatorname{im}\theta^{*,\mathrm{good}}_{D,\vartheta,j};\, \eta_\ell),$$

and, for every pair p of D that is of rank one for j, with r_p the index set of the rows of $O_p(\vartheta)$,

$$\mathsf{Den}_\iota(O_p(\vartheta), e^{\otimes}_{r_p, X-1}, \mathbb{C}\cdot\Xi_p(\vartheta);\, \eta_\ell).$$

(F3) *For every lift index $1 \le j \le m(D)$, every pair p of D and every sign $s \in \{1, -1\}$: if p is ramified, then the family $(\ell, \iota) \mapsto \varrho_p(\theta^j_{\ell,\iota}, s)^{-1}$ of complex numbers has height at most $x(D)^{999}$.*

LEAN `exists_isFitFamily` · `Data.IsFitFamily`

Here $B_0(D) = \mathrm{budget}(D)/(m(D)k(D))$; $O_D(\vartheta, j)$ is the j-th auxiliary pair of the twist ϑ, $e_0(D)$ the good idempotent and $\theta^{*,\mathrm{good}}_{D,\vartheta,j}$ the adjoint theta lift on the good forms (Definition 3.3); $O_p(\vartheta)$ is the auxiliary pair of the L-value data of p and ϑ, of type (r_p, n) with $\#r_p = 1$, $e^{\otimes}_{r_p,X-1}$ the idempotent of the tuple with all labels $X-1$ (written $e^{\otimes}_{r,c}$ for an index set r and a tuple c, as in the note *Primitive labels*) and $\Xi_p(\vartheta)$ the lift of the trivial character; $\mathsf{Den}_\iota(O, e_0, Z;\, \beta)$ is the bound on the denominator of the orthogonal projection to Z of Definition 3.1; $\varrho_p(\vartheta, s)$ is the ratio of L-values

$$\varrho_p(\vartheta, s) = \frac{\left|L^{(\mathrm{Ch}\cup Q(\vartheta))}\left(z_p, (\nu_p\vartheta^s)^{-1}\right)\right|}{\left|L^{(\mathrm{Ch})}\left(z_p, \nu_p^{-1}\right)\right|},$$

where the upper index names a set of finite places at which, and at whose conjugates, the Euler factors are removed, and Ch is the set of the chosen places; and a family $y = (y_{\ell,\iota})$ of complex numbers *has height at most c* if there are $H \in \mathbb{C}^\times$, a real number α and real numbers $(\eta_l)_{l\in\mathbb{N}}$ with $|\sigma(H)| \le e^\alpha$ for all field automorphisms σ of $\mathbb{C}$, $|\iota(H\, y_{\ell,\iota})| \le e^{\eta_\ell}$ for all (ℓ,ι), and $\alpha + \sum_{l<M \text{ prime}} \eta_l \le c$ for all large M (note *Heights*). The letter η_l is that of the overview of this handout, for the budget of (F2) and for the local bounds of a height; the lift index is written j here and in the formal text, and i in the overview.

Theorem A is the statement *Twists* of the overview of this handout. The following points of its reading were checked against the formal text by the second reader of this write-up, whose reading this note follows. The twist $\theta_{\ell,\iota}$ may depend on ι at a fixed ℓ. The whole family is chosen before the lift index, the pair and the sign are quantified. In (F2) the numbers η_l depend on j only (the proof even gives the same for all j), the same η_ℓ serves every ι over ℓ, both the image of the good lift and every rank-one line, and the twist in the bound is the j-th power $\theta_{\ell,\iota}^j$, whose auxiliary support is that of $\theta_{\ell,\iota}$. In (F3) it is the *inverse* ratio whose height is bounded, with one multiplier H for all embeddings.

Summary. A denominator of the orthogonal projection of integral forms to the image of a theta lift is a congruence between a lift and a form that is not a lift. This note proves that for each pair (ℓ,ι) there is a twist, with its first m powers, at which these denominators are within a budget whose sum over all primes is at most $B_0(D)$ (Theorem A). The chain of the proof: the Galois representations of the lifts are a row plus a string of powers of the cyclotomic character, the row serving all twists (Theorem E); one anticyclotomic twist makes the Selmer modules of all rows short and keeps the L-values in check (Theorem F, from the second note); Ribet's lemma, fed with a prime triangle (Theorem D, by reciprocity laws) and the bound for the odd part of Tate–Shafarevich groups, gives a Hecke element of bounded valuation (Theorem C); and that element bounds the denominator (Theorem B).

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles and that Lean's kernel accepted the proofs; an independent replay is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note is the third and last on the unit *The choice of the twists*: it treats the declarations of its last nine files and recalls, from the notes *Columns and rows as Galois representations* and *Short Selmer groups after one anticyclotomic twist*, the definitions it needs; together 92 of the unit's 205 declarations appear here. All statements are complete. The proofs of the short lemmas are complete, and the proofs of Theorems A, E and F follow the formal proofs step by step; the proofs of Theorems B, C, D and of Lemma 7.2 are outlines of formal proofs of a hundred to a hundred and fifty lines, and say so. Results of the background library and of other units are quoted as Inputs with their owners. Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 92 declarations.

1 The other main statements

The conventions and the recalled notions are in §2, the definitions of this note in §3. Throughout, ℓ is a prime number and $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ a fixed isomorphism of fields.

Theorem B (the denominator of the projection to lifts). *Let P be parameters, D admissible data for P, and ϑ a twist of D with $\mathsf{AuxTw}(\Sigma_{\mathrm{bad}}(D,\ell), Q(\vartheta); \mathrm{av}^-_\vartheta(\cdot), \mathrm{cr}_\ell(D))$. Let r, n be finite sets, O an auxiliary pair of type (r,n) over $E(D)$, e_0 an element of the complex group algebra of the group $\prod_i \mathrm{GL}_r(\mathbb{F}_{w_i})$ of the chosen places of O, and Z a subspace of $\mathcal{F}(O)$. Suppose that Z is contained in the span of the set of the $y \in Z$ with the following two properties:*

(a) *there is a family c of polynomials with $y \in \operatorname{im}\mathrm{pr}_O(c)$, $\mathsf{Apt}(O, c; \vartheta)$ and $\operatorname{im}\mathrm{pr}_O(c) \subseteq \operatorname{im}\mathrm{act}_O(e_0)$;*

(b) *there are Hecke data z such that y lifts z on O, a homomorphism $R\colon G_{E(D)} \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$ and an auxiliary pair V of type (r,n) over $E(D)$ with $\mathsf{HasRow}(V, z; R)$ and $\mu_O = \mu_V \cdot \vartheta$, and elements $\gamma \in G_{E(D)}$, $s' \in \bar{\mathbb{Z}}_\ell$ with $\mathsf{Short}(\mathrm{av}^-_\vartheta(\cdot), R; Q(\vartheta), \lfloor r(D)\cdot \mathrm{rb}_\ell(D)\rfloor + 1; \gamma, s')$.*

Then $\mathsf{Den}_\iota(O, e_0, Z; \mathrm{rb}_\ell(D)\cdot\log\ell)$.

LEAN `denominatorLe_rowBudget`

Theorem C (Ribet's method for columns). *Let P, D be as in Theorem B, K a finite totally ordered set, Y, n, r finite sets, $\nu \in \mathbb{N}$, $w = (w_k)_{k\in K}$ finite places of $E(D)$, $\tau = (\tau_k)_k$, $\phi = (\phi_k)_k$ families in $G_{E(D)}$, $u = (u_x)_{x\in Y}$ homomorphisms $u_x\colon G_{E(D)} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$, V an auxiliary pair of type (r,n) over $E(D)$ with rank difference $t_V = \#n - \#r$, ϑ a twist and c a family of polynomials with $\mathsf{Apt}(V, c; \vartheta)$. Let $\mathcal{H}'$ be a type, $\psi\colon G_{E(D)} \to \bar{\mathbb{Q}}_\ell$ a multiplicative homomorphism and Q a finite set of finite places with $\mathsf{AuxTw}(\Sigma_{\mathrm{bad}}(D,\ell), Q; \psi, \mathrm{cr}_\ell(D))$. Let $\mathrm{ty}\colon K \to \{e : 0 < e < t_V\}$ be monotone, $a \in G_{E(D)}$ and $h \in \mathbb{N}$ with $h \le v_\ell(\mathrm{ord}_{\mathrm{ch}}(D))$ and $\mathsf{Tri}_\nu(w, \tau, \phi; a; t_V, \mathrm{ty}; h_2(D)h, h_2(D)^{22}h)$. Let $\mathcal{S}$ be a $\bar{\mathbb{Z}}_\ell$-subalgebra of the functions $Y \to \bar{\mathbb{Q}}_\ell[X_k : k \in K]$. Suppose that for every $x \in Y$*

$$\mathsf{LabCol}(V; t_V, \mathcal{H}', Q \cup \bar{Q} \cup w(K) \cup w(\bar{K}); (\mathrm{tl}_V(c,i)(X-1)^{t_V})_i; u_x)$$

and $(u_x(g) - \psi(g))(u_x(g) - 1) = 0$ for all $g \in I_{Q\cup\bar{Q}}$. Let $\mathcal{X} \subseteq Y$, and for $x \in \mathcal{X}$ let $\mathcal{H}_x$ be a type, $R_x\colon G_{E(D)} \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$ a homomorphism, $\gamma_x \in G_{E(D)}$ and $y_x \in \bar{\mathbb{Z}}_\ell$; let $m_0 \in \mathbb{N}$ with $\lfloor r(D)\cdot\mathrm{rb}_\ell(D)\rfloor < m_0$. Suppose that for every $x \in \mathcal{X}$: $\mathsf{RowRep}_D(t_V, \mathcal{H}_x; R_x)$, $\mathsf{Short}(\psi, R_x; Q, m_0; \gamma_x, y_x)$, and $\operatorname{tr} u_x(g) = \psi(g)\operatorname{tr} R_x(g) + \sum_{e<t_V} \mathrm{cyc}(g)^{-e}$ for all g. Let $\mathcal{J}$ be an $\mathcal{S}$-submodule of the functions $Y \to \bar{\mathbb{Q}}_\ell[X_k : k \in K]$ and $\varepsilon = (\varepsilon_k)_{k\in K}$ polynomials over $\bar{\mathbb{Z}}_\ell$ with $\mathsf{Trld}(\mathcal{J}; u; \ell^\nu; \tau, \phi, \mathrm{id}, \varepsilon)$ and $\varepsilon_k(1) = 1$ for all k. Let $\pi \in \mathcal{X}$ and θ' a finitely supported function $G_{E(D)} \to \mathbb{N}$, and let $\beta(x) = \det(\sum_g \mathrm{rn}_{\gamma_\pi, a}(y_\pi, \ell^{m_0 + h_2(D)h}, \theta')(g)\,(u_x(g) - 1))$ for $x \in Y$. Then there is $d' \in \bar{\mathbb{Q}}_\ell$ with

$$\mathsf{Rib}(\mathcal{J}; \beta; \pi, d') \qquad \text{and} \qquad v_\ell(\mathrm{ord}_{\mathrm{ch}}(D)^3 \cdot d') \le \#r \cdot \mathrm{rb}_\ell(D).$$

LEAN `ribet_columns`

Theorem D (triangular Taylor–Wiles primes). *Let E be a CM field as in the conventions, t, N, y, h natural numbers with $N \geq 1$, $\gamma \in G_E$, and W a set of Frobenius data of E. Suppose that the image of E in $\bar{\mathbb{Q}}$ lies in the cyclotomic field of conductor N; that $N + t < y$; that $\mathrm{cyc}(\gamma)^a - 1$ divides ℓ^h in $\mathbb{Z}_\ell$ for all $0 < a < t$; that $\ell \leq y^2$ or $h = 0$; that E has at least four infinite places; and that for all $\nu \in \mathbb{N}$ and $g \in G_E$ there is $\delta \in G_E$ with $\mathrm{cyc}(\delta) = 1$ such that W contains most Frobenius data near $\delta^{\ell^\nu} g$. Then there are a finite totally ordered set J and a monotone map $\mathrm{ty} \colon J \to \{e : 0 < e < t\}$ such that for every $\nu > 0$ there is a system $(w_j, \mathfrak{P}_j, \phi_j)_{j \in J}$ of Taylor–Wiles places of E for the modulus ℓ^ν with*

$$\mathsf{Tri}_\nu((w_j)_j, (\tau_{\mathfrak{P}_j})_j, (\phi_j)_j;\, \gamma^{-1};\, t, \mathrm{ty};\, yh, y^{22}h)$$

and $(w_j, \mathfrak{P}_j, \phi_j) \in W$ for every j.

LEAN `exists_isPrimeTriangle`

Theorem E (the good eigensystems have rows). *Let P, D be as in Theorem B, ℓ a prime number and $\iota \colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ an isomorphism of fields, p a function from the pairs of a finite place of $E(D)$ and a natural number to $\mathbb{C}$, and j a lift index $(1 \leq j \leq m(D))$. Suppose that the space of the good forms on $W_{m(D)}(D)$ meets $\mathrm{Eig}_D(W_{m(D)}(D); p)$ non-trivially. Then there is a homomorphism $T \colon G_{E(D)} \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$, with r the index set of the rows of D, such that $\mathsf{HasRow}(O_D(1, j), p; T)$.*

LEAN `exists_hasRow_auxPair`

Theorem F (one m-fold twist for finitely many rows). *Let P, D be as in Theorem B, ℓ a prime number and $\iota \colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ an isomorphism of fields. Let I be a finite set, $(r_i)_{i \in I}$ finite sets, $R_i \colon G_{E(D)} \to \mathrm{GL}_{r_i}(\bar{\mathbb{Q}}_\ell)$ homomorphisms such that for every i there are t_i and a type $\mathcal{H}_i$ with $\mathsf{RowRep}_D(t_i, \mathcal{H}_i; R_i)$, and $M \in \mathbb{N}$. Then there is an m-fold twist τ of D such that for every lift index j $(1 \leq j \leq m(D))$, with $\vartheta = \tau^j$:*

(1) $\mathsf{AuxTw}(\Sigma_{\mathrm{bad}}(D, \ell), Q(\vartheta);\, \mathrm{av}^-_\vartheta(\cdot), \mathrm{cr}_\ell(D))$;

(2) *for every i there are $\gamma \in G_{E(D)}$ and $y \in \bar{\mathbb{Z}}_\ell$ with* $\mathsf{Short}(\mathrm{av}^-_\vartheta(\cdot), R_i;\, Q(\vartheta), M;\, \gamma, y)$;

(3) *for every pair p of D and sign s: if p is ramified, then $\left|\iota(\varrho_p(\vartheta, s))^{-1}\right| \leq \exp(9\,\mathrm{cr}_\ell(D) \log \ell)$.*

LEAN `exists_mFoldTwist_selmerLengthLe`

2 Conventions and recalled notions

Conventions. The conventions are those of the first note of the unit. The natural numbers are $0, 1, 2, \ldots$. ℓ is a prime number (also l, when several primes occur); $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell$, $\bar{\mathbb{Q}}_\ell$, $\bar{\mathbb{Z}}_\ell$, $\mathbb{C}_\ell$ and $G_{\mathbb{Q}_\ell} = \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ are as usual, with $|\ell| = 1/\ell$; $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ is a fixed isomorphism of fields; families indexed by the pairs (ℓ, ι) carry both as indices. $\bar{\mathbb{Q}}$ is a fixed algebraic closure of $\mathbb{Q}$ with a fixed complex conjugation c. A *CM field as in the conventions* is a CM number field E with a fixed embedding in $\bar{\mathbb{Q}}$ that contains a primitive root of unity of order 5040, with its CM type and its Weil data (Chapter VII); $G_E = \mathrm{Gal}(\bar{\mathbb{Q}}/E)$; E^+ is its maximal totally real subfield. P denotes parameters and D admissible data for P; $E(D)$ is a CM field as in the conventions. The letters n, r, d, σ, I, J, K, Y, X, T, when they index matrices or families, denote finite sets; $\#n$ is the number of elements. A *coefficient ring* is the ring of integers $\mathcal{O}$ of a finite Galois extension of $\mathbb{Q}_\ell$ inside $\bar{\mathbb{Q}}_\ell$; "for all large $\mathcal{O}$" means: for all coefficient rings containing a suitable one. V also denotes a hermitian space with s a parameter of forms on it, as in Chapter VII, and then $\mathcal{M}_D(V)$ is the space of the forms of principal level at the chosen places on V, which is assumed finite-dimensional where this is said.

2.1. Background. The background notions are those of §2 of the first note and of §2.1 of the second: finite places, norms $\mathrm{N}(w)$, conjugates $\bar{w}$, *split* places ($\bar{w} \neq w$), the sets $\Sigma_M(E)$; places $\mathfrak{P}$ of $\bar{\mathbb{Q}}$, the chosen place $\mathfrak{P}_w$ over w, groups $G_{\mathfrak{P}}$, $I_{\mathfrak{P}}$, I_S, Frobenius elements and tame generators $\tau_{\mathfrak{P}}$; g^{c} and $g^{[\pm]}$; cyc and cyc_M; characters from $\mathrm{U}(1)$; relations, $\mathrm{Ob}(L; B)$, Selmer modules $\mathrm{Sel}^L_S(\varrho; B)$; lengths len and $\mathrm{nlen}_{\mathcal{O}}$; the valuation v_ℓ; Tate twists $\mathcal{O}(-e)$; lattices of representations; Galois lattices and their classes; characters of $\mathrm{U}(1)$ with weight and tame part, avatars; tame characters, twists and pairs $p = (\nu_p, z_p)$ of D, with the weight $-c_p$ of ν_p and its tame part ψ_p; chosen places; auxiliary pairs O of type (r, n) with their chosen places w_i, characters μ_O, ψ_O, a_O, forms $\mathcal{F}(O)$ and Hecke operators $H_w(X)$; labels and the idempotents $e^{\otimes}_{r,c}$ of the tuples c of labels, in the group algebra of $\prod_i \mathrm{GL}_r(\mathbb{F}_{w_i})$; eigenforms, Hecke polynomials and attached representations; base levels. Further:

- A *Frobenius datum* of E is a triple $(w, \mathfrak{P}, \phi)$ of a finite place of prime norm, a place of $\bar{\mathbb{Q}}$ over it and an element ϕ whose inverse is an $\mathrm{N}(w)$-Frobenius at $\mathfrak{P}$ (a geometric Frobenius element). A set W of Frobenius data *contains most Frobenius data* if every non-empty open subset of G_E contains a non-empty open subset U such that, for all but finitely many finite places w, every Frobenius datum $(w, \mathfrak{P}, \phi)$ with $\phi \in U$ belongs to W; such sets form a filter. The set W *contains most Frobenius data near* g, for $g \in G_E$, if it contains the intersection of a set that contains most Frobenius data with the set of the data whose element ϕ lies in some neighbourhood of g. A statement holds *for most Frobenius data* (near g) if the set of the data for which it holds has this property. A system of *Taylor–Wiles places* of E for the modulus M, indexed by J, is a family of Frobenius data $(w_j, \mathfrak{P}_j, \phi_j)$ with the w_j pairwise distinct, no $\bar{w}_i$ among them, and $M \mid \mathrm{N}(w_j) - 1$.
- An *m-fold twist* of D is a twist ϑ such that $\psi^j_{\vartheta,w} \neq 1$ for all $w \in Q(\vartheta)$ and $1 \leq j \leq m(D)$; its powers ϑ^j, $1 \leq j \leq m(D)$, are twists with the same set $Q(\vartheta)$; $\mathrm{N}(Q(\vartheta))$ is the product of the norms of its places.

- For a form on O the *integrality at a ring homomorphism* $\mathbb{C} \to \mathbb{C}_\ell$ is that of the overview of this handout (the values lie in the weight lattice at each point); the *integrality at* ι is the integrality at ι followed by $\bar{\mathbb{Q}}_\ell \subseteq \mathbb{C}_\ell$. $\langle y, y' \rangle$ is the Petersson product of $\mathcal{F}(O)$, $\langle y, y' \rangle_\sigma$ its form at an automorphism σ of $\mathbb{C}$ and $\|y\|_\sigma^2 = \langle y, y \rangle_\sigma$; pr_Z is the orthogonal projection to a subspace Z.
- θ_O is the theta lift from the forms of O to forms on its hermitian space of rank $\#r$ (Chapter VII); a form *lifts* Hecke data z on O if at the split places off the places of O it is spherical with the Hecke polynomial of the string times that of z (Chapter VII; the notion is used in Definition 3.21 of the first note).
- For admissible data D: $W_j(D)$ is the hermitian space of the j-th single lift, with index set of rank $r(D)$; $e_0(D)$ the good idempotent in the group algebra of $\mathsf{G}(D) = \prod_w \mathrm{GL}_r(\mathbb{F}_w)$ over the chosen places, the sum of the $e^{\otimes}_{r,c}$ over the good tuple families c (r the index set of the rows); the *good forms* on a space are the image of $e_0(D)$ in its forms of principal level; $O_D(\vartheta, j)$ the j-th auxiliary pair of a twist; $O_p(\vartheta)$, $\Xi_p(\vartheta)$ the auxiliary pair and the lift of the trivial character of the L-value data of a pair p and a twist; $x(D)$ the first scale and budget(D) the budget (Chapters III, V, VII).
- $\mathrm{Sha}^{\mathrm{odd}}(\ell, E, \mathcal{O}, e, \nu)$ is the module of §2.1 of the second note.

Definition 2.1 (notions recalled from the first two notes)**.** The following are defined in the notes *Columns and rows as Galois representations* (numbers 3.x) and *Short Selmer groups after one anticyclotomic twist* (numbers 3.x of that note, marked II).

(1) (3.1, 3.2, 3.5, 3.6) $\mathcal{Z}_u(\varrho;\, a)$: the cocycles for ϱ modulo a that obey the relations among the $u_x(g) - 1$. $\mathrm{I}_r(i)$ is r for $i = *$ and a point for $i = e < t$; $\mathrm{blk}_T(i)(\cdot)$ is T for $i = *$ and $\mathcal{O}(-e)$ for $i = e$. $\mathcal{R}_{\mathrm{cris}}(u)$ is the set of the crystalline relations of u, and $\mathrm{bl}(u;\, w, M)$ the length of $\mathcal{Z}_w(u;\, \ell^M)$ modulo its elements that obey $\mathcal{R}_{\mathrm{cris}}(u)$.

(2) (3.7–3.9) $\mathrm{sgn}_{\mathrm{CM}}(\sigma) = \pm 1$ according as the complex embedding σ belongs to the CM type or not. $\mathsf{RowType}_D(r, t;\, \mathcal{H})$: the conditions (R1)–(R4) on a type. $\mathrm{lam}(h, b;\, l)$ is the exponent of l in the integer $(2b)!\,(\prod_{b-30h^3<k\le b+30h^3} k)^{h^3}$, and

$$\mathrm{loss}_D(l) = 10^6\,\frac{m(D)}{k(D)}\,\big(b_1(D)\,v_l(g(D)) + \mathrm{lam}(h_2(D), b_1(D);\, l)\big).$$

(3) (3.10, 3.11, 3.13) $G_{E(D)} = \mathrm{Gal}(\bar{\mathbb{Q}}/E(D))$. The set $\Sigma_g(D)$ consists of the places above $g(D)$, and $\Sigma_{\mathrm{bad}}(D, \ell)$ of the places containing $\ell g(D) N(D)$. For a tame character ϑ, $\mathrm{av}^-_\vartheta(g)$ is the value at g of the avatar of ϑ^{-1}. Finally $\mathrm{cr}_\ell(D) = v_\ell(\mathrm{ord}_{\mathrm{ch}}(D)) + 2^{-\ell}$.

(4) (3.12, 3.14–3.16) $\mathsf{CSD}_t(R)$: R is continuous with non-degenerate trace form and $\mathrm{tr}\, R(g^{\mathsf{c}}) = \mathrm{cyc}(g)^{-(t-1)}\,\mathrm{tr}\, R(g^{-1})$. $\mathsf{Short}(\psi, R;\, Q, m;\, \gamma, z)$: for all large $\mathcal{O}$ some lattice u of R and $\psi' = \psi$ make $\varrho = \psi' u$ scalar $= z$ at γ with $z - 1$ a unit, $\mathrm{cyc}(\gamma) = 1$, and the cocycles vanishing at γ in $\mathrm{Sel}^{\mathcal{R}_{\mathrm{cris}}(\varrho)}_{\Sigma_\ell(E(D))\cup\Sigma_g(D)\cup Q\cup\bar{Q}}(\varrho;\, \mathcal{O}/\ell^m)$ have normalised length at most $(10 f(D) \# n + 1)\mathrm{cr}_\ell(D)$. $\mathsf{ColRep}_D(t, \mathcal{H}, S;\, R)$ and $\mathsf{RowRep}_D(t, \mathcal{H};\, R)$: columns of level S and rows.

(5) (3.17–3.22) $\chi^{-1}_{1,O}$ the normalising character and $\mathrm{nm}_O(R) = \chi^{-1}_{1,O} \cdot R$; $\mathrm{tl}_O(c, i)$ the twisted label; $\mathsf{Apt}(O, c;\, \vartheta)$: the clauses (A1)–(A6); $\mathrm{pr}_O(c)$ the action of $e^{\otimes}_{r,c}$ on $\mathcal{F}(O)$; $\mathsf{HasRow}(O, z;\, R)$: R is a row and the normalised representations of the eigenforms lifting z with character $\mu_O \vartheta$ have the traces

$\mathrm{av}^{-}_{\vartheta}(g)\,\mathrm{tr}\,R(g) + \sum_{e<\#n-\#r} \mathrm{cyc}(g)^{-e}$; $\mathsf{LabCol}(O;\,k,\mathcal{H},S;\,c;\,R)$: a column labelled by the c_i at the chosen places.

(6) (3.25, 3.27–3.30) $\mathsf{Auxil}(S;\,I)$; $\mathrm{Ev}(\mathcal{O},S;\,\nu,e)$; $\mathsf{New}_{\nu}(w,\tau,\phi)$; $\tau^{[k]}=\tau_j^{[b]}$ for $k=(j,b)$; $\mathsf{Tri}_{\nu}(w,\tau,\phi;\,a;\,t,\mathrm{ty};\,h,c)$: the clauses (T1)–(T5); $\mathsf{AuxTw}(S,Q;\,\psi,c)$: Q is auxiliary for S, ψ is from U(1) relative to $\Sigma_{\ell}(E)$ with modulus Q, and $\sum_{\mathfrak{q}\in Q} v_{\ell}(\psi(\sigma_{\mathfrak{q}})-1)\le c$ at tame generators.

(7) (II: 3.3–3.6) The normaliser $\mathrm{rn}_{\gamma,a}(y,p,\theta)(\cdot)$ is the function $\delta_{\gamma a}-y\delta_a+p\theta$. The triangle cost $\mathsf{TriCost}(Z;\,\tau,\phi;\,I;\,C,b)$ consists of the clauses (C1), (C2). $\mathsf{TrId}(\mathcal{J};\,u;\,p;\,\tau,\phi,k,\varepsilon)$: $\mathcal{S}$ contains the variables and the coefficients of the characteristic polynomials, $\mathcal{J}$ the families $p\,\mathrm{tr}\,u_x(g)$ and the tie relations of traces; $\mathsf{Rib}(\mathcal{J};\,\beta;\,\pi,c)$: $\beta\in\mathcal{S}$ vanishes on $\mathcal{X}$, and some $f\in\mathcal{S}$ with $f\beta\in\mathcal{J}$ has on $\mathcal{X}$ no monomial properly dividing $\prod X_j$ and the coefficient c of $\prod X_j$ at π.

LEAN `obeyingCocycles` · `RowBlockIndex` · `blockRep` · `crysRels` · `blockLoss` · `cmSign` · `IsRowType` · `lambdaLoss` · `loss` · `AbsGal` · `placesAboveG` · `badPlaces` · `Data.TameChar.invAvatar` · `crudeUnit` · `IsConjSelfDual` · `SelmerLengthLe` · `IsColumn` · `IsRow` · `chiOneInv` · `normalisedRep` · `twistedLabel` · `IsApt` · `tupleProj` · `HasRow` · `IsLabelledColumn` · `IsAuxiliary` · `evenClasses` · `IsNewPlaces` · `withConjInv` · `IsPrimeTriangle` · `IsAuxTwist` · `ribetNormaliser` · `HasTriangleCost` · `IsTraceIdeal` · `IsRibetElement` · defined in the first two notes of the unit

3 The definitions of this note

Definition 3.1 (tests and denominators)**.** Let E be a CM field as in the conventions, r, n finite sets, O an auxiliary pair of type (r,n) over E, e_0 an element of the complex group algebra of $\prod_i \mathrm{GL}_r(\mathbb{F}_{w_i})$, and Z a subspace of $\mathcal{F}(O)$.

(a) $\mathrm{act}_O(e_0)$ is the endomorphism of $\mathcal{F}(O)$ by which e_0 acts.

(b) A form $y\in\mathcal{F}(O)$ is *tested by* (O,e_0,Z) if for every finite place w of E with $\bar{w}\neq w$ that lies outside the places of O and their conjugates, y is orthogonal to

$$\mathrm{im}\,\mathrm{act}_O(e_0)\ \cap\ \ker H_w(X_w)\ \cap\ Z, \qquad X_w=\left(\nu_w\,\mathrm{N}(w)^{(\#n-\#r)-1}\right)^{-1},$$

with ν_w the Frobenius value of a_O in rank $\#r$ at w and the exponent truncated at 0.

(c) For a prime number l and $\Xi\in\mathcal{F}(O)$, $\mathsf{TestForm}(O;\,l;\,\Xi)$ holds if Ξ is integral at every ring homomorphism $\mathbb{C}\to\mathbb{C}_l$ and is tested by $(O,e^{\otimes}_{r,X-1},\ker\theta_O)$, with $e^{\otimes}_{r,X-1}$ the idempotent of the tuple of labels all equal to $X-1$.

(d) For an isomorphism of fields $\iota\colon\mathbb{C}\to\bar{\mathbb{Q}}_{\ell}$ and a real number β, $\mathsf{Den}_{\iota}(O,e_0,Z;\,\beta)$ holds if, with $d=\dim Z$, for all families $(y_a)_{a<d}$ and $(y'_b)_{b<d}$ of forms in $\mathcal{F}(O)$ such that every y_a is integral at every ring homomorphism $\mathbb{C}\to\mathbb{C}_{\ell}$, and every y'_b is integral at ι and tested by $(O,e_0,Z^{\perp})$:

$$\left|\iota(\det(\langle y_a,\mathrm{pr}_Z\,y'_b\rangle)_{a,b})\right|\ \le\ \exp(d\cdot\#r\cdot\beta).$$

LEAN `AuxPair.groupAlgAction` · `AuxPair.IsTested` · `AuxPair.IsTestForm` · `DenominatorLe`

Definition 3.2 (the Hecke algebra away from g and its eigenparts)**.** Let D be admissible data.

(a) For a finite place w of $E(D)$, $\mathsf{SplitOff}_D(w)$ holds if $\bar{w}\neq w$ and neither w nor $\bar{w}$ is a chosen place above $g(D)$.

(b) The *Hecke algebra away from g*, written $\mathbb{T}(D)$, is the algebra of the families $(T_V)_V$ of endomorphisms of the spaces $\mathcal{M}_D(V)$, indexed by the hermitian spaces V of rank $r(D)$ with a parameter s, that are given, on all V at once, by one element of the group algebra of the group of finite adelic points trivial at the chosen places.

(c) For a pair $x = (w, i)$ of a finite place and a natural number, $\mathrm{T}_D(x) \in \mathbb{T}(D)$ acts on every $\mathcal{M}_D(V)$ by the Hecke operator $T_{w,i}$ if $\mathsf{SplitOff}_D(w)$, and is 0 otherwise.

(d) For a space V and a function p from the pairs (w, i) to $\mathbb{C}$, the *p-part* $\mathrm{Eig}_D(V; p)$ is the space of the $y \in \mathcal{M}_D(V)$ on which each $\mathrm{T}_D(x)$ acts by the scalar $p(x)$.

(e) For V, p and a chosen place x above $g(D)$, $\mathrm{Lab}_D(V, p; x)$ is the set of the primitive labels c of degree $r(D)$ over the residue field of x whose idempotent, acting at x, does not kill $\mathrm{Eig}_D(V; p)$.

(f) For V and $T \in \mathbb{T}(D)$, $\mathrm{hg}_{D,V}(T)$ is the restriction of T_V to the good forms on V (it preserves them by Lemma 4.8: T_V commutes with the action of $e_0(D)$); $T \mapsto \mathrm{hg}_{D,V}(T)$ is multiplicative.

LEAN `Data.IsSplitOffLevel` · `Data.heckeAlgebra` · `Data.heckeOperator` · `Data.eigenPart` · `Data.occurringLabels` · `Data.heckeOnGood`

Definition 3.3 (the adjoint lifts)**.** Let D be admissible data, ϑ a tame character of D and j a natural number. By Lemma 4.11 there is a linear map $L_{\vartheta,j}$ from the forms of the auxiliary pair $O_D(\vartheta, j)$ to $\mathcal{M}_D(W_j(D))$ which, for $1 \le j \le m(D)$, is the theta lift $\theta_{O_D(\vartheta,j)}$; the formal text chooses one. For $F \in \mathcal{M}_D(W_j(D))$, $\theta^*_{D,\vartheta,j}(F)$ is the image of F under the adjoint of $L_{\vartheta,j}$, a form of $O_D(\vartheta, j)$; and $\theta^{*,\mathrm{good}}_{D,\vartheta,j}$ is the restriction of $\theta^*_{D,\vartheta,j}(\cdot)$ to the good forms on $W_j(D)$. For a natural number z, $\mathrm{I}^{(1)}(z)$ is the index set with $12 + 4 + 4 + (2z - 20)$ elements used for the group $\mathrm{U}(2z)$ ($2z$ elements when $z \ge 10$).

LEAN `Data.auxLift` · `Data.goodLift` · `Data.RankOneIndex`

Definition 3.4 (ramified pairs, pairs of rank one, the budget for congruences)**.** Let D be admissible data and p a pair of D. Then $\mathsf{Ram}(p)$ (*p is ramified*) holds if, for every finite place w, $\psi_{p,w} \neq 1$ exactly when w is chosen. For a natural number j, $\mathsf{RkOne}(p; j)$ (*p is of rank one for j*) holds if p has lift rank $\mathrm{I}^{(1)}(z_p)$ (Chapter VII), $m(D) < z_p \le r(D)$, $1 \le j \le m(D)$, $\psi_p = \omega_D^j$, and for every infinite place v: $c_p(v) \le w_{D,j}(v)$ and $w_{D,j}(v) + r(D) \le c_p(v) + t(D)$. The *budget for congruences* is $B_0(D) = \mathrm{budget}(D)/(m(D)\,k(D))$.

LEAN `Data.Pair.IsRamified` · `Data.Pair.IsRankOne` · `Data.B₀`

The predicate "fit for D" of Theorem A is the definition of the formal text: a family $\theta = (\theta_{\ell,\iota})$ of m-fold twists of D, indexed by the pairs (ℓ, ι), is fit if (F1), (F2) and (F3) hold; the three clauses are its three fields, and the box of Theorem A is its statement.

Definition 3.5 (the budget of a prime)**.** Let D be admissible data and ℓ a natural number. Let $X_D \neq 0$ be the integer that Input 4 provides for the conductor $N_1(D)d$, the number $t(D)$ and the field $E(D)$ (the formal text chooses one). Then

$$\mathrm{rb}_\ell(D) = h_2(D)^{27}\big(\mathrm{cr}_\ell(D) + v_\ell(X_D)\big) + 4\,f(D)\,\mathrm{loss}_D(\ell).$$

LEAN `rowBudget`

Definition 3.6 (reciprocity laws)**.** Let E be a CM field as in the conventions and $r' \in \mathbb{N}$.

(a) A Frobenius datum $x = (w_x, \mathfrak{P}_x, \phi_x)$ of E is *split for* r' if w_x does not contain ℓ and $\mathrm{cyc}_{\ell^{r'}}(\phi_x) = 1$.

(b) For a finite place w of E, w^+ is the place of E^+ below w.

(c) Let $e \in \mathbb{N}$, Φ a subset of G_E, T a finite set of finite places of E^+ and $z\colon G_E \to \mathbb{Z}/\ell^{r'}$ a function. Then z is a *law for* (r', e, Φ, T) if: z is locally constant; for all g and all a, b with $\mathrm{cyc}_{\ell^{r'}}(a) = \mathrm{cyc}_{\ell^{r'}}(b) = 1$,

$$z(gag^{-1}b) = \mathrm{cyc}_{\ell^{r'}}(g)^{e+1} z(a) + z(b);$$

$z(b^{\mathsf{c}}) = -z(b)$ for all such b; and for every finite family $(x_j)_{j\in J}$ of Frobenius data that are split for r', with the places $w^+_{x_j}$ pairwise distinct and outside T, every index j with $z(\phi_{x_i}) = 0$ for all $i \neq j$, every coefficient ring $\mathcal{O}$, every $c \in \mathrm{Ev}(\mathcal{O}, \{w_{x_i}\}_i; r', e)$ that vanishes on Φ and every natural number n' divisible by $2z(\phi_{x_j})$ in $\mathbb{Z}/\ell^{r'}$: $n' \cdot c(\tau_{\mathfrak{P}_{x_j}}) = 0$.

LEAN `IsSplitPlace` · `primeBelowPlus` · `IsLaw`

Definition 3.7 (central operators on forms of L-value data)**.** Let D be admissible data, n a finite set and Y' L-value data over $E(D)$ for n (Chapter VII), with auxiliary pair $O_{Y'}$. For z in the finite adelic points of $\mathrm{U}(1)$, $\mathrm{cen}_{Y'}(z)$ is the translation of the forms of $O_{Y'}$ by the central element z, and $\mathrm{tc}_{Y'}(z) = b_{Y'}(\det z) \cdot \mathrm{cen}_{Y'}(z)$, with $b_{Y'}$ the second character of the auxiliary pair.

LEAN `LValueData.centralRep` · `LValueData.twistedCentralOp`

4 Forms, Hecke operators and the test

Lemma 4.1 (small facts)**.** *Let O be an auxiliary pair of type (r, n) over a CM field E as in the conventions, and D admissible data.* (a) *For an automorphism σ of $\mathbb{C}$, forms φ, φ', a split place w outside the places of O and their conjugates and $X \in \mathbb{C}$: $\langle H_w(X)\varphi, \varphi'\rangle_\sigma = \langle \varphi, H_{\bar w}(X^*)\varphi'\rangle_\sigma$, with $X^* = \sigma^{-1}(\overline{\sigma(X)})$.* (b) *For $g \in \prod_i \mathrm{GL}_r(\mathbb{F}_{w_i})$, the scalar $\prod_i \psi_{O,w_i}(\det g_i)$ is a root of unity. The image of the action of a group algebra element is stable under every endomorphism that commutes with the action of the group; and a central idempotent acts self-adjointly for every $\langle\cdot,\cdot\rangle_\sigma$.* (c) *For a space V as in the conventions, a function p and $y \in \mathrm{Eig}_D(V; p)$, the translates of y by the group algebra of $\mathsf{G}(D)$ lie in $\mathrm{Eig}_D(V; p)$. If $\mathcal{M}_D(V)$ is finite-dimensional, then for every subspace N' only finitely many p have $N' \cap \mathrm{Eig}_D(V; p) \neq 0$.* (d) *For a finite place w of E: $\bar w^+ = w^+$.*

Proof. (a) is the adjoint formula for the Hecke operators at a split place (Chapter VII). (b): the characters $\psi_{O,w}$ have finite order; the second assertion was used in Lemma 5.8 of the first note; the third follows from Lemma 4.2. (c) follows from Lemma 4.8, and from the independence of the simultaneous eigenspaces. (d): an element of E^+ equals its conjugate. □

LEAN `LevelPlaces.LevelForm.innerAt_heckePolyOnForms` · `AuxPair.isOfFinOrder_prod_psi_det` · `AuxPair.range_groupAlgAction_mem_invtSubmodule` · `AuxPair.innerAt_groupAlgAction` · `Data.principalRep_mem_eigenPart` · `Data.finite_eigenPart` · `primeBelowPlus_star`

Lemma 4.2. *Let E be a CM field as in the conventions, r, n finite sets and O an auxiliary pair of type (r,n) over E. For every automorphism σ of $\mathbb{C}$, every $g \in \prod_i \mathrm{GL}_r(\mathbb{F}_{w_i})$ and all forms $x, y \in \mathcal{F}(O)$: $\langle g \cdot x, g \cdot y\rangle_\sigma = \langle x, y\rangle_\sigma$.*

Proof. The action of g is translation by a lift of g times the scalar $\prod_i \psi_{O,w_i}(\det g_i)$. Translation preserves the pairing, and the scalar contributes its absolute value squared after σ, which is 1 for a root of unity. □

LEAN `AuxPair.innerAt_formAction`

Lemma 4.3. *Let E, r, n, O be as in Lemma 4.2. Let w and u be finite places of E with $\bar{w} \neq w$ and $\bar{u} \neq u$, both outside the places of O and their conjugates, let $X, X' \in \mathbb{C}$ and $y \in \mathcal{F}(O)$. Then $H_w(X)H_u(X')y = H_u(X')H_w(X)y$.*

Proof. The Hecke operators $T_{w,i}$ and $T_{u,i'}$ commute on the vectors spherical at both places, and the two sides are the same double sum. □

LEAN `AuxPair.heckePolyOnForms_comm`

Lemma 4.4 (projection to a tested space does not increase norms)**.** *Let E, r, n, O be as in Lemma 4.2, with its Weil data. Let w be a finite place with $\bar{w} \neq w$ outside the places of O and their conjugates, e_0' an element of the group algebra and Z' a subspace of $\mathcal{F}(O)$. Suppose that e_0' is a central idempotent, that $Z' \subseteq \operatorname{im}\mathrm{act}_O(e_0') \cap \ker H_w(X_w)$, and that y is tested by $(O, e_0', Z'^{\perp})$. Then $\|\mathrm{pr}_{Z'} y\|_\sigma^2 \leq \|y\|_\sigma^2$ for every automorphism σ of $\mathbb{C}$.*

Proof. The test at w says that y is orthogonal to $\operatorname{im}\mathrm{act}_O(e_0') \cap \ker H_w(X_w) \cap Z'^{\perp}$. By Lemma 4.1(a),(b) the orthogonal complement of $\operatorname{im} e_0' \cap \ker H_w(X_w)$ is $\ker e_0' + \operatorname{im} H_{\bar{w}}(X_w^*)$; hence $y = z + x$ with $z \in Z'$ and $x \in \ker e_0' + \operatorname{im} H_{\bar{w}}(X_w^*)$. For every σ, the form $\langle\cdot,\cdot\rangle_\sigma$ pairs x to zero with Z' (again by the adjoint formulas, as Z' lies in the image of e_0' and is killed by $H_w(X_w)$; Lemma 4.3 is used to move the operators). So $\mathrm{pr}_{Z'} y = z$ and $\|y\|_\sigma^2 = \|z\|_\sigma^2 + \|x\|_\sigma^2 \geq \|z\|_\sigma^2$, the norm $\|x\|_\sigma^2$ being a sum of non-negative terms over the classes. □

LEAN `AuxPair.normSqAt_starProjection_le`

Lemma 4.5. *Let D be admissible data, V a space as in the conventions, p a function on the pairs (w,i), $y \in \mathrm{Eig}_D(V; p)$ and w a finite place with $\mathsf{SplitOff}_D(w)$. Then y is spherical at w, and the Hecke polynomial at w acts on y through $X \mapsto \sum_i (-X)^i\, \mathrm{N}(w)^{i(i-1)/2}\, p(w,i)$.*

Proof. y has principal level at the chosen places only, so it is fixed by the maximal compact subgroup at w, and $T_{w,i}$ acts on it by $p(w,i)$ by the definition of the p-part. □

LEAN `Data.isSphericalWith_of_mem_eigenPart`

Lemma 4.6. *Let P be parameters, D admissible data for P and x a chosen place above $g(D)$, with residue field of q elements. Then $9 \mid q-1$, $M_\omega(P,\mu(D))^2 \mid q-1$ and $M_\omega(P,\mu(D)) + 1 < q$, where $M_\omega(P,\mu(D)) = (N_0 d)^{\mu(D)+6}$ is the integer of Chapter III.*

Proof. $q = g_i(D)$ for the prime below x, and the three facts are properties of the primes $g_i(D)$ of admissible data (Chapter III). □

LEAN `Data.dvd_card_resField_sub_one`

Lemma 4.7 (separating eigensystems)**.** *Let D be admissible data, q' a function on the pairs (w,i) and F a finite set of such functions. Then there is $b \in \mathbb{T}(D)$ such that for every $d' \in F \cup \{q'\}$, every space V as in the conventions and every $y \in \mathrm{Eig}_D(V;\, d')$: $b_V(y) = y$ if $d' = q'$, and $b_V(y) = 0$ otherwise.*

Proof. By induction on F. For a further system $e' \neq q'$ choose x with $e'(x) \neq q'(x)$ and replace b by $b - (e'(x) - q'(x))^{-1}\, b\, (\mathrm{T}_D(x) - q'(x))$: it acts as before on the parts where b acts by 0 or on the q'-part, and kills the e'-part. □

LEAN `Data.exists_hecke_proj_eigenPart`

Lemma 4.8. *Let D be admissible data, V a space as in the conventions, $T \in \mathbb{T}(D)$ and x an element of the group algebra of $\mathsf{G}(D)$. Then T_V commutes with the action of x on $\mathcal{M}_D(V)$.*

Proof. T_V is given by translations by elements trivial at the chosen places, x by translations by elements supported at the chosen places; such translations commute. □

LEAN `Data.commute_heckeAlgebra_principalRep`

Lemma 4.9. *Let D be admissible data and V a space as in the conventions such that $\mathcal{M}_D(V)$ is finite-dimensional. Then $\mathcal{M}_D(V)$ is the sum of the spaces $\mathrm{Eig}_D(V;\, p)$ over all functions p.*

Proof. Right translations are unitary for the Petersson product, and the commutation and adjoint formulas for the spherical Hecke operators at the places w with $\mathsf{SplitOff}_D(w)$ make them a commuting family of normal operators on the finite-dimensional space $\mathcal{M}_D(V)$ (the adjoint of an operator at w is one at $\bar{w}$). Such a family has a basis of joint eigenvectors (an input of the unit on the Weil representation), and a joint eigenvector lies in the p-part of its eigenvalues. □

LEAN `Data.iSup_eigenPart`

Lemma 4.10 (the theta lift and the action of the chosen group)**.** *Let E, r, n, O be as in Lemma 4.2, with its Weil data. Let k be an element of the stabiliser of the standard lattice of the hermitian space of O and s' an element of $\prod_i \mathrm{GL}_r(\mathbb{F}_{w_i})$ such that k has the reduction s' at the chosen places. Then for every form $\Xi \in \mathcal{F}(O)$ and every adelic point g: $(\theta_O \Xi)(g\,k) = \theta_O(s' \cdot \Xi)(g)$.*

Proof. Write the lift as the pairing of Ξ with the theta kernel. The pairing is equivariant, and the Weil representation of k acts on the kernel, in the variable of Ξ, through the reduction s' and the character that enters the action of the chosen group (an input of the unit on forms, Chapter VII). □

LEAN `AuxPair.theta_mul_eq_theta_formAction`

Lemma 4.11. *Let D be admissible data, ϑ a tame character of D and j a natural number. Then there is a linear map L' from $\mathcal{F}(O_D(\vartheta, j))$ to $\mathcal{M}_D(W_j(D))$ such that, if $1 \leq j \leq m(D)$, then $L'(\Xi) = \theta_{O_D(\vartheta,j)}\Xi$ for every Ξ.*

Proof. If j is not a lift index, take $L' = 0$. Otherwise the theta lift of an algebraic form of the auxiliary pair is a form of the right type on $W_j(D)$ (Chapter VII), and it has principal level at the chosen places by Lemma 4.10 with $s' = 1$. □

LEAN `Data.exists_theta_to_principalForms`

5 Rows exist, and the twist for rows: proofs of Theorems E and F

Lemma 5.1 (one row serves all twists)**.** *Let D be admissible data, ℓ a prime number and $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ an isomorphism of fields, r, n finite sets, O an auxiliary pair of type (r,n) over $E(D)$, and c a family of polynomials with $\mathsf{Apt}(O,c;\,1)$, 1 the trivial twist. Let b be Hecke data (a function of a finite place and a natural number), $\mathcal{H}$ a type and $R\colon G_{E(D)} \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$ a homomorphism. Suppose that $\mathsf{LabCol}(O;\,\#r,\mathcal{H},\emptyset;\,c;\,R)$; that for every $g \in G_{E(D)}$, $\iota(b(w,1))$ tends to $\operatorname{tr} R(g)$ along the Frobenius data $(w,\mathfrak{P},\phi)$ with ϕ near g (in the filter of most Frobenius data near g); and that $\mathsf{RowType}_D(\#r,\#n-\#r;\,\mathcal{H}^\flat)$, where $\mathcal{H}^\flat(\sigma)$ is $\mathcal{H}(\sigma)$ shifted by $-((\#r-(\#n-\#r))/2-\operatorname{sgn}_{\mathrm{CM}}(\sigma)\,e_O([\sigma]))$, with e_O the weight of μ_O. Then there is a homomorphism $T\colon G_{E(D)} \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$ with $\mathsf{HasRow}(O,b;\,T)$.*

Proof. By (A3), $\#n$ is even; put $a' = (\#r-(\#n-\#r))/2$. Let T be the twist of R by the avatar of $\mu_O^{-1}[0,a']$. By Lemma 9.4 of the first note it is a labelled column with exponent $\#n-\#r$, the type $\mathcal{H}^\flat$ and the labels of c scaled by the name of the tame part of μ_O^{-1}, that is the twisted labels $\mathrm{tl}_O(c,\cdot)$; by Theorem B of the first note and the hypothesis on $\mathcal{H}^\flat$ it is a row. For the trace identity of $\mathsf{HasRow}(O,b;\,T)$, let ϑ, V, y, H, S be as there. Both sides are continuous, so it suffices to compare limits along Frobenius data near a given g. At a datum $(w,\mathfrak{P},\phi)$ with w split, prime to ℓ and off the places, the trace of $S(\phi)$ is close to minus the second coefficient of the Hecke polynomial of y, which is that of the string $\prod_{i<\#n-\#r}(1-q^i\nu Y)$ times the polynomial of $b(w,\cdot)$ at $q^{\#n-\#r}\nu Y/\nu'$ (the definition of lifting, with Y the variable; $q=\mathrm{N}(w)$, and ν,ν' the Frobenius values of a_V in the two ranks). Multiplying by $\chi_{1,V}^{-1}(\phi)=\iota(\nu)^{-1}$ (Lemma 5.4 of the first note), the string contributes $\sum_{e<\#n-\#r} q^e = \sum_e \mathrm{cyc}(\phi)^{-e}$, and the other factor contributes $\iota(b(w,1))$ times the values at ϕ of the avatars of ϑ^{-1} and of $\mu_O^{-1}[0,a']$; this is a computation with the values of the characters at the uniformiser. Passing to the limit gives $\mathrm{av}_\vartheta^-(g)\operatorname{tr} T(g) + \sum_e \mathrm{cyc}(g)^{-e}$. □

LEAN `exists_row_for_all_twists`

Proof of Theorem E. Let z be a non-zero good form on $W_{m(D)}(D)$ in the p-part. Its image under the passage to algebraic forms (Chapter VII) is a form of some weight κ on the standard definite space of rank $r(D)$, of principal level at the chosen places, and an eigenform whose Hecke polynomial at w with $\mathsf{SplitOff}_D(w)$ is that of $p(w,\cdot)$ (Lemma 4.5). As $e_0(D)z = z \neq 0$ and $e_0(D)$ is the sum of the $e^{\otimes}_{r,c}$ over the good tuple families, some good family c has $e^{\otimes}_{r,c}z \neq 0$; by Lemma 6.1(a) of the first note, $\mathsf{Apt}(O_D(1,j),c;\,1)$. Let R be a representation attached to z (Input 3 of the first note). At each chosen place the form z has the parahoric type cut out by the corresponding factor of $e^{\otimes}_{r,c}$, hence the product label of c_i with no further blocks; by Lemma 9.5 of the first note, $\mathsf{LabCol}(O_D(1,j);\,r(D),\mathcal{H}_{\kappa,r(D)},\emptyset;\,c;\,R)$. The traces of R are the limits of the Hecke values $\iota(p(w,1))$ (attachment), and the shifted type satisfies the conditions of a row type: this is the comparison of the Harish-Chandra parameter of κ with the weights of the data, an input of the unit on the rows (Chapter VII). Lemma 5.1 gives T. □

Proof of Theorem F. By Lemma 5.5(c) of the first note there is a chosen place w in $\Sigma_g(D)\setminus\Sigma_\ell(E(D))$ with $\mathrm{N}(w)=g_i(D)$. For each ramified pair p and sign s, Theorem D of the first note gives a family $(\nu^{p,s},\mathrm{val}^{p,s})$ for $\Sigma_{\mathrm{bad}}(D,\ell)$; there are finitely many pairs. By Theorem C of the second note there are a coefficient

ring $\mathcal{O}$, lattices u_i of the R_i and γ_i, β_i such that u_i is balanced, in the sense of Definition 3.33 of the first note, for the prime ℓ, the set $\Sigma = \Sigma_\ell(E(D)) \cup \Sigma_g(D)$, the relations $\mathcal{R}_{\mathrm{cris}}(u_i)$, the pair (γ_i, β_i) and the precision M, and with the normalised length of $\beta_i + \sum_b v_{\mathcal{O}}(\det(u_i(\tau_w^{[b]}) - 1))$ at most $10f(D)\#r_i\,\mathrm{cr}_\ell(D)$. Apply Theorem A of the second note with $S = \Sigma$, $S_0 = \Sigma_{\mathrm{bad}}(D, \ell)$, $z = \tau_w$, the bounds $B_i = 10f(D)\#r_i\mathrm{cr}_\ell(D)$, $c = \mathrm{cr}_\ell(D)$ (by Theorem D(1) of the first note and Lemma 5.7(a) there, $v_\ell(\nu_i(\tau_w) - 1) \le v_\ell(\mathrm{N}(w) - 1) \le \mathrm{cr}_\ell(D)$), $\eta = \mathrm{cr}_\ell(D) > 0$ and $m = m(D)$. It gives a finite set Q and ψ such that for every power $1 \le d \le m(D)$ (the lift index j of the statement): $\mathsf{AuxTw}(\Sigma_{\mathrm{bad}}(D, \ell), Q; \psi^d, \mathrm{cr}_\ell(D))$, the values of the families at (Q, ψ^d) have valuations at most $v_\ell(\mathrm{val}(\emptyset, 1)) + 9\mathrm{cr}_\ell(D)$, and the Selmer modules of the $\psi^d u_i$ modulo ℓ^M have normalised length at most $(10f(D)\#r_i + 1)\mathrm{cr}_\ell(D)$ over all large coefficient rings. By Lemma 8.3 of the first note, $\psi = \mathrm{av}_\tau^-(\cdot)$ for a tame character τ with $Q(\tau) = Q$, and then $\psi^d = \mathrm{av}_{\tau^d}^-(\cdot)$. The character τ is an m-fold twist: if $\psi^d_{\tau,w'} = 1$ at some $w' \in Q$ and some $1 \le d \le m(D)$, then ψ^d would be trivial at the tame generator at w', so that $v_\ell(\psi^d(\sigma_{w'}) - 1) = \infty$, against the bound $\mathrm{cr}_\ell(D)$ in the auxiliary twist for that power d. This uses conclusion (2) of Theorem A of the second note for every power $d \le m(D)$, not only for $d = 1$; its proof there chooses the roots of unity so that all $\zeta_v^d - 1$ have small valuation. Now (1) holds by construction; (3) follows from Theorem D(2) of the first note with $a = 9\mathrm{cr}_\ell(D)$; and (2) is the Selmer bound, read as $\mathsf{Short}(\mathrm{av}_\vartheta^-(\cdot), R_i; Q(\vartheta), M; \gamma_i, y_i)$ with y_i the scalar by which $u_i(\gamma_i)$ acts: the Selmer module with the relations $\mathcal{R}_{\mathrm{cris}}(u_i) \cup \{\delta_{\gamma_i}\}$ carried to a larger coefficient ring contains the cocycles vanishing at γ_i in the Selmer module with the relations $\mathcal{R}_{\mathrm{cris}}(\psi' u_i)$, because the crystalline cocycles are compatible with the extension of the coefficients and ψ' is trivial on the decomposition groups above ℓ; $\psi'(\gamma_i) = 1$ because γ_i is inertial at a place of Σ prime to ℓ, where a character from $\mathrm{U}(1)$ with modulus Q is trivial. □

6 Ribet's element for columns: proof of Theorem C

Lemma 6.1 (obeying cocycles are Selmer cocycles)**.** *Let D be admissible data, $\mathcal{O}$ a coefficient ring, Y, n, r finite sets, $u = (u_x)_{x \in Y}$ homomorphisms $u_x \colon G_{E(D)} \to M_n(\mathcal{O})$, S a set of finite places of $E(D)$, $\varrho \colon G_{E(D)} \to M_r(\mathcal{O})$ a homomorphism, $b \in \mathcal{O}$ nonzero, L a set of relations and $c \colon G_{E(D)} \to (\mathcal{O}/b)^r$ a function. Suppose that every u_x is continuous, that $u_x(g) = 1$ for all $g \in I_{S^\complement}$ and all x, that $c \in \mathcal{Z}_u(\varrho; b)$ and that $c \in \mathrm{Ob}(L; \mathcal{O}/b)$. Then $c \in \mathrm{Sel}_S^L(\varrho; \mathcal{O}/b)$.*

Proof. By Lemma 4.1 of the first note there are k and a module Y_0, with injective first projection, containing all $((u_x(g) - 1)_x + b^k v, c(g))$. If $u_x(g) = 1$ for all x, then $(0, c(g)) \in Y_0$, so $c(g) = 0$: c is unramified outside S. If $u_x(h) \equiv u_x(g)$ modulo b^k for all x, the two points have the same first coordinate for a suitable v, so $c(h) = c(g)$; this condition is open because the u_x are continuous: c is locally constant. □

LEAN `obeyingCocycles.mem_selmerModule`

Lemma 6.2 (doubling the tied primes)**.** *Let D be admissible data, $\mathcal{O}$ a coefficient ring, K, Y sets, n a finite set, $t, \nu \in \mathbb{N}$, $w = (w_k)_{k \in K}$ finite places of $E(D)$, $\tau, \phi \colon K \to G_{E(D)}$, $u = (u_x)_{x \in Y}$ homomorphisms $G_{E(D)} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$, $\mathcal{S}$ a subalgebra and $\mathcal{J}$ an $\mathcal{S}$-submodule of the functions $Y \to \bar{\mathbb{Q}}_\ell[X_k : k \in K]$, and $\varepsilon = (\varepsilon_k)_k$ polynomials. Suppose $\mathsf{New}_\nu(w, \tau, \phi)$, $\mathsf{CSD}_t(u_x)$ for all x, and $\mathsf{TrId}(\mathcal{J}; u; \ell^\nu; \tau, \phi, \mathrm{id}, \varepsilon)$. Then*

$\mathsf{TrId}(\mathcal{J};\, u;\, \ell^\nu;\, \tau^{[\cdot]}, \phi^{[\cdot]}, \mathrm{pr}_1, \varepsilon)$*, where the families are indexed by the pairs* (k, b) *of an index and a sign and* $\mathrm{pr}_1(k, b) = k$*.*

Proof. Only the relations at the indices $(k, -)$ are new. By conjugate self-duality, the trace of $u_x((h^{[-]})^i g)$ is $\mathrm{cyc}(\cdot)^{-(t-1)}$ of that element times the trace of $u_x(h^i (g^{\mathsf{c}})^{-1})$. By Lemma 7.2 of the first note, $\mathrm{cyc}(\tau_k^{[-]}) = 1$ and $\mathrm{cyc}(\phi_k^{[-]})^{-e} \equiv 1$ modulo ℓ^ν. So the new relation at g is $\mathrm{cyc}(g)^{-(t-1)}$ times the old relation at $(g^{\mathsf{c}})^{-1}$, up to X_k times a combination of the families $\ell^\nu \operatorname{tr} u_x(\cdot)$; all of these lie in $\mathcal{J}$. □

LEAN `isTraceIdeal_withConjInv`

Lemma 6.3 (crystalline losses above ℓ are affordable)**.** *Let* D *be admissible data,* Y*,* n*,* r *finite sets,* $t \in \mathbb{N}$*,* V *an auxiliary pair of type* (r, n) *over* $E(D)$*,* ϑ *a twist and* c *a family of polynomials with* $\mathsf{Apt}(V, c; \vartheta)$*. Let* $\mathcal{H}$*,* $\mathcal{H}'$ *be types,* S *a set of finite places,* $R \colon G_{E(D)} \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$ *a homomorphism with* $\mathsf{RowRep}_D(t, \mathcal{H};\, R)$*, and* $R' = (R'_x)_{x \in Y}$ *homomorphisms* $G_{E(D)} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ *with* $\mathsf{LabCol}(V;\, t, \mathcal{H}', S;\, (\mathrm{tl}_V(c, i)(X-1)^t)_i;\, R'_x)$ *for all* x*. Let* ϵ *be a bijection from the disjoint union of the sets* $\mathrm{I}_r(i)$*,* $i \in \{*\} \sqcup \{0, \dots, t-1\}$*, to* n*, and* $\pi \in Y$*. Then for all large* $\mathcal{O}$*: for all homomorphisms* $T, T_0 \colon G_{E(D)} \to M_r(\mathcal{O})$*, every family* $u = (u_x)_{x \in Y}$ *of homomorphisms* $G_{E(D)} \to M_n(\mathcal{O})$ *and every family* $(M_i)_i$ *of natural numbers such that* $u_\pi(g)$ *is the block-diagonal matrix of the* $\mathrm{blk}_T(i)(g)$ *transported by* ϵ*, every* u_x *is a lattice of* R'_x*,* T_0 *is a lattice of* R*, and* T *and* T_0 *agree on* $G_{\mathfrak{P}_v}$ *for every* $v \in \Sigma_\ell(E(D))$*:*

$$\sum_i \mathrm{nlen}_{\mathcal{O}}(\mathrm{bl}(\mathrm{blk}_T(i)(\cdot);\, u, M_i)) \;\le\; 4\,\#r \cdot f(D) \cdot \mathrm{loss}_D(\ell).$$

Proof. Let $\mathcal{O}$ be large enough for Theorem E of the first note at every place v above ℓ, with the chart of v. A function obeys the crystalline relations of a block if and only if it obeys, for each v, the relations of the crystalline cocycles at v (Lemma 4.2(a) of the first note); so each block loss is at most the sum over v of the lengths of the obeying cocycles modulo those obeying the relations at v. By Lemma 7.3 of the second note, applied to the restrictions to $G_{\mathfrak{P}_v}$ and to a class that contains the local lattices of all u_x and of the block, this is at most the length of the crystalline cocycles of the twisted dual of the block modulo the cup-orthogonal of the class. The class is that of Lemma 7.2 of the second note: by that lemma the local lattices of the labelled columns u_x belong to it, with the jumps of $\mathcal{H}'$, the tame type at the places above $g(D)$ coming from Lemma 9.2 of the first note and Lemma 7.1 of the second (the eigenvalues of inertia are roots of unity of bounded order); the block T agrees above ℓ with a lattice of the row, and the string blocks have the single jump e. Theorem E of the first note bounds each term by $[G_{\mathbb{Q}_\ell} : j_v(G_{\mathfrak{P}_v})] \cdot \mathrm{rk} \cdot \mathrm{loss}_D(\ell)/x$, with $x = 1$ for the row block of rank $\#r$ and $x = t(D)$ for each of the $t \le t(D)$ string blocks. Summing over the blocks gives at most $2\#r\ \mathrm{loss}_D(\ell)$ times the index, and the sum of the indices over v is at most the degree $[E(D) : \mathbb{Q}] = 2f(D)$ (Lemma 4.4 of the first note). □

LEAN `sum_blockLoss_le`

Proof of Theorem C. The formal proof has a hundred lines; this is its outline. *Lattices.* Choose one coefficient ring $\mathcal{O}'$ over which: each row R_x, $x \in \mathcal{X}$, has a lattice with the Selmer bound of $\mathsf{Short}(\psi, R_x;\, Q, m_0;\, \gamma_x, y_x)$; each column u_x has a lattice u'_x; and for $x \in \mathcal{X}$ this lattice is block diagonal, a lattice T_x of ψR_x and the string $\mathcal{O}'(0), \dots, \mathcal{O}'(-(t_V - 1))$. The last point is the theorem of Brauer and Nesbitt applied

to the trace identity of the hypothesis: the row is absolutely irreducible because its lattices span all matrices (W3), and the characters cyc^{-e} are distinct. By Lemma 6.3 the block losses of T_π and of the string, at any moduli, total at most $4\#r\, f(D)\, \mathrm{loss}_D(\ell)$.

The application of Ribet's lemma. Apply Theorem D of the second note over $A = \mathcal{O}'$, with the points Y, the subset $\mathcal{X}$, the block $\varrho_x = T_x$, the characters $\xi_e = \mathrm{cyc}^{-e}$ ($\xi_0 = 1$), $p = \ell^\nu$, the doubled families $\tau^{[\cdot]}$, $\phi^{[\cdot]}$ with the trace ideal of Lemma 6.2, the element a of the triangle with $C = \ell^{h_2(D)h}$ (clause (T3)), $p_0 = \ell^{m_0}$, the elements γ_x and scalars y_x of the Selmer bounds ($y_x - 1$ a unit, $\mathrm{cyc}(\gamma_x) = 1$), and, as elements δ_v with polynomials o_v, two kinds of inertial elements, each with both signs. The first are the tame generators at the places of Q: they are killed by $(X - \psi)(X - 1)$, and $o_v = X - \psi(\delta_v)$; the values $o_v(1)$ have total valuation at most $2\mathrm{cr}_\ell(D)$ by the auxiliary twist. The second are the generators that Lemma 9.2 of the first note provides for the inertia above $g(D)$ off ℓ. Their images under every u_x are killed by $X^{\mathrm{ord}_{\mathrm{ch}}(D)} - 1$: the eigenvalues are roots of unity of orders dividing some $g_i(D)^d - 1$ with $1 \le d \le r(D)$ (Lemma 9.2(1) of the first note), and these numbers divide $\mathrm{ord}_{\mathrm{ch}}(D)$ (Lemma 5.5(a) there). So

$$o_v = 1 + X + \cdots + X^{\mathrm{ord}_{\mathrm{ch}}(D)-1} \qquad \text{satisfies} \qquad (o_v \cdot (X-1))(u_x(\delta_v)) = 0,$$

and $o_v(1) = \mathrm{ord}_{\mathrm{ch}}(D)$ has valuation at most $\mathrm{cr}_\ell(D)$. (The killing polynomial of Lemma 9.2 divided by $X - 1$ would not do: for a labelled column it contains the root 1 with the multiplicity t_V of the string, so the quotient vanishes at 1 as soon as $t_V \ge 2$.) Hypothesis (e) holds with b_0 the Selmer length bound plus the loss of the row block: by Lemma 6.1 the obeying cocycles modulo ℓ^{m_0} that obey the crystalline relations and vanish at the tame generators are Selmer cocycles for T_π unramified outside $\Sigma_\ell(E(D)) \cup \Sigma_g(D) \cup Q \cup \bar{Q}$ (Lemma 7.3 of the first note), and $m_0 > r(D)\, \mathrm{rb}_\ell(D)$ exceeds that bound. Hypothesis (g) holds by Theorem E of the second note for the prime triangle, with b_e the loss of the e-th string block plus the length of $\mathrm{Sha}^{\mathrm{odd}}(\ell, E(D), \mathcal{O}', e, \nu)$, which is at most the valuation of the integer X_D (Input 4), and with the finite set

$$s' = s \cup \{\tau_k^{[b]} : k \in K,\ b = \pm\}, \qquad s \text{ the set of clause (T2)},$$

a cocycle that vanishes at the $\tau_k^{[b]}$ is unramified at the w_k and their conjugates (Lemma 7.3 of the first note), and then (T2) detects it on s.

The bound. Theorem D of the second note gives c with $\mathsf{Rib}(\mathcal{J};\, \beta;\, \pi, c)$ and

$$v_\ell(c) \le b_0 + \sum_e b_e + t_V\Big(\sum_v v_\ell(o_v(1)) + 2\#s' \cdot h_2(D)h\Big) + (\#K + 1)h_2(D)h,$$

where $\#s' \le \#s + 2\#K$. Each term is at most a multiple of $\mathrm{cr}_\ell(D)$, of $v_\ell(X_D)$ or of $\mathrm{loss}_D(\ell)$. The Selmer bound is $(10f(D)\#r + 1)\mathrm{cr}_\ell(D)$. The losses total $4\#r f(D)\, \mathrm{loss}_D(\ell)$. The number h is at most $v_\ell(\mathrm{ord}_{\mathrm{ch}}(D))$, hence at most $\mathrm{cr}_\ell(D)$, and $(\#K + \#s)h_2(D)h \le h_2(D)^{22}h$ by (T2). The sum of the $v_\ell(o_v(1))$ is at most $2\mathrm{cr}_\ell(D)$ for the places of Q plus $\mathrm{cr}_\ell(D)$ for each of the at most $2 \cdot 4f(D)$ generators above $g(D)$ (Lemma 5.5(b) of the first note). Finally $t_V \le t(D)$ and $f(D)$ are bounded by powers of $h_2(D)$ (Chapter III). With $3v_\ell(\mathrm{ord}_{\mathrm{ch}}(D))$ for the factor $\mathrm{ord}_{\mathrm{ch}}(D)^3$, the total is at most $\#r\,(h_2(D)^{27}(\mathrm{cr}_\ell(D) + v_\ell(X_D)) + 4f(D)\, \mathrm{loss}_D(\ell))$. The comparison of the constants was not re-derived here. □

Input 4 (a bound for the odd part of Tate–Shafarevich groups of abelian CM fields; background library, unit of the cyclotomic fields). Let $N \geq 1$ and t be natural numbers (t may be 0) and E a CM field whose fixed embedding lands in the cyclotomic field of conductor N. Then there is a natural number $X \neq 0$ with $X \leq 2^{4(N+t+3)^{27}} \cdot h_E$, h_E the class number, such that for every prime l, every discrete valuation ring A of characteristic 0 that is finite and free over $\mathbb{Z}_l$, has finite quotients and contains the roots of unity of order the exponent of $(\mathbb{Z}/N)^\times$, every $e < t$ and every n': $\operatorname{len}_A \operatorname{Sha}^{\mathrm{odd}}(l, E, A, e, n') \leq v_A(X)$. For $e = 0$ the module is cut out by vanishing on the inertia above l, with no oddness condition; for $e > 0$ by oddness under conjugation and vanishing on the kernel of the cyclotomic character in the decomposition groups above l. *References* (second reading): C. Greither, *Class groups of abelian fields, and the main conjecture*, Theorem 3.2, p. 469; B. Mazur, A. Wiles, *Class fields of abelian extensions of* $\mathbb{Q}$, pp. 179–184. *Relation:* an explicit finite-level estimate. Greither's theorem is an identity of characteristic ideals in an Iwasawa algebra; the single integer, uniform in the prime, the twist and the precision, and its explicit size need further estimates, which the library carries out (it treats $e = 0$ separately and uses Stickelberger bounds and finite detecting sets). No printed statement with this uniform bound has been identified.

7 Reciprocity laws and prime triangles: proof of Theorem D

Lemma 7.1. *Let E be a CM field as in the conventions, $\mathcal{O}$ a coefficient ring, S a set of finite places, $r', e \in \mathbb{N}$, $c \in \mathrm{Ev}(\mathcal{O}, S;\, r', e)$ and $b \in \mathcal{O}/\ell^{r'}$. Suppose that $u'(b) = 0$ for every additive map $u' \colon \mathcal{O}/\ell^{r'} \to \mathbb{Z}/\ell^{r'}$ such that $u' \circ c$ is a continuous cocycle of G_E for the character $\mathrm{cyc}_{\ell^{r'}}^{-e}$. Then $b = 0$.*

Proof. The module $\mathcal{O}/\ell^{r'}$ is killed by $\ell^{r'}$, so the additive maps to $\mathbb{Z}/\ell^{r'}$ separate its points; and for every such map $u' \circ c$ is a continuous cocycle for $\mathrm{cyc}_{\ell^{r'}}^{-e}$, because c is a locally constant cocycle for $\mathcal{O}(-e)$ and u' is $\mathbb{Z}$-linear. □

LEAN `eq_zero_of_mem_evenClasses`

Lemma 7.2 (a reciprocity law with prescribed zeros)**.** *Let E be a CM field as in the conventions, r', e, k natural numbers, I a finite set, $(x_i)_{i \in I}$ Frobenius data of E and $(y_i)_{i<k}$ elements of G_E. Suppose that $0 < r'$, that E has at least four infinite places, that $k \leq \#I$, that every x_i is split for r', and that the places $w_{x_i}^+$ are pairwise distinct. Then there are a function $z \colon G_E \to \mathbb{Z}/\ell^{r'}$ and a finite set T of finite places of E^+ such that z is a law for $(r', e, \{\phi_{x_i}\}_i, T)$, $z(y_i) = 0$ for all $i < k$, and $z(g)$ is a unit for some g with $\mathrm{cyc}_{\ell^{r'}}(g) = 1$.*

Proof. Outline of a formal proof of 140 lines. Pass to E^+, a totally real field: G_E has index 2 in G_{E^+} and contains no complex conjugation; choose a section of the restriction and let $M' = \mathbb{Z}/\ell^{r'}$ with G_{E^+} acting through the corresponding lift of cyc^{-e}, which is trivial on complex conjugations. Apply the relations at split primes (Input 1 of the second note, with $K = E^+$, $m = \ell^{r'}$ and $X = M'$; its Selmer groups carry the condition of vanishing on the complex conjugations, which is not empty here). The set S consists of the places of E^+ above ℓ and the places $w_{x_i}^+$; the local condition at a place above ℓ is the vanishing on the kernel of the character, and at

the places $w^{+}_{x_i}$ it is the strict one: vanishing on the whole decomposition group (not "no condition"). The subgroup U' is an open normal subgroup inside the kernel of the character and of $\mathrm{cyc}_{\ell r'}$. This gives a group of relation families (n_i) among the values of Selmer cocycles at the elements γ_i, each of which extends, by Input 1(b), to an equivariant family on U' that satisfies the reciprocity relations at further split places outside the finite set T'. A count of the ℓ-torsion of the Selmer group (the same input for $M'[\ell]$ with two further primes from Chebotarev's theorem, a procyclic generator of the character, the hypotheses $k \le \#I$ and on the four infinite places) shows that the relation families have rank greater than k; so there is a primitive one, with a unit entry, that vanishes at the y_i. Let $z(g)$ be its component at the class of the restriction of g^{-1}. Local constancy, the rule for $gag^{-1}b$ and the oddness come from the equivariance of the extended family; the reciprocity clause is the relation of the family at new split places, applied to the cocycle obtained from an even class through Lemma 7.1 (the factor 2 comes from the passage from E to E^{+}). □

LEAN `exists_isLaw`

Proof of Theorem D. Outline of a formal proof of a hundred lines. By Input 5 there is a finite set $\mathfrak{p} \subseteq G_E$ with at most y^{20} elements that detects the cocycles of all $\mathcal{O}(-e)$ unramified outside ℓ. Let J be the set of the pairs (e, i) with $0 < e < t$ and $i \in \mathfrak{p}$, in lexicographic order, and ty the first projection. Fix $\nu > 0$. By recursion along J one chooses, for $j = (e, i_0)$, a Frobenius datum $(w_j, \mathfrak{P}_j, \phi_j) \in W$, a law z_j of twist e (Lemma 7.2) and its exceptional set, as follows. The law z_j is chosen to vanish at points that realise a basis of the values of the earlier laws of the same twist, and to be a unit at some point. An element u' with $\mathrm{cyc}_{\ell^\nu}(u') = 1$ is found at which every earlier law vanishes (the laws of another twist e' by the factor $m^e - m^{e'}$, with m the class of $\mathrm{cyc}(\gamma)$), while $z_j(u')$ is a unit times $\prod_{e' \neq e}(m^e - m^{e'})$. By the hypothesis on W there is a Frobenius datum in W whose Frobenius element is close to $\delta^{\ell^\nu} \cdot g_{i_0} g_{i_0}^{\mathsf{c}} \cdot u'(u'^{\mathsf{c}})^{-1}$, with $g_i = i^{-1}\gamma\, i\, \gamma^{-1}$, outside the earlier primes and exceptional sets; there $z_j(\phi_j) = 2 z_j(u')$, the earlier laws vanish, and every even class c of twist e satisfies $2c(g_{i_0}) = c(\phi_j)$. The places are Taylor–Wiles places for ℓ^ν because the Frobenius elements fix the ℓ^ν-th roots of unity. The clauses: (T1) by construction; (T2) with $s = \mathfrak{p}$, the cost $(\#J + \#\mathfrak{p})yh \le y^{22}h$ following from $\#\mathfrak{p} \le y^{20}$ and $t < y$; (T3) from the hypothesis that $\mathrm{cyc}(\gamma)^a - 1$ divides ℓ^h, with room for the factor ℓ^{3h}; (T4) from the reciprocity clauses of the laws, the divisor $2z_j(\phi_j)$ dividing ℓ^{yh}; (T5) from the detection by $\mathfrak{p}$ and Sah's lemma: an even class that vanishes at the ϕ_j of type e vanishes, up to the factor 2, at the g_i, $i \in \mathfrak{p}$, which gives the coboundary identity $\delta(a) \cdot 2z(g) = \delta(g) \cdot 2z(a)$ with $a = \gamma^{-1}$. □

Input 5 (finitely many points detect the cocycles unramified outside ℓ; background library, unit of the cyclotomic fields). Let $N \ge 1$, E a number field whose fixed embedding lands in the cyclotomic field of conductor N, and l a prime. Then there is a finite set $\mathfrak{p} \subseteq G_E$, with at most y^{20} elements for every y with $N \le y$ and $l \le y^2$, such that for every discrete valuation ring A of characteristic 0 that is finite and free over $\mathbb{Z}_l$ with finite quotients, and all $n', e \in \mathbb{N}$, a cocycle in $\mathrm{Sel}^{\emptyset}_{\Sigma_l(E)}(A(-e);\, A/l^{n'})$ that vanishes on $\mathfrak{p}$ is zero. *Reference:* none; the catalogue of this write-up records that the library proves it by generators of ray class groups and a count through Minkowski's bound, and that no published source is given.

8 The denominator bound: proof of Theorem B

Lemma 8.1. *Let P be parameters and D admissible data for P. Then there are a natural number $h \le v_\ell(\mathrm{ord}_{\mathrm{ch}}(D))$, with $\ell \le h_2(D)^2$ or $h = 0$, and $\gamma \in G_{E(D)}$ such that $\mathrm{cyc}(\gamma)^a - 1$ divides ℓ^h in $\mathbb{Z}_\ell$ for every $0 < a < t(D)$.*

Proof. If $h_2(D)^2 < \ell$, take $h = 0$ and for γ an element of an inertia group above ℓ whose cyclotomic character reduces to a generator of $(\mathbb{Z}/\ell)^\times$; it fixes $E(D)$, which lies in a cyclotomic field of conductor less than ℓ and prime to it, and $\mathrm{cyc}(\gamma)^a - 1$ is a unit because $a < t(D) < \ell - 1$. Otherwise take $h = v_\ell(\mathrm{ord}_{\mathrm{ch}}(D))$ and for γ a Frobenius element at a chosen place of norm $g_i(D)$ prime to ℓ (Lemma 5.5(c) of the first note): then $\mathrm{cyc}(\gamma)^a - 1 = g_i(D)^a - 1$ divides $\mathrm{ord}_{\mathrm{ch}}(D)$ (Lemma 5.5(a) there), which divides ℓ^h in $\mathbb{Z}_\ell$. □

LEAN `exists_cycChar_pow_sub_one_dvd`

Proof of Theorem B. Outline of a formal proof of 120 lines. Let $\mathcal{A}$ be the set of the y of the hypothesis; its span is Z. Let $d = \dim Z$ and (y_a), (y'_b) be as in Definition 3.1(d). By Lemma 4.5 of the first note, applied with $V = \mathcal{F}(O)$, the map j to smooth forms, the linear forms $\eta_a = \langle y_a, \cdot \rangle$ and $m = \mathrm{ord}_{\mathrm{ch}}(D)^3$, it suffices to find, for each non-zero $p_1 \in \mathcal{A}$ with its family c of labels, a linear map G on $\mathcal{F}(O)$ that acts by a scalar (times j) on every element of $\mathcal{A}$, whose scalar λ at p_1 satisfies $v_\ell(\mathrm{ord}_{\mathrm{ch}}(D)^3\lambda) \le \#r \ \mathrm{rb}_\ell(D)$, and such that $\mathrm{ord}_{\mathrm{ch}}(D)^3\langle y_a, w' \rangle$ is ℓ-integral whenever $j(w') = G(\mathrm{pr}_Z y'_b)$. The exponent of the conclusion is then $d \cdot \#r \cdot \mathrm{rb}_\ell(D) \log \ell$.

The base level and the eigensystems. Theorem C of the first note, for O, c and ϑ, gives a base level B containing the image of $\mathrm{pr}_O(c)$, such that $\mathrm{ord}_{\mathrm{ch}}(D) \cdot \mathrm{pr}_O(c)$ preserves ℓ-integrality, and whose eigensystems x at levels off $\Sigma_{\mathrm{bad}}(D, \ell)$ have normalised representations u_x that are labelled columns with the quadratic relation on the inertia above $Q(\vartheta) \cup Q(\bar{\vartheta})$. The elements of $\mathcal{A}$ with the label c are eigenvectors (they lift Hecke data); their eigensystems form the set $\mathcal{X}$, and by hypothesis (b) each has a row R_x with the trace identity of $\mathsf{HasRow}(V, z; R_x)$ and the Selmer bound. An eigensystem on which every Galois element has the eigenvalue 1 without being in $\mathcal{X}$ is a *weak lift*.

The primes. Let h and γ be as in Lemma 8.1. Theorem D is applied to $E = E(D)$ in the cyclotomic field of conductor $N_1(D)d$, with $t = \#n - \#r$ and $y = h_2(D)$, and to the set W of the Frobenius data $(w, \mathfrak{P}, \phi)$ with three properties. First, the twisted row ψR_{p_1}, with $\psi = \mathrm{av}^-_\vartheta(\cdot)$ and R_{p_1} the row of p_1, satisfies $|\det(1 - \psi(\phi)R_{p_1}(\phi))| \ge 1$: no eigenvalue of $\psi(\phi)R_{p_1}(\phi)$ is congruent to 1 modulo the maximal ideal. Secondly, w lies off $\Sigma_{\mathrm{bad}}(D, \ell)$, off its conjugates and off the level of the base level. Thirdly, the weak lifts that are congruent to p_1 (their characteristic polynomials are congruent to those of p_1 coefficient by coefficient) have at w their Hecke string (Lemma 10.4 of the first note). The hypothesis of Theorem D on W holds: by Lemma 10.2 of the first note, applied to $T = \psi R_{p_1}$ and to the powers $\delta = \gamma_0^i$ of the scalar inertial element γ_0 of (W4), on which ψ is trivial and cyc $= 1$, the first property holds for most Frobenius data near every $\delta^{\ell^\nu} g$; the other two hold for most Frobenius data. Theorem D gives for every ν Taylor–Wiles places with $\mathsf{Tri}_\nu(w, \tau, \phi; \gamma^{-1}; t, \mathrm{ty}; h_2(D)h, h_2(D)^{22}h)$.

Ribet's element and the operator. Let $\mathcal{S}$ be the algebra generated by the Hecke operators of the base level at the level of the Taylor–Wiles places, with the tie parameters as variables, acting on the eigensystems, and $\mathcal{J}$ the module of the

families that kill the forms of that level; the trace ideal condition holds by the theory of the note on patching (unit of the ties at auxiliary primes). Theorem C, with a normaliser θ' chosen so that Ribet's determinant β does not vanish on the eigensystems that are neither lifts nor weak lifts, and with ν large with respect to that determinant, gives $f \in \mathcal{S}$ with $f\beta \in \mathcal{J}$ and the coefficient d'. An input of the note on patching turns f into an operator E_s on the forms of the base level, diagonal on the eigensystems, with eigenvalue the coefficient d' on the lift p_1, such that $\operatorname{ord}_{\mathrm{ch}}(D)^2 E_s$ preserves ℓ-integrality on the forms orthogonal to the weak lifts that are not lifts. Put $G = E_s \circ \operatorname{pr}_O(c)$.

The test. It remains to see that $\operatorname{pr}_O(c)(y'_b - \operatorname{pr}_Z y'_b)$ is killed by E_s, that is, has no component on the lifts of $\mathcal{X}$ and on the weak lifts congruent to them. This is where the test of Definition 3.1(b) is used: a weak lift is killed by the Hecke polynomial $H_w(X_w)$ at a suitable split place w (its Hecke polynomial has the string of roots of Lemma 10.3 of the first note, and X_w is the inverse of the last root), it lies in the image of e_0, and y'_b is orthogonal to such forms inside $Z^\perp$. In detail, Lemma 6.4 of the first note reduces the claim to the isotypic vectors of the space

$$\operatorname{im} \operatorname{pr}_O(c) \;\cap\; Z^\perp \;\cap\; \big(\operatorname{im} \operatorname{act}_O(e_0) \cap \ker H_w(X_w)\big)^\perp,$$

which contains $\operatorname{pr}_O(c)(y'_b - \operatorname{pr}_Z y'_b)$ by the test and is stable under the Hecke operators (Lemmas 5.7(c) and 5.8 of the first note, Lemma 4.3). An isotypic vector of this space that E_s does not kill is a weak lift; hence it is killed by $H_w(X_w)$ (Lemma 10.3 of the first note) and lies in the image of e_0, so it is orthogonal to itself, and is zero. (Lemma 4.4 is not used here.) The orders of the stabilisers divide $\operatorname{ord}_{\mathrm{ch}}(D)$, which accounts for the third factor. □

9 The fit family: proof of Theorem A

Proof of Theorem A. Step 1: rows. For every (ℓ, ι) and every pair p of rank one for an index j, Lemma 5.1 gives a row $R'_{p,j}$ for the auxiliary pair $O_p(1)$ and the Hecke data of the trivial lift: it is applied to the trivial character of rank one, a labelled column with labels $X - 1$ (Lemma 6.1(b) of the first note) and the type $\{0\}$, whose shift is a row type by the inequalities of a pair of rank one. For every lift index j and every function p' whose part meets the good forms on $W_{m(D)}(D)$ non-trivially, Theorem E gives a row $R_{j,p'}$ for $O_D(1, j)$; by Lemma 4.1(c) there are finitely many such p', and there are finitely many pairs. Note that the rows $R_{j,p'}$ are built from eigensystems found on the one space $W_{m(D)}(D)$, for every j; Step 4 meets eigensystems on $W_j(D)$, and Input 6 below connects the two.

Step 2: the twists. For each pair (ℓ, ι), Theorem F, applied to the finitely many rows of Step 1 and to $M = \lfloor r(D)\ \mathrm{rb}_\ell(D) \rfloor + 1$, gives an m-fold twist $\theta_{\ell,\iota}$. This is the family.

Step 3: two budgets. For every finite set T of prime numbers,

$$\sum_{l \in T} \mathrm{rb}_l(D) \log l \le B_0(D) \qquad \text{and} \qquad \sum_{l \in T} 9\, \mathrm{cr}_l(D) \log l \le x(D)^{999}.$$

Indeed $\sum_{l \in T} v_l(a) \log l \le \log a$ for an integer $a \ge 1$, which bounds the contributions of $\operatorname{ord}_{\mathrm{ch}}(D)$, of X_D (by Input 4 and the bound of the class number by the discriminant) and of $g(D)$ and the factorials in $\mathrm{loss}_D(l)$; $\sum_l 2^{-l} \log l$ is bounded; and the resulting

sums of logarithms are at most $B_0(D)/2$ each, resp. $x(D)^{999}$, by the inequalities of the data (Chapter III).

Step 4: the clauses. (F1): by Theorem F(1) for $j = 1$, $Q(\theta_{\ell,\iota})$ is auxiliary for $\Sigma_{\mathrm{bad}}(D, \ell)$, so ℓ is prime to its norms (Lemma 8.2 of the first note). (F2): take $\eta_l = \mathrm{rb}_l(D) \log l$ for all j; the first budget of Step 3 is the condition on the sums. Fix j and (ℓ, ι), and let $\vartheta = \theta^j_{\ell,\iota}$; by Theorem F(1) it satisfies the hypothesis of Theorem B.

The good lift. The image of $\theta^{*,\mathrm{good}}_{D,\vartheta,j}$ is spanned by the images of the eigenvectors F in the parts $\mathrm{Eig}_D(W_j(D);\, p')$ of the good forms (Lemma 4.9), cut by the idempotents $e^{\otimes}_{r,c}$ of the good families (inputs of Chapter VII). Consider such an image that is not zero. It lies in $\operatorname{im} \mathrm{pr}_{O_D(\vartheta,j)}(c)$, with $\mathsf{Apt}(O_D(\vartheta, j), c;\, \vartheta)$ (Lemma 6.1(a) of the first note) and $\operatorname{im} e^{\otimes}_{r,c} \subseteq \operatorname{im} e_0(D)$; this is hypothesis (a) of Theorem B. It lifts the Hecke data p' (the adjoint lift of an eigenform, Chapter VII). The eigensystem p' occurs in the good forms on $W_j(D)$, since F is a non-zero good form in its part. By Input 6, applied with the lift indices j and $m(D)$, it occurs in the good forms on $W_{m(D)}(D)$ as well; so the row $R_{j,p'}$ of Step 1 is defined. It is a row for $V = O_D(1, j)$, whose character differs from that of $O_D(\vartheta, j)$ by ϑ, and the Selmer bound is Theorem F(2), the row being among those of Step 1; this is hypothesis (b). So Theorem B gives

$$\mathsf{Den}_\iota(O_D(\vartheta, j), e_0(D), \operatorname{im} \theta^{*,\mathrm{good}}_{D,\vartheta,j};\, \eta_\ell).$$

The lines of rank one. For a pair p of rank one for j, the line $\mathbb{C}\,\Xi_p(\vartheta)$ is spanned by one vector, which lies in the image of the idempotent with labels $X - 1$ (Lemma 6.1(c) of the first note, with (b) for aptness), lifts the Hecke data of the trivial lift, and has the row $R'_{p,j}$; Theorem B applies in the same way, and no transfer is needed.

(F3): by Theorem F(3), for every (ℓ, ι), $|\iota(\varrho_p(\theta^j_{\ell,\iota}, s)^{-1})| \le \exp(9\mathrm{cr}_\ell(D) \log \ell)$; with $H = 1$, $\alpha = 0$ and the local bounds $9\mathrm{cr}_l(D) \log l$ for the primes l, the second budget of Step 3 gives the height bound (note *Heights*). □

Input 6 (eigensystems of good forms on the spaces of two single lifts; unit *Good idempotents, labels and the transfer of traces*, Chapter VII: note *Good forms and rows*, Theorems 2(b) and 3). Let P be parameters, D admissible data for P, j and j' lift indices ($1 \le j, j' \le m(D)$), and p a function from the pairs of a finite place of $E(D)$ and a natural number to $\mathbb{C}$. (a) *Rigid labels.* For every such p and every chosen place x above $g(D)$, the set $\mathrm{Lab}_D(W_j(D), p;\, x)$ has at most one element. (b) *Transfer.* If the space of the good forms on $W_{j'}(D)$ meets $\mathrm{Eig}_D(W_{j'}(D);\, p)$ only in 0, then the space of the good forms on $W_j(D)$ meets $\mathrm{Eig}_D(W_j(D);\, p)$ only in 0. Equivalently: an eigensystem that occurs in the good forms on $W_j(D)$ occurs in the good forms on $W_{j'}(D)$. Part (b) is proved in that note from part (a) for both spaces and from the equality of the traces of the Hecke operators on the good forms of two unitary groups that differ at the real places; that equality rests on a comparison of trace formulas, an input of the background library there. The spaces $W_j(D)$ and $W_{j'}(D)$ have different signatures, so (b) is not a formal consequence of the definitions.

10 A bound for rank one

Lemma 10.1 (the central operators)**.** *Let D be admissible data, n a finite set, Y' L-value data over $E(D)$ for n and z a finite adelic point of* $\mathrm{U}(1)$*.* (a) *For forms x, y of $O_{Y'}$: $\langle x, \mathrm{tc}_{Y'}(z)y\rangle = \langle \mathrm{tc}_{Y'}(z^{-1})x, y\rangle$.* (b) *For every adelic point g, $\mathrm{tc}_{Y'}(z^{-1})$ carries the kernel form of Y' at g to the kernel form at gz.* (c) $\mathrm{tc}_{Y'}(z)\,\Xi_{Y'} = \Xi_{Y'}$*, where $\Xi_{Y'}$ is the lift of the trivial character.*

Proof. (a) The scalar $b_{Y'}(\det z)$ has absolute value 1, and translation by a central element preserves the pairing. (b) Compare the values at each point: the Weil representation of the scalar pair attached to $\det z^{-1}$ is the value of the character times the translation (an input of Chapter VII). (c) The lift of the trivial character is the class sum of the kernel forms, also of the translated ones, which (b) maps back. □

LEAN `LValueData.inner_twistedCentralOp` · `LValueData.twistedCentralOp_kernelForm` · `LValueData.twistedCentralOp_trivialLift`

Lemma 10.2 (the denominator of a rank-one line, in terms of two test forms)**.** *Let D be admissible data, n a finite set, Y' L-value data over $E(D)$ for n, (ℓ, ι) a pair of a prime and an isomorphism $\mathbb{C} \to \bar{\mathbb{Q}}_\ell$, β a real number and Ξ, Ξ' forms of $O_{Y'}$. Suppose that $\mathsf{Den}_\iota(O_{Y'}, e^{\otimes}_{r,X-1}, \mathbb{C}\cdot\Xi_{Y'}; \beta)$ (r the index set of the rows of $O_{Y'}$), that the stabilisers in the rational points of* $\mathrm{U}(1)$ *of all classes at the level of the chosen places of Y' are trivial, and that $\mathsf{TestForm}(O_{Y'}; \ell; \Xi)$ and $\mathsf{TestForm}(O_{Y'}; \ell; \Xi')$. Then, with h' the number of classes of* $\mathrm{U}(1)$ *at that level,*

$$\left|\iota\left(h' \cdot \langle \Xi, \mathrm{pr}_{\mathbb{C}\Xi_{Y'}} \Xi'\rangle\right)\right| \le e^{\beta}.$$

Proof. If $\Xi_{Y'} = 0$ the left side is 0. Otherwise the line has dimension 1, and the type of $O_{Y'}$ has $\#r = 1$. Choose representatives $s_1, \dots, s_{h'}$ of the classes that are integral above ℓ. Apply the definition of the denominator bound with $d = 1$, $y_0 = \Xi$ and $y_0' = \sum_c \mathrm{tc}_{Y'}(s_c^{-1})\Xi'$. By Lemma 10.1(a),(c) each term contributes $\langle \Xi, \mathrm{pr}\,\Xi'\rangle$, so the determinant is $h'\langle \Xi, \mathrm{pr}\,\Xi'\rangle$. The form y_0' is integral at ι because the central operators at integral points preserve integrality; it is tested by $(O_{Y'}, e^{\otimes}_{r,X-1}, (\mathbb{C}\Xi_{Y'})^{\perp})$ because the central operators commute with the idempotent and the Hecke polynomials, and because the sum over the classes of the central translates of a form orthogonal to $\Xi_{Y'}$ lies in $\ker \theta_{O_{Y'}}$ (the theta lift to U(1) pairs a form with the kernel forms, Lemma 10.1(b); the stabilisers are trivial), so that the test of Ξ' against $\ker \theta_{O_{Y'}}$ applies. □

LEAN `DenominatorLe.rank_one_bound`

11 Where the results are used

Theorem A is used once, in the proof of Theorem C of the handout (integral forms are long): the use stands in the file of the endgame on long forms, which the overview's map places in Chapter II. There the fit family provides, for each (ℓ, ι), the twist at which the volume of the chain is compared with the volume at the trivial twist; (F1) makes the index of the rows available at $\theta_{\ell,\iota}$, (F2) bounds the denominators of the Gram determinants of the single lifts and of the rank-one lines, and (F3) the heights of the ratios of L-values. Lemma 10.2 is not used by Theorem A; it is used in the unit of the rows (Chapter VII), in the height of the Hecke values, where the clause on

the rank-one lines of (F2) enters through it. Lemma 4.4 is not used by Theorem A or by Theorem B either; it is used in the unit of the rows (Chapter VII), in the height of the volume of a lift. The definitions of the Hecke algebra away from g and of the adjoint lifts (Definitions 3.2, 3.3) are also used by the notes on the volume of the chain.

12 Notes on the formal text

1. *Letters.* Besides the changes of the first two notes: the formal text writes `E` for the element e_0 of the group algebra in Theorem B, `O` for an auxiliary pair, `Y` for L-value data (here Y'), `r` for the exponent r' of Definition 3.6, `p` for a function of eigenvalues as well as for a pair, `s` for the scalar s' of Theorem B, `θ` both for the family of twists and for the function θ' of Theorem C, `d` for the coefficient d', and `B₀` for $B_0(D)$.

2. *Arguments.* The arguments were read from the text of the files; none was confirmed by Lean's own answer, since the artifact was not compiled here. In several declarations the program lists as arguments variables that occur only as names of fields or as bound names: a place w in the definitions of split Frobenius data and of laws and in the ramified and rank-one pairs; the field and bound variables in the definitions of the Hecke algebra and its relatives; and, in the definition of a fit family, the prime, the isomorphism, the group algebra element, the auxiliary pair and the index, which are bound inside its clauses. Lean presumably takes for a fit family only the data and the family θ. These definitions are therefore written out in words here (Definitions 3.1(b), 3.2(b), 3.6 and the box of Theorem A), with every listed variable introduced where it is bound. The arguments "a prime, with the isomorphism ι" of Lemma 5.1 and Theorem E (their statements use ι; the calls in the last file pass it), and "the data" of Lemma 4.6, whose statement begins with a local definition, come from declarations of variables before the statements; the dossier lists them from the source text since its last correction, and they stand in the statements here. In the definitions of the maps $\theta^*_{D,\vartheta,j}(\cdot)$ and $\mathrm{hg}_{D,V}(\cdot)$ the argument of the map is counted as an argument. All this is to be confirmed with Lean's own answer when the artifact is built.

3. *Theorem A.* The formal statement is "for all data there is θ that is a fit family"; the index set is the set of the pairs of a prime and an isomorphism. The clause (F2) of the formal definition quantifies the lift index first, then the function γ, then the pair (ℓ, ι), then the pair of rank one; (F3) quantifies the lift index, the pair and the sign before the hypothesis "ramified". The reading of these clauses, of the height and of the Euler factors of the ratio follows the second reader's message on the fit family; I compared it with the formal definition and found no difference.

4. *Hecke operators.* The Hecke operators away from the chosen places form a commuting family of *normal* operators (Lemma 4.9); the dossier calls them unitary, which would be a further condition (second reader).

5. *Input 4.* The catalogue row for the bound of the odd part asked $t \geq 1$ and called the module "the odd part" without restriction; the second reader's verdict,

followed here, corrects both: t is any natural number, and for $e = 0$ there is no oddness condition.

6. *Imports copied by name.* For 211 of the declarations of other units that this part of the dossier quotes, the reading in words was copied from other dossiers of the workspace; where such a declaration is stated here as an Input, its statement is that of the first two notes or of the catalogue.

7. *Outlines.* The proofs of Theorems B, C, D and of Lemma 7.2 are outlines of long formal proofs (of 120, 100, 100 and 140 lines); I compared the hypotheses used at each stage with the formal text and with the reader's step-by-step rendering, but did not re-derive the estimates. In particular the comparison of constants at the end of Theorem C, the recursion of Theorem D, the count of torsion in Lemma 7.2 and the construction of the operator in Theorem B (which uses the note on patching) are taken from the formal text.

8. *What the final theorems use.* Lemma 10.2, Lemma 4.4 and Definition 3.7 are not used by Theorem A; they are used in the unit of the rows (Chapter VII).

9. *The transfer between the spaces of the lifts.* The formal proof of Theorem A builds the rows of the good eigensystems from the space $W_{m(D)}(D)$ and uses them for eigensystems found on $W_j(D)$; it passes from one to the other by two results of the unit on good forms (Input 6), in the contrapositive form of part (b). They are imports of this unit and do not appear in the concordance, which lists the declarations of the unit only.

10. *Theorem C.* At the inertia above $g(D)$ the polynomials o_v are the sums $1 + X + \cdots + X^{\mathrm{ord}_{\mathrm{ch}}(D)-1}$, and the finite set of Ribet's lemma is the set of clause (T2) enlarged by the tame generators $\tau_k^{[\pm]}$. The formal proof ends with a comparison of the resulting bound with $\#r \cdot \mathrm{rb}_\ell(D)$, in which the first term carries the coefficient $4 + t(D)\,(2 + 10 f(D) + 6 h_2(D)^{22}) + h_2(D)^{22} + h_2(D)$; I did not expand that comparison.

How this note was written. From the dossier of the unit (its third chunk): the statements in words and the structured proofs, which a program compared line by line with the source (975 leaf steps in the 96 proofs of the whole unit), and from the second reader's message on the fit family, which takes precedence. I read in the Lean text: the file of the definitions of this note completely; the last file (the proof of Theorem A) completely; the files on the twist for rows, on the existence of rows and on rank one completely; and the statements, with the surrounding definitions, of the four long files (Ribet's element, reciprocity laws, prime triangles, the denominator bound), whose proofs I followed in the reader's rendering. In particular the proofs of the three lemmas taken from the file of Ribet's element (Lemmas 6.1, 6.2, 6.3) were written from the reader's rendering, not from the Lean text; the referee of this chapter compared them with the Lean source. In the revision of this note I read in the Lean text the passages of the four long proofs that the revision touches: the choice of the polynomials and of the finite set in Theorem C, the set of Frobenius data and the test in Theorem B, the local conditions in Lemma 7.2, and the call of the transfer in Theorem A. The statements of this note were compared with the Lean statements argument by argument, and the order of the quantifiers is that of the formal text. I did not open the proofs of the inputs. The references of Inputs

4 and 5 and their places are those of the second reading of this write-up and of its catalogue; I did not check them in the literature.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Definition 2.1	`IsConjSelfDual`	AptTupleDefs.lean	21–24
Definition 2.1	`crudeUnit`	AptTupleDefs.lean	27–27
Definition 2.1	`SelmerLengthLe`	AptTupleDefs.lean	34–38
Definition 2.1	`IsColumn`	AptTupleDefs.lean	43–45
Definition 2.1	`IsRow`	AptTupleDefs.lean	50–59
Definition 2.1	`chiOneInv`	AptTupleDefs.lean	68–68
Definition 2.1	`normalisedRep`	AptTupleDefs.lean	82–82
Definition 2.1	`twistedLabel`	AptTupleDefs.lean	86–87
Definition 2.1	`IsApt`	AptTupleDefs.lean	92–100
Definition 2.1	`tupleProj`	AptTupleDefs.lean	104–104
Definition 2.1	`HasRow`	AptTupleDefs.lean	108–110
Definition 2.1	`IsLabelledColumn`	AptTupleDefs.lean	114–116
Definition 2.1	`obeyingCocycles`	BlocksAndCocycles.lean	16–17
Definition 2.1	`RowBlockIndex`	BlocksAndCocycles.lean	41–41
Definition 2.1	`blockRep`	BlocksAndCocycles.lean	50–51
Definition 2.1	`IsAuxiliary`	Corepresentable.lean	55–55
Definition 2.1	`evenClasses`	Corepresentable.lean	136–138
Definition 2.1	`IsNewPlaces`	Corepresentable.lean	142–143
Definition 2.1	`withConjInv`	Corepresentable.lean	147–147
Definition 2.1	`IsPrimeTriangle`	Corepresentable.lean	152–162
Definition 2.1	`IsAuxTwist`	Corepresentable.lean	166–167
Lemma 8.1	`exists_cycChar_pow_sub_one_dvd`	DenominatorBound.lean	26–40
Theorem B	`denominatorLe_rowBudget`	DenominatorBound.lean	47–166
Definition 2.1	`crysRels`	DeterminantBound.lean	43–43
Definition 2.1	`blockLoss`	DeterminantBound.lean	47–48
Definition 2.1	`cmSign`	DeterminantBound.lean	56–56
Definition 2.1	`IsRowType`	DeterminantBound.lean	64–70
Definition 2.1	`lambdaLoss`	DeterminantBound.lean	74–75
Definition 2.1	`loss`	DeterminantBound.lean	78–79
Lemma 4.1	`LevelPlaces.LevelForm.innerAt_heckePolyOnForms`	FitTwists.lean	19–24
Definition 3.1	`AuxPair.groupAlgAction`	FitTwists.lean	35–35
Definition 3.1	`AuxPair.IsTested`	FitTwists.lean	39–40
Definition 3.1	`AuxPair.IsTestForm`	FitTwists.lean	44–45
Definition 3.1	`DenominatorLe`	FitTwists.lean	50–53
Lemma 4.1	`AuxPair.isOfFinOrder_prod_psi_det`	FitTwists.lean	59–61
Lemma 4.2	`AuxPair.innerAt_formAction`	FitTwists.lean	66–70
Lemma 4.1	`AuxPair.range_groupAlgAction_mem_invtSubmodule`	FitTwists.lean	72–74
Lemma 4.3	`AuxPair.heckePolyOnForms_comm`	FitTwists.lean	78–82
Lemma 4.1	`AuxPair.innerAt_groupAlgAction`	FitTwists.lean	86–88
Lemma 4.4	`AuxPair.normSqAt_starProjection_le`	FitTwists.lean	96–112
Definition 3.2	`Data.IsSplitOffLevel`	FitTwists.lean	117–117
Definition 3.2	`Data.heckeAlgebra`	FitTwists.lean	122–127
Definition 3.2	`Data.heckeOperator`	FitTwists.lean	131–136
Definition 3.2	`Data.eigenPart`	FitTwists.lean	140–140
Lemma 4.5	`Data.isSphericalWith_of_mem_eigenPart`	FitTwists.lean	145–148
Definition 3.2	`Data.occurringLabels`	FitTwists.lean	152–153
Lemma 4.6	`Data.dvd_card_resField_sub_one`	FitTwists.lean	155–158
Lemma 4.7	`Data.exists_hecke_proj_eigenPart`	FitTwists.lean	162–167
Lemma 4.8	`Data.commute_heckeAlgebra_principalRep`	FitTwists.lean	169–173

In this note	Lean declaration	File	Lines
Definition 3.2	`Data.heckeOnGood`	FitTwists.lean	177–179
Lemma 4.1	`Data.principalRep_mem_eigenPart`	FitTwists.lean	181–183
Lemma 4.1	`Data.finite_eigenPart`	FitTwists.lean	187–188
Lemma 4.9	`Data.iSup_eigenPart`	FitTwists.lean	192–197
Lemma 4.10	`AuxPair.theta_mul_eq_theta_formAction`	FitTwists.lean	205–212
Lemma 4.11	`Data.exists_theta_to_principalForms`	FitTwists.lean	216–222
Definition 3.3	`Data.auxLift`	FitTwists.lean	226–227
Definition 3.3	`Data.RankOneIndex`	FitTwists.lean	231–231
Definition 3.3	`Data.goodLift`	FitTwists.lean	237–237
Definition 3.4	`Data.Pair.IsRamified`	FitTwists.lean	241–241
Definition 3.4	`Data.Pair.IsRankOne`	FitTwists.lean	245–247
Definition 3.4	`Data.B₀`	FitTwists.lean	250–250
Theorem A	`Data.IsFitFamily`	FitTwists.lean	256–263
Theorem A	`exists_isFitFamily`	FitTwistsExist.lean	24–65
Definition 2.1	`AbsGal`	LevelPlacesChosen.lean	25–25
Definition 2.1	`placesAboveG`	LevelPlacesChosen.lean	28–28
Definition 2.1	`badPlaces`	LevelPlacesChosen.lean	32–32
Definition 2.1	`Data.TameChar.invAvatar`	LevelPlacesChosen.lean	59–60
Definition 3.7	`LValueData.centralRep`	RankOneDenominator.lean	13–14
Definition 3.7	`LValueData.twistedCentralOp`	RankOneDenominator.lean	18–18
Lemma 10.1	`LValueData.inner_twistedCentralOp`	RankOneDenominator.lean	20–24
Lemma 10.1	`LValueData.twistedCentralOp_kernelForm`	RankOneDenominator.lean	28–32
Lemma 10.1	`LValueData.twistedCentralOp_trivialLift`	RankOneDenominator.lean	34–36
Lemma 10.2	`DenominatorLe.rank_one_bound`	RankOneDenominator.lean	43–66
Lemma 7.1	`eq_zero_of_mem_evenClasses`	ReciprocityLaw.lean	12–17
Definition 3.6	`IsSplitPlace`	ReciprocityLaw.lean	24–24
Definition 3.6	`primeBelowPlus`	ReciprocityLaw.lean	27–27
Definition 3.6	`IsLaw`	ReciprocityLaw.lean	32–40
Lemma 4.1	`primeBelowPlus_star`	ReciprocityLaw.lean	42–43
Lemma 7.2	`exists_isLaw`	ReciprocityLaw.lean	48–195
Definition 2.1	`ribetNormaliser`	RibetCramer.lean	79–80
Definition 3.5	`rowBudget`	RibetElement.lean	36–37
Lemma 6.1	`obeyingCocycles.mem_selmerModule`	RibetElement.lean	42–50
Lemma 6.2	`isTraceIdeal_withConjInv`	RibetElement.lean	55–70
Lemma 6.3	`sum_blockLoss_le`	RibetElement.lean	80–114
Theorem C	`ribet_columns`	RibetElement.lean	122–223
Definition 2.1	`HasTriangleCost`	RibetFirstOrder.lean	20–24
Definition 2.1	`IsTraceIdeal`	RibetFirstOrder.lean	67–71
Definition 2.1	`IsRibetElement`	RibetFirstOrder.lean	75–77
Lemma 5.1	`exists_row_for_all_twists`	RowsExist.lean	29–55
Theorem E	`exists_hasRow_auxPair`	RowsExist.lean	61–89
Theorem D	`exists_isPrimeTriangle`	TrianglePrimes.lean	24–123
Theorem F	`exists_mFoldTwist_selmerLengthLe`	TwistForRows.lean	14–59

The definition of a fit family and its existence, as written in the artifact.

```
structure Data.IsFitFamily (θ : PrimePadicIso → D.MFoldTwist) where
  coprime ι : Nat.Coprime ι.1.1 (θ ι).levelNorm
  denominator_le j : ∃ γ, (∀ T : Finset ℕ, (∀ l ∈ T, l.Prime) → T.sum γ ≤ D.B₀) ∧ ∀ ι, DenominatorLe
    ι.2 _ D.e₀ (D.goodLift ((θ ι).powTwist j) j).range (γ ι.1) ∧ ∀ p : D.Pair, ∀ H : p.IsRankOne j,
    DenominatorLe ι.2 _ (tupleIdem _ fun _ ↦
    .X - 1) (ℂ · (p.lValueData H.1 ((θ ι).powTwist j)).trivialLift) (γ ι.1)
  height_le j (p : D.Pair) s : p.IsRamified → HeightLe (D.x ^ 999) fun ι ↦
    (p.ratio ((θ ι).powTwist j) s)⁻¹

lemma exists_isFitFamily {P} (D : Data P) : ∃ θ, D.IsFitFamily θ
```

In these texts `PrimePadicIso` is the set of the pairs (ℓ, ι), `ι.1.1` the prime and `ι.2` the isomorphism; `D.MFoldTwist` the set of the m-fold twists, `levelNorm` the product $\mathrm{N}(Q(\cdot))$, `powTwist j` the j-th power; `D.B₀` is $B_0(D)$, `D.e₀` the good idempotent, `goodLift` the adjoint lift on good forms, `lValueData` and `trivialLift` give $O_p(\vartheta)$ and $\Xi_p(\vartheta)$, `tupleIdem _ fun _ ↦ .X - 1` is $e^{\otimes}_{r_p, X-1}$, `HeightLe` the height bound and `ratio` is ϱ_p. The formal definition of the denominator bound (Definition 3.1(d)) is not reproduced: it contains a sign, for the orthogonal complement, that the fonts of this note lack; its place is in the table above.

Notation

ℓ, l, ι, (ℓ, ι), $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell$, $\bar{\mathbb{Q}}_\ell$, $\bar{\mathbb{Z}}_\ell$, $\mathbb{C}_\ell$, $\mathbb{C}_l$, $\mathbb{Z}_l$, $\mathbb{Q}_l$, $G_{\mathbb{Q}_\ell}$ primes, the isomorphism $\mathbb{C} \to \bar{\mathbb{Q}}_\ell$, the pair that indexes the fit family, the ℓ-adic objects

$\bar{\mathbb{Q}}$, c, g^{c}, $g^{[b]}$, E, E^+, G_E, h_E as in the first note; the maximal real subfield; the class number

P, D, $E(D)$, $m(D)$, $r(D)$, $k(D)$, $t(D)$, $f(D)$, $g_i(D)$, $g(D)$, $N(D)$, $h_2(D)$, $b_1(D)$, $N_1(D)$, d, $w_{D,j}(v)$, $\mathrm{ord}_{\mathrm{ch}}(D)$, $x(D)$, $\mathrm{budget}(D)$, $\mu(D)$, N_0, $M_\omega(P, \mu(D))$, ω_D parameters, data and its numbers

$\mathsf{G}(D)$, $e_0(D)$, $W_j(D)$, $\mathcal{M}_D(V)$, $\mathbb{T}(D)$ the chosen group, the good idempotent, the space of the j-th lift, forms of principal level, the Hecke algebra away from g

w, v, u, $\mathfrak{q}$, $\bar{w}$, $\mathrm{N}(w)$, $\mathbb{F}_w$, $\mathfrak{P}$, $\mathfrak{P}_w$, $G_{\mathfrak{P}}$, $I_{\mathfrak{P}}$, $\tau_{\mathfrak{P}}$, $\Sigma_M(E)$, I_S, $S^{\mathbb{C}}$ places and the attached groups (u also a family of representations)

cyc, cyc_M, v_ℓ, v_ℓ, v_l, v_A, len, nlen, $\mathcal{O}$, $\mathcal{O}'$, $\mathcal{O}(-e)$, A characters, valuations, lengths, coefficient rings

$\mathrm{Sel}^L_S(\varrho; B)$, $\mathrm{Ob}(L; B)$, $\mathrm{Sha}^{\mathrm{odd}}$, δ_g Selmer modules, obeying functions, the module of Input 4, indicator relations

ϑ, τ, $\theta = (\theta_{\ell,\iota})$, $Q(\vartheta)$, $\mathrm{N}(Q(\vartheta))$, $\psi_{\vartheta,w}$ twists, m-fold twists and the fit family; supports and their norms (τ also a family of tame generators)

p, ν_p, z_p, c_p, ψ_p, $\varrho_p(\vartheta, s)$, $L^{(S)}(z, \nu)$, Ch pairs; ratios of L-values; L-functions without the Euler factors at a set; the chosen places

O, V, $O_D(\vartheta, j)$, $O_p(\vartheta)$, $O_{Y'}$, Y', $\mathcal{F}(O)$, w_i, μ_O, ψ_O, a_O, e_O, $b_{Y'}$, θ_O auxiliary pairs, L-value data, forms, characters, the weight, the theta lift

Ξ, Ξ', $\Xi_p(\vartheta)$, $\Xi_{Y'}$, F forms; the lifts of the trivial character

$\langle y, y' \rangle$, $\langle y, y' \rangle_\sigma$, $\|y\|_\sigma$, pr_Z, $Z^\perp$, im, ker, dim Petersson products, norms, orthogonal projection and complement

$e^{\otimes}_{r,c}$, $e^{\otimes}_{r,X-1}$, r_p, e_0, e'_0, $H_w(X)$, X_w, X^*, ν_w, $T_{w,i}$, T_V idempotents; Hecke operators; the test value

$\mathrm{act}_O(e_0)$, $\mathsf{TestForm}(O; l; \Xi)$, $\mathsf{Den}_\iota(O, e_0, Z; \beta)$ Definition 3.1

$\mathsf{SplitOff}_D(w)$, $\mathrm{T}_D(x)$, $\mathrm{Eig}_D(V; p)$, $\mathrm{Lab}_D(V, p; x)$, $\mathrm{hg}_{D,V}(T)$ Definition 3.2

$\theta^*_{D,\vartheta,j}(F)$, $\theta^{*,\mathrm{good}}_{D,\vartheta,j}$, $\mathrm{I}^{(1)}(z)$, $L_{\vartheta,j}$ Definition 3.3

$\mathsf{Ram}(p)$, $\mathsf{RkOne}(p; j)$, $B_0(D)$, $\mathrm{rb}_\ell(D)$, X_D Definitions 3.4, 3.5

w^+, $\mathrm{cen}_{Y'}$, $\mathrm{tc}_{Y'}(z)$ Definitions 3.6, 3.7

$\mathcal{Z}_u(\varrho; a)$, $\mathrm{I}_r(i)$, $\mathrm{blk}_T(i)(g)$, $\mathcal{R}_{\mathrm{cris}}(u)$, $\mathrm{bl}(u; w, M)$ recalled, Definition 2.1(1)

$\mathrm{sgn}_{\mathrm{CM}}(\sigma)$, $\mathsf{RowType}_D(r, t; \mathcal{H})$, $\mathrm{lam}(h, b; l)$, $\mathrm{loss}_D(l)$ recalled, Definition 2.1(2)

$G_{E(D)}$, $\Sigma_g(D)$, $\Sigma_{\mathrm{bad}}(D, \ell)$, $\mathrm{av}^-_\vartheta(g)$, $\mathrm{cr}_\ell(D)$ recalled, Definition 2.1(3)

$\mathsf{CSD}_t(R)$, $\mathsf{Short}(\psi, R; Q, m; \gamma, z)$, $\mathsf{ColRep}_D(t, \mathcal{H}, S; R)$, $\mathsf{RowRep}_D(t, \mathcal{H}; R)$ recalled, Definition 2.1(4)

$\chi^{-1}_{1,O}$, $\mathrm{nm}_O(R)$, $\mathrm{tl}_O(c, i)$, $\mathsf{Apt}(O, c; \vartheta)$, $\mathrm{pr}_O(c)$, $\mathsf{HasRow}(O, z; R)$, $\mathsf{LabCol}(O; k, \mathcal{H}, S; c; R)$ recalled, Definition 2.1(5)

$\mathsf{Auxil}(S; I)$, $\mathrm{Ev}(\mathcal{O}, S; \nu, e)$, $\mathsf{New}_\nu(w, \tau, \phi)$, $\tau^{[k]}$, $\mathsf{AuxTw}(S, Q; \psi, c)$ recalled, Definition 2.1(6)

$\mathsf{Tri}_\nu(w, \tau, \phi; a; t, \mathrm{ty}; h, c)$ prime triangles, Definition 2.1(6)

$\mathrm{rn}_{\gamma,a}(y, p, \theta)(g)$, $\mathsf{TriCost}(Z; \tau, \phi; I; C, b)$, $\mathsf{TrId}(\mathcal{J}; u; p; \tau, \phi, k, \varepsilon)$, $\mathsf{Rib}(\mathcal{J}; \beta; \pi, c)$ recalled, Definition 2.1(7)

R, R', R_x, R'_x, R_i, T, T_0, T_x, u_x, u'_x, ϱ, ψ, ξ_e, $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{H}_x$, $\mathcal{H}_i$, $\mathcal{H}^\flat$, $\mathcal{H}_{\kappa,N}$, κ representations, lattices, characters and types, a weight (T also a finite set of primes or places and a Hecke element)

$\mathcal{S}$, $\mathcal{J}$, $\mathcal{X}$, ε_k, β, π, θ', d', f, id, pr_1, ty, t_V, m_0 the objects of Ribet's lemma in Theorem C (β also a real bound)

γ, γ_x, γ_i, ϕ, ϕ_j, ϕ_x, δ, δ_v, o_v, g, g_i, a, u', s, s', z, y, y_x, y_i, k elements of Galois groups, scalars, signs, forms and adelic points, as said at each place (γ also the function of (F2), z also a law or Hecke data, y also a number in Theorem D, k also a number or an adelic element)

x, x_i, w_x, $\mathfrak{P}_x$, W, U, J, K, I, Y, $\mathfrak{p}$, $\mathcal{A}$, Z, Z', N' Frobenius data, sets of them, an open set of Galois elements, index sets, a detecting set,

spanning sets and subspaces (x also an eigensystem, a pair (w, i) or a form)

n, r, r', r_i, t, t_i, e, e', ν, h, h', N, M, M_i, M', j, n', i, b, c, q, q', d, m, a' finite sets and natural numbers (c also a family of labels or a cocycle, b also Hecke data, a sign or a ring element, q', d', e' also functions of eigenvalues, m also the number in Lemma 4.5 of the first note)

H, α, η_l, σ, c' the multiplier and bound of a height; the budget of (F2) and the local bounds of a height; an automorphism of $\mathbb{C}$ or a complex embedding

B, E_s, G, λ, p_1, p', η_a, y_a, y'_b, w', L, L', S, Y_0, C, p_0, b_0, b_e, X, X_k, ζ, Λ, ϵ objects of the proofs: a base level, operators, scalars, linear forms and forms, linear maps, sets of places, the constants of Ribet's lemma, variables

$\mathrm{U}(1)$, $\mathrm{U}(2z)$, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$ unitary groups; the complex and real numbers; the natural numbers

s', γ_0, j', K, U', T', γ_i, n_i in the outlines and in Input 6: the enlarged finite set of Theorem C (also a scalar in Theorem B); the inertial element of (W4); a second lift index; the objects of Input 1 of the second note

Φ, Σ, φ, φ' the set on which a law tests even classes; the set $\Sigma_\ell(E(D)) \cup \Sigma_g(D)$; forms in Lemma 4.1

References

The references are those named by the second reading of this write-up, with the places given there; I did not open them.

C. Greither, *Class groups of abelian fields, and the main conjecture*, Ann. Inst. Fourier 42 (1992).

B. Mazur, A. Wiles, *Class fields of abelian extensions of* $\mathbb{Q}$, Invent. Math. 76 (1984).

A Taylor–Wiles prime at a time: ties of tame inertia to Frobenius and the denominator of the projection

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. Let ℓ be a prime and let w_j, $j \in J$, be auxiliary (Taylor–Wiles) places of a CM field, of prime norms $q_j \equiv 1$ modulo ℓ^ν. The finite group $\prod_j \mathrm{GL}_n(\mathbb{F}_{w_j})$ acts on the algebraic modular forms of principal level at the w_j on a definite unitary group of rank n. This note proves two things about the integral structure of these forms; in both, C is a common multiple of the orders of the stabilisers. At one place w_j there is an integral operator F, supported at that place, such that F times the trace of tame inertia minus one equals the sum over $\mathrm{GL}_n(\mathbb{F}_{w_j})$, which is the trace to the level without w_j, times the trace of a polynomial in Frobenius: this is the *tie*, and it holds in the Galois representations of all Hecke eigensystems at once, modulo ℓ^ν and up to the factor $2 \cdot n!$. Moreover F commutes with the action of $\mathrm{GL}_n(\mathbb{F}_{w_j})$ modulo ℓ^ν, up to the factor C and up to an error that vanishes after the trace to the base level followed by a Hecke element supported on the eigensystems whose Hecke polynomial contains a full string of roots. Taking one tie at each place, the top coefficient of Ribet's element, a polynomial in one variable for each place, gives a Hecke element that carries the projection of an integral form to a chosen set of eigensystems back into the integral forms, up to the factor C (Theorem A). The argument is algebra in group rings: a transfer computation in GL_n of a finite field and a first-order lemma on weighted sums over a finite group. It rests on Higman's criterion at auxiliary places, on the chain lemma and on local–global compatibility at pro-p Iwahori level, which are quoted.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles with Lean v4.35.0 against a pinned version of Mathlib and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note treats one unit of the artifact, the folder on one auxiliary prime at a time: all 10 of its 10 declarations. Theorem A, Propositions 4.4 and 5.4 and Lemmas 3.1 to 5.2 are proved in full from the Inputs of Section 2: two definitions of another unit of the argument, and results of the background library, which are stated in boxes and not proved here. Every statement below was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 10 declarations.

1 The statement

The twists of Chapter VIII are chosen so that certain Selmer groups are short. Ribet's method, which belongs to the unit *Selmer*, turns such a bound into an element of a Hecke algebra with one extra variable X_j for each auxiliary place: a relation among traces of Galois representations in which tame inertia at w_j is compared with X_j times a polynomial in Frobenius. The present unit is the bridge from that element to modular forms. In the words of the authors' comment at the head of the file, at one deep auxiliary level the trace to the old level is deformed in parameters, one for each prime, tying tame inertia to Frobenius, and the top coefficient of Ribet's Hecke element, a polynomial in them, clears the denominator of the projection to lifts up to a known factor. The objects are those of Settings 2.1 to 2.3 and of Definitions 2.4 to 2.7.

The auxiliary places of this note are the Taylor–Wiles places of the unit *Selmer* (the triangular arrangement of §5 of the handout's overview); they are chosen after the twist ϑ. They are not the primes of the set $Q(\vartheta)$ where the twist ramifies, and the letter Q of this note, the family of the auxiliary places, is not that set. The "ties" that the overview imposes at the primes of $Q(\vartheta)$ are a different condition from the tie of Proposition 5.4, which is a property of an operator at an auxiliary place.

Theorem A (the top coefficient of Ribet's element clears the denominator). *In Settings 2.1, 2.2 and 2.3, let $\beta\colon \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}_Q) \to \bar{\mathbb{Q}}_\ell$ be a function, $p \in \bar{\mathbb{Z}}_\ell[X]$ a polynomial with constant coefficient p_0, $\mathcal{X}$ a subset of $\mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}_Q)$ and π an element of $\mathcal{X}$, $\varepsilon = (\varepsilon_j)_{j\in J}$ a family of polynomials in $\bar{\mathbb{Z}}_\ell[X]$, $L \geq 1$ an integer, $\mu \in \bar{\mathbb{Q}}_\ell$ and $A \in [0,\infty]$. Suppose:*

(1) *for every $j \in J$ and every $\xi \in \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}_Q)$ that is an eigensystem of base level, has $p(\beta(\xi)) \neq 0$, and is such that $|T(\pi)| < 1$ for every $T \in \mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)$ with $T(\xi) = 0$: the Hecke polynomial h_{ξ,w_j} has a root string of length L and ratio q_j at $\omega(\mathrm{Fr}_j)^{-1}$ with cofactor dividing ε_j (Definition 2.5);*

(2) *the element $2 \cdot n! \cdot (p_0 C^2)^{|J|+1}$ divides ℓ^ν in $\bar{\mathbb{Z}}_\ell$;*

(3) *for every $\bar{\mathbb{Z}}_\ell$-subalgebra $\mathcal{S}$ of the ring of functions from $\mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}_Q)$ to $\bar{\mathbb{Q}}_\ell[X_j : j \in J]$ and every $\mathcal{S}$-submodule $\mathcal{J}$ of that ring such that $(\mathcal{S},\mathcal{J})$ is a trace ideal for $(\mathfrak{r},\ \ell^\nu,\ (\tau_j)_{j\in J},\ (\mathrm{Fr}_j)_{j\in J},\ \mathrm{id}_J,\ \varepsilon)$ (Definition 2.6), there is $\eta \in \bar{\mathbb{Q}}_\ell$ that is a Ribet coefficient of $(\mathcal{S},\mathcal{J})$ for β at $\pi \in \mathcal{X}$ (Definition 2.7) and has $\mathrm{ord}_\ell^+(\mu\eta) \leq A$.*

Then there is a function $s\colon \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}_Q) \to \bar{\mathbb{Q}}_\ell$ with $\mathrm{ord}_\ell^+(\mu\, s(\pi)) \leq A$ and the following property. For every base-level form $\varphi_0 \in \mathrm{Forms}_{\mathfrak{D}}(\emptyset)$, every natural number Δ such that $\Delta\varphi_0$ is integral, and every base-level form $\varphi \in \mathrm{Forms}_{\mathfrak{D}}(\emptyset)$ such that

(i) *$\langle s'\rangle\varphi = 0$ for every function $s'\colon \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}_Q) \to \bar{\mathbb{Q}}_\ell$ that vanishes on $\mathcal{X}$, and*

(ii) *$\langle s'\rangle\varphi_0 = \langle s'\rangle\varphi$ for every function $s'\colon \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}_Q) \to \bar{\mathbb{Q}}_\ell$ that vanishes at all ξ with $p(\beta(\xi)) = 0$,*

the form $\Delta \cdot C \cdot \langle s\rangle\varphi$ is integral.

LEAN `BaseLevel.exists_diagOp_mem_ellIntegralForms` · the formal text has in addition the two hypotheses of Setting 2.3, which the census does not list (Section 8)

Here $\langle s \rangle$ multiplies the forms of each eigensystem ξ by $\iota^{-1}(s(\xi))$, and ord_ℓ^+ is the ℓ-adic valuation truncated at 0. Condition (i) says that φ is supported on the eigensystems in $\mathcal{X}$, and (ii) that φ is the part of φ_0 on the eigensystems where $p(\beta)$ does not vanish. So φ is a projection of φ_0, which has denominators in general, and the theorem bounds the denominator at π: after multiplication by C, the element s clears it, and $s(\pi)$ is as small as Ribet's top coefficient η allows. Condition (1) is the string condition, imposed on the eigensystems ξ of base level that are congruent to π in the sense that every Hecke element vanishing at ξ has absolute value less than 1 at π; the unit *Selmer* verifies it where it applies the theorem.

2 The setting

Conventions. Rings have a unit and need not be commutative; a homomorphism from a group to a ring is a homomorphism into its multiplicative monoid. For a natural number M the group $\mathbb{Z}/M$ is written additively; the formal text writes it multiplicatively. For a finite group and a subgroup H of it with a homomorphism α from H to a commutative group, Tr_α is the transfer of α to the whole group; its formula is recalled in the proof of Proposition 4.4. In Sections 3 and 4 the implicit arguments of the declarations are introduced in each statement. In the other statements the ambient objects are those of Setting 2.1: a prime number ℓ with an isomorphism of fields ι, a CM field E with a CM type, embedded in $\bar{\mathbb{Q}}$, an index set of n elements, a hermitian space V, a weight κ, a finite set J, a natural number ν, a natural number C and a base datum $\mathfrak{D}$; and, where Setting 2.2 is cited, the condition $n \geq 1$, that V is standard, the Weil data of E, a homomorphism ω and a family $\mathfrak{r}$. Matrices of size n have rows and columns numbered from 1 to n; the formal text numbers them from 0.

The three Settings collect the objects of the background library on which the unit is built. They are stated as the unit's dossier reads them; the notes on the units of the library that own them (their numbers are given) define them in full.

Setting 2.1 (forms, levels, Hecke algebra, auxiliary places; from units 57, 102, 122, 123 and 130 of the library)**.** *Coefficients.* ℓ is a prime number and $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ a fixed isomorphism of fields, where $\bar{\mathbb{Q}}_\ell$ is an algebraic closure of $\mathbb{Q}_\ell$; $\bar{\mathbb{Z}}_\ell$ is the ring of integers of $\bar{\mathbb{Q}}_\ell$, and $|\cdot|$ is the absolute value of $\bar{\mathbb{Q}}_\ell$ with $|\ell| = \ell^{-1}$. The ring $\bar{\mathbb{Z}}_\ell$ maps to $\mathbb{C}$ by ι^{-1}, and through this map it acts on every complex vector space.

The group. E is a CM field with a CM type (a complex embedding chosen for each infinite place), embedded in an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$, and $G_E = \mathrm{Gal}(\bar{\mathbb{Q}}/E)$. For a finite place w of E, the completion is E_w, its ring of integers $\mathcal{O}_w$ with maximal ideal $\mathfrak{p}_w$, the residue field $\mathbb{F}_w$ has q_w elements, and $\bar{w}$ is the complex conjugate place. V is a positive definite hermitian space over E with coordinates in an index set of n elements, κ is a weight, G_f is the group of finite adelic points of the unitary group of V, which has a standard maximal compact subgroup. $\mathrm{Forms}_\kappa(V)$ is the complex vector space of the forms of weight κ on V (algebraic modular forms), on which G_f acts by right translation. At a place $w \neq \bar{w}$ the group $\mathrm{GL}_n(E_w)$ embeds in G_f, and loc_w is the resulting representation of $\mathrm{GL}_n(E_w)$ on $\mathrm{Forms}_\kappa(V)$; the *local group algebra* at w is the $\bar{\mathbb{Z}}_\ell$-algebra of operators on $\mathrm{Forms}_\kappa(V)$ generated by the translations by the elements of G_f supported at w and $\bar{w}$.

Base datum. C is a natural number and $\mathfrak{D}$ a *base datum* for (V, κ, C): a finite set of finite places of E; a space of forms that is stable under every operator commuting with the translations by the components, at these places and their conjugates, of the standard maximal compact subgroup, and on which some open subgroup acts through a character by its components there; and such that for every $\gamma \in G_{\mathrm{f}}$ the order of the finite group of the rational points that lie in the conjugate by γ of the standard maximal compact subgroup divides C.

Levels. For a family $\mathbf{w}$ of finite places of E, $\mathrm{Forms}_{\mathfrak{D}}(\mathbf{w})$ is the space of the forms in the space of $\mathfrak{D}$ that are fixed by the principal congruence subgroup at the places of $\mathbf{w}$ of the standard maximal compact subgroup, taken in its components outside the places of $\mathfrak{D}$ and their conjugates; it is finite-dimensional. For the empty family this is the space $\mathrm{Forms}_{\mathfrak{D}}(\emptyset)$ of the *base-level forms*, which is contained in every $\mathrm{Forms}_{\mathfrak{D}}(\mathbf{w})$. A form is *integral* if ι maps the coordinates of its values at the points of G_{f} that are integral above ℓ into $\bar{\mathbb{Z}}_\ell$. $\mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w})$ is the $\bar{\mathbb{Z}}_\ell$-algebra of the $\mathbb{C}$-linear endomorphisms of $\mathrm{Forms}_{\mathfrak{D}}(\mathbf{w})$ that map integral forms to integral forms.

Eigensystems. An *isotypic space* is a subspace of $\mathrm{Forms}_{\kappa}(V)$ of the following kind: for some system of Hecke eigenvalues at all finite places, it consists of the forms that at almost all split places w are fixed by $\mathrm{GL}_n(\mathcal{O}_w)$ and are eigenvectors of the spherical Hecke operators at w with the eigenvalues of the system. Each isotypic space ξ has a chosen such system, and $\mathsf{h}_{\xi,w} \in \mathbb{C}[X]$ is the *Hecke polynomial* of ξ at w, of degree n, formed from the chosen eigenvalues $\theta_{\xi,w,i} \in \mathbb{C}$ at w:

$$\mathsf{h}_{\xi,w} = \sum_{i=0}^{n} (-1)^i \, q_w^{i(i-1)/2} \, \theta_{\xi,w,i} \, X^{n-i}.$$

$\mathrm{Sys}_{\mathfrak{D}}(\mathbf{w})$ is the set of the isotypic spaces that contain a non-zero form of $\mathrm{Forms}_{\mathfrak{D}}(\mathbf{w})$, the *eigensystems of level* $\mathbf{w}$; it is finite, and $\mathrm{Sys}_{\mathfrak{D}}(\emptyset) \subseteq \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w})$ is the set of the *eigensystems of base level*.

Diagonal operators and the Hecke algebra. For a function $s \colon \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}) \to \bar{\mathbb{Q}}_\ell$, $\langle s \rangle$ is a chosen endomorphism of $\mathrm{Forms}_{\kappa}(V)$ that multiplies every form of each $\xi \in \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w})$ by $\iota^{-1}(s(\xi))$; it preserves $\mathrm{Forms}_{\mathfrak{D}}(\mathbf{w})$. The *Hecke algebra* $\mathbb{T}_{\mathfrak{D}}(\mathbf{w})$ is the $\bar{\mathbb{Z}}_\ell$-algebra of the functions $T \colon \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}) \to \bar{\mathbb{Q}}_\ell$ for which the restriction of $\langle T \rangle$ to $\mathrm{Forms}_{\mathfrak{D}}(\mathbf{w})$ lies in $\mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w})$; for such T the symbol $\langle T \rangle$ also denotes this restriction, and $T \mapsto \langle T \rangle$ is a homomorphism of $\bar{\mathbb{Z}}_\ell$-algebras from $\mathbb{T}_{\mathfrak{D}}(\mathbf{w})$ to $\mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w})$.

Auxiliary places. J is a finite set, ν a natural number, and Q is a family of *Taylor–Wiles places* for (E, J, ℓ^ν): for each $j \in J$ a finite place w_j of E whose norm q_j is a prime number, a place of $\bar{\mathbb{Q}}$ above w_j, and an element $\mathrm{Fr}_j \in G_E$ that is a geometric Frobenius element at that place; the places w_j are distinct, no conjugate $\bar{w}_i$ is among them, and ℓ^ν divides $q_j - 1$ for every j. $\tau_j \in G_E$ is the chosen generator of tame inertia at the place above w_j, and $\mathbf{w}_Q = (w_j)_{j \in J}$. The group $\prod_{j \in J} \mathrm{GL}_n(\mathbb{F}_{w_j})$ acts on $\mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$ by translation through lifts; this is a homomorphism

$$\mathrm{res}_{\mathfrak{D},Q} \colon \prod_{j \in J} \mathrm{GL}_n(\mathbb{F}_{w_j}) \longrightarrow \mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w}_Q), \qquad \text{and} \qquad \mathrm{Nm}_{\mathfrak{D},Q} = \sum_{\gamma} \mathrm{res}_{\mathfrak{D},Q}(\gamma)$$

is the sum over the whole group. For $j \in J$ and $\gamma \in \mathrm{GL}_n(\mathbb{F}_{w_j})$, $\gamma^{(j)}$ is the element of the product with γ at j and 1 elsewhere. $\mathrm{Loc}_{\mathfrak{D}}(Q, j)$ is the $\bar{\mathbb{Z}}_\ell$-subalgebra of $\mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w}_Q)$ of the operators that are restrictions of elements of the local group algebra at w_j.

Setting 2.2 (the twisted family of Galois representations; from units 37, 80, 102 and 106)**.** In Setting 2.1, suppose $n \geq 1$, that V is *standard* (all its signs are 1 and its adelic point is the identity) and that E carries the Weil data of the library. The Weil data contain a primitive 5040-th root of unity of E; consequently a finite place w of E with $\bar{w} = w$ has norm r^2 for an integer $r \equiv 3$ modulo 4, and a finite place of prime norm has $w \neq \bar{w}$ (background library, elementary). Every isotypic space ξ has a chosen homomorphism $\varrho_\xi \colon G_E \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$: a representation attached to the Hecke polynomials of ξ if one exists (Input 3), and arbitrary otherwise. Let $\omega \colon G_E \to \bar{\mathbb{Z}}_\ell$ be a homomorphism and let $\mathfrak{r} = (\mathfrak{r}_\xi)_\xi$ be a family of homomorphisms $\mathfrak{r}_\xi \colon G_E \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$, one for each isotypic space ξ, such that

$$\mathfrak{r}_\xi(\gamma) = \omega(\gamma) \cdot \varrho_\xi(\gamma) \qquad \text{for every isotypic space } \xi \text{ and every } \gamma \in G_E.$$

Setting 2.3 (auxiliary places off level and of trivial character)**.** In Settings 2.1 and 2.2, suppose:

(a) for every $j \in J$ the place w_j is *off level* for $\mathfrak{D}$ at the base level: neither w_j nor $\bar{w}_j$ is a place of $\mathfrak{D}$, the prime ℓ does not lie under w_j, and for every $\xi \in \mathrm{Sys}_{\mathfrak{D}}(\emptyset)$ the base-level forms in ξ are fixed by $\mathrm{GL}_n(\mathcal{O}_{w_j})$ and are eigenvectors of the spherical Hecke operators at w_j with the chosen eigenvalues $\theta_{\xi,w_j,i}$ of ξ;

(b) $\omega(\tau_j) = 1$ for every $j \in J$.

Definition 2.4 (the truncated valuation; unit 92)**.** For $z \in \bar{\mathbb{Q}}_\ell$ put $\mathrm{ord}_\ell^+(0) = \infty$ and, for $z \neq 0$,

$$\mathrm{ord}_\ell^+(z) = \max(0, -\log_\ell |z|) \in [0, \infty).$$

So $\mathrm{ord}_\ell^+(\ell) = 1$, and $\mathrm{ord}_\ell^+(z) = 0$ whenever $|z| \geq 1$: for instance $\mathrm{ord}_\ell^+(1/\ell) = 0$. This is not the ℓ-adic valuation, which is negative at $1/\ell$; the two agree on $\bar{\mathbb{Z}}_\ell$. If $|z| \leq |z'|$ then $\mathrm{ord}_\ell^+(z') \leq \mathrm{ord}_\ell^+(z)$.

Definition 2.5 (root strings; unit 33)**.** Let $G, G' \in \mathbb{C}[X]$, let q and L be natural numbers and $\theta \in \mathbb{C}$. The polynomial G *has a root string of length L and ratio q at θ with cofactor dividing G'* if there is a polynomial $G_1 \in \mathbb{C}[X]$ that divides G' and satisfies

$$G(\theta X) = (X-1)(X-q)\cdots(X-q^{L-1}) \cdot G_1.$$

For $\theta \neq 0$ the numbers $\theta, \theta q, \ldots, \theta q^{L-1}$ are then roots of G. Polynomials over $\bar{\mathbb{Z}}_\ell$ and elements of $\bar{\mathbb{Z}}_\ell$ are used here through the map $\iota^{-1} \colon \bar{\mathbb{Z}}_\ell \to \mathbb{C}$.

Definition 2.6 (trace ideals; unit 138 of the argument)**.** Let I be a set, $\mathcal{S}$ a $\bar{\mathbb{Z}}_\ell$-subalgebra of the ring of functions from I to the polynomial ring $\bar{\mathbb{Q}}_\ell[X_j : j \in J]$, and $\mathcal{J}$ an $\mathcal{S}$-submodule of that ring. Let further a group be given, a family $\mathfrak{r} = (\mathfrak{r}_i)_{i \in I}$ of maps from the group to the $n \times n$ matrices over $\bar{\mathbb{Q}}_\ell$, an element $y \in \bar{\mathbb{Z}}_\ell$, two families (τ_j) and (Fr_j) of elements of the group, indexed by a set with a map $j \mapsto j^\flat$ to J, and a family $\varepsilon = (\varepsilon_j)_{j \in J}$ of polynomials in $\bar{\mathbb{Z}}_\ell[X]$. The pair $(\mathcal{S}, \mathcal{J})$ is a *trace ideal for* $(\mathfrak{r},\ y,\ (\tau_j),\ (\mathrm{Fr}_j),\ j \mapsto j^\flat,\ \varepsilon)$ if

(a) for every $j \in J$ the constant function with value X_j lies in $\mathcal{S}$;

(b) for every element γ of the group and every natural number e, the function that sends $i \in I$ to the coefficient of X^e in the characteristic polynomial of $\mathfrak{r}_i(\gamma)$, a constant polynomial, lies in $\mathcal{S}$;

(c) for every element γ of the group, the function $i \mapsto y \cdot \operatorname{tr} \mathfrak{r}_i(\gamma)$ lies in $\mathcal{J}$, and so does, for every index j of the two families, the function

$$i \;\longmapsto\; \operatorname{tr}\big((\mathfrak{r}_i(\tau_j) - 1)\,\mathfrak{r}_i(\gamma)\big) - X_{j^\flat} \cdot \operatorname{tr}\big(((X-1)\,\varepsilon_{j^\flat})(\mathfrak{r}_i(\mathrm{Fr}_j)) \cdot \mathfrak{r}_i(\gamma)\big).$$

In Theorem A the set I is $\mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}_Q)$, the group is G_E, the family $\mathfrak{r}$ is that of Setting 2.2 restricted to $\mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}_Q)$, the two families are indexed by J itself and $j^\flat = j$. The formal definition, and that of Definition 2.7, is made for any commutative ring in the place of $\bar{\mathbb{Z}}_\ell$, any commutative algebra over it in the place of $\bar{\mathbb{Q}}_\ell$ and matrices over a ring between the two; only the case stated here is used.

Definition 2.7 (Ribet coefficients; unit 138 of the argument)**.** Let I, $\mathcal{S}$ and $\mathcal{J}$ be as in Definition 2.6, let $\beta\colon I \to \bar{\mathbb{Q}}_\ell$ be a function, $\mathcal{X}$ a subset of I and $\pi \in \mathcal{X}$. Write $\mathbf{1} \in \mathbb{N}^J$ for the exponent of the monomial $\prod_{j\in J} X_j$, and $\mathbf{d} < \mathbf{1}$ for the exponents $\mathbf{d} \neq \mathbf{1}$ of the monomials that divide it. An element $\eta \in \bar{\mathbb{Q}}_\ell$ is a *Ribet coefficient of* $(\mathcal{S}, \mathcal{J})$ *for* β *at* $\pi \in \mathcal{X}$ if the function β, with constant polynomials as values, lies in $\mathcal{S}$ and there is $\Phi \in \mathcal{S}$ such that: the product $\Phi \cdot \beta$ lies in $\mathcal{J}$; for every $i \in \mathcal{X}$, $\beta(i) = 0$ and the coefficient of the polynomial $\Phi(i)$ at $X^{\mathbf{d}}$ vanishes for every $\mathbf{d} < \mathbf{1}$; and the coefficient of $\Phi(\pi)$ at $\prod_{j\in J} X_j$ is η.

Definition 2.8 (two subgroups at a place; units 30 and 122)**.** Let w be a finite place of a number field E and $n \geq 1$. Inside $\mathrm{GL}_n(\mathcal{O}_w)$, the subgroup Par_w consists of the matrices whose first column lies in $\mathfrak{p}_w$ below its first entry, and the *pro-p Iwahori subgroup* $\mathrm{Iw}_w \subseteq \mathrm{Par}_w$ of the matrices whose reduction modulo $\mathfrak{p}_w$ is upper triangular with 1 on the diagonal. More generally a function from $\{1, \dots, n\}$ to $\mathbb{N}$ has a *block parahoric subgroup*: the matrices in $\mathrm{GL}_n(\mathcal{O}_w)$ whose entry in row i and column i' lies in $\mathfrak{p}_w$ whenever the function is smaller at i' than at i; Par_w belongs to the function that is 0 at 1 and 1 elsewhere, and $\mathrm{GL}_n(\mathcal{O}_w)$ to the function 0. For $\gamma \in \mathrm{GL}_n(E_w)$ and a representation of $\mathrm{GL}_n(E_w)$, U_γ is the Hecke operator of the double coset $\mathrm{Iw}_w\,\gamma\,\mathrm{Iw}_w$: the sum of the operators of representatives of the cosets $\gamma'\,\mathrm{Iw}_w$ contained in it. For a uniformiser ϖ of E_w, U_ϖ is this operator for the diagonal matrix with entries $\varpi, 1, \dots, 1$.

Inputs from the background library, 1: the forms. They are taken for granted, as the dossier of this unit reads them. $\mathbf{w}$ is a family of finite places of E indexed by a finite set.

Input 1 (units 102 and 130; elementary). In Setting 2.1: (a) $\mathrm{Forms}_{\mathfrak{D}}(\mathbf{w})$ is the direct sum of the spaces $\xi \cap \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w})$, $\xi \in \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w})$. (b) An element of the local group algebra at a place w maps every isotypic space into itself, and it commutes with the translation by every element of G_{f} that is trivial at w and $\bar{w}$. (c) If ℓ does not lie under w, the local group algebra at w maps integral forms to integral forms; if $w \neq \bar{w}$, the spherical Hecke operators at w belong to it. (d) The Hecke algebra $\mathbb{T}_{\mathfrak{D}}(\mathbf{w})$ is closed in the space of all functions from the finite set $\mathrm{Sys}_{\mathfrak{D}}(\mathbf{w})$ to $\bar{\mathbb{Q}}_{\ell}$. (e) All but finitely many finite places w of E are off level for $\mathfrak{D}$ at $\mathbf{w}$: condition (a) of Setting 2.3 holds for them, with $\mathbf{w}$ in the place of the empty family.

Input 2 (unit 130; the action at the auxiliary places; (a) to (e) elementary). In Setting 2.1: (a) a form $\varphi \in \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$ is of base level if and only if $\mathrm{res}_{\mathfrak{D},Q}(\gamma)\varphi = \varphi$ for every $\gamma \in \prod_j \mathrm{GL}_n(\mathbb{F}_{w_j})$. (b) Every element of $\mathrm{Loc}_{\mathfrak{D}}(Q,j)$ commutes with $\langle T \rangle$ for every $T \in \mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)$. Suppose moreover that the places w_j are off level (condition (a) of Setting 2.3) and let $j \in J$. Then: (c) for every $\gamma \in \mathrm{GL}_n(\mathcal{O}_{w_j})$ with reduction $\bar{\gamma}$ and every $\varphi \in \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$, $\mathrm{res}_{\mathfrak{D},Q}(\bar{\gamma}^{(j)})\varphi = \mathrm{loc}_{w_j}(\gamma)\varphi$; (d) $\mathrm{res}_{\mathfrak{D},Q}(\gamma^{(j)}) \in \mathrm{Loc}_{\mathfrak{D}}(Q,j)$ for every $\gamma \in \mathrm{GL}_n(\mathbb{F}_{w_j})$; (e) if $Y \in \mathrm{Loc}_{\mathfrak{D}}(Q,j)$ has all its values fixed by Iw_{w_j}, then for every $\gamma \in \mathrm{GL}_n(E_{w_j})$ and every $G \in \bar{\mathbb{Z}}_{\ell}[X]$ there is $Y' \in \mathrm{Loc}_{\mathfrak{D}}(Q,j)$ with $Y'(\varphi) = G(U_\gamma)(Y(\varphi))$ for all $\varphi \in \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$, where U_γ is taken in loc_{w_j}; (f) *(Higman's criterion up to C)* for every subgroup Γ of $\prod_j \mathrm{GL}_n(\mathbb{F}_{w_j})$ there is $\Psi \in \mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w}_Q)$ with

$$\sum_{\gamma \in \Gamma} \mathrm{res}_{\mathfrak{D},Q}(\gamma^{-1}) \cdot \Psi \cdot \mathrm{res}_{\mathfrak{D},Q}(\gamma) = C$$

(the formal statement is for a finite group with an injective homomorphism to the product, which is the same). *Reference for* (f), as the catalogue of the handout gives it: R. Taylor, A. Wiles, Ann. of Math. 141 (1995); L. Clozel, M. Harris, R. Taylor, Publ. IHÉS 108 (2008); D. G. Higman, Duke Math. J. 21 (1954); place not confirmed. *Relation* (second reader): classical freeness assumes trivial stabilisers; the declaration constructs a trace equal to a stabiliser multiple.

Inputs from the background library, continued.

Input 3 (units 3, 5, 80 and 102; attached Galois representations). In Setting 2.2, let ξ be an isotypic space that contains a non-zero form. Then ϱ_ξ is attached to the Hecke polynomials of ξ in the sense of the library, and in particular: for every $\gamma \in G_E$ and every natural number e, the coefficient of X^e in $\det(X - \varrho_\xi(\gamma))$ is the limit of the ι-images of the coefficients of X^e in the polynomials $\mathsf{h}_{\xi,w}$, along the filter of most Frobenius data $(w, \cdot, \mathrm{Fr})$ whose Frobenius element tends to γ. This filter is proper, its data live at places of prime norm, and a property that holds for the data at all but finitely many places holds along it. *Reference*: G. Chenevier, M. Harris, Construction of automorphic Galois representations II, Camb. J. Math. 1 (2013), Theorem 3.2.3. *Relation*: a weaker form (closeness at most Frobenius elements in place of equality at all unramified split places); the second reader derives it from exact unramified compatibility after base change and a decomposition of the discrete constituents.

Input 4 (unit 102; elementary). In Setting 2.1 the space $\mathrm{Forms}_\kappa(V)$ carries a positive definite hermitian form $(\varphi' \mid \varphi)$ that is invariant under right translation by G_{f}. In particular the operators $\mathrm{loc}_w(\gamma)$ and, on $\mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$, the operators $\mathrm{res}_{\mathfrak{D},Q}(\gamma)$ preserve it.

Inputs from the background library, 2: one split place. Here w is a finite place of E with $w \neq \bar{w}$, ϖ is a uniformiser of E_w and $u \in E_w$ a unit of $\mathcal{O}_w$.

Input 5 (unit 33; the chain lemma, second part). In Setting 2.1, let $\varphi' \in \mathrm{Forms}_\kappa(V)$ be non-zero, fixed by $\mathrm{GL}_n(\mathcal{O}_w)$ and an eigenvector of the spherical Hecke operators at w with eigenvalues θ_i, and let $G = \sum_{i=0}^{n} (-1)^i q_w^{i(i-1)/2} \theta_i X^{n-i}$. Let $\theta \in \mathbb{C}$ be non-zero, e a natural number and $G' \in \mathbb{C}[X]$, and put $G''(X) = (\theta X - q_w^e) \cdot G'(\theta X)$. If G has a root string of length $e+1$ and ratio q_w at θ^{-1} with cofactor dividing G', then

$$(\varphi' \mid G''(U_\varpi)\, \varphi) = 0 \qquad \text{for every } \mathrm{Iw}_w\text{-fixed } \varphi \in \mathrm{Forms}_\kappa(V),$$

where U_ϖ is taken in loc_w. The library proves this for every smooth representation of $\mathrm{GL}_n(E_w)$ with an invariant positive definite hermitian form. *Reference*, as the catalogue gives it: M. Tadić, Ann. Sci. ENS 19 (1986); J. Bernstein, Lecture Notes in Math. 1041 (1984); place not confirmed, and the orthogonality stated here has no cited source.

Input 6 (unit 30; elementary). For every function $\theta \colon E_w \to \mathbb{C}$, the operator U_ϖ of loc_w maps into itself the space of the Iw_w-fixed forms on which every $\gamma \in \mathrm{Par}_w$ acts by the scalar $\theta(\gamma_{11})$, where γ_{11} is the first entry of γ.

Input 7 (unit 89; the local Artin map). Let a place of $\bar{\mathbb{Q}}$ above a finite place w of E be given, with its decomposition group in G_E, and let Art_w be the local Artin map from $E_w^\times$ to the abelianisation of that group, normalised so that uniformisers go to geometric Frobenius elements. (a) If τ is the chosen generator of tame inertia at that place, there is a unit u of $\mathcal{O}_w$ with $\mathrm{Art}_w(u)$ the class of τ whose reduction generates $\mathbb{F}_w^\times$. (b) If $\mathrm{Fr} \in G_E$ is a geometric Frobenius element at that place, there is a uniformiser ϖ with $\mathrm{Art}_w(\varpi)$ the class of Fr. *Reference*: J.-P. Serre, Local Fields (1979), Chapters XII to XIV. *Relation*: weaker forms (existence statements in the image of local reciprocity).

Inputs from the background library, 2, continued. The place w, the uniformiser ϖ and the unit u are as before.

Input 8 (unit 32; local–global compatibility at tame level). In Setting 2.2, suppose that ℓ does not lie under w, and let a place of $\bar{\mathbb{Q}}$ above w be given together with elements Fr and τ of its decomposition group such that $\mathrm{Art}_w(\varpi)$ is the class of Fr, $\mathrm{Art}_w(u)$ is the class of τ, Fr is a geometric Frobenius element and τ lies in the inertia group. Let ξ be an isotypic space and $\varphi \in \xi$.
(a) Suppose $\varphi \neq 0$ is Iw_w-fixed and every γ in the block parahoric subgroup of some function acts on φ by a scalar $\theta(\gamma) \in \mathbb{C}$. Let θ_i be the value of θ at the diagonal matrix with u at the place i and 1 elsewhere, and suppose that $\theta_i = \theta_{i'}$ only if the function has the same value at i and i'. Then every polynomial over $\bar{\mathbb{Q}}_\ell$ that vanishes at all $\iota(\theta_i)$ vanishes at the matrix $\varrho_\xi(\tau)$; the matrices $\varrho_\xi(\mathrm{Fr})$ and $\varrho_\xi(\tau)$ commute; and there are characters $\theta'_1, \dots, \theta'_n$ of $E_w^\times$, forming an exponent of loc_w with an exponent functional that does not vanish on φ, such that for every natural number e the polynomial $\prod_i (X - \iota(\theta'_i(\varpi)\theta_i^e))$ vanishes at $\varrho_\xi(\mathrm{Fr})\varrho_\xi(\tau)^e$.
(b) Suppose φ is Iw_w-fixed and every $\gamma \in \mathrm{Par}_w$ acts on φ by the scalar $\theta(\gamma_{11})$, for a function $\theta\colon E_w \to \mathbb{C}$ with $\theta(u) \neq 0$ and $\theta(u)^{n!} \neq 1$. Then for every $G \in \mathbb{C}[X]$ and every $n \times n$ matrix y over $\bar{\mathbb{Q}}_\ell$,

$$\iota^{-1}\,\mathrm{tr}\big((\varrho_\xi(\tau) - 1)\,y\big) \cdot G(U_\varpi)\varphi = \iota^{-1}\,\mathrm{tr}\big(G^\iota(\varrho_\xi(\mathrm{Fr}))\,(\varrho_\xi(\tau) - 1)\,y\big) \cdot \varphi,$$

where $G^\iota \in \bar{\mathbb{Q}}_\ell[X]$ is the image of G under ι. *Reference*, as the catalogue gives it: R. Taylor, T. Yoshida, J. Amer. Math. Soc. 20 (2007); M. Harris, R. Taylor (2001); place not confirmed. *Relation* (second reader): for (a), the published comparison away from ℓ needs a local type and exponent calculation to recover these simultaneous matrix identities; for (b), a matrix-weighted identity for a specific Hecke operator requires additional parahoric linear algebra. Both comparisons are open.

Inputs 1(a) and 4 are used in the following form. Two endomorphisms of $\mathrm{Forms}_{\mathfrak{D}}(\mathbf{w})$ that agree on $\xi \cap \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w})$ for every $\xi \in \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w})$ are equal; and since $\langle T \rangle$ multiplies the forms of ξ by $\iota^{-1}(T(\xi))$, a form $\varphi \in \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w})$ with components $\varphi_\xi \in \xi$ has $\langle T \rangle \varphi = \sum_\xi \iota^{-1}(T(\xi))\varphi_\xi$.

3 The line stabiliser in GL_n of a finite field

Lemma 3.1 (a character of the line stabiliser whose transfer is a function of the determinant). *Let E be a number field, w a finite place of E, $n \geq 1$ an integer and M a natural number that divides $q_w - 1$. Let $u \in E_w$ be a unit of $\mathcal{O}_w$ whose reduction $\bar{u}$ generates the group $\mathbb{F}_w^\times$. Then there are a subgroup H of $\mathrm{GL}_n(\mathbb{F}_w)$, a homomorphism $\alpha\colon H \to \mathbb{Z}/M$, two functions $\lg, \delta\colon E_w \to \mathbb{Z}/M$ and an element $\gamma_0 \in \mathrm{GL}_n(\mathbb{F}_w)$ with the following properties, in which $\bar{\gamma} \in \mathrm{GL}_n(\mathbb{F}_w)$ is the reduction of $\gamma \in \mathrm{GL}_n(\mathcal{O}_w)$, with first entry γ_{11}.*

(a) *For every $\gamma \in \mathrm{GL}_n(\mathcal{O}_w)$: $\bar{\gamma} \in H$ if and only if $\gamma \in \mathrm{Par}_w$; if $\bar{\gamma} \in H$ then $\alpha(\bar{\gamma}) = \lg(\gamma_{11})$; $\mathrm{Tr}_\alpha(\bar{\gamma}) = \delta(\det\gamma)$; and if $\gamma \in \mathrm{Iw}_w$ then $\mathrm{Tr}_\alpha(\bar{\gamma}) = 0$ and $\alpha(\bar{\gamma}) = 0$.*
(b) *$\lg(u) = 1$, $\mathrm{Tr}_\alpha(\gamma_0) = \delta(u)$, and $n! \cdot \mathrm{Tr}_\alpha(\gamma_0) = n!$ in $\mathbb{Z}/M$.*

Proof. Write $\Gamma = \mathrm{GL}_n(\mathbb{F}_w)$. Let H be the stabiliser in Γ of the first coordinate line: the matrices h whose first column vanishes below its first entry h_{11}. For $h, h' \in H$ the first entry of hh' is $h_{11}h'_{11}$, so $h \mapsto h_{11}$ is a homomorphism from H to $\mathbb{F}_w^\times$. This group is cyclic of order $q_w - 1$ and generated by $\bar{u}$; since M divides $q_w - 1$ there is a homomorphism $\lg_0 \colon \mathbb{F}_w^\times \to \mathbb{Z}/M$ with $\lg_0(\bar{u}) = 1$. Put $\alpha(h) = \lg_0(h_{11})$, and for $z \in \mathbb{F}_w^\times$ let $\delta_0(z) = \mathrm{Tr}_\alpha(\mathrm{diag}(z, 1, \dots, 1))$; this is a homomorphism from $\mathbb{F}_w^\times$ to $\mathbb{Z}/M$.

The transfer is $\delta_0 \circ \det$. Both Tr_α and $\delta_0 \circ \det$ are homomorphisms from Γ to a commutative group, and Γ is generated by the diagonal matrices and the transvections: the matrices that differ from the identity matrix by one entry $z \in \mathbb{F}_w$, in a row i and a column $i' \neq i$. A transvection has determinant 1. For fixed $i \neq i'$ the transvection with entry $z + z'$ is the product of those with entries z and z', so the transfer of the transvection with entry z is a homomorphism in z from the additive group of $\mathbb{F}_w$, of order q_w, to $\mathbb{Z}/M$; so q_w times each of its values is 0, and since $q_w \equiv 1$ modulo M the values themselves are 0. A diagonal matrix $\mathrm{diag}(z_1, \dots, z_n)$ is the product of the matrices with z_i at the place i and 1 elsewhere. Such a matrix is conjugate, by a permutation matrix, to $\mathrm{diag}(z_i, 1, \dots, 1)$, and the transfer, a homomorphism to a commutative group, is constant on conjugacy classes. So $\mathrm{Tr}_\alpha(\mathrm{diag}(z_1, \dots, z_n)) = \sum_i \delta_0(z_i) = \delta_0(z_1 \cdots z_n)$. Hence $\mathrm{Tr}_\alpha = \delta_0 \circ \det$ on Γ.

The functions and the element. For a unit z of $\mathcal{O}_w$ with reduction $\bar{z}$ put $\lg(z) = \lg_0(\bar{z})$ and $\delta(z) = \delta_0(\bar{z})$; at the other elements of E_w the two functions take arbitrary values, which play no role. Let $\gamma_0 = \mathrm{diag}(\bar{u}, 1, \dots, 1)$.

Property (a). Let $\gamma \in \mathrm{GL}_n(\mathcal{O}_w)$. Its reduction lies in H exactly when the first column of γ lies in $\mathfrak{p}_w$ below the first entry, that is, when $\gamma \in \mathrm{Par}_w$. In that case γ_{11} is a unit and $\alpha(\bar{\gamma}) = \lg_0(\bar{\gamma}_{11}) = \lg(\gamma_{11})$. The determinant commutes with reduction, so $\mathrm{Tr}_\alpha(\bar{\gamma}) = \delta_0(\det \bar{\gamma}) = \delta(\det \gamma)$. If $\gamma \in \mathrm{Iw}_w$, then $\bar{\gamma}$ is upper triangular with 1 on the diagonal: it lies in H, its determinant is 1 and its first entry is 1, whence $\mathrm{Tr}_\alpha(\bar{\gamma}) = \delta_0(1) = 0$ and $\alpha(\bar{\gamma}) = \lg_0(1) = 0$.

Property (b). $\lg(u) = \lg_0(\bar{u}) = 1$ and $\mathrm{Tr}_\alpha(\gamma_0) = \delta_0(\bar{u}) = \delta(u)$ by the definitions. For the last equation consider the scalar matrix $\bar{u} \cdot 1_n$. It is central in Γ and lies in H, so for every representative x of a coset xH one has $(\bar{u} \cdot 1_n)x = x\,(\bar{u} \cdot 1_n)$, and formula (1) for the transfer, in the proof of Proposition 4.4, gives $\mathrm{Tr}_\alpha(\bar{u} \cdot 1_n) = [\Gamma : H] \cdot \alpha(\bar{u} \cdot 1_n) = [\Gamma : H]$. The index of H is the number of lines in $\mathbb{F}_w^n$, which is $1 + q_w + \cdots + q_w^{n-1} \equiv n$ modulo M. On the other hand $\mathrm{Tr}_\alpha(\bar{u} \cdot 1_n) = \delta_0(\bar{u}^n) = n \cdot \delta_0(\bar{u})$. So $n \cdot \mathrm{Tr}_\alpha(\gamma_0) = n$ in $\mathbb{Z}/M$, and multiplication by $(n-1)!$ gives $n! \cdot \mathrm{Tr}_\alpha(\gamma_0) = n!$. □

LEAN `exists_line_stabilizer_log` · the formal text takes the unit as an element of $E_w^\times$ with a proof that it lies in the unit group of $\mathcal{O}_w$

4 Weighted sums over a finite group and the first-order lemma

Definition 4.1 (weighted sums)**.** Let Γ be a finite group, R a ring and M a natural number. For homomorphisms $\rho \colon \Gamma \to R$ and $\chi \colon \Gamma \to \mathbb{Z}/M$ and a function $f \colon \mathbb{Z}/M \to \mathbb{Z}$, the *weighted sum* is

$$\mathsf{W}_{\rho,\chi}(f) = \sum_{\gamma \in \Gamma} f(\chi(\gamma)) \cdot \rho(\gamma) \ \in R.$$

For fixed ρ and χ it is additive in f, and $\mathsf{W}_{\rho,\chi}(\,\cdot\,)$ is this additive map from the group of the functions $\mathbb{Z}/M \to \mathbb{Z}$ to R. The constant function with value 1 is written 1, and for $z' \in \mathbb{Z}/M$ or $z' \in \mathbb{N}$ the function $z \mapsto f(z - z')$ is written $f(\,\cdot - z')$.

LEAN `weightedSum`

Lemma 4.2 (translation of the weights)**.** *Let Γ be a finite group, R a ring, M a natural number, $\rho\colon \Gamma \to R$ and $\chi\colon \Gamma \to \mathbb{Z}/M$ homomorphisms, $f\colon \mathbb{Z}/M \to \mathbb{Z}$ a function and $\gamma \in \Gamma$. Then*

$$\rho(\gamma) \cdot \mathsf{W}_{\rho,\chi}(f) = \mathsf{W}_{\rho,\chi}(f(\,\cdot - \chi(\gamma))).$$

Proof. $\rho(\gamma) \sum_{\gamma'} f(\chi(\gamma'))\rho(\gamma') = \sum_{\gamma'} f(\chi(\gamma'))\rho(\gamma\gamma')$, and the substitution $\gamma'' = \gamma\gamma'$ turns this into $\sum_{\gamma''} f(\chi(\gamma'') - \chi(\gamma))\rho(\gamma'')$. □

LEAN `mul_weightedSum`

Lemma 4.3 (weights that agree modulo M)**.** *Let R be a ring, S a ring, M a natural number and $\sigma\colon R \to S$ a ring homomorphism with $M \cdot 1_S = 0$. Let I be a set, B an additive map from the group of the functions $I \to \mathbb{Z}$ to R, and $f, f'\colon I \to \mathbb{Z}$ two functions with $f'(i) \equiv f(i)$ modulo M for every $i \in I$. Then $\sigma(B(f)) = \sigma(B(f'))$.*

Proof. There is a function $f''\colon I \to \mathbb{Z}$ with $f - f' = M \cdot f''$, so

$$\sigma(B(f)) - \sigma(B(f')) = M \cdot \sigma(B(f'')) = 0.$$

□

LEAN `map_eq_of_weights_modEq`

In the next statement $\partial_e f = f(\,\cdot - e) - f$ is the difference of a function $f\colon \mathbb{Z}/M \to \mathbb{Z}$ with step e, and $\rho|_H$ is the restriction of ρ to a subgroup H.

Proposition 4.4 (the first-order lemma)**.** *Let Γ be a finite group, R a ring, M a natural number, $\rho\colon \Gamma \to R$ a homomorphism, H a subgroup of Γ, S a ring and $\sigma\colon R \to S$ a ring homomorphism with $M \cdot 1_S = 0$. Let $\alpha\colon H \to \mathbb{Z}/M$ be a homomorphism and B an additive map from the group of the functions $\mathbb{Z}/M \to \mathbb{Z}$ to R such that*

$$B(f) \cdot \rho(h) = B(f(\,\cdot - \alpha(h))) \qquad \text{for all } f \text{ and all } h \in H.$$

Let $\mathcal{A}$ be a subring of R that contains $\rho(\gamma)$ for every $\gamma \in \Gamma$ and $B(f)$ for every f, and let $(c_i)_{i\in I}$ be a family of elements of R, indexed by a set I, each of which commutes with every element of $\mathcal{A}$. Let $\Psi, \Pi \in R$, $\gamma_0 \in \Gamma$ and $C, e \in \mathbb{N}$ be such that

(H1) $\sum_{h\in H} \rho(h^{-1})\,\Psi\,\rho(h) = C$;
(H2) $\rho(h) \cdot B(1) = B(1)$ *for every* $h \in H$;
(H3) $\Pi \cdot B(1) = 0$;
(H4) $\Pi \cdot \rho(\gamma) = \Pi$ *for every* $\gamma \in \Gamma$;
(H5) *multiplication by the order* $|\Gamma|$ *is injective on* R;
(H6) $e \cdot \mathrm{Tr}_\alpha(\gamma_0) = e$ *in* $\mathbb{Z}/M$.

Then there is $F \in \mathcal{A}$ with the following two properties.

(i) *For every $\gamma \in \Gamma$ there is $Z \in R$ with*

$$\sigma\big(C \cdot F \cdot (\rho(\gamma) - 1)\big) = \sigma\Big(\Big(\sum_{\gamma'\in\Gamma} \rho(\gamma')\Big) \cdot Z\Big) \qquad \text{and} \qquad \Pi \cdot Z = 0.$$

(ii) *For all $i, i' \in I$ and $y \in R$: if*

$$B(\partial_e f)\cdot c_i = \mathsf{W}_{\rho|_H,\alpha}(\partial_e f)\cdot c_{i'} \quad \text{for every } f, \qquad \text{and}$$
$$\mathsf{W}_{\rho,\mathrm{Tr}_\alpha}(f)\cdot c_{i'} = (\rho(\gamma_0)-1)\cdot \mathsf{W}_{\rho,\mathrm{Tr}_\alpha}(f)\cdot y \quad \text{for every } f,$$

then

$$2e\cdot\sigma(F)\,\sigma(c_i) = 2e\cdot\sigma\Big(\sum_{\gamma\in\Gamma}\rho(\gamma)\Big)\,\sigma(y).$$

Proof. Coset sums. Fix a set $\Sigma \subseteq \Gamma$ of representatives of the left cosets γH. For $\gamma \in \Gamma$ let $[\gamma] \in \Sigma$ be the representative of γH and $(\gamma)_H = [\gamma]^{-1}\gamma \in H$, so that $\gamma = [\gamma]\cdot(\gamma)_H$. Put

$$Y_+ = \sum_{x\in\Sigma}\rho(x), \qquad Y_- = \sum_{x\in\Sigma}\rho(x^{-1}), \qquad \mathrm{Nm} = \sum_{\gamma\in\Gamma}\rho(\gamma),$$

and let $v\colon \mathbb{Z}/M \to \mathbb{Z}$ send a residue to its representative of least absolute value, so that $v(z)\equiv z$ modulo M. The transfer of α is, by definition and independently of Σ,

$$\mathrm{Tr}_\alpha(\gamma) = \sum_{x\in\Sigma}\alpha\big((\gamma x)_H\big) \qquad (\gamma\in\Gamma). \tag{1}$$

For a function $f_0\colon H\to\mathbb{Z}$ and $\gamma'\in\Gamma$,

$$Y_+\cdot\Big(\sum_{h\in H} f_0(h)\rho(h)\Big)\cdot\rho(\gamma'^{-1}) = \sum_{\gamma\in\Gamma} f_0\big((\gamma\gamma')_H\big)\,\rho(\gamma). \tag{2}$$

Indeed the left side is the sum of $f_0(h)\rho(xh\gamma'^{-1})$ over $(x,h)\in\Sigma\times H$, and $(x,h)\mapsto xh\gamma'^{-1}$ is a bijection from $\Sigma\times H$ to Γ whose inverse sends γ to $([\gamma\gamma'],(\gamma\gamma')_H)$. For $f_0=1$ and $\gamma'=1$ this reads $Y_+\sum_{h\in H}\rho(h)=\mathrm{Nm}$. Hence, by (H2) and then (H1) with h^{-1} in the place of h,

$$\mathrm{Nm}\cdot\Psi\cdot B(1) = Y_+\sum_{h\in H}\rho(h)\,\Psi\,\rho(h^{-1})\,B(1) = C\cdot Y_+\cdot B(1). \tag{3}$$

The operator. Let $F = Y_+\cdot B(-v)\cdot Y_-$. It lies in $\mathcal{A}$, because $\mathcal{A}$ is a subring that contains the $\rho(\gamma)$ and $B(-v)$.

Property (i). Let $\gamma\in\Gamma$. The map $x\mapsto[\gamma x]$ is a permutation of Σ, and $[\gamma x]^{-1}\gamma = (\gamma x)_H\,x^{-1}$; therefore $Y_-\,\rho(\gamma) = \sum_x \rho((\gamma x)_H)\,\rho(x^{-1})$, and by the hypothesis on B

$$F\cdot\rho(\gamma) = Y_+\sum_{x\in\Sigma} B\big(-v(\,\cdot - \alpha((\gamma x)_H))\big)\,\rho(x^{-1}).$$

For $z'\in\mathbb{Z}/M$ the functions $-v(\,\cdot - z')$ and $-v + v(z')\cdot 1$ agree modulo M at every point, so $\sigma(B(-v(\,\cdot - z'))) = \sigma(B(-v)) + v(z')\,\sigma(B(1))$ by Lemma 4.3. Hence

$$\sigma\big(F\cdot(\rho(\gamma)-1)\big) = \sigma\Big(Y_+\cdot B(1)\cdot\sum_{x\in\Sigma} v\big(\alpha((\gamma x)_H)\big)\,\rho(x^{-1})\Big).$$

Multiply by C and use (3): property (i) holds with

$$Z = \Psi\cdot B(1)\cdot\sum_{x\in\Sigma} v\big(\alpha((\gamma x)_H)\big)\,\rho(x^{-1}).$$

It remains to see that $\Pi\cdot\Psi\cdot B(1)=0$. By (H4), $\Pi\cdot\mathrm{Nm} = |\Gamma|\cdot\Pi$ and $\Pi\cdot Y_+ = [\Gamma:H]\cdot\Pi$. Multiplying (3) by Π on the left gives

$$|\Gamma|\cdot\Pi\Psi B(1) = C\,[\Gamma:H]\cdot\Pi B(1) = 0$$

by (H3), and (H5) gives $\Pi\Psi B(1) = 0$.

Property (ii). Let i, i' and y satisfy the two hypotheses of (ii). First, by (2) for $f_0 = v \circ \alpha$ and $\gamma' = x$, summed over $x \in \Sigma$,

$$Y_+ \cdot \mathsf{W}_{\rho|_H,\alpha}(v) \cdot Y_- = \sum_{\gamma\in\Gamma}\Big(\sum_{x\in\Sigma} v(\alpha((\gamma x)_H))\Big)\rho(\gamma),$$

and by (1) the inner sum is congruent to $v(\mathrm{Tr}_\alpha(\gamma))$ modulo M. So Lemma 4.3, applied to functions on Γ, gives

$$\sigma(Y_+ \cdot \mathsf{W}_{\rho|_H,\alpha}(v) \cdot Y_-) = \sigma(\mathsf{W}_{\rho,\mathrm{Tr}_\alpha}(v)) : \tag{4}$$

conjugating by the representatives sums the weights over the cosets. Secondly, take in the first hypothesis the function $f = -v \cdot (v + e)$. Since $v(z - e) \equiv v(z) - e$ modulo M,

$$\partial_e f(z) \equiv -(v(z) - e)\, v(z) + v(z)\, (v(z) + e) = 2e \cdot v(z) \pmod M$$

for every z: the function f is a discrete antiderivative of $2e \cdot v$, and this is where the factor 2 of the statement comes from. So, by Lemma 4.3,

$$\sigma(B(\partial_e f)) = 2e \cdot \sigma(B(v)) \qquad \text{and} \qquad \sigma(\mathsf{W}_{\rho|_H,\alpha}(\partial_e f)) = 2e \cdot \sigma(\mathsf{W}_{\rho|_H,\alpha}(v)).$$

Multiply the first hypothesis by Y_+ on the left and by Y_- on the right, apply σ, and move c_i and $c_{i'}$ to the right of $Y_- \in \mathcal{A}$, with which they commute. By the definition of F and by (4) this gives

$$-2e \cdot \sigma(F)\, \sigma(c_i) = 2e \cdot \sigma(\mathsf{W}_{\rho,\mathrm{Tr}_\alpha}(v))\, \sigma(c_{i'}) = \sigma(\mathsf{W}_{\rho,\mathrm{Tr}_\alpha}(2e \cdot v) \cdot c_{i'}).$$

Thirdly, by the second hypothesis for the function $2e \cdot v$ and by Lemma 4.2,

$$\mathsf{W}_{\rho,\mathrm{Tr}_\alpha}(2e \cdot v) \cdot c_{i'} = \mathsf{W}_{\rho,\mathrm{Tr}_\alpha}(2e \cdot (v(\,\cdot - \mathrm{Tr}_\alpha(\gamma_0)) - v)) \cdot y,$$

and $2e\,(v(z - \mathrm{Tr}_\alpha(\gamma_0)) - v(z)) \equiv -2e \cdot \mathrm{Tr}_\alpha(\gamma_0) \equiv -2e$ modulo M by (H6). By Lemma 4.3 once more, σ of the right side is

$$\sigma(\mathsf{W}_{\rho,\mathrm{Tr}_\alpha}(-2e \cdot 1))\, \sigma(y) = -2e \cdot \sigma(\mathrm{Nm})\, \sigma(y).$$

The last two displayed equations before this one together give $2e \cdot \sigma(F)\sigma(c_i) = 2e \cdot \sigma(\mathrm{Nm})\sigma(y)$. □

LEAN `one_prime_first_order` · the formal text writes the family as a map from an index type and takes Ψ, Π, γ_0, C and e as implicit arguments

5 The tie at one auxiliary place

Lemma 5.1 (testing on eigenvectors)**.** *In Setting 2.1, let $j \in J$, let $T, T' \in \mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)$, let Γ be a finite group, $\rho\colon \Gamma \to \mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w}_Q)$ a homomorphism with $\rho(\gamma) \in \mathrm{Loc}_{\mathfrak{D}}(Q, j)$ for every $\gamma \in \Gamma$, $\chi\colon \Gamma \to \mathbb{Z}/\ell^\nu$ a homomorphism, $f\colon \mathbb{Z}/\ell^\nu \to \mathbb{Z}$ a function and $Y \in \mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w}_Q)$. Let $e \in \mathbb{Z}/\ell^\nu$ and $\theta \in \mathbb{C}$, let $\mathcal{B}$ be a $\mathbb{C}$-linear map from $\mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$ to $\mathrm{Forms}_\kappa(V)$ such that $Y(\varphi) = \mathcal{B}(\mathsf{W}_{\rho,\chi}(f)\varphi)$ for every $\varphi \in \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$, and let $\tilde f\colon \mathbb{Z}/\ell^\nu \to \mathbb{C}$ be a function with $\tilde f(z - e) - \theta \cdot \tilde f(z) = f(z)$ for every z. Suppose that*

$$\iota^{-1}(T(\xi)) \cdot \varphi = \iota^{-1}(T'(\xi)) \cdot \mathcal{B}(\varphi)$$

for every $\xi \in \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}_Q)$*, every additive character* $\psi\colon \mathbb{Z}/\ell^\nu \to \mathbb{C}^\times$ *and every non-zero* $\varphi \in \xi \cap \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$ *such that* $\rho(\gamma)\varphi = \psi(\chi(\gamma))\cdot\varphi$ *for all* $\gamma \in \Gamma$ *and* $\psi(e) \neq \theta$*. Then*

$$\mathsf{W}_{\rho,\chi}(f)\cdot\langle T\rangle = Y\cdot\langle T'\rangle \qquad \text{in } \mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w}_Q).$$

Proof. By Input 1(a) it suffices to compare the two sides on a form $\varphi \in \xi \cap \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$, for each $\xi \in \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}_Q)$. For a function $f'\colon \mathbb{Z}/\ell^\nu \to \mathbb{C}$ put

$$\varphi[f'] = \sum_{\gamma\in\Gamma}\bigl(f'(\chi(\gamma)-e) - \theta\cdot f'(\chi(\gamma))\bigr)\cdot\rho(\gamma)\varphi.$$

This is linear in f', and $\varphi[\tilde{f}] = \mathsf{W}_{\rho,\chi}(f)\varphi$. The left side of the assertion sends φ to $\iota^{-1}(T(\xi))\cdot\varphi[\tilde{f}]$ and the right side to $\iota^{-1}(T'(\xi))\cdot Y(\varphi) = \iota^{-1}(T'(\xi))\cdot\mathcal{B}(\varphi[\tilde{f}])$. So it suffices to prove

$$\iota^{-1}(T(\xi))\cdot\varphi[f'] = \iota^{-1}(T'(\xi))\cdot\mathcal{B}(\varphi[f'])$$

for all f', and by linearity for the additive characters $f' = \psi$, which form a basis of the functions. Now

$$\varphi[\psi] = (\psi(-e)-\theta)\cdot\sum_{\gamma\in\Gamma}\psi(\chi(\gamma))\,\rho(\gamma)\varphi, \qquad \rho(\gamma')\,\varphi[\psi] = \psi^{-1}(\chi(\gamma'))\cdot\varphi[\psi] \quad (\gamma'\in\Gamma),$$

the second by the substitution $\gamma \mapsto \gamma'\gamma$ in the sum, and $\varphi[\psi] \in \xi$ by Input 1(b), since each $\rho(\gamma)$ is the restriction of an element of the local group algebra at w_j. If $\varphi[\psi] = 0$, both sides of the equation for $f' = \psi$ vanish. This is the case whenever $\psi^{-1}(e) = \theta$; it may also be the case otherwise. If $\varphi[\psi] \neq 0$, then $\psi^{-1}(e) = \psi(-e) \neq \theta$, and the hypothesis, applied to ξ, to the character ψ^{-1} and to the form $\varphi[\psi]$, is the equation. □

LEAN `BaseLevel.weightedSum_ext` · the formal text does not use that V is standard, the Weil data or $n \geq 1$

Lemma 5.2 (coefficients of characteristic polynomials lie in the Hecke algebra)**.** *In Settings 2.1 and 2.2, let* $\mathbf{w}$ *be a family of finite places of* E *indexed by* J*, let* $\gamma \in G_E$ *and let* e *be a natural number. Then the function on* $\mathrm{Sys}_{\mathfrak{D}}(\mathbf{w})$ *that sends* ξ *to the coefficient of* X^e *in* $\det(X - \mathfrak{r}_\xi(\gamma))$ *belongs to* $\mathbb{T}_{\mathfrak{D}}(\mathbf{w})$*.*

Proof. Since $\mathfrak{r}_\xi(\gamma) = \omega(\gamma)\varrho_\xi(\gamma)$, the coefficient is $\omega(\gamma)^{n-e}$ times the coefficient of X^e in $\det(X - \varrho_\xi(\gamma))$ for $e \leq n$, and it is 0 for $e > n$. The Hecke algebra is a $\bar{\mathbb{Z}}_\ell$-algebra and $\omega(\gamma) \in \bar{\mathbb{Z}}_\ell$, so it suffices to treat ϱ. Every $\xi \in \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w})$ contains a non-zero form, so by Input 3 the function in question is the limit, in the space of functions on the finite set $\mathrm{Sys}_{\mathfrak{D}}(\mathbf{w})$, of the functions $\xi \mapsto \iota(\text{coefficient of } X^e \text{ in } \mathsf{h}_{\xi,w})$ along a proper filter on Frobenius data at places w of prime norm. The Hecke algebra is closed (Input 1(d)), and all but finitely many places are off level for $\mathfrak{D}$ at $\mathbf{w}$ (Input 1(e)). So it suffices to show that the function belongs to $\mathbb{T}_{\mathfrak{D}}(\mathbf{w})$ for a place w of prime norm that is off level. Such a place has $w \neq \bar{w}$, its norm being a prime (Setting 2.2; this is where the root of unity of the Weil data is used). The coefficient of X^e in $\mathsf{h}_{\xi,w}$ is an integer, depending only on w, n and e, times the chosen eigenvalue $\theta_{\xi,w,n-e}$ of the spherical Hecke operator of index $n-e$ at w. Because w is off level, the forms in $\xi \cap \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w})$ are eigenvectors of that operator with this eigenvalue, for every ξ; so by Input 1(a) the diagonal operator of the function $\xi \mapsto \iota(\theta_{\xi,w,n-e})$ agrees on $\mathrm{Forms}_{\mathfrak{D}}(\mathbf{w})$ with the spherical Hecke operator. That operator lies in the local group algebra at w and ℓ does not lie under w, so it maps integral forms to integral forms (Input 1(c)). Hence the function belongs to $\mathbb{T}_{\mathfrak{D}}(\mathbf{w})$, and so does its multiple by an integer. □

LEAN BaseLevel.charpoly_mem_heckeAlgebra

Definition 5.3 (trace elements)**.** In Settings 2.1 and 2.2, let $\mathbf{w}$ be a family of finite places of E indexed by J. For a polynomial $G \in \bar{\mathbb{Z}}_\ell[X]$ and elements $\gamma', \gamma \in G_E$, the *trace element* $\mathfrak{t}_\mathfrak{r}(G;\, \gamma', \gamma)$ is the function

$$\mathfrak{t}_\mathfrak{r}(G;\, \gamma', \gamma)\colon \ \mathrm{Sys}_\mathfrak{D}(\mathbf{w}) \to \bar{\mathbb{Q}}_\ell, \qquad \xi \longmapsto \mathrm{tr}\big(G(\mathfrak{r}_\xi(\gamma')) \cdot \mathfrak{r}_\xi(\gamma)\big).$$

It belongs to $\mathbb{T}_\mathfrak{D}(\mathbf{w})$: the matrix $G(\mathfrak{r}_\xi(\gamma'))\mathfrak{r}_\xi(\gamma)$ is a combination, with the coefficients of G, of the matrices $\mathfrak{r}_\xi(\gamma'^{\,i}\gamma)$, and the trace of $\mathfrak{r}_\xi(\gamma'')$ is minus the coefficient of X^{n-1} in $\det(X - \mathfrak{r}_\xi(\gamma''))$, to which Lemma 5.2 applies. The index $\mathfrak{r}$ stands for the family $\mathfrak{r}$ together with the relation $\mathfrak{r}_\xi(\gamma) = \omega(\gamma) \cdot \varrho_\xi(\gamma)$ of Setting 2.2, which is the first argument of the formal definition. For instance $\mathfrak{t}_\mathfrak{r}(X-1;\, \tau_j, \gamma)$ sends ξ to $\mathrm{tr}((\mathfrak{r}_\xi(\tau_j) - 1)\mathfrak{r}_\xi(\gamma))$, and $\mathfrak{t}_\mathfrak{r}(1;\, 1, \gamma)$ sends ξ to $\mathrm{tr}\,\mathfrak{r}_\xi(\gamma)$.

LEAN BaseLevel.traceElem

Proposition 5.4 (the tie of tame inertia to Frobenius at one auxiliary place)**.** *In Settings 2.1, 2.2 and 2.3, let $j \in J$, let $\varepsilon \in \bar{\mathbb{Z}}_\ell[X]$ be a polynomial, $L \geq 1$ an integer and $T \in \mathbb{T}_\mathfrak{D}(\mathbf{w}_Q)$. Suppose that for every $\xi \in \mathrm{Sys}_\mathfrak{D}(\mathbf{w}_Q)$ that is an eigensystem of base level and has $T(\xi) \neq 0$, the Hecke polynomial h_{ξ,w_j} has a root string of length L and ratio q_j at $\omega(\mathrm{Fr}_j)^{-1}$ with cofactor dividing ε* (Definition 2.5)*. Let S be a ring and $\sigma\colon \mathrm{End}^{\mathrm{int}}_\mathfrak{D}(\mathbf{w}_Q) \to S$ a ring homomorphism with $\ell^\nu \cdot 1_S = 0$. Put*

$$\rho_j(\gamma) = \mathrm{res}_{\mathfrak{D},Q}(\gamma^{(j)}) \quad \textit{for } \gamma \in \mathrm{GL}_n(\mathbb{F}_{w_j}), \qquad \textit{and} \qquad \mathrm{Nm}_j = \sum_{\gamma \in \mathrm{GL}_n(\mathbb{F}_{w_j})} \rho_j(\gamma).$$

Then there is $F \in \mathrm{Loc}_\mathfrak{D}(Q, j)$ with the following two properties.

(i) *For every $\gamma \in \mathrm{GL}_n(\mathbb{F}_{w_j})$ there is $Z \in \mathrm{End}^{\mathrm{int}}_\mathfrak{D}(\mathbf{w}_Q)$ with*

$$\sigma\big(C \cdot F \cdot (\rho_j(\gamma) - 1)\big) = \sigma(\mathrm{Nm}_j \cdot Z) \quad \textit{and} \quad \langle T\rangle \cdot \mathrm{Nm}_{\mathfrak{D},Q} \cdot Z = 0 \ \textit{ in } \mathrm{End}^{\mathrm{int}}_\mathfrak{D}(\mathbf{w}_Q).$$

(ii) *For every $\gamma \in G_E$, the element $y = \sigma\big(\langle \mathfrak{t}_\mathfrak{r}(X-1;\, \tau_j, \gamma)\rangle\big)$ of S satisfies*

$$\sigma(\mathrm{Nm}_j) \cdot y = 0 \qquad \textit{and} \qquad 2 \cdot n! \cdot \sigma(F) \cdot y = 2 \cdot n! \cdot \sigma(\mathrm{Nm}_j) \cdot \sigma\big(\langle \mathfrak{t}_\mathfrak{r}((X-1)\,\varepsilon;\, \mathrm{Fr}_j, \gamma)\rangle\big).$$

The trace elements are taken at the level $\mathbf{w}_Q$.

LEAN BaseLevel.exists_tie · the formal statement ends at line 103 of its file: the census cuts it at the binding of ρ_j

In words: modulo ℓ^ν and up to the factor $2 \cdot n!$, the operator F at w_j times the trace of $(\tau_j - 1)\gamma$ is the sum Nm_j over $\mathrm{GL}_n(\mathbb{F}_{w_j})$, the trace to the level without w_j, times the trace of $(\mathrm{Fr}_j - 1)\,\varepsilon(\mathrm{Fr}_j)\,\gamma$. The tame generator and the Frobenius element enter two different trace elements: the polynomial $X - 1$ is evaluated at τ_j, and $(X-1)\varepsilon$ at Fr_j. Property (i) measures how far F is from commuting with the action of $\mathrm{GL}_n(\mathbb{F}_{w_j})$: up to the factor C and modulo ℓ^ν, the operator $F \cdot (\rho_j(\gamma) - 1)$ is a multiple of Nm_j by an operator that $\langle T\rangle \cdot \mathrm{Nm}_{\mathfrak{D},Q}$ annihilates. This is the form in which the authors' comment, "ties hold on forms and weak lifts do not move", enters the proof of Theorem A (Step 8).

Proof. Write $w = w_j$ and $q = q_j$. By Input 7 there are a unit u of $\mathcal{O}_w$ whose reduction generates $\mathbb{F}_w^\times$ and whose image under Art_w is the class of τ_j, and a uniformiser ϖ

of E_w whose image is the class of Fr_j. Since ℓ^ν divides $q-1$, Lemma 3.1 applies to (E, w, n, ℓ^ν, u); it gives a subgroup H of $\mathrm{GL}_n(\mathbb{F}_w)$, a homomorphism $\alpha\colon H \to \mathbb{Z}/\ell^\nu$, functions lg and δ and an element γ_0. Let

$$\varepsilon^+ = (X - q^{L-1}) \cdot \varepsilon \qquad \text{and} \qquad \varepsilon^\sharp(X) = \varepsilon^+(\omega(\mathrm{Fr}_j) \cdot X),$$

polynomials over $\bar{\mathbb{Z}}_\ell$. By Setting 2.3(a) the place w is off level, so ℓ does not lie under w and Input 2(c) to (f) are available; and $w \neq \bar{w}$.

Three remarks on fixed vectors.

(R1) For $\gamma \in \mathrm{GL}_n(\mathcal{O}_w)$ and every form φ in $\mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$ one has $\mathrm{loc}_w(\gamma)\varphi = \rho_j(\bar{\gamma})\varphi$ (Input 2(c)).

(R2) A form $\varphi \in \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$ that is fixed by $\rho_j(h)$ for all $h \in H$ with $\alpha(h) = 0$ and $\mathrm{Tr}_\alpha(h) = 0$ is fixed by Iw_w: an element of Iw_w lies in Par_w, its reduction h lies in H and has $\alpha(h) = 0 = \mathrm{Tr}_\alpha(h)$ by Lemma 3.1(a), and (R1) applies.

(R3) For every $f\colon \mathbb{Z}/\ell^\nu \to \mathbb{Z}$ and $\varphi \in \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$ the form $\mathsf{W}_{\rho_j|_H,\alpha}(f)\varphi$ is fixed by Iw_w: for $h \in H$ with $\alpha(h) = 0$, Lemma 4.2 gives

$$\rho_j(h) \cdot \mathsf{W}_{\rho_j|_H,\alpha}(f) = \mathsf{W}_{\rho_j|_H,\alpha}(f),$$

and (R2) applies.

The operators $B(f)$. The operator $\mathsf{W}_{\rho_j|_H,\alpha}(f)$ lies in $\mathrm{Loc}_{\mathfrak{D}}(Q, j)$ by Input 2(d), and its values are fixed by Iw_w by (R3). So Input 2(e) gives, for every f, an operator

$$B(f) \in \mathrm{Loc}_{\mathfrak{D}}(Q, j) \quad \text{with} \quad B(f)\varphi = \varepsilon^\sharp(U_\varpi)(\mathsf{W}_{\rho_j|_H,\alpha}(f)\varphi) \quad (\varphi \in \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)),$$

and B is additive in f. We apply Proposition 4.4 to the group $\Gamma = \mathrm{GL}_n(\mathbb{F}_w)$, the ring $R = \mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w}_Q)$, the modulus $M = \ell^\nu$, the homomorphism ρ_j, the subgroup H, the ring S with σ, the homomorphism α, the map B, the subring $\mathcal{A} = \mathrm{Loc}_{\mathfrak{D}}(Q, j)$, the family $(\langle T' \rangle)_{T' \in \mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)}$ indexed by the Hecke algebra, a Higman operator Ψ for the subgroup $\{h^{(j)} : h \in H\}$ (Input 2(f)), the element $\Pi = \langle T \rangle \cdot \mathrm{Nm}_{\mathfrak{D},Q}$, the element γ_0, the integer C and $e = n!$. Its hypotheses hold:

- *The hypothesis on B.* For $h \in H$, the substitution $h' \mapsto h'h$ in the sum gives

$$\mathsf{W}_{\rho_j|_H,\alpha}(f) \cdot \rho_j(h) = \mathsf{W}_{\rho_j|_H,\alpha}(f(\cdot - \alpha(h)));$$

  apply $\varepsilon^\sharp(U_\varpi)$.
- *The subring and the family.* $\rho_j(\gamma) \in \mathrm{Loc}_{\mathfrak{D}}(Q, j)$ by Input 2(d), $B(f) \in \mathrm{Loc}_{\mathfrak{D}}(Q, j)$ by construction, and every $\langle T' \rangle$ commutes with $\mathrm{Loc}_{\mathfrak{D}}(Q, j)$ by Input 2(b).
- (H1) is the property of Ψ, and (H6) is Lemma 3.1(b).
- (H2). Let $h \in H$, $\varphi \in \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$ and $\varphi' = \mathsf{W}_{\rho_j|_H,\alpha}(1)\varphi$. By (R3) and Lemma 4.2 the form φ' is fixed by Iw_w and by every $\rho_j(h')$, $h' \in H$; by (R1) and Lemma 3.1(a) it is fixed by Par_w. By Input 6 for the constant function 1, the operator U_ϖ preserves the space of such forms, hence so does $\varepsilon^\sharp(U_\varpi)$. So $B(1)\varphi$ is fixed by Par_w; as reduction maps Par_w onto H, (R1) gives $\rho_j(h)B(1)\varphi = B(1)\varphi$.
- (H4). $\mathrm{Nm}_{\mathfrak{D},Q} \cdot \mathrm{res}_{\mathfrak{D},Q}(\gamma^{(j)}) = \mathrm{Nm}_{\mathfrak{D},Q}$, a sum over a group.
- (H5). R is a ring of endomorphisms of a complex vector space.

Hypothesis (H3), $\langle T\rangle \cdot \mathrm{Nm}_{\mathfrak{D},Q} \cdot B(1) = 0$, is the heart of the matter and is proved next. Granting it, the proposition gives $F \in \mathrm{Loc}_{\mathfrak{D}}(Q, j)$ whose property (i) is property (i) above.

Proof of (H3)*: the chain lemma.* Let $\varphi \in \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$ and $\varphi_1 = \langle T\rangle\ \mathrm{Nm}_{\mathfrak{D},Q}\ B(1)\varphi$; we show $(\varphi_1 \mid \varphi_1) = 0$ for the invariant form of Input 4. The operator $\langle T\rangle$ commutes with $B(1)$ and with every $\mathrm{res}_{\mathfrak{D},Q}(\gamma)$, a product of operators in the algebras $\mathrm{Loc}_{\mathfrak{D}}(Q, j')$ (Input 2(b) and (d)). So $\varphi_1 = \mathrm{Nm}_{\mathfrak{D},Q}\ B(1)\langle T\rangle\varphi$ and also $\varphi_1 = \langle T\rangle\varphi_2$ with $\varphi_2 = \mathrm{Nm}_{\mathfrak{D},Q}\ B(1)\varphi$. The form φ_2 is fixed by every $\mathrm{res}_{\mathfrak{D},Q}(\gamma)$, so it is of base level (Input 2(a)), and by Input 1(a) for the empty family it is a sum of forms $\varphi' \in \xi \cap \mathrm{Forms}_{\mathfrak{D}}(\emptyset)$, $\xi \in \mathrm{Sys}_{\mathfrak{D}}(\emptyset)$. For such a form $\langle T\rangle\varphi' = \iota^{-1}(T(\xi))\varphi'$. So it suffices to show $(\varphi' \mid \varphi_1) = 0$ for every $\xi \in \mathrm{Sys}_{\mathfrak{D}}(\emptyset)$ with $T(\xi) \neq 0$ and every non-zero $\varphi' \in \xi \cap \mathrm{Forms}_{\mathfrak{D}}(\emptyset)$. Such a φ' is fixed by every $\mathrm{res}_{\mathfrak{D},Q}(\gamma)$, and these operators preserve the form, hence

$$(\varphi' \mid \varphi_1) = \sum_{\gamma} (\varphi' \mid \mathrm{res}_{\mathfrak{D},Q}(\gamma)B(1)\langle T\rangle\varphi) = \Big|\prod_{j'\in J} \mathrm{GL}_n(\mathbb{F}_{w_{j'}})\Big| \cdot (\varphi' \mid \varepsilon^\sharp(U_\varpi)\varphi_3),$$
$$\text{where} \qquad \varphi_3 = \mathsf{W}_{\rho_j|_H,\alpha}(1)\langle T\rangle\varphi,$$

and φ_3 is fixed by Iw_w by (R3). By Setting 2.3(a) the form φ' is fixed by $\mathrm{GL}_n(\mathcal{O}_w)$ and is an eigenvector of the spherical Hecke operators at w with the chosen eigenvalues of ξ, so its Hecke polynomial is $\mathsf{h}_{\xi,w}$. By the hypothesis of the proposition, $\mathsf{h}_{\xi,w}$ has a root string of length $L = (L-1)+1$ and ratio q at $\omega(\mathrm{Fr}_j)^{-1}$ with cofactor dividing ε, and $\omega(\mathrm{Fr}_j)$ is a unit of $\bar{\mathbb{Z}}_\ell$, hence not zero. The chain lemma (Input 5, with $\theta = \omega(\mathrm{Fr}_j)$, $e = L-1$ and $G' = \varepsilon$) gives $(\varphi' \mid G''(U_\varpi)\varphi_3) = 0$ for $G''(X) = (\omega(\mathrm{Fr}_j)X - q^{L-1}) \cdot \varepsilon(\omega(\mathrm{Fr}_j)X) = \varepsilon^\sharp(X)$. This proves (H3).

Property (ii). Fix $\gamma \in G_E$. For $G \in \bar{\mathbb{Z}}_\ell[X]$ put

$$T_G = \mathsf{t}_{\mathfrak{r}}(G;\, \mathrm{Fr}_j, \gamma) \qquad \text{and} \qquad T_G^\tau = \mathsf{t}_{\mathfrak{r}}(G;\, \mathrm{Fr}_j, \tau_j\gamma) - \mathsf{t}_{\mathfrak{r}}(G;\, \mathrm{Fr}_j, \gamma),$$

elements of $\mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)$; so

$$T_G^\tau(\xi) = \mathrm{tr}\big(G(\mathfrak{r}_\xi(\mathrm{Fr}_j))\,(\mathfrak{r}_\xi(\tau_j) - 1)\,\mathfrak{r}_\xi(\gamma)\big) \qquad \text{and} \qquad T_1^\tau = \mathsf{t}_{\mathfrak{r}}(X-1;\, \tau_j, \gamma).$$

Recall $\mathfrak{r}_\xi(\tau_j) = \varrho_\xi(\tau_j)$ by Setting 2.3(b).

Claim A: where GL_n *acts through the determinant, the tame generator acts as a diamond operator.* For every $G \in \bar{\mathbb{Z}}_\ell[X]$ and every $f\colon \mathbb{Z}/\ell^\nu \to \mathbb{Z}$,

$$\mathsf{W}_{\rho_j,\mathrm{Tr}_\alpha}(f) \cdot \langle T_G^\tau\rangle = (\rho_j(\gamma_0) - 1) \cdot \mathsf{W}_{\rho_j,\mathrm{Tr}_\alpha}(f) \cdot \langle T_G\rangle. \tag{5}$$

This is Lemma 5.1 for the group $\mathrm{GL}_n(\mathbb{F}_w)$ with ρ_j and $\chi = \mathrm{Tr}_\alpha$, with $e = 0$, $\theta = 0$ and $\tilde{f} = f$, with $\mathcal{B} = \rho_j(\gamma_0) - 1$ and $Y = (\rho_j(\gamma_0) - 1)\mathsf{W}_{\rho_j,\mathrm{Tr}_\alpha}(f)$, and with the Hecke elements T_G^τ and T_G. Its hypothesis is checked as follows. Let $\xi \in \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}_Q)$, let ψ be an additive character and $\varphi \in \xi \cap \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$ a non-zero form with $\rho_j(\gamma')\varphi = \psi(\mathrm{Tr}_\alpha(\gamma'))\varphi$ for all $\gamma' \in \mathrm{GL}_n(\mathbb{F}_w)$; the condition $\psi(0) \neq 0$ is empty. By (R2) the form φ is fixed by Iw_w, and by (R1) and Lemma 3.1(a) every $\gamma' \in \mathrm{GL}_n(\mathcal{O}_w)$ acts on it by the scalar $\psi(\delta(\det\gamma'))$, which is $\psi(\delta(u))$ at each diagonal matrix with one entry u and the others 1. Input 8(a), for the function 0, whose block parahoric subgroup is $\mathrm{GL}_n(\mathcal{O}_w)$, shows that the polynomial $X - \iota(\psi(\delta(u)))$ vanishes at $\varrho_\xi(\tau_j)$: this matrix is

the scalar $\iota(\psi(\delta(u)))$. Hence $T_G^\tau(\xi) = (\iota(\psi(\delta(u))) - 1) \cdot T_G(\xi)$. On the other hand $(\rho_j(\gamma_0) - 1)\varphi = (\psi(\mathrm{Tr}_\alpha(\gamma_0)) - 1)\varphi = (\psi(\delta(u)) - 1)\varphi$ by Lemma 3.1(b). So $\iota^{-1}(T_G^\tau(\xi))\varphi = \iota^{-1}(T_G(\xi)) \cdot (\rho_j(\gamma_0) - 1)\varphi$, as required.

The first assertion of (ii). Take $G = 1$ and $f = 1$ in (5). Then $\mathsf{W}_{\rho_j, \mathrm{Tr}_\alpha}(1) = \mathrm{Nm}_j$ and $(\rho_j(\gamma_0) - 1)\,\mathrm{Nm}_j = 0$, so $\mathrm{Nm}_j \cdot \langle T_1^\tau \rangle = 0$ already in $\mathrm{End}_{\mathfrak{D}}^{\mathrm{int}}(\mathbf{w}_Q)$, and $\sigma(\mathrm{Nm}_j) \cdot y = 0$.

Claim B: on the eigenvectors of the line stabiliser, the trace against $\tau_j - 1$ turns U_ϖ into Frobenius. For every $f\colon \mathbb{Z}/\ell^\nu \to \mathbb{Z}$,

$$B(\partial_{n!} f) \cdot \langle T_1^\tau \rangle = \mathsf{W}_{\rho_j|_H, \alpha}(\partial_{n!} f) \cdot \langle T_{\varepsilon^+}^\tau \rangle. \tag{6}$$

This is Lemma 5.1 for the group H with $\rho_j|_H$ and $\chi = \alpha$, with $e = n!$, $\theta = 1$, the integral function $\partial_{n!} f$ and $\tilde{f} = f$, with $\mathcal{B}$ the restriction of $\varepsilon^\sharp(U_\varpi)$ to $\mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$ and $Y = B(\partial_{n!} f)$, and with the Hecke elements $T_{\varepsilon^+}^\tau$ and T_1^τ. To check its hypothesis let $\xi \in \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}_Q)$, let ψ be an additive character with $\psi(n!) \neq 1$ and $\varphi \in \xi \cap \mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$ a non-zero form with $\rho_j(h)\varphi = \psi(\alpha(h))\varphi$ for all $h \in H$. By (R2) the form φ is fixed by Iw_w, and by (R1) and Lemma 3.1(a) every $\gamma' \in \mathrm{Par}_w$ acts on it by the scalar $\psi(\lg(\gamma'_{11}))$. The function $\theta = \psi \circ \lg$ on E_w has $\theta(u) = \psi(1) \neq 0$ and $\theta(u)^{n!} = \psi(n!) \neq 1$. Input 8(b), for the image of $\varepsilon^\sharp$ in $\mathbb{C}[X]$ and the matrix $\mathfrak{r}_\xi(\gamma)$, gives

$$\iota^{-1}\,\mathrm{tr}\big((\varrho_\xi(\tau_j) - 1)\mathfrak{r}_\xi(\gamma)\big) \cdot \varepsilon^\sharp(U_\varpi)\varphi = \iota^{-1}\,\mathrm{tr}\big(\varepsilon^\sharp(\varrho_\xi(\mathrm{Fr}_j))\,(\varrho_\xi(\tau_j) - 1)\,\mathfrak{r}_\xi(\gamma)\big) \cdot \varphi.$$

Here

$$\varepsilon^\sharp(\varrho_\xi(\mathrm{Fr}_j)) = \varepsilon^+(\omega(\mathrm{Fr}_j)\varrho_\xi(\mathrm{Fr}_j)) = \varepsilon^+(\mathfrak{r}_\xi(\mathrm{Fr}_j)),$$

so the equation reads

$$\iota^{-1}(T_1^\tau(\xi)) \cdot \mathcal{B}(\varphi) = \iota^{-1}(T_{\varepsilon^+}^\tau(\xi)) \cdot \varphi,$$

as required.

The tie. Apply property (ii) of Proposition 4.4 to the indices T_1^τ and $T_{\varepsilon^+}^\tau$ of the family and to the element $\langle T_{\varepsilon^+} \rangle$ of R: its first hypothesis is (6) and its second is (5) for $G = \varepsilon^+$. It gives

$$2 \cdot n! \cdot \sigma(F) \cdot \sigma(\langle T_1^\tau \rangle) = 2 \cdot n! \cdot \sigma(\mathrm{Nm}_j) \cdot \sigma(\langle T_{\varepsilon^+} \rangle).$$

Finally $T_{\varepsilon^+} = \mathfrak{t}_{\mathfrak{r}}(X\varepsilon;\, \mathrm{Fr}_j, \gamma) - q^{L-1} \cdot \mathfrak{t}_{\mathfrak{r}}(\varepsilon;\, \mathrm{Fr}_j, \gamma)$, and $q^{L-1} \equiv 1$ modulo ℓ^ν because ℓ^ν divides $q - 1$; as ℓ^ν vanishes in S, $\sigma(\langle T_{\varepsilon^+} \rangle) = \sigma(\langle \mathfrak{t}_{\mathfrak{r}}((X-1)\varepsilon;\, \mathrm{Fr}_j, \gamma) \rangle)$. □

6 Proof of Theorem A

Proof of Theorem A. All trace elements are taken at the level $\mathbf{w}_Q$. A multi-index is an element $\mathbf{d}$ of $\mathbb{N}^J$, with components $\mathbf{d}_j$. Multi-indices are ordered componentwise, $|\mathbf{d}|$ is the sum of the components, $\mathbf{1}$ has all components 1, $\mathbf{1}_j$ has the component 1 at j and 0 elsewhere, and $X^{\mathbf{d}} = \prod_j X_j^{\mathbf{d}_j}$.

Step 1: separating elements. Call $\xi \in \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}_Q)$ *separated from* π if there is $T \in \mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)$ with $T(\xi) = 0$ and $|T(\pi)| \geq 1$; choose such an element T_ξ for each separated ξ, and let T_* be the product of these finitely many elements. Then $|T_*(\pi)| \geq 1$, and if $T_*(\xi) \neq 0$ then ξ is not separated from π: every $T \in \mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)$ with $T(\xi) = 0$ has $|T(\pi)| < 1$.

Step 2: Ribet's element. Let ev be the map from the polynomial ring $\mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)[X_j : j \in J]$ to the ring of functions from $\mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}_Q)$ to $\bar{\mathbb{Q}}_\ell[X_j : j \in J]$ that evaluates the coefficients at each eigensystem; it is an injective homomorphism of $\bar{\mathbb{Z}}_\ell$-algebras. Let $\mathcal{S}$ be its image, let $\mathfrak{t}$ be the ideal of $\mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)[X_j : j \in J]$ generated by the elements

$$\ell^\nu \cdot \mathfrak{t}_{\mathfrak{r}}(1;\, 1, \gamma) \qquad \text{and}$$
$$\mathfrak{t}_{\mathfrak{r}}(X - 1;\, \tau_j, \gamma) - X_j \cdot \mathfrak{t}_{\mathfrak{r}}((X-1)\,\varepsilon_j;\, \mathrm{Fr}_j, \gamma) \qquad (\gamma \in G_E,\ j \in J),$$

and let $\mathcal{J} = \mathrm{ev}(\mathfrak{t})$, an $\mathcal{S}$-submodule. The pair $(\mathcal{S}, \mathcal{J})$ is a trace ideal for

$$(\mathfrak{r},\ \ell^\nu,\ (\tau_j)_{j\in J},\ (\mathrm{Fr}_j)_{j\in J},\ \mathrm{id}_J,\ \varepsilon) :$$

the variables lie in $\mathcal{S}$, the coefficients of the characteristic polynomials lie in $\mathcal{S}$ by Lemma 5.2 at the level $\mathbf{w}_Q$, and the functions of Definition 2.6(c) are the images under ev of the generators of $\mathfrak{t}$. By hypothesis (3) there is a Ribet coefficient $\eta \in \bar{\mathbb{Q}}_\ell$ of $(\mathcal{S}, \mathcal{J})$ for β at $\pi \in \mathcal{X}$ with $\mathrm{ord}_\ell^+(\mu\eta) \le A$. Unfolding Definition 2.7: the function β is the image under ev of a polynomial, hence, comparing constant terms, $\beta \in \mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)$; and there is a polynomial $\Phi = \sum_{\mathbf{d}} \Phi_{\mathbf{d}} X^{\mathbf{d}}$ with coefficients $\Phi_{\mathbf{d}} \in \mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)$ such that, ev being injective,

$$\beta \cdot \Phi \in \mathfrak{t}; \qquad \beta(\xi) = 0 \ \text{ and } \ \Phi_{\mathbf{d}}(\xi) = 0 \quad (\xi \in \mathcal{X},\ \mathbf{d} < \mathbf{1}); \qquad \Phi_{\mathbf{1}}(\pi) = \eta. \tag{7}$$

Step 3: the function s. Put

$$T_0 = T_* \cdot p(\beta) \in \mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q) \qquad \text{and} \qquad s = T_*^{|J|+1} \cdot \Phi_{\mathbf{1}}.$$

Then $|\mu\, s(\pi)| \ge |\mu\, \eta|$ by Step 1, so $\mathrm{ord}_\ell^+(\mu\, s(\pi)) \le \mathrm{ord}_\ell^+(\mu\eta) \le A$ (Definition 2.4). It remains to prove the property of s. Fix φ_0, Δ and φ as in the statement; both forms lie in $\mathrm{Forms}_{\mathfrak{D}}(\mathbf{w}_Q)$.

Step 4: constants. By hypothesis (2) there is $\upsilon \in \bar{\mathbb{Z}}_\ell$ with

$$\ell^\nu = 2 \cdot n! \cdot (p_0 C^2)^{|J|+1} \cdot \upsilon. \tag{8}$$

Since $p(\beta) - p_0$ is a multiple of β in $\mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)$, there is $T' \in \mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)$ with

$$(C^2\, T_0)^{|J|+1} = (p_0 C^2)^{|J|+1} \cdot T_*^{|J|+1} + \beta \cdot T'. \tag{9}$$

By Input 2(f) for the whole group $\prod_j \mathrm{GL}_n(\mathbb{F}_{w_j})$ there is $\Psi \in \mathrm{End}_{\mathfrak{D}}^{\mathrm{int}}(\mathbf{w}_Q)$ with

$$\sum_{\gamma} \mathrm{res}_{\mathfrak{D},Q}(\gamma^{-1})\, \Psi\, \mathrm{res}_{\mathfrak{D},Q}(\gamma) = C.$$

Step 5: the module that φ_0 spans over the integral operators, modulo ℓ^ν. Let

$$\mathfrak{n} = \{\Upsilon \in \mathrm{End}_{\mathfrak{D}}^{\mathrm{int}}(\mathbf{w}_Q) :\ \Upsilon(\varphi_0) = \ell^\nu \cdot \Upsilon'(\varphi_0) \text{ for some } \Upsilon' \in \mathrm{End}_{\mathfrak{D}}^{\mathrm{int}}(\mathbf{w}_Q)\}.$$

This is a left ideal of $\mathrm{End}_{\mathfrak{D}}^{\mathrm{int}}(\mathbf{w}_Q)$, and it contains $\ell^\nu \cdot \mathrm{End}_{\mathfrak{D}}^{\mathrm{int}}(\mathbf{w}_Q)$. Let

$$\mathcal{M} = \mathrm{End}_{\mathfrak{D}}^{\mathrm{int}}(\mathbf{w}_Q)/\mathfrak{n},$$

a left module over $\mathrm{End}_{\mathfrak{D}}^{\mathrm{int}}(\mathbf{w}_Q)$ on which ℓ^ν vanishes; let $\sigma\colon \mathrm{End}_{\mathfrak{D}}^{\mathrm{int}}(\mathbf{w}_Q) \to S$ be the action, with S the ring of the $\bar{\mathbb{Z}}_\ell$-linear endomorphisms of $\mathcal{M}$, and write $[\Upsilon] \in \mathcal{M}$ for

the class of Υ. (The map $\Upsilon \mapsto \Upsilon(\varphi_0)$ identifies $\mathcal{M}$ with the lattice $\mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w}_Q)\cdot\varphi_0$ modulo ℓ^ν.) We claim: for every $T'' \in \mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)$ that is a multiple of T_0 in $\mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)$,

$$\begin{aligned} \sigma(\langle T''\rangle)\,\sigma(\mathrm{Nm}_{\mathfrak{D},Q})\,[\Psi] = 0 \quad &\Longleftrightarrow \\ \text{there is } \Upsilon \in \mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w}_Q) \text{ with } \ell^\nu\cdot\Upsilon(\varphi_0) &= C\cdot\langle T''\rangle\varphi. \end{aligned} \tag{10}$$

Indeed the left side says that $\langle T''\rangle\,\mathrm{Nm}_{\mathfrak{D},Q}\,\Psi \in \mathfrak{n}$, that is, $\langle T''\rangle\,\mathrm{Nm}_{\mathfrak{D},Q}\,\Psi(\varphi_0) = \ell^\nu\Upsilon(\varphi_0)$ for some Υ. The form φ_0 is of base level, hence fixed by every $\mathrm{res}_{\mathfrak{D},Q}(\gamma)$ (Input 2(a)), so

$$\mathrm{Nm}_{\mathfrak{D},Q}\,\Psi(\varphi_0) = \sum_{\gamma}\mathrm{res}_{\mathfrak{D},Q}(\gamma)\Psi\,\mathrm{res}_{\mathfrak{D},Q}(\gamma^{-1})\varphi_0 = C\cdot\varphi_0$$

by the property of Ψ. And $\langle T''\rangle\varphi_0 = \langle T''\rangle\varphi$ by hypothesis (ii), because T'' is a multiple of $p(\beta)$ and so vanishes wherever $p(\beta)$ does.

Step 6: the ties. Let $j \in J$. Proposition 5.4 applies to j, to the polynomial ε_j, to L, to the Hecke element T_0 and to $\sigma\colon\ \mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w}_Q) \to S$: if $\xi \in \mathrm{Sys}_{\mathfrak{D}}(\mathbf{w}_Q)$ is of base level and $T_0(\xi) \neq 0$, then $p(\beta(\xi)) \neq 0$ and $T_*(\xi) \neq 0$, so by Step 1 every T with $T(\xi) = 0$ has $|T(\pi)| < 1$, and hypothesis (1) gives the root string. We obtain $F_j \in \mathrm{Loc}_{\mathfrak{D}}(Q, j)$ such that, with ρ_j and Nm_j as in the proposition:

(i) for every $\gamma \in \mathrm{GL}_n(\mathbb{F}_{w_j})$ there is $Z \in \mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w}_Q)$ with

$$\sigma\big(C\cdot F_j\cdot(\rho_j(\gamma)-1)\big) = \sigma(\mathrm{Nm}_j\cdot Z) \qquad \text{and} \qquad \langle T_0\rangle\cdot\mathrm{Nm}_{\mathfrak{D},Q}\cdot Z = 0;$$

(ii) for every $\gamma \in G_E$, the two Hecke elements

$$T_1 = \mathtt{t}_{\mathfrak{r}}(X-1;\,\tau_j,\gamma) \qquad \text{and} \qquad T_2 = \mathtt{t}_{\mathfrak{r}}((X-1)\,\varepsilon_j;\,\mathrm{Fr}_j,\gamma)$$

satisfy

$$\sigma(\mathrm{Nm}_j)\,\sigma(\langle T_1\rangle) = 0 \qquad \text{and} \qquad 2\,n!\cdot\sigma(F_j)\,\sigma(\langle T_1\rangle) = 2\,n!\cdot\sigma(\mathrm{Nm}_j)\,\sigma(\langle T_2\rangle).$$

Step 7: the operators $\Xi_{\mathbf{d}}$. The places w_j are distinct and none is conjugate to another, so the images under σ of $\mathrm{Loc}_{\mathfrak{D}}(Q,j)$ and $\mathrm{Loc}_{\mathfrak{D}}(Q,j')$ commute for $j \neq j'$ (Input 1(b)). For $\mathbf{d} \in \mathbb{N}^J$ let

$$\Xi_{\mathbf{d}} = \prod_{j\in J}\sigma(S_j(\mathbf{d})) \in S, \qquad \text{where} \quad S_j(\mathbf{d}) = \begin{cases} F_j & \text{if } \mathbf{d}_j = 0,\\ \mathrm{Nm}_j & \text{if } \mathbf{d}_j = 1,\\ 0 & \text{if } \mathbf{d}_j \geq 2. \end{cases}$$

In the authors' words, $\Xi_{\mathbf{d}}$ is the coefficient complementary to $\mathbf{d}$ of the family $\prod_j(\mathrm{Nm}_j + X_jF_j)$. Four properties:

(a) $\Xi_{\mathbf{d}}$ commutes with $\sigma(\langle T\rangle)$ for every $T \in \mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)$ (Input 2(b));

(b) $\Xi_{\mathbf{1}} = \sigma(\mathrm{Nm}_{\mathfrak{D},Q})$, because $\mathrm{res}_{\mathfrak{D},Q}$ is multiplicative and the product over j of the sums Nm_j is the sum over the product group;

(c) $\Xi_{\mathbf{d}} = 0$ unless $\mathbf{d} \leq \mathbf{1}$;

(d) for $j \in J$ and $y, y' \in S$:

$$\text{if} \quad \sigma(F_j)\,y = \sigma(\mathrm{Nm}_j)\,y' \quad \text{and} \quad \sigma(\mathrm{Nm}_j)\,y = 0, \qquad \text{then} \quad \Xi_{\mathbf{d}}\cdot y = \Xi_{\mathbf{d}+\mathbf{1}_j}\cdot y'.$$

For (d), write $\Xi_{\mathbf{d}} = \Xi' \cdot \sigma(S_j(\mathbf{d}))$, where Ξ' is the product of the other factors, the same for $\mathbf{d}$ and $\mathbf{d}+\mathbf{1}_j$. If $\mathbf{d}_j = 0$ then $\Xi_{\mathbf{d}}\, y = \Xi'\sigma(F_j)y = \Xi'\sigma(\mathrm{Nm}_j)y' = \Xi_{\mathbf{d}+\mathbf{1}_j} y'$; if $\mathbf{d}_j = 1$ then $\Xi_{\mathbf{d}}\, y = \Xi'\sigma(\mathrm{Nm}_j)y = 0 = \Xi_{\mathbf{d}+\mathbf{1}_j} y'$; if $\mathbf{d}_j \geq 2$ both sides vanish.

Step 8: the lifts do not move. We prove by induction on the integer $i \geq 0$: for every $\mathbf{d} \in \mathbb{N}^J$ with $|\mathbf{d}|+i \geq |J|+1$ and every $\Upsilon \in \mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w}_Q)$ with $\sigma(\mathrm{Nm}_{\mathfrak{D},Q})\,\sigma(\langle T_0\rangle)\,[\Upsilon] = 0$,

$$\sigma\big(\langle (C^2\, T_0)^i\rangle\big)\,\Xi_{\mathbf{d}}\,[\Upsilon] = 0. \tag{11}$$

For $i = 0$, $|\mathbf{d}| \geq |J|+1 > |\mathbf{1}|$, so $\mathbf{d}$ is not $\leq \mathbf{1}$ and $\Xi_{\mathbf{d}} = 0$ by (c). Suppose the assertion holds for i, and let $\mathbf{d}$ and Υ be as above with $i+1$ in the place of i. Put

$$\Pi_i = \sigma\big(\langle (C^2\, T_0)^i\rangle\big) \cdot \Xi_{\mathbf{d}} \cdot C \;\in S.$$

We claim $\Pi_i \cdot \sigma(\mathrm{res}_{\mathfrak{D},Q}(\gamma)) = \Pi_i$ for every $\gamma \in \prod_j \mathrm{GL}_n(\mathbb{F}_{w_j})$. The elements with this property form a subgroup, so it suffices to take $\gamma = \gamma_j^{(j)}$ with $j \in J$ and $\gamma_j \in \mathrm{GL}_n(\mathbb{F}_{w_j})$, and to show $\Pi_i \cdot \sigma(\rho_j(\gamma_j) - 1) = 0$. Let Z be as in Step 6(i) for γ_j. By (d), with $y = C \cdot \sigma(\rho_j(\gamma_j) - 1)$ and $y' = \sigma(Z)$ (the condition $\sigma(\mathrm{Nm}_j)y = 0$ holds because $\mathrm{Nm}_j\, \rho_j(\gamma_j) = \mathrm{Nm}_j$),

$$\Pi_i \cdot \sigma(\rho_j(\gamma_j) - 1) = \sigma\big(\langle (C^2\, T_0)^i\rangle\big) \cdot \Xi_{\mathbf{d}+\mathbf{1}_j} \cdot \sigma(Z).$$

For every $\Upsilon' \in \mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w}_Q)$ the operator $Z\Upsilon'$ satisfies, by Step 6(i),

$$\sigma(\mathrm{Nm}_{\mathfrak{D},Q})\,\sigma(\langle T_0\rangle)\,[Z\Upsilon'] = [\mathrm{Nm}_{\mathfrak{D},Q}\,\langle T_0\rangle\, Z\,\Upsilon'] = [\langle T_0\rangle\ \mathrm{Nm}_{\mathfrak{D},Q}\ Z\,\Upsilon'] = 0,$$

the second equality because $\langle T_0\rangle$ commutes with every $\mathrm{res}_{\mathfrak{D},Q}(\gamma)$ (Input 2(b) and (d)), and $|\mathbf{d}+\mathbf{1}_j| + i \geq |J|+1$. So the induction hypothesis gives

$$\sigma\big(\langle (C^2\, T_0)^i\rangle\big)\,\Xi_{\mathbf{d}+\mathbf{1}_j}\,[Z\Upsilon'] = 0 \qquad \text{for every } \Upsilon' \in \mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w}_Q),$$

which proves the claim. By the claim and the property of Ψ,

$$\Pi_i \cdot C = \sum_{\gamma} \Pi_i\, \sigma(\mathrm{res}_{\mathfrak{D},Q}(\gamma^{-1}))\,\sigma(\Psi)\,\sigma(\mathrm{res}_{\mathfrak{D},Q}(\gamma)) = \Pi_i \cdot \sigma(\Psi) \cdot \Xi_{\mathbf{1}},$$

using (b). Therefore, by (a),

$$\sigma\big(\langle (C^2\, T_0)^{i+1}\rangle\big)\,\Xi_{\mathbf{d}}\,[\Upsilon] = \Pi_i \cdot C \cdot \sigma(\langle T_0\rangle)\,[\Upsilon] = \Pi_i\,\sigma(\Psi)\,\sigma(\mathrm{Nm}_{\mathfrak{D},Q})\,\sigma(\langle T_0\rangle)\,[\Upsilon] = 0.$$

Step 9: the family kills the ideal. Let Λ be the additive map from $\mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)[X_j : j \in J]$ to S with

$$\Lambda\Big(\sum_{\mathbf{d}} T_{\mathbf{d}}\, X^{\mathbf{d}}\Big) = \sum_{\mathbf{d} \leq \mathbf{1}} \Xi_{\mathbf{d}} \cdot \sigma(\langle T_{\mathbf{d}}\rangle),$$

$$\text{so that} \quad \Lambda(T \cdot X^{\mathbf{d}}) = \Xi_{\mathbf{d}} \cdot \sigma(\langle T\rangle) \ \text{ for all } T \text{ and } \mathbf{d}$$

by (c). We claim $2\,n! \cdot \Lambda$ vanishes on $\mathfrak{t}$. The ideal $\mathfrak{t}$ is spanned additively by the products of a monomial $T \cdot X^{\mathbf{d}}$ with a generator. For a generator $\ell^{\nu} \cdot \mathfrak{t}_{\mathfrak{r}}(1;\, 1, \gamma)$ the value of Λ is a multiple of ℓ^{ν}, which vanishes in S. For a generator $T_1 - X_j\, T_2$, in the notation of Step 6(ii),

$$\Lambda\big(T\, X^{\mathbf{d}}\,(T_1 - X_j T_2)\big) = \Xi_{\mathbf{d}}\,\sigma(\langle T_1\rangle)\,\sigma(\langle T\rangle) - \Xi_{\mathbf{d}+\mathbf{1}_j}\,\sigma(\langle T_2\rangle)\,\sigma(\langle T\rangle),$$

and $2\,n!$ times this vanishes by (d), applied to the elements

$$y = 2\,n! \cdot \sigma(\langle T_1\rangle)\,\sigma(\langle T\rangle) \qquad \text{and} \qquad y' = 2\,n! \cdot \sigma(\langle T_2\rangle)\,\sigma(\langle T\rangle),$$

whose two conditions are the two assertions of Step 6(ii).

Step 10: conclusion. By (7), $T' \cdot \beta \cdot \Phi$ lies in $\mathfrak{t}$. In (8) the product $\upsilon \cdot 2\,n! \cdot (p_0 C^2)^{|J|+1}$ is ℓ^ν, which vanishes in S. So by Step 9 and (9),

$$0 = \upsilon \cdot 2\,n! \cdot \Lambda(T' \beta\, \Phi)\,[\Psi] = \upsilon \cdot 2\,n! \cdot \sum_{\mathbf{d} \le \mathbf{1}} \sigma\big(\langle (C^2\,T_0)^{|J|+1} \rangle\big)\, \Xi_{\mathbf{d}}\, \sigma(\langle \Phi_{\mathbf{d}} \rangle)\,[\Psi].$$

Let $\mathbf{d} < \mathbf{1}$. By (7) the element $\Phi_{\mathbf{d}}$ vanishes on $\mathcal{X}$, so $\langle \Phi_{\mathbf{d}} \rangle \varphi = 0$ by hypothesis (i), and (10) for $T'' = T_0 \Phi_{\mathbf{d}}$, with $\Upsilon = 0$, gives $\sigma(\mathrm{Nm}_{\mathfrak{D},Q})\, \sigma(\langle T_0 \rangle)\, [\langle \Phi_{\mathbf{d}} \rangle \Psi] = 0$. By Step 8 with $i = |J| + 1$ the term of $\mathbf{d}$ in the sum vanishes. Only $\mathbf{d} = \mathbf{1}$ remains, and by (a) and (b)

$$\sigma(\langle T'' \rangle)\, \sigma(\mathrm{Nm}_{\mathfrak{D},Q})\,[\Psi] = 0 \qquad \text{for} \quad T'' = \upsilon \cdot 2\,n! \cdot (C^2\,T_0)^{|J|+1} \cdot \Phi_{\mathbf{1}},$$

a multiple of T_0. By (10) there is $\Upsilon \in \mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w}_Q)$ with $\ell^\nu \cdot \Upsilon(\varphi_0) = C \cdot \langle T'' \rangle \varphi$. By (9) and (8), $T'' = \ell^\nu \cdot s + \upsilon \cdot 2\,n! \cdot \beta T'\, \Phi_{\mathbf{1}}$, and the second term vanishes on $\mathcal{X}$, so that its diagonal operator kills φ by hypothesis (i). Hence $\ell^\nu \cdot \Upsilon(\varphi_0) = \ell^\nu \cdot C \cdot \langle s \rangle \varphi$, and $\Upsilon(\varphi_0) = C \cdot \langle s \rangle \varphi$. Since Υ maps integral forms to integral forms and $\Delta \varphi_0$ is integral, $\Delta \cdot C \cdot \langle s \rangle \varphi = \Upsilon(\Delta \varphi_0)$ is integral. □

7 Where the results are used

Outside this unit only two of its declarations are named: Theorem A and Lemma 5.2. Both are applied once, in the unit *Selmer*, in its bound for the denominator of the projection of integral forms to lifts. There the homomorphism ω is the inverse of a character with values in $\bar{\mathbb{Z}}_\ell$ that is trivial on the tame generators, $\mathfrak{r}$ is the family of the normalised Galois representations, the places Q come from a triangular arrangement of Taylor–Wiles primes, hypothesis (1) comes from the Hecke polynomials of the weak lifts at the Frobenius elements of these primes, and hypothesis (3) is the output of Ribet's method. This is what the text of that file shows; the note on the unit *Selmer* states it. The other eight declarations serve Theorem A only.

8 Notes on the formal text

Hypotheses that the census does not list. The formal Theorem A takes the two hypotheses of Setting 2.3: the file declares them as section hypotheses and includes them, in line 87, for the declarations that follow, and the formal proof uses both, through Proposition 5.4. The census reads the arguments from the text and attaches that line to the next declaration only, which is Proposition 5.4, because two modifier lines stand between the inclusion and that declaration. The statement above therefore has two hypotheses more than the census records. Lean's own answer was not obtained: nothing is compiled in this workspace. In the same way the census does not list, for Lemma 5.1, the natural number C on which the base datum depends, because the lemma binds another variable of the same Lean name, the homomorphism χ.

A statement that the census cuts. The formal statement of Proposition 5.4 binds ρ_j and Nm_j inside its type before the existential quantifier. The census ends a statement at the first such binding, so its range (lines 96 to 99) stops before the conclusion; the statement was read to its end in line 103.

Where the formal statements are stronger or weaker than the ones above.

- In Proposition 5.4, the first assertion of (ii) holds before σ is applied, as the proof shows; the formal statement asserts it after σ, as above. In (i) the vanishing $\langle T\rangle \cdot \mathrm{Nm}_{\mathfrak{D},Q} \cdot Z = 0$ is asserted in $\mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w}_Q)$ itself, in the formal text and above.
- In Theorem A the function s that the proof constructs is an element of the Hecke algebra $\mathbb{T}_{\mathfrak{D}}(\mathbf{w}_Q)$; the formal statement, like the one above, only asserts a function. Hypothesis (3) quantifies over all pairs $(\mathcal{S}, \mathcal{J})$; the proof uses it for one pair.
- Lemma 5.1 is stated above in Setting 2.1 only. The formal lemma indeed does not use that V is standard, the Weil data or $n \geq 1$: the file removes these three from it. Its last hypothesis, the relation between $\tilde{f}$ and f, has a default proof in the formal text and is supplied by automation at both uses.
- Lemma 5.2 and Definition 5.3 need neither the auxiliary places Q nor the number ν of Setting 2.1; they are stated for an arbitrary family $\mathbf{w}$ indexed by J, an implicit argument of the formal text.
- Proposition 4.4 and Lemmas 4.2 and 4.3 hold for every natural number M, also for $M = 0$. The file declares $M \neq 0$ as a standing instance and removes it from each of these lemmas. Whether the formal definition of the weighted sum carries this instance cannot be read off the text, since its term does not use it; in every use here $M = \ell^\nu \geq 1$.
- In Lemma 3.1 the functions lg and δ are defined on all of E_w through a choice; only their values at units of $\mathcal{O}_w$ matter. Hypothesis (H5) of Proposition 4.4 is stated in the formal text as regularity of the natural number $|\Gamma|$ on R.
- The string length L is an implicit argument of Proposition 5.4 and of Theorem A, with the hypothesis $L \neq 0$.

What is not used. Of Input 8(a) only the first assertion is used, and only for the function 0. The first part of the chain lemma, a lower bound for the absolute value of a root at the end of a string, is not used in this unit. In Lemma 3.1 every assertion is used.

The valuation. The function ord^+_ℓ of Definition 2.4 is truncated: it is 0 at every element of absolute value at least 1. The second reader pointed this out against a reading that normalised it at ℓ alone; in Theorem A it is applied to $\mu\eta$ and $\mu\, s(\pi)$, where μ is an unrestricted element of $\bar{\mathbb{Q}}_\ell$, and to an upper bound $A \geq 0$, with which the truncation is compatible.

Two corrections of the second reader. The second reader compared the dossier of this unit with the Lean text and corrected two points, both followed here. In the proof of Lemma 5.1 the form $\varphi[\psi]$ vanishes whenever $\psi^{-1}(e) = \theta$, and it may vanish otherwise; the hypothesis is applied in the non-zero case, where $\psi^{-1}(e) \neq \theta$ follows. And the valuation is the truncated one. The second reader also asked that the tie

keep both the sum over the group and the factor $2 \cdot n!$, and that the two different trace elements be kept apart; Proposition 5.4 does so.

What the authors' comments name as related work. The head of the second file says that auxiliary tame level and first-order derivatives in diamond operators occur in the literature in other settings and names A. Wiles (1990), E. Lecouturier (2019) and A. Dotto and J. Thorne (2026); for the chain lemma it names M. Tadić (1986), W. Casselman (1980) and L. Clozel, M. Harris and R. Taylor (2008, Lemma 3.1.3 and §3.4). These attributions are the authors'; the references at the end repeat them as the file gives them, and the catalogue of the handout says which places were confirmed.

How this note was written. From the unit's dossier: the reader's statements of the 10 declarations and of the imports, and the six structured proofs, every step; and from the second reader's message on this unit. The two Lean files of the unit were read in full, statements and proofs, and each statement above was compared with its Lean text argument by argument. In the library the following were read in the Lean text, as statements and definitions only: the base datum, the levels, the integral forms, the eigensystems and the Taylor–Wiles places; the diagonal operators and the Hecke algebra; the operators at the auxiliary places with Inputs 2(c) to (f); the root strings and the chain lemma; the two statements of Input 8 to their ends, which the census cuts; the two definitions of Definitions 2.6 and 2.7; the valuation; the two statements of Input 7; the filter of Input 3. No proof of the library was opened. The reader marked three places as least certain. They were derived again from the Lean lines: the linear form in the proof of hypothesis (H3) of the first-order lemma inside Proposition 5.4 (lines 132 to 151: the form $\varphi' \mapsto (\langle T \rangle \varphi' \mid \varphi_1)$, tested on base-level eigenforms, as the reader says); the induction of Step 8 (lines 264 to 279); and Steps 9 and 10 (lines 280 to 304), including the cancellation of the factors $(p_0 C^2)^{|J|+1}$ and v. The reader's claims agree with the Lean text in all three. The arguments above follow the formal ones, with these differences: the transfer of a central element is computed from formula (1) and not quoted; the module of Step 5 is presented as a quotient by a left ideal, which is what the formal construction is; and the product map through which the formal text defines the operators Ξ_{d} has no name here. The program check of the unit's dossier covered 122 leaf steps of 6 proofs. Nothing was compiled for this note.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Lemma 3.1	`exists_line_stabilizer_log`	OnePrime.lean	15–50
Definition 4.1	`weightedSum`	OnePrime.lean	61–61
Lemma 4.2	`mul_weightedSum`	OnePrime.lean	64–65
Lemma 4.3	`map_eq_of_weights_modEq`	OnePrime.lean	72–74
Proposition 4.4	`one_prime_first_order`	OnePrime.lean	81–118
Lemma 5.1	`BaseLevel.weightedSum_ext`	OnePrimeTie.lean	44–59
Lemma 5.2	`BaseLevel.charpoly_mem_heckeAlgebra`	OnePrimeTie.lean	66–77
Definition 5.3	`BaseLevel.traceElem`	OnePrimeTie.lean	81–85
Proposition 5.4	`BaseLevel.exists_tie`	OnePrimeTie.lean	96–178
Theorem A	`BaseLevel.exists_diagOp_mem_ellIntegralForms`	OnePrimeTie.lean	190–304

Theorem A and the first-order lemma as they are written in the artifact.

```
lemma BaseLevel.exists_diagOp_mem_ellIntegralForms (b : D.Eigensystem Q.w → PadicAlgCl ℓ) (p :
  EllAdicIntegers[X]) (𝒳 : Set (D.Eigensystem Q.w)) (π : 𝒳) e {t} (ht : t ≠ 0) (he :
  ∀ j (x : D.Eigensystem Q.w), x.1 ∈ D.Eigensystem ![] → aeval (b x) p ≠ 0 → (∀ u : D.heckeAlgebra
  Q.w, u.1 x = 0 → ‖u.1 π‖ < 1) → HasRootString (x.1.heckePolyAt (Q.w j)) ((e j).map (algebraMap
  EllAdicIntegers ℂ)) (Q.w j).q (algebraMap EllAdicIntegers ℂ (a (Q.frobData j).frob))⁻¹ t) (hv :
  (2 * (card n)! : ℕ) * (p.coeff 0 * c ^ 2) ^ (card J + 1) ∣ ℓ ^ v) {m γ} (rib : ∀ S 𝒥,
  IsTraceIdeal (S := S) (Y := D.Eigensystem Q.w) 𝒥 (N ·.1 · |>.1) (ℓ ^ v : EllAdicIntegers)
  (Q.frobData · |>.isPlaceOver.tameGenerator) (Q.frobData · |>.frob) (·) e → ∃ d,
  IsRibetElement 𝒥 b π d ∧ ellVal (m * d) ≤ γ) : ∃ s, ellVal (m * s π.1) ≤
  γ ∧ ∀ l ∈ D.formsOfLevel ![], ∀ d : ℕ, d • l ∈ ellIntegralForms → ∀ y ∈ D.formsOfLevel ![],
  (∀ s, (∀ x ∈ 𝒳, s x = 0) → diagOp s y = 0) → (∀ s, (∀ x, aeval (b x) p = 0 → s x = 0) →
  diagOp s l = diagOp s γ) → d • c • diagOp s y ∈ ellIntegralForms

lemma one_prime_first_order (a : H →* Multiplicative (ZMod m)) (B : _ →+ R) (hB : ∀ f (p : H),
  B f * ρ p = B (f <| · - (a p).toAdd)) (𝒜 : Subring R) (hρ : ∀ g, ρ g ∈ 𝒜) (h𝒜 : ∀ f, B f ∈ 𝒜) {A}
  {ac : A → R} (hc : ∀ {x}, x ∈ 𝒜 → ∀ t, Commute (ac t) x) {Ψ L γ} {n k : ℕ} (hΨ : ∑ p : H,
  ρ ↑p⁻¹ * Ψ * ρ p = n) (hX : ∀ p ∈ H, ρ p * B 1 = B 1) (h : L * B 1 = 0) (hL : ∀ g, L * ρ g =
  L) (tf : IsSMulRegular R (Fintype.card Γ)) (hn : k * (a.transfer γ).toAdd = k) : ∃ F ∈ 𝒜, (∀ g,
  ∃ Z, σ (n • (F * (ρ g - 1))) = σ ((∑ g, ρ g) * Z) ∧ L * Z = 0) ∧ ∀ {t t' r}, (∀ f : ZMod m → ℤ,
  B ((f <| · - k) - f) * ac t = weightedSum (ρ.comp H.subtype) a ((f <| · - k) - f) * ac t') →
  (∀ f, weightedSum ρ a.transfer f * ac t' = (ρ γ - 1) * weightedSum ρ a.transfer f * r) →
  (2 * k) • (σ F * σ (ac t)) = (2 * k) • (σ (∑ g, ρ g) * σ r)
```

In these texts `EllAdicIntegers` is $\bar{\mathbb{Z}}_\ell$, `PadicAlgCl ℓ` is $\bar{\mathbb{Q}}_\ell$ and `Ω` is $\bar{\mathbb{Q}}$; `D` is the base datum $\mathfrak{D}$ and `c` the number C; `Q.w` is the family $\mathbf{w}_Q$ and `![]` the empty family; `D.Eigensystem`, `D.heckeAlgebra`, `D.integralEnd`, `D.formsOfLevel`, `D.localOps` and `D.residueRep` are $\mathrm{Sys}_{\mathfrak{D}}$, $\mathbb{T}_{\mathfrak{D}}$, $\mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}$, $\mathrm{Forms}_{\mathfrak{D}}$, $\mathrm{Loc}_{\mathfrak{D}}$ and $\mathrm{res}_{\mathfrak{D},Q}$; `diagOp` and `heckeAction` are both written $\langle\cdot\rangle$; `ellVal` is ord^+_ℓ; `card n` is the rank n; `N` is the family $\mathfrak{r}$ and `a` the homomorphism ω; `b`, `e`, `t`, `m` and `γ` of Theorem A are β, ε, L, μ and A; `l`, `d` and `y` in its conclusion are φ_0, Δ and φ; `HasRootString`, `IsTraceIdeal` and `IsRibetElement` are the notions of Definitions 2.5, 2.6 and 2.7. In the first-order lemma `a`, `ac`, `L`, `γ`, `n`, `k` and `r` are α, the family (c_i), Π, γ_0, C, e and y, and `a.transfer` is Tr_α. In the formal Proposition 5.4 the ring called `R` is the ring S above.

Notation

- ℓ, ι, $\bar{\mathbb{Q}}_\ell$, $\bar{\mathbb{Z}}_\ell$ a prime number; the fixed isomorphism $\mathbb{C} \to \bar{\mathbb{Q}}_\ell$; an algebraic closure of $\mathbb{Q}_\ell$; its ring of integers
- $|z|$, $\mathrm{ord}^+_\ell(z)$ the absolute value of $\bar{\mathbb{Q}}_\ell$ with $|\ell| = \ell^{-1}$; the truncated valuation (Definition 2.4)
- E, $\bar{\mathbb{Q}}$, G_E a number field, from Setting 2.1 on a CM field; an algebraic closure of $\mathbb{Q}$; the Galois group of $\bar{\mathbb{Q}}$ over E
- w, $\bar{w}$, E_w, $\mathcal{O}_w$, $\mathfrak{p}_w$, $\mathbb{F}_w$, q_w a finite place of E; its conjugate; the completion; its integers; their maximal ideal; the residue field; its number of elements
- n, 1_n the rank, the size of the matrices; the identity matrix
- V, κ, G_{f}, $\mathrm{Forms}_\kappa(V)$ the hermitian space; the weight; the finite adelic points of its unitary group; the forms
- loc_w, Par_w, Iw_w the representation of $\mathrm{GL}_n(E_w)$ on the forms; two subgroups of $\mathrm{GL}_n(\mathcal{O}_w)$ (Definition 2.8)
- ϖ, u, U_γ, U_ϖ a uniformiser; a unit of $\mathcal{O}_w$; Hecke operators of double cosets of Iw_w
- C, $\mathfrak{D}$ a natural number; a base datum for (V, κ, C); in Proposition 4.4, C is any natural number
- $\mathbf{w}$, $\emptyset$, $\mathrm{Forms}_{\mathfrak{D}}(\mathbf{w})$, $\mathrm{End}^{\mathrm{int}}_{\mathfrak{D}}(\mathbf{w})$ a family of finite places; the empty family; the forms of level $\mathbf{w}$; their integral endomorphisms
- ξ, π, $\mathrm{Sys}_{\mathfrak{D}}(\mathbf{w})$, $\mathcal{X}$ isotypic spaces, eigensystems; the eigensystems of level $\mathbf{w}$; a set of eigensystems
- $\mathsf{h}_{\xi,w}$, $\theta_{\xi,w,i}$ the Hecke polynomial of ξ at w; the chosen Hecke eigenvalues
- θ, θ_i, θ'_i a complex number, or a function with complex values; in Input 8, characters
- s, s', $\langle s\rangle$ functions from eigensystems to $\bar{\mathbb{Q}}_\ell$; the diagonal operator

T, T', T'', T_ξ, T_*, T_0, T_1, T_2, T_G, T_G^τ, $T_{\mathbf{d}}$ elements of a Hecke algebra $\mathbb{T}_{\mathfrak{D}}(\mathrm{w})$

ϑ, $Q(\vartheta)$, r the twist of the handout's overview and the set of the primes where it ramifies (Section 1 only); an integer with $r \equiv 3$ modulo 4 (Setting 2.2)

J, j, j', ν, Q, w_j, q_j, $\mathbf{w}_Q$ a finite set; its elements; an exponent of ℓ; the Taylor–Wiles places; the place of index j; its norm; the family of the places

Fr_j, τ_j, Art_w the geometric Frobenius element and the tame generator at w_j; the local Artin map

$\mathrm{res}_{\mathfrak{D},Q}$, $\mathrm{Nm}_{\mathfrak{D},Q}$, $\gamma^{(j)}$, $\mathrm{Loc}_{\mathfrak{D}}(Q,j)$ the action of $\prod_j \mathrm{GL}_n(\mathbb{F}_{w_j})$; the sum over this group; the element with γ at j; the operators at w_j

ρ_j, Nm_j the action of $\mathrm{GL}_n(\mathbb{F}_{w_j})$; the sum over this group

ϱ_ξ, ω, $\mathfrak{r}$, $\mathfrak{r}_\xi$, $\mathfrak{r}_i$ the chosen Galois representation; a homomorphism $G_E \to \bar{\mathbb{Z}}_\ell$; the twisted family; its members

$\mathfrak{t}_{\mathfrak{r}}(G;\gamma',\gamma)$ the trace element (Definition 5.3)

G, G', G'', G_1, G^ι, X, X_j polynomials in X; the variable; the variables of Ribet's element

Γ, γ, γ', γ'', γ_0, γ_j, $\bar\gamma$, γ_{11} a finite group; elements of a group; the distinguished element of Lemma 3.1; a reduction; a first entry

H, h, h', h_{11}, α, Tr_α a subgroup; its elements; a first entry; a homomorphism $H \to \mathbb{Z}/M$; its transfer

M, z, z', z_i, $\bar z$ a natural number, the modulus; arguments of functions: residues, elements of a field; a reduction

$\lg$, $\lg_0$, δ, δ_0 the functions of Lemma 3.1 and the homomorphisms that define them

R, S, σ, 1_S rings; a ring homomorphism $R \to S$; the unit of S

ρ, χ, ψ a homomorphism from a group to a ring; a homomorphism to $\mathbb{Z}/M$; an additive character

f, f', f'', f_0, $\tilde f$ functions with integral or complex values

$\mathsf{W}_{\rho,\chi}(f)$, $\partial_e f$ the weighted sum (Definition 4.1); the difference $f(\cdot - e) - f$

B, $\mathcal{A}$, I, i, i', c_i an additive map on functions; a subring; an index set; indices, also of rows and columns; a commuting family

Ψ, Π, Π_i, F, F_j, Z, y, y' elements of a ring: a Higman operator; an invariant element; the operator of the tie; elements

e, L a natural number or its class: a step, an exponent; the length of a root string

Σ, x, $[\gamma]$, $(\gamma)_H$, Y_+, Y_-, Nm, v representatives of cosets; one of them; the representative and the H-part of γ; the two coset sums; the sum over Γ; the least absolute residue

Y, Y', $\mathcal{B}$ operators; a linear map (Lemma 5.1)

φ, φ', $\varphi_0, \dots, \varphi_3$, $\varphi[f']$, $(\varphi' \mid \varphi)$ forms; the invariant hermitian form

β, p, p_0, ε, ε_j, ε^+, $\varepsilon^\sharp$ a function on eigensystems; a polynomial; its constant coefficient; polynomials over $\bar{\mathbb{Z}}_\ell$

μ, A, η, Δ, υ an element of $\bar{\mathbb{Q}}_\ell$; a bound in $[0,\infty]$; a Ribet coefficient; a natural number; an element of $\bar{\mathbb{Z}}_\ell$

$\mathcal{S}$, $\mathcal{J}$, Φ, $\Phi_{\mathbf{d}}$, $\mathfrak{t}$, ev a subalgebra; a submodule; Ribet's polynomial; its coefficients; the ideal of the ties; evaluation

$\mathbf{d}$, $\mathbf{d}_j$, $|\mathbf{d}|$, $\mathbf{1}$, $\mathbf{1}_j$, $X^{\mathbf{d}}$ a multi-index; its components; their sum; the multi-indices of $\prod_j X_j$ and of X_j; a monomial

$\mathfrak{n}$, $\mathcal{M}$, $[\Upsilon]$, Υ, Υ' a left ideal; the quotient module; a class; integral operators

$\Xi_{\mathbf{d}}$, Ξ', $S_j(\mathbf{d})$, Λ the operators of Step 7; an additive map (Step 9)

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The places are those that the catalogue of the handout and the second reader's comparison give; where they say that a place is not confirmed, so does the text above. No reference was typed from memory.

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Exponents of elements of Kisin modules, the cost of deepening them, and cokernel killers

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. The last step of the choice of the twists bounds the loss that the conditions at ℓ cause in Selmer groups, by lifting crystalline extension classes in large weight. Its tool is local and p-adic: the Breuil–Kisin module of a crystalline lattice, seen inside $W(C^\flat) \otimes L$. This note measures an element of that module, with all Frobenius translates $\mathcal{E}_\nu = \varphi^\nu(\mathcal{E})$ of the Eisenstein element inverted, by an *exponent* at each $\mathcal{E}_\nu$ and each Hodge–Tate weight. Three things are proved. A Galois operator $(g-1)/(z\tau)$ raises the exponent at $\mathcal{E}_\nu$ by one at the price of an explicit power of p (the *cost*). The standard exponents of level l describe exactly the l-th Frobenius pullback of the Kisin module, and exponents add under equivariant pairings. Finally, for an extension of a trivial lattice by a crystalline one over an unramified base, $p^\kappa \mathcal{E}^\rho$ kills the cokernel of the map of Kisin modules as soon as all weights are at most $(p-3)(\kappa+\rho)$.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles with Lean v4.35.0 against a pinned version of Mathlib and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note is the first of three on one unit of the artifact, the folder on lifting; it treats 77 of the unit's 141 declarations: three files (exponent lattices, the Hodge expansion, cokernel killers) and four definitions of a fourth. Theorems A to D and every lemma below are proved from the Inputs of Section 2, which belong to the background library and are stated and not proved here; the main one is the frame theorem for the Kisin module of a crystalline lattice. Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 77 declarations.

1 The statements

The objects are those of Settings 2.1 and 2.2: a prime p, a group Γ over the absolute Galois group of $\mathbb{Q}_p$, a Kummer chart with ring $\mathfrak{S} \subset A_{\mathrm{inf}}$, coordinate z and Eisenstein element $\mathcal{E}$, and lattice data L with their periods. The exponent module $\mathfrak{X}_L(i;o)$ is Definition 2.4, the cost $\mathrm{cost}(\nu,n)$ is Definition 2.3.

Theorem A (the depth step). *In Setting 2.1, let $\nu \geq 0$ be an integer and let n be an integer with $n \neq 0$. Then there are $g \in \Gamma'$, a unit w of A_{inf} and elements $y, X \in A_{\mathrm{inf}}$ such that*

$$g \cdot \mathcal{E}_\nu = w\,\mathcal{E}_\nu, \qquad \mathcal{E}_\nu \mid g \cdot s - s \ \text{ in } A_{\mathrm{inf}} \text{ for every } s \in \mathfrak{S},$$
$$\mathcal{E}_\nu \mid y\,X - p^{\mathrm{cost}(\nu,n)}, \qquad z\tau \cdot X = w^n - 1.$$

LEAN `Lift.KummerChart.exists_depth_step`

Theorem B (standard exponents are Frobenius pullbacks). *In Setting 2.2, let $x \in W(C^\flat) \otimes L$, let i be a letter and let $l \geq 0$ be an integer. Then $x \in \mathfrak{X}_L(i; \mathrm{std}_l)$ if and only if $\varphi^{-l} x \in \mathfrak{M}^{\mathrm{inf}}(L)$ and $x \in (W(C^\flat) \otimes L)_i$.*

LEAN `Lift.KummerChart.JumpLattice.mem_withExponents_stdExponent_iff`

Theorem C (exponents add). *In Setting 2.2, let A, L, L'' be lattice data over $\mathcal{O}$ and let $b\colon A \times L \to L''$ be an $\mathcal{O}$-bilinear map of their lattices with $b(ga, gt) = g\,b(a,t)$ for all $g \in \Gamma$, $a \in A$, $t \in L$. Let i be a letter, let o_A, o, $o_{L''}$ be exponent functions, let $y \in \mathfrak{X}_A(i; o_A)$ and $x \in \mathfrak{X}_L(i; o)$, and suppose that*

$$o_{L''}(\nu, v) \ \leq \ o(\nu, u') + o_A(\nu, u)$$

for every $\nu \geq 0$ and all $u \in U_A(i^{p^{-\nu}})$, $u' \in U_L(i^{p^{-\nu}})$, $v \in U_{L''}(i^{p^{-\nu}})$ with $u + u' \leq v$. Then $b_W(y, x) \in \mathfrak{X}_{L''}(i; o_{L''})$.

LEAN `Lift.KummerChart.JumpLattice.wittPairing_mem_withExponents`

Theorem D (a cokernel killer over an unramified chart). *In Setting 2.2, suppose that the Kummer chart is unramified and that $p \geq 5$. Let S be a side over $\mathcal{O}$, let i be a letter and let $\kappa, \rho \geq 0$ be integers such that $u \leq (p-3)(\kappa + \rho)$ for every letter j and every $u \in U_{Y_S}(j)$. Then $\mathrm{Kill}_S(i; \kappa, \rho)$: the element $p^\kappa \mathcal{E}^\rho$ multiplies the part at i of $A_{\mathrm{inf}} \otimes C_S$ into the image of $\mathfrak{X}_{\tilde{Y}_S}(i; \mathrm{std}_0)$.*

LEAN `Lift.KummerChart.PairingSide.killsCokernel_of_isUnramified`

Theorem A is the arithmetic heart: the cost $v_p(n) + \mathfrak{d} + 3\nu$ is what one Galois step costs. Theorem B is the bridge between the exponents and Kisin's module, and makes the definition of the exponents usable. Theorem C is what lets Part 2 pair a class with a dual class. Theorem D is the first source of cokernel killers; Part 3 has a second, for tame types.

2 The setting

Conventions. Throughout, p is a prime number. The declarations of this unit take the following ambient objects as implicit or instance arguments; they are fixed by Settings 2.1 and 2.2 and named again in each statement that needs them: the prime p; the group Γ over $G_{\mathbb{Q}_p}$; the Kummer chart $\mathfrak{c}$ for Γ; the coefficient ring $\mathcal{O}$ with its root of unity ω; lattice data L, L', L'', A over $\mathcal{O}$; a side S; a letter i; integers l, ν, n, κ, r; exponent functions o, o'; elements x, y of $W(C^\flat)\otimes L$; a period μ; an element g of Γ'. Rings are commutative with 1. For an integer n, $v_p(n)$ is the exponent of p in n, with the convention $v_p(0)=0$. A unit w of a ring has all integral powers w^n, $n\in\mathbb{Z}$. Tensor products without a subscript are over $\mathbb{Z}_p$. An *exponent function* is a function $o\colon \mathbb{N}\times\mathbb{Z}\to\mathbb{Z}$, written $(\nu,u)\mapsto o(\nu,u)$; inequalities between exponent functions are meant pointwise. The Lean text calls the artifact's period ring *Bcris*; it is Colmez's ring $B_{\max}$, and this note writes $B_{\max}$.

Setting 2.1 (period rings and a Kummer chart; from the background library)**.** *Period rings.* C is the completion of an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_C$; $G_{\mathbb{Q}_p}=\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. $\mathcal{O}_C^\flat$ is the tilt of $\mathcal{O}_C$, with fraction field $C^\flat$; $A_{\mathrm{inf}}=W(\mathcal{O}_C^\flat)\subset W(C^\flat)$ are the rings of Witt vectors, with Frobenius automorphism φ and the action of $G_{\mathbb{Q}_p}$; $[\,\cdot\,]$ is the Teichmüller lift; $\theta\colon A_{\mathrm{inf}}\to\mathcal{O}_C$ is Fontaine's map. Fix $\varepsilon\in\mathcal{O}_C^\flat$, a compatible system of p-power roots of unity whose component of level one is a primitive p-th root of unity, and put

$$\tau=[\varepsilon^{1/p}]-1,\qquad \xi=\sum_{0\le a<p}[\varepsilon^{1/p}]^a,\qquad \text{so that}\quad \xi\,\tau=[\varepsilon]-1;$$

ξ generates the kernel of θ. B_{dR}^+ is the completion of $A_{\mathrm{inf}}[1/p]$ along the kernel of θ, a complete discrete valuation ring with residue field C, and B_{dR} is its fraction field; for $n\in\mathbb{Z}$, $\mathrm{Fil}^n B_{\mathrm{dR}}$ is the n-th power of the maximal ideal of B_{dR}^+, so that $\mathrm{Fil}^0 B_{\mathrm{dR}}=B_{\mathrm{dR}}^+$. The group $G_{\mathbb{Q}_p}$ acts on B_{dR} and preserves each $\mathrm{Fil}^n B_{\mathrm{dR}}$.

Three further period rings are used, defined as the formal text defines them. $A_{\max}$ is the p-adic completion of the ring $A_{\mathrm{inf}}[X]/(pX-\xi)$, that is, of A_{inf} with an element ξ/p adjoined;

$$B^+=A_{\max}[1/p],\qquad B_{\max}=B^+[1/([\varepsilon]-1)].$$

Frobenius φ and the group $G_{\mathbb{Q}_p}$ act on these three rings, compatibly with their actions on A_{inf}. The maps $A_{\max}\to B^+\to B_{\max}$ are injective, and there is an injective homomorphism of A_{inf}-algebras $B_{\max}\to B_{\mathrm{dR}}$ that commutes with $G_{\mathbb{Q}_p}$ and maps B^+ into B_{dR}^+; the formal text obtains B_{dR}^+ as the completion of B^+ along ξ. Frobenius is not defined on B_{dR}. In this note "the image of $A_{\max}$" is its image in B^+, and an element of B_{dR} "lies in the image of $B_{\max}$" if it is a value of the map $B_{\max}\to B_{\mathrm{dR}}$. The rings $A_{\max}$ and $B_{\max}$ are those of Colmez (P. Colmez, *Théorie d'Iwasawa des représentations de de Rham d'un corps local*, Ann. of Math. 148 (1998), §III.2: $A_{\max}$ by restricted power series in ξ/p, and $B_{\max}=A_{\max}[1/p,1/t]$; by his Lemma III.3.9 the quotient of his element t by $[\varepsilon]-1$ is a unit of $A_{\max}$, so the two localizations agree). This comparison is the second reader's. The ring B_{cris} defined with divided powers is a different ring and is not used.

For $\nu\ge 0$, the composite of $\varphi^{-\nu}\colon A_{\mathrm{inf}}\to A_{\mathrm{inf}}$ with $A_{\mathrm{inf}}\to B_{\mathrm{dR}}$ is again written $\varphi^{-\nu}\colon A_{\mathrm{inf}}\to B_{\mathrm{dR}}$, and $\theta_\nu=\theta\circ\varphi^{-\nu}\colon A_{\mathrm{inf}}\to\mathcal{O}_C$.

The group. Γ is a group with a homomorphism $j_\Gamma\colon \Gamma \to G_{\mathbb{Q}_p}$ whose image is open; it acts on the rings above through j_Γ, and $\chi\colon \Gamma \to \mathbb{Z}_p^\times$ is the cyclotomic character composed with j_Γ. A property holds *near* 1 if it holds for all $g \in \Gamma$ with $j_\Gamma(g)$ in some neighbourhood of 1. The group $\mathbb{Z} \times \Gamma$ acts on $W(C^\flat)$ by $(a,g)\cdot c = \varphi^a(j_\Gamma(g)c)$; write φ for $(1,1)$ and g for $(0,g)$.

The chart. A *chart* consists of an integer $h \geq 1$, the ring $\mathbb{W} = W(\mathbb{F}_{p^h})$ of Witt vectors of the field $\mathbb{F}_{p^h} = \{c \in \mathcal{O}_C^\flat : c^{p^h} = c\}$, a homomorphism of $\mathbb{W}$-algebras $\mathbb{W}[[X]] \to A_{\mathrm{inf}}$ that sends X to a Teichmüller lift z which is not a unit, a natural number $e_{\mathfrak{c}}$ and a unit A of $\mathbb{W}[[X]]$ such that the image $\mathcal{E}$ of $X^{e_{\mathfrak{c}}}\mathsf{A} + p$ is ξ times a unit of A_{inf}. Its ring $\mathfrak{S}$ is the image of $\mathbb{W}[[X]]$; it contains z and $\mathcal{E}$, is stable under φ, and $\mathcal{E}_\nu = \varphi^\nu(\mathcal{E}) \in \mathfrak{S}$ for $\nu \geq 0$. The integer h is the *residue degree*, $e_{\mathfrak{c}}$ the *index*, and

$$\mathfrak{d} = 0 \ \text{ if } e_{\mathfrak{c}} = 1, \qquad \mathfrak{d} = v_p(e_{\mathfrak{c}}) + 1 \ \text{ otherwise.}$$

A *Kummer chart* $\mathfrak{c}$ for Γ is a chart together with a subgroup Γ' of Γ, of index prime to p, that contains all elements near 1, such that $j_\Gamma(g)$ fixes $\mathbb{W}$ and $\theta(z)$ for every $g \in \Gamma'$, every element of $G_{\mathbb{Q}_p}$ that fixes $\mathbb{W}$ and $\theta(z)$ lies in $j_\Gamma(\Gamma')$, and for every integer $a \geq 0$ some $g \in \Gamma'$ has $g \cdot z = [\varepsilon]^a z$. It is *unramified* if $\mathcal{E} = z + p$, $e_{\mathfrak{c}} = 1$ and $\mathfrak{d} = 0$. The field F is the subfield of $\bar{\mathbb{Q}}_p$ fixed by $j_\Gamma(\Gamma')$. The ring $\mathfrak{S}_B$ is the $\mathfrak{S}$-algebra of the elements of B^+ whose multiplication commutes with every $\mathfrak{S}$-linear endomorphism of B^+ that preserves the image of A_{max}; it contains the image of $\mathfrak{S}$. For $\nu \geq 0$, F_ν is the subfield of B_{dR} generated over F by the products of an element of $\mathfrak{S}_B$ and an element of $\varphi^{-\nu}(\mathfrak{S})$; so $\varphi^{-\nu}(\mathfrak{S}) \subseteq F_\nu$ and $F_0 \subseteq F_\nu$. Finally $\mathrm{co}_0\colon A_{\mathrm{inf}} \to \mathfrak{S}$ is a fixed $\mathfrak{S}$-linear map that is the identity on $\mathfrak{S}$, one of the coordinates $(\mathrm{co}_a)_a$ of a fixed topological basis of A_{inf} over $\mathfrak{S}$ for the (p,z)-adic topology; an element of A_{inf} all of whose coordinates vanish is 0.

Setting 2.2 (lattice data, periods and Kisin modules; from the background library)**.** In Setting 2.1:

Letters and coefficients. A *letter* is an element of $\mathbb{F}_{p^h}^\times$; the letters are permuted by $i \mapsto i^p$, and $i^{p^{-\nu}}$ is the letter whose p^ν-th power is i. A *coefficient ring* is the ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_p$ together with an element $\omega \in \mathcal{O}$ with $\omega^{p^h-1} = 1$.

Lattices. A *lattice* over $\mathcal{O}$ is a finite free $\mathcal{O}$-module with an $\mathcal{O}$-linear action of Γ. For a lattice T, $W(C^\flat) \otimes T$ carries the action $(a,g)(c \otimes t) = \varphi^a(j_\Gamma(g)c) \otimes gt$ of $\mathbb{Z} \times \Gamma$, and $A_{\mathrm{inf}} \otimes T$ is the A_{inf}-submodule spanned by the $1 \otimes t$. A $\mathbb{Z}_p$-bilinear map $b\colon T \times T' \to T''$ has a $W(C^\flat)$-bilinear extension to these modules. For a letter i, $(W(C^\flat) \otimes T)_i$ is the A_{inf}-submodule on which $1 \otimes \omega$ acts as $[i]$, and

$$\mathrm{pr}_i = \frac{1}{p^h - 1} \sum_{0 \leq a < p^h - 1} \left([i]^{-1}(1 \otimes \omega)\right)^a$$

is the projector onto it. The lattice T is *potentially crystalline* if the elements of $B_{\mathrm{max}} \otimes T$ fixed near 1 span it over B_{max}, and *crystalline on* Γ' if those fixed by all of Γ' do. For a $\mathbb{Z}_p$-algebra map $\gamma\colon \mathcal{O} \to C$ and $u \in \mathbb{Z}$, the *multiplicity of the Hodge–Tate jump u of T at γ* is the dimension over $\bar{\mathbb{Q}}_p$ of the space of γ-semilinear maps $\lambda\colon T \to C$ with $\lambda(gt) = \chi(g)^{-u}\, g\lambda(t)$ for each t and all g near 1. $Z^1_{\mathrm{f}}(T)$ is the $\mathcal{O}$-module of the maps $c\colon \Gamma \to T$ for which some $y \in B_{\mathrm{max}} \otimes T$ has $1 \otimes c(g) = gy - y$ for all g (the crystalline cocycles).

Lattice data. A *lattice datum* L over $\mathcal{O}$ is a lattice, also written L, that is crystalline on Γ', together with a finite set $U_L(i) \subset \mathbb{Z}$ for every letter i, such that $U_L(i)$ contains every u that is a Hodge–Tate jump of L of non-zero multiplicity at some $\gamma\colon \mathcal{O} \to C$ with $\gamma(\omega) = \theta([i])$. (The formal structure also records a jump set for each γ, contained in the corresponding $U_L(i)$.) An *equivariant map* $f\colon L \to L'$ of lattice data is a Γ-equivariant $\mathcal{O}$-linear map of the lattices; $1 \otimes f$ is its extension $W(C^\flat) \otimes L \to W(C^\flat) \otimes L'$. The *jump bound* $h_L \geq 0$ is the least integer a such that all Hodge–Tate jumps of L, as a $\mathbb{Z}_p$-representation of Γ', lie in $[-a, a]$.

Periods. $\mathrm{Per}(L)$ is the set of the A_{inf}-linear maps $\mu\colon W(C^\flat) \otimes L \to B_{\mathrm{dR}}$ with $\mu(1 \otimes gt) = g \cdot \mu(1 \otimes t)$ for $t \in L$, $g \in \Gamma'$. For a letter i and $u \in \mathbb{Z}$, $\mathrm{Per}_{i,u}(L)$ is the set of the $\mu \in \mathrm{Per}(L)$ with $\mu(\mathrm{pr}_i(1 \otimes t)) \in \mathrm{Fil}^{-u} B_{\mathrm{dR}}$ for all $t \in L$ (the periods *of type u at i*). For $\nu \geq 0$ the map $x \mapsto \mu(\varphi^{-\nu}x)$ is semilinear along $\varphi^{-\nu}\colon A_{\mathrm{inf}} \to B_{\mathrm{dR}}$. The *fraction span* of L is the set of the $x \in W(C^\flat) \otimes L$ that a non-zero element of A_{inf} multiplies into $A_{\mathrm{inf}} \otimes L$.

Kisin modules. $\mathfrak{M}^{\mathrm{inf}}(L)$ is the A_{inf}-module of the $x \in W(C^\flat) \otimes L$ such that $\tau^b x \in A_{\mathrm{inf}} \otimes L$ for some integer $b \geq 0$, and $\mu(\varphi^{a+1}x) \in \mathrm{Fil}^0 B_{\mathrm{dR}}$ for all integers $a \geq 0$ and all $\mu \in \mathrm{Per}(L)$. $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{loc}}(L)$ is the A_{inf}-module of the x that some finite product of the $\mathcal{E}_\nu$ multiplies into $\mathfrak{M}^{\mathrm{inf}}(L)$. For an A_{inf}-submodule Q of $W(C^\flat) \otimes L$, its *points over F_0* are the $x \in Q$ with $\mu(x) \in F_0$ for all $\mu \in \mathrm{Per}(L)$; they form an $\mathfrak{S}$-module. $\mathfrak{M}(L)$ is the $\mathfrak{S}$-module of the points of $\mathfrak{M}^{\mathrm{inf}}(L)$ over F_0 (the *Kisin module*), and $\mathfrak{M}_{\mathrm{loc}}(L)$ that of the points of $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{loc}}(L)$ over F_0. For $l \geq 0$, std_l is the exponent function with $\mathrm{std}_l(\nu, u) = 0$ for $\nu < l$ and $\mathrm{std}_l(\nu, u) = -u$ for $\nu \geq l$.

Definition 2.3 (the cost of a step)**.** In Setting 2.1, for integers $\nu \geq 0$ and n,

$$\mathrm{cost}(\nu, n) \;=\; v_p(n) + \mathfrak{d} + 3\nu.$$

LEAN `Lift.KummerChart.cost`

Definition 2.4 (elements with exponents)**.** In Setting 2.2, for a lattice datum L, a letter i and an exponent function o, $\mathfrak{X}_L(i; o)$ is the set of the $x \in \mathfrak{M}^{\mathrm{inf}}_{\mathrm{loc}}(L) \cap (W(C^\flat) \otimes L)_i$ such that, for every $\nu \geq 0$,

$$\mu(\varphi^{-\nu}x) \in \mathrm{Fil}^{\,o(\nu,u)} B_{\mathrm{dR}} \qquad \text{for all } u \in U_L(i^{p^{-\nu}}) \text{ and all } \mu \in \mathrm{Per}_{i^{p^{-\nu}},u}(L).$$

It is an A_{inf}-submodule of $W(C^\flat) \otimes L$. For an integer $l \geq 0$,

$$\mathfrak{N}_{l,i}(L) \;=\; \mathfrak{M}_{\mathrm{loc}}(L) \cap \mathfrak{X}_L(i; \mathrm{std}_l),$$

an $\mathfrak{S}$-module.

LEAN `Lift.KummerChart.JumpLattice.withExponents` · `Lift.KummerChart.JumpLattice.locLetterPart`

Definition 2.5 (parts, invariants, operations on exponents)**.** In Setting 2.2, let L, A, L'' be lattice data.

(i) For an A_{inf}-submodule Q of $W(C^\flat) \otimes L$, $(Q)_{\mathfrak{S}}$ is the $\mathbb{Z}_p$-module of the elements of $\mathfrak{M}_{\mathrm{loc}}(L)$ that lie in Q, and $\mathrm{Inv}(Q)$ is the $\mathbb{Z}_p$-module of the $x \in W(C^\flat) \otimes L$ with $gx - x \in Q$ for all $g \in \Gamma'$. (The module $\mathfrak{M}_{\mathrm{loc}}(L)$ is a $\mathbb{Z}_p$-module because $\mathfrak{S}$ is a $\mathbb{Z}_p$-algebra; the formal text records this as an instance.)

(ii) For exponent functions o^{lo}, o^{hi}, an integer $\nu \geq 0$ and $u \in \mathbb{Z}$,

$$\mathrm{imp}(o^{\mathrm{lo}}, o^{\mathrm{hi}})(\nu, u) = o^{\mathrm{hi}}(\nu, u) \ \text{ if } o^{\mathrm{lo}}(\nu, u) > 0,$$
$$\mathrm{imp}(o^{\mathrm{lo}}, o^{\mathrm{hi}})(\nu, u) = \min(o^{\mathrm{hi}}(\nu, u), 0) \ \text{ otherwise.}$$

(iii) For integers $l \geq 0$ and n, an exponent function o, an integer $\nu \geq 0$ and $u \in \mathbb{Z}$,

$$\mathrm{fr}_{l,n}(o)(\nu, u) = n \ \text{ if } \nu < l, \qquad \mathrm{fr}_{l,n}(o)(\nu, u) = o(\nu - l, u) \ \text{ if } \nu \geq l.$$

(iv) For an $\mathcal{O}$-bilinear map $b\colon A \times L \to L''$ of the lattices, $y \in W(C^\flat) \otimes A$ and $x \in W(C^\flat) \otimes L$, $b_W(y, x) \in W(C^\flat) \otimes L''$ is the value at (y, x) of the $W(C^\flat)$-bilinear extension of b.

LEAN `Lift.KummerChart.JumpLattice.locKisinPart` · `Lift.KummerChart.JumpLattice.invariantsMod` · `Lift.KummerChart.JumpLattice.improvedExponent` · `Lift.KummerChart.JumpLattice.frobExponent` · `Lift.KummerChart.JumpLattice.wittPairing`

Definition 2.6 (sides)**.** In Setting 2.2, a *side* over $\mathcal{O}$ is a triple $S = (Y_S, C_S, \alpha_S)$ of a lattice datum Y_S over $\mathcal{O}$ with $0 \in U_{Y_S}(i)$ for every letter i, a lattice C_S over $\mathcal{O}$, and an $\mathcal{O}$-linear map $\alpha_S\colon C_S \to Z^1_{\mathrm{f}}(Y_S)$. For a side S:

(i) $\tilde{Y}_S$ is the lattice datum on $Y_S \times C_S$ with the action $g(y, c) = (gy + \alpha_S(c)(g),\, c)$ and $U_{\tilde{Y}_S}(i) = U_{Y_S}(i)$ for every letter i: the extension of the module C_S, with trivial action, by Y_S along the cocycles α_S. It is crystalline on Γ', and its non-zero jumps are jumps of Y_S (Input I11).

(ii) C_S^{tr} is the lattice datum formed by the module C_S with the trivial action of Γ and $U_{C_S^{\mathrm{tr}}}(i) = \{0\}$ for every letter i.

(iii) $\mathrm{pr}_S^C\colon \tilde{Y}_S \to C_S^{\mathrm{tr}}$ is the equivariant map $(y, c) \mapsto c$.

LEAN `Lift.KummerChart.PairingSide` · `Lift.KummerChart.PairingSide.extLattice` · `Lift.KummerChart.PairingSide.trivLattice` · `Lift.KummerChart.PairingSide.projTriv`

Definition 2.7 (windows)**.** (i) For integers $\eta, \beta, W \geq 0$, a sign $\mathsf{s} \in \{1, -1\}$ and a finite set U of integers, $U \subseteq [-\eta, \eta] \cup \mathsf{s} \cdot [\beta, \beta + W]$ means that every $u \in U$ satisfies $-\eta \leq u \leq \eta$ or $\beta \leq \mathsf{s}u \leq \beta + W$.

(ii) For integers n, $\nu \geq 0$ and u, $\mathrm{rs}_n(\nu, u) = n - u$ if $\nu = 0$ and $\mathrm{rs}_n(\nu, u) = -u$ if $\nu > 0$.

(iii) For integers $n \geq 0$ and c, $\mathcal{W}_n(c) = [-n, n] \cup [c - n, c + n]$, a finite set of integers.

(iv) For integers $p, n \geq 0$ and β, $\mathrm{NM}(p; n, \beta)$ means that every $a \in \mathcal{W}_n(\beta) \cup \mathcal{W}_n(-\beta)$ that is divisible by p is 0.

LEAN `Lift.IsInJumpWindow` · `Lift.raisedAtE` · `Lift.twoWindows` · `Lift.NoMultipleNear`

Definition 2.8 (constants, the lifting ideal, cokernel killers)**.** In Setting 2.2, let S be a side over $\mathcal{O}$ and i a letter.

(i) For $c \in C_S$, $\mathrm{cst}_{S,i}(c) = \mathrm{pr}_i(1 \otimes c) \in W(C^\flat) \otimes C_S$; the map $c \mapsto \mathrm{cst}_{S,i}(c)$ is $\mathbb{Z}_p$-linear.

(ii) For integers $r, l \geq 0$, $\mathfrak{a}_S(r; l, i)$ is the ideal of A_{inf} generated by r and by the $a \in A_{\mathrm{inf}}$ with

$$a \cdot \mathrm{pr}_i(A_{\mathrm{inf}} \otimes C_S) \ \subseteq \ (1 \otimes \mathrm{pr}_S^C)(\mathfrak{X}_{\tilde{Y}_S}(i; \mathrm{std}_l)).$$

(iii) For integers $\kappa, \rho \geq 0$, $\mathrm{Kill}_S(i;\kappa,\rho)$ means $p^\kappa \mathcal{E}^\rho \in \mathfrak{a}_S(0;0,i)$; in words, $p^\kappa \mathcal{E}^\rho$ *kills the cokernel of* S *at* i.

LEAN `Lift.KummerChart.PairingSide.constantAt` · `Lift.KummerChart.PairingSide.liftIdeal` · `Lift.KummerChart.PairingSide.KillsCokernel`

Inputs from the background library. The following results are taken for granted. Each is a declaration of the background library, read in its Lean text; the number in brackets is the unit that owns it. All are in Setting 2.2, for a lattice datum L.

I1. *Divisibility and filtration* [83]. For $r \in A_{\mathrm{inf}}$ and $\nu \geq 0$: $\mathcal{E}_\nu \mid r$ if and only if $\theta_\nu(r) = 0$, if and only if $\varphi^{-\nu}(r) \in \mathrm{Fil}^1 B_{\mathrm{dR}}$. The image $\varphi^{-\nu}(\mathcal{E}_\nu)$ generates the maximal ideal of B_{dR}^+. If $\mathcal{E}_\nu \nmid r$, then $\varphi^{-\nu}(r)$ is a unit of B_{dR}^+. Moreover $\mathcal{E}_\nu \nmid \mathcal{E}_{\nu'}$ for $\nu \neq \nu'$, $\mathcal{E}_\nu \nmid z\tau$, $\mathcal{E}_\nu \nmid p$, $\mathcal{E}_\nu \mid \varphi^l(\tau)$ for $\nu < l$, $z\tau \mid \varphi^\nu(z\tau)$, $\theta(z) \mid p$ and $\tau \mid \xi - p$; p divides neither z nor τ, so $z\tau$ and the $\mathcal{E}_\nu$ are units of $W(C^\flat)$. Elementary.

I2. *Galois action on the chart* [83]. For $g \in G_{\mathbb{Q}_p}$, $g \cdot \mathcal{E}$ is $\mathcal{E}$ times a unit of A_{inf}. If g fixes $\mathbb{W}$ and $d \in A_{\mathrm{inf}}$ divides $gz - z$, then d divides $gs - s$ for every $s \in \mathfrak{S}$; if moreover g fixes $\theta(z)$, it fixes $\theta(s)$ for every $s \in \mathfrak{S}$. For $g \in \Gamma'$ and $s \in \mathfrak{S}$, $z\tau\mathcal{E}$ divides $gs - s$. Elementary.

I3. *A derivative* [83]. There is a polynomial P_0 over $\mathbb{W}$ with $\mathcal{E} \mid P_0(z)$ in $\mathfrak{S}$ and $\theta(P_0'(z)) \mid p^{\mathfrak{d}}$. The handout's catalogue relates it to the different of a totally ramified extension (J.-P. Serre, *Local Fields*, Ch. III, §6; place not confirmed).

I4. *Lifting the exponent* [83]. Let $f\colon A_{\mathrm{inf}} \to R$ be a surjective ring homomorphism, $a \in A_{\mathrm{inf}}$, and $v \in A_{\mathrm{inf}}$ a non-unit dividing $a - 1$ with $p\, f(a-1)\, f(v) \mid f(a-1)^p$. Then $f(a^n - 1) \mid p^{v_p(n)} f(a-1)$ for every integer $n \geq 1$. Also: for $n \geq 1$ there are $\alpha, \beta \in A_{\mathrm{inf}}$ with $\tau\mathcal{E}\,(p^{v_p(n)} + \beta z) = \alpha\,([\varepsilon]^n - 1)$. A divisibility form of the lemma on lifting the exponent of a prime in $a^n - 1$. Elementary.

I5. *Residues modulo* $\mathcal{E}_\nu$ [83]. For $\nu \geq 0$, every $s \in \mathfrak{S}$ is congruent modulo $\mathcal{E}_\nu$ to $\sum_{a < p^\nu e_{\mathfrak{c}}} w_a z^a$ with $w_a \in \mathbb{W}$. Moreover $z^{e_{\mathfrak{c}}}$ lies in the ideal $(\mathcal{E}, p)$ of $\mathfrak{S}$, and $\theta_\nu([\varepsilon])$ is a primitive p^ν-th root of unity. Elementary.

Here and in the two boxes that follow, "Elementary" is the class that the handout's catalogue of the background library gives to the declarations behind an Input: short arguments from the definitions, for which no reference is sought.

Inputs from the background library, continued.

I6. *Tate* [58]. Let $V \in B_{\mathrm{dR}}$ and let M be a B_{dR}^{+}-submodule of B_{dR}. If $gV - V \in M$ for all $g \in \Gamma'$, there is $c \in F$ with $V - c \in M$. An element c of $\bar{\mathbb{Q}}_p$ lies in $\mathrm{Fil}^n B_{\mathrm{dR}}$ if and only if $c = 0$ or $n \leq 0$. An element of B_{dR} fixed by Γ' lies in $\mathrm{Fil}^0 B_{\mathrm{dR}} \cap F_0$. The first statement is, in the second reader's comparison, a weaker form of the degree-zero parts of Tate's two theorems (J. Tate, *p-Divisible Groups*, §3.3, Theorems 1 and 2; P. Colmez, *Théorie d'Iwasawa des représentations de de Rham d'un corps local*, §III.1).

I7. *Values in the fields F_ν* [83]. *(a)* Let $f\colon A_{\mathrm{inf}} \to \mathfrak{S}$ be $\mathfrak{S}$-linear, $r_k \in A_{\mathrm{inf}}$ and $v_k \in F_\nu$, finitely many, and M a B_{dR}^{+}-submodule of B_{dR}. If $\sum_k \varphi^{-\nu}(r_k)v_k \in M$, then $\sum_k \varphi^{-\nu}(f(r_k))v_k \in M$. *(b)* Let $V \in F_\nu \cap \mathrm{Fil}^0 B_{\mathrm{dR}}$ and $g, t_1, \dots, t_m \in \Gamma'$. If $\mathcal{E}_\nu \mid g\sum_k t_k s - \sum_k t_k s$ for all $s \in \mathfrak{S}$, then $g\sum_k t_k V - \sum_k t_k V \in \mathrm{Fil}^1 B_{\mathrm{dR}}$. Part (b) is elementary; for part (a), which the catalogue classes as substantial and without a source: reference to be confirmed.

I8. *The frame theorem* [83]. *(a)* If $v_1, \dots, v_m$ are points of the fraction span over F_0 and $c_k \in A_{\mathrm{inf}}$ with $\sum_k c_k v_k = 0$, then $\sum_k f(c_k)v_k = 0$ for every $\mathfrak{S}$-linear $f\colon A_{\mathrm{inf}} \to \mathfrak{S}$. *(b)* $\mathfrak{M}^{\mathrm{inf}}(L) = A_{\mathrm{inf}} \cdot \mathfrak{M}(L)$. *(c)* $\tau^{h_L}\mathfrak{M}^{\mathrm{inf}}(L) \subseteq A_{\mathrm{inf}} \otimes L$ and $\tau^{h_L}(A_{\mathrm{inf}} \otimes L) \subseteq \mathfrak{M}^{\mathrm{inf}}(L)$. *(d)* $\mathcal{E}^{h_L}\varphi(x) \in \mathfrak{M}(L)$ for $x \in \mathfrak{M}(L)$. *(e)* Let $d \in A_{\mathrm{inf}}$ be such that d divides r whenever d divides $\mathcal{E}^a r$ in B^+. If $x \in \mathfrak{M}^{\mathrm{inf}}(L)$ and $\mathcal{E}^{2h_L}\mu(x) \in B^+ d$ for every crystalline period μ (a period in $\mathrm{Per}(L)$ whose values on $1 \otimes L$ lie in the image of $B_{\max}$), then $x \in d\,\mathfrak{M}^{\mathrm{inf}}(L)$; and $\mathcal{E}^{h_L}\mu(x)$ lies in the image of $\mathfrak{S}_B$ for such μ and $x \in \mathfrak{M}(L)$. The element $d = z\tau$ has the divisibility property, and $z\tau \mid gs - s$ in B^+ for $g \in \Gamma'$, $s \in \mathfrak{S}_B$. This is the artifact's form of the existence and the finite height of the Breuil–Kisin module of a crystalline lattice; the file names it a known result (Kisin lattice of a crystalline lattice). Reference to be confirmed.

Inputs from the background library, concluded.

I9. *Periods* [82, 83]. *(a)* For $\mu \in \mathrm{Per}(L)$, x in the fraction span and $g \in \Gamma'$: $\mu(gx) = g \cdot \mu(x)$. The fraction span contains $\mathfrak{M}^{\mathrm{inf}}(L)$ and is stable under $\mathbb{Z} \times \Gamma$, as is $A_{\mathrm{inf}} \otimes L$; $\mathfrak{M}^{\mathrm{inf}}(L)$ is stable under Γ'; $W(C^\flat) \otimes L$ has no A_{inf}-torsion. *(b)* For an equivariant $f\colon L \to L'$: $\mu \circ (1 \otimes f) \in \mathrm{Per}(L)$ for $\mu \in \mathrm{Per}(L')$, and $1 \otimes f$ maps $\mathfrak{M}^{\mathrm{inf}}(L)$ into $\mathfrak{M}^{\mathrm{inf}}(L')$; $\mu \circ \mathrm{pr}_i \in \mathrm{Per}(L)$ for $\mu \in \mathrm{Per}(L)$. *(c)* $\varphi^\nu x \in (W(C^\flat) \otimes L)_i$ if and only if $x \in (W(C^\flat) \otimes L)_{ip^{-\nu}}$; pr_i commutes with Γ' and with $1 \otimes f$. *(d)* If $u \notin U_L(i)$, then $\mathrm{Per}_{i,u}(L) \subseteq \mathrm{Per}_{i,u-1}(L)$. *(e)* Every $\mathbb{Z}_p$-linear map $L \to B_{\mathrm{dR}}$ is $t \mapsto \mu(1 \otimes t)$ for an A_{inf}-linear $\mu\colon W(C^\flat) \otimes L \to B_{\mathrm{dR}}$; two such maps that agree on $1 \otimes L$ agree on the fraction span. Elementary.

I10. *A Hodge basis* [87]. Let $(e_k)_k$ be a $\mathbb{Z}_p$-basis of L. There are integers κ_k and an invertible matrix H over B_{dR}, the product of an invertible matrix over B_{dR}^+ and of a diagonal matrix with entries generating $\mathrm{Fil}^{\kappa_k} B_{\mathrm{dR}}$, such that $\mathsf{R}(g) \cdot g(\mathsf{H}) = \mathsf{H}$ for all $g \in \Gamma'$, where $\mathsf{R}(g)$ is the matrix of g on L in the basis. This is the Hodge–Tate decomposition of a potentially crystalline lattice; in the second reader's comparison the passage from potential crystallinity to Hodge–Tate weights is classical for field coefficients (J.-M. Fontaine, *Représentations p-adiques semi-stables*, Astérisque 223, Exposé III, Theorem 3.8(i), Remark 3.9(ii), §5.1.7) and the formulation for coefficient fibres is not located.

I11. *Extensions and twists* [86, 87]. For a side S, the lattice $\tilde{Y}_S$ is crystalline on Γ', and for $u \neq 0$ the multiplicity of the jump u of $\tilde{Y}_S$ at any γ is at most that of Y_S. If Γ acts on a lattice through the c-th power of χ, its jumps are all equal to $-c$. Both rest on Tate's vanishing of the invariants of the non-trivial twists of C (Input I6). Reference to be confirmed.

I12. *A radical* [59]. If $a \in A_{\mathrm{inf}}$ is not a unit and $d \in A_{\mathrm{inf}}$ is not divisible by p, then a lies in the radical of the ideal (p, d). The ring A_{inf} is p-adically complete, and p does not divide τ. Elementary.

3 The depth steps

Throughout this section the setting is Setting 2.1. By Input I1, the ring homomorphism θ_ν has kernel $\mathcal{E}_\nu A_{\mathrm{inf}}$; it commutes with $G_{\mathbb{Q}_p}$, so this ideal is stable under $G_{\mathbb{Q}_p}$.

Proof of Theorem A. The elements. Choose $\gamma \in \Gamma'$ with $\gamma z = [\varepsilon]^{p^\nu} z$. By Input I2, $\gamma\mathcal{E} = w_0\mathcal{E}$ for a unit w_0 of A_{inf}, and $z\tau\mathcal{E}$ divides $\gamma\mathcal{E} - \mathcal{E} = (w_0 - 1)\mathcal{E}$; as $\mathcal{E} \neq 0$ and A_{inf} is a domain, $z\tau \mid w_0 - 1$. For every integer $n \neq 0$, $w_0 - 1$ divides $w_0^n - 1$ (for $n < 0$ write $w_0^n - 1 = -w_0^n(w_0^{-n} - 1)$). Since $z\tau \mid \varphi^\nu(z\tau)$ (Input I1), $z\tau$ divides $\varphi^\nu(w_0^n - 1) = w^n - 1$, where $w = \varphi^\nu(w_0)$. Let $g = \gamma$ and let X be the quotient, so that $z\tau X = w^n - 1$. Then $g\mathcal{E}_\nu = \varphi^\nu(\gamma\mathcal{E}) = w\mathcal{E}_\nu$, because the actions of φ and of $G_{\mathbb{Q}_p}$ commute. Next $\theta_\nu([\varepsilon]^{p^\nu} - 1) = \theta([\varepsilon] - 1) = 0$, so $\mathcal{E}_\nu$ divides $\gamma z - z = ([\varepsilon]^{p^\nu} - 1)z$, and therefore $\gamma s - s$ for every $s \in \mathfrak{S}$, by Input I2 (γ fixes $\mathbb{W}$).

Reduction. It remains to find y with $\mathcal{E}_\nu \mid yX - p^{\mathrm{cost}(\nu,n)}$. It suffices that $\theta_\nu(X)$ divides $p^{\mathrm{cost}(\nu,n)}$ in $\mathcal{O}_C$: then, θ being surjective, there is y_0 with $\theta(y_0)\theta_\nu(X) = p^{\mathrm{cost}(\nu,n)}$, and

$y = \varphi^{\nu}(y_0)$ has $\theta_\nu(yX - p^{\mathrm{cost}(\nu,n)}) = 0$. Now $\theta_\nu(z\tau) \neq 0$ (Input I1) and $\theta_\nu(z\tau)\,\theta_\nu(X) = \theta_\nu(w^n - 1) = \theta(w_0^n - 1)$. So it suffices to prove

$$\theta(w_0^n - 1) \;\mid\; p^{\,v_p(n)+\mathfrak{d}+3\nu} \cdot \theta_\nu(z\tau) \qquad \text{in } \mathcal{O}_C. \tag{1}$$

The term $v_p(n)$. Write $w_0 - 1 = z\tau d$ with $d \in A_{\mathrm{inf}}$. The number $\theta(\tau) + 1$ is a p-th root of unity, so the binomial theorem gives $\theta(\tau)^p = -p\,\theta(\tau)\,c$ for some $c \in \mathcal{O}_C$. Hence

$$\theta(w_0 - 1)^p = -p\,c\,\theta(\tau)\,\theta(z)^p\theta(d)^p \qquad \text{is divisible by}$$
$$p \cdot \theta(w_0 - 1) \cdot \theta(z) = p\,\theta(\tau)\,\theta(z)^2\theta(d),$$

because $p \geq 2$. So Input I4 applies to $f = \theta$, $a = w_0$ and the non-unit $v = z$, with the exponent $|n|$: $\theta(w_0^{|n|} - 1)$ divides $p^{v_p(n)}\theta(w_0 - 1)$, and $w_0^n - 1$ is $w_0^{|n|} - 1$ times a unit.

The term $\mathfrak{d} + \nu$. Let P_0 be the polynomial of Input I3, with $P_0(z) = \mathcal{E}G$, $G \in \mathfrak{S}$. Taylor's formula gives $r \in A_{\mathrm{inf}}$ with

$$P_0(\gamma z) = P_0(z) + P_0'(z)\,(\gamma z - z) + r\,(\gamma z - z)^2.$$

The left side is $\gamma(P_0(z)) = w_0\mathcal{E}\,\gamma(G)$, as γ fixes $\mathbb{W}$. On the right, $\gamma z - z = ([\varepsilon]^{p^\nu} - 1)z$ and $[\varepsilon]^{p^\nu} - 1 = ([\varepsilon] - 1)\,\sigma_\nu$ with $\sigma_\nu = \sum_{a<p^\nu}[\varepsilon]^a$, while $[\varepsilon] - 1 = \xi\tau = \mathcal{E}\tau\mathsf{u}$ for a unit u. Dividing by $\mathcal{E}$:

$$w_0\,\gamma(G) - G = P_0'(z)\,\tau\mathsf{u}\,\sigma_\nu z + r\,\tau\mathsf{u}\,\sigma_\nu z \cdot ([\varepsilon]^{p^\nu} - 1)z.$$

Apply θ. The last term vanishes; $\theta(\gamma G) = \theta(G)$ by Input I2; and $\theta(\sigma_\nu) = p^\nu$, since $\theta([\varepsilon]) = 1$. Therefore $\theta(G)\,\theta(w_0 - 1) = \theta(P_0'(z))\,\theta(\mathsf{u})\,p^\nu\,\theta(z\tau)$, and $\theta(w_0 - 1)$ divides $p^{\mathfrak{d}+\nu}\,\theta(z\tau)$.

The term 2ν. If $\nu = 0$ then $\theta_\nu = \theta$. If $\nu \geq 1$, then $\theta(z) \mid p$ and $\theta(\tau) \mid p$ (Input I1: τ divides $\xi - p$ and $\theta(\xi) = 0$), so $\theta(z\tau)$ divides p^2, hence $p^{2\nu}\theta_\nu(z\tau)$. In both cases $\theta(z\tau) \mid p^{2\nu}\theta_\nu(z\tau)$. The three divisibilities give (1). □

Lemma 3.1 (the depth step at exponent 0)**.** *In Setting 2.1, let $\nu \geq 0$ be an integer and put $m = p^{\mathrm{cost}(\nu,0)}$. There are $t_1, \ldots, t_m \in \Gamma'$ such that, for every $g \in \Gamma'$ and every $s \in \mathfrak{S}$,*

$$\mathcal{E}_\nu \;\mid\; g \cdot \sum_{k=1}^{m} t_k \cdot s - \sum_{k=1}^{m} t_k \cdot s \qquad \text{in } A_{\mathrm{inf}}.$$

LEAN `Lift.KummerChart.exists_depth_step_zero`

Proof. For $1 \leq k \leq m$ choose $t_k \in \Gamma'$ with $t_k z = [\varepsilon]^{k-1}z$. Divisibility by $\mathcal{E}_\nu$ is the vanishing of θ_ν, which commutes with Γ'; so by Input I5 it suffices to treat $s = wz^a$ with $w \in \mathbb{W}$, and to show that $\theta_\nu(g\sum_k t_k s) = \theta_\nu(\sum_k t_k s)$. Each t_k fixes w, so $\theta_\nu(t_k s) = \theta_\nu(w)\theta_\nu(z)^a\varrho^{(k-1)a}$, where $\varrho = \theta_\nu([\varepsilon])$ is a primitive p^ν-th root of unity. If $p^\nu \nmid a$, then $\varrho^a \neq 1$ and $(\varrho^a)^m = 1$, because p^ν divides $m = p^{\mathfrak{d}+3\nu}$; the geometric sum $\sum_{k=1}^{m}\varrho^{(k-1)a}$ vanishes, and both sides are 0. If $a = p^\nu a'$, then $z^a = \varphi^\nu(z^{a'})$ and, for every $\gamma \in \Gamma'$, $\theta_\nu(\gamma s) = \theta(\gamma\varphi^{-\nu}(w))\,\theta(\gamma z^{a'}) = \theta_\nu(s)$, because γ fixes $\mathbb{W}$ and $\theta(z)$. So $\theta_\nu(\sum_k t_k s) = m\,\theta_\nu(s)$, and the same computation with gt_k in place of t_k gives the same value. □

4 Exponent modules: first properties

Throughout, the setting is Setting 2.2; L and L' are lattice data over $\mathcal{O}$.

Lemma 4.1 (the localized modules)**.** *Let $x \in W(C^\flat) \otimes L$, let $\nu, l \geq 0$ and n be integers, let $g \in \Gamma'$ and let $f\colon L \to L'$ be an equivariant map.*

(a) $\mathcal{E}_\nu x \in \mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$ *if and only if* $x \in \mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$.
(b) $\mathcal{E}_\nu^n x \in \mathfrak{M}_{\mathrm{loc}}(L)$ *if and only if* $x \in \mathfrak{M}_{\mathrm{loc}}(L)$.
(c) *If* $x \in \mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$, *then* $(1 \otimes f)(x) \in \mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L')$; *if* $x \in \mathfrak{M}_{\mathrm{loc}}(L)$, *then* $(1 \otimes f)(x) \in \mathfrak{M}_{\mathrm{loc}}(L')$.
(d) *If* $x \in \mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$, *then* $gx \in \mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$.
(e) $\mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$ *is contained in the fraction span of* L.
(f) *If* $x \in \mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$ *and* $\mu \in \mathrm{Per}(L)$, *then* $\mu(\varphi^{-\nu}(gx - x)) = g \cdot \mu(\varphi^{-\nu}x) - \mu(\varphi^{-\nu}x)$.
(g) *If* $x \in \mathfrak{M}_{\mathrm{loc}}(L)$, *then* $\varphi^l x \in \mathfrak{M}_{\mathrm{loc}}(L)$.
(h) *The* A_{inf}*-span of* $\mathfrak{M}_{\mathrm{loc}}(L)$ *is* $\mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$.
(i) *If* $x \in \mathfrak{M}_{\mathrm{loc}}(L)$ *and* $\mu \in \mathrm{Per}(L)$, *then* $\mu(\varphi^{-\nu}x) \in F_\nu$.

LEAN `Lift.KummerChart.JumpLattice.frobE_smul_mem_kisinLocAinf_iff` · `Lift.KummerChart.JumpLattice.frobEUnit_zpow_smul_mem_kisinLoc` · `Lift.KummerChart.JumpLattice.Hom.wittMap_mem_kisinLocAinf` · `Lift.KummerChart.JumpLattice.Hom.wittMap_mem_kisinLoc` · `Lift.KummerChart.JumpLattice.gal_smul_mem_kisinLocAinf` · `Lift.KummerChart.JumpLattice.kisinLocAinf_le_fracSpan` · `Lift.KummerChart.JumpLattice.valueAt_smul_sub` · `Lift.KummerChart.JumpLattice.phi_pow_mem_kisinLoc` · `Lift.KummerChart.JumpLattice.span_kisinLoc` · `Lift.KummerChart.JumpLattice.valueAt_mem_valueField` · `Lift.KummerChart.JumpLattice.valueAt_zero` (the identity $\mu(\varphi^0 x) = \mu(x)$)

Proof. Several proofs go by *induction on the localization*: a property that holds on $\mathfrak{M}^{\mathrm{inf}}(L)$ and passes from $\mathcal{E}_\nu x$ to x, for every ν, holds on $\mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$.

(a) holds by the definition of $\mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$ as a saturation. (b) The units c of $W(C^\flat)$ with $c\,\mathfrak{M}_{\mathrm{loc}}(L) = \mathfrak{M}_{\mathrm{loc}}(L)$ form a group, so it suffices to show that $\mathcal{E}_\nu x \in \mathfrak{M}_{\mathrm{loc}}(L)$ if and only if $x \in \mathfrak{M}_{\mathrm{loc}}(L)$. For $\mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}$ this is (a); and $\mu(\mathcal{E}_\nu x) = \mathcal{E}_\nu \mu(x)$ lies in F_0 exactly when $\mu(x)$ does, because the image of $\mathcal{E}_\nu$ in B_{dR} is a non-zero element of F_0.

(c) $1 \otimes f$ is $W(C^\flat)$-linear and maps $\mathfrak{M}^{\mathrm{inf}}(L)$ into $\mathfrak{M}^{\mathrm{inf}}(L')$, hence $\mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$ into $\mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L')$; and $\mu \circ (1 \otimes f) \in \mathrm{Per}(L)$ for $\mu \in \mathrm{Per}(L')$ (Input I9), which gives the values in F_0. (d) By induction on the localization: $\mathfrak{M}^{\mathrm{inf}}(L)$ is stable under Γ', and $g(\mathcal{E}_\nu x) = g(\mathcal{E}_\nu)\, gx$ with $g(\mathcal{E}_\nu)$ equal to $\mathcal{E}_\nu$ times a unit of A_{inf}, so (a) applies. (e) By the same induction from $\mathfrak{M}^{\mathrm{inf}}(L)$, which lies in the fraction span. (f) g commutes with $\varphi^{-\nu}$, the fraction span is stable under $\mathbb{Z} \times \Gamma$, and $\mu(gx') = g\mu(x')$ on it (Input I9).

(g) It suffices to treat $l = 1$. By induction on the localization, applied to the points over F_0: for $x \in \mathfrak{M}(L)$, $\mathcal{E}^{h_L}\varphi(x) \in \mathfrak{M}(L)$ by the frame theorem (Input I8(d)), so $\varphi(x) \in \mathfrak{M}_{\mathrm{loc}}(L)$ by (b); and $\varphi(\mathcal{E}_\nu x) = \mathcal{E}_{\nu+1}\varphi(x)$, so (b) passes the property from $\mathcal{E}_\nu x$ to x.

(h) By induction on the localization: $\mathfrak{M}^{\mathrm{inf}}(L) = A_{\mathrm{inf}} \cdot \mathfrak{M}(L)$ (Input I8(b)), and if $\mathcal{E}_\nu x = \sum_k a_k v_k$ with $v_k \in \mathfrak{M}_{\mathrm{loc}}(L)$, then $x = \sum_k a_k\, \mathcal{E}_\nu^{-1} v_k$ with $\mathcal{E}_\nu^{-1} v_k \in \mathfrak{M}_{\mathrm{loc}}(L)$ by (b).

(i) By (e) there is $r \neq 0$ in A_{inf} with $r\varphi^{-\nu}x \in A_{\mathrm{inf}} \otimes L$; then $\tau^{h_L} r\varphi^{-\nu}x \in \mathfrak{M}^{\mathrm{inf}}(L)$ (Input I8(c)), and by (h) it is $\sum_k a_k v_k$ with $a_k \in A_{\mathrm{inf}}$, $v_k \in \mathfrak{M}_{\mathrm{loc}}(L)$. Apply φ^{ν}:

$$-\varphi^{\nu}(\tau^{h_L} r)\, x + \sum_k \varphi^{\nu}(a_k)\, \varphi^{\nu} v_k = 0,$$

a relation with coefficients in A_{inf} between x and the $\varphi^{\nu}v_k$, which lie in $\mathfrak{M}_{\mathrm{loc}}(L)$ by (g). By Input I8(a) it persists when a coordinate co_a is applied to the coefficients; choose a with $c = \mathrm{co}_a(\varphi^{\nu}(\tau^{h_L} r)) \neq 0$. Applying $\mu \circ \varphi^{-\nu}$ gives $\varphi^{-\nu}(c)\, \mu(\varphi^{-\nu}x) = \sum_k \varphi^{-\nu}(\mathrm{co}_a(\varphi^{\nu} a_k))\, \mu(v_k)$. Here $\mu(v_k) \in F_0 \subseteq F_{\nu}$ and $\varphi^{-\nu}(\mathfrak{S}) \subseteq F_{\nu}$; and $\varphi^{-\nu}(c) \neq 0$ in B_{dR}, because φ is an automorphism of A_{inf} and the map $A_{\mathrm{inf}} \to B_{\mathrm{dR}}^{+}$ is injective. So $\mu(\varphi^{-\nu}x)$ lies in the field F_{ν}. □

Lemma 4.2 (the exponent modules)**.** *Let $x \in W(C^{\flat}) \otimes L$, let i be a letter and let o, o' be exponent functions.*

(a) *Let $x \in \mathfrak{X}_L(i; o)$. Suppose that, for every $\nu \geq 0$, every $u \in U_L(i^{p^{-\nu}})$ with $o(\nu, u) < o'(\nu, u)$ and every $\mu \in \mathrm{Per}_{i^{p^{-\nu}},u}(L)$, one has $\mu(\varphi^{-\nu}x) \in \mathrm{Fil}^{o'(\nu,u)} B_{\mathrm{dR}}$. Then $x \in \mathfrak{X}_L(i; o')$.*

(b) *Let $x \in \mathfrak{X}_L(i; o)$. If $o'(\nu, u) \leq o(\nu, u)$ for all $\nu \geq 0$ and $u \in U_L(i^{p^{-\nu}})$, then $x \in \mathfrak{X}_L(i; o')$.*

(c) *Let $\nu \geq 0$, $u \in \mathbb{Z}$ and $\mu \in \mathrm{Per}_{i^{p^{-\nu}},u}(L)$. If $x \in A_{\mathrm{inf}} \otimes L$ and $x \in (W(C^{\flat}) \otimes L)_i$, then $\mu(\varphi^{-\nu}x) \in \mathrm{Fil}^{-u} B_{\mathrm{dR}}$.*

(d) *Let $f\colon L \to L'$ be an equivariant map with $U_{L'}(j) \subseteq U_L(j)$ for every letter j. If $x \in \mathfrak{X}_L(i; o)$, then $(1 \otimes f)(x) \in \mathfrak{X}_{L'}(i; o)$.*

(e) *Let $r \in A_{\mathrm{inf}}$ with $\mathcal{E}_{\nu} \nmid r$ for all $\nu \geq 0$, and let $x \in \mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$. If $rx \in \mathfrak{X}_L(i; o)$, then $x \in \mathfrak{X}_L(i; o)$.*

(f) *Let $\nu_0 \geq 0$ and n be integers and $x \in \mathfrak{X}_L(i; o)$. If $o'(\nu, u) \leq o(\nu, u) + n$ for $\nu = \nu_0$ and $o'(\nu, u) \leq o(\nu, u)$ for $\nu \neq \nu_0$, for all $u \in U_L(i^{p^{-\nu}})$, then $\mathcal{E}_{\nu_0}^{n} x \in \mathfrak{X}_L(i; o')$.*

(g) *If $x \in \mathfrak{X}_L(i; o)$ and $g \in \Gamma'$, then $gx - x \in \mathfrak{X}_L(i; o)$.*

LEAN `Lift.KummerChart.JumpLattice.mem_withExponents_of_mem_hodgeFil` · `Lift.KummerChart.JumpLattice.mem_withExponents_of_le` · `Lift.KummerChart.JumpLattice.valueAt_mem_hodgeFil_of_mem_ainfLattice` · `Lift.KummerChart.JumpLattice.Hom.map_mem_withExponents` · `Lift.KummerChart.JumpLattice.mem_withExponents_of_smul` · `Lift.KummerChart.JumpLattice.frobEUnit_zpow_mem_withExponents` · `Lift.KummerChart.JumpLattice.smul_sub_mem_withExponents`

Proof. (a) and (b) hold because the filtration decreases. (c) The element $\varphi^{-\nu}x$ lies in $A_{\mathrm{inf}} \otimes L$ and in the part at $i^{p^{-\nu}}$ (Input I9), so it is an A_{inf}-combination of elements $\mathrm{pr}_{i^{p^{-\nu}}}(1 \otimes t)$, on which μ takes values in the B_{dR}^{+}-module $\mathrm{Fil}^{-u} B_{\mathrm{dR}}$. (d) Membership in $\mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}$ is Lemma 4.1(c); $1 \otimes f$ commutes with $1 \otimes \omega$ and with φ; and for $\mu \in \mathrm{Per}_{j,u}(L')$ with $u \in U_{L'}(j) \subseteq U_L(j)$ the period $\mu \circ (1 \otimes f)$ lies in $\mathrm{Per}_{j,u}(L)$, because $1 \otimes f$ commutes with pr_j. (e) r is a non-zero-divisor on $W(C^{\flat}) \otimes L$ and commutes with $1 \otimes \omega$, so x lies in the part at i; and $\mu(\varphi^{-\nu}(rx)) = \varphi^{-\nu}(r)\mu(\varphi^{-\nu}x)$ with $\varphi^{-\nu}(r)$ a unit of B_{dR}^{+} (Input I1). (f) $\mathcal{E}_{\nu_0}^{n} x$ lies in $\mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$ by Lemma 4.1(a) and in the part at i, and $\mu(\varphi^{-\nu}(\mathcal{E}_{\nu_0}^{n} x)) = \varphi^{-\nu}(\mathcal{E}_{\nu_0})^{n} \mu(\varphi^{-\nu}x)$; the factor $\varphi^{-\nu}(\mathcal{E}_{\nu_0})$ is a uniformizer of B_{dR}^{+} for $\nu = \nu_0$ and a unit otherwise (Input I1). (g) By Lemma 4.1(d) and (f): $gx - x$ lies in $\mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$, in the part at i, and its values are $g\mu(\varphi^{-\nu}x) - \mu(\varphi^{-\nu}x)$, in the Galois-stable module $\mathrm{Fil}^{o(\nu,u)} B_{\mathrm{dR}}$. □

Lemma 4.3 (division by $z\tau$)**.** *Let $g \in \Gamma'$ and $x \in \mathfrak{M}_{\mathrm{loc}}(L)$.*

(a) $(z\tau)^{-1}(gx - x) \in \mathfrak{M}_{\mathrm{loc}}^{\inf}(L)$.

(b) *Let i be a letter and o an exponent function. If $gx - x \in \mathfrak{X}_L(i; o)$, then $(z\tau)^{-1}(gx - x) \in \mathfrak{X}_L(i; o)$.*

LEAN `Lift.KummerChart.JumpLattice.smul_sub_mem_zTau_kisinLocAinf` · `Lift.KummerChart.JumpLattice.smul_sub_mem_zTauUnit_smul_of_deep`

Proof. (a) By induction on the localization, for the points over F_0. Let $x \in \mathfrak{M}(L)$. Then $gx - x \in \mathfrak{M}^{\inf}(L)$, and for a crystalline period μ we have $\mu(gx - x) = g\mu(x) - \mu(x)$, and $\mathcal{E}^{h_L}\mu(x) = s$ lies in the image of $\mathfrak{S}_B$. Since $z\tau$ divides $gs - s$ and $g\mathcal{E} - \mathcal{E}$ in B^+, the element $\mathcal{E}^{2h_L}(g\mu(x) - \mu(x))$ is a multiple of $z\tau$; so $gx - x \in z\tau\mathfrak{M}^{\inf}(L)$ by Input I8(e). For the step, let $\mathcal{E}_\nu x$ satisfy the claim, with $x \in \mathfrak{M}_{\mathrm{loc}}(L)$. Then

$$g(\mathcal{E}_\nu x) - \mathcal{E}_\nu x = (g\mathcal{E}_\nu - \mathcal{E}_\nu)\, gx + \mathcal{E}_\nu(gx - x),$$

and $z\tau$ divides $g\mathcal{E}_\nu - \mathcal{E}_\nu$ in $A_{\inf}$ (Input I2) while $gx \in \mathfrak{M}_{\mathrm{loc}}^{\inf}(L)$; so $(z\tau)^{-1}\mathcal{E}_\nu(gx - x) \in \mathfrak{M}_{\mathrm{loc}}^{\inf}(L)$, and Lemma 4.1(a) removes $\mathcal{E}_\nu$. (b) follows from (a) and Lemma 4.2(e) with $r = z\tau$ (Input I1). □

Lemma 4.4 (values at a period of any type)**.** *Let $\nu \geq 0$, let i be a letter, o an exponent function, $u \in \mathbb{Z}$, $x \in \mathfrak{X}_L(i; o)$ and $\mu \in \mathrm{Per}_{i^{p^{-\nu}},u}(L)$. Let M be a B_{dR}^+-submodule of B_{dR} with $\mathrm{Fil}^{o(\nu,v)} B_{\mathrm{dR}} \subseteq M$ for every $v \in U_L(i^{p^{-\nu}})$ with $v \leq u$. Then $\mu(\varphi^{-\nu}x) \in M$.*

LEAN `Lift.KummerChart.JumpLattice.valueAt_mem_of_hodgeFil_le` · the integer u need not lie in $U_L(i^{p^{-\nu}})$

Proof. Put $j = i^{p^{-\nu}}$ and suppose $\mu(\varphi^{-\nu}x) \notin M$. We show by descending induction that $\mu \in \mathrm{Per}_{j,v}(L)$ for every $v \leq u$. If this holds for v and $v \in U_L(j)$, then $\mu(\varphi^{-\nu}x) \in \mathrm{Fil}^{o(\nu,v)} B_{\mathrm{dR}} \subseteq M$, which is excluded; so $v \notin U_L(j)$, and $\mu \in \mathrm{Per}_{j,v-1}(L)$ by Input I9(d). Now choose $r \neq 0$ in $A_{\inf}$ with $rx \in A_{\inf} \otimes L$ (Lemma 4.1(e)). By Lemma 4.2(c), $\varphi^{-\nu}(r)\,\mu(\varphi^{-\nu}x) \in \mathrm{Fil}^{-v} B_{\mathrm{dR}}$ for every $v \leq u$, so it is 0; as $\varphi^{-\nu}(r) \neq 0$, $\mu(\varphi^{-\nu}x) = 0 \in M$: a contradiction. □

5 The retraction

Lemma 5.1 (a retraction onto the module over $\mathfrak{S}$)**.** *In Setting 2.2, for a lattice datum L there is an $\mathfrak{S}$-linear map $r\colon \mathfrak{M}_{\mathrm{loc}}^{\inf}(L) \to \mathfrak{M}_{\mathrm{loc}}(L)$ with $r(ax) = \mathrm{co}_0(a)\, x$ for all $a \in A_{\inf}$ and $x \in \mathfrak{M}_{\mathrm{loc}}(L)$.*

LEAN `Lift.KummerChart.JumpLattice.exists_coordRetract`

Proof. By Lemma 4.1(h), $\mathfrak{M}_{\mathrm{loc}}^{\inf}(L)$ is the image of $A_{\inf} \otimes_{\mathfrak{S}} \mathfrak{M}_{\mathrm{loc}}(L)$. The $\mathfrak{S}$-linear map $a \otimes x \mapsto \mathrm{co}_0(a)x$ on the tensor product factors through this image if and only if every relation $\sum_k a_k x_k = 0$ with $x_k \in \mathfrak{M}_{\mathrm{loc}}(L)$ implies $\sum_k \mathrm{co}_0(a_k)x_k = 0$; this is Input I8(a), the elements of $\mathfrak{M}_{\mathrm{loc}}(L)$ being points of the fraction span over F_0. □

Definition 5.2 (the retraction)**.** In Setting 2.2, for a lattice datum L, the map ret_L from $\mathfrak{M}_{\mathrm{loc}}^{\inf}(L)$ to $\mathfrak{M}_{\mathrm{loc}}(L)$ is a chosen map with the property of Lemma 5.1.

LEAN `Lift.KummerChart.JumpLattice.coordRetract`

Lemma 5.3 (properties of the retraction)**.** *In Setting 2.2, let L, L' be lattice data.*

(a) *If* $v_1, \dots, v_m \in \mathfrak{M}_{\mathrm{loc}}(L)$, $a_1, \dots, a_m \in A_{\mathrm{inf}}$ *and* $y \in \mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$ *with* $\sum_k a_k v_k = y$, *then* $\mathrm{ret}_L(y) = \sum_k \mathrm{co}_0(a_k) v_k$. *If* $x \in \mathfrak{M}_{\mathrm{loc}}(L)$, *then* $\mathrm{ret}_L(x) = x$.

(b) *Let* $f\colon L \to L'$ *be an equivariant map,* $y \in \mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$ *and* $y' \in \mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L')$ *with* $(1 \otimes f)(y) = y'$. *Then* $(1 \otimes f)(\mathrm{ret}_L(y)) = \mathrm{ret}_{L'}(y')$.

(c) *Let* $n \in \mathbb{Z}$, $l \geq 0$, $\mu \in \mathrm{Per}(L)$, $y \in \mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$ *and* $v \in F_l$. *If* $\mu(\varphi^{-l} y) - v \in \mathrm{Fil}^n B_{\mathrm{dR}}$, *then* $\mu(\varphi^{-l} \mathrm{ret}_L(y)) - v \in \mathrm{Fil}^n B_{\mathrm{dR}}$.

(d) *Let* i *be a letter,* o *an exponent function and* $y \in \mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$. *If* $y \in \mathfrak{X}_L(i; o)$, *then* $\mathrm{ret}_L(y) \in \mathfrak{X}_L(i; o)$.

LEAN `Lift.KummerChart.JumpLattice.coordRetract_sum` · `Lift.KummerChart.JumpLattice.coordRetract_of_mem_kisinLoc` · `Lift.KummerChart.JumpLattice.Hom.map_coordRetract` · `Lift.KummerChart.JumpLattice.valueAt_coordRetract_sub_mem_hodgeFil` · `Lift.KummerChart.JumpLattice.coordRetract_mem_withExponents`

Proof. (a) is the defining property, with $\mathrm{co}_0(1) = 1$. (b) Write $y = \sum_k a_k v_k$ with $v_k \in \mathfrak{M}_{\mathrm{loc}}(L)$ (Lemma 4.1(h)). Then $y' = \sum_k a_k (1 \otimes f)(v_k)$, with $(1 \otimes f)(v_k) \in \mathfrak{M}_{\mathrm{loc}}(L')$, and both sides equal $\sum_k \mathrm{co}_0(a_k)(1 \otimes f)(v_k)$ by (a). (c) With y written as in (b),

$$\mu(\varphi^{-l} y) - v = \sum_k \varphi^{-l}(a_k)\, \mu(\varphi^{-l} v_k) + \varphi^{-l}(1) \cdot (-v),$$

and the values $\mu(\varphi^{-l} v_k)$ lie in F_l by Lemma 4.1(i). Input I7(a) with $f = \mathrm{co}_0$ replaces each coefficient a_k by $\mathrm{co}_0(a_k)$ and keeps 1. (d) $\mathrm{ret}_L(y) \in \mathfrak{M}_{\mathrm{loc}}(L) \subseteq \mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$. Multiplication by ω is an equivariant map $L \to L$, so by (b) and the $\mathfrak{S}$-linearity of ret_L, $(1 \otimes \omega)\,\mathrm{ret}_L(y) = \mathrm{ret}_L((1 \otimes \omega) y) = \mathrm{ret}_L([i] y) = [i]\,\mathrm{ret}_L(y)$, as $[i] \in \mathbb{W}$. The values are in the right steps of the filtration by (c) with $v = 0$. □

6 Deepening

Lemma 6.1 (small facts)**.** *In Setting 2.2:*

(a) *If the Kummer chart is unramified and* n *is an integer such that* $p \mid n$ *implies* $n = 0$, *then* $\mathrm{cost}(\nu, n) \leq 3\nu$ *for every* $\nu \geq 0$.

(b) *Every element* c *of* F, *seen in* B_{dR}, *satisfies* $g \cdot c = c$ *for all* $g \in \Gamma'$.

LEAN `Lift.KummerChart.JumpLattice.cost_le_of_isUnramified` · `Lift.KummerChart.JumpLattice.galMonoid_smul_localFieldOf`

Proof. (a) $v_p(n) = 0$ and $\mathfrak{d} = 0$. (b) F is the field fixed by $j_\Gamma(\Gamma')$. □

Lemma 6.2 (deep elements have better exponents)**.** *In Setting 2.2, let* L *be a lattice datum,* i *a letter,* o^{lo} *and* o^{hi} *exponent functions, and* $x \in \mathfrak{X}_L(i; o^{\mathrm{lo}})$. *If* $x \in \mathrm{Inv}(\mathfrak{X}_L(i; o^{\mathrm{hi}}))$, *then* $x \in \mathfrak{X}_L(i; \mathrm{imp}(o^{\mathrm{lo}}, o^{\mathrm{hi}})(\cdot, \cdot))$.

LEAN `Lift.KummerChart.JumpLattice.mem_withExponents_of_deep`

Proof. Fix $\nu \geq 0$, $u \in U_L(i^{p^{-\nu}})$ and $\mu \in \mathrm{Per}_{i^{p^{-\nu}}, u}(L)$, and write $a = o^{\mathrm{lo}}(\nu, u)$, $b = o^{\mathrm{hi}}(\nu, u)$ and $V = \mu(\varphi^{-\nu} x)$. Then $V \in \mathrm{Fil}^a B_{\mathrm{dR}}$, and $gV - V = \mu(\varphi^{-\nu}(gx - x)) \in \mathrm{Fil}^b B_{\mathrm{dR}}$ for all $g \in \Gamma'$ (Lemma 4.1(f)). By Input I6 there is $c \in F$ with $V - c \in \mathrm{Fil}^b B_{\mathrm{dR}}$. Then $c \in \mathrm{Fil}^{\min(a,b)} B_{\mathrm{dR}}$, so $c = 0$ or $\min(a, b) \leq 0$. If $a > 0$: either $c = 0$, or $b \leq 0$ and $c \in \mathrm{Fil}^0 B_{\mathrm{dR}} \subseteq \mathrm{Fil}^b B_{\mathrm{dR}}$; in both cases $V \in \mathrm{Fil}^b B_{\mathrm{dR}}$. If $a \leq 0$: $V - c$ and c lie in $\mathrm{Fil}^{\min(b,0)} B_{\mathrm{dR}}$. □

Proposition 6.3 (one step deeper, at an explicit cost)**.** *In Setting 2.2, let L be a lattice datum, $\nu \geq 0$ an integer and n an integer. There are an integer $m \geq 0$, elements $t_1, \dots, t_m \in \Gamma'$ and $y \in A_{\mathrm{inf}}$ such that for every $x \in \mathfrak{M}_{\mathrm{loc}}(L)$ and every $\mu \in \mathrm{Per}(L)$: if there is $c \in F$ with $\mu(\varphi^{-\nu}x) - c \in \mathrm{Fil}^n B_{\mathrm{dR}}$, then there is $c' \in F$ with*

$$\mu\Big(\varphi^{-\nu}\Big(p^{\mathrm{cost}(\nu,n)}x + \sum_{k=1}^{m} y\,(z\tau)^{-1}(t_k x - x)\Big)\Big) - c' \;\in\; \mathrm{Fil}^{n+1} B_{\mathrm{dR}}.$$

LEAN `Lift.KummerChart.JumpLattice.exists_reaches_depth_succ`

Proof. *The case $n \neq 0$.* Take g, w, y_0, X from Theorem A for (ν, n), and $m = 1$, $t_1 = g$, $y = -y_0$. Let $V = \mu(\varphi^{-\nu}x)$ and $V - c \in \mathrm{Fil}^n B_{\mathrm{dR}}$. Put $\pi_\nu = \varphi^{-\nu}(\mathcal{E}_\nu)$, a uniformizer of B_{dR}^+ in F_ν, and $V - c = \pi_\nu^n V_0$; then $V_0 \in F_\nu \cap \mathrm{Fil}^0 B_{\mathrm{dR}}$, by Lemma 4.1(i). With $\hat{w} = \varphi^{-\nu}(w)$ we have $g\pi_\nu = \hat{w}\pi_\nu$ and $gc = c$, so

$$gV - V = \pi_\nu^n(\hat{w}^n\, gV_0 - V_0) = \pi_\nu^n((\hat{w}^n - 1)V_0 + \hat{w}^n(gV_0 - V_0)).$$

Here $gV_0 - V_0 \in \mathrm{Fil}^1 B_{\mathrm{dR}}$ by Input I7(b), applied to the single element 1 and to the divisibility $\mathcal{E}_\nu \mid gs - s$ of Theorem A; and $\hat{w}^n - 1 = \varphi^{-\nu}(z\tau)\,\varphi^{-\nu}(X)$. As $\varphi^{-\nu}(z\tau)$ is a unit of B_{dR}^+, Lemma 4.1(f) gives

$$\mu(\varphi^{-\nu}((z\tau)^{-1}(gx - x))) \;\equiv\; \pi_\nu^n \varphi^{-\nu}(X) V_0 \pmod{\mathrm{Fil}^{n+1} B_{\mathrm{dR}}}.$$

Multiply by $\varphi^{-\nu}(y_0)$ and use $\varphi^{-\nu}(y_0 X) \equiv p^{\mathrm{cost}(\nu,n)}$ modulo $\mathrm{Fil}^1 B_{\mathrm{dR}}$: the value of μ at $\varphi^{-\nu}(y_0(z\tau)^{-1}(gx - x))$ is congruent to $p^{\mathrm{cost}(\nu,n)}(V - c)$. So the element of the statement has value congruent to $p^{\mathrm{cost}(\nu,n)}V - p^{\mathrm{cost}(\nu,n)}(V - c) = p^{\mathrm{cost}(\nu,n)}c$, and $c' = p^{\mathrm{cost}(\nu,n)}c$ serves.

The case $n = 0$. Take the $m = p^{\mathrm{cost}(\nu,0)}$ elements t_k of Lemma 3.1 and $y = z\tau$. The element of the statement is $mx + \sum_k (t_k x - x) = \sum_k t_k x$, with value $\sum_k t_k V$, where $V = \mu(\varphi^{-\nu}x) \in F_\nu \cap \mathrm{Fil}^0 B_{\mathrm{dR}}$. By Lemma 3.1 and Input I7(b), $g\sum_k t_k V - \sum_k t_k V \in \mathrm{Fil}^1 B_{\mathrm{dR}}$ for all $g \in \Gamma'$; Input I6 gives c'. □

7 The Hodge expansion and the proof of Theorem B

Throughout, the setting is Setting 2.2. The space $\mathrm{Fun}(L)$ of all A_{inf}-linear maps $W(C^\flat) \otimes L \to B_{\mathrm{dR}}$ is a vector space over B_{dR}.

Lemma 7.1 (testing a value on a Hodge basis)**.** *Let L be a lattice datum, x an element of the fraction span of L, i a letter, $n \in \mathbb{Z}$ and $\mu \in \mathrm{Fun}(L)$. Suppose that $D(\mu) \cdot \mu'(x) \in \mathrm{Fil}^n B_{\mathrm{dR}}$ for every $\kappa \in \mathbb{Z}$, every $\mu' \in \mathrm{Per}_{i,\kappa}(L)$ and every B_{dR}-linear map $D\colon \mathrm{Fun}(L) \to B_{\mathrm{dR}}$ with the two properties*

- (D1) *if $\lambda \in \mathrm{Fun}(L)$, $a \in \mathbb{Z}$ and $\lambda(1 \otimes t) \in \mathrm{Fil}^a B_{\mathrm{dR}}$ for all $t \in L$, then $D(\lambda) \in \mathrm{Fil}^{a+\kappa} B_{\mathrm{dR}}$;*
- (D2) *if $\lambda, \lambda' \in \mathrm{Fun}(L)$, $g \in \Gamma'$ and $\lambda'(1 \otimes gt) = g \cdot \lambda(1 \otimes t)$ for all $t \in L$, then $D(\lambda') = g \cdot D(\lambda)$.*

Then $\mu(x) \in \mathrm{Fil}^n B_{\mathrm{dR}}$.

LEAN `Lift.KummerChart.JumpLattice.mem_hodgeFil_of_hodge_basis`

Proof. Take a $\mathbb{Z}_p$-basis $(e_k)_k$ of L and κ_k, H as in Input I10. By Input I9(e) there are $\mu_k \in \mathrm{Fun}(L)$ with $\mu_k(1 \otimes e_a) = (\mathsf{H}^{-1})_{ka}$. For $\lambda \in \mathrm{Fun}(L)$ put $D_k(\lambda) = \sum_a \lambda(1 \otimes e_a)\mathsf{H}_{ak}$. Then λ and $\sum_k D_k(\lambda)\mu_k$ agree on the basis, hence on the fraction span, so $\mu(x) = \sum_k D_k(\mu)\,\mu_k(x)$. It suffices to check that each triple (κ_k, μ_k, D_k) is admissible. The identity $\mathsf{R}(g)\,g(\mathsf{H}) = \mathsf{H}$ says exactly that μ_k is Γ'-equivariant on $1 \otimes L$ and that D_k has (D2). The entries of the k-th row of H^{-1} lie in $\mathrm{Fil}^{-\kappa_k} B_{\mathrm{dR}}$; as $\mathrm{pr}_i(1 \otimes t) \in A_{\mathrm{inf}} \otimes L$, this gives $\mu_k \in \mathrm{Per}_{i,\kappa_k}(L)$. The entries of the k-th column of H lie in $\mathrm{Fil}^{\kappa_k} B_{\mathrm{dR}}$, which gives (D1). □

Lemma 7.2 (Frobenius shifts the exponents)**.** *Let L be a lattice datum, i a letter, $l \geq 0$ an integer, o an exponent function and $x \in W(C^\flat) \otimes L$. If $x \in \mathfrak{X}_L(i^{p^{-l}}; o)$, then $\varphi^l x \in \mathfrak{X}_L(i; \mathrm{fr}_{l,0}(o)(\cdot,\cdot))$.*

LEAN `Lift.KummerChart.JumpLattice.phi_pow_mem_withExponents`

Proof. $\varphi^l x$ lies in $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{loc}}(L)$ by Lemma 4.1(g) and (h), and in the part at i. Let $\mu \in \mathrm{Per}_{i^{p^{-\nu}},u}(L)$. For $\nu \geq l$, $\mu(\varphi^{-\nu}\varphi^l x) = \mu(\varphi^{-(\nu-l)}x) \in \mathrm{Fil}^{o(\nu-l,u)} B_{\mathrm{dR}}$, the letter being $(i^{p^{-l}})^{p^{-(\nu-l)}}$. For $\nu < l$ we need $\mu(\varphi^{a+1}x) \in \mathrm{Fil}^0 B_{\mathrm{dR}}$ with $a = l - \nu - 1 \geq 0$: this holds on $\mathfrak{M}^{\mathrm{inf}}(L)$ by definition, and it passes from $\mathcal{E}_{\nu'}x$ to x because $\mu(\varphi^{a+1}(\mathcal{E}_{\nu'}x)) = \mathcal{E}_{\nu'+a+1}\,\mu(\varphi^{a+1}x)$ and the image of $\mathcal{E}_{\nu'+a+1}$ in B_{dR} is a unit of B^+_{dR} (Input I1, with index 0). □

Proof of Theorem B. Write $x = \varphi^l x_0$ and $j = i^{p^{-l}}$; then $x \in (W(C^\flat) \otimes L)_i$ if and only if $x_0 \in (W(C^\flat) \otimes L)_j$.

From the Kisin module to the exponents. Let $x_0 \in \mathfrak{M}^{\mathrm{inf}}(L)$ lie in the part at j. Then $\tau^b x_0 \in A_{\mathrm{inf}} \otimes L$ for some b, and for $\mu \in \mathrm{Per}_{j^{p^{-\nu}},u}(L)$ Lemma 4.2(c) gives $\varphi^{-\nu}(\tau)^b \mu(\varphi^{-\nu}x_0) \in \mathrm{Fil}^{-u} B_{\mathrm{dR}}$, with $\varphi^{-\nu}(\tau)$ a unit of B^+_{dR}. So $x_0 \in \mathfrak{X}_L(j; \mathrm{std}_0)$, and Lemma 7.2 gives $x \in \mathfrak{X}_L(i; \mathrm{fr}_{l,0}(\mathrm{std}_0)(\cdot,\cdot)) = \mathfrak{X}_L(i; \mathrm{std}_l)$.

From the exponents to the Kisin module. Let $x \in \mathfrak{X}_L(i; \mathrm{std}_l)$. We must show that x_0 is killed into $A_{\mathrm{inf}} \otimes L$ by a power of τ, and that $\mu(\varphi^{a+1}x_0) \in \mathrm{Fil}^0 B_{\mathrm{dR}}$ for $a \geq 0$, $\mu \in \mathrm{Per}(L)$.

First. We show that every $x' \in \mathfrak{X}_L(i; \mathrm{std}_l)$ is multiplied into $A_{\mathrm{inf}} \otimes L$ by a power of $\varphi^l(\tau)$; applying φ^{-l} then gives the claim for x_0. This goes by induction on the localization. On $\mathfrak{M}^{\mathrm{inf}}(L)$ it holds because τ divides $\varphi^l(\tau)$. Suppose it holds for $\mathcal{E}_\nu x'$, where $x' \in \mathfrak{X}_L(i; \mathrm{std}_l)$: so $c\,\mathcal{E}_\nu x' \in A_{\mathrm{inf}} \otimes L$ with c a power of $\varphi^l(\tau)$. If $\nu < l$, then $\mathcal{E}_\nu$ divides $\varphi^l(\tau)$ (Input I1), and a higher power works for x'. If $\nu \geq l$, write $c\,\mathcal{E}_\nu x' = \sum_k b_k(1 \otimes e_k)$ in a $\mathbb{Z}_p$-basis of L and let $\lambda_k \in \mathrm{Fun}(L)$ extend the k-th coordinate. Then

$$\varphi^{-\nu}(b_k) = \lambda_k(\varphi^{-\nu}(c\,\mathcal{E}_\nu x')) = \varphi^{-\nu}(c)\,\varphi^{-\nu}(\mathcal{E}_\nu)\,\lambda_k(\varphi^{-\nu}x').$$

We claim $\lambda_k(\varphi^{-\nu}x') \in \mathrm{Fil}^0 B_{\mathrm{dR}}$. By Lemma 7.1, at the letter $i^{p^{-\nu}}$ and for the element $\varphi^{-\nu}x'$ of the fraction span, it suffices that $D(\lambda_k)\mu'(\varphi^{-\nu}x') \in \mathrm{Fil}^0 B_{\mathrm{dR}}$ for admissible (κ, μ', D): by (D1), $D(\lambda_k) \in \mathrm{Fil}^\kappa B_{\mathrm{dR}}$, and by Lemma 4.4 with $M = \mathrm{Fil}^{-\kappa} B_{\mathrm{dR}}$, $\mu'(\varphi^{-\nu}x') \in \mathrm{Fil}^{-\kappa} B_{\mathrm{dR}}$, since $\mathrm{std}_l(\nu, v) = -v \geq -\kappa$ for $v \leq \kappa$. So $\varphi^{-\nu}(b_k) \in \mathrm{Fil}^1 B_{\mathrm{dR}}$, that is $\mathcal{E}_\nu \mid b_k$, and $c\,x' \in A_{\mathrm{inf}} \otimes L$ because $W(C^\flat) \otimes L$ has no A_{inf}-torsion.

Second. Let $\mu \in \mathrm{Per}(L)$ and $a \geq 0$; apply Lemma 7.1 to μ, the element $\varphi^{a+1}x_0$ and the letter $j^{p^{a+1}}$, with $n = 0$. For admissible (κ, μ', D): $D(\mu)$ is fixed by Γ' by (D2), because μ is a period, so $D(\mu) \in \mathrm{Fil}^0 B_{\mathrm{dR}}$ (Input I6). And $\varphi^{a+1}x \in$

$\mathfrak{X}_L(i^{p^{a+1}}; \mathrm{fr}_{a+1,0}(\mathrm{std}_l)(\cdot,\cdot))$ by Lemma 7.2, with $\mathrm{fr}_{a+1,0}(\mathrm{std}_l)(l,v) = 0$ for every v; so $\mu'(\varphi^{a+1}x_0) = \mu'(\varphi^{-l}\varphi^{a+1}x) \in \mathrm{Fil}^0 B_{\mathrm{dR}}$ by Lemma 4.4 with $M = \mathrm{Fil}^0 B_{\mathrm{dR}}$. □

Lemma 7.3 (projections to a letter)**.** *Let L be a lattice datum, $x \in W(C^\flat) \otimes L$, i a letter and $l \geq 0$.*

(a) *If $\varphi^{-l}x \in \mathfrak{M}^{\mathrm{inf}}(L)$, then $\mathrm{pr}_i(x) \in \mathfrak{X}_L(i; \mathrm{std}_l)$.*
(b) *If x is a point over F_0 of $\varphi^l\mathfrak{M}^{\mathrm{inf}}(L)$, that is $\varphi^{-l}x \in \mathfrak{M}^{\mathrm{inf}}(L)$ and $\mu(x) \in F_0$ for all $\mu \in \mathrm{Per}(L)$, then $\mathrm{pr}_i(x) \in \mathfrak{N}_{l,i}(L)$.*

LEAN Lift.KummerChart.JumpLattice.letterProj_mem_withExponents · Lift.KummerChart.JumpLattice.letterProj_mem_of_valueFieldPoints

Proof. (a) pr_i is a combination, with coefficients in A_{inf}, of powers of $1 \otimes \omega$, and multiplication by ω is an equivariant map $L \to L$; so $\varphi^{-l}\,\mathrm{pr}_i(x) \in \mathfrak{M}^{\mathrm{inf}}(L)$ (Input I9(b)), and Theorem B applies. (b) By (a), and because $\mu \circ \mathrm{pr}_i \in \mathrm{Per}(L)$. □

8 Exponents add: the proof of Theorem C

Lemma 8.1 (pairings)**.** *In Setting 2.2, let A, L, L'' be lattice data and $b\colon A \times L \to L''$ an $\mathcal{O}$-bilinear map.*

(a) *For a letter i, $x \in W(C^\flat)\otimes L$ and $y \in W(C^\flat)\otimes A$: $b_W(\mathrm{pr}_i(y), x) = \mathrm{pr}_i(b_W(y,x))$.*
(b) *Let T be a multiplicative subset of A_{inf} and let N_A, N_L, $N_{L''}$ be A_{inf}-submodules of $W(C^\flat) \otimes A$, $W(C^\flat) \otimes L$, $W(C^\flat) \otimes L''$. If some element of T multiplies y into N_A, some element of T multiplies x into N_L, and $b_W(y', x') \in N_{L''}$ for all $y' \in N_A$, $x' \in N_L$, then some element of T multiplies $b_W(y,x)$ into $N_{L''}$.*

LEAN Lift.KummerChart.JumpLattice.wittPairing_letterProj_left · Lift.KummerChart.JumpLattice.wittPairing_mem_saturateBy

Proof. (a) b is $\mathcal{O}$-bilinear, so its extension carries $1 \otimes \omega$ in the first variable to $1 \otimes \omega$ on $W(C^\flat) \otimes L''$. (b) By bilinearity. □

Proof of Theorem C. A criterion. Let $\mu \in \mathrm{Per}(L'')$, $m \in \mathbb{Z} \times \Gamma$, a letter j, $u \in \mathbb{Z}$ and $n \in \mathbb{Z}$; let x' lie in the fraction span of L and $y' \in W(C^\flat) \otimes A$. We claim that $\mu(m \cdot b_W(y', x')) \in \mathrm{Fil}^n B_{\mathrm{dR}}$ provided that

$$\lambda(my') \cdot \mu'(mx') \in \mathrm{Fil}^n B_{\mathrm{dR}} \tag{2}$$

for all $\kappa \in \mathbb{Z}$, all $\mu' \in \mathrm{Per}_{j,\kappa}(L)$ and all $\lambda \in \mathrm{Per}(A)$ such that $\lambda \in \mathrm{Per}_{j,u-\kappa}(A)$ if $\mu \in \mathrm{Per}_{j,u}(L'')$. Indeed, the equivariance of b gives $m \cdot b_W(y', x') = b_W(my', mx')$. Apply Lemma 7.1 to the lattice datum L, the element mx' and the functional $\mu \circ b_W(my', \cdot)$. For an admissible triple (κ, μ', D) put $\lambda(a) = D(\mu \circ b_W(a, \cdot))$ for $a \in W(C^\flat) \otimes A$. This is an A_{inf}-linear map; it is a period of A by (D2) and the equivariance of b and μ; and if $\mu \in \mathrm{Per}_{j,u}(L'')$, then by Lemma 8.1(a) the functional $\mu \circ b_W(\mathrm{pr}_j(1\otimes a), \cdot)$ takes values in $\mathrm{Fil}^{-u} B_{\mathrm{dR}}$ on $1 \otimes L$, so $\lambda(\mathrm{pr}_j(1\otimes a)) \in \mathrm{Fil}^{-u+\kappa} B_{\mathrm{dR}}$ by (D1): λ has type $u - \kappa$ at j. So (2) is what Lemma 7.1 asks.

The localized module. By Lemma 8.1(b) for the multiplicative set generated by the $\mathcal{E}_\nu$, then for the powers of τ, it suffices to see that $b_W(y', x') \in \mathfrak{M}^{\mathrm{inf}}(L'')$ for $y' \in \mathfrak{M}^{\mathrm{inf}}(A)$, $x' \in \mathfrak{M}^{\mathrm{inf}}(L)$. The pairing maps $(A_{\mathrm{inf}} \otimes A) \times (A_{\mathrm{inf}} \otimes L)$ into $A_{\mathrm{inf}} \otimes L''$. And for $\mu \in \mathrm{Per}(L'')$, $a \geq 0$, the criterion with $m = \varphi^{a+1}$ and $n = 0$ applies: $\lambda(\varphi^{a+1}y')$ and $\mu'(\varphi^{a+1}x')$ lie in $\mathrm{Fil}^0 B_{\mathrm{dR}}$ by the definition of the Kisin modules.

The letter. $y = \mathrm{pr}_i(y)$, so $b_W(y,x) = \mathrm{pr}_i(b_W(y,x))$ by Lemma 8.1(a).

The exponents. Let $\nu \geq 0$, $j = i^{p^{-\nu}}$, $v \in U_{L''}(j)$ and $\mu \in \mathrm{Per}_{j,v}(L'')$. By the criterion with $m = \varphi^{-\nu}$, $u = v$ and $n = o_{L''}(\nu, v)$, it suffices that $\lambda(\varphi^{-\nu}y)\,\mu'(\varphi^{-\nu}x) \in \mathrm{Fil}^{o_{L''}(\nu,v)} B_{\mathrm{dR}}$ for $\mu' \in \mathrm{Per}_{j,\kappa}(L)$ and $\lambda \in \mathrm{Per}_{j,v-\kappa}(A)$. Apply Lemma 4.4 to y and λ, with the module $M_1 = \{a \in B_{\mathrm{dR}} : a\,\mu'(\varphi^{-\nu}x) \in \mathrm{Fil}^{o_{L''}(\nu,v)} B_{\mathrm{dR}}\}$: we must show $\mathrm{Fil}^{o_A(\nu,u)} B_{\mathrm{dR}} \subseteq M_1$ for $u \in U_A(j)$, $u \leq v - \kappa$. Fix such u and $a \in \mathrm{Fil}^{o_A(\nu,u)} B_{\mathrm{dR}}$, and apply Lemma 4.4 to x and μ', with $M_2 = \{c \in B_{\mathrm{dR}} : a\,c \in \mathrm{Fil}^{o_{L''}(\nu,v)} B_{\mathrm{dR}}\}$: for $u' \in U_L(j)$ with $u' \leq \kappa$ we have $u + u' \leq v$, so $o_{L''}(\nu,v) \leq o(\nu,u') + o_A(\nu,u)$ and $a \cdot \mathrm{Fil}^{o(\nu,u')} B_{\mathrm{dR}} \subseteq \mathrm{Fil}^{o_{L''}(\nu,v)} B_{\mathrm{dR}}$. □

9 The lifting ideal and cokernel killers

Throughout, the setting is Setting 2.2 and S is a side over $\mathcal{O}$.

Lemma 9.1 (counting in two windows)**.** *Let $n, p, d, m, H \geq 0$ be integers and c an integer. If $n \leq p\,m$ and $|c| + n \leq H$, then*

$$\sum_{a \in \mathcal{W}_n(c)} (v_p(a) + d) \;\leq\; (4m+2)\,\lfloor \log_p H \rfloor + (4n+2)\,d.$$

LEAN `Lift.sum_twoWindows_le` · here p is any natural number, $v_p(0) = 0$, and $\lfloor \log_p H \rfloor$ is read as 0 when $p \leq 1$ or $H = 0$

Proof. The set $\mathcal{W}_n(c)$ has at most $4n+2$ elements, which accounts for the terms d. A non-zero $a \in \mathcal{W}_n(c)$ has $|a| \leq H$, so $v_p(a) \leq \lfloor \log_p H \rfloor$; and $v_p(a) \neq 0$ only if $a = pa'$, where a' lies in $\mathcal{W}_m(\lfloor c/p \rfloor)$, a set with at most $4m+2$ elements. □

Lemma 9.2 (cyclotomic twists)**.** *Let L be a lattice datum, $c \geq 0$ an integer, i a letter and $x \in \mathfrak{X}_L(i; \mathrm{std}_0)$. If $gt = \chi(g)^c t$ for all $g \in \Gamma$ and $t \in L$, then $\tau^c x \in A_{\mathrm{inf}} \otimes L$.*

LEAN `Lift.KummerChart.JumpLattice.tau_pow_smul_mem_ainfLattice_of_chi`

Proof. By Theorem B with $l = 0$, $x \in \mathfrak{M}^{\mathrm{inf}}(L)$; by the frame theorem $\tau^{h_L} x \in A_{\mathrm{inf}} \otimes L$; and $h_L \leq c$ because all jumps of L are $-c$ (Input I11). □

Lemma 9.3 (the lifting ideal: formal properties)**.** *Let $r, l, n, c, a \geq 0$ be integers, i a letter and $s \in A_{\mathrm{inf}}$.*

(a) *$s \in \mathfrak{a}_S(r; l, i)$ if and only if there is $t \in A_{\mathrm{inf}}$ such that for every $c' \in C_S$ some $x \in \mathfrak{X}_{\tilde{Y}_S}(i; \mathrm{std}_l)$ has $(1 \otimes \mathrm{pr}_S^C)(x) = (s - rt) \cdot \mathrm{cst}_{S,i}(c')$.*

(b) *$p^\kappa \in \mathfrak{a}_S(p^\kappa; l, i)$ for every integer $\kappa \geq 0$.*

(c) *$\varphi^l\ \mathrm{cst}_{S,i}(c') = \mathrm{cst}_{S,i^{p^l}}(c')$ for $c' \in C_S$.*

(d) *If $s \in \mathfrak{a}_S(r; 0, i)$ and $u \leq n$ for every $u \in U_{Y_S}(i)$, then $s\,\mathcal{E}^n \in \mathfrak{a}_S(r; 1, i)$.*

(e) *If $p^c \in \mathfrak{a}_S(p^{c+1}; l, i)$, then $\mathfrak{a}_S(p^{c+1}; l, i) \subseteq \mathfrak{a}_S(0; l, i)$.*

(f) *$\mathrm{cst}_{S,i}(c') \in \mathfrak{N}_{l,i}(C_S^{\mathrm{tr}})$ for $c' \in C_S$.*

(g) *$\varphi^l(s) \in \mathfrak{a}_S(r; l, i^{p^l})$ if and only if $s \in \mathfrak{a}_S(r; 0, i)$.*

(h) *$p^c z^{pa} \in \mathfrak{a}_S(r; 1, i^p)$ if and only if $p^c z^a \in \mathfrak{a}_S(r; 0, i)$.*

LEAN `Lift.KummerChart.PairingSide.mem_liftIdeal_iff` · `Lift.KummerChart.PairingSide.p_pow_mem_liftIdeal_self` · `Lift.KummerChart.PairingSide.phi_pow_smul_constantAt` · `Lift.KummerChart.PairingSide.mul_E_pow_mem_liftIdeal_one` · `Lift.KummerChart.PairingSide.liftIdeal_le_of_p_pow_mem` ·

Lift.KummerChart.PairingSide.constantAt_mem_locLetterPart · Lift.KummerChart.PairingSide.phi_pow_mem_liftIdeal_iff · Lift.KummerChart.PairingSide.p_pow_mul_z_pow_mem_liftIdeal_iff

Proof. (a) $A_{\mathrm{inf}} \otimes C_S$ is spanned over A_{inf} by the $1 \otimes c'$, and the image of $\mathfrak{X}_{\tilde{Y}_S}(i; \mathrm{std}_l)$ is an A_{inf}-module. (b) It is the generator. (c) φ maps the Teichmüller lift of i to that of i^p, and fixes $1 \otimes \omega$ and $1 \otimes c'$.

(d) Let x lift $(s - rt)\,\mathrm{cst}_{S,i}(c')$ as in (a), with $l = 0$. By Lemma 4.2(f) with $\nu_0 = 0$, $\mathcal{E}^n x$ has the exponents $n - u \geq 0 = \mathrm{std}_1(0, u)$ at $\nu = 0$ and $-u = \mathrm{std}_1(\nu, u)$ at $\nu \geq 1$, for $u \in U_{Y_S}(i)$; it lifts $(s\mathcal{E}^n - rt\mathcal{E}^n)\,\mathrm{cst}_{S,i}(c')$.

(e) Put $J = \mathfrak{a}_S(0; l, i)$; then $\mathfrak{a}_S(p^{c+1}; l, i) = (p^{c+1}) + J$. The hypothesis says $(p^c) \subseteq p \cdot (p^c) + J$. As A_{inf} is p-adically complete, p lies in its Jacobson radical, and Nakayama's lemma gives $(p^c) \subseteq J$.

(f) The element $1 \otimes c'$ lies in $A_{\mathrm{inf}} \otimes C_S$, is fixed by φ, and every period takes a Γ'-fixed value on it, which lies in $\mathrm{Fil}^0 B_{\mathrm{dR}} \cap F_0$ (Input I6); so it is a point over F_0 of $\varphi^l \mathfrak{M}^{\mathrm{inf}}(C_S^{\mathrm{tr}})$, and Lemma 7.3(b) applies.

(g) By (a), Theorem B and (c): $x \in \mathfrak{X}_{\tilde{Y}_S}(i^{p^l}; \mathrm{std}_l)$ if and only if $\varphi^{-l} x \in \mathfrak{X}_{\tilde{Y}_S}(i; \mathrm{std}_0)$, and φ^l is a bijection that commutes with $1 \otimes \mathrm{pr}_S^C$ and fixes the integer r. (h) is (g) for $l = 1$, since $\varphi(z) = z^p$. □

Lemma 9.4 (lifts over $\mathfrak{S}$)**.** *Let $r, l \geq 0$ be integers, i a letter and $f \in \mathfrak{S}$ with $f \in \mathfrak{a}_S(r; l, i)$. Then there is $t \in \mathfrak{S}$ such that for every $y \in \mathfrak{N}_{l,i}(C_S^{\mathrm{tr}})$ some $x \in \mathfrak{N}_{l,i}(\tilde{Y}_S)$ has $(1 \otimes \mathrm{pr}_S^C)(x) = (f - rt)\,y$.*

LEAN Lift.KummerChart.PairingSide.exists_lift_of_mem_liftIdeal

Proof. Write $f = rt_0 + s$ with $t_0 \in A_{\mathrm{inf}}$ and s in the colon ideal of Definition 2.8(ii), and put $t = \mathrm{co}_0(t_0)$. Let $y \in \mathfrak{N}_{l,i}(C_S^{\mathrm{tr}})$. As $U_{C_S^{\mathrm{tr}}}(j) = \{0\}$, y has the exponents std_0, so $y \in A_{\mathrm{inf}} \otimes C_S$ by Lemma 9.2 with $c = 0$, and $y = \mathrm{pr}_i(y)$. Hence $sy = (1 \otimes \mathrm{pr}_S^C)(x_1)$ with $x_1 \in \mathfrak{X}_{\tilde{Y}_S}(i; \mathrm{std}_l)$. Put $x = \mathrm{ret}_{\tilde{Y}_S}(x_1)$; it lies in $\mathfrak{N}_{l,i}(\tilde{Y}_S)$ by Lemma 5.3(d), and by Lemma 5.3(b) and (a), $(1 \otimes \mathrm{pr}_S^C)(x) = \mathrm{ret}_{C_S^{\mathrm{tr}}}(sy) = \mathrm{co}_0(s)\,y = (f - rt)\,y$. □

Lemma 9.5 (differentiating in the lifting ideal)**.** *Let $r \geq 0$ be an integer, i a letter, $f \in \mathfrak{S}$, $s, \alpha, \beta \in A_{\mathrm{inf}}$ and $g \in \Gamma'$. If $f \in \mathfrak{a}_S(r; 1, i)$ and $z\tau\mathcal{E}\,(s + \beta f) = \alpha\,(gf - f)$, then $s \in \mathfrak{a}_S(r; 1, i)$.*

LEAN Lift.KummerChart.PairingSide.mem_liftIdeal_of_derivation

Proof. Take $t \in \mathfrak{S}$ from Lemma 9.4 with $l = 1$, and $v \in A_{\mathrm{inf}}$ with $gt - t = z\tau\mathcal{E}v$ (Input I2). Let $c' \in C_S$, $y = \mathrm{cst}_{S,i}(c')$, which lies in $\mathfrak{N}_{1,i}(C_S^{\mathrm{tr}})$ by Lemma 9.3(f) and is fixed by g, and let $x \in \mathfrak{N}_{1,i}(\tilde{Y}_S)$ with $(1 \otimes \mathrm{pr}_S^C)(x) = (f - rt)y$. Then $(1 \otimes \mathrm{pr}_S^C)(gx - x) = ((gf - f) - r\,z\tau\mathcal{E}v)y$, and

$$x' = \alpha\,\mathcal{E}^{-1}(z\tau)^{-1}(gx - x) - \beta x \qquad \text{has} \qquad (1 \otimes \mathrm{pr}_S^C)(x') = \big(s - r(\alpha v - \beta t)\big)\,y.$$

By Lemma 9.3(a) it remains to see that $x' \in \mathfrak{X}_{\tilde{Y}_S}(i; \mathrm{std}_1)$. The element $gx - x$ has the exponents std_1 (Lemma 4.2(g)). At $\nu = 0$ it has more: for a period μ of type u at i, $V = \mu(x)$ lies in $F_0 \cap \mathrm{Fil}^0 B_{\mathrm{dR}}$, and $gV - V \in \mathrm{Fil}^1 B_{\mathrm{dR}}$ by Input I7(b), since $\mathcal{E} \mid gs' - s'$ for $s' \in \mathfrak{S}$. So $gx - x \in \mathfrak{X}_{\tilde{Y}_S}(i; \mathrm{fr}_{1,1}(\mathrm{std}_0)(\cdot, \cdot))$ by Lemma 4.2(a), and so does $(z\tau)^{-1}(gx - x)$ by Lemma 4.3(b). Lemma 4.2(f) with $\nu_0 = 0$, $n = -1$ then gives the exponents std_1 for $\mathcal{E}^{-1}(z\tau)^{-1}(gx - x)$. □

Lemma 9.6 (from level one to level zero, and removing $\mathcal{E}$)**.** *Let $r, a, n, c, \kappa, b \geq 0$ be integers and i a letter.*

(a) *If $pa \leq n$, $p^c z^n \in \mathfrak{a}_S(r; 1, i^p)$ and $c + v_p(n!) \leq \kappa + v_p((pa)!)$, then $p^\kappa z^a \in \mathfrak{a}_S(r; 0, i)$.*

(b) *If $p^c z^a \mathcal{E}^n \in \mathfrak{a}_S(r; 1, i)$, $c + \mathfrak{d}\, n + v_p(n!) \leq b$ and $a + n \leq \kappa$, then $p^b z^\kappa \in \mathfrak{a}_S(r; 1, i)$.*

LEAN `Lift.KummerChart.PairingSide.mem_liftIdeal_zero_of_mem_one` · `Lift.KummerChart.PairingSide.mem_liftIdeal_of_mul_E_pow`

Proof. (a) By induction on $n \geq pa$. If $n = pa$, then $c \leq \kappa$ and Lemma 9.3(h) applies. For the step from n to $n+1$, let $f = p^c z^{n+1} \in \mathfrak{a}_S(r; 1, i^p)$ with $c + v_p((n+1)!) \leq \kappa + v_p((pa)!)$. Choose $g \in \Gamma'$ with $gz = [\varepsilon]z$, and α, β with $\tau\mathcal{E}(p^{v_p(n+1)} + \beta z) = \alpha([\varepsilon]^{n+1} - 1)$ (Input I4). Then $gf - f = p^c([\varepsilon]^{n+1} - 1)z^{n+1}$ and

$$z\tau\mathcal{E}\,(p^{c+v_p(n+1)} z^n + \beta f) = \alpha\,(gf - f).$$

By Lemma 9.5, $p^{c+v_p(n+1)} z^n \in \mathfrak{a}_S(r; 1, i^p)$, and the induction hypothesis applies to it, because $c + v_p(n+1) + v_p(n!) = c + v_p((n+1)!)$.

(b) By induction on n; for $n = 0$ the element $p^b z^\kappa$ is a multiple of $p^c z^a$. For the step, let $f = p^c z^a \mathcal{E}^{n+1} \in \mathfrak{a}_S(r; 1, i)$. Take g, w, y, X from Theorem A for $(0, n+1)$, and write $yX - p^{\mathrm{cost}(0,n+1)} = \mathcal{E}b_0$, $gz = z(1 + \tau\mathcal{E}v)$ (Input I2) and $(1+\tau\mathcal{E}v)^a = 1 + \tau\mathcal{E}vt'$. Then

$$gf - f = p^c z^a \mathcal{E}^{n+1}((1 + \tau\mathcal{E}vt')w^{n+1} - 1) = p^c z^a \mathcal{E}^{n+1}\,\tau(zX + \mathcal{E}vt'w^{n+1}),$$

and a direct computation gives

$$zy \cdot (gf - f) = z\tau\mathcal{E}\,(s_1 + \beta_1 f),$$
$$s_1 = p^{c+\mathrm{cost}(0,n+1)} z^{a+1} \mathcal{E}^n, \qquad \beta_1 = zb_0 + yvt'w^{n+1}.$$

By Lemma 9.5, $s_1 \in \mathfrak{a}_S(r; 1, i)$. The induction hypothesis applies to it, with $a+1$ in place of a and $c' = c + \mathrm{cost}(0, n+1)$ in place of c: indeed $a + 1 + n \leq \kappa$ and

$$c' + \mathfrak{d}n + v_p(n!) = c + v_p(n+1) + \mathfrak{d} + \mathfrak{d}n + v_p(n!) = c + \mathfrak{d}(n+1) + v_p((n+1)!) \leq b.$$

□

Lemma 9.7 (a power of the coordinate, modulo p^κ)**.** *Let $n \geq 0$ be an integer with $u \leq n$ for every letter j and every $u \in U_{Y_S}(j)$, and let $\kappa \geq 0$ be an integer. Then there is an integer $a \geq 0$ with $z^a \in \mathfrak{a}_S(p^\kappa; 0, j)$ for every letter j and $a\,p \leq e_{\mathfrak{c}}\,(n + \kappa) + a$.*

LEAN `Lift.KummerChart.PairingSide.exists_z_pow_mem_liftIdeal`

Proof. Let I be the intersection of the ideals $\mathfrak{a}_S(p^\kappa; 0, j)$ over the letters j. It contains p^κ. It contains τ^h with $h = h_{\tilde{Y}_S}$: for $c' \in C_S$, $\tau^h(1 \otimes (0, c')) \in \mathfrak{M}^{\inf}(\tilde{Y}_S)$ by the frame theorem, its projection to the letter j has the exponents std_0 by Lemma 7.3(a), and it maps to $\tau^h\,\mathrm{cst}_{S,j}(c')$. Since z is not a unit and $p \nmid \tau$, z lies in the radical of (p, τ) (Input I12), hence in the radical of I. Let a be the least integer with $z^a \in I$, and put $N = e_{\mathfrak{c}}(n + \kappa) + a$ and $a' = \lfloor N/p \rfloor$. It suffices to show $z^{a'} \in I$: then $a \leq a'$, that is $ap \leq N$.

Fix a letter i. By Lemma 9.3(d), $z^a\mathcal{E}^n \in \mathfrak{a}_S(p^\kappa;1,i^p)$, and $p^\kappa z^a$ lies there too. Since $z^{e_\mathfrak{c}} \in (\mathcal{E},p)$ (Input I5), $z^{e_\mathfrak{c}(n+\kappa)} \in (\mathcal{E},p)^{n+\kappa} \subseteq (\mathcal{E}^n,p^\kappa)$, and so $z^N \in \mathfrak{a}_S(p^\kappa;1,i^p)$. Now Lemma 9.6(a), with $c=0$, exponent 0 of p in the conclusion, and $pa' \le N$, gives $z^{a'} \in \mathfrak{a}_S(p^\kappa;0,i)$: its numerical condition $v_p(N!) \le v_p((pa')!)$ holds because no multiple of p lies in $(pa',N]$. □

Proof of Theorem D. Here $\mathcal{E}=z+p$, $e_\mathfrak{c}=1$ and $\mathfrak{d}=0$. Put $q=\kappa+\rho$. By Lemma 9.7 for $n=(p-3)q$ and the modulus p^{q+1} there is b with $z^b \in \mathfrak{a}_S(p^{q+1};0,j)$ for all letters j and $(p-1)\,b \le (p-2)\,q+1$. We claim that, for every integer $a \ge 0$,

$$p^{\max(q-a,0)}z^a \in \mathfrak{a}_S(p^{q+1};0,i). \tag{3}$$

Put $a_1=\max(a,b)$. Then $z^{a_1} \in \mathfrak{a}_S(p^{q+1};0,i)$, so $z^{pa_1} \in \mathfrak{a}_S(p^{q+1};1,i^p)$ by Lemma 9.3(h), and Lemma 9.6(a) with $c=0$ and $n=pa_1$ gives (3) as soon as $v_p((pa_1)!) \le \max(q-a,0)+v_p((pa)!)$. For $a \ge b$ this is clear. For $a<b$, since $v_p((pa)!)=a+v_p(a!)$, it follows from $b+v_p(b!) \le q$; this is clear for $b=0$, and $(p-1)v_p(b!) \le b-1$ for $b \ge 1$, so

$$(p-1)^2(b+v_p(b!)) \le (p-1)(pb-1) \le p(p-2)q+1 < (p-1)^2(q+1).$$

For $a=0$, (3) reads $p^q \in \mathfrak{a}_S(p^{q+1};0,i)$, so $\mathfrak{a}_S(p^{q+1};0,i) \subseteq \mathfrak{a}_S(0;0,i)$ by Lemma 9.3(e). Finally

$$p^\kappa\mathcal{E}^\rho = \sum_{a=0}^{\rho}\binom{\rho}{a}\,p^{\,q-a}z^a,$$

and each term lies in $\mathfrak{a}_S(p^{q+1};0,i)$ by (3). □

10 Where the results are used

Parts 2 and 3 of the notes on this unit (*Cup products of crystalline classes vanish up to a power of p: the walk along Frobenius orbits*, and the note on few low weights and tame types) use everything above. Proposition 6.3 and Lemma 6.2 are iterated in Part 2 over finite sets of steps (ν,n), the sum of whose costs is the price of a deep point; Theorem C pairs a point of a side with a point of the dual side; Theorem D feeds the walk along Frobenius orbits that bounds cup products of crystalline classes; and Lemma 9.1 and the lifting ideal with its differentiation lemma return in Part 3, in the estimates for cut windows and in the cokernel killer for tame types. Outside this unit nothing of this note is used directly.

11 Notes on the formal text

Arguments as the text gives them. The arguments of the declarations were read from the Lean text and from the census of the workspace; they were not printed by Lean. The census reads a bound variable of the definition of the exponent module as an explicit argument (an index ν): the definition has the two explicit arguments i and o and the implicit L, as every use shows, and the note writes all three. In the definition of the constants the census counts the element c to which the linear map is applied as an argument, and the note writes it so. In the pairing $b_W(y,x)$ the census reads the arrows of the bilinear map as two arguments, which are y and x. The declarations of the three files take the prime, the group and the chart as instance arguments through the types of their other arguments, also where the census does not list them.

What the formal statements contain beyond the natural ones. A lattice datum records a jump set at every embedding of $\mathcal{O}$, of which only the union over a letter, $U_L(i)$, is used here. A side carries an action of Γ on its module C_S that is never used: the extension and the trivial datum give C_S the trivial action. The sets $U_L(i)$ are bounds: they contain the jumps and may be larger; this is why Lemma 4.4 allows a type u outside the set. The localized modules invert all $\mathcal{E}_\nu$, $\nu \geq 0$, at once, and the exponent at each of them is a separate condition. In Definition 2.7(iv) and Lemma 9.1 the letter p is an arbitrary natural number. Theorem D assumes $p \geq 5$; we did not examine whether the formal proof needs this, and the estimate given above uses only $p \geq 3$. Theorem A is stated for all $n \neq 0$, negative ones included; the cost depends only on $v_p(n)$.

Assemblies and small results. Lemma 4.1, Lemma 4.2, Lemma 5.3, Lemma 6.1, Lemma 8.1 and Lemma 9.3 collect several declarations each; the Lean names are listed under them. The identity $\mu(\varphi^0 x) = \mu(x)$ is a declaration of its own in the formal text and is not stated here.

The background. The Inputs are stated as their Lean texts read; the frame theorem (Input I8) has further clauses that this note does not use (a spanning statement up to a power of p, used in Part 2). The period ring is Colmez's $B_{\max}$ under the name *Bcris*. The second reader of the workspace compared the p-adic Hodge inputs with the literature: for Input I6 she finds a weaker form of Tate's theorems, for Input I10 a classical statement for field coefficients whose formulation here is not located. We did not confirm a place in the literature for the other Inputs.

How this note was written. From the reader's dossier of the unit: the statements of all members and the structured proofs of the declarations treated here. The three Lean files and the first part of the fourth were read in full, together with the Lean definitions of the background that the Settings restate (the chart, the Kummer chart, lattice data, periods, the Kisin modules, the frame theorem). The proof of Theorem A was read line by line in the Lean text, because the reader had marked its bookkeeping as least certain; the account above of where the three parts of the cost come from is ours, and it agrees with the reader's steps. The proofs of Theorem B, of Lemma 7.1 and of Theorem C were also rebuilt from the Lean terms. The arithmetic at the end of the proof of Theorem D is ours and replaces an automated step of the formal proof. The program check of the unit's dossier covered 375 leaf steps of 69 proofs.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Definition 2.7	`Lift.IsInJumpWindow`	CokernelKillers.lean	17–18
Definition 2.7	`Lift.raisedAtE`	CokernelKillers.lean	23–23
Definition 2.7	`Lift.twoWindows`	CokernelKillers.lean	27–27
Definition 2.7	`Lift.NoMultipleNear`	CokernelKillers.lean	30–30
Lemma 9.1	`Lift.sum_twoWindows_le`	CokernelKillers.lean	32–38
Lemma 9.2	`Lift.KummerChart.JumpLattice.tau_pow_smul_mem_ainfLattice_of_chi`	CokernelKillers.lean	46–51
Definition 2.8	`Lift.KummerChart.PairingSide.constantAt`	CokernelKillers.lean	57–58
Definition 2.8	`Lift.KummerChart.PairingSide.liftIdeal`	CokernelKillers.lean	62–64

In this note	Lean declaration	File	Lines
Definition 2.8	`Lift.KummerChart.PairingSide.KillsCokernel`	CokernelKillers.lean	68–68
Lemma 9.3	`Lift.KummerChart.PairingSide.mem_liftIdeal_iff`	CokernelKillers.lean	73–76
Lemma 9.3	`Lift.KummerChart.PairingSide.p_pow_mem_liftIdeal_self`	CokernelKillers.lean	78–79
Lemma 9.3	`Lift.KummerChart.PairingSide.phi_pow_smul_constantAt`	CokernelKillers.lean	84–85
Lemma 9.3	`Lift.KummerChart.PairingSide.mul_E_pow_mem_liftIdeal_one`	CokernelKillers.lean	87–90
Lemma 9.3	`Lift.KummerChart.PairingSide.liftIdeal_le_of_p_pow_mem`	CokernelKillers.lean	95–99
Lemma 9.3	`Lift.KummerChart.PairingSide.constantAt_mem_locLetterPart`	CokernelKillers.lean	101–104
Lemma 9.3	`Lift.KummerChart.PairingSide.phi_pow_mem_liftIdeal_iff`	CokernelKillers.lean	106–111
Lemma 9.3	`Lift.KummerChart.PairingSide.p_pow_mul_z_pow_mem_liftIdeal_iff`	CokernelKillers.lean	113–115
Lemma 9.4	`Lift.KummerChart.PairingSide.exists_lift_of_mem_liftIdeal`	CokernelKillers.lean	120–127
Lemma 9.5	`Lift.KummerChart.PairingSide.mem_liftIdeal_of_derivation`	CokernelKillers.lean	132–144
Lemma 9.6	`Lift.KummerChart.PairingSide.mem_liftIdeal_zero_of_mem_one`	CokernelKillers.lean	149–157
Lemma 9.6	`Lift.KummerChart.PairingSide.mem_liftIdeal_of_mul_E_pow`	CokernelKillers.lean	162–174
Lemma 9.7	`Lift.KummerChart.PairingSide.exists_z_pow_mem_liftIdeal`	CokernelKillers.lean	178–188
Theorem D	`Lift.KummerChart.PairingSide.killsCokernel_of_isUnramified`	CokernelKillers.lean	192–202
Definition 2.6	`Lift.KummerChart.PairingSide`	CupProductBound.lean	17–18
Definition 2.6	`Lift.KummerChart.PairingSide.extLattice`	CupProductBound.lean	24–27
Definition 2.6	`Lift.KummerChart.PairingSide.trivLattice`	CupProductBound.lean	31–32
Definition 2.6	`Lift.KummerChart.PairingSide.projTriv`	CupProductBound.lean	36–36
Definition 2.3	`Lift.KummerChart.cost`	ExponentLattices.lean	24–26
Theorem A	`Lift.KummerChart.exists_depth_step`	ExponentLattices.lean	32–64
Lemma 3.1	`Lift.KummerChart.exists_depth_step_zero`	ExponentLattices.lean	68–81
Definition 2.4	`Lift.KummerChart.JumpLattice.withExponents`	ExponentLattices.lean	88–90
Definition 2.4	`Lift.KummerChart.JumpLattice.locLetterPart`	ExponentLattices.lean	94–94
Lemma 4.1	`Lift.KummerChart.JumpLattice.valueAt_zero`	ExponentLattices.lean	99–99
Lemma 4.1	`Lift.KummerChart.JumpLattice.frobE_smul_mem_kisinLocAinf_iff`	ExponentLattices.lean	101–102
Lemma 4.1	`Lift.KummerChart.JumpLattice.frobEUnit_zpow_smul_mem_kisinLoc`	ExponentLattices.lean	104–108
Lemma 4.1	`Lift.KummerChart.JumpLattice.Hom.wittMap_mem_kisinLocAinf`	ExponentLattices.lean	110–111
Lemma 4.1	`Lift.KummerChart.JumpLattice.Hom.wittMap_mem_kisinLoc`	ExponentLattices.lean	113–114
Lemma 4.1	`Lift.KummerChart.JumpLattice.gal_smul_mem_kisinLocAinf`	ExponentLattices.lean	116–120
Lemma 4.2	`Lift.KummerChart.JumpLattice.mem_withExponents_of_mem_hodgeFil`	ExponentLattices.lean	122–125
Lemma 4.2	`Lift.KummerChart.JumpLattice.mem_withExponents_of_le`	ExponentLattices.lean	127–128
Lemma 4.2	`Lift.KummerChart.JumpLattice.valueAt_mem_hodgeFil_of_mem_ainfLattice`	ExponentLattices.lean	130–134

In this note	Lean declaration	File	Lines
Lemma 4.2	`Lift.KummerChart.JumpLattice.Hom.map_mem_withExponents`	ExponentLattices.lean	139–142
Lemma 4.2	`Lift.KummerChart.JumpLattice.mem_withExponents_of_smul`	ExponentLattices.lean	147–150
Lemma 4.2	`Lift.KummerChart.JumpLattice.frobEUnit_zpow_mem_withExponents`	ExponentLattices.lean	155–161
Lemma 4.1	`Lift.KummerChart.JumpLattice.kisinLocAinf_le_fracSpan`	ExponentLattices.lean	163–164
Lemma 4.1	`Lift.KummerChart.JumpLattice.valueAt_smul_sub`	ExponentLattices.lean	166–168
Lemma 4.2	`Lift.KummerChart.JumpLattice.smul_sub_mem_withExponents`	ExponentLattices.lean	170–172
Lemma 4.3	`Lift.KummerChart.JumpLattice.smul_sub_mem_zTau_kisinLocAinf`	ExponentLattices.lean	176–188
Lemma 4.1	`Lift.KummerChart.JumpLattice.phi_pow_mem_kisinLoc`	ExponentLattices.lean	190–195
Lemma 4.1	`Lift.KummerChart.JumpLattice.span_kisinLoc`	ExponentLattices.lean	197–201
Lemma 4.1	`Lift.KummerChart.JumpLattice.valueAt_mem_valueField`	ExponentLattices.lean	203–214
Lemma 5.1	`Lift.KummerChart.JumpLattice.exists_coordRetract`	ExponentLattices.lean	216–221
Definition 5.2	`Lift.KummerChart.JumpLattice.coordRetract`	ExponentLattices.lean	225–225
Lemma 5.3	`Lift.KummerChart.JumpLattice.coordRetract_sum`	ExponentLattices.lean	227–229
Lemma 5.3	`Lift.KummerChart.JumpLattice.coordRetract_of_mem_kisinLoc`	ExponentLattices.lean	231–232
Lemma 5.3	`Lift.KummerChart.JumpLattice.Hom.map_coordRetract`	ExponentLattices.lean	234–238
Lemma 4.4	`Lift.KummerChart.JumpLattice.valueAt_mem_of_hodgeFil_le`	ExponentLattices.lean	243–252
Lemma 4.3	`Lift.KummerChart.JumpLattice.smul_sub_mem_zTauUnit_smul_of_deep`	ExponentLattices.lean	256–258
Definition 2.5	`Lift.KummerChart.JumpLattice.locKisinPart`	HodgeExpansion.lean	23–23
Definition 2.5	`Lift.KummerChart.JumpLattice.invariantsMod`	HodgeExpansion.lean	27–28
Lemma 6.1	`Lift.KummerChart.JumpLattice.cost_le_of_isUnramified`	HodgeExpansion.lean	30–31
Definition 2.5	`Lift.KummerChart.JumpLattice.improvedExponent`	HodgeExpansion.lean	35–36
Lemma 6.2	`Lift.KummerChart.JumpLattice.mem_withExponents_of_deep`	HodgeExpansion.lean	41–46
Lemma 6.1	`Lift.KummerChart.JumpLattice.galMonoid_smul_localFieldOf`	HodgeExpansion.lean	48–49
Proposition 6.3	`Lift.KummerChart.JumpLattice.exists_reaches_depth_succ`	HodgeExpansion.lean	54–74
Definition 2.5	`Lift.KummerChart.JumpLattice.wittPairing`	HodgeExpansion.lean	82–83
Lemma 8.1	`Lift.KummerChart.JumpLattice.wittPairing_letterProj_left`	HodgeExpansion.lean	87–88
Lemma 8.1	`Lift.KummerChart.JumpLattice.wittPairing_mem_saturateBy`	HodgeExpansion.lean	90–92
Definition 2.5	`Lift.KummerChart.JumpLattice.frobExponent`	HodgeExpansion.lean	96–98
Lemma 7.1	`Lift.KummerChart.JumpLattice.mem_hodgeFil_of_hodge_basis`	HodgeExpansion.lean	102–118
Lemma 5.3	`Lift.KummerChart.JumpLattice.valueAt_coordRetract_sub_mem_hodgeFil`	HodgeExpansion.lean	120–125

In this note	Lean declaration	File	Lines
Lemma 5.3	`Lift.KummerChart.JumpLattice.coordRetract_mem_withExponents`	HodgeExpansion.lean	127–132
Lemma 7.2	`Lift.KummerChart.JumpLattice.phi_pow_mem_withExponents`	HodgeExpansion.lean	134–143
Theorem B	`Lift.KummerChart.JumpLattice.mem_withExponents_stdExponent_iff`	HodgeExpansion.lean	148–174
Lemma 7.3	`Lift.KummerChart.JumpLattice.letterProj_mem_withExponents`	HodgeExpansion.lean	176–179
Lemma 7.3	`Lift.KummerChart.JumpLattice.letterProj_mem_of_valueFieldPoints`	HodgeExpansion.lean	181–183
Theorem C	`Lift.KummerChart.JumpLattice.wittPairing_mem_withExponents`	HodgeExpansion.lean	188–207

Two statements as the artifact has them:

```
lemma Lift.KummerChart.exists_depth_step (hn : n ≠ 0) : ∃ g : galMonoid, ∃ ω : _ˣ, ∃ υ X,
  g • (k.frobE a).1 = ↑ω * k.frobE a ∧ (∀ s : k.S, (k.frobE a).1 ∣ g • s - s) ∧
  (k.frobE a).1 ∣ υ * X - p ^ cost a n ∧ zTau * X = (ω ^ n).1 - 1

lemma mem_withExponents_stdExponent_iff : x ∈ withExponents i (stdExponent l) ↔
  x ∈ frobKisinAinf l ∧ x ∈ letterPart i
```

Legend. The formal names of the objects and the letters of this note: the chart `k` is $\mathfrak{c}$, its ring `k.S` is $\mathfrak{S}$, `k.E` is $\mathcal{E}$ and `k.frobE a` is $\mathcal{E}_\nu$; `k.f` is h, `k.e` is $e_{\mathfrak{c}}$ and `differentExp` is $\mathfrak{d}$; `galMonoid` is Γ' acting through $(0, g)$; `zTau` is $z\tau$; `hodgeFil n` is $\mathrm{Fil}^n B_{\mathrm{dR}}$; `valueField a` is F_ν and `localFieldOf` is F; `coord₀` is co_0; `forms` and `formsOfType` are $\mathrm{Per}(L)$ and $\mathrm{Per}_{i,u}(L)$; `valueAt a μ x` is $\mu(\varphi^{-\nu}x)$; `kisinAinf`, `kisinLocAinf`, `kisinLoc`, `kisinModule` are $\mathfrak{M}^{\mathrm{inf}}(L)$, $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{loc}}(L)$, $\mathfrak{M}_{\mathrm{loc}}(L)$, $\mathfrak{M}(L)$; `stepBack i a` is $i^{p^{-\nu}}$; `stdExponent l` is std_l; `ζ` of the coefficients is ω.

Notation

- ℓ the prime of Chapter VIII of the handout; it is p here
- p a prime number; in Definition 2.7(iv) and Lemma 9.1 any natural number
- $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\bar{\mathbb{Q}}_p$, C, $\mathcal{O}_C$ the p-adic numbers and integers, an algebraic closure, its completion and the integers of the latter
- $G_{\mathbb{Q}_p}$ the absolute Galois group of $\mathbb{Q}_p$
- $\mathcal{O}_C^\flat$, $C^\flat$, A_{inf}, $W(C^\flat)$ the tilt, its fraction field, their rings of Witt vectors
- φ, θ, θ_ν, $[\cdot]$ Frobenius, Fontaine's map, $\theta \circ \varphi^{-\nu}$, the Teichmüller lift
- ε, τ, ξ a system of p-power roots of unity in the tilt, $[\varepsilon^{1/p}] - 1$, a generator of $\ker\theta$
- B^+_{dR}, B_{dR}, $\mathrm{Fil}^n B_{\mathrm{dR}}$ the de Rham period rings and the filtration
- A_{max}, B^+, B_{max} Colmez's period rings (Setting 2.1): the p-adic completion of $A_{\mathrm{inf}}[X]/(pX - \xi)$, $A_{\mathrm{max}}[1/p]$, $B^+[1/([\varepsilon] - 1)]$
- Γ, j_Γ, χ a group over $G_{\mathbb{Q}_p}$, its map, the cyclotomic character
- $\mathfrak{c}$, Γ' the Kummer chart and its subgroup
- h, $\mathbb{W}$, $\mathbb{F}_{p^h}$, X, A the residue degree, $W(\mathbb{F}_{p^h})$, the field, the variable and the unit series of the chart
- $\mathfrak{S}$, z, $\mathcal{E}$, $\mathcal{E}_\nu$ the ring of the chart, its coordinate, the Eisenstein element, $\varphi^\nu(\mathcal{E})$
- $e_{\mathfrak{c}}$, $\mathfrak{d}$ the index of the chart; 0 or $v_p(e_{\mathfrak{c}}) + 1$
- F, F_ν, $\mathfrak{S}_B$ the field fixed by Γ', the value fields, the ring of Setting 2.1
- co_0, co_a coordinates $A_{\mathrm{inf}} \to \mathfrak{S}$
- $\mathcal{O}$, ω the coefficient ring and its root of unity
- i, j letters, elements of $\mathbb{F}_{p^h}^\times$
- L, L', L'', A, T, T', T'' lattice data and lattices
- $U_L(i)$, h_L the jump set at a letter, the jump bound
- pr_i, $(W(C^\flat) \otimes L)_i$ the projector to the letter i and its image
- $\mathrm{Per}(L)$, $\mathrm{Per}_{i,u}(L)$, $\mathrm{Fun}(L)$ periods, periods of type u at i, all A_{inf}-linear functionals
- μ, μ', μ_k, λ, λ', λ_k periods and functionals; in Setting 2.2 λ is a semilinear map

$\mathfrak{M}^{\mathrm{inf}}(L)$, $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{loc}}(L)$, $\mathfrak{M}_{\mathrm{loc}}(L)$, $\mathfrak{M}(L)$ Kisin modules over A_{inf}, localized, over $\mathfrak{S}$ localized, over $\mathfrak{S}$

std_l the standard exponents of level l

o, o', o_A, $o_{L''}$, o^{lo}, o^{hi} exponent functions

$\mathrm{cost}(\nu, n)$ the cost of a step

$\mathfrak{X}_L(i; o)$, $\mathfrak{N}_{l,i}(L)$ elements with exponents; those in $\mathfrak{M}_{\mathrm{loc}}(L)$ with standard exponents

$(Q)_{\mathfrak{S}}$, $\mathrm{Inv}(Q)$ the elements of $\mathfrak{M}_{\mathrm{loc}}(L)$ in Q; the invariants modulo Q

$\mathrm{imp}(o^{\mathrm{lo}}, o^{\mathrm{hi}})(\nu, u)$, $\mathrm{fr}_{l,n}(o)(\nu, u)$ improved and shifted exponents

$b_W(y, x)$ the extension of a bilinear map

ret_L the retraction

S, Y_S, C_S, α_S a side and its parts

$\tilde{Y}_S$, C_S^{tr}, pr_S^C the extension, the trivial datum, the projection

$Z^1_{\mathrm{f}}(T)$ the crystalline cocycles

$U \subseteq [-\eta, \eta] \cup \mathsf{s} \cdot [\beta, \beta + W]$ the set U lies in a low and a far window

$\mathrm{rs}_n(\nu, u)$, $\mathcal{W}_n(c)$, $\mathrm{NM}(p; n, \beta)$ exponents raised at $\mathcal{E}$; two windows; no multiple of p near

$\mathrm{cst}_{S,i}(c)$, $\mathfrak{a}_S(r; l, i)$, $\mathrm{Kill}_S(i; \kappa, \rho)$ constants, the lifting ideal, a cokernel killer

ν, ν_0, ν', l indices of Frobenius translates and levels

u, u', v, κ, κ_k jumps and types; κ is also an exponent of p, v also an element of A_{inf} or of a field

n, m, a, a', a_1, b, c, d, k, q, r, H, N, W, η, β, ρ integers; b is also a bilinear map, c also an element, r also a map or an element

s a sign

g, γ, t_k elements of Γ' or of $G_{\mathbb{Q}_p}$; γ is also a map $\mathcal{O} \to C$

w, w_0, $\hat{w}$, u units

x, x', x_0, x_1, y, y', y_0, X, s, s', t, t_0, t', f, G, P_0, α, β, s_1, β_1, a_k, b_k, b_0, c_k, v_k, w_a, e_k elements of rings and modules; f is also a map, t also an element of a lattice

V, V_0, π_ν, ϱ, σ_ν elements of B_{dR}, a uniformizer, a root of unity, a sum

D, D_k, H, $\mathsf{R}(g)$ coordinates in a Hodge basis, the Hodge matrix, the matrix of g

M, M_1, M_2, Q, N_A, N_L, $N_{L''}$, I, J submodules and ideals

c' an element of C_S or of F; in the proof of Lemma 9.6 an integer

Cup products of crystalline classes vanish up to a power of p: the walk along Frobenius orbits

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. Two crystalline lattices with a pairing into the Tate twist modulo p^M, and a crystalline class of each: this note shows that the cup product of the two classes becomes a coboundary after multiplication by an explicit power of p that does not depend on the ranks. The classes are represented by points of the localized Kisin modules of the extensions they define; a point has exponents and a depth, and Part 1 says what it costs to deepen it. A pair of elements is *safe* if its pairing lies in the Witt vectors of the maximal ideal of the tilt, and safety of four pairs gives the vanishing, by the Artin–Schreier–Witt sequence and a restriction to a subgroup of index prime to p. Safety is proved from exponents alone, with a point of a third extension that carries the pairing. The points are moved along the Frobenius orbits of the letters from chosen cuts; this is the walk. A lemma for long gaps between cuts and a relation among few elements of the same weight are proved for use in Part 3.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles with Lean v4.35.0 against a pinned version of Mathlib and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note is the second of three on one unit of the artifact, the folder on lifting; it treats 44 of the unit's 141 declarations: the file on the walk and nine declarations of the file on the cup product bound. Theorems A to D and the lemmas are proved from the results of Part 1 (the note *Exponents of elements of Kisin modules, the cost of deepening them, and cokernel killers*), which are restated in a box, and from Inputs of the background library, stated and not proved. Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 44 declarations.

1 The statements

The objects are those of Settings 2.1 and 2.2: a Kummer chart, lattice data, sides $S = (Y_S, C_S, \alpha_S)$ with their extensions $\tilde{Y}_S$, and a pairing Π of two sides modulo p^M. Safety is Definition 2.4, points are Definition 2.5, deep points are Definition 2.6.

Theorem A (four safe pairs give a coboundary). *In Setting 2.2, let S, S' be sides over $\mathcal{O}$ and Π a pairing of S and S'. Let $e, e_1, e_2, e_3 \geq 0$ be integers with $e_1 + e_2 + e_3 \leq e$, let $c \in C_S$ and $c' \in C_{S'}$, and let $x, w \in W(C^\flat) \otimes \tilde{Y}_S$ and $x' \in W(C^\flat) \otimes \tilde{Y}_{S'}$ satisfy*

$$(1 \otimes \mathrm{pr}_S^C)(x) = (1 \otimes \mathrm{pr}_S^C)(w) = p^{e_1}(1 \otimes c), \qquad (1 \otimes \mathrm{pr}_{S'}^C)(x') = p^{e_2}(1 \otimes c').$$

Suppose that for every $g \in \Gamma'$ the following four conditions hold: $\mathrm{Safe}_\Pi(e_3; gx - x, g(hx' - x'))$ for every $h \in \Gamma'$; $\mathrm{Safe}_\Pi(e_3; w - x, gx' - x')$; $\mathrm{Safe}_\Pi(e_3; gw - w, g(\varphi x' - x'))$; and $\mathrm{Safe}_\Pi(e_3; \varphi x - w, g\varphi x' - \varphi x')$. Then $\mathrm{Cup}_\Pi(e; c, c')$.

LEAN `Lift.Pairing.ScaledCupBounds.of_isSafe`

Theorem B (the walk). *In Setting 2.2, let S, S' be sides over $\mathcal{O}$, Π a pairing of S and S', $c \in C_S$, $c' \in C_{S'}$, and $\kappa, e, e' \geq 0$ integers. Let cut and Q be properties of letters. Suppose that for every letter i some $i^{p^{-a}}$ with an integer $a > 0$ has the property cut, and let $\mathsf{l}(i)$ be the least such a. Let o, o', z, z' be families of exponent functions indexed by the letters. Suppose that for every letter i with the property cut:*

(i) *S has a deep point over $p^e c$ at i against S', in the sense of Definition 2.6, for the data $(\mathsf{cut}, \mathsf{l}(i), o, o', z)$ and the proposition $\mathsf{Q}(i)$;*

(ii) *S' has a deep point over $p^{e'} c'$ at i against S, for the data $(\mathsf{cut}, \mathsf{l}(i), o', o, z)$ and the proposition "not $\mathsf{Q}(i)$";*

(iii) *$\mathrm{Pt}_{S_\Pi^\vee}(1, i; \kappa; z(i), z'(i))$ is not empty;*

(iv) *for every $\nu \geq 0$, the function $u \mapsto z(i)(\nu, u)$ is non-increasing.*

Then $\mathrm{Cup}_\Pi(E; c, c')$ for every integer E with $e + e' + \kappa \leq E$.

LEAN `Lift.KummerChart.PairingSide.SidePairing.scaledCupBounds_of_points_at_cuts`

Theorem C (rank-free exponents at large weights). *In Setting 2.2, let S, S' be sides over $\mathcal{O}$ and Π a pairing of S and S'. Let $\rho, \kappa, \eta, W, H \geq 0$ be integers, and for every letter i let $\beta(i) \geq 0$ be an integer and $\mathsf{s}(i) \in \{1, -1\}$ a sign. Suppose:*

(i) *for every letter i: $\mathrm{Kill}_{S_\Pi^\vee}(i; \kappa, \rho)$, $\mathrm{Kill}_S(i; \kappa, \rho)$ and $\mathrm{Kill}_{S'}(i; \kappa, \rho)$;*

(ii) *for every letter i: $\mathrm{NM}(p; 6(\rho + \eta + W), \beta(i))$;*

(iii) *for every letter i: $U_{Y_S}(i) \subseteq [-\eta, \eta] \cup -\mathsf{s}(i) \cdot [\beta(i), \beta(i) + W]$ and $U_{Y_{S'}}(i) \subseteq [-\eta, \eta] \cup \mathsf{s}(i) \cdot [\beta(i), \beta(i) + W]$;*

(iv) *either no element of $U_{Y_S}(i) \cup U_{Y_{S'}}(i)$ lies in $[-\rho, 0)$, for any letter i; or the Kummer chart is unramified, $p \geq 5$, $\eta \geq 1$ and $U_{Y_{S'}}(i) \subseteq [-\eta, 0]$ for every letter i;*

(v) *$3\,(\kappa + (48(\rho + \eta + W) + 4)\,\mathfrak{d}) \leq H$.*

Then $\mathrm{Cup}_\Pi(H; c, c')$ for all $c \in C_S$ and $c' \in C_{S'}$.

LEAN `Lift.KummerChart.PairingSide.SidePairing.scaledCupBounds_of_killsCokernel`

Theorem D (the long gap). *In Setting 2.1, suppose that the Kummer chart is unramified. Let S be a side over $\mathcal{O}$, i a letter, $l > 0$ an integer and $\kappa, \rho, \gamma, \delta, N, A \geq 0$ integers such that* $\mathrm{Kill}_S(i^{p^{-l}}; \kappa, 0)$, $\mathrm{Kill}_S(i; 0, \rho)$,

$$N = \rho + \gamma \sum_{0 \leq a < l} p^a, \qquad u \leq \gamma \ \text{ for all } a < l \text{ and } u \in U_{Y_S}(i^{p^{-a}}),$$
$$N < A, \qquad 3\, e_{\mathfrak{c}} \leq p^l, \qquad e_{\mathfrak{c}} A \leq p^l \delta.$$

Then for every $c \in C_S$ the set $\mathrm{Pt}_S(c, i; \kappa + 4\delta; \mathrm{sh}_l(o_1)(\cdot,\cdot), \mathrm{sh}_l(o_2)(\cdot,\cdot))$ *is not empty, where $o_1(u) = \min(A, -u - N)$ and $o_2(u) = A$ for $u \in \mathbb{Z}$.*

LEAN `Lift.KummerChart.PairingSide.pointsOver_nonempty_of_long_gap`

Theorem C is the result that Part 3 uses most: the exponent H depends on the windows and on the killers, not on the ranks of the lattices. Theorem D produces a deep point when the cuts are far apart.

2 The setting

Conventions. Throughout, p is a prime number. The declarations of this unit take the following ambient objects as implicit or instance arguments; they are fixed by Settings 2.1 and 2.2 and named again in each statement that needs them: the prime p; the group Γ over $G_{\mathbb{Q}_p}$; the Kummer chart $\mathfrak{c}$ for Γ; the coefficient ring $\mathcal{O}$; a lattice datum L; sides S, S'; a pairing Π; elements $c \in C_S$ and $c' \in C_{S'}$; a letter i; integers l, κ, ρ, e, e', n, A; exponent functions o, o', o_1, o_2, o_3; elements x, w, y of the modules $W(C^\flat) \otimes \tilde{Y}_S$; an element g of Γ'. Tensor products without a subscript are over $\mathbb{Z}_p$; $v_p(n)$ is the exponent of p in an integer n, with $v_p(0) = 0$. An *exponent function* is a function $o \colon \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$; inequalities and minima of exponent functions are pointwise. The Lean text calls the artifact's period ring *Bcris*; it is Colmez's ring $B_{\max}$.

Setting 2.1 (the base: chart, lattice data, sides; from the background library and from Part 1)**.** The notation is that of Settings 2.1 and 2.2 and Definitions 2.3 to 2.8 of Part 1, which this Setting recalls.

Rings and the chart. $A_{\mathrm{inf}} \subset W(C^\flat)$ are the Witt vectors of the tilt $\mathcal{O}_C^\flat$ of the integers of the completed algebraic closure C of $\mathbb{Q}_p$ and of its fraction field, with Frobenius φ; $W(\mathfrak{m}^\flat)$ is the ideal of A_{inf} of the Witt vectors with components in the maximal ideal $\mathfrak{m}^\flat$ of the tilt; $\theta \colon A_{\mathrm{inf}} \to \mathcal{O}_C$ is Fontaine's map; B_{dR} is the field of de Rham periods with its filtration $\mathrm{Fil}^n B_{\mathrm{dR}}$, $n \in \mathbb{Z}$; $\tau = [\varepsilon^{1/p}] - 1$. The rings $A_{\max}$, $B^+ = A_{\max}[1/p]$ and $B_{\max} = B^+[1/([\varepsilon] - 1)]$ are those of Setting 2.1 of Part 1: $A_{\max}$ is the p-adic completion of the ring obtained from A_{inf} by adjoining the quotient by p of the generator $([\varepsilon] - 1)/\tau$ of the kernel of θ. Frobenius and $G_{\mathbb{Q}_p}$ act on them, and $B_{\max}$ has an injective homomorphism to B_{dR} that commutes with $G_{\mathbb{Q}_p}$; a lattice T is potentially crystalline, or crystalline on a group, if the elements of $B_{\max} \otimes T$ fixed near 1, or by the group, span it, and $Z^1_{\mathrm{f}}(T)$ consists of the maps $c \colon \Gamma \to T$ with $1 \otimes c(g) = gy - y$ for some $y \in B_{\max} \otimes T$ (Part 1, Setting 2.2). Γ is a group with a homomorphism $j_\Gamma \colon \Gamma \to G_{\mathbb{Q}_p}$ of open image, and χ its cyclotomic character; $\mathbb{Z} \times \Gamma$ acts on $W(C^\flat)$ through φ and j_Γ. A Kummer chart $\mathfrak{c}$ for Γ is fixed: its ring $\mathfrak{S} \subset A_{\mathrm{inf}}$, a quotient of $\mathbb{W}[[X]]$ with $\mathbb{W} = W(\mathbb{F}_{p^h})$, its coordinate z, the Teichmüller lift of a

non-unit of the tilt, its Eisenstein element $\mathcal{E}$ with $\mathcal{E}_\nu = \varphi^\nu(\mathcal{E})$, its index $e_{\mathfrak{c}}$ and the number $\mathfrak{d}$, its subgroup Γ' of index prime to p, which contains all elements near 1, the field F fixed by Γ', the fields $F_\nu \subset B_{\mathrm{dR}}$, and the retraction $\mathrm{co}_0 \colon A_{\mathrm{inf}} \to \mathfrak{S}$. The chart is *unramified* if $\mathcal{E} = z + p$, $e_{\mathfrak{c}} = 1$, $\mathfrak{d} = 0$. The *letters* are the elements of $\mathbb{F}_{p^h}^\times$, with Teichmüller lifts $[i]$.

Lattice data. $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$ with a root of unity ω, $\omega^{p^h-1} = 1$. A lattice datum L over $\mathcal{O}$ is a finite free $\mathcal{O}$-module with an $\mathcal{O}$-linear action of Γ, crystalline on Γ', with finite sets $U_L(i) \subset \mathbb{Z}$ that contain its Hodge–Tate jumps at the embeddings above the letter i. It has its module $W(C^\flat) \otimes L$ with the action of $\mathbb{Z} \times \Gamma$, the lattice $A_{\mathrm{inf}} \otimes L$, the parts $(W(C^\flat) \otimes L)_i$ with projectors pr_i, the periods $\mathrm{Per}(L)$ and $\mathrm{Per}_{i,u}(L)$, the Kisin modules $\mathfrak{M}^{\mathrm{inf}}(L)$, $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{loc}}(L)$, $\mathfrak{M}_{\mathrm{loc}}(L)$, the retraction $\mathrm{ret}_L \colon \mathfrak{M}^{\mathrm{inf}}_{\mathrm{loc}}(L) \to \mathfrak{M}_{\mathrm{loc}}(L)$, and for a letter i and an exponent function o the module $\mathfrak{X}_L(i;o)$ of the elements of $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{loc}}(L) \cap (W(C^\flat) \otimes L)_i$ with $\mu(\varphi^{-\nu}x) \in \mathrm{Fil}^{o(\nu,u)} B_{\mathrm{dR}}$ for all $\nu \geq 0$, $u \in U_L(i^{p^{-\nu}})$ and $\mu \in \mathrm{Per}_{i^{p^{-\nu}},u}(L)$. Further $\mathfrak{N}_{l,i}(L) = \mathfrak{M}_{\mathrm{loc}}(L) \cap \mathfrak{X}_L(i; \mathrm{std}_l)$, with $\mathrm{std}_l(\nu, u) = 0$ for $\nu < l$ and $-u$ for $\nu \geq l$; $(Q)_{\mathfrak{S}}$ is the set of the elements of $\mathfrak{M}_{\mathrm{loc}}(L)$ in a submodule Q; $\mathrm{Inv}(Q)$ is the set of the x with $gx - x \in Q$ for all $g \in \Gamma'$; $\mathrm{cost}(\nu, n) = v_p(n) + \mathfrak{d} + 3\nu$; $\mathrm{imp}(o^{\mathrm{lo}}, o^{\mathrm{hi}})(\nu, u)$ is $o^{\mathrm{hi}}(\nu, u)$ if $o^{\mathrm{lo}}(\nu, u) > 0$ and $\min(o^{\mathrm{hi}}(\nu, u), 0)$ otherwise; $\mathrm{fr}_{l,n}(o)(\nu, u)$ is n for $\nu < l$ and $o(\nu - l, u)$ for $\nu \geq l$; and $b_W(y, x)$ is the value of the $W(C^\flat)$-bilinear extension of an $\mathcal{O}$-bilinear map b of lattices.

Sides. A side $S = (Y_S, C_S, \alpha_S)$ over $\mathcal{O}$ consists of a lattice datum Y_S with $0 \in U_{Y_S}(i)$ for all letters i, a lattice C_S, and an $\mathcal{O}$-linear map α_S from C_S to the crystalline cocycles $Z^1_{\mathrm{f}}(Y_S)$. Its extension $\tilde{Y}_S$ is $Y_S \times C_S$ with $g(y, c) = (gy + \alpha_S(c)(g), c)$ and the sets $U_{Y_S}(i)$; C_S^{tr} is C_S with the trivial action and the sets $\{0\}$; $\mathrm{pr}^C_S \colon \tilde{Y}_S \to C_S^{\mathrm{tr}}$ is the projection. For $c \in C_S$, $\mathrm{cst}_{S,i}(c) = \mathrm{pr}_i(1 \otimes c)$; $\mathfrak{a}_S(r; l, i)$ is the lifting ideal, and $\mathrm{Kill}_S(i; \kappa, \rho)$ means $p^\kappa \mathcal{E}^\rho \in \mathfrak{a}_S(0; 0, i)$. Finally $\mathrm{rs}_n(\nu, u)$ is $n - u$ for $\nu = 0$ and $-u$ for $\nu > 0$; $\mathcal{W}_n(c) = [-n, n] \cup [c - n, c + n]$; $\mathrm{NM}(p; n, \beta)$ says that p divides no non-zero element of $\mathcal{W}_n(\beta) \cup \mathcal{W}_n(-\beta)$; and $U \subseteq [-\eta, \eta] \cup \mathsf{s} \cdot [\beta, \beta + W]$ says that every $u \in U$ has $|u| \leq \eta$ or $\beta \leq \mathsf{s}u \leq \beta + W$.

Setting 2.2 (pairings and the cup product; from the background library)**.** In Setting 2.1:

Twists and duals. For a lattice T over $\mathcal{O}$, $T^\vee(1) = \mathrm{Hom}_{\mathcal{O}}(T, \mathcal{O}(1))$ is the twisted dual, where $\mathcal{O}(1)$ is $\mathcal{O}$ with Γ acting through χ; for lattices T, T', $T \otimes_{\mathcal{O}} T'$ carries the diagonal action.

Pairings. For lattices Y, Y' over $\mathcal{O}$, a *pairing* of Y and Y' consists of an integer $M \geq 0$, an $\mathcal{O}$-linear map $\psi \colon Y \otimes_{\mathcal{O}} Y' \to \mathcal{O}(1)$, a $\mathbb{Z}_p$-linear map $\mathrm{tr} \colon \mathcal{O} \to \mathbb{Z}_p$, and a crystalline cocycle $c_\Pi \in Z^1_{\mathrm{f}}((Y \otimes_{\mathcal{O}} Y')^\vee(1))$ with $p^M c_\Pi(g) = g\psi - \psi$ for all $g \in \Gamma$. Its *pairing modulo* p^M is

$$\langle y, y' \rangle = \mathrm{tr}(\psi(y \otimes y')) \bmod p^M \ \in \ \mathbb{Z}/p^M, \qquad \langle gy, gy' \rangle = \chi(g)\, \langle y, y' \rangle.$$

A *pairing of sides* S and S' is a pairing of the lattices Y_S and $Y_{S'}$. The *third pairing* of Π is the $\mathcal{O}$-bilinear map

$$\mathsf{p}_\Pi \colon ((Y \otimes_{\mathcal{O}} Y')^\vee(1) \times \mathcal{O}) \times (Y \otimes_{\mathcal{O}} Y') \to \mathcal{O}(1), \qquad ((\lambda, a), t) \mapsto p^M \lambda(t) + a\, \psi(t);$$

it is Γ-equivariant when the first factor carries the action of the extension of $\mathcal{O}$ along $a \mapsto a\, c_\Pi$.

The module of values. $\mathbb{M}_M = W(C^\flat) \otimes \mathbb{Z}/p^M(1)$, with $\mathbb{Z} \times \Gamma$ acting on the second factor through χ; its *small part* $\mathfrak{s}_M$ is the image of $W(\mathfrak{m}^\flat) \cdot (1 \otimes 1)$, an A_{inf}-submodule stable under $\mathbb{Z} \times \Gamma$.

The cup product. Fix an injective homomorphism e_M from $\mathbb{Z}/p^M$ to the multiplicative group of $\bar{\mathbb{Q}}_p$. For a topology on Γ and maps $a \colon \Gamma \to Y$, $a' \colon \Gamma \to Y'$, the *cup product of a and a' under Π bounds* if the function $(g, h) \mapsto \mathsf{e}_M(\langle a(g), g\, a'(h) \rangle)$ on $\Gamma \times \Gamma$ is the coboundary of a continuous 1-cochain of Γ with values in $\bar{\mathbb{Q}}_p^\times$, on which Γ acts through j_Γ.

Definition 2.3 (the objects attached to a pairing of sides)**.** In Setting 2.2, let S, S' be sides over $\mathcal{O}$ and Π a pairing of S and S', modulo p^M.

(i) $\mathrm{pr}_S^Y \colon W(C^\flat) \otimes \tilde{Y}_S \to W(C^\flat) \otimes Y_S$ is the $W(C^\flat)$-linear map induced by $(y, c) \mapsto y$, and $\mathrm{in}_S^Y \colon Y_S \to \tilde{Y}_S$ is the equivariant map $y \mapsto (y, 0)$.

(ii) $\mathrm{pr}_S^{C,\mathrm{loc}}$ is the $\mathbb{Z}_p$-linear map $\mathfrak{M}_{\mathrm{loc}}(\tilde{Y}_S) \to W(C^\flat) \otimes C_S$, $x \mapsto (1 \otimes \mathrm{pr}_S^C)(x)$.

(iii) $\mathsf{T}_{S,S'}$ is the lattice datum $Y_S \otimes_{\mathcal{O}} Y_{S'}$ with the sets $U_{Y_S}(i) + U_{Y_{S'}}(i)$ of sums.

(iv) $\mathcal{O}(1)$ is the lattice datum on $\mathcal{O}$ with Γ acting through χ and the set $\{-1\}$ at every letter.

(v) $S_\Pi^\vee$ is the side whose lattice datum is $(Y_S \otimes_{\mathcal{O}} Y_{S'})^\vee(1)$ with the sets $\{0\} \cup \{-v-1 : v \in U_{Y_S}(i) + U_{Y_{S'}}(i)\}$, whose lattice is $\mathcal{O}$ with the trivial action, and whose cocycle map is $a \mapsto a\, c_\Pi$.

(vi) b_Π is the $W(C^\flat)$-bilinear extension $(W(C^\flat) \otimes Y_S) \times (W(C^\flat) \otimes Y_{S'}) \to \mathbb{M}_M$ of the pairing modulo p^M; its value at (y, y') is written $\mathsf{b}_\Pi(y, y')$.

That these are lattice data and a side rests on Input J1. The type of the pairings of S and S' is a declaration of its own in the formal text.

LEAN `Lift.KummerChart.PairingSide.wittFst` · `Lift.KummerChart.PairingSide.inclKer` · `Lift.KummerChart.PairingSide.projTrivLoc` · `Lift.KummerChart.PairingSide.tensorLattice` · `Lift.KummerChart.tateLattice` · `Lift.Pairing.thirdSide` · `Lift.Pairing.wittPairing` · `Lift.KummerChart.PairingSide.SidePairing`

Definition 2.4 (safe pairs; cup products that bound after scaling)**.** In Setting 2.2, let S, S' be sides over $\mathcal{O}$ and Π a pairing of S and S', modulo p^M, and let $e \geq 0$ be an integer.

(i) For $w \in W(C^\flat) \otimes \tilde{Y}_S$ and $y \in W(C^\flat) \otimes \tilde{Y}_{S'}$, $\mathrm{Safe}_\Pi(e; w, y)$ means

$$p^e\, \mathsf{b}_\Pi(\mathrm{pr}_S^Y(w), \mathrm{pr}_{S'}^Y(y)) \in \mathfrak{s}_M.$$

(ii) For $c \in C_S$ and $c' \in C_{S'}$, $\mathrm{Cup}_\Pi(e; c, c')$ means that, for every topology on Γ for which j_Γ is an open embedding, the cup product of $\alpha_S(c)$ and $p^e \alpha_{S'}(c')$ under Π bounds.

LEAN `Lift.Pairing.IsSafe` · `Lift.Pairing.ScaledCupBounds`

Definition 2.5 (deep elements, costs, points)**.** In Setting 2.1, let S be a side over $\mathcal{O}$, i a letter and o, o_1, o' exponent functions.

(i) $\mathfrak{X}_S^0(i; o)$ is the A_{inf}-module of the $x \in \mathfrak{X}_{\tilde{Y}_S}(i; o)$ with $(1 \otimes \mathrm{pr}_S^C)(x) = 0$.

(ii) $\mathrm{Deep}_S(i; o)$ is the $\mathbb{Z}_p$-module of the $x \in \mathfrak{M}_{\mathrm{loc}}(\tilde{Y}_S)$ such that $(1 \otimes \mathrm{pr}_S^C)(x) = \mathrm{cst}_{S,i}(c)$ for some $c \in C_S$ and $x \in \mathrm{Inv}(\mathfrak{X}_{\tilde{Y}_S}(i; o))$.

(iii) For an integer $e \geq 0$ and an additive map A from $\mathfrak{M}_{\mathrm{loc}}(\tilde{Y}_S)$ to itself, $\mathrm{Cost}_S(\mathsf{A}) \leq e$ means that $\mathsf{A}(x) - p^e x \in (\mathfrak{X}^0_S(j; o''))_{\mathfrak{S}}$ for every letter j, every exponent function o'' and every $x \in \mathrm{Deep}_S(j; o'')$; in words, A *has cost* e.

(iv) For $c \in C_S$ and an integer $e \geq 0$, $\mathrm{Pt}_S(c, i; e; o_1, o')$ is the set of the $x \in \mathfrak{M}_{\mathrm{loc}}(\tilde{Y}_S)$ with $(1 \otimes \mathrm{pr}^C_S)(x) = p^e \, \mathrm{cst}_{S,i}(c)$, $x \in \mathfrak{X}_{\tilde{Y}_S}(i; o_1)$ and $x \in \mathrm{Inv}(\mathfrak{X}_{\tilde{Y}_S}(i; o'))$: the *points over* $p^e c$ *at* i *with exponents* o_1 *and depth* o'.

LEAN `Lift.KummerChart.PairingSide.kerWithExponents` · `Lift.KummerChart.PairingSide.deepElements` · `Lift.KummerChart.PairingSide.Reaches` · `Lift.KummerChart.PairingSide.pointsOver`

Definition 2.6 (a deep point against another side)**.** In Setting 2.1, let S, S' be sides over $\mathcal{O}$, $c \in C_S$, i a letter, $e, l \geq 0$ integers, cut a property of letters, o, o', z families of exponent functions indexed by the letters, and q a proposition. *S has a deep point over $p^e c$ at i against S', for the data $(\mathsf{cut}, l, o, o', z)$ and the proposition q*, if there are an exponent function d and an integer $r \geq 0$ such that $\mathrm{Pt}_S(c, i; e; o(i), d)$ is not empty, $i^{p^{-r}}$ has the property cut, and for all $\nu \geq 0$, $u \in U_{Y_S}(i^{p^{-\nu}})$ and $v \in U_{Y_{S'}}(i^{p^{-\nu}})$:

(a) $-z(i)(\nu, -u - v - 1) - d(\nu, u) < \min(-v,\ o'(i)(\nu, v))$;

(b) if q holds, then $\mathrm{std}_l(\nu, u) \leq o(i)(\nu, u)$ and

$$-\mathrm{fr}_{r,0}(z(i^{p^{-r}}))(\nu, -u - v - 1) - d(\nu, u) < \min(o'(i)(\nu, v),\ \mathrm{fr}_{l,0}(o'(i^{p^{-l}}))(\nu, v)).$$

LEAN `Lift.KummerChart.PairingSide.HasDeepPoint`

Definition 2.7 (exponents for the long gap)**.** In Setting 2.1:

(i) For an exponent function o, integers $\nu, \nu' \geq 0$ and $u, v \in \mathbb{Z}$, $\mathrm{up}_{\nu,u}(o)(\nu', v) = o(\nu', v) + 1$ if $(\nu', v) = (\nu, u)$, and $\mathrm{up}_{\nu,u}(o)(\nu', v) = o(\nu', v)$ otherwise.

(ii) For an integer $l \geq 0$, a function $o_1 \colon \mathbb{Z} \to \mathbb{Z}$, $\nu \geq 0$ and $u \in \mathbb{Z}$, $\mathrm{sh}_l(o_1)(\nu, u) = \mathrm{std}_l(\nu, u) + o_1(u)$ if $\nu = 0$, and $\mathrm{sh}_l(o_1)(\nu, u) = \mathrm{std}_l(\nu, u)$ otherwise.

(iii) For a lattice datum L, integers $l, A \geq 0$ and a letter i, $\mathfrak{K}_{l,A,i}(L)$ is the $\mathbb{Z}_p$-module of the $x \in \mathfrak{N}_{l,i}(L)$ with $x \in \mathrm{Inv}(\mathfrak{X}_L(i; \mathrm{sh}_l(o_2)(\cdot, \cdot)))$, where o_2 is the constant function with value A.

LEAN `Lift.KummerChart.JumpLattice.raiseExponent` · `Lift.KummerChart.JumpLattice.shiftedStd` · `Lift.KummerChart.JumpLattice.constantsModE`

Inputs from Part 1. The following results of the note *Exponents of elements of Kisin modules, the cost of deepening them, and cokernel killers* are used; L is a lattice datum, S a side, i a letter, o, o' exponent functions, $g \in \Gamma'$.

P1. *Localized modules* (Lemma 4.1). $\mathcal{E}_\nu^n x \in \mathfrak{M}_{\mathrm{loc}}(L)$ if and only if $x \in \mathfrak{M}_{\mathrm{loc}}(L)$; $\mathfrak{M}_{\mathrm{loc}}(L)$ is stable under φ; $\mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$ is stable under Γ' and is the A_{inf}-span of $\mathfrak{M}_{\mathrm{loc}}(L)$; $1 \otimes f$ maps $\mathfrak{M}_{\mathrm{loc}}(L)$ into $\mathfrak{M}_{\mathrm{loc}}(L')$ for an equivariant f; and $\mu(\varphi^{-\nu}(gx - x)) = g\mu(\varphi^{-\nu}x) - \mu(\varphi^{-\nu}x)$ for $x \in \mathfrak{M}_{\mathrm{loc}}^{\mathrm{inf}}(L)$, $\mu \in \mathrm{Per}(L)$.

P2. *Exponent modules* (Lemma 4.2). $\mathfrak{X}_L(i;o) \subseteq \mathfrak{X}_L(i;o')$ if $o' \le o$ on the jump sets; to pass from o to a larger o' it suffices to check the values where $o < o'$; $\mathcal{E}_{\nu_0}^n$ maps $\mathfrak{X}_L(i;o)$ into the module with exponents raised by n at ν_0; $gx - x \in \mathfrak{X}_L(i;o)$ for $x \in \mathfrak{X}_L(i;o)$; and an equivariant $f\colon L \to L'$ with $U_{L'}(j) \subseteq U_L(j)$ for all j maps $\mathfrak{X}_L(i;o)$ into $\mathfrak{X}_{L'}(i;o)$.

P3. *Division by $z\tau$* (Lemma 4.3). If $x \in \mathfrak{M}_{\mathrm{loc}}(L)$ and $gx - x \in \mathfrak{X}_L(i;o)$, then $(z\tau)^{-1}(gx - x) \in \mathfrak{X}_L(i;o)$.

P4. *Values at any type* (Lemma 4.4). If $x \in \mathfrak{X}_L(i;o)$, $\mu \in \mathrm{Per}_{i^{p^{-\nu}},u}(L)$ and a B_{dR}^+-module M contains $\mathrm{Fil}^{o(\nu,v)} B_{\mathrm{dR}}$ for all $v \in U_L(i^{p^{-\nu}})$ with $v \le u$, then $\mu(\varphi^{-\nu}x) \in M$.

P5. *The retraction* (Lemmas 5.1 and 5.3). ret_L is $\mathfrak{S}$-linear, the identity on $\mathfrak{M}_{\mathrm{loc}}(L)$, with $\mathrm{ret}_L(ax) = \mathrm{co}_0(a)x$ for $a \in A_{\mathrm{inf}}$, $x \in \mathfrak{M}_{\mathrm{loc}}(L)$; it commutes with $1 \otimes f$ for equivariant f; it maps $\mathfrak{X}_L(i;o)$ into itself; and if $\mu(\varphi^{-l}y) - v \in \mathrm{Fil}^n B_{\mathrm{dR}}$ with $v \in F_l$, then $\mu(\varphi^{-l}\,\mathrm{ret}_L(y)) - v \in \mathrm{Fil}^n B_{\mathrm{dR}}$.

Inputs from Part 1, continued.

P6. *Deepening* (Lemma 6.2 and Proposition 6.3). If $x \in \mathfrak{X}_L(i;o^{\mathrm{lo}}) \cap \mathrm{Inv}(\mathfrak{X}_L(i;o^{\mathrm{hi}}))$, then $x \in \mathfrak{X}_L(i;\mathrm{imp}(o^{\mathrm{lo}},o^{\mathrm{hi}})(\cdot,\cdot))$. For integers $\nu \ge 0$ and n there are $t_1,\dots,t_m \in \Gamma'$ and $y \in A_{\mathrm{inf}}$ such that, for $x \in \mathfrak{M}_{\mathrm{loc}}(L)$ and $\mu \in \mathrm{Per}(L)$: if $\mu(\varphi^{-\nu}x) - c \in \mathrm{Fil}^n B_{\mathrm{dR}}$ for some $c \in F$, then $\mu(\varphi^{-\nu}(p^{\mathrm{cost}(\nu,n)}x + \sum_k y(z\tau)^{-1}(t_k x - x))) - c' \in \mathrm{Fil}^{n+1} B_{\mathrm{dR}}$ for some $c' \in F$.

P7. *Theorem B and its corollaries* (Theorem B, Lemmas 7.2 and 7.3). $x \in \mathfrak{X}_L(i;\mathrm{std}_l)$ if and only if $\varphi^{-l}x \in \mathfrak{M}^{\mathrm{inf}}(L)$ and $x \in (W(C^\flat) \otimes L)_i$. If $x \in \mathfrak{X}_L(i^{p^{-l}};o)$, then $\varphi^l x \in \mathfrak{X}_L(i;\mathrm{fr}_{l,0}(o)(\cdot,\cdot))$. If x is a point over F_0 of $\varphi^l\mathfrak{M}^{\mathrm{inf}}(L)$, then $\mathrm{pr}_i(x) \in \mathfrak{N}_{l,i}(L)$.

P8. *Theorem C* (exponents add). For an equivariant $\mathcal{O}$-bilinear map $b\colon A \times L \to L''$ of lattice data, $y \in \mathfrak{X}_A(i;o_A)$ and $x \in \mathfrak{X}_L(i;o)$: if $o_{L''}(\nu,v) \le o(\nu,u') + o_A(\nu,u)$ whenever $u \in U_A(i^{p^{-\nu}})$, $u' \in U_L(i^{p^{-\nu}})$, $v \in U_{L''}(i^{p^{-\nu}})$ and $u + u' \le v$, then $b_W(y,x) \in \mathfrak{X}_{L''}(i;o_{L''})$. Moreover $b_W(\mathrm{pr}_i(y),x) = \mathrm{pr}_i(b_W(y,x))$.

P9. *The lifting ideal* (Lemmas 9.2, 9.3 and 9.4, Theorem D). If Γ acts on L through χ^c and $x \in \mathfrak{X}_L(i;\mathrm{std}_0)$, then $\tau^c x \in A_{\mathrm{inf}} \otimes L$. $\mathrm{cst}_{S,i}(c) \in \mathfrak{N}_{l,i}(C_S^{\mathrm{tr}})$ and $\varphi^l\,\mathrm{cst}_{S,i}(c) = \mathrm{cst}_{S,i^{p^l}}(c)$. $\varphi^l(s) \in \mathfrak{a}_S(r;l,i^{p^l})$ if and only if $s \in \mathfrak{a}_S(r;0,i)$. If $f \in \mathfrak{S}$ lies in $\mathfrak{a}_S(0;l,i)$, then every $y \in \mathfrak{N}_{l,i}(C_S^{\mathrm{tr}})$ has an $x \in \mathfrak{N}_{l,i}(\tilde{Y}_S)$ with $(1 \otimes \mathrm{pr}_S^C)(x) = fy$. If the chart is unramified, $p \ge 5$ and all $u \in U_{Y_S}(j)$ are $\le (p-3)(\kappa+\rho)$, then $\mathrm{Kill}_S(i;\kappa,\rho)$.

Inputs from the background library. Each is a declaration of the background library, read in its Lean text; the number in brackets is the unit that owns it.

J1. *Stability of the class* [81, 86]. The tensor product of two lattice data, the twisted dual of a lattice datum with the negatives of its jump sets shifted by -1, and $\mathcal{O}(c)$ with the set $\{-c\}$ are lattice data. The third pairing p_Π is Γ-equivariant. Reference to be confirmed.

J2. *Artin–Schreier–Witt over the tilt* [82]. Let $M \geq 0$. *(a)* p^M kills $\mathbb{M}_M$, and every $d \in \mathbb{M}_M$ is $\varphi d' - d'$ for some d'. *(b)* If $\varphi d - d \in \mathfrak{s}_M$, then $d - 1 \otimes k \in \mathfrak{s}_M$ for some $k \in \mathbb{Z}/p^M$. *(c)* If $1 \otimes k \in \mathfrak{s}_M$, then $k = 0$. *(d)* Let T be a $\mathbb{Z}_p$-representation of Γ such that for each $t \in T$, $gt - t \in p^M T$ for g near 1, and let $\Lambda \colon W(C^\flat) \otimes T \to \mathbb{M}_M$ be $W(C^\flat)$-linear. Then for each $v \in W(C^\flat) \otimes T$, $\Lambda(gv - v) \in \mathfrak{s}_M$ for g near 1. In the handout's catalogue this is a weaker form of the first step of Herr's complex (L. Herr, *Sur la cohomologie galoisienne des corps p-adiques*, Bull. Soc. Math. France 126 (1998), 563–600; place not confirmed); in the second reader's comparison the Artin–Schreier–Witt sequence (L. Illusie, *Complexe de de Rham–Witt*, §2.7) explains (a), and the statements about the small part and (d) are not located in the literature.

J3. *Crystalline cocycles are small near* 1 [87]. For a potentially crystalline lattice T, $c \in Z^1_{\mathrm{f}}(T)$ and $M \geq 0$: $c(g) \in p^M T$ for g near 1. For $t \in T$ the map $g \mapsto gt - t$ lies in $Z^1_{\mathrm{f}}(T)$. Reference to be confirmed.

J4. *Restriction and the index* [95]. For an open subgroup Δ of finite index in a topological group and a class in the second continuous cohomology whose restriction to Δ vanishes, the index times the class is 0. Elementary in the handout's catalogue, which names the classical statement behind it: corestriction after restriction is multiplication by the index (J.-P. Serre, *Local Fields*, Ch. VII; place within the chapter not confirmed, and the passage to continuous cochains of a topological group not located).

Inputs from the background library, continued.

J5. *Tate* [58]. If $V \in B_{\mathrm{dR}}$, M is a B_{dR}^{+}-submodule of B_{dR} and $gV - V \in M$ for all $g \in \Gamma'$, there is $c \in F$ with $V - c \in M$. (A weaker form of Tate's theorems; see Part 1, Input I6.)

J6. *Periods of one type* [82]. For a letter i and $u \in \mathbb{Z}$ there are $r \le m_L(i,u)$ periods $\mu_1, \dots, \mu_r \in \mathrm{Per}_{i,u}(L)$ such that every $\mu \in \mathrm{Per}_{i,u}(L)$ is a combination of them with coefficients in $\bar{\mathbb{Q}}_p$, modulo $\mathrm{Per}_{i,u-1}(L)$; here $m_L(i,u)$ is the total multiplicity of the jump u of L over the embeddings above i. The handout's catalogue calls it the authors' analogue for Kisin modules of statements on Wach modules (L. Berger, *Limites de représentations cristallines*, Compositio Math. 140 (2004), §III; N. Wach, Bull. Soc. Math. France 124 (1996); place not confirmed). Also: every $\mathbb{Z}_p$-linear map $L \to B_{\mathrm{dR}}$ extends to an A_{inf}-linear map on $W(C^\flat) \otimes L$, and $\mu \circ \mathrm{pr}_i \in \mathrm{Per}(L)$ for $\mu \in \mathrm{Per}(L)$.

J7. *Auerbach's lemma* [general library]. Let U be a finite-dimensional space of functions with values in C on a finite set. Then there are points $s_1, \dots, s_q$ of the set, q the dimension, and a basis $\mathsf{z}_1, \dots, \mathsf{z}_q$ of U such that $\mathsf{z}_b(s_{b'})$ is 1 for $b = b'$ and 0 otherwise, and $|\mathsf{z}_b(k)| \le 1$ for every point k; so $\mathsf{v}(k) = \sum_b \mathsf{z}_b(k)\,\mathsf{v}(s_b)$ for all $\mathsf{v} \in \mathsf{U}$. This is a special case of the declaration, which allows a compact set of linear forms. (H. Auerbach, thesis, Lwów 1930; place not confirmed; in the second reader's comparison a determinant-maximization extension of Auerbach's lemma.)

J8. *The frame theorem: spanning* [83]. Let $l, A, \delta \ge 0$ with $3e_{\mathfrak{c}} \le p^l$ and $e_{\mathfrak{c}} A \le p^l \delta$, and let x be a point over F_0 of $\varphi^l \mathfrak{M}^{\mathrm{inf}}(L)$. Then $p^{4\delta}x$ is a combination, with coefficients in $\mathfrak{S}$, of points y over F_0 of $\varphi^l \mathfrak{M}^{\mathrm{inf}}(L)$ with $g\mu(y) - \mu(y) \in \mathrm{Fil}^A B_{\mathrm{dR}}$ for all $\mu \in \mathrm{Per}(L)$, $g \in \Gamma'$. Reference to be confirmed (the file calls the frame theorem a known result: the Kisin lattice of a crystalline lattice).

J9. *Residues* [83]. If the chart is unramified, every $s \in \mathfrak{S}$ is $w + \mathcal{E}t$ with $w \in \mathbb{W}$ and $t \in \mathfrak{S}$. The ring $\mathbb{W}$ is free of rank h over $\mathbb{Z}_p$, its elements are fixed by Γ', and the difference of the Teichmüller lifts of two distinct letters is a unit. The coordinate z lies in $W(\mathfrak{m}^\flat)$. Elementary.

3 Four safe pairs give a coboundary

Throughout this section the setting is Setting 2.2; S, S' are sides over $\mathcal{O}$ and Π is a pairing of S and S', modulo p^M.

Lemma 3.1 (the action on the components)**.** *Let $n = (a, g) \in \mathbb{Z} \times \Gamma$.*

(a) *For $x \in W(C^\flat) \otimes \tilde{Y}_S$, $\mathrm{pr}_S^Y(n \cdot x) = n \cdot \mathrm{pr}_S^Y(x) + (1 \otimes \alpha_S(\cdot)(g))(n \cdot (1 \otimes \mathrm{pr}_S^C)(x))$, where $\alpha_S(\cdot)(g) \colon C_S \to Y_S$ is the map $c \mapsto \alpha_S(c)(g)$.*

(b) *For $x \in W(C^\flat) \otimes Y_S$ and $y \in W(C^\flat) \otimes Y_{S'}$: $n \cdot \mathsf{b}_\Pi(x, y) = \mathsf{b}_\Pi(n \cdot x, n \cdot y)$.*

LEAN `Lift.wittFst_smul` · `Lift.Pairing.smul_wittPairing`

Proof. (a) On $c_0 \otimes (y, c)$ both sides are $nc_0 \otimes (gy + \alpha_S(c)(g))$. (b) The pairing modulo p^M satisfies $\langle gy, gy' \rangle = \chi(g)\langle y, y' \rangle$, and φ acts on the first factors only. □

In particular, if $(1\otimes \mathrm{pr}_S^C)(x) = p^{e_1}(1\otimes c)$, then for $g\in\Gamma$

$$\mathrm{pr}_S^Y(gx) = g\,\mathrm{pr}_S^Y(x) + p^{e_1}(1\otimes \alpha_S(c)(g)), \qquad \mathrm{pr}_S^Y(\varphi x) = \varphi\,\mathrm{pr}_S^Y(x). \tag{1}$$

Proof of Theorem A. Fix a topology on Γ for which j_Γ is an open embedding. Write $a=\alpha_S(c)$, $a'=\alpha_{S'}(c')$, $\xi=\mathrm{pr}_S^Y(x)$, $\xi'=\mathrm{pr}_{S'}^Y(x')$, $\xi_w=\mathrm{pr}_S^Y(w)$, $\mathsf{a}_g = 1\otimes a(g)$, $\mathsf{a}'_g = 1\otimes a'(g)$, and $\mathsf{B}(y,y') = p^{e_3}\mathsf{b}_\Pi(y,y')$. The maps a, a' are cocycles: $\mathsf{a}_{gh} = \mathsf{a}_g + g\mathsf{a}_h$. By (1), for $g,h\in\Gamma'$,

$$\mathrm{pr}_S^Y(gx-x) = g\xi-\xi+p^{e_1}\mathsf{a}_g, \qquad \mathrm{pr}_{S'}^Y(g(hx'-x')) = g(h\xi'-\xi')+p^{e_2}g\mathsf{a}'_h,$$

and similarly for w and for $\varphi x'$, whose components in $W(C^\flat)\otimes C_S$, $W(C^\flat)\otimes C_{S'}$ are again $p^{e_1}(1\otimes c)$ and $p^{e_2}(1\otimes c')$.

The cochain. By Input J2(a) there is $y_0\in\mathbb{M}_M$ with $\varphi y_0 - y_0 = \mathsf{B}(\xi_w,\varphi\xi'-\xi')$. For $g\in\Gamma'$ put

$$D_g = p^{e_1}\mathsf{B}(\mathsf{a}_g, g\xi') - \mathsf{B}(\xi, g\xi'-\xi') - p^{e_2}\mathsf{B}(\xi,\mathsf{a}'_g) + (gy_0-y_0) \;\in\; \mathbb{M}_M.$$

A direct computation with Lemma 3.1(b) and (1) gives

$$\begin{aligned}\varphi D_g - D_g = {}& \mathsf{B}(\mathrm{pr}_S^Y(gw-w), \mathrm{pr}_{S'}^Y(g(\varphi x'-x'))) \\ & - \mathsf{B}(\mathrm{pr}_S^Y(\varphi x-w), \mathrm{pr}_{S'}^Y(g\varphi x'-\varphi x')) - \mathsf{B}(\mathrm{pr}_S^Y(w-x), \mathrm{pr}_{S'}^Y(gx'-x')) :\end{aligned}$$

on the right the terms with ξ_w combine to $\mathsf{B}(g\xi_w, g(\varphi\xi'-\xi')) - \mathsf{B}(\xi_w,\varphi\xi'-\xi') = (g-1)(\varphi y_0 - y_0)$, the terms with a_g to $p^{e_1}\mathsf{B}(\mathsf{a}_g, g(\varphi\xi'-\xi'))$, those with a'_g to $-p^{e_2}\mathsf{B}(\varphi\xi-\xi,\mathsf{a}'_g)$, and the rest to $-\mathsf{B}(\varphi\xi, g\varphi\xi'-\varphi\xi') + \mathsf{B}(\xi, g\xi'-\xi')$. The three terms on the right lie in $\mathfrak{s}_M$ by the last three hypotheses. By Input J2(b) there is $k(g)\in\mathbb{Z}/p^M$ with $D_g - 1\otimes k(g)\in\mathfrak{s}_M$.

The coboundary. For $g,h\in\Gamma'$ the cocycle relations give

$$gD_h - D_{gh} + D_g = -p^{e_1}\mathsf{B}(\mathsf{a}_g, g(h\xi'-\xi')) - \mathsf{B}(g\xi-\xi, g(h\xi'-\xi')) - p^{e_2}\mathsf{B}(g\xi-\xi, g\mathsf{a}'_h),$$

the terms with y_0 cancelling. On the other hand the first hypothesis says that

$$\mathsf{B}(g\xi-\xi+p^{e_1}\mathsf{a}_g,\; g(h\xi'-\xi')+p^{e_2}g\mathsf{a}'_h)\in\mathfrak{s}_M.$$

Adding the two, and using that $\mathfrak{s}_M$ is stable under Γ:

$$p^{e_1+e_2+e_3}(1\otimes\langle a(g), g\,a'(h)\rangle) - 1\otimes(\chi(g)k(h)-k(gh)+k(g))\in\mathfrak{s}_M,$$

so by Input J2(c)

$$p^{e_1+e_2+e_3}\langle a(g), g\,a'(h)\rangle = \chi(g)k(h)-k(gh)+k(g) \qquad (g,h\in\Gamma'), \tag{2}$$

and, multiplying by $r = p^{e-e_1-e_2-e_3}$,

$$p^e\langle a(g), g\,a'(h)\rangle = r\,(\chi(g)k(h)-k(gh)+k(g)) \qquad (g,h\in\Gamma'). \tag{3}$$

Continuity. By Input J3, $a(g)\in p^MY_S$ and $a'(g)\in p^MY_{S'}$ for g near 1, and $gt-t\in p^MY_{S'}$ near 1 for each $t\in Y_{S'}$. Hence the function

$$\mathsf{c}(g,h) = \mathsf{e}_M(p^e\langle a(g), g\,a'(h)\rangle) \qquad (g,h\in\Gamma)$$

is locally constant on $\Gamma \times \Gamma$; it is a 2-cocycle, being a cup product of cocycles. For h near 1 we have $D_h \in \mathfrak{s}_M$: the terms with a_h, a'_h vanish because p^M kills $\mathbb{M}_M$, and $\mathsf{B}(\xi, h\xi' - \xi')$ and $hy_0 - y_0$ lie in $\mathfrak{s}_M$ by Input J2(d), applied to $Y_{S'}$ and to $\mathbb{Z}/p^M(1)$. So $k(h) = 0$ for h near 1 (Input J2(c)); the left side of (2) vanishes for such h, because $a'(h) \in p^M Y_{S'}$; and then (2) shows $k(gh) = k(g)$: the cochain k on Γ' is locally constant.

Conclusion. The subgroup Γ' is open, because it contains the elements near 1. By (3) the restriction of c to Γ' is the coboundary of the continuous cochain $\mathsf{e}_M \circ (rk)$. By Input J4, the index of Γ' kills the class of c. This class is also killed by p^M, and the index is prime to p; so the class is trivial, which is the assertion. □

4 Points: formal properties, strictness and deepening

Throughout, the setting is Setting 2.1; S is a side over $\mathcal{O}$.

Lemma 4.1 (small facts)**.** *Let i be a letter and o, o_1, o_2, o_3 exponent functions.*

(a) *For $y \in W(C^\flat) \otimes Y_S$: $\mathrm{pr}_S^Y((1 \otimes \mathrm{in}_S^Y)(y)) = y$ and $(1 \otimes \mathrm{pr}_S^C)((1 \otimes \mathrm{in}_S^Y)(y)) = 0$.*

(b) *Let $e \geq 0$ be an integer, A an additive map with $\mathrm{Cost}_S(\mathsf{A}) \leq e$, and N a $\mathbb{Z}_p$-submodule of $\mathfrak{M}_{\mathrm{loc}}(\tilde{Y}_S)$ with $(\mathfrak{X}_S^0(i;o))_{\mathfrak{S}} \subseteq N$. If $x \in N$ and $x \in \mathrm{Deep}_S(i;o)$, then $\mathsf{A}(x) \in N$.*

(c) $(\mathfrak{X}_S^0(i;o))_{\mathfrak{S}} \subseteq \mathrm{Deep}_S(i;o)$.

(d) *Let $c \in C_S$, $e \geq 0$, $x \in \mathrm{Pt}_S(c, i; e; o_1, o)$ and $g \in \Gamma'$. Then $(z\tau)^{-1}(gx - x) \in \mathfrak{X}_S^0(i;o)$.*

(e) *Let $c \in C_S$, $e \geq 0$, $x \in \mathrm{Pt}_S(c, i; e; o_1, o)$ and $w \in \mathrm{Pt}_S(c, i; e; o_2, o_3)$. Then $w - x \in \mathfrak{X}_S^0(i; \min(o_1, o_2))$.*

LEAN `Lift.KummerChart.PairingSide.wittFst_inclKer` · `Lift.KummerChart.PairingSide.Reaches.mem_of_stable` · `Lift.KummerChart.PairingSide.kerWithExponents_le_deepElements` · `Lift.KummerChart.PairingSide.smul_sub_mem_kerWithExponents` · `Lift.KummerChart.PairingSide.sub_mem_kerWithExponents_of_pointsOver` · `Lift.KummerChart.PairingSide.stepBack_zero` (the identity $i^{p^0} = i$)

Proof. (a) is clear. (b) $\mathsf{A}(x) = p^e x + (\mathsf{A}(x) - p^e x)$. (c) Such an element lies over $0 = \mathrm{cst}_{S,i}(0)$, and its Galois differences keep the exponents (Input P2). (d) By Input P3; the image in $W(C^\flat) \otimes C_S$ is 0 because g fixes $\mathrm{cst}_{S,i}(c)$. (e) By Input P2, and both points lie over the same element. □

Lemma 4.2 (safety is checked letter by letter)**.** *In Setting 2.2, let S, S' be sides, Π a pairing of S and S', $\kappa \geq 0$ an integer, and for every letter j let $u_j \in (W(C^\flat) \otimes \tilde{Y}_S)_j$ and $v_j \in (W(C^\flat) \otimes \tilde{Y}_{S'})_j$. If $\mathrm{Safe}_\Pi(\kappa; u_j, v_j)$ for every letter j, then $\mathrm{Safe}_\Pi(\kappa; \sum_j u_j, \sum_j v_j)$.*

LEAN `Lift.KummerChart.PairingSide.isSafe_sum`

Proof. The form ψ is $\mathcal{O}$-bilinear, so the pairing modulo p^M satisfies $\langle \omega y, y' \rangle = \langle y, \omega y' \rangle$. For letters $j \neq j'$ this gives $([j] - [j'])\, \mathsf{b}_\Pi(\mathrm{pr}_S^Y(u_j), \mathrm{pr}_{S'}^Y(v_{j'})) = 0$, and $[j] - [j']$ is a unit (Input J9). So the cross terms vanish. □

Lemma 4.3 (extending a period to the extension)**.** *Let M be a B_{dR}^+-submodule of B_{dR} and $\mu \in \mathrm{Per}(Y_S)$ with $\mu(1 \otimes t) \in M$ for all $t \in Y_S$. Then there is $\mu' \in \mathrm{Per}(\tilde{Y}_S)$ with $\mu'(A_{\mathrm{inf}} \otimes \tilde{Y}_S) \subseteq M$ and $\mu' \circ (1 \otimes \mathrm{in}_S^Y) = \mu$.*

LEAN `Lift.KummerChart.PairingSide.exists_extend_form`

Proof. Let $\Lambda\colon B_{\max}\otimes Y_S\to B_{\mathrm{dR}}$ be the $B_{\max}$-linear map with $\Lambda(1\otimes t)=\mu(1\otimes t)$; it commutes with Γ'. For $c\in C_S$ choose $y_c\in B_{\max}\otimes Y_S$ with $1\otimes\alpha_S(c)(g)=gy_c-y_c$ for all g. Then $g\Lambda(y_c)-\Lambda(y_c)=\mu(1\otimes\alpha_S(c)(g))\in M$ for $g\in\Gamma'$, so by Input J5 there is $a_c\in F$ with $\Lambda(y_c)-a_c\in M$. Take a $\mathbb{Z}_p$-basis (e_k) of C_S, let $\mu_C\colon W(C^\flat)\otimes C_S\to B_{\mathrm{dR}}$ be A_{inf}-linear with $\mu_C(1\otimes e_k)=\Lambda(y_{e_k})-a_{e_k}$ (Input J6), and put $\mu'=\mu\circ\mathrm{pr}_S^Y+\mu_C\circ(1\otimes\mathrm{pr}_S^C)$. Its values on $A_{\mathrm{inf}}\otimes\tilde Y_S$ lie in M. It is a period: for $g\in\Gamma'$, $\mu'(1\otimes g(t,c))=g\mu(1\otimes t)+\mu(1\otimes\alpha_S(c)(g))+\mu_C(1\otimes c)$, and $\mu(1\otimes\alpha_S(c)(g))=(g-1)\mu_C(1\otimes c)$ by linearity in c, the a_{e_k} being fixed by Γ'. □

Lemma 4.4 (strictness)**.** *Let i be a letter, o an exponent function and $w\in\mathfrak{X}^0_S(i;o)$. Then there is $y\in\mathfrak{X}_{Y_S}(i;o)$ with $(1\otimes\mathrm{in}_S^Y)(y)=w$; and if $w\in\mathfrak{M}_{\mathrm{loc}}(\tilde Y_S)$, then $y\in\mathfrak{M}_{\mathrm{loc}}(Y_S)$.*

LEAN `Lift.KummerChart.PairingSide.exists_inclKer_eq_of_projTriv_eq_zero`

Proof. The sequence $0\to W(C^\flat)\otimes Y_S\to W(C^\flat)\otimes\tilde Y_S\to W(C^\flat)\otimes C_S\to 0$ is exact, so $w=(1\otimes\mathrm{in}_S^Y)(y)$ with $y=\mathrm{pr}_S^Y(w)$, and y lies in the part at i. Some product π of the $\mathcal{E}_\nu$ has $\pi w\in\mathfrak{M}^{\mathrm{inf}}(\tilde Y_S)$; then $\tau^a\pi y\in A_{\mathrm{inf}}\otimes Y_S$ for some a, and for $\mu\in\mathrm{Per}(Y_S)$, extended to μ' by Lemma 4.3 with $M=B_{\mathrm{dR}}$, and every $b\ge 0$, $\mu(\varphi^{b+1}\pi y)=\mu'(\varphi^{b+1}\pi w)\in\mathrm{Fil}^0 B_{\mathrm{dR}}$. So $y\in\mathfrak{M}^{\mathrm{inf}}_{\mathrm{loc}}(Y_S)$. For the exponents let $j=i^{p^{-\nu}}$, $u\in U_{Y_S}(j)$ and $\mu\in\mathrm{Per}_{j,u}(Y_S)$. The period $\mu\circ\mathrm{pr}_j$ takes values in $\mathrm{Fil}^{-u}B_{\mathrm{dR}}$ on $1\otimes Y_S$; its extension μ' of Lemma 4.3 lies in $\mathrm{Per}_{j,u}(\tilde Y_S)$, and $\mu(\varphi^{-\nu}y)=\mu'(\varphi^{-\nu}w)\in\mathrm{Fil}^{o(\nu,u)}B_{\mathrm{dR}}$. Finally, if $w\in\mathfrak{M}_{\mathrm{loc}}(\tilde Y_S)$, then $(1\otimes\mathrm{in}_S^Y)(\mathrm{ret}_{Y_S}(y))=\mathrm{ret}_{\tilde Y_S}(w)=w$ by Input P5, so $\mathrm{ret}_{Y_S}(y)=y$ and $y\in\mathfrak{M}_{\mathrm{loc}}(Y_S)$. □

Lemma 4.5 (a point from a cokernel killer)**.** *Let i be a letter, $\kappa,\rho\ge 0$ integers with $\mathrm{Kill}_S(i;\kappa,\rho)$, and $c\in C_S$. Then $\mathrm{Pt}_S(c,i;\kappa;\mathrm{rs}_{-\rho}(\cdot,\cdot),\mathrm{rs}_{-\rho}(\cdot,\cdot))$ is not empty.*

LEAN `Lift.KummerChart.PairingSide.pointsOver_nonempty_of_killsCokernel`

Proof. By Input P9 there is $x_1\in\mathfrak{N}_{0,i}(\tilde Y_S)$ over $p^\kappa\mathcal{E}^\rho\,\mathrm{cst}_{S,i}(c)$. Then $x=\mathcal{E}^{-\rho}x_1$ lies in $\mathfrak{M}_{\mathrm{loc}}(\tilde Y_S)$, over $p^\kappa\,\mathrm{cst}_{S,i}(c)$, with the exponents std_0 lowered by ρ at $\nu=0$ (Inputs P1 and P2); its Galois differences have the same exponents. □

Proposition 4.6 (deepening along a set of steps)**.** *Let I be a finite set of pairs (ν,n) of an integer $\nu\ge 0$ and an integer n. Then there is an additive map $\mathsf{A}\colon\mathfrak{M}_{\mathrm{loc}}(\tilde Y_S)\to\mathfrak{M}_{\mathrm{loc}}(\tilde Y_S)$ with $\mathrm{Cost}_S(\mathsf{A})\le\sum_{(\nu,n)\in I}\mathrm{cost}(\nu,n)$ such that for every letter i, all exponent functions o, o' and every $x\in\mathrm{Deep}_S(i;o)$: if I contains every (ν,n) with $o(\nu,u)\le n<o'(\nu,u)$ for some $u\in U_{Y_S}(i^{p^{-\nu}})$, then $\mathsf{A}(x)\in\mathrm{Deep}_S(i;o')$.*

LEAN `Lift.KummerChart.PairingSide.exists_reaches_deep`

Proof. By induction on I, adding at each stage a pair whose second component is maximal. For $I=\emptyset$ take the identity: then $o'\le o$ on the jump sets. Let $I=I_0\cup\{(\nu_0,n_0)\}$ with $n\le n_0$ for all $(\nu,n)\in I_0$, and let A_0 serve for I_0. Take $t_1,\dots,t_m$ and y from Input P6 for (ν_0,n_0) and the lattice datum $\tilde Y_S$, and put

$$\mathsf{B}(x)=p^{\mathrm{cost}(\nu_0,n_0)}x+\sum_k\mathrm{ret}_{\tilde Y_S}\big(y\,(z\tau)^{-1}(t_kx-x)\big),\qquad\mathsf{A}=\mathsf{B}\circ\mathsf{A}_0.$$

Cost. For $x \in \mathrm{Deep}_S(i;o)$ each $(z\tau)^{-1}(t_k x - x)$ lies in $\mathfrak{X}_{\tilde{Y}_S}(i;o)$ (Input P3) and maps to 0 in $W(C^\flat) \otimes C_S$, and ret keeps both properties (Input P5); so B has cost $\mathrm{cost}(\nu_0, n_0)$. Costs add under composition, by Lemma 4.1(b) and (c).

Depth. Let $x \in \mathrm{Deep}_S(i;o)$ and let I contain the steps from o to o'. Let o'' be o' lowered to n_0 at the places (ν_0, u) where $o'(\nu_0, u) = n_0 + 1$. By the maximality of n_0, I_0 contains the steps from o to o'', so $x_1 = \mathsf{A}_0(x) \in \mathrm{Deep}_S(i;o'')$, and $\mathsf{B}(x_1) \in \mathrm{Deep}_S(i;o'')$ by Lemma 4.1(b). By Input P2 it remains to check the Galois differences of $\mathsf{B}(x_1)$ at the places where $o'' < o'$: so let u with $o'(\nu_0, u) = n_0 + 1$ and $\mu \in \mathrm{Per}_{i^{p^{-\nu_0}},u}(\tilde{Y}_S)$. The value $V = \mu(\varphi^{-\nu_0} x_1)$ has $gV - V \in \mathrm{Fil}^{n_0} B_{\mathrm{dR}}$ for all $g \in \Gamma'$, so $V - c \in \mathrm{Fil}^{n_0} B_{\mathrm{dR}}$ for some $c \in F$ (Input J5). By Input P6, the value of μ at $\varphi^{-\nu_0}$ of $p^{\mathrm{cost}(\nu_0,n_0)} x_1 + \sum_k y (z\tau)^{-1}(t_k x_1 - x_1)$ is congruent to some $c' \in F$ modulo $\mathrm{Fil}^{n_0+1} B_{\mathrm{dR}}$, and by Input P5 the same holds for its retraction $\mathsf{B}(x_1)$. As c' is fixed by Γ', the Galois differences of that value lie in $\mathrm{Fil}^{n_0+1} B_{\mathrm{dR}}$. □

Lemma 4.7 (points after deepening; Frobenius)**.** *Let $c \in C_S$, i a letter, $e, e', l \geq 0$ integers and o, o', o_1 exponent functions.*

(a) *Let $x \in \mathrm{Pt}_S(c, i; e; o_1, o)$ and let I be a finite set of pairs (ν, n) that contains every (ν, n) with $o(\nu, u) \leq n < o'(\nu, u)$ for some $u \in U_{Y_S}(i^{p^{-\nu}})$. If $e + \sum_{(\nu,n)\in I} \mathrm{cost}(\nu, n) \leq e'$, then $\mathrm{Pt}_S(c, i; e'; \mathrm{imp}(o_1, o')(\cdot,\cdot), o')$ is not empty.*

(b) *If $w \in \mathrm{Pt}_S(c, i^{p^{-l}}; e; o, o')$, then $\varphi^l w \in \mathrm{Pt}_S(c, i; e; \mathrm{fr}_{l,0}(o)(\cdot,\cdot), \mathrm{fr}_{l,1}(o')(\cdot,\cdot))$.*

(c) *Let $n \geq 0$ be an integer and β an integer with $\mathrm{NM}(p; n, \beta)$, and let o^{lo}, o^{hi} be exponent functions. Suppose that $\mathrm{Pt}_S(c, i; \kappa; o^{\mathrm{lo}}, o)$ is not empty, for an integer $\kappa \geq 0$, and that for all $\nu \geq 0$, $u \in U_{Y_S}(i^{p^{-\nu}})$ and integers a with $o(\nu, u) \leq a < o^{\mathrm{hi}}(\nu, u)$ one has $\nu = 0$ and $a \in \mathcal{W}_n(\beta) \cup \mathcal{W}_n(-\beta)$. Then $\mathrm{Pt}_S(c, i; \kappa + (8n + 4)\mathfrak{d}; \mathrm{imp}(o^{\mathrm{lo}}, o^{\mathrm{hi}})(\cdot,\cdot), o^{\mathrm{hi}})$ is not empty.*

LEAN `Lift.KummerChart.PairingSide.pointsOver_nonempty_of_costs` · `Lift.KummerChart.PairingSide.phi_pow_smul_mem_pointsOver` · `Lift.KummerChart.PairingSide.pointsOver_nonempty_of_noMultipleNear`

Proof. (a) Let A be the map of Proposition 4.6 for I, of cost ϵ, and $x_2 = p^{e'-e-\epsilon} x$. It lies over a constant, in $\mathrm{Deep}_S(i;o)$ and in $\mathrm{Deep}_S(i;o_1)$. So $\mathsf{A}(x_2)$ lies over $p^{e'} \mathrm{cst}_{S,i}(c)$, in $\mathfrak{X}_{\tilde{Y}_S}(i;o_1)$ (Lemma 4.1(b)) and in $\mathrm{Deep}_S(i;o')$; Input P6 improves its exponents.

(b) $\mathfrak{M}_{\mathrm{loc}}$ is stable under φ, $\varphi^l \mathrm{cst}_{S,i^{p^{-l}}}(c) = \mathrm{cst}_{S,i}(c)$, and the exponents shift by Input P7. For the depth, $(z\tau)^{-1}(gw - w) \in \mathfrak{X}_{\tilde{Y}_S}(i^{p^{-l}}; o')$ by Lemma 4.1(d), so its image under φ^l has the exponents $\mathrm{fr}_{l,0}(o')(\cdot,\cdot)$; and $g\varphi^l w - \varphi^l w = \varphi^l(z\tau) \cdot \varphi^l((z\tau)^{-1}(gw - w))$, where $\varphi^l(\tau)$ is divisible by $\mathcal{E}_\nu$ for every $\nu < l$, which raises the exponents at these ν by 1.

(c) Apply (a) with $I = \{(0, a) : a \in \mathcal{W}_n(\beta) \cup \mathcal{W}_n(-\beta)\}$. This set has at most $8n + 4$ elements, and $\mathrm{cost}(0, a) = v_p(a) + \mathfrak{d} = \mathfrak{d}$ for each of them, because p divides none of them except possibly 0. □

5 Safety from exponents

Throughout, the setting is Setting 2.2; S, S' are sides over $\mathcal{O}$ and Π a pairing of S and S' modulo p^M. Write $\mathsf{m}\colon \mathcal{O}\times Y_S\to Y_S$ for the scalar multiplication, $\mathsf{t}\colon Y_S\times Y_{S'}\to Y_S\otimes_{\mathcal{O}} Y_{S'}$ for the tensor map, and $\bar{\mathrm{tr}}\colon \mathcal{O}(1)\to \mathbb{Z}/p^M(1)$ for tr followed by reduction.

Lemma 5.1 (the pairing through the third side)**.** *Let i be a letter. Then:*

(a) $\mathsf{m}_W(\mathrm{cst}_{S_\Pi^\vee,i}(1),w)=\mathrm{pr}_i(w)$ *for every* $w\in W(C^\flat)\otimes Y_S$;

(b) *for every* $n\in W(C^\flat)\otimes \tilde{Y}_{S_\Pi^\vee}$, $w\in W(C^\flat)\otimes Y_S$ *and* $y\in W(C^\flat)\otimes Y_{S'}$,

$$(1\otimes\bar{\mathrm{tr}})\Big(\mathsf{p}_{\Pi W}(n,\mathsf{t}_W(w,y))\Big)=\mathsf{b}_\Pi\Big(\mathsf{m}_W((1\otimes \mathrm{pr}^C_{S_\Pi^\vee})(n),w),\ y\Big).$$

LEAN `Lift.KummerChart.PairingSide.wittPairing_constantAt_and_thirdPairing` · the formal statement binds the extension of the scalar multiplication by a local definition before the two assertions

Proof. Both are identities of $W(C^\flat)$-multilinear maps; check them on elementary tensors. (a) is Input P8, last sentence. For (b), with $n=1\otimes(\lambda,a)$: $\bar{\mathrm{tr}}(p^M\lambda(t\otimes t')+a\psi(t\otimes t'))=\langle at,t'\rangle$. □

Proposition 5.2 (safety from exponents)**.** *Let i be a letter, $l,\kappa\ge 0$ integers and o, o', o_2, o_3 exponent functions. Let $n\in \mathrm{Pt}_{S_\Pi^\vee}(1,i^{p^{-l}};\kappa;o_2,o_3)$, and suppose that $u\mapsto o_2(\nu,u)$ is non-increasing for every $\nu\ge 0$. Let $w\in\mathfrak{X}^0_S(i;o)$ and $y\in\mathfrak{X}^0_{S'}(i;o')$, and suppose that*

$$0<\mathrm{fr}_{l,0}(o_2)(\nu,-u-u'-1)+o(\nu,u)+o'(\nu,u') \tag{4}$$

for all $\nu\ge 0$, $u\in U_{Y_S}(i^{p^{-\nu}})$ and $u'\in U_{Y_{S'}}(i^{p^{-\nu}})$. Then $\mathrm{Safe}_\Pi(\kappa;z\tau\cdot w,y)$ *and* $\mathrm{Safe}_\Pi(\kappa;w,z\tau\cdot y)$.

LEAN `Lift.KummerChart.PairingSide.SidePairing.isSafe_of_exponents`

Proof. By Lemma 4.4, $w=(1\otimes \mathrm{in}^Y_S)(w_0)$ and $y=(1\otimes\mathrm{in}^Y_{S'})(y_0)$ with $w_0\in\mathfrak{X}_{Y_S}(i;o)$, $y_0\in\mathfrak{X}_{Y_{S'}}(i;o')$. By Lemma 4.7(b), $n_1=\varphi^l n$ is a point of $S_\Pi^\vee$ at i over $p^\kappa\cdot 1$, with the exponents $o_4=\mathrm{fr}_{l,0}(o_2)(\cdot,\cdot)$, again non-increasing in u.

Exponents. Apply Input P8 to the tensor map t:

$$\mathsf{t}_W(w_0,y_0)\in\mathfrak{X}_{\mathsf{T}_{S,S'}}(i;o_5)\qquad\text{with}\qquad o_5(\nu,v)=1-o_4(\nu,-v-1).$$

The condition is $o_5(\nu,v)\le o(\nu,u)+o'(\nu,u')$ for $u+u'\le v$; it follows from (4), because o_4 is non-increasing in its second argument and $-v-1\le -u-u'-1$. Apply it again to the third pairing p_Π, with values in $\mathcal{O}(1)$:

$$q=\mathsf{p}_{\Pi W}(n_1,\mathsf{t}_W(w_0,y_0))\ \in\ \mathfrak{X}_{\mathcal{O}(1)}(i;\mathrm{std}_0);$$

the condition is $\mathrm{std}_0(\nu,-1)=1\le o_5(\nu,v)+o_4(\nu,u_3)$ for jumps u_3 of the third side with $u_3+v\le -1$, which holds because $o_4(\nu,u_3)\ge o_4(\nu,-v-1)$. By Input P9 with the first power of χ, $\tau q\in A_{\mathrm{inf}}\otimes\mathcal{O}(1)$.

The pairing. By Lemma 5.1 and $(1\otimes\mathrm{pr}^C_{S_\Pi^\vee})(n_1)=p^\kappa\,\mathrm{cst}_{S_\Pi^\vee,i}(1)$,

$$(1\otimes\bar{\mathrm{tr}})(q)=p^\kappa\,\mathsf{b}_\Pi(\mathrm{pr}_i(w_0),y_0)=p^\kappa\,\mathsf{b}_\Pi(w_0,y_0).$$

Hence $p^\kappa\mathsf{b}_\Pi(z\tau w_0,y_0)=z\cdot(1\otimes\bar{\mathrm{tr}})(\tau q)$. The element $(1\otimes\bar{\mathrm{tr}})(\tau q)$ is an A_{inf}-combination of elements $1\otimes k$, and $z\in W(\mathfrak{m}^\flat)$ (Input J9); so the product lies in $\mathfrak{s}_M$.

As $\mathrm{pr}^Y_S(w) = w_0$ and $\mathrm{pr}^Y_{S'}(y) = y_0$ (Lemma 4.1(a)) and the pairing is bilinear, both assertions follow. □

Lemma 5.3 (safety of a Galois difference against a translate)**.** *In the situation of Proposition 5.2, with the point n, the functions o, o', o_2, o_3, $y \in \mathfrak{X}^0_{S'}(i; o')$ and (4) as there, let $c \in C_S$, $e \geq 0$ an integer, o_1 an exponent function and $x \in \mathrm{Pt}_S(c, i; e; o_1, o)$. Then $\mathrm{Safe}_\Pi(\kappa; gx - x, g \cdot y)$ for every $g \in \Gamma'$.*

LEAN `Lift.KummerChart.PairingSide.SidePairing.isSafe_smul_sub`

Proof. $(z\tau)^{-1}(gx - x) \in \mathfrak{X}^0_S(i; o)$ by Lemma 4.1(d), and $gy = (gy - y) + y \in \mathfrak{X}^0_{S'}(i; o')$ by Input P2. Apply Proposition 5.2 to these two elements. □

6 The walk

Proof of Theorem B. For each letter j with the property cut (a *cut*) choose: from (i), an exponent function d_j, an integer r_j and a point $x_j \in \mathrm{Pt}_S(c, j; e; o(j), d_j)$; from (ii), d'_j, r'_j and $x'_j \in \mathrm{Pt}_{S'}(c', j; e'; o'(j), d'_j)$; from (iii), a point $n_j \in \mathrm{Pt}_{S^\vee_\Pi}(1, j; \kappa; z(j), z'(j))$. For a letter j let $\mathsf{l} = \mathsf{l}(j)$ and $j^- = j^{p^{-\mathsf{l}}}$, the nearest cut strictly before j on its Frobenius orbit, and put

$$X^\perp_j = \varphi^{\mathsf{l}} x_{j^-}, \qquad X^\top_j = x_j \ \text{ if } j \text{ is a cut}, \quad X^\top_j = X^\perp_j \ \text{ otherwise},$$

$$X^{\mathsf{Q}}_j = x_j \ \text{ if } j \text{ is a cut and } \mathsf{Q}(j), \quad X^{\mathsf{Q}}_j = X^\perp_j \ \text{ otherwise},$$

and define $X'^\perp_j$, $X'^\top_j$ in the same way from the x'_j. By Lemma 4.7(b) these are points at j over $p^e c$, respectively $p^{e'} c'$; a point moved from j^- has the exponents $\mathrm{fr}_{\mathsf{l},0}(o(j^-))(\cdot,\cdot)$ and the depth $\mathrm{fr}_{\mathsf{l},1}(d_{j^-})(\cdot,\cdot)$. Let $X^\top = \sum_j X^\top_j$, and likewise $X^\perp$, X^{Q}, $X'^\top$, $X'^\perp$. The component of $\varphi X^\top$ at j is $\varphi(X^\top_{j^{1/p}})$, and the nearest cut at or before $j^{p^{-1}}$ is the nearest cut strictly before j; hence $\varphi X^\top = X^\perp$ and $\varphi X'^\top = X'^\perp$.

We apply Theorem A to $x = X^\top$, $w = X^{\mathsf{Q}}$, $x' = X'^\top$, with $(e_1, e_2, e_3) = (e, e', \kappa)$. The three elements lie over $p^e(1 \otimes c)$, $p^e(1 \otimes c)$, $p^{e'}(1 \otimes c')$, because $\sum_j \mathrm{cst}_{S,j}(c) = 1 \otimes c$. By Lemma 4.2 the four safety conditions may be checked on the components at each letter j, which lie in the parts at j. Fix j and $g \in \Gamma'$.

First condition: $(gX^\top_j - X^\top_j,\ g(hX'^\top_j - X'^\top_j))$. Let j° be the cut from which $X^\top_j$ was moved, at distance $a \geq 0$. Apply Lemma 5.3 at j with the third point n_{j° (and $l = a$), the point $X^\top_j$ of depth $\mathrm{fr}_{a,1}(d_{j^\circ})(\cdot,\cdot)$, and the element $hX'^\top_j - X'^\top_j$ of $\mathfrak{X}^0_{S'}(j; \mathrm{fr}_{a,0}(o'(j^\circ))(\cdot,\cdot))$. Condition (4) reads, for $\nu < a$, $0 < 0 + 1 + 0$; and for $\nu \geq a$ it is clause (a) of Definition 2.6 for S at j°, at the index $\nu - a$.

The other conditions involve $X^{\mathsf{Q}}_j - X^\top_j$, $X'^\perp_j - X'^\top_j$ and $X^\perp_j - X^{\mathsf{Q}}_j$. If j is not a cut, all three vanish and there is nothing to prove. Let j be a cut.

Case $\mathsf{Q}(j)$. Then $X^{\mathsf{Q}}_j = X^\top_j = x_j$; the second condition is trivial. The third is $(gx_j - x_j,\ g(X'^\perp_j - x'_j))$: by Lemma 4.1(e), $X'^\perp_j - x'_j \in \mathfrak{X}^0_{S'}(j; \min(\mathrm{fr}_{\mathsf{l},0}(o'(j^-))(\cdot,\cdot), o'(j)))$, and Lemma 5.3 applies with the point x_j of depth d_j and the third point at the cut $j^{p^{-r_j}}$; condition (4) is the second inequality of clause (b) for S at j. The fourth is $(X^\perp_j - x_j,\ gX'^\perp_j - X'^\perp_j)$: here $X^\perp_j - x_j \in \mathfrak{X}^0_S(j; \min(\mathrm{fr}_{\mathsf{l},0}(o(j^-))(\cdot,\cdot), o(j)))$ and $(z\tau)^{-1}(gX'^\perp_j - X'^\perp_j)$ has the exponents $\mathrm{fr}_{\mathsf{l},1}(d'_{j^-})(\cdot,\cdot)$; Proposition 5.2 applies with the third point at j^-. Condition (4) holds for $\nu < \mathsf{l}$ because $o(j)(\nu, u) \geq \mathrm{std}_{\mathsf{l}}(\nu, u) = 0$ by the first inequality of clause (b) for S at j; and for $\nu \geq \mathsf{l}$ by clause (a) for S' at j^-,

which has the term $\min(-u, o(j^-)(\nu - \mathsf{l}, u))$ on its right, together with $o(j)(\nu, u) \geq \mathrm{std}_{\mathsf{l}}(\nu, u) = -u$.

Case not $\mathsf{Q}(j)$. Then $X_j^{\mathsf{Q}} = X_j^{\perp}$; the fourth condition is trivial, and the other two are the mirror images of the previous case with the roles of S and S' exchanged: the second is $(X_j^{\perp} - x_j,\ gx'_j - x'_j)$, treated by Proposition 5.2 with the depth d'_j of x'_j and the third point at the cut $j^{p^{-r'_j}}$, by the second inequality of clause (b) for S' at j; the third is $(gX_j^{\perp} - X_j^{\perp},\ g(X_j'^{\perp} - x'_j))$, treated by Lemma 5.3 with the third point at j^-, by clause (a) for S at j^- and the first inequality of clause (b) for S' at j. □

Proof of Theorem C. We apply Theorem B with every letter a cut, so that $\mathsf{l}(i) = 1$ for all i, with $e = \kappa + (48(\rho + \eta + W) + 4)\mathfrak{d}$ and an integer $e' \leq e$, and with $z(i) = z'(i) = \mathrm{rs}_{-\rho}(\cdot, \cdot)$, which is non-increasing in u; the third points exist by Lemma 4.5 and hypothesis (i). Then $e + e' + \kappa \leq H$ by (v). Put $n_0 = 6(\rho + \eta + W)$ and $D = \mathrm{rs}_{2\rho+\eta}(\cdot, \cdot)$. Write i^- for $i^{p^{-1}}$.

First alternative of (iv). Take $\mathsf{Q}(i)$ to be $\mathsf{s}(i) = 1$. For S at i, Lemma 4.5 gives a point over $p^{\kappa}c$ with exponents and depth $\mathrm{rs}_{-\rho}(\cdot, \cdot)$, and Lemma 4.7(c) deepens it to D: the steps are at $\nu = 0$, from $-\rho - u$ to $2\rho + \eta - u$, and they lie in $[-n_0, n_0]$ for $|u| \leq \eta$ and within n_0 of $\pm\beta(i)$ for u in the far window. The result is a point over $p^e c$ with depth D and exponents $o(i) = \mathrm{imp}(\mathrm{rs}_{-\rho}(\cdot, \cdot), D)(\cdot, \cdot)$, and $r = 0$; the same for S', giving $o'(i)$ and $e' = e$. Explicitly $o(i)(\nu, u) = -u$ for $\nu \geq 1$, and $o(i)(0, u)$ is $2\rho + \eta - u$ for $u < -\rho$ and $\min(2\rho + \eta - u, 0)$ for $u \geq -\rho$. It remains to check clauses (a) and (b) of Definition 2.6. At $\nu = 0$ the left side of (a) is $-\rho - \eta - v - 1$, which is smaller than $-v$, than $2\rho + \eta - v$, and than 0 when $v \geq -\rho$; at $\nu \geq 1$ it is $-v - 1 < -v = o'(i)(\nu, v)$. So (a) holds for both sides. In (b), for the side that leads at i, the inequality $\mathrm{std}_1(0, u) = 0 \leq o(i)(0, u)$ asks $u \leq 2\rho + \eta$ when $u \geq -\rho$: the leading side has its far window on the negative side, so its jumps are at most η. The second inequality of (b) asks in addition, at $\nu = 0$, that $-\rho - \eta - v - 1 < 0$, which holds because the jumps v of the other side are at least $-\eta$; and at $\nu = 1$ that $-v - 1 < o'(i^{p^{-1}})(0, v)$, which fails exactly when $-\rho \leq v < 0$: this is excluded by the alternative. For $\nu \geq 2$ it reads $-v - 1 < -v$.

Second alternative of (iv). Now $\mathfrak{d} = 0$. Take $\mathsf{Q}(i)$ false for all i, so that S' leads everywhere. The points of S and of the third side are as before. For S', whose jumps are all ≤ 0, Input P9 gives $\mathrm{Kill}_{S'}(j; 0, 0)$ for every letter j, and Lemma 4.5 a point over c' at i^- with exponents and depth $\mathrm{rs}_0(\cdot, \cdot)$. Deepen it to D by Lemma 4.7(c): the steps are at $\nu = 0$ and lie in $[0, 2\rho + 2\eta)$. Apply φ (Lemma 4.7(b)); the depth becomes 1 at $\nu = 0$ and $D(\nu - 1, u)$ at $\nu \geq 1$. Deepen once more at $\nu = 0$, from 1 to $2\rho + \eta - u$. The result is a point over c' at i, so that $e' = 0$, with depth d' and exponents $o'(i)$ given by

$$d'(0, u) = d'(1, u) = 2\rho + \eta - u, \qquad o'(i)(0, u) = 0,$$
$$o'(i)(1, u) = 2\rho + \eta - u \ \text{ for } u < 0, \qquad o'(i)(1, 0) = 0,$$

and $d'(\nu, u) = o'(i)(\nu, u) = -u$ for $\nu \geq 2$. Take $r' = 1$ at the letters with $\mathsf{s}(i) = 1$ and $r' = 0$ at the others.

Clause (a) for S. The left side is $-\rho - \eta - v - 1$ at $\nu = 0$ and $-v - 1$ at $\nu \geq 1$, as before. At $\nu = 0$ the right side is $\min(-v, 0)$, and $-\rho - \eta - v - 1 < 0$ because $v \geq -\eta$. At $\nu = 1$ it is $\min(-v, o'(i)(1, v))$, which is larger than $-v - 1$ for $v < 0$ and equals 0 for $v = 0$. At $\nu \geq 2$ it is $-v$.

Clause (a) for S'. For $u' \in U_{Y_{S'}}$, $v \in U_{Y_S}$ the left side is $-\rho-\eta-v-1$ at $\nu=0$, $-2\rho-\eta-v-1$ at $\nu=1$ and $-v-1$ at $\nu\ge 2$. It is smaller than $-v$, and smaller than $o(i)(\nu,v)$: at $\nu\ge 1$ this is $-v$, and at $\nu=0$ it was checked in the first alternative.

Clause (b) for S'. First, $\mathrm{std}_1(\nu,u)\le o'(i)(\nu,u)$: at $\nu=0$ both are 0, and at $\nu\ge 1$ the left side is $-u$. Second, the right side is the minimum of $o(i)(\nu,v)$ and of the function equal to 0 at $\nu=0$ and to $o(i^-)(\nu-1,v)$ at $\nu\ge 1$. If $\mathsf{s}(i)=-1$, the left side is that of clause (a). At $\nu=0$ we need $-\rho-\eta-v-1<0$: the far window of S at i is on the positive side, so $v\ge-\eta$. At $\nu=1$ the left side is $-2\rho-\eta-v-1$ and the right side is $\min(o(i)(1,v),\,o(i^-)(0,v))$, with $o(i)(1,v)=-v$. For $v<-\rho$ the right side is $\min(-v,2\rho+\eta-v)=-v$, larger than the left side; for $v\ge-\rho$ it is $\min(-v,2\rho+\eta-v,0)=\min(-v,0)$, and the left side is smaller than $-v$ and negative. At $\nu\ge 2$ we need $-v-1<-v$. If $\mathsf{s}(i)=1$, the third point is moved from i^-, and the left side is $-(2\rho+\eta-u')$ at $\nu=0$, $-\rho-\eta-v-1$ at $\nu=1$ and $-v-1$ at $\nu\ge 2$. At $\nu=0$ it is negative, because $\eta\ge 1$ and $u'\le 0$, while $\min(o(i)(0,v),0)=0$: the far window of S at i is on the negative side, so $v\le\eta$ and $o(i)(0,v)\ge 0$. At $\nu=1$ we need $-\rho-\eta-v-1<\min(-v,\,o(i^-)(0,v))$, which holds as in the previous case, and at $\nu\ge 2$ again $-v-1<-v$. □

7 The long gap

Throughout, the setting is Setting 2.1.

Proposition 7.1 (a relation among elements of the same weight)**.** *Suppose that the Kummer chart is unramified. Let L be a lattice datum, o an exponent function, i a letter, $u\in\mathbb{Z}$ and $n\ge 0$ an integer with $m_L(i,u)\le n$, where $m_L(i,u)$ is the total multiplicity of the jump u of L above i. Let $\nu_0\ge 0$ be an integer and j a letter with $j^{p^{-\nu_0}}=i$, and suppose that $o(\nu_0,u)<o(\nu_0,v)$ for every $v\in U_L(i)$ with $v<u$. Let $x_0,\dots,x_{hn}\in\mathfrak{M}_{\mathrm{loc}}(L)$ with $x_k\in\mathfrak{X}_L(j;o)$ for all k. Then there are $c_0,\dots,c_{hn}\in\mathbb{Z}_p$, one of them equal to 1, with*

$$\sum_k c_k x_k \;\in\; (\mathfrak{X}_L(j;o_+))_{\mathfrak{S}} + (\mathfrak{X}_L(j;\mathrm{up}_{\nu_0,u}(o)(\cdot,\cdot)))_{\mathfrak{S}},$$

where $o_+(\nu,v)=o(\nu,v)+1$ for $\nu=0$ and $o_+(\nu,v)=o(\nu,v)$ otherwise.

LEAN `Lift.KummerChart.JumpLattice.exists_comb_of_jumpMultiplicity_le`

Proof. Let $\mu_1,\dots,\mu_r$, $r\le n$, be the periods of Input J6 for (i,u), and $a=o(\nu_0,u)$. By Input P4 and the hypothesis on o, $\mu_m(\varphi^{-\nu_0}x_k)\in\mathrm{Fil}^a B_{\mathrm{dR}}$; write it as $\pi^a t_{km}$ with a fixed uniformizer π and $t_{km}\in B^+_{\mathrm{dR}}$, with residue $\bar t_{km}\in C$. By Auerbach's lemma (Input J7) applied to the column space of the matrix $(\bar t_{km})$, there are indices $s_1,\dots,s_q$, $q\le r$, and $\bar a_{kb}=\mathsf{z}_b(k)\in\mathcal{O}_C$ with $\bar t_{km}=\sum_b \bar a_{kb}\bar t_{s_b m}$ for all k, m. Lift $\bar a_{kb}$ to $a_{kb}\in A_{\mathrm{inf}}$ along θ and put $y_k=x_k-\sum_b\varphi^{\nu_0}(a_{kb})x_{s_b}\in\mathfrak{X}_L(j;o)$. Then $\mu_m(\varphi^{-\nu_0}y_k)\in\mathrm{Fil}^{a+1}B_{\mathrm{dR}}$ for all m; since every period of type u at i is a combination of the μ_m, with coefficients in $\bar{\mathbb{Q}}_p$, modulo periods of type $u-1$, on which the value lies in $\mathrm{Fil}^{a+1}B_{\mathrm{dR}}$ by Input P4, we get $y_k\in\mathfrak{X}_L(j;\mathrm{up}_{\nu_0,u}(o)(\cdot,\cdot))$, and the same for $\mathrm{ret}_L(y_k)=x_k-\sum_b\mathrm{co}_0(\varphi^{\nu_0}a_{kb})x_{s_b}$. Write $\mathrm{co}_0(\varphi^{\nu_0}a_{kb})=w_{kb}+\mathcal{E}y_{kb}$ with $w_{kb}\in\mathbb{W}$ (Input J9). The $hn+1$ vectors $(w_{kb})_b\in\mathbb{W}^q$ lie in a free $\mathbb{Z}_p$-module of rank $hq\le hn$,

so they satisfy a non-trivial relation over $\mathbb{Z}_p$; dividing by a coefficient of least valuation we get $c_k \in \mathbb{Z}_p$, one of them 1, with $\sum_k c_k w_{kb} = 0$ for all b. Then

$$\sum_k c_k x_k = \sum_k c_k \, \mathrm{ret}_L(y_k) + \mathcal{E} \cdot \sum_{k,b} c_k \, y_{kb} \, x_{s_b},$$

and multiplication by $\mathcal{E}$ raises the exponents at $\nu = 0$ by 1. □

Lemma 7.2 (coefficients of an expansion in powers of $\mathcal{E}$)**.** *Let L be a lattice datum, $l > 0$ and $A \geq 0$ integers, i a letter, and let $\mathsf{w} \colon \mathfrak{S} \to \mathbb{W}$, $\mathsf{t}_0 \colon \mathfrak{S} \to \mathfrak{S}$ be maps with $s - \mathsf{w}(s) = \mathcal{E}\,\mathsf{t}_0(s)$ for all $s \in \mathfrak{S}$. Let $v_1, \dots, v_m \in \mathfrak{K}_{l,A,i}(L)$, $c \in \mathfrak{K}_{l,A,i}(L)$, $n \geq 0$ an integer, $s_1, \dots, s_m \in \mathfrak{S}$, f a polynomial with integer coefficients, with f_n its coefficient of degree n, and $o_1 \colon \mathbb{Z} \to \mathbb{Z}$ a function with $o_1(u) \leq A$ for all u. Suppose that $\sum_a s_a v_a - f(\mathcal{E})\, c \in \mathfrak{X}_L(i; \mathrm{sh}_l(o_1)(\cdot,\cdot))$. Then the element*

$$d = \sum_a \mathsf{w}(\mathsf{t}_0^n(s_a)) v_a - f_n \, c$$

lies in $\mathfrak{K}_{l,A,i}(L)$ and in $\mathfrak{X}_L(i; \mathrm{sh}_l(o_3)(\cdot,\cdot))$, where $o_3(u) = 0$ if $o_1(u) \leq n$ and $o_3(u) = A$ otherwise.

LEAN `Lift.KummerChart.JumpLattice.coeff_mem_constantsModE` · the formal statement introduces d by a local definition inside the statement

Proof. The module $\mathfrak{K}_{l,A,i}(L)$ is stable under $\mathbb{W}$, whose elements are fixed by Γ'; so d lies in it. By induction on n. Write $s_a = \mathsf{w}(s_a) + \mathcal{E}\mathsf{t}_0(s_a)$ and $f = f_0 + X f_1$:

$$\sum_a s_a v_a - f(\mathcal{E}) c = d_0 + \mathcal{E}\, y, \qquad d_0 = \sum_a \mathsf{w}(s_a) v_a - f_0 c, \quad y = \sum_a \mathsf{t}_0(s_a) v_a - f_1(\mathcal{E}) c.$$

Here y has the exponents std_l, so $\mathcal{E}y$ has them raised by 1 at $\nu = 0$, and d_0 has the exponents $\mathrm{sh}_l(\min(1, o_1))(\cdot,\cdot)$. Its Galois differences have the exponent A at $\nu = 0$. By Input P6, d_0 has at $\nu = 0$ the exponent A where $o_1(u) \geq 1$ and $\min(A, 0) = 0$ elsewhere: this is the case $n = 0$. For the step, d_0 has the exponents $\mathrm{sh}_l(o_1)(\cdot,\cdot)$, hence so has $\mathcal{E}y$, and y has the exponents $\mathrm{sh}_l(o_1 - 1)(\cdot,\cdot)$. The induction hypothesis for $(\mathsf{t}_0(s_a))_a$, f_1, $o_1 - 1$ and n gives the claim for $n + 1$. □

Lemma 7.3 (projections that are constant modulo a power of $\mathcal{E}$)**.** *Let L be a lattice datum, $l > 0$ and $A \geq 0$ integers, i a letter and $x \in W(C^\flat) \otimes L$. Suppose that x is a point over F_0 of $\varphi^l \mathfrak{M}^{\mathrm{inf}}(L)$ and that $g\mu(x) - \mu(x) \in \mathrm{Fil}^A B_{\mathrm{dR}}$ for all $\mu \in \mathrm{Per}(L)$ and $g \in \Gamma'$. Then $\mathrm{pr}_i(x) \in \mathfrak{K}_{l,A,i}(L)$.*

LEAN `Lift.KummerChart.JumpLattice.letterProj_mem_constantsModE`

Proof. $\mathrm{pr}_i(x) \in \mathfrak{N}_{l,i}(L)$ by Input P7. Its Galois differences have the exponents std_l; at $\nu = 0$, where $\mathrm{std}_l = 0$ because $l > 0$, the value of a period μ on $g\,\mathrm{pr}_i(x) - \mathrm{pr}_i(x)$ is $(g-1)(\mu \circ \mathrm{pr}_i)(x) \in \mathrm{Fil}^A B_{\mathrm{dR}}$. Input P2 concludes. □

Proof of Theorem D. A multiple in the Frobenius pullback. By Lemma 4.5 there is $y \in \mathrm{Pt}_S(c, i; 0; \mathrm{rs}_{-\rho}(\cdot,\cdot), \mathrm{rs}_{-\rho}(\cdot,\cdot))$. As the chart is unramified, $\mathcal{E}_a = z^{p^a} + p = (\mathcal{E} - p)^{p^a} + p$. Let

$$f = X^\rho \prod_{0 \leq a < l} \left((X - p)^{p^a} + p\right)^\gamma,$$

a monic polynomial of degree N with integer coefficients; then $f(\mathcal{E})y = \mathcal{E}^\rho \prod_{a<l} \mathcal{E}_a^\gamma \, y$ has the exponents $\gamma - u$ at $\nu < l$ and $-u$ at $\nu \geq l$ (Input P2). Since the jumps at

these ν are at most γ, it has the exponents std_l, so it is a point over F_0 of $\varphi^l \mathfrak{M}^{\mathrm{inf}}(\tilde{Y}_S)$ (Input P7).

Spanning. By Input J8, $p^{4\delta} f(\mathcal{E}) y = \sum_a r_a v_a$ with $r_a \in \mathfrak{S}$ and points v_a whose values are Γ'-invariant modulo $\mathrm{Fil}^A B_{\mathrm{dR}}$. Projecting to the letter i and using Lemma 7.3: $p^{4\delta} f(\mathcal{E}) y = \sum_a r_a \,\mathrm{pr}_i(v_a)$ with $\mathrm{pr}_i(v_a) \in \mathfrak{K}_{l,A,i}(\tilde{Y}_S)$.

The coefficient of degree N. Choose w, t_0 as in Lemma 7.2 (Input J9) and put

$$d = \sum_a \mathsf{w}(\mathsf{t}_0^N(r_a))\ \mathrm{pr}_i(v_a).$$

Apply the lemma with $c = 0$, $f = 0$ and $o_1(u) = \min(A, \gamma - u)$: d lies in $\mathfrak{K}_{l,A,i}(\tilde{Y}_S)$ and has at $\nu = 0$ the exponent A where $\gamma - u > N$, and 0 elsewhere. Apply it also in C_S^{tr}, to the images of the $\mathrm{pr}_i(v_a)$, to the element $\mathrm{cst}_{S,i}(c)$, the polynomial $p^{4\delta} f$ and the constant function A: the hypothesis holds because y lies over $\mathrm{cst}_{S,i}(c)$, and as f is monic of degree $N < A$ the conclusion is that

$$a_0 = (1 \otimes \mathrm{pr}_S^C)(d) - p^{4\delta}\,\mathrm{cst}_{S,i}(c)$$

has the exponents $\mathrm{sh}_l(o_2)(\cdot,\cdot)$ in C_S^{tr}, o_2 the constant A. So $\mathcal{E}^{-A} a_0 \in \mathfrak{N}_{l,i}(C_S^{\mathrm{tr}})$.

The point. From $\mathrm{Kill}_S(i^{p^{-l}}; \kappa, 0)$ we get $p^\kappa \in \mathfrak{a}_S(0; l, i)$, so by Input P9 there is $v \in \mathfrak{N}_{l,i}(\tilde{Y}_S)$ over $p^\kappa \mathcal{E}^{-A} a_0$. The element $x = p^\kappa d - \mathcal{E}^A v$ lies in $\mathfrak{M}_{\mathrm{loc}}(\tilde{Y}_S)$, over $p^{\kappa+4\delta}\,\mathrm{cst}_{S,i}(c)$. At $\nu = 0$ its exponent is at least $\min(A, -u - N)$: the term $\mathcal{E}^A v$ has A, and $p^\kappa d$ has A or, where $\gamma - u \le N$, the exponent $0 \ge -u - N$. Elsewhere the exponents are std_l. The Galois differences of both terms have the exponents $\mathrm{sh}_l(o_2)(\cdot,\cdot)$. □

8 Where the results are used

Part 3 of the notes on this unit applies Theorem B with cuts chosen on the Frobenius orbits of the letters and windows around them, Theorem C for lattices whose weights lie in a low and a far window, Theorem D and Proposition 7.1 for classes with few low weights, and the lemmas on points for the cokernel killer of tame types. Its two results, a length bound and an exponent, are the inputs of the bound for Selmer groups at ℓ in the unit on Selmer groups.

9 Notes on the formal text

Arguments as the text gives them. The arguments were read from the Lean text and the census of the workspace, and were not printed by Lean. The census reads a bound variable as an explicit argument in the definition of the cost of a map (a letter) and in the statement of Proposition 4.6; the note states what the Lean text binds. The property of Definition 2.6 has eleven arguments, which the note writes out in words. Two statements contain a local definition before their conclusion, which a program that cuts a statement at its first definition sign misses: Lemma 5.1 and Lemma 7.2; both were read in the source in full. The hypotheses of Proposition 5.2 and Lemma 5.3 are section hypotheses of the formal text, included in these two declarations by name.

Differences from the natural statements. A scaled cup product is asked to bound for every topology on Γ that makes j_Γ an open embedding; no topology is part of the

data. In Theorem B the function l is an argument together with the hypothesis that it gives the nearest cut; so every Frobenius orbit of letters must contain a cut, and the statement of Theorem B says so. The families z' of depths of the third points occur in the hypotheses of Theorem B and are not used in its proof. In Definition 2.6 the integer r and the property that $i^{p^{-r}}$ is a cut locate the third point used with clause (b). In Theorem C the same pair (κ, ρ) kills the three cokernels at every letter. In Proposition 7.1 the formal text takes $hn + 1$ elements indexed from 0.

Arithmetic. In the proofs of Theorems B and C the formal text settles the comparisons of exponents by a decision procedure for linear arithmetic. The note gives them by hand: in Theorem B as the reduction of each safety condition to a clause of Definition 2.6, in Theorem C as the explicit inequalities of the two alternatives. In the second alternative of Theorem C the hypothesis $p \geq 5$ enters only through the cokernel killer of Part 1, and $\eta \geq 1$ only at $\nu = 0$ for the letters with $\mathsf{s}(i) = 1$.

The background. The period ring is Colmez's $B_{\max}$. For the Artin–Schreier–Witt input the second reader of the workspace finds the Frobenius surjectivity in the literature and no source for the full statement used here; for Auerbach's lemma a determinant-maximization extension; for Tate's theorem a weaker form. No place in the literature was confirmed for the frame theorem and for the stability of the class of lattice data.

How this note was written. From the reader's dossier of the unit: the statements of all members and the structured proofs of the declarations treated here. The two Lean files were read in full. The proofs of Theorem A and of Theorem B, which the reader had marked as least certain, were read line by line in the Lean text: the identities for D_g in the proof of Theorem A were recomputed by hand, and the four non-trivial safety conditions of the walk, with the lemma and the third point used in each, were matched with the four tactic blocks of the formal proof. The inequalities in the proof of Theorem C, in both alternatives, were worked out by hand for this note; the formal proof leaves them to automation, so the account of where the windows, the excluded interval and the hypothesis $\eta \geq 1$ enter is ours. The proof of Theorem D and of Proposition 7.1 follow the formal proofs. The program check of the unit's dossier covered 375 leaf steps of 69 proofs.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Definition 2.3	`Lift.KummerChart.PairingSide.wittFst`	CupProductBound.lean	40–40
Definition 2.3	`Lift.KummerChart.PairingSide.tensorLattice`	CupProductBound.lean	44–45
Definition 2.3	`Lift.Pairing.thirdSide`	CupProductBound.lean	51–53
Definition 2.3	`Lift.Pairing.wittPairing`	CupProductBound.lean	57–58
Definition 2.4	`Lift.Pairing.IsSafe`	CupProductBound.lean	62–62
Definition 2.4	`Lift.Pairing.ScaledCupBounds`	CupProductBound.lean	66–67
Lemma 3.1	`Lift.wittFst_smul`	CupProductBound.lean	69–71
Lemma 3.1	`Lift.Pairing.smul_wittPairing`	CupProductBound.lean	73–74
Theorem A	`Lift.Pairing.ScaledCupBounds.of_isSafe`	CupProductBound.lean	83–122
Definition 2.3	`Lift.KummerChart.tateLattice`	LongGap.lean	25–27
Lemma 4.1	`Lift.KummerChart.PairingSide.stepBack_zero`	LongGap.lean	37–37
Definition 2.3	`Lift.KummerChart.PairingSide.inclKer`	LongGap.lean	40–40
Definition 2.3	`Lift.KummerChart.PairingSide.projTrivLoc`	LongGap.lean	44–45

In this note	Lean declaration	File	Lines
Definition 2.5	`Lift.KummerChart.PairingSide.deepElements`	LongGap.lean	50–51
Definition 2.5	`Lift.KummerChart.PairingSide.kerWithExponents`	LongGap.lean	55–56
Definition 2.5	`Lift.KummerChart.PairingSide.Reaches`	LongGap.lean	61–62
Definition 2.3	`Lift.KummerChart.PairingSide.SidePairing`	LongGap.lean	65–65
Definition 2.5	`Lift.KummerChart.PairingSide.pointsOver`	LongGap.lean	69–70
Definition 2.6	`Lift.KummerChart.PairingSide.HasDeepPoint`	LongGap.lean	75–79
Lemma 4.1	`Lift.KummerChart.PairingSide.wittFst_inclKer`	LongGap.lean	88–89
Lemma 4.2	`Lift.KummerChart.PairingSide.isSafe_sum`	LongGap.lean	91–100
Lemma 4.3	`Lift.KummerChart.PairingSide.exists_extend_form`	LongGap.lean	105–120
Lemma 4.1	`Lift.KummerChart.PairingSide.Reaches.mem_of_stable`	LongGap.lean	122–124
Lemma 4.1	`Lift.KummerChart.PairingSide.kerWithExponents_le_deepElements`	LongGap.lean	126–127
Lemma 4.4	`Lift.KummerChart.PairingSide.exists_inclKer_eq_of_projTriv_eq_zero`	LongGap.lean	132–148
Lemma 4.5	`Lift.KummerChart.PairingSide.pointsOver_nonempty_of_killsCokernel`	LongGap.lean	153–159
Proposition 4.6	`Lift.KummerChart.PairingSide.exists_reaches_deep`	LongGap.lean	165–191
Lemma 4.7	`Lift.KummerChart.PairingSide.pointsOver_nonempty_of_costs`	LongGap.lean	197–206
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Theorem B	`Lift.KummerChart.PairingSide.SidePairing.scaledCupBounds_of_points_at_cuts`	LongGap.lean	273–309
Lemma 4.7	`Lift.KummerChart.PairingSide.pointsOver_nonempty_of_noMultipleNear`	LongGap.lean	317–323
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Definition 2.7	`Lift.KummerChart.JumpLattice.raiseExponent`	LongGap.lean	359–361
Proposition 7.1	`Lift.KummerChart.JumpLattice.exists_comb_of_jumpMultiplicity_le`	LongGap.lean	366–404
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Definition 2.7	`Lift.KummerChart.JumpLattice.constantsModE`	LongGap.lean	421–422
Lemma 7.2	`Lift.KummerChart.JumpLattice.coeff_mem_constantsModE`	LongGap.lean	430–448
Lemma 7.3	`Lift.KummerChart.JumpLattice.letterProj_mem_constantsModE`	LongGap.lean	453–458
Theorem D	`Lift.KummerChart.PairingSide.pointsOver_nonempty_of_long_gap`	LongGap.lean	463–496

Two statements as the artifact has them:

```
lemma Pairing.ScaledCupBounds.of_isSafe {e₁ e₂ e₃ c c'} x w x' (he : e₁ + e₂ + e₃ ≤ e) (hx :
  S.projTriv.wittMap x = p ^ e₁ • .mk 1 c) (hw : S.projTriv.wittMap w = p ^ e₁ • .mk 1 c) (hx' :
  S'.projTriv.wittMap x' = p ^ e₂ • .mk 1 c') (H : ∀ g : k.galMonoid, (∀ h :
  k.galMonoid, P.IsSafe e₃ (g • x - x) (g • (h • x' - x'))) ∧ P.IsSafe e₃ (w - x) (g • x' - x') ∧
  P.IsSafe e₃ (g • w - w) (g • (φ • x' - x')) ∧ P.IsSafe e₃ (φ • x - w) (g • φ • x' - φ • x')) :
  P.ScaledCupBounds e c c'

def HasDeepPoint (cut : _ → Prop) l (o o' z : _ → ℕ → ℤ → ℤ) (Q : Prop) := ∃ d r,
    (S.pointsOver c' i' e (o i') d).Nonempty ∧ cut (i'.stepBack r) ∧ ∀ a, ∀ u ∈ S.U (i'.stepBack a),
  ∀ v ∈ S'.U (i'.stepBack a), -z i' a (-u - v - 1) - d a u < min (-v) (o' i' a v) ∧
  (Q → stdExponent l a u ≤ o i' a u ∧ -frobExponent r 0 (z (i'.stepBack r)) a (-u - v - 1) - d a u <
  min (o' i' a v) (frobExponent l 0 (o' (i'.stepBack l)) a v))
```

Legend. As in Part 1; in addition `P` or `h` is the pairing Π and `P.m` is M; `smallPart` is $\mathfrak{s}_M$; `zmodPowToUnits` is e_M; `thirdPairing` is p_Π; `toZp` is tr; `cut`, `Q` and `l` of the walk are cut, Q and l; the integers `s`, `a` that are exponents of $\mathcal{E}$ are ρ, `b` is β, `ε` is `s`, and `g` of the long gap is γ.

Notation

ℓ the prime of Chapter VIII of the handout; it is p here

p, $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\bar{\mathbb{Q}}_p$, C, $\mathcal{O}_C$, $G_{\mathbb{Q}_p}$ the prime, the p-adic numbers and integers, an algebraic closure, its completion, the integers, the Galois group

A_{inf}, $W(C^\flat)$, $W(\mathfrak{m}^\flat)$, $\mathcal{O}_C^\flat$, $C^\flat$, $\mathfrak{m}^\flat$ Witt vectors of the tilt, of its fraction field, of its maximal ideal

φ, θ, ε, τ, $[\cdot]$ Frobenius, Fontaine's map, a system of roots of unity, $[\varepsilon^{1/p}]-1$, the Teichmüller lift

B_{dR}, B_{dR}^+, $\mathrm{Fil}^n B_{\mathrm{dR}}$ the de Rham period rings and the filtration

$A_{\max}$, B^+, $B_{\max}$ Colmez's period rings (Setting 2.1; Part 1, Setting 2.1): a p-adic completion, $A_{\max}[1/p]$, $B^+[1/([\varepsilon]-1)]$

Γ, Γ', j_Γ, χ, Δ the group, the subgroup of the chart, the map to $G_{\mathbb{Q}_p}$, the cyclotomic character, an open subgroup

$\mathfrak{c}$, $\mathfrak{S}$, $\mathbb{W}$, h, X, z, $\mathcal{E}$, $\mathcal{E}_\nu$, $e_{\mathfrak{c}}$, $\mathfrak{d}$ the chart, its ring, $W(\mathbb{F}_{p^h})$, the residue degree, the variable, the coordinate, the Eisenstein element and its translates, the index, the different exponent

F, F_ν, co_0 the field fixed by Γ', the value fields, the retraction $A_{\mathrm{inf}} \to \mathfrak{S}$

$\mathcal{O}$, ω, $\mathcal{O}(1)$ coefficients, the root of unity, the Tate twist

i, j, j', i^-, j^-, j° letters; $i^- = i^{p^{-1}}$; the nearest cut before j, the cut from which a point is moved

L, L', L'', A, T, T', Y, Y' lattice data and lattices; A is also an integer

$U_L(i)$, $m_L(i,u)$ the jump set and the multiplicity of a jump at a letter

pr_i, $\mathrm{Per}(L)$, $\mathrm{Per}_{i,u}(L)$ projector and periods

$\mathfrak{M}^{\mathrm{inf}}(L)$, $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{loc}}(L)$, $\mathfrak{M}_{\mathrm{loc}}(L)$, ret_L Kisin modules and the retraction

$\mathfrak{X}_L(i;o)$, $\mathfrak{N}_{l,i}(L)$, $(Q)_{\mathfrak{S}}$, $\mathrm{Inv}(Q)$ exponent modules; elements of $\mathfrak{M}_{\mathrm{loc}}$; invariants modulo Q

std_l, $\mathrm{cost}(\nu,n)$, $\mathrm{imp}(o^{\mathrm{lo}},o^{\mathrm{hi}})(\nu,u)$, $\mathrm{fr}_{l,n}(o)(\nu,u)$ standard exponents, cost, improved and shifted exponents

$b_W(y,x)$ the extension of a bilinear map b

S, S', Y_S, C_S, α_S sides and their parts

$\tilde{Y}_S$, C_S^{tr}, pr_S^C, pr_S^Y, in_S^Y, $\mathrm{pr}_S^{C,\mathrm{loc}}$ extension, trivial datum, projections, inclusion

$\mathrm{cst}_{S,i}(c)$, $\mathfrak{a}_S(r;l,i)$, $\mathrm{Kill}_S(i;\kappa,\rho)$ constants, lifting ideal, cokernel killer

$\mathrm{rs}_n(\nu,u)$, $\mathcal{W}_n(c)$, $\mathrm{NM}(p;n,\beta)$, $U \subseteq [-\eta,\eta] \cup \mathsf{s}\cdot[\beta,\beta+W]$ raised exponents and windows

Π, M, ψ, tr, $\overline{\mathrm{tr}}$, c_Π, $\langle\ ,\ \rangle$, p_Π a pairing and its parts; the third pairing

$\mathbb{M}_M$, $\mathfrak{s}_M$, e_M the module of values, its small part, the embedding of $\mathbb{Z}/p^M$

$\mathsf{T}_{S,S'}$, $\mathcal{O}(1)$, $S_\Pi^\vee$, b_Π tensor datum, Tate datum, third side, extended pairing

$\mathrm{Safe}_\Pi(e;w,y)$, $\mathrm{Cup}_\Pi(e;c,c')$ a safe pair; the scaled cup product bounds

$\mathfrak{X}^0_S(i;o)$, $\mathrm{Deep}_S(i;o)$, $\mathrm{Cost}_S(\mathsf{A}) \le e$, $\mathrm{Pt}_S(c,i;e;o_1,o')$ kernel with exponents, deep elements, cost of a map, points

$\mathrm{up}_{\nu,u}(o)(\nu',v)$, $\mathrm{sh}_l(o_1)(\nu,u)$, $\mathfrak{K}_{l,A,i}(L)$ an exponent raised at one place; shifted standard exponents; constants modulo $\mathcal{E}^A$

cut, Q, q, $\mathsf{l}(i)$ properties of letters, a proposition, the distance to the nearest cut

o, o', o'', $o_1,\dots,o_5$, o_+, o^{lo}, o^{hi}, d, d', d_j, d'_j, D, $z(i)$, $z'(i)$ exponent functions and families of them; o_1, o_2, o_3 are also functions of u alone; z is also the coordinate

ν, ν', ν_0, l, u, u', u_3, v indices, levels, jumps; v is also an element

e, e', e_1, e_2, e_3, E, κ, ρ, ϵ, H, N, W, η, β, γ, δ, n, n_0, m, q, r, r', r_j, r'_j, a, b, k integers; n is also a pair or a point, q also an element, r also a power of p, k also a cochain or an index, b also a bilinear map

s a sign

g, h, t_k elements of Γ; h is also the residue degree

x, x', x_1, x_2, x_j, x'_j, x_k, w, w_0, y, y', y_0, y_c, y_k, y_{kb}, n_j, n_1, v_j, u_j, v_a, a_0, c, c', c_0, c_k, s, s_a, r_a, f, f_n, f_0, f_1, t, t', λ, π elements of modules, lattices and rings; f is also an equivariant map

$X_j^\top$, $X_j^\perp$, X_j^Q, $X_j'^\top$, $X_j'^\perp$ and their sums the points of the walk

ξ, ξ', ξ_w, a_g, a'_g, B, D_g, y_0, $k(g)$, c the objects of the proof of Theorem A; B is also a map in Proposition 4.6

μ, μ', μ_C, μ_m, Λ, a_c, V periods, a linear map, elements of F and of B_{dR}

A, A_0, I, I_0 additive maps and sets of steps

m, t scalar multiplication and the tensor map

t_{km}, $\bar{t}_{km}$, a_{kb}, $\bar{a}_{kb}$, w_{kb}, s_b, U, z_b, v the matrices of the proof of Proposition 7.1 and the objects of Auerbach's lemma

w, t_0 remainder and quotient of the division by $\mathcal{E}$

M, N, Q submodules; M is also the exponent of the pairing, N also an integer

Crystalline classes with few low weights, sparse large weights and tame types: a length bound and rank-free exponents

One unit of the Lean artifact grh-compact, read back into mathematics

Summary. This note draws the consequences of Parts 1 and 2 that the bound for Selmer groups at ℓ uses. For a crystalline lattice L over an unramified base whose weights lie in a low window and a far window, the crystalline classes of L whose cup product with every crystalline class of every dual-type lattice bounds have, inside all crystalline classes, a colength that is linear in the number of low weights and in the rank, with an explicit constant; and all crystalline classes have this property when no weight is small and negative. For lattices of tame inertial type over $\mathbb{Q}_p$, a power of p whose exponent does not depend on the ramification index makes every crystalline class orthogonal in this sense: the cokernel of Kisin modules is killed by a power of p read off a polynomial that kills the tame generator. Two further rank-free exponents are proved, for weights of two sizes and for sparse large weights, from a general statement with cuts on the Frobenius orbits.

What this note is. The artifact grh-compact is a body of Lean code: the compact edition of a formal proof, due to L. Alpöge with the AI system Claude, of the Riemann hypothesis and of its generalisation to Dirichlet L-functions. That headline is a claim. The authors report that every file compiles with Lean v4.35.0 against a pinned version of Mathlib and that Lean's kernel accepted the proofs; an independent build is recorded in the handout's record of checks when it reports. No mathematician has read the argument through, and nothing here says that the Riemann hypothesis is proved. This note is the third of three on one unit of the artifact, the folder on lifting; it treats 19 of the unit's 141 declarations, those of the last three files. Theorems A and B, Proposition 3.3 and the results on tame charts are proved in full from Parts 1 and 2 and from Inputs of the background library, which are stated and not proved. Proposition 5.2 is proved in full as well, including the final comparisons of exponents, which the formal text settles by a decision procedure for linear arithmetic and which are carried out by hand here; Theorems C and D are deduced from it. No proof of this note is partial. Two Inputs of the background library, the crystalline criterion K3 and the finiteness of the module of crystalline cocycles K5, are statements of the formal library that are close to, but not the same as, published theorems; the boxes say exactly how. Every statement was compared with its Lean text, argument by argument. The Lean text read is that of commit `72fead10`; Appendix A is the concordance of 19 declarations.

1 The statements

The objects are those of Setting 2.1: the Kummer chart with its letters, lattices over a coefficient ring $\mathcal{O}$, classes $\mathrm{Cl}(\mathcal{F}, \mathsf{K}; U)$ of lattices, the module $L^{\perp_Q}$ of the maps orthogonal to a class, and the notation of Parts 1 and 2.

Theorem A (few low weights). *In Setting 2.1, let Γ carry a topology for which j_Γ is an open embedding. Let $\mathfrak{c}$ be a Kummer chart for Γ that is unramified, with residue degree h, let $\omega \in \mathcal{O}$ with $\omega^{p^h-1} = 1$, let L be a lattice over $\mathcal{O}$, and let $U = (U(\gamma))_\gamma$ and $U' = (U'(\gamma))_\gamma$ be families of finite sets of integers indexed by the $\mathbb{Z}_p$-algebra maps $\gamma\colon \mathcal{O} \to C$. Suppose $p \geq 5$. Let $\rho, \eta, W \geq 0$ be integers, and for every letter i let $\beta(i) \geq 0$ be an integer and $\mathsf{s}(i)$ a sign, such that for every letter i*

$$\beta(i) + W + \eta \leq (p-3)\rho, \qquad \eta \geq 1,$$
$$6(\rho + \eta + W) < \beta(i), \qquad \mathrm{NM}(p; 6(\rho+\eta+W), \beta(i)),$$

$\mathcal{U}_{\mathfrak{c},\omega}(U)(i) \subseteq [-\eta, \eta] \cup -\mathsf{s}(i) \cdot [\beta(i), \beta(i)+W]$ and $\mathcal{U}_{\mathfrak{c},\omega}(U')(i) \subseteq [-\eta, \eta] \cup \mathsf{s}(i) \cdot [\beta(i), \beta(i)+W]$. Suppose that $L \in \mathrm{Cl}(\top, \bot; U')$. Then:

1. *if no element of $U(\gamma) \cup U'(\gamma)$ lies in $[-\rho, 0)$, for any γ, or if $U'(\gamma) \subseteq [-\eta, 0]$ for every γ, then $Z^1_{\mathrm{f}}(L) \subseteq L^{\perp \mathrm{Cl}(\top,\bot;U)}$;*
2. *for every integer $m_0 \geq 0$ such that the multiplicities of the jumps $u \in [-\eta, \eta]$ of L at γ have sum at most m_0, for every γ,*

$$\begin{aligned} &\mathrm{len}_{\mathcal{O}}(Z^1_{\mathrm{f}}(L)/(Z^1_{\mathrm{f}}(L) \cap L^{\perp \mathrm{Cl}(\top,\bot;U)})) \\ \leq\ & 11\, h\, (m_0(\rho+1) + \mathrm{rk}_{\mathcal{O}}(L)\,(W+1)) \cdot \mathrm{len}_{\mathcal{O}}(\mathcal{O}/p\mathcal{O}). \end{aligned}$$

LEAN `Lift.KummerChart.few_low_jumps`

Theorem B (tame types: an exponent independent of the ramification index). *In Setting 2.1, let Γ carry a topology for which j_Γ is an open embedding, and suppose that j_Γ is surjective. Let L be a lattice over $\mathcal{O}$, U and U' families of finite sets of integers indexed by the maps $\gamma\colon \mathcal{O} \to C$, $\mathcal{F}$ a filter on Γ, K a relation between elements of Γ and polynomials over $\mathcal{O}$, and $\mathsf{r}, \mathsf{n} \geq 0$ integers. Write K^- for the relation $(s, Q) \mapsto \mathsf{K}(s^{-1}, Q)$. Suppose that $L \in \mathrm{Cl}(\mathcal{F}, \mathsf{K}^-; U')$, and that $\mathrm{Tame}_p(\mathsf{r}, \mathsf{n}; \mathsf{K}(s, \cdot))$ for every $s \in \Gamma$ with $j_\Gamma(s)$ in the inertia group. Let $\eta, W, \beta \geq 0$ be integers and s a sign with $U(\gamma) \subseteq [-\eta, \eta] \cup -\mathsf{s} \cdot [\beta, \beta+W]$ and $U'(\gamma) \subseteq [-\eta, \eta] \cup \mathsf{s} \cdot [\beta, \beta+W]$ for every γ, and suppose*

$$\beta + 6(W+\eta) + \mathsf{r}^2(\mathsf{n}+1)^2 + 2 < p.$$

Then for every $a \in Z^1_{\mathrm{f}}(L)$,

$$p^{3\mathsf{r}^2(\mathsf{n}+1)^2 + 150(\eta+W+1)} \cdot a \ \in\ L^{\perp \mathrm{Cl}(\mathcal{F},\mathsf{K};U)}.$$

LEAN `Lift.tame_types_exponent`

Theorem C (sparse thick letters). *In Setting 2.6, of whose objects only S, S', Π, thick, s, β, η, V, W, X, B, J and G are used, assume of its conditions only* (W1), (W2), (W5), (W6), (W7) *and* (W12). *Suppose moreover that the Kummer chart is unramified and $p \geq 7$, or that $\mathsf{s}(i) = \mathsf{s}(i')$ for all letters i, i'; that*

$$G + V \leq \eta, \qquad 11\,(p^J + 1)\,W \leq B, \qquad 4\,(J + 1 + \mathfrak{d} + \lfloor \log_p(2B) \rfloor)\,W \leq X;$$

and that for every letter i and all integers $a, a' \in [1, J]$: if i and i^{p^a} are thick, then $i^{p^{a+a'}}$ is not thick. Then $\mathrm{Cup}_\Pi(10^4 X(J+1); c, c')$ for all $c \in C_S$ and $c' \in C_{S'}$.

LEAN `Lift.KummerChart.PairingSide.SidePairing.scaledCupBounds_sparse_thick`

Theorem D (two sizes). *In Setting 2.6, of whose objects only S, S', Π, thick, s, β, η, V, W, B and G are used, assume of its conditions only* (W1), (W2), (W7) *and* (W12). *Suppose moreover that the Kummer chart is unramified and $p \geq 7$, and let $K_0 \geq 0$ be an integer with*

$$5W \leq G, \qquad G \leq B, \qquad B < K_0\,p^2, \qquad 2G < K_0\,p, \qquad W \leq K_0.$$

Then $\mathrm{Cup}_\Pi(10^4 K_0(\lfloor \log_p(2B) \rfloor + 1); c, c')$ for all $c \in C_S$ and $c' \in C_{S'}$.

LEAN `Lift.KummerChart.PairingSide.SidePairing.scaledCupBounds_two_sizes`

The unit on Selmer groups uses these four statements and nothing else of the three parts. In all four the exponent or the length does not involve the rank of a partner lattice, and in Theorem B not the ramification index of the field that makes the lattices crystalline.

2 The setting

Conventions. Throughout, p is a prime number. The declarations of this unit take the following ambient objects as implicit or instance arguments; they are fixed by the Settings and named again in each statement that needs them: the prime p; the group Γ over $G_{\mathbb{Q}_p}$, in the last file with a topology; the Kummer chart $\mathfrak{c}$ for Γ; the coefficient ring $\mathcal{O}$; a lattice L or a lattice datum L; sides S, S' and a pairing Π; a letter i; exponent functions o, o'; the integers and functions of Settings 2.5 and 2.6. Tensor products without a subscript are over $\mathbb{Z}_p$; $v_p(n)$ is the exponent of p in an integer n, with $v_p(0) = 0$; $\lfloor \log_p n \rfloor$ is the largest $a \geq 0$ with $p^a \leq n$, and 0 for $n = 0$. A *sign* is 1 or -1. Exponent functions are functions $\mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$. The Lean text calls the artifact's period ring *Bcris*; it is Colmez's ring $B_{\max}$.

Setting 2.1 (the base; from the background library and from Parts 1 and 2)**.** The notation is that of Parts 1 and 2, which this Setting recalls, with some further notions of the background library.

The chart and lattice data. C is the completed algebraic closure of $\mathbb{Q}_p$; $A_{\mathrm{inf}} \subset W(C^\flat)$, φ, θ and B_{dR} with $\mathrm{Fil}^n B_{\mathrm{dR}}$ are the period rings of Part 1. As there (Part 1, Setting 2.1), $A_{\max}$ is the p-adic completion of $A_{\mathrm{inf}}[X]/(pX - \xi)$, where $\xi = \sum_{0 \leq a < p} [\varepsilon^{1/p}]^a$ generates the kernel of θ; $B^+ = A_{\max}[1/p]$; and $B_{\max} = B^+[1/([\varepsilon] - 1)]$. Frobenius φ

and $G_{\mathbb{Q}_p}$ act on these rings; the maps $A_{\max} \to B^+ \to B_{\max}$ are injective, and $B_{\max}$ has an injective homomorphism to B_{dR} that commutes with $G_{\mathbb{Q}_p}$ and maps B^+ into $\mathrm{Fil}^0 B_{\mathrm{dR}}$. A lattice T is *potentially crystalline* if the elements of $B_{\max} \otimes T$ fixed near 1 span it over $B_{\max}$, and *crystalline* on a group if those fixed by the group do. Γ is a group with a homomorphism $j_\Gamma \colon \Gamma \to G_{\mathbb{Q}_p}$ of open image, χ its cyclotomic character; an element $s \in \Gamma$ is *inertial* if $j_\Gamma(s)$ lies in the inertia group. A Kummer chart $\mathfrak{c}$ for Γ has its ring $\mathfrak{S} \subset A_{\mathrm{inf}}$, a quotient of $\mathbb{W}[[X]]$ with $\mathbb{W} = W(\mathbb{F}_{p^h})$, its residue degree h, its coordinate z, its Eisenstein element $\mathcal{E}$ with $\mathcal{E}_\nu = \varphi^\nu(\mathcal{E})$, its index $e_{\mathfrak{c}}$, the number $\mathfrak{d}$ (equal to 0 if $e_{\mathfrak{c}} = 1$ and to $v_p(e_{\mathfrak{c}}) + 1$ otherwise), and its subgroup Γ' of index prime to p containing the elements near 1; it is *unramified* if $\mathcal{E} = z + p$, $e_{\mathfrak{c}} = 1$, $\mathfrak{d} = 0$. The *letters* are the elements of $\mathbb{F}_{p^h}^\times$. $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$; with an element $\omega \in \mathcal{O}$, $\omega^{p^h-1} = 1$, it is a coefficient ring for the chart. A *lattice* over $\mathcal{O}$ is a finite free $\mathcal{O}$-module with an $\mathcal{O}$-linear action of Γ; $Z^1_{\mathrm{f}}(L)$ is the $\mathcal{O}$-module of its crystalline cocycles, and for $\gamma \colon \mathcal{O} \to C$ and $u \in \mathbb{Z}$ the jump u of L at γ has a multiplicity. A *lattice datum* is a lattice crystalline on Γ' with finite sets $U_L(i) \subset \mathbb{Z}$, one for each letter, containing its jumps at the γ with $\gamma(\omega) = \theta([i])$; $m_L(i,u)$ is the sum of the multiplicities of the jump u over these γ. It has the modules $\mathfrak{M}^{\mathrm{inf}}(L)$, $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{loc}}(L)$, $\mathfrak{M}_{\mathrm{loc}}(L)$ and the Kisin module $\mathfrak{M}(L)$, the periods $\mathrm{Per}(L)$, the jump bound h_L, the projectors pr_i onto the parts $(W(C^\flat) \otimes L)_i$, and for an exponent function o the module $\mathfrak{X}_L(i;o)$ of the elements of $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{loc}}(L) \cap (W(C^\flat) \otimes L)_i$ whose periods at $i^{p^{-\nu}}$ and u, after $\varphi^{-\nu}$, lie in $\mathrm{Fil}^{o(\nu,u)} B_{\mathrm{dR}}$; a larger o gives a smaller module. Further: $\mathrm{std}_l(\nu,u) = 0$ for $\nu < l$ and $-u$ for $\nu \geq l$; $\mathfrak{N}_{l,i}(L) = \mathfrak{M}_{\mathrm{loc}}(L) \cap \mathfrak{X}_L(i;\mathrm{std}_l)$; $(Q)_{\mathfrak{S}}$ is the set of the elements of $\mathfrak{M}_{\mathrm{loc}}(L)$ in a submodule Q; $\mathrm{Inv}(Q)$ is the set of the x with $gx - x \in Q$ for all $g \in \Gamma'$; $\mathrm{cost}(\nu,n) = v_p(n) + \mathfrak{d} + 3\nu$; $\mathrm{imp}(o,o')(\nu,u)$ is $o'(\nu,u)$ if $o(\nu,u) > 0$ and $\min(o'(\nu,u),0)$ otherwise; $\mathrm{fr}_{l,n}(o)(\nu,u)$ is n for $\nu < l$ and $o(\nu-l,u)$ for $\nu \geq l$; $\mathrm{up}_{\nu,u}(o)(\nu',v)$ is $o(\nu',v)+1$ if $(\nu',v) = (\nu,u)$ and $o(\nu',v)$ otherwise; and $\mathrm{sh}_l(o_1)(\nu,u)$, for a function o_1 of u, is $\mathrm{std}_l(\nu,u) + o_1(u)$ for $\nu = 0$ and $\mathrm{std}_l(\nu,u)$ otherwise.

Sides and points. A side $S = (Y_S, C_S, \alpha_S)$ over $\mathcal{O}$ consists of a lattice datum Y_S with $0 \in U_{Y_S}(i)$ for all i, a lattice C_S and an $\mathcal{O}$-linear map $\alpha_S \colon C_S \to Z^1_{\mathrm{f}}(Y_S)$. Its extension $\tilde{Y}_S$ is $Y_S \times C_S$ with $g(y,c) = (gy + \alpha_S(c)(g), c)$ and the sets $U_{Y_S}(i)$; C_S^{tr} is C_S with the trivial action and the sets $\{0\}$; $\mathrm{in}_S^Y \colon Y_S \to \tilde{Y}_S$ and $\mathrm{pr}_S^C \colon \tilde{Y}_S \to C_S^{\mathrm{tr}}$ are the inclusion and the projection, and $\mathrm{pr}_S^{C,\mathrm{loc}}$ is the map $1 \otimes \mathrm{pr}_S^C$ on $\mathfrak{M}_{\mathrm{loc}}(\tilde{Y}_S)$. For $c \in C_S$, $\mathrm{cst}_{S,i}(c) = \mathrm{pr}_i(1 \otimes c)$; $\mathfrak{a}_S(r;l,i)$ is the lifting ideal of A_{inf} (Part 1, Definition 2.8; its characterization is Input Q3(a)); $\mathrm{Kill}_S(i;\kappa,\rho)$ means $p^\kappa \mathcal{E}^\rho \in \mathfrak{a}_S(0;0,i)$. $\mathfrak{X}^0_S(i;o)$ is the module of the $x \in \mathfrak{X}_{\tilde{Y}_S}(i;o)$ with $(1 \otimes \mathrm{pr}_S^C)(x) = 0$; $\mathrm{Deep}_S(i;o)$ is the $\mathbb{Z}_p$-module of the $x \in \mathfrak{M}_{\mathrm{loc}}(\tilde{Y}_S)$ such that $(1 \otimes \mathrm{pr}_S^C)(x) = \mathrm{cst}_{S,i}(c)$ for some $c \in C_S$ and $x \in \mathrm{Inv}(\mathfrak{X}_{\tilde{Y}_S}(i;o))$; and $\mathrm{Pt}_S(c,i;e;o_1,o')$ is the set of the $x \in \mathfrak{M}_{\mathrm{loc}}(\tilde{Y}_S)$ with $(1 \otimes \mathrm{pr}_S^C)(x) = p^e \,\mathrm{cst}_{S,i}(c)$, $x \in \mathfrak{X}_{\tilde{Y}_S}(i;o_1)$ and $x \in \mathrm{Inv}(\mathfrak{X}_{\tilde{Y}_S}(i;o'))$, the points over $p^e c$ at i with exponents o_1 and depth o'. An additive map A of $\mathfrak{M}_{\mathrm{loc}}(\tilde{Y}_S)$ *has cost* e, written $\mathrm{Cost}_S(\mathsf{A}) \leq e$, if $\mathsf{A}(x) - p^e x \in (\mathfrak{X}^0_S(j;o))_{\mathfrak{S}}$ for all letters j, exponent functions o and $x \in \mathrm{Deep}_S(j;o)$. A pairing Π of sides S and S' is a pairing of the lattices Y_S and $Y_{S'}$ (Part 2, Setting 2.2); it has its third side $S^\vee_\Pi$, whose lattice datum is the Tate twist of the dual of $Y_S \otimes_{\mathcal{O}} Y_{S'}$ with the jump sets $\{0\} \cup \{-u-v-1\}$, $u \in U_{Y_S}(i)$, $v \in U_{Y_{S'}}(i)$; and $\mathrm{Cup}_\Pi(e;c,c')$ means that, for every topology on Γ for which j_Γ is an open embedding, the cup product of $\alpha_S(c)$ and $p^e \alpha_{S'}(c')$ under Π bounds. The windows: $\mathrm{rs}_n(\nu,u)$ is $n-u$ for $\nu = 0$ and $-u$ for $\nu > 0$; $\mathcal{W}_n(c) = [-n,n] \cup [c-n, c+n]$; $\mathrm{NM}(p;n,\beta)$ says

that p divides no non-zero element of $\mathcal{W}_n(\beta)\cup\mathcal{W}_n(-\beta)$; and $U\subseteq[-\eta,\eta]\cup\mathsf{s}\cdot[\beta,\beta+W]$ says that every $u\in U$ has $|u|\le\eta$ or $\beta\le\mathsf{s}u\le\beta+W$.

The linear Frobenius. Let T be a potentially crystalline lattice, $s\in\Gamma$ and $m\ge0$ such that $j_\Gamma(s)^{-1}$ acts on the residue field of $\bar{\mathbb{Q}}_p$ as the p^m-th power (for $m=0$: s is inertial). Then $\beta^T_{s,m}$ is the $B_{\max}$-linear endomorphism of $B_{\max}\otimes T$ that agrees with $\varphi^m j_\Gamma(s)\otimes s$ on the elements fixed near 1. A polynomial Q over $\mathcal{O}$ *kills* (s,m) *on* T if T is potentially crystalline and $Q(\beta^T_{s,m})=0$, the coefficients acting through T.

Classes. Let $\mathcal{F}$ be a filter on Γ, K a relation between elements of Γ and polynomials over $\mathcal{O}$, and $U=(U(\gamma))_\gamma$ a family of finite sets of integers. A lattice Y over $\mathcal{O}$ lies in the *class* $\mathrm{Cl}(\mathcal{F},\mathsf{K};U)$ if the elements of $B_{\max}\otimes Y$ that are fixed by $j_\Gamma(g)\otimes g$ for all g near 1 and for all g in some member of $\mathcal{F}$ span $B_{\max}\otimes Y$; every jump of Y at γ with non-zero multiplicity lies in $U(\gamma)$; and Q kills $(s,0)$ on Y whenever $\mathsf{K}(s,Q)$ and s is inertial. Here $\top$ is the filter whose only member is Γ, so that the first condition for $\mathcal{F}=\top$ says that Y is crystalline on Γ, and $\bot$ is the empty relation.

Orthogonality. A pairing of lattices Y and L and the statement that the cup product of two maps $a\colon\Gamma\to Y$, $c\colon\Gamma\to L$ under it bounds, for a topology on Γ, are as in Part 2, Setting 2.2. For a topology on Γ, a class $\mathcal{Q}$ of lattices over $\mathcal{O}$ and a lattice L, $L^{\perp\mathcal{Q}}$ is the $\mathcal{O}$-module of the maps $c\colon\Gamma\to L$ such that for every $Y\in\mathcal{Q}$, every pairing of Y and L and every $a\in Z^1_{\mathrm{f}}(Y)$ the cup product of a and c bounds.

Lengths. For a ring R and submodules N, N' of an R-module, $\mathrm{len}_R(N/(N\cap N'))$ is the length of the R-module $N/(N\cap N')$, a natural number or ∞.

Definition 2.2 (constants that lift to deep points)**.** In Setting 2.1, for a side S over $\mathcal{O}$, a letter i, an integer n and an exponent function o, $\mathrm{DC}_S(i;n,o)$ is the image under $\mathrm{pr}_S^{C,\mathrm{loc}}$ of the $\mathbb{Z}_p$-module of the $x\in\mathfrak{M}_{\mathrm{loc}}(\tilde{Y}_S)$ with $x\in\mathfrak{X}_{\tilde{Y}_S}(i;\mathrm{rs}_n(\cdot,\cdot))$ and $x\in\mathrm{Deep}_S(i;o)$; it is a $\mathbb{Z}_p$-submodule of $W(C^\flat)\otimes C_S$.

LEAN `Lift.KummerChart.PairingSide.deepConstants`

Definition 2.3 (tame charts)**.** In Setting 2.1, a *tame chart* for Γ is a Kummer chart $\mathfrak{c}$ for Γ together with: an element $\varrho\in\mathbb{W}$ whose reduction modulo p is a primitive $e_{\mathfrak{c}}$-th root of unity; an inertial element $\sigma\in\Gamma$, the *tame generator*, such that for every $k\ge0$, σ^k fixes $\mathbb{W}$ and $\sigma^k z=\varrho^k z$; and an element $\varpi\in\bar{\mathbb{Q}}_p$ with $\varpi^{e_{\mathfrak{c}}}=p$ that is fixed by $j_\Gamma(g)$ for all $g\in\Gamma'$; such that p does not divide $e_{\mathfrak{c}}$ and Γ' is normal in Γ.

LEAN `Lift.TameChart`

Definition 2.4 (tame types; the universal side)**.** In Setting 2.1:

(i) For a natural number p, integers $\mathsf{r},\mathsf{n}\ge0$ and a property K_0 of polynomials over $\mathcal{O}$, $\mathrm{Tame}_p(\mathsf{r},\mathsf{n};\mathsf{K}_0)$ means that K_0 holds for $X^{M_0}-1$, where M_0 is the least common multiple of the p^a-1 for $1\le a\le\mathsf{r}$, and for the image in $\mathcal{O}[X]$ of some monic polynomial over $\mathbb{Z}_p$ of degree at most n.

(ii) For a Kummer chart $\mathfrak{c}$, $\omega\in\mathcal{O}$ with $\omega^{p^h-1}=1$, a family $U=(U(\gamma))_\gamma$ of finite sets of integers and a letter i,

$$\mathcal{U}_{\mathfrak{c},\omega}(U)(i)=\{0\}\cup\bigcup_{\gamma(\omega)=\theta([i])}U(\gamma).$$

(iii) Let $\mathfrak{c}$ and ω be as in (ii), $\mathcal{F}$ a filter on Γ every member of which contains Γ', and L a lattice with $L\in\mathrm{Cl}(\mathcal{F},\bot;U)$. The *universal side* $\mathsf{S}^{\mathrm{u}}_{\mathfrak{c},\omega}(\mathcal{F};L;U)$ is the

side over $(\mathcal{O},\omega)$ whose lattice datum is L with the sets $\mathcal{U}_{\mathfrak{c},\omega}(U)(i)$, whose lattice is $Z^1_{\mathrm{f}}(L)$ with the trivial action, and whose cocycle map is the identity. (It is a lattice datum because L is crystalline on Γ'; and $Z^1_{\mathrm{f}}(L)$ is finitely generated by Input K5 and has no torsion, so it is free.)

LEAN `Lift.IsTameType` · `Lift.KummerChart.letterJumps` · `Lift.KummerChart.universalSide`

Setting 2.5 (few low weights)**.** In Setting 2.1, the Kummer chart is unramified, S is a side over $\mathcal{O}$, $\rho,\eta,W\ge 0$ are integers, and for every letter i there are integers $\beta(i),n_{\mathrm{lo}}(i),n_{\mathrm{all}}(i)\ge 0$ and a sign $\mathsf{s}(i)$. The following conditions are referred to by their names.

(F1) $p\ge 5$.

(F2) $\beta(i)+W+\eta\le(p-3)\rho$ for every letter i.

(F3) $6(\rho+\eta+W)<\beta(i)$ for every letter i.

(F4) $\mathrm{NM}(p;6(\rho+\eta+W),\beta(i))$ for every letter i.

(F5) $U_{Y_S}(i)\subseteq[-\eta,\eta]\cup\mathsf{s}(i)\cdot[\beta(i),\beta(i)+W]$ for every letter i.

(F6) $\sum_{u\in[-\eta,\eta]} h\,m_{Y_S}(i,u)\le n_{\mathrm{lo}}(i)$ and $\sum_{u\in U_{Y_S}(i)} h\,m_{Y_S}(i,u)\le n_{\mathrm{all}}(i)$ for every letter i.

Setting 2.6 (cuts and windows)**.** In Setting 2.1, S and S' are sides over $\mathcal{O}$ and Π is a pairing of S and S'; cut and thick are properties of letters; for every letter i there are a sign $\mathsf{s}(i)$ and integers $\rho(i),\beta(i)\ge 0$; and $\eta,V,W,X,\kappa,B,n,J,\rho_0,G\ge 0$ are integers. The following conditions are referred to by their names; each statement says which of them it assumes.

(W1) $V+4\eta\le W$.

(W2) $U_{Y_S}(i)\subseteq[-\eta,\eta]\cup-\mathsf{s}(i)\cdot[\beta(i),\beta(i)+V]$ and $U_{Y_{S'}}(i)\subseteq[-\eta,\eta]\cup\mathsf{s}(i)\cdot[\beta(i),\beta(i)+V]$ for every letter i.

(W3) $(J+1+\mathfrak{d}+\lfloor\log_p(2B)\rfloor)\,n+\mathfrak{d}\,G\le X$.

(W4) $12\,(J+1)\,X<\kappa$.

(W5) $e_{\mathfrak{c}}\cdot 2B\le p^{J+1}X$.

(W6) $8\,e_{\mathfrak{c}}\le p^{J+1}$.

(W7) $\eta\ge 1$.

(W8) $W\le n$, $W\le G$, and $\beta(i)+\rho(i)+W\le 2B$ for every letter i.

(W9) $G\le(p-1)\,n$.

(W10) For every letter i with the property cut: either $\rho(i)=0$ and $\beta(i^{p^a})+W\le G$ for all $a\in[1,J]$; or $\rho(i)<\beta(i)$, the chart is unramified, $p\ge 5$ and $\rho_0\le\kappa+\rho(i)$.

(W11) If $p\ge 5$, then $\beta(i)+W\le(p-3)\rho_0$ for every letter i.

(W12) $\beta(i)=B$ if i is thick, and $\beta(i)=G$ otherwise.

Inputs from Parts 1 and 2. From Part 1 (*Exponents of elements of Kisin modules, the cost of deepening them, and cokernel killers*) and Part 2 (*Cup products of crystalline classes vanish up to a power of p: the walk along Frobenius orbits*); L is a lattice datum, S a side.

Q1. *Exponent modules* (Part 1, Lemmas 4.1, 4.2, Theorem B, Lemma 7.3). Monotonicity in the exponents; $\mathcal{E}^n_{\nu_0}$, for an integer n of either sign, raises the exponents at ν_0 by n and preserves $\mathfrak{M}_{\mathrm{loc}}(L)$; if $rx \in \mathfrak{X}_L(i;o)$ with $x \in \mathfrak{M}^{\mathrm{inf}}_{\mathrm{loc}}(L)$ and $\mathcal{E}_\nu \nmid r$ for all ν, then $x \in \mathfrak{X}_L(i;o)$; $gx - x \in \mathfrak{X}_L(i;o)$ for $x \in \mathfrak{X}_L(i;o)$, $g \in \Gamma'$; $x \in \mathfrak{X}_L(i;\mathrm{std}_0)$ if and only if $x \in \mathfrak{M}^{\mathrm{inf}}(L) \cap (W(C^\flat) \otimes L)_i$; $\mathrm{pr}_i(x) \in \mathfrak{X}_L(i;\mathrm{std}_0)$ for $x \in \mathfrak{M}^{\mathrm{inf}}(L)$; the map $1 \otimes \mathrm{in}^Y_S$ sends $\mathfrak{X}_{Y_S}(i;o)$ into $\mathfrak{X}_{\tilde{Y}_S}(i;o)$ and $\mathfrak{M}_{\mathrm{loc}}(Y_S)$ into $\mathfrak{M}_{\mathrm{loc}}(\tilde{Y}_S)$; and $\mathrm{cost}(\nu, n) \le 3\nu$ if the chart is unramified and p does not divide n or $n = 0$ (Part 1, Lemma 6.1).

Q2. *Deep elements* (Part 1, Lemma 6.2). If $x \in \mathfrak{X}_L(i;o)$ and $x \in \mathrm{Inv}(\mathfrak{X}_L(i;o'))$, then $x \in \mathfrak{X}_L(i;\mathrm{imp}(o,o')(\cdot,\cdot))$.

Q3. *The lifting ideal* (Part 1, Lemmas 9.1, 9.3 to 9.7, Theorem D). *(a)* $s \in \mathfrak{a}_S(r;l,i)$ if and only if there is $t \in A_{\mathrm{inf}}$ such that every $(s - rt)\,\mathrm{cst}_{S,i}(c)$ is the image of an element of $\mathfrak{X}_{\tilde{Y}_S}(i;\mathrm{std}_l)$. *(b)* If $p^c \in \mathfrak{a}_S(p^{c+1};l,i)$, then $\mathfrak{a}_S(p^{c+1};l,i) \subseteq \mathfrak{a}_S(0;l,i)$. *(c)* $p^c z^{pa} \in \mathfrak{a}_S(r;1,i^p)$ if and only if $p^c z^a \in \mathfrak{a}_S(r;0,i)$; if $s \in \mathfrak{a}_S(r;0,i)$ and $u \le n$ for all $u \in U_{Y_S}(i)$, then $s\,\mathcal{E}^n \in \mathfrak{a}_S(r;1,i)$. *(d)* If $f \in \mathfrak{S}$ lies in $\mathfrak{a}_S(r;l,i)$, there is $t \in \mathfrak{S}$ such that every $y \in \mathfrak{N}_{l,i}(C^{\mathrm{tr}}_S)$ has an $x \in \mathfrak{N}_{l,i}(\tilde{Y}_S)$ over $(f - rt)y$; and $\mathrm{cst}_{S,i}(c) \in \mathfrak{N}_{l,i}(C^{\mathrm{tr}}_S)$. *(e)* If $pa \le n$, $p^c z^n \in \mathfrak{a}_S(r;1,i^p)$ and $c + v_p(n!) \le \kappa + v_p((pa)!)$, then $p^\kappa z^a \in \mathfrak{a}_S(r;0,i)$. *(f)* If $p^c z^a \mathcal{E}^n \in \mathfrak{a}_S(r;1,i)$, $c + \mathfrak{d}n + v_p(n!) \le b$ and $a + n \le \kappa$, then $p^b z^\kappa \in \mathfrak{a}_S(r;1,i)$. *(g)* If $u \le n$ for all letters j and $u \in U_{Y_S}(j)$, then for every $\kappa \ge 0$ there is $a \ge 0$ with $z^a \in \mathfrak{a}_S(p^\kappa;0,j)$ for all j and $a\,p \le e_{\mathfrak{c}}(n + \kappa) + a$. *(h)* If the chart is unramified, $p \ge 5$ and $u \le (p-3)(\kappa + \rho)$ for all j and $u \in U_{Y_S}(j)$, then $\mathrm{Kill}_S(i;\kappa,\rho)$. *(i)* $\sum_{a \in \mathcal{W}_n(c)} (v_p(a) + d) \le (4m+2)\lfloor \log_p H \rfloor + (4n+2)d$ if $n \le pm$ and $|c| + n \le H$.

Inputs from Parts 1 and 2, continued.

Q4. *Points* (Part 2, Lemmas 4.1, 4.4, 4.5, 4.7, Proposition 4.6). *(a)* If $\mathrm{Kill}_S(i;\kappa,\rho)$, then $\mathrm{Pt}_S(c,i;\kappa;\mathrm{rs}_{-\rho}(\cdot,\cdot),\mathrm{rs}_{-\rho}(\cdot,\cdot))$ is not empty. *(b)* For a finite set I of steps (ν,n) there is an additive map A of $\mathfrak{M}_{\mathrm{loc}}(\tilde{Y}_S)$ with $\mathsf{A}(x)-p^{\epsilon}x \in (\mathfrak{X}^0_S(i;o))_{\mathfrak{S}}$ for $x \in \mathrm{Deep}_S(i;o)$, where $\epsilon=\sum_{(\nu,n)\in I}\mathrm{cost}(\nu,n)$, and $\mathsf{A}(x) \in \mathrm{Deep}_S(i;o')$ if I contains all steps from o to o'; such a map preserves every $\mathbb{Z}_p$-submodule of $\mathfrak{M}_{\mathrm{loc}}(\tilde{Y}_S)$ that contains $(\mathfrak{X}^0_S(i;o))_{\mathfrak{S}}$, on its elements in $\mathrm{Deep}_S(i;o)$; and $(\mathfrak{X}^0_S(i;o))_{\mathfrak{S}} \subseteq \mathrm{Deep}_S(i;o)$. *(c)* A point of $\mathrm{Pt}_S(c,i;e;o_1,o)$ gives a point of $\mathrm{Pt}_S(c,i;e';\mathrm{imp}(o_1,o')(\cdot,\cdot),o')$ if I contains the steps from o to o' and $e+\epsilon\le e'$; in particular, if $\mathrm{NM}(p;n,\beta)$ and all these steps are at $\nu=0$ and in $\mathcal{W}_n(\beta)\cup\mathcal{W}_n(-\beta)$, one may take $e'=e+(8n+4)\mathfrak{d}$. *(d)* φ^l maps $\mathrm{Pt}_S(c,i^{p^{-l}};e;o,o')$ into $\mathrm{Pt}_S(c,i;e;\mathrm{fr}_{l,0}(o)(\cdot,\cdot),\mathrm{fr}_{l,1}(o')(\cdot,\cdot))$. *(e)* (Strictness.) Every $w\in\mathfrak{X}^0_S(i;o)$ is the image of a $y\in\mathfrak{X}_{Y_S}(i;o)$, and $y\in\mathfrak{M}_{\mathrm{loc}}(Y_S)$ if $w\in\mathfrak{M}_{\mathrm{loc}}(\tilde{Y}_S)$.

Q5. *The walk* (Part 2, Theorems B, C and D, Definition 2.6). Theorem B: deep points of S and S' against each other at every cut, with a point of the third side, give $\mathrm{Cup}_{\Pi}(E;c,c')$. Theorem C: the rank-free exponent $H\ge 3(\kappa+(48(\rho+\eta+W)+4)\mathfrak{d})$ for sides with jumps in opposite windows and three cokernels killed by $p^{\kappa}\mathcal{E}^{\rho}$. Theorem D: the long gap. They are cited with their hypotheses where used.

Q6. *A relation among elements of one weight* (Part 2, Proposition 7.1). Unramified chart; $m_L(i,u)\le n$; $j^{p^{-\nu_0}}=i$; $o(\nu_0,u)<o(\nu_0,v)$ for $v\in U_L(i)$, $v<u$; $x_0,\dots,x_{hn}\in\mathfrak{M}_{\mathrm{loc}}(L)\cap\mathfrak{X}_L(j;o)$. Then some combination $\sum_k c_k x_k$ with $c_k\in\mathbb{Z}_p$, one of them 1, lies in $(\mathfrak{X}_L(j;o_+))_{\mathfrak{S}}+(\mathfrak{X}_L(j;\mathrm{up}_{\nu_0,u}(o)(\cdot,\cdot)))_{\mathfrak{S}}$, where o_+ is o raised by 1 at $\nu=0$.

Inputs from the background library. Each is a declaration of the background library, read in its Lean text; the number in brackets is the unit that owns it.

K1. *Relative lengths* [57]. For $\mathbb{Z}_p$-submodules: *(a)* $\operatorname{len}_{\mathbb{Z}_p}(N/(N\cap N'')) \le \operatorname{len}_{\mathbb{Z}_p}(N/(N\cap N')) + \operatorname{len}_{\mathbb{Z}_p}(N'/(N'\cap N''))$; *(b)* $\operatorname{len}_{\mathbb{Z}_p}(N/(N\cap f^{-1}(N'))) = \operatorname{len}_{\mathbb{Z}_p}(f(N)/(f(N)\cap N'))$ for a linear map f; *(c)* $\operatorname{len}_{\mathbb{Z}_p}(N/(N\cap\bigcap_k N_k)) \le \sum_k \operatorname{len}_{\mathbb{Z}_p}(N/(N\cap N_k))$ for finitely many N_k; *(d)* if $p^a N \subseteq N'$ and any $m+1$ elements x_k of N with $px_k \in N'$ have a combination $\sum_k c_k x_k \in N'$ with $c_k \in \mathbb{Z}_p$, one of them 1, then $\operatorname{len}_{\mathbb{Z}_p}(N/(N\cap N')) \le a\,m$. Elementary.

K2. *The linear Frobenius* [86]. For potentially crystalline lattices: *(a)* $\beta^T_{st,m+m'} = \beta^T_{s,m}\,\beta^T_{t,m'}$, and $\beta^T_{g,0} = 1$ for inertial g near 1; *(b)* a $\mathbb{Z}_p$-linear map $f\colon T \to T'$ that commutes with s and with all g near 1 satisfies $(1\otimes f)\,\beta^T_{s,m} = \beta^{T'}_{s,m}\,(1\otimes f)$; $\beta^T_{s,0} = 1$ for inertial s if T is crystalline on Γ; *(c)* an element of $B_{\max}$ fixed near 1 is fixed by every inertial element. In the second reader's comparison, Fontaine gives the Weil action on the periods of a finite-dimensional representation (J.-M. Fontaine, *Représentations ℓ-adiques potentiellement semi-stables*, Astérisque 223, Exposé VIII, §§2.3.2–2.3.3, §2.3.7), and the $B_{\max}$-linear extension on arbitrary modules is not located. *Relation*: not settled; the existence of the linear extension is part of what the formal library proves, and no published statement of it in this generality is known to us.

K3. *A crystalline criterion* [86]. If T is potentially crystalline and $\beta^T_{s,0} = 1$ for every inertial $s \in \Gamma$, then T is crystalline on Γ. *What is quoted* is this statement of the formal library, for any group Γ over $G_{\mathbb{Q}_p}$ of open image, any module T over a commutative $\mathbb{Z}_p$-algebra, and with $B_{\max}$ as period ring. *Relation to the literature*: a weaker form of Fontaine's criterion, namely only the implication from "inertia acts trivially on the potentially crystalline periods" to "crystalline"; the place is not confirmed. In the second reader's comparison, unramified descent and the Weil inertia account for the classical criterion for finite-dimensional representations (Fontaine, *Représentations p-adiques semi-stables*, Astérisque 223, Exposé III, §§1.8.5–1.8.6, §5.6.8(iii); Exposé VIII, §2.3), and a published descent for arbitrary modules and for the spanning condition over $B_{\max}$ is not located. So the Input is not a citation of a published theorem: it is a statement that the formal library proves, close to a classical one.

Inputs from the background library, continued.

K4. *Duals of tensor products* [86]. Let s be inertial, Y and L lattices in classes with relations K, K', and q, q' monic polynomials over $\mathbb{Z}_p$ with $\mathsf{K}(s,q)$, $\mathsf{K}'(s,q')$. Then a monic polynomial over $\mathbb{Z}_p$ of degree at most $\deg q \cdot \deg q'$ kills $(s^{-1},0)$ on $(Y \otimes_{\mathcal{O}} L)^{\vee}(1)$. Reference to be confirmed.

K5. *Crystalline cocycles and jumps* [86, 87]. For a potentially crystalline lattice T: $Z^1_{\mathrm{f}}(T)$ is a finite $\mathcal{O}$-module, and for each γ the multiplicities of the jumps of T at γ have sum $\mathrm{rk}_{\mathcal{O}}(T)$. For $u \neq 0$ the multiplicity of the jump u of $\tilde{Y}_S$ at γ is at most that of Y_S. *What is quoted* for the first statement is the finite generation, over the coefficient ring, of the module of the *integral* crystalline cocycles defined with $B_{\max}$; the formal library deduces it from a rank formula for that module, which it proves itself. *Relation to the literature*: the Bloch–Kato local dimension formula (S. Bloch, K. Kato, *L-functions and Tamagawa numbers of motives*, §3.7.2–3, Corollary 3.8.4) concerns H^1_{f} of the rational representation, defined with B_{cris}. The passage from it to the module of integral cocycles with $B_{\max}$ needs three things: the count of the coboundaries, the comparison of the invariants over $B_{\max}$ and over B_{cris} (Colmez, *Théorie d'Iwasawa des représentations de de Rham d'un corps local*, §III.4, for finite-dimensional representations), and the continuity of crystalline cocycles. The second reader sketches this bridge and marks it as not located in the literature; it is not made here. So the first statement is a special case of the Bloch–Kato formula only after that bridge. Of it this note uses the finite generation alone (Definition 2.4); the rank formula is not used. The second statement is classical for field coefficients (Fontaine, Exposé III, Theorem 3.8(i), Remark 3.9(ii), §5.1.7), its formulation for coefficient fibres not located; the third is elementary.

K6. *Tame inertia* [89, 92]. Let e be prime to p, $y \in \bar{\mathbb{Q}}_p$ with y^e a uniformizer of $\mathbb{Q}_p$, and f a homomorphism from the inertia group of $G_{\mathbb{Q}_p}$ to a monoid, with open kernel and $f(g)^e = 1$ for all g. Then f is trivial on the stabilizer of y. The file names it Hilbert's ramification theory for infinite Galois extensions; reference to be confirmed.

Inputs from the background library, concluded.

K7. *The frame theorem with a divisor; Galois and the Kisin module* [83, 82]. Let L be a lattice datum. A *crystalline period* is a period $\mu \in \mathrm{Per}(L)$ whose values on $1 \otimes L$ are given by a $\mathbb{Z}_p$-linear map $L \to B_{\max}$. *(a)* Let $d \in A_{\inf}$ be such that d divides r in $A_{\inf}$ whenever d divides $\mathcal{E}^m r$ in B^+. If $x \in \mathfrak{M}^{\inf}(L)$ and $\mathcal{E}^{2h_L}\mu(x) \in B^+ d$ for every crystalline period μ, then $x \in d\,\mathfrak{M}^{\inf}(L)$. The coordinate z is such a d. *(b)* $\mathcal{E}^{h_L}\mu(x)$ lies in the image of $\mathfrak{S}_B$ for every crystalline period μ and $x \in \mathfrak{M}(L)$. *(c)* If $g \in G_{\mathbb{Q}_p}$ fixes $\mathbb{W}$, then z divides $gs - s$ in B^+ for every $s \in \mathfrak{S}_B$. *(d)* Let $g \in G_{\mathbb{Q}_p}$, $d \in A_{\inf}$ and $m \geq 0$; if d divides $gs - s$ in B^+ for all $s \in \mathfrak{S}_B$ and $b \in B_{\mathrm{dR}}$ has $\mathcal{E}^m b$ in the image of $\mathfrak{S}_B$, then $\mathcal{E}^{2m}(gb - b) \in B^+ d$. *(e)* For $g \in \Gamma$ with $g\Gamma' g^{-1} \subseteq \Gamma'$: $g\,\mathfrak{M}^{\inf}(L) \subseteq \mathfrak{M}^{\inf}(L)$, and $g^{-1} \circ \mu \circ g \in \mathrm{Per}(L)$ for every $\mu \in \mathrm{Per}(L)$. *(f)* A $\mathbb{Z}_p$-linear combination of periods that vanishes on $1 \otimes L$ vanishes on $\mathfrak{M}^{\inf}(L)$. Here $\mathfrak{S}_B$ is the ring of Part 1, Setting 2.1. Reference to be confirmed (the Kisin lattice of a crystalline lattice).

K8. *Existence of charts* [83]. *(a)* For $h \geq 1$, $x \in \mathcal{O}_C$ and a unit v with $x^e v = -p$ there is a chart with residue degree h, index e, coordinate z with $\theta(z) = x$, and constant unit series. *(b)* If a closed subgroup of $G_{\mathbb{Q}_p}$ acts transitively on the n-th roots of p for all $n > 0$, and $u, v \in \mathcal{O}_C^\flat$ have untilts whose e-th power is p, then some element of the subgroup sends $[u]$ to $[v]$. The inertia group has this property. *(c)* Every $s \in \mathfrak{S}$ is $w + zt$ with $w \in \mathbb{W}$, $t \in \mathfrak{S}$; and if z divides $\mathcal{E}^a r$ in B^+, then z divides r in $A_{\inf}$. Elementary.

3 Constants that lift to deep points

Throughout this section the setting is Setting 2.5. The chart is unramified, so $\mathcal{E} = z + p$, $\mathfrak{d} = 0$ and $\mathrm{cost}(\nu, n) = v_p(n) + 3\nu$. For a letter i and an exponent function o write

$$\mathsf{P}_i(o) = (\mathfrak{X}_{\tilde{Y}_S}(i; \mathrm{rs}_0(\cdot,\cdot)))_{\mathfrak{S}} \cap \mathrm{Deep}_S(i; o), \qquad \mathsf{P}_i^0(o) = (\mathfrak{X}_S^0(i; o))_{\mathfrak{S}},$$

so that $\mathrm{DC}_S(i; 0, o) = \mathrm{pr}_S^{C,\mathrm{loc}}(\mathsf{P}_i(o))$ by Definition 2.2. Write i^- for the letter $i^{p^{-1}}$.

Lemma 3.1 (deepening at one Frobenius index: the loss of constants)**.** *In Setting 2.5, assume of its conditions only* (F6)*. Let i, i' be letters and $\nu_0 \geq 0$ an integer with $i^{p^{-\nu_0}} = i'$, let o, o' be exponent functions and $A, B \geq 0$ integers. Suppose that:*

(i) *for every $v \in \mathbb{Z}$, p divides no integer n with $o(0,v) < n < o(0,v) + p^{\nu_0}$;*

(ii) *for every $\nu \geq 0$ and $u \in U_{Y_S}(i^{p^{-\nu}})$: $-u \leq o(\nu, u)$, and $o'(\nu, u) \leq o(\nu, u)$ if $\nu \neq \nu_0$;*

(iii) *the function $u \mapsto o'(\nu_0, u)$ is non-increasing on $U_{Y_S}(i')$;*

(iv) *for every $u \in U_{Y_S}(i')$: $\sum_{o(\nu_0,u) \leq n < o'(\nu_0,u)} \mathrm{cost}(\nu_0, n) \leq A + B_u$, where $B_u = B$ if $|u| \leq \eta$ and $B_u = 0$ otherwise.*

Then

$$\mathrm{len}_{\mathbb{Z}_p}\big(\mathrm{DC}_S(i; 0, o)/(\mathrm{DC}_S(i; 0, o) \cap \mathrm{DC}_S(i; 0, o'))\big) \;\leq\; A \cdot n_{\mathrm{all}}(i') + B \cdot n_{\mathrm{lo}}(i').$$

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Proof. Multiplying (iv) by $h\, m_{Y_S}(i',u)$ and summing over $u \in U_{Y_S}(i')$ gives, by (F6),

$$\sum_{(u,n)\in I} h\, m_{Y_S}(i',u)\ \mathrm{cost}(\nu_0,n) \ \le\ A\, n_{\mathrm{all}}(i') + B\, n_{\mathrm{lo}}(i'),$$
$$I = \{(u,n) : u \in U_{Y_S}(i'),\ o(\nu_0,u) \le n < o'(\nu_0,u)\}.$$

We prove that the relative length is at most the left side, by induction on the number of elements of I, for all o at once.

If no $v \in U_{Y_S}(i')$ has $o(\nu_0,v) < o'(\nu_0,v)$, then $o' \le o$ on all jump sets by (ii), so $\mathrm{Deep}_S(i;o) \subseteq \mathrm{Deep}_S(i;o')$ (Input Q1) and the relative length is 0. Otherwise let u be the least such v and let o^+ be the function $\mathrm{up}_{\nu_0,u}(o)(\cdot,\cdot)$. The hypotheses hold for (o^+,o'): (i) because $o^+(0,\cdot)$ differs from $o(0,\cdot)$ only if $\nu_0 = 0$, when the intervals of (i) are empty; (ii), (iii) and (iv) at once. Its set of steps is $I \setminus \{(u,o(\nu_0,u))\}$. By Input K1(a) and the induction hypothesis it remains to show

$$\mathrm{len}_{\mathbb{Z}_p}(\mathrm{DC}_S(i;0,o)/(\mathrm{DC}_S(i;0,o)\cap \mathrm{DC}_S(i;0,o^+))) \ \le\ \mathrm{cost}(\nu_0,o(\nu_0,u))\cdot h\, m_{Y_S}(i',u). \tag{1}$$

By (ii) and Input Q1, $\mathsf{P}^0_i(o) \subseteq (\mathfrak{X}_{\tilde{Y}_S}(i;\mathrm{rs}_0(\cdot,\cdot)))_{\mathfrak{S}}$, and $\mathsf{P}^0_i(o) \subseteq \mathrm{Deep}_S(i;o)$ (Input Q4(b)); so $\mathsf{P}^0_i(o) \subseteq \mathsf{P}_i(o)$, and $\mathrm{pr}_S^{C,\mathrm{loc}}$ vanishes on $\mathsf{P}^0_i(o)$. By Input K1(b) the left side of (1) is the relative length of $\mathsf{P}_i(o)$ and the preimage of $\mathrm{pr}_S^{C,\mathrm{loc}}(\mathsf{P}_i(o^+))$, which contains $N' = \mathsf{P}_i(o^+) + \mathsf{P}^0_i(o)$. We apply Input K1(d) to $N = \mathsf{P}_i(o)$ and N', with the exponent $\mathrm{cost}(\nu_0,o(\nu_0,u))$ and with $m = h\, m_{Y_S}(i',u)$.

The power of p. Let $x \in \mathsf{P}_i(o)$. The only step from o to o^+ is $(\nu_0,o(\nu_0,u))$. Let A be the additive map of Input Q4(b) for this one step, of cost $\epsilon = \mathrm{cost}(\nu_0,o(\nu_0,u))$. Then $\mathsf{A}(x) \in \mathrm{Deep}_S(i;o^+)$, $\mathsf{A}(x) - p^\epsilon x \in \mathsf{P}^0_i(o)$, and $\mathsf{A}(x)$ lies in $(\mathfrak{X}_{\tilde{Y}_S}(i;\mathrm{rs}_0(\cdot,\cdot)))_{\mathfrak{S}}$, a $\mathbb{Z}_p$-module that contains $\mathsf{P}^0_i(o)$. So $p^\epsilon x \in N'$.

The relations. Let $x_0,\dots,x_m \in \mathsf{P}_i(o)$ with $p\,x_k \in N'$, and write $p\,x_k = x'_k + w_k$ with $x'_k \in \mathsf{P}_i(o^+)$ and $w_k \in \mathsf{P}^0_i(o)$. By strictness (Input Q4(e)), $w_k = (1\otimes \mathrm{in}_S^Y)(y_k)$ with $y_k \in \mathfrak{M}_{\mathrm{loc}}(Y_S)\cap \mathfrak{X}_{Y_S}(i;o)$. By the minimality of u and by (iii), every $v \in U_{Y_S}(i')$ with $v < u$ has $o(\nu_0,v) \ge o'(\nu_0,v) \ge o'(\nu_0,u) > o(\nu_0,u)$. So Input Q6 applies to the lattice datum Y_S, the letters i' and i, the jump u with $n = m_{Y_S}(i',u)$, and the elements y_k: there are $c_k \in \mathbb{Z}_p$, one of them equal to 1, and

$$\sum_k c_k y_k = y' + y'', \qquad y' \in (\mathfrak{X}_{Y_S}(i;o_+))_{\mathfrak{S}},\quad y'' \in (\mathfrak{X}_{Y_S}(i;o^+))_{\mathfrak{S}},$$

where o_+ is o raised by 1 at $\nu = 0$. Put $x^* = \sum_k c_k x_k \in \mathsf{P}_i(o)$ and $w = (1\otimes \mathrm{in}_S^Y)(y') \in \mathsf{P}^0_i(o_+)$. We show $x^* \in N'$.

First, $p\,x^* - w \in \mathrm{Deep}_S(i;o^+)$: it lies in $\mathfrak{M}_{\mathrm{loc}}(\tilde{Y}_S)$ over a constant, and

$$p\,x^* - w = \sum_k c_k x'_k + (1\otimes \mathrm{in}_S^Y)(y''),$$

where each x'_k is invariant modulo $\mathfrak{X}_{\tilde{Y}_S}(i;o^+)$ and the last term lies in $\mathfrak{X}_{\tilde{Y}_S}(i;o^+)$, a module stable under $x \mapsto gx - x$ for $g \in \Gamma'$ (Input Q1).

Secondly, let I_2 be the set of the pairs $(0,n)$ with $o(0,v) < n < o(0,v) + p^{\nu_0}$ for some $v \in U_{Y_S}(i)$. By (i) each of them has $\mathrm{cost}(0,n) = v_p(n) = 0$; so the additive map B of Input Q4(b) for I_2 has cost 0. Since $w \in \mathsf{P}^0_i(o_+) \subseteq \mathrm{Deep}_S(i;o_+)$, the element $\mathsf{B}(w)$ lies in $\mathsf{P}^0_i(o_+)$ and in $\mathrm{Deep}_S(i;o_2)$, where $o_2(0,v) = o(0,v) + p^{\nu_0}$ and $o_2(\nu,v) = o(\nu,v)$ for

$\nu > 0$: the steps from o_+ to o_2 are in I_2. By Input Q2, $\mathsf{B}(w) \in \mathfrak{X}_{\tilde{Y}_S}(i; \mathrm{imp}(o_+, o_2)(\cdot,\cdot))$. Now $\mathrm{imp}(o_+, o_2)(0,v) - p^{\nu_0} \geq o(0,v)$: this is clear if $o(0,v) \geq 0$ or $o(0,v) + p^{\nu_0} \leq 0$, and the remaining case $o(0,v) < 0 < o(0,v) + p^{\nu_0}$ is excluded by (i), as p divides 0. Hence, by Input Q1,

$$w_3 = \mathcal{E}^{-p^{\nu_0}}\,\mathsf{B}(w) \;\in\; \mathsf{P}_i^0(o).$$

Since $\mathcal{E} = z + p$ and $\mathcal{E}_{\nu_0} = z^{p^{\nu_0}} + p$, we have $\mathcal{E}^{p^{\nu_0}} = \mathcal{E}_{\nu_0} + p\,r$ with $r \in \mathfrak{S}$. Put $w_4 = r\,w_3 - (\mathsf{B}(x^*) - x^*) \in \mathsf{P}_i^0(o)$; the second term lies in $\mathsf{P}_i^0(o)$ because B has cost 0. As B is additive,

$$p\,(x^* - w_4) = p\,\mathsf{B}(x^*) - p\,r\,w_3 = \mathsf{B}(p\,x^* - w) + \mathcal{E}^{p^{\nu_0}} w_3 - p\,r\,w_3 = \mathsf{B}(p\,x^* - w) + \mathcal{E}_{\nu_0}\,w_3.$$

Here $\mathsf{B}(p\,x^* - w) \in \mathrm{Deep}_S(i; o^+)$ (Input Q4(b), for the module $\mathrm{Deep}_S(i; o^+)$ itself), and $\mathcal{E}_{\nu_0} w_3 \in \mathfrak{X}_{\tilde{Y}_S}(i; o^+)$ because multiplication by $\mathcal{E}_{\nu_0}$ raises the exponents at ν_0 by 1. So for every $g \in \Gamma'$ the element $p\,(g(x^* - w_4) - (x^* - w_4))$ lies in $\mathfrak{X}_{\tilde{Y}_S}(i; o^+)$; as no $\mathcal{E}_\nu$ divides p, Input Q1 gives $g(x^* - w_4) - (x^* - w_4) \in \mathfrak{X}_{\tilde{Y}_S}(i; o^+)$. Therefore $x^* - w_4 \in \mathsf{P}_i(o^+)$ and $x^* \in N'$. □

Lemma 3.2 (the colength of the constants that lift)**.** *In Setting 2.5, assume* (F1) *to* (F6)*. For a letter i let o_i be the following exponent function: $o_i = \mathrm{rs}_{2\rho+\eta}(\cdot,\cdot)$ if $\mathsf{s}(i) = 1$; and if $\mathsf{s}(i) = -1$,*

$$o_i(0,u) = \beta(i) + W + 2\rho, \qquad o_i(1,u) = W + 1 - u + 2\rho_u, \qquad o_i(\nu,u) = -u \;\; (\nu \geq 2),$$

where $\rho_u = \rho$ if $|u| \leq \eta$ and $\rho_u = 0$ otherwise. Then for every letter i, with N_i the image of the map $c \mapsto \mathrm{cst}_{S,i}(c)$ on C_S,

$$\mathrm{len}_{\mathbb{Z}_p}(N_i/(N_i \cap \mathrm{DC}_S(i; 0, o_i))) \;\leq\; (2\rho+1)\,n_{\mathrm{lo}}(i) + 3\,(W+1)\,n_{\mathrm{all}}(i^-) + 6\rho\,n_{\mathrm{lo}}(i^-).$$

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Proof. By (F2), (F3) and (F5), $\rho \geq 1$, every jump of Y_S is at most $(p-3)\rho$, and a jump u at i with $|u| > \eta$ has $|u| \geq \beta(i) > 6(\rho + \eta + W)$.

A supply of points. Let $s \in \{0, \rho\}$, $d \in C_S$ and n an integer with

$$n - u \leq \mathrm{imp}(\mathrm{rs}_{s-\rho}(\cdot,\cdot), \mathrm{rs}_{2\rho+\eta}(\cdot,\cdot))(0,u) \qquad \text{for all } u \in U_{Y_S}(i). \tag{2}$$

Then $p^s\,\mathrm{cst}_{S,i}(d) \in \mathrm{DC}_S(i; n, \mathrm{rs}_{2\rho+\eta}(\cdot,\cdot))$. Indeed $\mathrm{Kill}_S(i; s, \rho - s)$ by Input Q3(h), which gives a point over $p^s d$ with exponents and depth $\mathrm{rs}_{s-\rho}(\cdot,\cdot)$ (Input Q4(a)); the steps to the depth $\mathrm{rs}_{2\rho+\eta}(\cdot,\cdot)$ are at $\nu = 0$ and lie in $\mathcal{W}_{6(\rho+\eta+W)}(\beta(i)) \cup \mathcal{W}_{6(\rho+\eta+W)}(-\beta(i))$, so by (F4) and Input Q4(c), with $\mathfrak{d} = 0$, there is a point over $p^s d$ with exponents $\mathrm{imp}(\mathrm{rs}_{s-\rho}(\cdot,\cdot), \mathrm{rs}_{2\rho+\eta}(\cdot,\cdot))(\cdot,\cdot)$ and depth $\mathrm{rs}_{2\rho+\eta}(\cdot,\cdot)$; at $\nu \geq 1$ these exponents are $-u$, and (2) is the comparison at $\nu = 0$.

The low weights. We show, for every integer t with $-\eta \leq t \leq 0$,

$$\mathrm{len}_{\mathbb{Z}_p}(N_i/(N_i \cap \mathrm{DC}_S(i; t, \mathrm{rs}_{2\rho+\eta}(\cdot,\cdot)))) \;\leq\; \rho \sum_{-\eta \leq u < t} h\,m_{Y_S}(i,u). \tag{3}$$

For $t = -\eta$ the supply with $s = 0$ and $n = -\eta$ gives $N_i \subseteq \mathrm{DC}_S(i; -\eta, \mathrm{rs}_{2\rho+\eta}(\cdot,\cdot))$: in (2) the right side is $2\rho + \eta - u$ for $u < -\rho$, and $\min(2\rho + \eta - u, 0)$ for $u \geq -\rho$, when $u \geq -\eta$ because no jump lies in $[-\rho, -\eta)$. Let $-\eta \leq t < 0$ and suppose (3) for t. By Input K1(a) and (b) it suffices to bound by $\rho \cdot h\,m_{Y_S}(i,t)$ the relative

length of $N = (\mathfrak{X}_{\tilde{Y}_S}(i;\mathrm{rs}_t(\cdot,\cdot)))_{\mathfrak{S}} \cap \mathrm{Deep}_S(i;\mathrm{rs}_{2\rho+\eta}(\cdot,\cdot))$ and the preimage N' of $\mathrm{DC}_S(i;t+1,\mathrm{rs}_{2\rho+\eta}(\cdot,\cdot))$ under $\mathrm{pr}_S^{C,\mathrm{loc}}$; we use Input K1(d). For $x \in N$ over $\mathrm{cst}_{S,i}(d)$, the supply with $s=\rho$ and $n=t+1\le 0$ gives $p^\rho x \in N'$: the right side of (2) is now $2\rho+\eta-u$ for $u<0$ and $\min(2\rho+\eta-u,0) \ge t+1-u$ for $u\ge 0$. Let $x_0,\dots,x_m \in N$ with $m = h\,m_{Y_S}(i,t)$. The multiplicity of the jump $t\neq 0$ of $\tilde{Y}_S$ is at most that of Y_S (Input K5), and $\mathrm{rs}_t(0,t) = 0 < \mathrm{rs}_t(0,v)$ for $v<t$; so by Input Q6, for the lattice datum $\tilde{Y}_S$ and $\nu_0=0$, a combination $x^* = \sum_k c_k x_k$ with $c_k \in \mathbb{Z}_p$, one of them 1, lies in $\mathfrak{X}_{\tilde{Y}_S}(i;o^+)$, where o^+ is $\mathrm{rs}_t(\cdot,\cdot)$ raised by 1 at $(0,t)$ (the other module of Input Q6 is contained in this one). By Input Q2, $x^* \in \mathfrak{X}_{\tilde{Y}_S}(i;\mathrm{imp}(o^+,\mathrm{rs}_{2\rho+\eta}(\cdot,\cdot))(\cdot,\cdot))$, and $\mathrm{imp}(o^+,\mathrm{rs}_{2\rho+\eta}(\cdot,\cdot))(0,u) \ge t+1-u$: for $u\le t$ the left side is $2\rho+\eta-u$, and for $u>t$ it is at least $\min(2\rho+\eta-u,0) \ge t+1-u$. So $x^* \in (\mathfrak{X}_{\tilde{Y}_S}(i;\mathrm{rs}_{t+1}(\cdot,\cdot)))_{\mathfrak{S}}$ and $x^* \in N'$. This proves (3); for $t=0$ its right side is at most $\rho\,n_{\mathrm{lo}}(i)$ by (F6).

If $\mathsf{s}(i)=1$ the lemma is proved. Let $\mathsf{s}(i)=-1$, and let N be the least multiple of p with $\mathsf{N} \ge \beta(i)+W+2\rho$.

The index 0. Apply Lemma 3.1 with $\nu_0=0$, $i'=i$, $A=0$, $B=\rho+1$, $o=\mathrm{rs}_{2\rho+\eta}(\cdot,\cdot)$ and $o'(0,u)=\mathsf{N}$, $o'(\nu,u)=-u$ for $\nu\ge 1$. Hypotheses (i) to (iii) are clear. For (iv) let $u \in U_{Y_S}(i)$. If $|u|\le\eta$, the sum is at most $\sum_{0\le n\le \mathsf{N}} v_p(n) = v_p(\mathsf{N}!)$, and by Legendre's formula and (F2)

$$(p-1)\,v_p(\mathsf{N}!) \le \mathsf{N}-1 \le \beta(i)+W+2\rho+p-2 \le (p-1)(\rho+1).$$

If $|u|>\eta$, then $-u \ge \beta(i)$, and the sum runs over $\beta(i)+2\rho+\eta \le n < \mathsf{N}$. A multiple n of p in this range has $n \le \mathsf{N}-p < \beta(i)+W+2\rho$, so it lies in $\mathcal{W}_{6(\rho+\eta+W)}(\beta(i))$ and is not zero, against (F4); so the sum is 0. Hence $\mathrm{len}_{\mathbb{Z}_p}(\mathrm{DC}_S(i;0,o)/(\mathrm{DC}_S(i;0,o)\cap \mathrm{DC}_S(i;0,o'))) \le (\rho+1)\,n_{\mathrm{lo}}(i)$.

The index 1. Apply Lemma 3.1 with $\nu_0=1$, $i'=i^-$, $A=3(W+1)$, $B=6\rho$, the function o' just defined in the place of o, and o_i in the place of o'. (i): there is no multiple of p strictly between N and $\mathsf{N}+p$. (ii): $-u\le\mathsf{N}$ for $u\ge-\beta(i)-W$, and $o_i \le o'$ at $\nu\neq 1$. (iii): for $u<u'$ in $U_{Y_S}(i^-)$ one has $o_i(1,u) \ge o_i(1,u')$ unless $|u'|\le\eta<|u|$, when $u'-u \ge \beta(i^-)-\eta > 2\rho$. (iv): the sum has $W+1+2\rho_u$ terms $\mathrm{cost}(1,n)$ with $-u \le n < W+1-u+2\rho_u$; these n lie in $\mathcal{W}_{6(\rho+\eta+W)}(\beta(i^-))\cup\mathcal{W}_{6(\rho+\eta+W)}(-\beta(i^-))$, so by (F4) and Input Q1 each term is at most 3. Hence $\mathrm{len}_{\mathbb{Z}_p}(\mathrm{DC}_S(i;0,o')/(\mathrm{DC}_S(i;0,o')\cap \mathrm{DC}_S(i;0,o_i))) \le 3(W+1)\,n_{\mathrm{all}}(i^-)+6\rho\,n_{\mathrm{lo}}(i^-)$.

The three bounds add up, by Input K1(a), to the bound of the lemma. □

Proposition 3.3 (few low weights: a submodule of small colength on which cup products bound)**.** *In Setting 2.5, assume* (F1) *to* (F6)*, and suppose* $\eta\ge 1$*. Then there is a* $\mathbb{Z}_p$*-submodule* C *of* C_S *with*

$$\mathrm{len}_{\mathbb{Z}_p}(C_S/(C_S\cap\mathsf{C})) \;\le\; 11\sum_i \big(n_{\mathrm{lo}}(i)\,(\rho+1)+n_{\mathrm{all}}(i)\,(W+1)\big),$$

the sum over the letters, such that for every side T *over* $\mathcal{O}$*, every pairing* Π *of* T *and* S *and every* $c\in C_T$*: if* $U_{Y_T}(i) \subseteq [-\eta,\eta]\cup -\mathsf{s}(i)\cdot[\beta(i),\beta(i)+W]$ *for every letter* i*, then* $\mathrm{Cup}_\Pi(0;c,d)$ *for every* $d\in\mathsf{C}$*.*

LEAN `Lift.KummerChart.PairingSide.exists_scaledCupBounds_few_low_jumps`

Proof. Let C be the set of the $d \in C_S$ with $\mathrm{cst}_{S,i}(d) \in \mathrm{DC}_S(i;0,o_i)$ for every letter i, with o_i as in Lemma 3.2. By Input K1(b) and (c) and Lemma 3.2,

$$\begin{aligned}\mathrm{len}_{\mathbb{Z}_p}(C_S/(C_S\cap\mathsf{C})) &\le \sum_i \big((2\rho+1)\,n_{\mathrm{lo}}(i) + 3(W+1)\,n_{\mathrm{all}}(i^-) + 6\rho\,n_{\mathrm{lo}}(i^-)\big)\\ &= \sum_i \big((8\rho+1)\,n_{\mathrm{lo}}(i) + 3(W+1)\,n_{\mathrm{all}}(i)\big),\end{aligned}$$

since $i \mapsto i^-$ permutes the letters; this is at most the bound stated.

Let T, Π, c be as in the statement and $d \in \mathsf{C}$. We apply Theorem B of Part 2 (Input Q5) to the pairing Π of T and S with $\kappa = e = e' = E = 0$, with every letter a cut, so that $\mathsf{I}(i) = 1$, with $\mathsf{Q}(i)$ the statement $\mathsf{s}(i) = 1$, and with the families

$$o^T = \mathrm{imp}(\mathrm{rs}_{-\rho}(\cdot,\cdot), \mathrm{rs}_{2\rho+\eta}(\cdot,\cdot))(\cdot,\cdot), \qquad o_i^S = \mathrm{imp}(\mathrm{rs}_0(\cdot,\cdot), o_i)(\cdot,\cdot),$$
$$\mathsf{z} = \mathrm{imp}(\mathrm{rs}_{-\rho}(\cdot,\cdot), \mathrm{rs}_{\rho}(\cdot,\cdot))(\cdot,\cdot), \qquad \mathsf{z}' = \mathrm{rs}_{\rho}(\cdot,\cdot),$$

the first, third and fourth being the same at all letters. Explicitly, at $\nu \ge 1$ the functions o^T and z are $u \mapsto -u$, and

$$o^T(0,u) = \begin{cases} 0, & -\rho \le u \le 2\rho+\eta,\\ 2\rho+\eta-u, & \text{otherwise,}\end{cases} \qquad \mathsf{z}(0,s) = \begin{cases} 0, & |s| \le \rho,\\ \rho - s, & |s| > \rho;\end{cases}$$

in particular $\mathsf{z}(\nu,\cdot)$ is non-increasing and

$$\mathsf{z}(0,s) \ge -s-\rho \quad \text{for all } s. \tag{4}$$

For the jumps v of Y_S at j, $o_j^S(0,v)$ is $2\rho+\eta-v$ or 0 if $\mathsf{s}(j) = 1$, the value 0 only for $v \ge 0$; it is $\beta(j)+W+2\rho$ for $v < 0$ and 0 for $v \ge 0$ if $\mathsf{s}(j) = -1$; and

$$o_j^S(\nu,v) \ge -v \ \text{ and } \ o_j(\nu,v) \ge -v \qquad \text{for all } \nu \ge 0 \text{ and } v \in U_{Y_S}(j^{p^{-\nu}}), \tag{5}$$

because $v \ge -\beta(j) - W$ at $\nu = 0$.

The points. By (F2) every jump of T, of S and of the third side is at most $(p-3)\rho$: those of the third side are 0 or $-u-v-1$ with u, v jumps of T and S at the same letter, in opposite windows. So $\mathrm{Kill}_T(i;0,\rho)$ and $\mathrm{Kill}_{S_\Pi^\vee}(i;0,\rho)$ (Input Q3(h)), and as in the proof of Lemma 3.2, by (F4) and Input Q4(a) and (c) with $\mathfrak{d} = 0$, the sets $\mathrm{Pt}_T(c,i;0;o^T,\mathrm{rs}_{2\rho+\eta}(\cdot,\cdot))$ and $\mathrm{Pt}_{S_\Pi^\vee}(1,i;0;\mathsf{z},\mathsf{z}')$ are not empty: the steps of the third side are within $\rho+2\eta+W+1$ of 0 or $\pm\beta(i)$, hence in the windows of (F4), because $\rho \ge 1$ by (F2) and (F3). (The formal statement assumes $\eta \ge 1$ and passes it to its automated arithmetic; no step of the argument given here uses it.) Since $d \in \mathsf{C}$, there is $x \in \mathsf{P}_i(o_i)$ over $\mathrm{cst}_{S,i}(d)$; by Input Q2 it is a point of $\mathrm{Pt}_S(d,i;0;o_i^S,o_i)$.

The deep point of T. We check Definition 2.6 of Part 2 for T against S at i, with the depth $\mathrm{rs}_{2\rho+\eta}(\cdot,\cdot)$ and $r = 0$. Let $\nu \ge 0$, u a jump of T and v a jump of S at $i^{p^{-\nu}}$, and $s = -u-v-1$. The left side of clause (a) is $s+u = -v-1$ for $\nu \ge 1$, and for $\nu = 0$ it is, by (4),

$$-\mathsf{z}(0,s) - (2\rho+\eta-u) \ \le \ -v-1-\rho-\eta.$$

Both are smaller than $-v$ and, by (5) and the list of the values of $o_i^S(0,v)$, smaller than $o_i^S(\nu,v)$. For clause (b) let $\mathsf{s}(i) = 1$. Then the jumps of T at i are at most η, so $o^T(0,u) \ge 0 = \mathrm{std}_1(0,u)$, and $o^T(\nu,u) = -u = \mathrm{std}_1(\nu,u)$ for $\nu \ge 1$. With $r = 0$ the

left side is that of (a); the additional bound $\mathrm{fr}_{1,0}(o^S_{i^-})(\nu,v)$ is 0 for $\nu=0$, where the left side is at most $-v-1-\rho-\eta<0$ because $v\ge-\eta$; and it is $o^S_{i^-}(\nu-1,v)\ge -v$ for $\nu\ge1$, by (5).

The deep point of S. We check the definition for S against T at i, for the proposition $\mathsf{s}(i)=-1$, with the depth o_i and $r=1$. Let v be a jump of S and u a jump of T at $i^{p^{-\nu}}$, $s=-u-v-1$. Clause (a) asks that $-\mathsf{z}(\nu,s)-o_i(\nu,v)<\min(-u,o^T(\nu,u))$. For $\nu\ge1$ the left side is at most $s+v=-u-1$, by (5), and $o^T(\nu,u)=-u$. For $\nu=0$ and $\mathsf{s}(i)=1$ it is at most $-u-1-\rho-\eta$ by (4). For $\nu=0$ and $\mathsf{s}(i)=-1$ it is $-\mathsf{z}(0,s)-\beta(i)-W-2\rho\le-u-1-\rho$, by (4) and $v\ge-\beta(i)-W$. In both cases it is smaller than $-u$, than $2\rho+\eta-u$, and than 0 when $u\ge-\rho$, hence smaller than $o^T(0,u)$.

Clause (b), for $\mathsf{s}(i)=-1$: $o^S_i(0,v)\ge0$ and $o^S_i(\nu,v)\ge-v$ for $\nu\ge1$, so $\mathrm{std}_1\le o^S_i$ on the jumps. It remains to show

$$-\mathrm{fr}_{1,0}(\mathsf{z})(\nu,s)-o_i(\nu,v)\ <\ \min\bigl(o^T(\nu,u),\ \mathrm{fr}_{1,0}(o^T)(\nu,u)\bigr).$$

For $\nu=0$ the left side is $-(\beta(i)+W+2\rho)$, the right side is $\min(o^T(0,u),0)$, and $o^T(0,u)\ge2\rho+\eta-\beta(i)-W$ or $o^T(0,u)=0$. For $\nu\ge2$ the left side is $s+v=-u-1$ and the right side is $-u$. Let $\nu=1$; the letter is i^-, the left side is $-\mathsf{z}(0,s)-(W+1-v+2\rho_v)$, the right side is $\min(-u,o^T(0,u))$. So we need

$$\mathsf{z}(0,s)>-s-W-2-2\rho_v\qquad\text{and}\qquad\mathsf{z}(0,s)>v-W-1-2\rho_v-o^T(0,u).$$

The first: if $|v|\le\eta$ it follows from (4); if $|v|>\eta$ and $|s|>\rho$ from $\mathsf{z}(0,s)=\rho-s$; and if $|v|>\eta$ and $|s|\le\rho$, then u is a far jump as well, in the opposite window, so $|u+v|\le W$, $s\ge-W-1$ and $\mathsf{z}(0,s)=0>-s-W-2$. The second: if $o^T(0,u)=2\rho+\eta-u$ it follows from (4). If $o^T(0,u)=0$, then $-\rho\le u\le2\rho+\eta$, so $|u|\le\eta$; for $|v|\le\eta$, (4) gives $\mathsf{z}(0,s)\ge u+v+1-\rho\ge v+1-2\rho$; for $|v|>\eta$ one has $|s|>\rho$ and $\mathsf{z}(0,s)=\rho+u+v+1\ge v+1$.

All hypotheses of Theorem B of Part 2 hold, and it gives $\mathrm{Cup}_\Pi(0;c,d)$. □

4 Tame charts

In this section $\mathfrak{c}$ is a tame chart for Γ (Definition 2.3), with its root of unity ϱ, its tame generator σ and its element ϖ. For a polynomial q over $\mathbb{Z}_p$ write $q=\sum_k q_kX^k$; it kills $(\sigma,0)$ on a lattice if its image in $\mathcal{O}[X]$ does.

Lemma 4.1 (a polynomial in the tame generator makes elements divisible by the coordinate)**.** *In Setting 2.1, let $\mathfrak{c}$ be a tame chart for Γ with tame generator σ, L a lattice datum over $\mathcal{O}$, and q a polynomial over $\mathbb{Z}_p$ with $q(\beta^L_{\sigma,0})=0$. Then for every $x\in\mathfrak{M}(L)$ there is $y\in\mathfrak{M}^{\inf}(L)$ with*

$$z\cdot y=\sum_{k=0}^{\deg q}q_k\,\sigma^k x.$$

LEAN `Lift.TameChart.exists_z_smul_eq_sum_tameGen`

Proof. Since Γ' is normal, each σ^k preserves $\mathfrak{M}^{\inf}(L)$, and $\mu_k=\sigma^{-k}\circ\mu\circ\sigma^k$ is a period for every period μ (Input K7(e)); so the right side w lies in $\mathfrak{M}^{\inf}(L)$. By Input K7(a) it suffices to show $\mathcal{E}^{2h_L}\mu(w)\in B^+z$ for every crystalline period μ. Let

$\lambda\colon L \to B_{\max}$ give the values of μ on $1 \otimes L$, and for $g \in \Gamma$ let $\Lambda_g\colon B_{\max} \otimes L \to B_{\max}$ be the $B_{\max}$-linear map with $\Lambda_g(1 \otimes t) = j_\Gamma(g)^{-1}\lambda(gt)$. Then

$$\Lambda_{g'}\big((j_\Gamma(g) \otimes g)\, y\big) = j_\Gamma(g)\, \Lambda_{g'g}(y) \qquad (g, g' \in \Gamma,\ y \in B_{\max} \otimes L), \tag{6}$$

and $\Lambda_{\sigma^k g} = \Lambda_{\sigma^k}$ for $g \in \Gamma'$, because μ commutes with the element $\sigma^k g \sigma^{-k}$ of Γ'.

Let $y \in B_{\max} \otimes L$ be fixed near 1. By (6) the element $\Lambda_{\sigma^k}(y)$ of $B_{\max}$ is fixed near 1, hence fixed by the inertial element σ^k (Input K2(c)). On such y, $(\beta^L_{\sigma,0})^k = \beta^L_{\sigma^k,0}$ acts as $j_\Gamma(\sigma^k) \otimes \sigma^k$ (Input K2(a)), so by (6) again

$$\Lambda_1\big((\beta^L_{\sigma,0})^k y\big) = j_\Gamma(\sigma^k)\, \Lambda_{\sigma^k}(y) = \Lambda_{\sigma^k}(y),$$
$$\text{and} \qquad \sum_k q_k \Lambda_{\sigma^k}(y) = \Lambda_1\big(q(\beta^L_{\sigma,0})\, y\big) = 0.$$

As L is potentially crystalline, such y span $B_{\max} \otimes L$; so $\sum_k q_k \Lambda_{\sigma^k} = 0$. On $1 \otimes t$ this says $\sum_k q_k \mu_k(1 \otimes t) = 0$, hence $\sum_k q_k \mu_k(x) = 0$ (Input K7(f)). Therefore

$$\mu(w) = \sum_k q_k\, \sigma^k(\mu_k(x)) = \sum_k q_k(\sigma^k \mu_k(x) - \mu_k(x)).$$

Each μ_k is a crystalline period, so $\mathcal{E}^{h_L}\mu_k(x)$ lies in the image of $\mathfrak{S}_B$ (Input K7(b)); σ^k fixes $\mathbb{W}$, so z divides $\sigma^k s - s$ for $s \in \mathfrak{S}_B$ (Input K7(c)); and Input K7(d) gives $\mathcal{E}^{2h_L}(\sigma^k \mu_k(x) - \mu_k(x)) \in B^+ z$. □

Lemma 4.2 (the extension is killed by the product with $X - 1$)**.** *In Setting 2.1, let $\mathfrak{c}$ be a tame chart for Γ with tame generator σ, S a side over $\mathcal{O}$, and q a polynomial over $\mathbb{Z}_p$ that kills $(\sigma, 0)$ on Y_S. Then*

$$(q \cdot (X - 1))(\beta^{\tilde{Y}_S}_{\sigma,0}) = 0.$$

LEAN `Lift.TameChart.aeval_linearFrob_extLattice_eq_zero`

Proof. The maps in^Y_S and pr^C_S commute with Γ, so the linear Frobenius commutes with $1 \otimes \mathrm{in}^Y_S$ and $1 \otimes \mathrm{pr}^C_S$, and it is the identity on $B_{\max} \otimes C^{\mathrm{tr}}_S$ (Input K2(b)). For $y \in B_{\max} \otimes \tilde{Y}_S$ the element $\beta^{\tilde{Y}_S}_{\sigma,0} y - y$ therefore maps to 0 in $B_{\max} \otimes C_S$, so it is $(1 \otimes \mathrm{in}^Y_S)(x)$ with $x \in B_{\max} \otimes Y_S$, and $q(\beta^{\tilde{Y}_S}_{\sigma,0})(1 \otimes \mathrm{in}^Y_S)(x) = (1 \otimes \mathrm{in}^Y_S)(q(\beta^{Y_S}_{\sigma,0})x) = 0$. □

Proposition 4.3 (tame charts: a cokernel killer independent of the index)**.** *In Setting 2.1, let $\mathfrak{c}$ be a tame chart for Γ with tame generator σ, S a side over $\mathcal{O}$, $n, m, \kappa \geq 0$ integers, i a letter, and q a monic polynomial over $\mathbb{Z}_p$ that kills $(\sigma, 0)$ on Y_S. Suppose that $u \leq n$ for every letter j and every $u \in U_{Y_S}(j)$, that*

$$e_{\mathfrak{c}} \leq p^m, \qquad n + \kappa + 2 < p, \qquad m\,(\deg q + 1) \leq \kappa.$$

Then $\mathrm{Kill}_S(i; \kappa, 0)$.

LEAN `Lift.TameChart.killsCokernel_of_isKilledBy`

Proof. Put $Q = q \cdot (X - 1) = \sum_k Q_k X^k$, monic of degree $\deg q + 1$, with $Q(1) = 0$, and $r = p^{\kappa+1}$. By Input Q3(g) there is $a_0 \geq 0$ with $z^{a_0} \in \mathfrak{a}_S(r; 0, i)$ and $a_0(p-1) \leq e_{\mathfrak{c}}(n + \kappa + 1)$; as $n + \kappa + 1 < p - 1$, $a_0 < e_{\mathfrak{c}}$. For $0 \leq a < e_{\mathfrak{c}}$ let N_a be the number of the integers k with $a < k < e_{\mathfrak{c}}$ and $Q(\varrho^k) \in p\mathbb{W}$. We show, by descending induction on a from a_0,

$$p^{mN_a} z^a \in \mathfrak{a}_S(r; 0, i) \qquad (0 \leq a \leq a_0). \tag{7}$$

For $a = a_0$ it holds. Let $0 < a \le a_0$ and suppose (7) for a; write $c = mN_a$.

If $Q(\varrho^a) \in p\mathbb{W}$, then $N_{a-1} = N_a + 1$. By Input Q3(c), $p^c z^{pa} \in \mathfrak{a}_S(r; 1, i^p)$, and by Input Q3(e), with $p(a-1) \le pa$, the element $p^{c+m} z^{a-1}$ lies in $\mathfrak{a}_S(r; 0, i)$ as soon as $c + v_p((pa)!) \le c + m + v_p((p(a-1))!)$, that is $1 + v_p(a) \le m$; and $p^{v_p(a)} \le a < e_{\mathfrak{c}} \le p^m$.

If $Q(\varrho^a)$ is a unit of $\mathbb{W}$, then $N_{a-1} = N_a$. Let $f = p^c z^a$. By Input Q3(d) there is $t_0 \in \mathfrak{S}$ such that for every $c' \in C_S$ some $x \in \mathfrak{N}_{0,i}(\tilde{Y}_S)$ lies over $(f - r\,t_0)\,\mathrm{cst}_{S,i}(c')$; write $t_0 = w + z\,t$ with $w \in \mathbb{W}$, $t \in \mathfrak{S}$ (Input K8(c)). By Input Q1, $x \in \mathfrak{M}^{\inf}(\tilde{Y}_S)$, and it is a point over F_0; so $x \in \mathfrak{M}(\tilde{Y}_S)$. By Lemmas 4.2 and 4.1 there is $y \in \mathfrak{M}^{\inf}(\tilde{Y}_S)$ with $z\,y = \sum_k Q_k \sigma^k x$. Now σ^k fixes $\mathbb{W}$, hence commutes with pr_i and fixes $\mathrm{cst}_{S,i}(c')$, and $\sigma^k z = \varrho^k z$; so, with $T = \sum_k Q_k \varrho^k \sigma^k(t) \in A_{\inf}$,

$$\begin{aligned} z \cdot (1 \otimes \mathrm{pr}_S^C)(\mathrm{pr}_i\, y) &= \sum_k Q_k (p^c \varrho^{ka} z^a - r\,w - r\,\varrho^k z\,\sigma^k(t))\,\mathrm{cst}_{S,i}(c') \\ &= z\,\big(Q(\varrho^a)\,p^c z^{a-1} - r\,T\big)\,\mathrm{cst}_{S,i}(c'), \end{aligned}$$

the term with w vanishing because $Q(1) = 0$. The coordinate z is a unit of $W(C^\flat)$, and $\mathrm{pr}_i\, y \in \mathfrak{X}_{\tilde{Y}_S}(i; \mathrm{std}_0)$ (Input Q1); so by Input Q3(a), $Q(\varrho^a)\,p^c z^{a-1} \in \mathfrak{a}_S(r; 0, i)$, and the unit can be removed.

This proves (7). For $a = 0$ it says $p^{mN_0} \in \mathfrak{a}_S(r; 0, i)$. The elements ϱ^k, $0 < k < e_{\mathfrak{c}}$, are distinct modulo p, and the reduction of Q is monic, so $N_0 \le \deg q + 1$ and $mN_0 \le \kappa$. Hence $p^\kappa \in \mathfrak{a}_S(p^{\kappa+1}; 0, i)$, and Input Q3(b) gives $p^\kappa \in \mathfrak{a}_S(0; 0, i)$. □

Proposition 4.4 (existence of tame charts)**.** *In Setting 2.1, suppose that j_Γ is surjective. Let $e \ge 0$ and $h_0 \ge 1$ be integers such that p does not divide h_0 and e divides $p^{h_0} - 1$. Then there is a tame chart for Γ whose index is e.*

LEAN `Lift.exists_tameChart`

Proof. Put $N = p^{h_0} - 1$; it is prime to p, and $e \ge 1$. Choose $\varpi \in \bar{\mathbb{Q}}_p$ with $\varpi^e = p$, an element $\varpi^\flat$ of the tilt $\mathcal{O}_C^\flat$ given by a compatible system of p-power roots of ϖ (its *untilt* is ϖ), and a primitive e-th root of unity a in the subfield $\mathbb{F}_{p^{h_0}}$ of the tilt. By Input K8(a), for $x = \varpi$ and $v = -1$, there is a chart with residue degree h_0, index e, constant unit series, and coordinate $z = [\varpi^\flat]$ with $\theta(z) = \varpi$.

The subgroup. Let χ_N be the character of $G_{\mathbb{Q}_p}$ on the N-th roots of unity, and $\Delta = \ker \chi_N \cap \mathrm{Stab}(\varpi)$, an open subgroup. An element of $G_{\mathbb{Q}_p}$ fixes $\mathbb{W}$ if and only if it lies in $\ker \chi_N$; so Δ is the group of the elements that fix $\mathbb{W}$ and $\theta(z)$. For $g \in G_{\mathbb{Q}_p}$ the quotient $g\varpi/\varpi$ is an e-th, hence an N-th root of unity. It follows that Δ is normal: for $\delta \in \Delta$ and $g \in G_{\mathbb{Q}_p}$, δ fixes $g^{-1}\varpi = (g^{-1}\varpi/\varpi)\,\varpi$, so $g\delta g^{-1}$ fixes ϖ. It also follows that $g^N \in \Delta$ for $g \in \ker \chi_N$, because g fixes the root of unity $g\varpi/\varpi$. Finally $g^{h_0} \in \ker \chi_N$ for every g: it acts on the N-th roots of unity as a power of the p^{h_0}-th power map, which is the identity. So every element of $G_{\mathbb{Q}_p}/\Delta$ has order dividing $h_0 N$, the order of this finite group divides a power of $h_0 N$, and it is prime to p. Put $\Gamma' = j_\Gamma^{-1}(\Delta)$; it is normal, of index prime to p since j_Γ is surjective, it contains the elements near 1, $j_\Gamma(\Gamma') = \Delta$, and its elements fix ϖ.

The elements of inertia. The inertia group is closed and acts transitively on the n-th roots of p for every $n > 0$; so by Input K8(b) and the surjectivity of j_Γ, for every $v \in \mathcal{O}_C^\flat$ whose untilt has e-th power p there is an inertial $s \in \Gamma$ with $s \cdot z = [v]$.

Inertial elements fix $\mathbb{W}$. For $v = \varepsilon^l \varpi^b$, $l \geq 0$, the element s fixes $\theta(z)$, so it lies in Γ' and $s \cdot z = [\varepsilon]^l z$: with Γ' the chart is a Kummer chart. For $v = a\,\varpi^b$ we get the tame generator σ, with $\sigma z = [a]z$ and, since σ fixes $\varrho = [a] \in \mathbb{W}$, $\sigma^k z = \varrho^k z$ for all k. The index e divides N, so p does not divide it. □

5 Cuts and windows

In this section the setting is Setting 2.6.

Lemma 5.1 (the cokernel killer at a cut)**.** *In Setting 2.6, assume* (W3)*,* (W4)*,* (W5)*,* (W6)*,* (W8)*,* (W9)*,* (W10) *and* (W11)*. Let T be a side over $\mathcal{O}$ such that for every letter j and every integer n' with $\beta(j) + W \leq n'$, every $u \in U_{Y_T}(j)$ satisfies $u \leq n'$. Then* $\mathrm{Kill}_T(i; \kappa, \rho(i))$ *for every letter i with the property* cut*.*

LEAN `Lift.KummerChart.PairingSide.killsCokernel_at_cut`

Proof. Let i be a cut. If the second alternative of (W10) holds, then by (W11) every jump of T is at most $(p-3)\rho_0 \leq (p-3)(\kappa + \rho(i))$, and Input Q3(h) gives the claim. Suppose the first: $\rho(i) = 0$ and $\beta(i^{p^a}) + W \leq G$ for $1 \leq a \leq J$. Put $r = p^{\kappa+1}$. By (W8) every jump of T is at most $2B$, so by Input Q3(g) there is $a_0 \geq 0$ with $z^{a_0} \in \mathfrak{a}_T(r; 0, j)$ for every letter j and $a_0(p-1) \leq e_{\mathfrak{c}}(2B + \kappa + 1)$.

Claim. Let $J' \leq J$, let j be a letter with $\beta(j^{p^a}) + W \leq G$ for $1 \leq a \leq J'$, and $x \geq 0$ an integer with $p\,a_0 \leq p^{J'+1}x$. Then $p^{J'X} z^{x+n} \in \mathfrak{a}_T(r; 0, j)$. For $J' = 0$, $a_0 \leq x$. Let $J' \geq 1$. The claim for $J'-1$, the letter j^p and the integer px gives $p^{(J'-1)X} z^{px+n} \in \mathfrak{a}_T(r; 0, j^p)$; the jumps of T at j^p are at most G, so $p^{(J'-1)X} z^{px+n} \mathcal{E}^G \in \mathfrak{a}_T(r; 1, j^p)$ (Input Q3(c)). By Legendre's formula and (W9), $v_p(G!) \leq n$, so $\mathfrak{d}\,G + v_p(G!) \leq X$ by (W3), and $px + n + G \leq p(x+n)$ by (W9); Input Q3(f) gives $p^{J'X} z^{p(x+n)} \in \mathfrak{a}_T(r; 1, j^p)$, and Input Q3(c) the claim.

Take $J' = J$, $j = i$ and $x_0 = 2X + \lfloor \kappa/4 \rfloor + 1$. By (W5) and (W6),

$$p\,a_0 \leq 2\,e_{\mathfrak{c}}\,(2B + \kappa + 1) \leq p^{J+1}(2X + \tfrac{\kappa+1}{4}) \leq p^{J+1} x_0,$$

so $p^{JX} z^m \in \mathfrak{a}_T(r; 0, i)$ with $m = x_0 + n$, and $p^{JX} z^{pm} \in \mathfrak{a}_T(r; 1, i^p)$. Now $v_p((pm)!) = m + v_p(m!) \leq 2m - 1$, and

$$JX + 2m - 1 = (J+4)X + 2n + 1 + 2\lfloor \kappa/4 \rfloor \leq \kappa,$$

because $n \leq X$ by (W3) and $\kappa - 2\lfloor \kappa/4 \rfloor \geq 6(J+1)X + 1$ by (W4). So Input Q3(e), with the exponent 0 of z, gives $p^\kappa \in \mathfrak{a}_T(p^{\kappa+1}; 0, i)$, and Input Q3(b) gives $p^\kappa \in \mathfrak{a}_T(0; 0, i)$. □

Proposition 5.2 (cuts with windows)**.** *In Setting 2.6, assume* (W1) *to* (W11)*, and let $E \geq 0$ be an integer. Suppose that:*

(i) *$\rho(i) + W \leq p\,n$ for every letter i;*

(ii) *$51\,\kappa \leq E$;*

(iii) *for every letter i there is an integer l with $0 < l \leq 2J + 1$ such that $i^{p^{-l}}$ has the property* cut*;*

(iv) *for every letter i and every integer $l \geq 0$ such that $i^{p^{-l}}$ has the property* cut *and no $i^{p^{-a}}$ with $0 < a < l$ has it, there is an integer $t \geq 0$ for which one of the following holds:*

(α) *for every* $0 \le a < l$: $\mathsf{s}(i^{p^{-a}}) = \mathsf{s}(i^{p^{-t}})$ *or* $\beta(i^{p^{-a}}) + V \le \eta$;
(β) *the chart is unramified,* $p \ge 5$, $\rho(i^{p^{-l}}) = 0$, $t + J + 1 = l$, $\rho(i) = 0$ *or* $t = 0$,

$$\rho_0 + G \sum_{0 \le a \le J} p^a < \beta(i^{p^{-t}}),$$

and for every $0 \le a < l$ *with* $a \ne t$: $\mathsf{s}(i^{p^{-a}}) \ne \mathsf{s}(i^{p^{-t}})$ *or* $\beta(i^{p^{-a}}) + V \le \eta$; *and* $\beta(i^{p^{-a}}) + W \le G$ *if* $t < a$.

Then $\mathrm{Cup}_\Pi(E; c, c')$ *for all* $c \in C_S$ *and* $c' \in C_{S'}$.

LEAN `Lift.KummerChart.PairingSide.SidePairing.scaledCupBounds_of_cut_windows`

Proof. The data of the walk. For a letter i let $\mathsf{l}(i)$ be the least $l > 0$ such that $i^{p^{-l}}$ is a cut; $\mathsf{l}(i) \le 2J + 1$ by (iii). Let $t(i)$ be an integer t of (iv) for i and $l = \mathsf{l}(i)$, and $\mathsf{s}^\circ(i) = \mathsf{s}(i^{p^{-t(i)}})$. Let $\eta_\nu = \eta$ for $\nu \le 2J$ and $\eta_\nu = 0$ for $\nu > 2J$. For a sign ϵ let

$$\mathsf{R}^\epsilon_i(\nu, u) = \max\bigl(\mathrm{std}_{\mathsf{l}(i)}(\nu, u),\, -u - \eta_\nu\bigr) \ \text{ if } \mathsf{s}^\circ(i) = \epsilon, \qquad \mathsf{R}^\epsilon_i(\nu, u) = -u - \eta_\nu \ \text{ otherwise.}$$

We apply Theorem B of Part 2 (Input Q5) with the property cut, with $\mathsf{Q}(i)$ the statement $\mathsf{s}^\circ(i) = 1$, with the families $o = \mathsf{R}^{+1}$, $o' = \mathsf{R}^{-1}$ and $z(i) = z'(i) = \mathrm{rs}_{-\rho(i)}(\cdot, \cdot)$, and with $e = e' = 25\kappa$; by (ii), $e + e' + \kappa \le E$. The functions $z(i)(\nu, \cdot)$ are non-increasing.

The third side. A jump of $S^\vee_\Pi$ at j is 0 or $-u - v - 1$ with u, v jumps of S and S' at j; by (W1) and (W2) it is at most $\beta(j) + W$. So Lemma 5.1 applies to the third side, and to S and S'. For a cut i, Input Q4(a) gives a point of $\mathrm{Pt}_{S^\vee_\Pi}(1, i; \kappa; z(i), z'(i))$.

The deep points. Let ϵ, ϵ' be signs and T, T' sides over $\mathcal{O}$ such that $U_{Y_T}(j) \subseteq [-\eta, \eta] \cup -\epsilon\mathsf{s}(j) \cdot [\beta(j), \beta(j) + V]$ and $U_{Y_{T'}}(j) \subseteq [-\eta, \eta] \cup \epsilon\mathsf{s}(j) \cdot [\beta(j), \beta(j) + V]$ for every letter j; let $c \in C_T$ and i a cut. We claim that T has a deep point over $p^{25\kappa}c$ at i against T', for the data $(\mathsf{cut}, \mathsf{l}(i), \mathsf{R}^\epsilon, \mathsf{R}^{\epsilon'}, z)$ and the proposition $\mathsf{s}^\circ(i) = \epsilon$. For $(T, T', \epsilon, \epsilon') = (S, S', 1, -1)$ and $(S', S, -1, 1)$ these are hypotheses (i) and (ii) of Theorem B of Part 2, by (W2).

Let I be the set of the pairs (ν, n) with $\nu < 4J + 2$ and $n \in \mathcal{W}_{W_\nu}(\pm\beta(i^{p^{-\nu}}))$, where $W_0 = \rho(i) + W$, $W_\nu = W$ for $\nu > 0$, and the sign is $+$ if and only if $\mathsf{s}(i^{p^{-\nu}}) = \epsilon$. By Input Q3(i), with (i) and (W8), the costs of the pairs of I with a given ν have sum at most

$$(4n + 2)\lfloor \log_p(2B) \rfloor + (4W_\nu + 2)(\mathfrak{d} + 3\nu) \ \le \ 72\,X,$$

by (W3) and $W \le n$; for $\nu = 0$ one uses that $\mathfrak{d} = 0$ if $\rho(i) \ne 0$, by (W10). So the costs of I have sum at most $(4J + 2) \cdot 72X$, and by (W4)

$$\kappa + 4X + (4J + 2) \cdot 72\,X \le 25\,\kappa. \tag{8}$$

Let $t = t(i)$. Two cases.

Case 1: $\mathsf{s}^\circ(i) \ne \epsilon$, *or alternative* ($\alpha$) *of* (iv) *holds.* By Lemma 5.1 and Input Q4(a) there is a point over $p^\kappa c$ at i with exponents and depth $\mathrm{rs}_{-\rho(i)}(\cdot, \cdot)$. Let $d(\nu, u) = \mathrm{rs}_{\rho(i)}(\nu, u) + \eta_{\lfloor \nu/2 \rfloor}$. A step (ν, n) from $\mathrm{rs}_{-\rho(i)}(\cdot, \cdot)$ to d has $\nu \le 4J + 1$ and $-u - (W_\nu - W) \le n < -u + (W_\nu - W) + \eta$ for a jump u of T at $i^{p^{-\nu}}$; as $-u$ lies in $[-\eta, \eta]$ or in $\epsilon\mathsf{s}(i^{p^{-\nu}}) \cdot [\beta, \beta + V]$ and $V + \eta \le W$, the step lies in I. By Input Q4(c) and (8) there is a point over $p^{25\kappa}c$ with depth d and exponents $\mathrm{imp}(\mathrm{rs}_{-\rho(i)}(\cdot, \cdot), d)(\cdot, \cdot)$. We take this depth d and $r = 0$; the letter i is a cut.

Case 2: $\mathsf{s}^\circ(i) = \epsilon$ *and* (α) *fails.* Then (β) holds: the chart is unramified, $p \geq 5$, $\mathsf{l}(i) = t + J + 1$, and $i^{p^{-\mathsf{l}(i)}}$ is a cut at which $\rho = 0$. Let $j = i^{p^{-t}}$. We apply Theorem D of Part 2 (Input Q5) to T at j with $l = J + 1$, $\gamma = G$, $\rho = \rho_0$, $N = \rho_0 + G\sum_{0 \leq a \leq J} p^a$, $A = 2B$ and $\delta = X$. Its hypotheses: $\mathrm{Kill}_T(j^{p^{-(J+1)}}; \kappa, 0)$ by Lemma 5.1; $\mathrm{Kill}_T(j; 0, \rho_0)$ by (W11) and Input Q3(h); the jumps of T at j are at most $\eta \leq G$, because $\mathsf{s}(j) = \epsilon$ puts the far window of T at j on the negative side, and those at $j^{p^{-a}}$, $0 < a \leq J$, are at most G by (β); $N < \beta(j) \leq 2B$ by (β) and (W8); $3e_{\mathfrak{c}} \leq p^{J+1}$ by (W6); $e_{\mathfrak{c}} \cdot 2B \leq p^{J+1}X$ by (W5). It gives a point of $\mathrm{Pt}_T(c, j; \kappa + 4X; \mathrm{sh}_{J+1}(o_1)(\cdot,\cdot), \mathrm{sh}_{J+1}(o_2)(\cdot,\cdot))$ with $o_1(u) = \min(2B, -u - N)$ and $o_2(u) = 2B$, and Input Q4(d) moves it by φ^t to a point over $p^{\kappa+4X}c$ at i with depth $\mathrm{fr}_{t,1}(\mathrm{sh}_{J+1}(o_2)(\cdot,\cdot))(\cdot,\cdot)$. Let $d(t,u) = 2B$ and $d(\nu,u) = \mathrm{std}_{t+J+1}(\nu,u) + 3\eta_{\lfloor \nu/2 \rfloor}$ for $\nu \neq t$. The steps from that depth to d have $\nu \leq 4J + 1$, $\nu \neq t$, and $0 \leq n < 3\eta$ or $-u \leq n < -u + 3\eta$; as $V + 4\eta \leq W$ they lie in I. By Input Q4(c) and (8) there is a point over $p^{25\kappa}c$ at i with depth d. We take this d and $r = \mathsf{l}(i)$.

The comparisons. In both cases it remains to compare exponents. Write o^* for the exponents of the point found. It must be shown that $o^* \geq \mathsf{R}_i^\epsilon$ on the jumps of T, so that the point lies in $\mathrm{Pt}_T(c, i; 25\kappa; \mathsf{R}_i^\epsilon, d)$ (Input Q1), and that the depth d and the integer r satisfy clauses (a) and (b) of Definition 2.6 of Part 2 against $\mathsf{R}^{\epsilon'}$ and the family z, for the proposition $\mathsf{s}^\circ(i) = \epsilon$. The formal text settles these inequalities by a decision procedure for linear arithmetic after a case distinction; here they are done by hand. Put $\mathsf{l} = \mathsf{l}(i)$; let $\nu \geq 0$, let u be a jump of T and v a jump of T' at $i^{p^{-\nu}}$, and $s = -u - v - 1$. By the choice of z, $z(i)(0,s) = -\rho(i) - s$ and $z(i)(\nu,s) = -s$ for $\nu \geq 1$. Three remarks are used.

(1) For every letter j, $\mathsf{R}_j^{\epsilon'}(\nu,v) \geq -v - \eta_\nu$, by the definition. Hence the right side of clause (a), which is $\min(-v, \mathsf{R}_i^{\epsilon'}(\nu,v))$, is at least $-v - \eta_\nu$; and the right side of the second inequality of clause (b), which is $\min(\mathsf{R}_i^{\epsilon'}(\nu,v), \mathrm{fr}_{\mathsf{l},0}(\mathsf{R}_{i'}^{\epsilon'})(\nu,v))$ with $i' = i^{p^{-\mathsf{l}}}$, is at least $\min(-v - \eta_\nu, 0)$ for $\nu < \mathsf{l}$ and at least $\min(-v - \eta_\nu, -v - \eta_{\nu-\mathsf{l}})$ for $\nu \geq \mathsf{l}$.

(2) $\eta_\nu \leq \eta_{\lfloor \nu/2 \rfloor}$ for all ν, and $\eta_{\nu-\mathsf{l}} \leq \eta_{\lfloor \nu/2 \rfloor}$ for $\nu \geq \mathsf{l}$: if $\lfloor \nu/2 \rfloor > 2J$, then $\nu \geq 4J + 2$ and $\nu - \mathsf{l} \geq 2J + 1$, because $\mathsf{l} \leq 2J + 1$. Moreover $\eta_\nu = \eta_{\lfloor \nu/2 \rfloor} = \eta$ for $\nu < \mathsf{l}$.

(3) By the windows of T and T': at a letter j' with $\mathsf{s}(j') = \epsilon$ the jumps of T are at most η and those of T' at least $-\eta$; at a letter with $\mathsf{s}(j') \neq \epsilon$ the jumps of T are at least $-\eta$ and those of T' at most η; at a letter with $\beta(j') + V \leq \eta$ all jumps lie in $[-\eta, \eta]$; and at every letter j' a jump outside $[-\eta, \eta]$, a *far* jump, has absolute value in $[\beta(j'), \beta(j') + V]$.

The comparisons in Case 1. Here $o^* = \mathrm{imp}(\mathrm{rs}_{-\rho(i)}(\cdot,\cdot), d)(\cdot,\cdot)$, that is,

$$o^*(0,u) = \begin{cases} \rho(i) - u + \eta, & u < -\rho(i), \\ \min(\rho(i) - u + \eta,\, 0), & u \geq -\rho(i), \end{cases}$$

$$o^*(\nu,u) = \begin{cases} -u + \eta_{\lfloor \nu/2 \rfloor}, & u < 0, \\ \min(-u + \eta_{\lfloor \nu/2 \rfloor},\, 0), & u \geq 0 \end{cases} \qquad (\nu \geq 1).$$

So $o^*(\nu,u) \geq -u$ for $\nu \geq 1$, and $o^*(\nu,u) \geq -u - \eta_\nu$ for all ν: at $\nu = 0$ this is clear for $u < -\rho(i)$ and for $u \geq -\eta$, and no jump of T at the cut i lies in $[-\rho(i), -\eta)$, because by (W10) either $\rho(i) = 0$, or $\rho(i) < \beta(i)$ while a jump $u < -\eta$ is far and has

$u \le -\beta(i)$. If $\mathsf{s}^\circ(i) \neq \epsilon$, then $\mathsf{R}^\epsilon_i(\nu,u) = -u - \eta_\nu$ and $o^* \ge \mathsf{R}^\epsilon_i$ is proved. If $\mathsf{s}^\circ(i) = \epsilon$, then (α) holds, and with (3) it gives

$$u \le \eta \quad \text{and} \quad v \ge -\eta \qquad \text{at every } \nu < \mathsf{l}. \tag{9}$$

At these ν the formulas give $o^*(\nu,u) \ge 0 = \mathrm{std}_\mathsf{l}(\nu,u)$, by (2); and at $\nu \ge \mathsf{l}$, $o^*(\nu,u) \ge -u = \mathrm{std}_\mathsf{l}(\nu,u)$. So $o^* \ge \max(\mathrm{std}_\mathsf{l}, -u-\eta_\nu) = \mathsf{R}^\epsilon_i$ on the jumps.

Clause (a), with $d(0,u) = \rho(i) - u + \eta$ and $d(\nu,u) = -u + \eta_{\lfloor \nu/2 \rfloor}$ for $\nu \ge 1$: the left side $-z(i)(\nu,s) - d(\nu,u)$ is $-v - 1 - \eta$ at $\nu = 0$ and $-v - 1 - \eta_{\lfloor \nu/2 \rfloor}$ at $\nu \ge 1$. By (2) it is smaller than $-v - \eta_\nu$, and (1) gives (a). Clause (b): let $\mathsf{s}^\circ(i) = \epsilon$. The first inequality, $\mathrm{std}_\mathsf{l} \le \mathsf{R}^\epsilon_i$, holds by the definition of R^ϵ_i. In the second, as $r = 0$, the left side is that of (a). At $\nu < \mathsf{l}$ it is $-v - 1 - \eta$, by (2), which is smaller than $-v - \eta$ and, by (9), negative. At $\nu \ge \mathsf{l}$ it is $-v - 1 - \eta_{\lfloor \nu/2 \rfloor}$, smaller than $-v - \eta_\nu$ and than $-v - \eta_{\nu - \mathsf{l}}$ by (2). By (1), clause (b) holds.

The comparisons in Case 2. Here $\mathsf{s}(j) = \epsilon$ for $j = i^{p^{-t}}$, $r = \mathsf{l} = t + J + 1$ and $\rho(i^{p^{-\mathsf{l}}}) = 0$; moreover $\rho(i) = 0$ unless $t = 0$, and in that case $j = i$. By (W1), (W8) and (β), $\eta \le W \le G \le N < \beta(j)$; so by (3) every jump of T or T' at j has absolute value at most $\beta(j) + V$, and by (W1), (W7) and (W8), applied to the letter j,

$$\beta(j) + V + \eta < \beta(j) + W \le 2B - \rho(j). \tag{10}$$

By (β) and (3),

$$u \ge -\eta \ \text{ and } \ v \le \eta \quad \text{at } \nu < \mathsf{l},\ \nu \neq t; \qquad u \le \eta \ \text{ and } \ v \ge -\eta \quad \text{at } \nu = t, \tag{11}$$

and a jump u of T at j with $u \ge -N$ is not far, since a far one has $u \le -\beta(j) < -N$; so it has $|u| \le \eta$. The point moved by φ^t has the exponents $\mathrm{fr}_{t,0}(\mathrm{sh}_{J+1}(o_1)(\cdot,\cdot))(\cdot,\cdot)$, which are 0 at $\nu < \mathsf{l}$, $\nu \neq t$, are $\min(2B, -u - N)$ at $\nu = t$, and are $-u$ at $\nu \ge \mathsf{l}$. As $d(\nu,u) = 3\eta \ge 0$ at $\nu < \mathsf{l}$, $\nu \neq t$, and $d(\nu,u) = -u + 3\eta_{\lfloor \nu/2 \rfloor}$ at $\nu \ge \mathsf{l}$, Input Q4(c) gives

$$o^*(\nu,u) = 0 \ \ (\nu < \mathsf{l},\ \nu \neq t), \qquad o^*(\nu,u) \ge -u \ \ (\nu \ge \mathsf{l}),$$

$$o^*(t,u) = \begin{cases} 2B, & u < -N, \\ 0, & u \ge -N. \end{cases}$$

This is at least $\mathsf{R}^\epsilon_i(\nu,u) = \max(\mathrm{std}_\mathsf{l}(\nu,u), -u - \eta_\nu)$. At $\nu < \mathsf{l}$, $\nu \neq t$: $-u - \eta \le 0$ by (11). At $\nu = t$: if $u < -N$, then $-u - \eta \le -u \le \beta(j) + V \le 2B$ by (10); if $u \ge -N$, then $|u| \le \eta$ and $-u - \eta \le 0$. At $\nu \ge \mathsf{l}$: $\mathrm{std}_\mathsf{l}(\nu,u) = -u$.

Clause (a). At $\nu < \mathsf{l}$, $\nu \neq t$, the left side is $s - 3\eta$; for $\nu = 0$ this uses $\rho(i) = 0$, which holds because then $t \neq 0$. It is smaller than $-v - \eta$, since $u \ge -\eta$ by (11). At $\nu = t$ the left side is $s - 2B$ if $t \ge 1$ and $\rho(i) + s - 2B$ if $t = 0$; it is smaller than $-v - \eta$ because $\rho(i) - u + \eta \le 2B$: indeed $-u \le \beta(j) + V$, and $\rho(i)$ is 0 or is $\rho(j)$, so (10) applies. At $\nu \ge \mathsf{l}$ the left side is $-v - 1 - 3\eta_{\lfloor \nu/2 \rfloor}$, smaller than $-v - \eta_\nu$ by (2). By (1), clause (a) holds.

Clause (b). The first inequality holds by the definition of R^ϵ_i, as $\mathsf{s}^\circ(i) = \epsilon$. For the second, $z(i^{p^{-\mathsf{l}}}) = \mathrm{rs}_0(\cdot,\cdot)$, so $\mathrm{fr}_{\mathsf{l},0}(z(i^{p^{-\mathsf{l}}}))(\nu,s)$ is 0 at $\nu < \mathsf{l}$ and $-s$ at $\nu \ge \mathsf{l}$. At $\nu < \mathsf{l}$ the left side is therefore $-d(\nu,u)$, and by (1) it suffices that $-d(\nu,u) < \min(-v - \eta, 0)$. At $\nu = t$ this reads $-2B < \min(-v - \eta, 0)$, and $v + \eta \le \beta(j) + V + \eta < 2B$ by (10). At $\nu \neq t$ it reads $-3\eta < \min(-v - \eta, 0)$, which holds because $v \le \eta$ by (11) and

$\eta \geq 1$ by (W7). At $\nu \geq \mathsf{l}$ the left side is $s - d(\nu, u) = -v - 1 - 3\eta_{\lfloor \nu/2 \rfloor}$, smaller than $-v - \eta_\nu$ and than $-v - \eta_{\nu-\mathsf{l}}$ by (2). By (1), clause (b) holds, and the claim is proved in both cases.

All hypotheses of Theorem B of Part 2 hold: every letter has a cut among the $i^{p^{-a}}$, $a > 0$, by (iii); its hypotheses (i) and (ii) are the claim; (iii) is the point of the third side; and (iv) was noted. It gives $\mathrm{Cup}_{\Pi}(E; c, c')$. □

Proof of Theorem C. By (W1) and (W7), $W \geq 4$; by the hypotheses $G \leq \eta \leq W$, $B \geq 11(p^J+1)W \geq 22W$ and $X \geq 16$. The objects cut, $\rho(\cdot)$, n, κ, ρ_0 of Setting 2.6 do not occur in the theorem; the proof chooses them, and it replaces G. So Proposition 5.2 is applied to another instance of Setting 2.6, with the same S, S', Π, s, β, η, V, W, X, B and J; condition (W12) need not hold in that instance, and it is not among the hypotheses of the Proposition. Call a letter i a *cut* if no i^{p^a} with $1 \leq a \leq J$ is thick. We apply Proposition 5.2 with this property, with $\rho(i) = 0$ for all i, with $2W$ in the place of n and of G, with $\rho_0 = \lfloor 3B/5 \rfloor$, $\kappa = 12(J+1)X + 1$ and $E = 10^4 X(J+1)$, and with B, J, X, V, W, η, β, s as they are. Conditions (W1), (W2), (W5), (W6), (W7) are assumed, (W4) holds by the choice of κ, and (W3) reads $(J + 1 + \mathfrak{d} + \lfloor \log_p(2B) \rfloor) \cdot 2W + \mathfrak{d} \cdot 2W \leq X$, which follows from the hypothesis on X. (W8): $\beta(i) + W \leq B + W \leq 2B$. (W9): $2W \leq (p-1) \cdot 2W$. (W10): at a cut i the letters i^{p^a}, $1 \leq a \leq J$, are not thick, so $\beta(i^{p^a}) + W = G + W \leq 2W$. (W11): $B + W \leq 2\lfloor 3B/5 \rfloor$, because $B \geq 5W + 8$. Hypothesis (i): $W \leq 2pW$. (ii): $51\kappa = 612(J+1)X + 51 \leq 10^4 X(J+1)$.

A thick letter that is not a cut. Let j be thick and not a cut. Then $j^{p^{-(J+1)}}$ is a cut. Indeed some $j^{p^{a'}}$, $1 \leq a' \leq J$, is thick; if $j^{p^{-(J+1)+b}}$ were thick for some $1 \leq b \leq J$, the hypothesis of the theorem for this letter and the exponents $a = J + 1 - b$ and a' would say that $j^{p^{a'}}$ is not thick.

Hypothesis (iii). Suppose that no $i^{p^{-l}}$, $1 \leq l \leq 2J+1$, is a cut. As $i^{p^{-(J+1)}}$ is not a cut, some $i^{p^{-a}}$ with $1 \leq a \leq J$ is thick; it is not a cut, so $i^{p^{-(a+J+1)}}$ is a cut, and $a + J + 1 \leq 2J + 1$.

Hypothesis (iv). Let i and l be as there. If $i^{p^{-a}}$ is thick with $0 < a < l$, then it is not a cut, so $i^{p^{-(a+J+1)}}$ is a cut and $l \leq a + J + 1$; and $l < a + J + 1$ is impossible, because the cut $i^{p^{-l}}$ would have the thick letter $i^{p^{-a}}$ among its translates by p^b, $1 \leq b \leq J$. So among the $i^{p^{-a}}$, $0 \leq a < l$, only those with $a = 0$ and $a = l - J - 1$ can be thick. If (α) holds with $t = 0$ we are done. Otherwise some $a < l$ has $\mathsf{s}(i^{p^{-a}}) \neq \mathsf{s}(i)$ and $\beta(i^{p^{-a}}) + V > \eta$; then $i^{p^{-a}}$ is thick, because $G + V \leq \eta$, and $a \neq 0$, so $a = l - J - 1$. Take $t = a$. The signs are not all equal, so the chart is unramified and $p \geq 7$; $\rho = 0$ everywhere; $t + J + 1 = l$; and

$$\rho_0 + 2W \sum_{0 \leq b \leq J} p^b < \tfrac{3}{5}B + 4Wp^J \leq \tfrac{3}{5}B + \tfrac{4}{11}B < B = \beta(i^{p^{-t}}).$$

For $b < l$, $b \neq t$: if $i^{p^{-b}}$ is thick then $b = 0$ and $\mathsf{s}(i) \neq \mathsf{s}(i^{p^{-t}})$; if it is not thick then $\beta(i^{p^{-b}}) + V = G + V \leq \eta$; and for $t < b$ it is not thick, so $\beta(i^{p^{-b}}) + W = G + W \leq 2W$. This is (β).

Proposition 5.2 gives $\mathrm{Cup}_{\Pi}(10^4 X(J+1); c, c')$. □

Proof of Theorem D. By (W1) and (W7), $W \geq 4$, so $G \geq 20$, $B \geq G > 0$ and $K_0 \geq 1$. Put $\lambda = \lfloor \log_p(2B) \rfloor$. For a letter i let $\rho(i) = \max(\lfloor B/p \rfloor - 7K_0, 0)$ if i^p is thick and

$\rho(i) = 0$ otherwise, and call i a *cut* if $\rho(i) < \beta(i)$. Since $\lfloor B/p \rfloor < B$ and $\beta(i) \in \{B, G\}$, a letter i is not a cut if and only if

$$i \text{ is not thick}, \qquad i^p \text{ is thick}, \qquad G + 7K_0 \le \lfloor B/p \rfloor. \tag{12}$$

As in the proof of Theorem C, Proposition 5.2 is applied to another instance of Setting 2.6, with the same S, S', Π, $\mathfrak{s}$, β, η, V, W and B, in which (W12) is not needed: with this property and this ρ, with K_0 in the place of n, $J = 1$, $\rho_0 = \lfloor B/p \rfloor + 12K_0$, $G + W$ in the place of G, $X = (2+\lambda)K_0$, $\kappa = 24X + 1$ and $E = 10^4 K_0(\lambda+1)$. The chart is unramified, so $e_{\mathfrak{c}} = 1$, $\mathfrak{d} = 0$. (W1), (W2), (W7) are assumed; (W3) and (W4) hold by the choices. (W5): $2B < 2K_0p^2 \le p^2X$. (W6): $8 \le p^2$. (W8): $W \le K_0$, and $\rho(i) + W \le \max(\lfloor B/p \rfloor - 6K_0, W) \le B$, so $\beta(i) + \rho(i) + W \le 2B$. (W9): $G + W \le G + K_0 < K_0(p/2+1) \le (p-1)K_0$. (W10): let i be a cut; if i^p is not thick, then $\rho(i) = 0$ and $\beta(i^p) + W = G + W$; if i^p is thick, then $\rho(i) < \beta(i)$, the chart is unramified, $p \ge 5$, and $\rho_0 \le \kappa + \rho(i)$ because $\rho(i) \ge \lfloor B/p \rfloor - 7K_0$ and $19K_0 \le \kappa$. (W11): with $B \le p\lfloor B/p \rfloor + p - 1$ and $\lfloor B/p \rfloor \le K_0 p - 1$,

$$(p-3)\rho_0 - (B + K_0) \;\ge\; (12p-37)K_0 - 3\lfloor B/p \rfloor - p + 1 \;\ge\; (9p-37)K_0 - p + 4 \;>\; 0.$$

Hypothesis (i): $\rho(i) + W = W \le pK_0$ if $\rho(i) = 0$, and $\rho(i) + W \le \lfloor B/p \rfloor - 6K_0 < pK_0$ otherwise. (ii): $51\kappa = 1224(2+\lambda)K_0 + 51 \le 10^4 K_0(\lambda+1)$. (iii): if $i^{p^{-1}}$ is not a cut, it is not thick by (12), so $i^{p^{-2}}$ is a cut, again by (12); thus $l \le 2$.

(iv): let i and l be as there; $l \le 2$. Take $t = 0$. For $l \le 1$, (α) holds. Let $l = 2$: $i^{p^{-1}}$ is not a cut, so by (12) it is not thick, i is thick and $G + 7K_0 \le \lfloor B/p \rfloor$. If (α) fails, then $\mathfrak{s}(i^{p^{-1}}) \ne \mathfrak{s}(i)$, and (β) holds: the chart is unramified and $p \ge 7$; $\rho(i^{p^{-2}}) = 0$ because $i^{p^{-1}}$ is not thick; $t + J + 1 = 2$; $\beta(i^{p^{-1}}) + W = G + W$; and, as $B - \lfloor B/p \rfloor \ge (p-1)(G + 7K_0)$, $W \le K_0$ and $2G < K_0 p$,

$$\begin{aligned} B - \rho_0 - (G+W)(1+p) \;&\ge\; (p-1)(G+7K_0) - 12K_0 - (p+1)(G+K_0) \\ &= (6p-20)K_0 - 2G \;>\; 0. \end{aligned}$$

Proposition 5.2 gives $\mathrm{Cup}_\Pi(10^4 K_0(\lambda+1); c, c')$. □

6 The two theorems on lattices

Lemma 6.1 (orthogonality from the universal sides)**.** *In Setting 2.1, let Γ carry a topology for which j_Γ is an open embedding. Let $\mathfrak{c}$ be a Kummer chart for Γ, $\omega \in \mathcal{O}$ with $\omega^{p^h - 1} = 1$, $\mathcal{F}$ a filter on Γ every member of which contains Γ', U and U' families of finite sets of integers indexed by the maps $\gamma\colon \mathcal{O} \to C$, $\mathcal{Q}$ a class of lattices over $\mathcal{O}$ contained in $\mathrm{Cl}(\mathcal{F}, \perp; U)$, L a lattice with $L \in \mathrm{Cl}(\mathcal{F}, \perp; U')$, $e \ge 0$ an integer and $x \in Z^1_{\mathrm{f}}(L)$. Suppose that $\mathrm{Cup}_\Pi(e; c, x)$ for every $Y \in \mathcal{Q}$, every pairing Π of the sides $\mathsf{S}^{\mathrm{u}}_{\mathfrak{c},\omega}(\mathcal{F}; Y; U)$ and $\mathsf{S}^{\mathrm{u}}_{\mathfrak{c},\omega}(\mathcal{F}; L; U')$, and every $c \in Z^1_{\mathrm{f}}(Y)$. Then $p^e x \in L^{\perp\mathcal{Q}}$.*

LEAN `Lift.KummerChart.mem_cupOrthogonal_of_scaledCupBounds`

Proof. A pairing of the two universal sides is a pairing of the lattices Y and L, their cocycle maps are the identity, and the topology of Γ is one of those of Definition 2.4 of Part 2. So the hypothesis says that the cup product of c and $p^e x$ bounds, for every $Y \in \mathcal{Q}$, every pairing and every $c \in Z^1_{\mathrm{f}}(Y)$; this is the definition of $L^{\perp\mathcal{Q}}$. □

Proof of Theorem A. The filter $\top$ has the single member Γ, which contains Γ'. For a lattice Y in $\mathrm{Cl}(\top,\bot;U)$ write $S_Y=\mathsf{S}^{\mathrm{u}}_{\mathfrak{c},\omega}(\top;Y;U)$, and $S_L=\mathsf{S}^{\mathrm{u}}_{\mathfrak{c},\omega}(\top;L;U')$. Their jump sets at i are $\mathcal{U}_{\mathfrak{c},\omega}(U)(i)$ and $\mathcal{U}_{\mathfrak{c},\omega}(U')(i)$, in the windows of the statement.

(1) Let $x\in Z^1_{\mathrm{f}}(L)$, Y in the class, Π a pairing of S_Y and S_L, and $c\in Z^1_{\mathrm{f}}(Y)$. We apply Theorem C of Part 2 (Input Q5) with $\kappa=0$, the given ρ, and $H=0$. Its hypothesis (i) holds by Input Q3(h): the jumps of S_Y, S_L and of the third side are at most $\beta(i)+W+\eta\le(p-3)\rho$. Hypotheses (ii) and (iii) are assumed. (iv): in the first case of (1) no element of the two jump sets lies in $[-\rho,0)$, as 0 does not; in the second, the chart is unramified, $p\ge5$, $\eta\ge1$, and $\mathcal{U}_{\mathfrak{c},\omega}(U')(i)\subseteq[-\eta,0]$. (v): $\mathfrak{d}=0$. So $\mathrm{Cup}_\Pi(0;c,x)$, and Lemma 6.1 with $e=0$ gives $x\in L^{\perp\mathrm{Cl}(\top,\bot;U)}$.

(2) For a letter i let g_i be the number of the maps γ with $\gamma(\omega)=\theta([i])$. The side S_L satisfies Setting 2.5 with the given $\rho,\eta,W,\beta,\mathsf{s}$ and

$$n_{\mathrm{lo}}(i)=h\,m_0\,g_i,\qquad n_{\mathrm{all}}(i)=h\,\mathrm{rk}_{\mathcal{O}}(L)\,g_i\,:$$

(F1) to (F5) are hypotheses of the theorem, and (F6) holds because $m_L(i,u)$ is the sum over these γ of the multiplicities of u, which for each γ have sum at most m_0 over $|u|\le\eta$ and sum $\mathrm{rk}_{\mathcal{O}}(L)$ over all u (Input K5). Let $\mathsf{C}\subseteq Z^1_{\mathrm{f}}(L)$ be the $\mathbb{Z}_p$-submodule of Proposition 3.3. For $d\in\mathsf{C}$, Y in the class, a pairing Π of S_Y and S_L and $c\in Z^1_{\mathrm{f}}(Y)$, the proposition gives $\mathrm{Cup}_\Pi(0;c,d)$; by Lemma 6.1, $\mathsf{C}\subseteq L^{\perp\mathrm{Cl}(\top,\bot;U)}$. Distinct letters have distinct $\theta([i])$, and there are $\mathrm{rk}_{\mathbb{Z}_p}(\mathcal{O})$ maps γ in all, so $\sum_i g_i\le\mathrm{rk}_{\mathbb{Z}_p}(\mathcal{O})$ and

$$\begin{aligned}\mathrm{len}_{\mathbb{Z}_p}\bigl(Z^1_{\mathrm{f}}(L)/(Z^1_{\mathrm{f}}(L)\cap L^{\perp\mathrm{Cl}(\top,\bot;U)})\bigr)&\le\mathrm{len}_{\mathbb{Z}_p}\bigl(Z^1_{\mathrm{f}}(L)/(Z^1_{\mathrm{f}}(L)\cap\mathsf{C})\bigr)\\&\le 11\,h\,\bigl(m_0(\rho+1)+\mathrm{rk}_{\mathcal{O}}(L)(W+1)\bigr)\cdot\mathrm{rk}_{\mathbb{Z}_p}(\mathcal{O}).\end{aligned}$$

Let $f_{\mathcal{O}}$ be the degree of the residue field of $\mathcal{O}$ over $\mathbb{F}_p$. The length over $\mathbb{Z}_p$ of an $\mathcal{O}$-module of finite length is $f_{\mathcal{O}}$ times its length over $\mathcal{O}$, and $\mathrm{rk}_{\mathbb{Z}_p}(\mathcal{O})=\mathrm{len}_{\mathbb{Z}_p}(\mathcal{O}/p\mathcal{O})=f_{\mathcal{O}}\,\mathrm{len}_{\mathcal{O}}(\mathcal{O}/p\mathcal{O})$. Dividing by $f_{\mathcal{O}}$ gives (2). □

Proof of Theorem B. The tame chart. Let M_0 be the least common multiple of the p^a-1, $1\le a\le\mathsf{r}$. It divides $p^{\mathsf{r}!}-1$, and p does not divide $\mathsf{r}!$ because $\mathsf{r}<p$. By Proposition 4.4 there is a tame chart $\mathfrak{c}$ for Γ with index $e_{\mathfrak{c}}=M_0$; let σ be its tame generator and ϖ its element. Then $e_{\mathfrak{c}}\le\prod_{a\le\mathsf{r}}p^a\le p^{\mathsf{r}^2}$, and $\mathfrak{d}\le1$: the index is prime to p, so $\mathfrak{d}$ is 0 or 1. As a coefficient root we take $\omega=1$.

The lattices are crystalline on Γ'. Let $\mathcal{F}'$ be the filter of the subsets of Γ that contain Γ'. We claim: if K_1 is a relation with $\mathrm{Tame}_p(\mathsf{r},\mathsf{n};\mathsf{K}_1(s,\cdot))$ for all inertial s, then every lattice Y in $\mathrm{Cl}(\mathcal{F},\mathsf{K}_1;U)$ lies in $\mathrm{Cl}(\mathcal{F}',\bot;U)$. Indeed Y is potentially crystalline. As j_Γ is bijective, $g\mapsto\beta^Y_{j_\Gamma^{-1}(g),0}$ is a map from the inertia group of $G_{\mathbb{Q}_p}$ to the endomorphisms of $B_{\max}\otimes Y$; it is a homomorphism of monoids and is trivial near 1 (Input K2(a)), and its M_0-th power is trivial, because $X^{M_0}-1$ kills $(s,0)$ on Y for inertial s by the definitions of the tame type and of the class. By Input K6, for $y=\varpi$ and the exponent $M_0=e_{\mathfrak{c}}$, it is trivial on the stabilizer of ϖ, which contains $j_\Gamma(s)$ for the inertial $s\in\Gamma'$. By Input K3, applied to the group Γ' with the restriction of j_Γ, for which the linear Frobenius of an inertial element is the same, Y is crystalline on Γ'. The condition on the jumps is unchanged, and the relation $\bot$ imposes nothing. The claim applies to K and to K^-, since the inverse of an inertial element is inertial.

The killers. By the hypothesis on K at σ and at σ^{-1} there are monic polynomials q, q' over $\mathbb{Z}_p$ of degree at most n with $\mathsf{K}(\sigma,q)$ and $\mathsf{K}(\sigma^{-1},q')$. Let Y be a lattice

in $\mathrm{Cl}(\mathcal{F},\mathsf{K};U)$. Then q kills $(\sigma,0)$ on Y and q' kills $(\sigma^{-1},0)$ on Y; and since $L\in\mathrm{Cl}(\mathcal{F},\mathsf{K}^{-};U')$, q' kills $(\sigma,0)$ on L and q kills $(\sigma^{-1},0)$ on L. By Input K4 for $s=\sigma^{-1}$, a monic polynomial q_3 over $\mathbb{Z}_p$ of degree at most n^2 kills $(\sigma,0)$ on $(Y\otimes_{\mathcal{O}}L)^{\vee}(1)$.

The walk. Let $S_Y=\mathsf{S}^{\mathrm{u}}_{\mathfrak{c},1}(\mathcal{F}';Y;U)$ and $S_L=\mathsf{S}^{\mathrm{u}}_{\mathfrak{c},1}(\mathcal{F}';L;U')$, let Π be a pairing of S_Y and S_L, $c\in Z^1_{\mathrm{f}}(Y)$ and $a\in Z^1_{\mathrm{f}}(L)$. The jump sets of the two sides are in the windows of the statement, with $\beta(i)=\beta$ and $\mathsf{s}(i)=\mathsf{s}$ for all i, and the jumps of the two sides and of the third side are at most $n=\beta+6(W+\eta)$. Put $\kappa=\mathsf{r}^2(\mathsf{n}+1)^2$. By Proposition 4.3 with $m=\mathsf{r}^2$, for the polynomials q, q' and q_3,

$$\mathrm{Kill}_{S_Y}(i;\kappa,0),\qquad \mathrm{Kill}_{S_L}(i;\kappa,0),\qquad \mathrm{Kill}_{S^{\vee}_{\Pi}}(i;\kappa,0)$$

for every letter i: $n+\kappa+2<p$ is the hypothesis, and $\mathsf{r}^2(\mathsf{n}^2+1)\le\kappa$. We apply Theorem C of Part 2 (Input Q5) with this κ, with $\rho=0$ and $H=3\kappa+150(\eta+W+1)$. (i) has just been shown. (ii): a non-zero element of $\mathcal{W}_{6(\eta+W)}(\beta)\cup\mathcal{W}_{6(\eta+W)}(-\beta)$ has absolute value at most $\beta+6(\eta+W)<p$. (iii) is assumed. (iv): the interval $[-0,0)$ is empty. (v): $3(\kappa+(48(\eta+W)+4)\mathfrak{d})\le 3\kappa+144(\eta+W)+12\le H$, as $\mathfrak{d}\le 1$. So $\mathrm{Cup}_{\Pi}(H;c,a)$, and Lemma 6.1, for the filter $\mathcal{F}'$, the class $\mathcal{Q}=\mathrm{Cl}(\mathcal{F},\mathsf{K};U)$ and $e=H$, gives $p^{H}a\in L^{\perp\mathrm{Cl}(\mathcal{F},\mathsf{K};U)}$. □

7 Where the results are used

The unit on Selmer groups of the handout bounds the loss at one place above ℓ by the four theorems, in its note *Columns and rows as Galois representations* (the lemma called *the crystalline case* there, and the proof of Theorem E of that note). Theorem B serves the tame case of that proof, in which ℓ is one of the primes of the second level of the data. The other three serve the crystalline case, through the lemma, whose integers Q, h and λ are those of that note (its h is not the residue degree of a chart): Theorem C (sparse thick letters) when $\ell\le Q$, where the signs are constant; Theorem D (two sizes) when $Q<\ell$ and $h^3\le\lambda$; and Theorem A (few low weights), with both of its assertions, when $Q<\ell$ and $\lambda<h^3$. Nothing else of the three parts is used there. Inside the unit, Proposition 3.3 is used only for Theorem A, the results on tame charts only for Theorem B, and Proposition 5.2 only for Theorems C and D.

A dictionary. The notes on the unit on Selmer groups write $\mathcal{Q}(F,K,U)$ for the class $\mathrm{Cl}(\mathcal{F},\mathsf{K};U)$, $Z_{\mathrm{f}}(L)$ for $Z^1_{\mathrm{f}}(L)$, "of tame type (ℓ,ϱ,M)" for $\mathrm{Tame}_{\ell}(\varrho,M;\cdot)$, f_k and e_k for the residue degree h and the index $e_{\mathfrak{c}}$ of a chart, ϵ for the signs s, and $B_{\mathrm{cris}}\otimes L$ for $B_{\max}\otimes L$, with the remark that the ring is Colmez's $B_{\max}$. "The note on crystalline lifts" there means the three parts on this unit; the theorems it quotes are Theorems A to D of this part.

8 Notes on the formal text

Arguments as the text gives them. The arguments were read from the Lean text and the census of the workspace, and were not printed by Lean. The hypotheses of Settings 2.5 and 2.6 are section hypotheses of the formal text; each declaration includes those that its statement here names, by the commands of the text that include and omit them, which were read. Several declarations carry implicit arguments that their statements do not use: a letter and an exponent function in Proposition 3.3, a

function of the letters in Theorem C, an integer in Lemma 5.1 and Theorem C, and the function β in Lemma 3.1. The universal side has four explicit arguments: the chart, the root, the hypothesis on the filter and the membership of the lattice in a class. Its notation shows the filter in the slot of the hypothesis on it, and the lattice with its jump sets in the slot of the membership. The statement of Lemma 5.1 is the formal one with its auxiliary integer n'.

The number $\mathfrak{d}$ for a tame chart. For a tame chart with index different from 1 the number $\mathfrak{d}$ is $v_p(e_{\mathfrak{c}}) + 1 = 1$, not 0. An earlier state of the reader's dossier put it equal to 0 in the proof of Theorem B; the second reader of the workspace corrected this, the dossier now says so, and the formal proof agrees: it unfolds the definition and uses only that p does not divide the index. The constant $150(\eta + W + 1)$ absorbs the term.

The product in Lemma 4.2. The polynomial is the product $q \cdot (X - 1)$, evaluated at the linear Frobenius; the factor $X - 1$ is what removes the constant w in the proof of Proposition 4.3.

The filter. The hypothesis of the formal text on the filter $\mathcal{F}$ says that every member of $\mathcal{F}$ contains Γ', not that Γ' is a member.

Lengths. In Theorem A the length is over $\mathcal{O}$, and the factor $\mathrm{len}_{\mathcal{O}}(\mathcal{O}/p\mathcal{O})$ is the ramification index of $\mathcal{O}$; the formal proof passes through lengths over $\mathbb{Z}_p$ as in the proof above.

Arithmetic. The formal text settles the comparisons of sizes and exponents by decision procedures. The note gives them by hand in Lemma 3.2, Proposition 3.3, Lemma 5.1, Proposition 5.2 and the proofs of Theorems A, B, C and D. In Proposition 5.2 the final comparisons use (W1), (W2), (W7), (W8), (W10), the choice of $t(i)$ in (iv) and, in Case 2, the inequality $N < \beta(j)$; of the family $\mathsf{R}^{\epsilon'}$ of the other side they use only the lower bound $-v - \eta_\nu$, as the formal proof does. The hypothesis $\eta \geq 1$ of Proposition 3.3 is not used by the argument given for it. Theorems C and D do not involve the objects cut, $\rho(\cdot)$, n, κ, ρ_0 of Setting 2.6 (Theorem D not J and X either): the formal statements omit the corresponding hypotheses, and their proofs choose these objects. In the proof of Lemma 3.2 the integer N is the least multiple of p that is at least $\beta(i) + W + 2\rho$.

The background. The period ring is Colmez's $B_{\max}$ (Setting 2.1). The Inputs K1 to K8 are statements of the background library; the authors report that Lean's kernel accepted their proofs, and they are not proved in this note. Two of them must not be read as citations of published theorems. The crystalline criterion (Input K3) is one implication of Fontaine's criterion, in a generality (arbitrary modules, any group of open image, the ring $B_{\max}$) for which the second reader finds no published statement; its place in the literature is not confirmed. The finiteness of the module of crystalline cocycles (Input K5) is a statement about integral cocycles defined with $B_{\max}$; the Bloch–Kato formula is about the rational cohomology defined with B_{cris}, and the bridge between the two (coboundaries, the comparison of the two period rings, continuity) is sketched by the second reader and not located in the literature. For the linear Frobenius the second reader finds the classical statement for representations over a field in Fontaine's exposés and no source for the form on arbitrary modules. No place in the literature was confirmed for the frame theorem, for the tame inertia statement in the form used, and for the killing of duals of tensor products.

How this note was written. From the reader's dossier of the unit: the statements of all members and the structured proofs of the declarations treated here; and from the second reader's two reports on the unit, whose corrections were applied. The three Lean files were read in full. The proofs of Lemma 3.1 and of Proposition 4.4, which the reader had marked as least certain, were read line by line in the Lean text: in the first, the identity $p(x^* - w_4) = \mathsf{B}(px^* - w) + \mathcal{E}_{\nu_0} w_3$ and the three places where hypothesis (i) enters were recomputed by hand; in the second, the normality of Δ and the bound for the exponent of the quotient. The verifications of the deep points in Proposition 3.3, the induction in Proposition 4.3 and the reductions of Theorems C and D to Proposition 5.2 were worked out by hand; the formal proofs leave them to automation. The comparisons at the end of the proof of Proposition 5.2 were first left to the formal text; the referee of this note carried them out and found that they also use (W10), and they were then checked once more and written into the proof. The program check of the unit's dossier covered 375 leaf steps of 69 proofs.

Appendix A. Concordance with the Lean artifact

In this note	Lean declaration	File	Lines
Definition 2.4	`Lift.IsTameType`	ExponentTheorems.lean	39–40
Definition 2.4	`Lift.KummerChart.letterJumps`	ExponentTheorems.lean	44–45
Definition 2.4	`Lift.KummerChart.universalSide`	ExponentTheorems.lean	50–52
Lemma 6.1	`Lift.KummerChart.mem_cupOrthogonal_of_scaledCupBounds`	ExponentTheorems.lean	56–59
Theorem A	`Lift.KummerChart.few_low_jumps`	ExponentTheorems.lean	69–99
Theorem B	`Lift.tame_types_exponent`	ExponentTheorems.lean	107–139
Definition 2.2	`Lift.KummerChart.PairingSide.deepConstants`	FewLowJumps.lean	20–21
Lemma 3.1	`Lift.KummerChart.PairingSide.relLength_deepConstants_le`	FewLowJumps.lean	35–94
Lemma 3.2	`Lift.KummerChart.PairingSide.exists_relLength_deepConstants_le`	FewLowJumps.lean	101–143
Proposition 3.3	`Lift.KummerChart.PairingSide.exists_scaledCupBounds_few_low_jumps`	FewLowJumps.lean	148–163
Definition 2.3	`Lift.TameChart`	SparseThickLetters.lean	27–33
Lemma 4.1	`Lift.TameChart.exists_z_smul_eq_sum_tameGen`	SparseThickLetters.lean	43–61
Lemma 4.2	`Lift.TameChart.aeval_linearFrob_extLattice_eq_zero`	SparseThickLetters.lean	67–76
Proposition 4.3	`Lift.TameChart.killsCokernel_of_isKilledBy`	SparseThickLetters.lean	81–116
Proposition 4.4	`Lift.exists_tameChart`	SparseThickLetters.lean	124–159
Lemma 5.1	`Lift.KummerChart.PairingSide.killsCokernel_at_cut`	SparseThickLetters.lean	178–194
Proposition 5.2	`Lift.KummerChart.PairingSide.SidePairing.scaledCupBounds_of_cut_windows`	SparseThickLetters.lean	199–243
Theorem C	`Lift.KummerChart.PairingSide.SidePairing.scaledCupBounds_sparse_thick`	SparseThickLetters.lean	253–270
Theorem D	`Lift.KummerChart.PairingSide.SidePairing.scaledCupBounds_two_sizes`	SparseThickLetters.lean	273–286

Two statements as the artifact has them:

```
lemma KummerChart.few_low_jumps (hk : IsUnramified) (hp : 5 ≤ p) {a n W} {b : _ → ℕ} {ε : _ →
  _} (ha : ∀ n, b n + W + n ≤ (p - 3) * a) (hn : 1 ≤ n) (hb : ∀ n, 6 * (a + n + W) < b n) (hs : ∀ n,
  NoMultipleNear p (6 * (a + n + W)) (b n)) (hL : InClass ⊤ ⊥ U' L) (hV : ∀ i, IsInJumpWindow n
  (-ε i) (b i) W (k.letterJumps ζ U i) ∧ IsInJumpWindow n (ε i) (b i) W (k.letterJumps ζ U' i)) :
```

```
((∀ g υ, υ ∈ U g ∨ υ ∈ U' g → υ ∉ Ico (-↑a) 0) ∨ (∀ g, U' g ⊆ Icc (-n) 0) → crysCocycles L.ρ ≤
  cupOrthogonal (InClass ⊤ ⊥ U) L) ∧ ∀ m, (∀ g, ∑ υ ∈ Icc (-↑n) ↑n, jump L.ρ g υ ≤ m) →
  relLength (crysCocycles L.ρ) (cupOrthogonal (InClass ⊤ ⊥ U) L) ≤
  ↑(11 * (k.f * (m * (a + 1) + finrank 0 L * (W + 1)))) * length 0 (0 / Ideal.span {(p : 0)})

lemma tame_types_exponent (hj : Surjective Γ.overGp.toGp) {F K} (hL : InClass F (K ·⁻¹) U' L) (hK :
  ∀ s, IsWeilPair (toGp s) 0 → IsTameType p r N (K s)) {n W b ε} (hU : ∀ g,
  IsInJumpWindow n (-ε) b W (U g) ∧ IsInJumpWindow n ε b W (U' g)) (hp : b + 6 * (W + n) +
  r ^ 2 * (N + 1) ^ 2 + 2 < p) : ∀ a ∈ crysCocycles L.ρ,
  (p : 0) ^ (3 * (r ^ 2 * (N + 1) ^ 2) + 150 * (n + W + 1)) • a ∈ cupOrthogonal (InClass F K U) L
```

Legend. As in Parts 1 and 2; in addition `InClass F K U` is the class $\mathrm{Cl}(\mathcal{F},\mathsf{K};U)$, `cupOrthogonal` is $L^{\perp_Q}$, `crysCocycles` is Z^1_{f}, `relLength` is the relative length, `linearFrob` is the linear Frobenius, `IsWeilPair (toGp s) 0` says that s is inertial, `jump` is the multiplicity of a jump, `k.f` is h and `k.e` is $e_{\mathfrak{c}}$, `differentExp` is $\mathfrak{d}$, `letterFrob` is $i \mapsto i^p$ and `stepBack i a` is $i^{p^{-a}}$; `zetaE`, `tameGen` and ϖ of a tame chart are ϱ, σ and ϖ. In the first file `a` is ρ, `b` is β, `ε` is `s`, `n` and `r` are n_{lo} and n_{all}, and in its first lemma `a` is ν_0 and `c`, `c'` are o, o'. In the second file `s`, `s₀` are ρ, ρ_0, `b` is B, `T` is the property of being thick, `K` is K_0, `e` is E, and `v` in the proof of Proposition 4.3 is a. In the third file `ζ` is ω, `r` and `N` are `r` and `n`, `m` is m_0 and `g` is γ.

Notation

ℓ the prime of Chapter VIII of the handout; it is p here

p, $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\bar{\mathbb{Q}}_p$, C, $\mathcal{O}_C$, $G_{\mathbb{Q}_p}$, $\mathbb{F}_p$ the prime, the p-adic numbers and integers, an algebraic closure, its completion, the integers, the Galois group, the prime field

A_{inf}, $W(C^\flat)$, $\mathcal{O}^\flat_C$ Witt vectors of the tilt and of its fraction field, the tilt

A_{max}, B^+, B_{max}, ξ Colmez's period rings (Setting 2.1; Part 1, Setting 2.1): the p-adic completion of $A_{\mathrm{inf}}[X]/(pX-\xi)$, $A_{\mathrm{max}}[1/p]$, $B^+[1/([\varepsilon]-1)]$; a generator of $\ker\theta$

φ, θ, ε, $[\cdot]$ Frobenius, Fontaine's map, a system of roots of unity in the tilt, the Teichmüller lift

B_{dR}, $\mathrm{Fil}^n B_{\mathrm{dR}}$ the field of de Rham periods and its filtration

Γ, Γ', j_Γ, χ, χ_N, Δ the group, the subgroup of the chart, the map to $G_{\mathbb{Q}_p}$, the cyclotomic character, the character on the N-th roots of unity, an open subgroup

$\mathfrak{c}$, $\mathfrak{S}$, $\mathfrak{S}_B$, $\mathbb{W}$, h, X, z, $\mathcal{E}$, $\mathcal{E}_\nu$, $e_{\mathfrak{c}}$, $\mathfrak{d}$, F_0 the chart, its ring, the ring of Part 1, $W(\mathbb{F}_{p^h})$, the residue degree, the variable, the coordinate, the Eisenstein element and its translates, the index, the different exponent, a value field

ϱ, σ, ϖ, $\varpi^\flat$ the root of unity, the tame generator and the root of p of a tame chart; an element of the tilt above ϖ

$\mathcal{O}$, ω, γ, $f_{\mathcal{O}}$ coefficients, the root of unity, a map $\mathcal{O}\to C$, the residue degree of $\mathcal{O}$

i, i', j, i^- letters; $i^- = i^{p^{-1}}$

L, Y, T, T' lattices and lattice data; T, T' are also sides

$Z^1_{\mathrm{f}}(L)$, $\mathrm{rk}_{\mathcal{O}}(L)$, $U_L(i)$, $m_L(i,u)$, h_L crystalline cocycles, the rank, the jump set and the multiplicity of a jump at a letter, the jump bound

U, U', $U(\gamma)$ families of finite sets of integers

pr_i, $\mathrm{Per}(L)$, $\mathfrak{M}^{\mathrm{inf}}(L)$, $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{loc}}(L)$, $\mathfrak{M}_{\mathrm{loc}}(L)$, $\mathfrak{M}(L)$ projector, periods, Kisin modules

$\mathfrak{X}_L(i;o)$, $\mathfrak{N}_{l,i}(L)$, $(Q)_{\mathfrak{S}}$, $\mathrm{Inv}(Q)$ exponent modules; elements of $\mathfrak{M}_{\mathrm{loc}}$; invariants modulo Q

std_l, $\mathrm{cost}(\nu,n)$, $\mathrm{imp}(o,o')(\nu,u)$, $\mathrm{fr}_{l,n}(o)(\nu,u)$, $\mathrm{up}_{\nu,u}(o)(\nu',v)$, $\mathrm{sh}_l(o_1)(\nu,u)$ standard exponents, cost, improved, shifted and raised exponents

S, S', Y_S, C_S, α_S, S_Y, S_L sides and their parts; universal sides

$\tilde{Y}_S$, C^{tr}_S, pr^C_S, in^Y_S, $\mathrm{pr}^{C,\mathrm{loc}}_S$ extension, trivial datum, projection, inclusion

$\mathrm{cst}_{S,i}(c)$, $\mathfrak{a}_S(r;l,i)$, $\mathrm{Kill}_S(i;\kappa,\rho)$ constants, lifting ideal, cokernel killer

$\mathrm{rs}_n(\nu,u)$, $\mathcal{W}_n(c)$, $\mathrm{NM}(p;n,\beta)$, $U\subseteq[-\eta,\eta]\cup\mathsf{s}\cdot[\beta,\beta+W]$ raised exponents and windows

Π, $S^\vee_\Pi$, $\mathrm{Cup}_\Pi(e;c,c')$ a pairing, its third side, the scaled cup product bounds

$\mathfrak{X}^0_S(i;o)$, $\mathrm{Deep}_S(i;o)$, $\mathrm{Cost}_S(\mathsf{A})\le e$, $\mathrm{Pt}_S(c,i;e;o_1,o')$ kernel with exponents, deep elements, cost of a map, points

$\beta^T_{s,m}$ the linear Frobenius

$\mathrm{Cl}(\mathcal{F},\mathsf{K};U)$, $\mathcal{F}$, $\mathcal{F}'$, K, K^-, K_0, K_1, $\top$, $\bot$, $\mathcal{Q}$ a class of lattices; filters; relations and a property of polynomials; the trivial filter, the empty relation; a class

$L^{\perp Q}$, $\mathrm{len}_R(N/(N\cap N'))$, len, R the orthogonal module, the relative length, length, a ring

$\mathrm{DC}_S(i;n,o)$, $\mathrm{Tame}_p(\mathsf{r},\mathsf{n};\mathsf{K}_0)$, $\mathcal{U}_{\mathfrak{c},\omega}(U)(i)$, $\mathsf{S}^{\mathrm{u}}_{\mathfrak{c},\omega}(\mathcal{F};L;U)$ constants that lift, tame types, jump sets at a letter, the universal side

cut, thick, Q, $\mathsf{l}(i)$, $t(i)$, $\mathsf{s}^\circ(i)$ properties of letters, the distance to the nearest cut, the integer of (iv), a sign

o, o', o^+, o_+, o_1, o_2, o_i, o^T, o_i^S, z, z', $z(i)$, $z'(i)$, d, R_i^ϵ exponent functions and families of them; o_1, o_2 are also functions of u alone; z is also the coordinate, d also an element

ν, ν', ν_0, l, u, v, s indices, levels, jumps; s is also an element of Γ

ρ, η, W, V, β, B, G, J, J', X, κ, ρ_0, ρ_u, W_ν, η_ν, n, n', K_0, E, H, λ the integers of the Settings and of the proofs; X is also the variable, λ also a linear map

n_{lo}, n_{all}, m_0, g_i, r, n, M_0, h_0, N, N, N_a bounds for multiplicities, the number of maps above a letter, the sizes of a tame type, a least common multiple, a residue degree, $p^{h_0}-1$, a multiple of p, a number of roots

s, ϵ, ϵ' signs; ϵ is also a cost

o^* the exponents of the point found, in the proof of Proposition 5.2

$\mathcal{Q}(F,K,U)$, $Z_{\mathrm{f}}(L)$, f_k, e_k, (ℓ,ϱ,M), B_{cris} the notation of the notes on Selmer groups (Section 7); B_{cris} is also the ring defined with divided powers, which is not used

A, B_u, a, a', a_0, b, c, c', e, e', k, m, r, t, x_0, γ, δ integers; B of Lemma 3.1 is an integer; a is also a root of unity and a cocycle, c an element, k, r, t also elements, δ an element of Δ

g, g', σ^k elements of Γ or of $G_{\mathbb{Q}_p}$

x, x'_k, x_k, x^*, y, y', y'', y_k, w, w_k, w_3, w_4, d, t_0, f elements of modules and rings

$\mathsf{P}_i(o)$, $\mathsf{P}_i^0(o)$, N, N', N_i, I, I_2 modules and sets of steps of Section 3; N is also an integer

C the submodule of C_S of Proposition 3.3

A, B additive maps

q, q', q_3, q_k, Q, Q_k, T polynomials and their coefficients; an element of A_{inf}

μ, μ_k, Λ_g periods and linear maps

$v_p(n)$, $\lfloor \log_p n \rfloor$, $\deg q$, $\mathrm{rk}_{\mathbb{Z}_p}(\mathcal{O})$, $\mathrm{Stab}(\varpi)$ the exponent of p in n, the integral logarithm, the degree, the rank over $\mathbb{Z}_p$, the stabilizer of ϖ in $G_{\mathbb{Q}_p}$

Catalogue of quoted results

The argument (the formal folder `New`) uses results of the background library (`Lib`) that its authors class as known. That class is their working classification: the background also holds definitions that belong to this proof. Each entry below says what kind of result it is (a definition, an elementary lemma, a named theorem of the literature, or a substantial result not identified with a named one), what it says, and, for a named theorem, the reference with its place and the relation of the formal statement to the published one. A reference whose place we could not confirm says so. Where the second reader has compared the formal statement with the published one, her verdict and sources replace the first reading (113 entries so far). Classes on the whole boundary: definition: 977, elementary: 1773, named: 110, substantial: 84.

Below: a summary by area, then every named theorem and every substantial result in full. The definitions and the elementary lemmas (1 773 and 977 entries) are listed with the rest in the complete catalogue, a separate document (`catalogue-full.pdf`).

area	files	definitions	elementary	named	substantial
Adelic	52	159	326	22	8
Analytic	3	1	6	4	0
CM	11	23	41	4	8
ClassField	44	100	170	12	4
Cohomology	21	24	31	0	0
Cyclotomic	2	0	1	1	1
DiscreteSeries	39	107	193	10	11
GLn	13	20	68	1	7
GaloisRep	18	16	21	8	1
General	63	178	334	11	5
LocalGL	15	19	32	5	5
PadicComparison	7	7	4	1	1
PadicHodge	32	107	176	10	7
Shimura	21	11	28	2	2
SiegelWeil	9	19	23	4	11
Weil	58	186	319	15	13

class	what it says	reference; relation	formal name

Adelic

class	what it says	reference; relation	formal name
substantial proof-specific	Unfolding at a finite place: let $\eta = \eta_0 * \cdots * \eta_{\{j-1\}}$, K(g) the congruence subgroup of level g, ϑ an s-finite measure and c an element of the unitary group. Then $(\eta * \vartheta)(K(g)c^{-1}) = \eta(K(g)) \cdot \vartheta(\mathcal{N}K(g)c^{-1})$, with $\mathcal{N} = \mathcal{N}_{<j}$ (Shape.Form.unipotentBelow).		IntegralPlace.ascConv_mconv_unfolding
substantial	Let $\zeta_R(\mu) \le 2$, $V \in GL_\iota(R)$, β a unit, and single out $s + k$ coordinates of $V^H w$, $k \ge \mu$. The integral of IntegralPlace.contentIndex^s of these coordinates over the w with βw integral is at most $\zeta_R(\mu) \cdot \mathrm{md}(\beta)^{-(|\iota| - s)}$.		IntegralPlace.lintegral_contentIndex_pow_le
substantial	Let $sm = 1$, $\beta = \bar{\beta}$, $\zeta_{\hat{O}}(\mu) \le 2$, $2\mu + 2 \le |\varkappa|$. Integrals of f(x_T) over x integral off T with $m(x^* x - \beta - a) \in \hat{O}$ are v times the same integrals at T, for a finite v with vol(codifferent E s)$\cdot \nu\{\Gamma : m\Gamma/\delta \in R\} \cdot v \le \zeta_{\hat{O}}(\mu) \mathrm{vol}(\hat{O})^{|\varkappa|}$.		IntegralPlace.mass_near_quadric_off_places_le
substantial	If $\zeta_R(\mu) \le 2$ and $|\iota| \ge \mu + 1$, then on $|\iota| + 1$ hyperbolic planes the measure $\delta(h(v,v))dv$ (Place.quadricMeasure) of the v with βv integral is at most $\zeta_R(\mu) \cdot \mathrm{md}(\beta)^{-(2|\iota|+1)}$.		IntegralPlace.quadricMeasure_cone_le
named	For stably unramified CM E, a hermitian square matrix b with entries in $\hat{O}$ and a left inverse over $\hat{O}$ satisfies $k^H b k = 1$ for some k in $GL_n(\hat{O})$.	C. T. C. Wall, On the classification of hermitian forms I, local classification around pp. 432–433, split and unramified cases; Theorem 2 for the lift from the residue field; Jacobowitz, Hermitian forms over local fields, American Journal of Mathematics 84 (1962), 441–465, general classification *second reading: Orthonormalization of a unimodular integral Hermitian matrix at all finite places.*	exists_orthonormal_$\hat{O}$

class	what it says	reference; relation	formal name
named	Witt over the integral finite adeles: two integral n × k matrices with orthonormal columns ($X^H X = Y^H Y = 1$) are related by some g in $K_0 = GL_n(\hat{O}) \cap G_f$: $gX = Y$.	First, Witt's extension theorem for quadratic spaces over semiperfect rings, Remark 2.1 and Corollary 4.9. Corollary 4.11 concerns generation by quasi-reflections *second reading: Integral extension of two orthonormal frames, applied separately on the finite-place orbits of conjugation.*	`witt_Ohat_of_orthonormal`
named	Witt's theorem: over a field with involution having some $z - \bar{z} \neq 0$, for invertible hermitian J, left invertible X, Y with $X^H J X = Y^H J Y$ satisfy $g\,X = Y$ for some g in U(J).	First, Witt's extension theorem for quadratic spaces over semiperfect rings, Remark 2.1 and Corollary 4.9, p. 16 of preprint *second reading: Witt extension in matrix coordinates for a nondegenerate Hermitian form with a nontrivial field involution.*	`witt`

class	what it says	reference; relation	formal name
named	A non-zero form in an isotypic space has an attached Galois representation in the artifact's sense: a continuous semisimple ρ : Gal(Ω/E) $\to$ GL_n($\bar{Q}$_ℓ) such that, for every $\eta > 0$, at most Frobenius elements (the filter of most degree-one Frobenii) the characteristic polynomial of ρ is coefficientwise within η, ℓ-adically, of the Hecke polynomial of the space at that place.	G. Chenevier and M. Harris, Construction of automorphic Galois representations, II, Theorem 3.2.3, cohomological conjugate-self-dual cuspidal GL representations; J.-P. Labesse, Changement de base CM et séries discrètes, §5.1, Corollaire 5.3 and the compact-infinity observation immediately preceding it; C. Mœglin and J.-L. Waldspurger, Le spectre résiduel de GL(n), introductory theorem, pp. 605–606, and §I.11; M. Harris, K.-W. Lan, R. Taylor and J. Thorne, On the rigid cohomology of certain Shimura varieties, Theorem A and Corollary 7.14, Tate-normalized compatibility at unramified primes; J. Newton and J. Thorne, Symmetric power functoriality for holomorphic modular forms, §1, Theorem 1.2, Corollary 1.3 and its proof, together with the conventions for rec$^{\intercal}$ before §1 *second reading: Coefficientwise ℓ-adic approximation on the Frobenius filter follows from exact unramified compatibility after definite-unitary base change and decomposition of the discrete GL constituents.*	`Isotypic.attached_galRep`

class	what it says	reference; relation	formal name
named	A form of Minkowski's second theorem: for y in GL_n(𝔸) there are a basis Y_j of E^n and integers x_j with $y^{-1}Y_j$ integral at finite places, of Euclidean length ≤ 2^{x_j} over all f infinite places together, and |det y|_𝔸 · 4^{f∑x_j} ≤ (8fn·rd)^{fn}, rd the root discriminant.	E. Bombieri and J. Vaaler, On Siegel's lemma, §§II.2–III.1, Theorem 3, equation (3.3), pp. 16–18, adelic successive minima; M. Henk, Successive minima and lattice points, Theorem 1.3 and its proof in §3, upper bound in Minkowski's second theorem for general real lattices *second reading: Bombieri–Vaaler Theorem 3 gives the claimed constant after passing from their product body to a joint Euclidean norm and rounding the radii to powers of two.*	`minkowski_second`
named	Witt's theorem for a hermitian space V: if X, Y are left invertible n × m matrices over E with $X^{H}JX = Y^{H}JY$, J the matrix of V, then gX = Y for some g in U(J)(E).	First, Witt's extension theorem for quadratic spaces over semiperfect rings, Corollary 4.9 *second reading: The field-level extension theorem evaluated on the Hermitian matrix of the supplied CM space.*	`Space.witt`

class	what it says	reference; relation	formal name
named	Unfolding: let H act measurably on X preserving the s-finite μ, with ν an s-finite inversion-invariant measure on H admitting a cutoff. For measurable f : X → [0,∞], integrating x ↦ ∫_H f(hx) dν over H\X gives ∫_X f dμ.	D. Gentner and G. Last, Palm pairs and the general mass-transport principle, §2, equations (2.5)–(2.9), proper Haar actions and normalized orbital measures; O. Kallenberg, Invariant measures and disintegrations with applications to Palm and related kernels, Probability Theory and Related Fields 139 (2007), 285–310. Publisher abstract. The numbered identities are given explicitly by Gentner–Last §2; S. Staton, Commutative Semantics for Probabilistic Programming, §4.2, Proposition 5, interchange for two s-finite measures and a nonnegative measurable function *second reading: The integral equality holds for arbitrary measurable actions with an inversion-invariant s-finite group measure and a cutoff. This exceeds the cited proper Haar-action setting.*	`lintegral_orbitMeasure_lintegral`

class	what it says	reference; relation	formal name
named	Unfolding: let H act measurably on X with a cutoff, ν s-finite and inversion invariant on H, μ s-finite and H-invariant on X. Then (x, h) ↦ hx is measurable and carries βμ × ν to μ, where βμ is orbitMeasure for the chosen cutoff β.	D. Gentner and G. Last, Palm pairs and the general mass-transport principle, §2, equations (2.5)–(2.9), proper Haar actions and normalized orbital measures; O. Kallenberg, Invariant measures and disintegrations with applications to Palm and related kernels, Probability Theory and Related Fields 139 (2007), 285–310. Publisher abstract. The numbered identities are given explicitly by Gentner–Last §2; S. Staton, Commutative Semantics for Probabilistic Programming, §4.2, Proposition 5, interchange for two s-finite measures and a nonnegative measurable function *second reading: The claimed Kallenberg specialization has incompatible measure hypotheses. The exact pushforward follows from s-finite Tonelli and the two invariances.*	`measurePreserving_orbitMeasure`
substantial	If the sizes fit ϰ (FitsRanks), then for complex n'_τ×n_τ matrices w_τ, z_τ: S_ϰ(w)* G' S_ϰ(z) = ∏_τ det(w_τ* z_τ)^{d_τ} · G S_ϰ(w*z), with G', G the Gram matrices for n', n and d_τ given by schurDetExp.	header: J. C. Jantzen, Representations of Algebraic Groups (2nd ed. 2003); R. Goodman, N. Wallach, Symmetry, Representations, and Invariants (2009), ch. 5	`PolyRep.schur_conjTranspose_weylGram_schur`

class	what it says	reference; relation	formal name
named	If l is twice the Harish-Chandra parameter of (ϰ, e), the product over w of the formal degrees formalDegree of l_w for the compact group U(n) equals the dimension of the representation of weight ϰ.	A. W. Knapp, Advanced Real Analysis, digital second edition, Chapter VI, Corollary 6.29(b), p. 249, Schur orthogonality for normalized Haar measure; Compact-group and polynomial-representation notes hosted by N. Wallach, Theorems 4.5 and 10.7, especially the Schur-character description of irreducible GL modules *second reading: The compact formal-degree identity is Schur orthogonality with probability Haar measure. The right side is a tableau-model dimension.*	`IsHCParam.prod_formalDegree`
named	Irreducibility: let each l_w be injective, v = highestVector ϰ l, and x range over the product of compact unitary groups. If ⟨v, xu⟩ = 0 for all x, or the linear form f kills all xv, then f(u) = 0.	A. W. Knapp, Advanced Real Analysis, digital second edition, Chapter VI, Corollary 6.29(b), p. 249, Schur orthogonality for normalized Haar measure; Compact-group and polynomial-representation notes hosted by N. Wallach, Theorems 4.5 and 10.7, especially the Schur-character description of irreducible GL modules *second reading: A vanishing consequence of Schur orthogonality for a unit highest vector in a finite tensor product of compact unitary representations.*	`highestVector_cyclic`

class	what it says	reference; relation	formal name
named	On a second countable locally compact group with a Haar measure μ that is also right invariant, every s-finite right invariant measure ν is dμ for some d in [0, ∞].	F. van Doorn, Formalized Haar Measure, §6, Lemma 11 and Theorem 12, regular invariant measures and a compact set of positive Haar measure; S. Staton, Commutative Semantics for Probabilistic Programming, §4.2, Proposition 5, interchange for two s-finite measures and a nonnegative measurable function *second reading: The full s-finite assertion extends regular Haar uniqueness and includes an infinite multiplier. There is a direct derivation from s-finite Tonelli.*	`exists_eq_smul_haar`
named	Unfolding: for a discrete subgroup Γ of a second countable locally compact group G with left Haar measure μ and f integrable, g ↦ ∑_{γ∈Γ} f(γg) is integrable on Γ\G, with integral ∫_G f dμ.	D. Gentner and G. Last, Palm pairs and the general mass-transport principle, §2, equations (2.5)–(2.9), proper Haar actions and normalized orbital measures; S. Staton, Commutative Semantics for Probabilistic Programming, §4.2, Proposition 5, interchange for two s-finite measures and a nonnegative measurable function *second reading: Unfolding for the proper left action of a countable discrete subgroup, with its counting measure and the ambient left Haar measure.*	`integral_quotMeasure_tsum`

class	what it says	reference; relation	formal name
named	For a natural number N ≠ 0 in the different of E, the lattice of V contains a basis of E^n of vectors of length at most 6N(fn)²/λ; f counts the infinite places, λ = min(1, first minimum of the lattice of V).	W. Banaszczyk, New bounds in some transference theorems in the geometry of numbers, Math. Ann. 296 (1993), Theorems (2.1) and (2.2), pp. 631–632; J. C. Lagarias, H. W. Lenstra Jr. and C. P. Schnorr, Korkin-Zolotarev bases and successive minima of a lattice and its reciprocal lattice, Theorems 2.4–2.5, pp. 336–337 *second reading: Banaszczyk's Euclidean transference bound plus the trace-dual calculation gives a smaller radius for an E-basis of lattice vectors.*	`Space.exists_short_basis`
named	The q-binomial theorem for subspaces: for a subspace W of a finite vector space over a field with q elements, ∑_{U ⊆ W} (−X)^{dim U} q^{binom(dim U, 2)} = ∏_{l < dim W} (1 − q^l X) for X ∈ ℂ.	Stanley, Enumerative Combinatorics, vol. 1, second edition, Proposition 1.7.2 and equation (1.87), including the finite-field proof *second reading: The finite q-binomial theorem in its subspace-counting form.*	`q_binomial_submodule`
named	The product formula: for a totally complex number field E and x in E^×, the principal idele of x has module ideleAbs equal to 1.	Milne, Algebraic Number Theory, Chapter 7, Theorem 7.15 *second reading: The number-field product formula for the principal idèle of a totally complex field.*	`ideleAbs_toAdele`
substantial	Let F be a form of level K(M), its type a cusp kernel, N ≠ 0 in the different, ϰ = cuspConst, 0 < δ ≤ 1, λ(g) < (δ/(Mϰ²))^n. If |F(gx) − F(g)| ≤ C for real x with all ‖x_w − 1‖ ≤ δ, then |F(g)| ≤ C.	header of Petersson: I. Satake, Sém. H. Cartan 10 (1957/58), exp. 9 bis; R. Godement, ibid., exp. 10; A. Borel, Park City lectures	`Space.IsForm.norm_le_of_firstMinimum_lt`

class	what it says	reference; relation	formal name
named	If e is a cusp kernel on G_∞, every form F of type e for V decays rapidly: |F| times any power of the height is bounded.	A. Borel, Automorphic forms on reductive groups, §§7.6–7.9 and 8.1–8.4, pp. 20–24 of the lecture text; S. D. Miller and W. Schmid, On the rapid decay of cuspidal automorphic forms, Theorem A and Theorems 2.23–2.24; G. Muić, Smooth cuspidal automorphic forms and integrable discrete series, §6, Lemma 6-3 and Theorem 6-4, and §7, Theorem 7-6 *second reading: Borel and Miller–Schmid establish rapid decay for cuspidal automorphic forms. A primary result starting from this full continuous convolution-kernel definition remains unlocated.*	`Space.IsForm.mem_rapidFunctions`
named	Poincaré series: for a cusp kernel e, x ∈ G and a level K (IsLevel), there is a K-fixed form k of type e on V with conj k(g) = ∑_{γ∈Γ} e((xγg)_∞)·1_K((xγg)_f) for all g ∈ G.	R. Godement, Série de Poincaré et Spitzenformen, Séminaire Henri Cartan 10, Exposé 10, §4, Théorèmes 5 and 5 bis; G. Muić, Smooth cuspidal automorphic forms and integrable discrete series, §6, Lemma 6-3 and Theorem 6-4, and §7, Theorem 7-6 *second reading: Godement's holomorphic Poincaré series and Muić's discrete-series constructions have extra hypotheses. Muić's compact-support argument explains the project's pointwise periodization.*	`Space.exists_kernel_form`

class	what it says	reference; relation	formal name
named	If e is a cusp kernel on G_∞ and K is an open subgroup of G_f, the forms of type e fixed by K make up a finite-dimensional complex vector space.	A. Borel, Automorphic forms on reductive groups, §§7.6–7.9 and 8.1–8.4, pp. 20–24 of the lecture text *second reading: Godement's finite-dimensionality lemma in Borel §8.3 applies after the project's uniform height bound and compactness criterion. Harish-Chandra's fixed-type theorem in §8.4 has different hypotheses.*	`Space.finite_levelSpace`
substantial	Unfolding in stages: i : L × H → G closed, L, H unimodular, Λ ⊂ H discrete, Γ = i(Λ), Re ρ ≥ 0, nonzero near a point. Some R ≠ 0 gives ∫_{Γ\G} F = Rc for continuous integrable F whenever each g has k in L × H with ∫_L ∫_{Λ\H} F(lakg) = ρ(g)c.	header of the Unimodular file: O. Kallenberg, Probab. Theory Related Fields 139 (2007); N. Bourbaki, Intégration VII (1963)	`exists_integral_stages_eq_mul`
substantial	Let Γ ≤ H ≤ G, Γ discrete, H closed, μ left invariant, F continuous, left Γ-invariant. If ∫ F(s j(x,b) x) dκ(b) = aF(sx) for s in H (κ a probability, j(x,b) in H) and ∫_H f(h j(x,b)) dh = d∫_H f with d ≠ a, then ∫_{Γ\G} F dμ = 0.		`integral_quotMeasure_eq_zero_of_mean`
named	If A is finite, killed by n and fixed by Q, and the index of Q is prime to n, then A has Higman's property DiscreteModule.IsHigmanProjective for Q.	Aschbacher, Kessar and Oliver, Fusion Systems in Algebra and Topology, representation-theory appendix, Theorem 4.2 (Higman's criterion), relative trace of the identity *second reading: An explicit relative-trace witness from an invertible subgroup index on the coefficient module.*	`DiscreteModule.FiniteKilledFixedInj.isHigmanProjective_of_coprime`

class	what it says	reference; relation	formal name
named	One-sided Poisson summation on the adeles of E: let F ≥ 0 be integrable and lower semicontinuous at the points of a finite set T ⊂ E. If S bounds all finite sums of $\lvert\int \psi(-\xi x)F(x)\,dx\rvert$ over ξ in E, then $\sum_{t\in T} F(t) \le S$.	Tate, Fourier Analysis in Number Fields and Hecke's Zeta-Functions, Theorem 4.1.4, Lemmas 4.2.1–4.2.4 and the normalization following Theorem 4.2.1 *second reading: A one-sided Poisson estimate for a nonnegative integrable function, with local lower semicontinuity replacing global regularity.*	`IsDualFamily.sum_le_of_fourier`
named	Landherr's theorem for spaces with a point: over stably unramified E (StablyUnram), any two spaces V, W with the same signs differ by a rational change of coordinates a and moving the point by some g in G.	Landherr, Äquivalenz Hermitescher Formen über einem beliebigen algebraischen Zahlkörper, pp. 246–247, especially the local–global equivalence on p. 247 *second reading: Landherr's local–global equivalence, expressed in the coordinates of two framed spaces.*	`Space.exists_eq_move`
named	Weak approximation for U(J): for a finite set S of finite places, a neighbourhood O of 1 in G and g in G, some rational γ has γg = hu with u in O and h in G_f trivial at S.	Platonov and Rapinchuk, Algebraic Groups and Number Theory, Chapter 7, §7.1, Proposition 7.3, weak approximation for smooth rational varieties *second reading: Weak approximation on a rational unitary group, with unconstrained finite components represented by a subgroup.*	`Space.weak_approximation`

Analytic

class	what it says	reference; relation	formal name
named	Landau's lemma, one-sided form: if E is as in the setting above with residues r, $\beta > 0$, re r(x) ≥ 0 for real $x > \beta$, and $E(v) \leq e\hat{}\{(\beta-1)v\}$ for large v, then $r(\rho) = 0$ for every ρ with re $\rho > \beta$ (Φ has no pole right of β).	D. V. Widder, The Laplace Transform, Chapter II, §5, Theorem 5b, p. 58. The 1946 second printing of the 1941 edition specifies a finite abscissa and a monotone determining function.; H. L. Montgomery and R. C. Vaughan, Multiplicative Number Theory I, §15.1, Lemma 15.1, p. 463. The locally Riemann-integrable formulation for a density of eventual constant sign. *second reading: corollary of Widder II.5b with a residue argument and a measurable density*	`IsPolar.landau`
named	$\tilde{L}(s, \chi) \neq 0$ for re $s \geq 1$: $L(s, \chi)$ has no zero on re $s \geq 1$, and $(s-1)L(s, \chi_0)$ does not vanish there either (at $s = 1$ it is the residue).	Mathlib at 56a005855ea17e0b0958f73ec801ae140b0a1679, DirichletCharacter.LFunction_ne_zero_of_one_le_re, Nonvanishing.lean, lines 395–400.; Same revision, DirichletCharacter.LFunctionTrivChar$_1$ and LFunctionTrivChar$_1$_apply_one_ne_zero, DirichletContinuation.lean, lines 332–344.; Compact definition of entireL, QuadField.lean, lines 29–31. *second reading: direct consequence of two pinned Mathlib results after splitting the principal case*	`entireL_no_zero`

class	what it says	reference; relation	formal name
named	Let M > 0 be a multiple of 4 and A a natural number. For all large W, every k ≥ 2 with M^k ≤ W^A and every U in [W, W^A], there is a prime p ≡ 1 (mod M^k) with U ≤ log p ≤ U + W.	Montgomery and Vaughan, Multiplicative Number Theory I, §11.3, Corollary 11.19, p. 381 (von Mangoldt sum), and Corollary 11.21, p. 382 (prime sum); Printed statements of Corollaries 11.19 and 11.21 *second reading: weaker consequence, derivation checked*	`exists_prime_modEq_one_log_mem`
named	Let β > 1/2, y > 0, and a a unit mod N such that every prime d ≡ a (mod N) satisfies d ≡ 7 (mod 8) and is prime to N. Then for some prime d ≡ a (mod N), no $\tilde{L}(s, \xi\cdot(\cdot/d))$ with ξ a character mod N vanishes in re s ≥ β, |im s| ≤ y.	M. Jutila, On mean values of Dirichlet polynomials with real characters, Acta Arith. 27 (1975), pp. 191–198. See the character family and Theorem 1 on p. 192 and Theorem 2, equation (6), on p. 193.; Compact proof, zero_density_quadTwist and exists_entireL_quadTwist_ne_zero, ZeroFreeTwist.lean, lines 147–206. *second reading: real-character density theorem with an additional estimate needed for simultaneous complex twists*	`exists_entireL_quadTwist_ne_zero`

class	what it says	reference; relation	formal name

CM

class	what it says	reference; relation	formal name
substantial proof-specific	The ring $\mathfrak{o}[[\varphi]]$ of integral power series in φ, viewed as series in z over $\mathbb{C}_p$, is stable under d/dz.		`CMLattice.deriv_mem_phiSeriesRing`
named proof-specific	For d a prime $\equiv 7 \bmod 8$, a CM lattice and any embedding $\iota\colon \mathbb{C} \to \mathbb{C}_p$, the invariant a_1 is p-adically integral: $\lvert\iota(a_1)\rvert \le 1$.	M. Deuring, Die Klassenkörper der komplexen Multiplikation (1958), English translation by C.-W. Chin, §13, p. 31 of the translation, explicitly uses integrality of the singular class invariant j(α).; D. A. Cox, Primes of the Form x^2+ny^2, §11, Theorem 11.1, p. 220, gives integrality for j of a proper fractional ideal of an imaginary quadratic order.; J. H. Silverman, Advanced Topics in the Arithmetic of Elliptic Curves, II §6, Theorem 6.1, p. 140, CM j-integrality; the Weierstrass invariant formulae are in The Arithmetic of Elliptic Curves, 2nd ed., III §1, p. 42.; Pinned data and the equation of the normalized cubic, CurveTheta.lean 81–92 and 307–321.; Pinned stronger unit-discriminant lemma, IntegralParameter.lean 102–106. *second reading: Integrality of this specified coefficient follows from CM integrality of j and the equation of the marked cubic.*	`CMLattice.norm_a₁_le_one`

class	what it says	reference; relation	formal name
substantial proof-specific	Tate's trick. Let d ≡ 7 mod 8 be prime, a: C → C′ an isogeny of degree N, y torsion modulo Λ′, h, h′ the heights CMLattice.naiveHeight at a p-adic place ι. If |ιg| ≤ h′^m on ℂ, then |ι((U′^m_y g)(az))| ≤ h(z)^{mN} for all z.	the technique (Tate's telescoping trick for local heights): J. Silverman, Advanced Topics in the Arithmetic of Elliptic Curves (1994), Ch. VI (file header)	`CMLattice.norm_translate_le_naiveHeight_pow`
substantial proof-specific	Read in ℂ_p, the Taylor series of ξ(z) = ℘(z+t) − ℘(t) lies in 𝔬[[φ]].		`CMLattice.taylorVia_translatedWp_mem_phiSeriesRing`
substantial proof-specific	If a complex number a satisfies aΛ ⊂ Λ' for the lattices of two CM data for d (a prime ≡ 7 mod 8), then a is p-adically integral: |ι(a)|_p ≤ 1.		`CMLattice.norm_le_one_of_isogeny`
substantial proof-specific	Let 0 ≠ δ ∈ $\boldsymbol{\mathcal{O}}$_K with δδ̄ = N and δy ∈ Λ. Then, read in ℂ_p, the series (U_yϑ)(δ̄z) lies in ϑ^N·𝔬[[φ]].	cf. K. Bannai, S. Kobayashi, Algebraic theta functions and the p-adic interpolation of Eisenstein–Kronecker numbers, Duke Math. J. 153 (2010), 2.14–2.16 (header of the IntegralParameter file)	`CMLattice.rescale_translate_eq_theta_pow_mul`
substantial proof-specific	In ℂ_p, ϑ = exp(−L) with L = O(z²), L″ = L_2 + ξ and √−d·L_2 integral. For each limit (a, N) of pairs (ιδ, ι(δδ̄)), δ ∈ $\boldsymbol{\mathcal{O}}$_K, the series φ(az), exp(NL − L(az)) and (NL − L(az))′ lie in 𝔬[[φ]].		`CMLattice.exists_log_theta`
substantial	Let L be σ-semilinear with values in $\mathcal{W}$_f, Lϱ(x) = ϱ(bx)L on $\mathcal{W}$_f, σψ₀(z) = ψ₀(b̄bz), R ∩ R̄ = ∅. Then b is a unit, Lω(g) = (σΔ(g)/Δ(g))ω(g)L on $\mathcal{W}$_f, and some c and unitary adele z have L vac = cω(z)vac, L(Ψ⊗vac) = cω(z)(σΨ⊗vac), Ψ Schwartz–Bruhat above R.	G. Shimura, Theta functions with complex multiplication, Duke Math. J. 43 (1976) 673–696	`Space.semilinear_intertwines_weil`

class	what it says	reference; relation	formal name
named proof-specific	Complex multiplication for theta functions. Let $\Lambda_C = \Omega_0\mathcal{O}_K$ and $C\hat{}\sigma$ = CMLattice.galConjugate. There are $\Omega \neq 0$ with area($C\hat{}\sigma$) = $\lvert\Omega\rvert^2\sqrt{d}/2$ and a positive integer multiple in $\Lambda_\{C\hat{}\sigma\}$, and $a_S \in K$ for finite $S \subset K$, clustering at a finite adele as S grows, with $\sigma(U_\{x\Omega_0\}\vartheta_C) = U_\{x\Omega a_S\}\vartheta_\{C\hat{}\sigma\}$ as Taylor series for $x \in S$.	G. Shimura, Introduction to the Arithmetic Theory of Automorphic Functions (1971), §5.3, Theorem 5.4, p. 117 (right page of scan), with the torsion diagram and the inverse idèle explicitly shown.; J. S. Milne, Complex Multiplication, v0.10 (2020), II §9, Theorem 9.17, pp. 80–81, finite idèle, type of the conjugate and torsion diagram.; K. Bannai and S. Kobayashi, Algebraic theta functions and the p-adic interpolation of Eisenstein–Kronecker numbers, §2.2, Theorem 2.9, pp. 25–26 in arXiv v4, and Theorem 2.12, pp. 26–27, for the translated sigma expansion.; Pinned σ, chosen conjugate, and equality of translated Taylor series on a finite set, ConjugateCurve.lean 11 and 69–81.; Pinned Taylor coefficient convention and normalized twisted theta, CurveTheta.lean 65–69 and 100–112. *second reading: The cited CM reciprocity theorem identifies the action on torsion. Equality of the full, normalized translated theta germs needs a further comparison.*	`CMLattice.complex_multiplication_theta`

class	what it says	reference; relation	formal name
named proof-specific	Let $Z = X + \cdots$ with $f(Z)$ integral for all $f \in \mathfrak{o}[[\varphi]]$, let $a\Lambda \subset \Lambda$ with $a\bar{a} = q$ odd, and let g be integral without constant term, $Z(g) = \iota(a)Z$. Then g has a unit coefficient in some degree $\leq q$.	J. H. Silverman, The Arithmetic of Elliptic Curves, 2nd ed., IV §7, Proposition 7.2(a,b), p. 133, Theorem 7.4 and Corollary 7.5, p. 134. The order of a nonzero formal isogeny is p^height, and for an elliptic isogeny equals its inseparable degree.; J. H. Silverman, Advanced Topics in the Arithmetic of Elliptic Curves, II §4, Proposition 4.4, pp. 124–126, for injectivity and preservation of isogeny degree at good reduction.; V. G. Drinfel'd, Elliptic modules (1974), §1(D), Proposition 1.7, pp. 566–567, finite-height classification and the endomorphism algebra.; Pinned derivative of φ and the normalized Weierstrass equation, CurveTheta.lean 310–321.; Pinned IsIsogeny as lattice inclusion and an area/degree equality, SigmaFunction.lean 47–50. *second reading: The unit-coefficient bound follows from the inseparable degree of the reduced isogeny, after identifying the integral local parameter.*	`CMLattice.exists_unit_coeff_degree_le`

class	what it says	reference; relation	formal name
named proof-specific	If for every f in $\mathfrak{o}[[\varphi]] \subset \mathbb{C}_p[[z]]$ the coefficient of z^{p−1} in f′ has absolute value below 1, then the coefficient of z^{p−1} in $\iota\xi$ has absolute value at least 1, and p does not split in $\mathbb{Q}(\sqrt{-d})$.	N. Katz and B. Mazur, Arithmetic Moduli of Elliptic Curves (1985), §12.4.1, especially (12.4.1.1) and (12.4.1.3), p. 354, for the Hasse coefficient of [p] and the restricted p-th power of the invariant derivation.; N. Katz, p-adic properties of modular schemes and modular forms, §2.0, pp. 97–98, gives the p-th iterate description of Frobenius on the invariant derivation.; M. Deuring, Die Typen der Multiplikatorenringe elliptischer Funktionenkörper, Abh. Math. Sem. Hansischen Univ. 14 (1941), introduction nos. 2–3, pp. 198–200; §11 no. 1, pp. 269–270, relates reduction to splitting in the CM field.; J. H. Silverman, The Arithmetic of Elliptic Curves, 2nd ed., V §3, Theorem 3.1(a), pp. 144–145, and V §4, Theorem 4.1(a), pp. 148–149 (equivalent supersingularity criteria and the characteristic-p cubic coefficient formula).; Pinned derivative in the integral formal parameter, IntegralParameter.lean 81–96 and 280–305.; Pinned coefficient recurrence and Legendre-symbol implication, FormalHeight.lean 137–162. *second reading: The premise detects vanishing of the Hasse invariant and accounts for nonsplitting. The second-kind coefficient estimate requires a separate calculation.*	`CMLattice.one_le_norm_coeff_translatedWp_nonsplit`

class	what it says	reference; relation	formal name
substantial proof-specific	Euler's addition formula in integral form: read in ℂ_p, the series φ(u + w) lies in 𝔬[[φ(u), φ(w)]] (the addition law of the CM curve is integral in the parameter φ).		`CMLattice.translatedWpDeriv_add_mem_multiPhiRing`

ClassField

class	what it says	reference; relation	formal name
substantial	Polarisation: suppose M(q(u^⊤Y)) = K(u) for all tuples u of unitary n×n matrices. Then ⟨q, ∏_τ det(rows r)^{#ι_τ}⟩·M(∏ det(η_{τk}Y_τ)) = ⟨K, ∏ det(η_{τk}X_τ)⟩ in the Fock pairing, X_τ being the columns r of the generic n×n matrix.		`fockPairing_polarisation`
named	Let ψ be trivial off a finite set S. If ∏_v Φ_v(u)^{e(v)} · ∏_{w∈S} ψ_w(u mod w) = 1 for every integer u of E with uū = 1, then some character of U(1)(𝔸), trivial on rational points, has weight e and tame part ψ.	A. Weil, On a certain type of characters of the idèle-class group of an algebraic number-field, Tokyo–Nikko proceedings (1956), pp. 1–7, especially pp. 2–3 on the unit and finite-index extension conditions, equation (3).; A. Raghuram, Notes on the arithmetic of Hecke L-functions, §3.1, equation (12) and Proposition 13, pp. 9–10 of the preprint.; Raghuram, §4.3.2, Proposition 25, p. 17. This existence theorem prescribes the pure infinity type only.; Pinned definitions of AutChar and HasWeight, CharacterDefs.lean, lines 16–60. *second reading: The cited full idèle-class character existence result has a different domain and compatibility condition*	`AutChar.exists_hasWeight_of_prod_eq_one`

class	what it says	reference; relation	formal name
named	Kronecker's theorem in a CM field E: an algebraic integer γ of E with $\gamma\bar{\gamma} = 1$ satisfies $\gamma^n = 1$, n the number of roots of unity in E.	L. Kronecker, Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten, J. reine angew. Math. 53 (1857), pp. 173–175, Satz I beginning on p. 173. The digitized Werke I reprint gives the theorem on p. 105. *second reading: Kronecker's first theorem for integral norm-one elements of a CM field, with the torsion-group exponent supplied*	`pow_torsionOrder_eq_one`

class	what it says	reference; relation	formal name
named	Let ν have weight $e \neq 0$ and some tame part, and let S be finite. Then the L-function of $x \mapsto \nu(x/\bar{x})$ without the factors at $S \cup \bar{S}$ is non-zero at every real $s \geq 1$.	D. Ramakrishnan and R. J. Valenza, Fourier Analysis on Number Fields (1999), §7.7, Theorem 7-28, p. 289 and (7.16), proof pp. 289–292, including the nonquadratic case. This gives the precise nonvanishing theorem and reduction to the finite L-function.; J. Tate, Fourier analysis in number fields and Hecke's zeta-functions, Cassels–Fröhlich, ch. XV, §4.4, Main Theorem 4.4.1, p. 339, for the global zeta-integral continuation and exceptional poles.; Pinned heckeCoeff and real extension, EulerProduct.lean 113–124. Eventual unramifiedness is in the same file at line 99.; Pinned definition of the partial product and doubled infinity type, AutCharLFunction.lean 49–69. *second reading: Real-axis specialization of Hecke nonvanishing for a nonzero angular weight vector, with a finite Euler-factor adjustment.*	`AutChar.HasWeight.L_ne_zero`
named	Chebotarev: for a non-empty open $W \subset \Gamma = \mathrm{Gal}(\bar{K}/K)$ and a finite set S of primes, there is $v \notin S$ of prime norm with a place above it all of whose arithmetic Frobenius elements lie in W.	Milne, Class Field Theory, v4.03, Chapter VIII §7, Theorem 7.4 *second reading: existence consequence of finite Galois Chebotarev, with a chosen place above the prime*	`chebotarev`

class	what it says	reference; relation	formal name
named	For A lying over w, if s in the decomposition group acts on residues as the (N w)^n-th power, n ≥ 0, then its class is the local Artin symbol of some x ∈ K_w^× of order −n.	J.-P. Serre, Local Fields, XIII §4, Proposition 13 and corollary, pp. 197–198, and XIV §6, Theorem 1, Corollary 2 and Remark 2, pp. 218–219. The map on the unramified quotient is F raised to the additive valuation. The modified abelian group allows the integral powers of F.; Pinned residue-action definition, Places/GaloisDefs.lean, lines 73–78.; Pinned uniformizer case, ArtinMapAtPlace.lean, lines 130–147. *second reading: Existence in the image of local reciprocity with a specified valuation, after translating geometric Frobenius*	`IsPlaceOver.exists_localArtin_eq`
named	For A above w, there is a unit u of K_w whose local Artin image is the class of the chosen tame generator of inertia and whose residue generates the multiplicative group of the residue field.	J.-P. Serre, Local Fields, XIII §4, Proposition 13 and its corollary, pp. 197–198, and XIV §6, Corollary 2(ii), p. 219. Units and inertia in the abelian reciprocity map.; Serre, Local Fields, XIV §2, Proposition 6, p. 208, and §3, Proposition 8 with finite-field corollary, pp. 210–211. The first identifies the Kummer ratio, the second computes its residue.; Pinned choice of γ and its use at n=Nw−1, ArtinMapAtPlace.lean, lines 234–276. *second reading: A simultaneous unit lift and primitive residue derived from two local reciprocity assertions*	`IsPlaceOver.exists_localArtin_eq_tameGenerator`

class	what it says	reference; relation	formal name
named	Existence theorem: for K totally complex, a continuous homomorphism ϑ from the finite ideles to a group C with a T_1 topology, trivial on K^×, is μ∘Art for a continuous homomorphism μ: Γ → C.	J. S. Milne, Class Field Theory, v4.03, V §5, Reciprocity Theorem 5.3 and Existence Theorem 5.5, pp. 178–179, and Remark 5.7(a), pp. 179–180 (surjectivity and the closed kernel).; Pinned finite-idèle factorization, including the T_1 assumption and quotient-map proof, Avatars.lean 157–172. *second reading: The finite-idèle factorization for a totally complex number field follows from the kernel description in global reciprocity.*	`exists_comp_artinMap`
substantial	Let M be finite, unramified outside S (primes dividing N), fixed by complex conjugations, with at most (#M)^k trivialised cocycles, and $B^1 \subset L_v$ on S. Then selmerZ1, ramification allowed at the primes of s, has #selmerZ1·(#M)^(#S+#∞) ≤ (#M)^k·∏_{v∈S} #(L_v ∩ Z^1), ∞ the infinite places of K.		`selmerZ1.card_mul_pow_le`
substantial	Relations at split primes. For U fixing M and μ_m, without complex conjugations, there are split primes, Frobenii φ_i and elements γ_i such that each family (n_i) with ∑ n_i(x(γ_i)) = 0 on Selmer cocycles x extends to an equivariant homomorphism on U giving such relations at other split primes. A bound on trivialised cocycles bounds their number below.		`selmerZ1.exists_split_primes_relations`

class	what it says	reference; relation	formal name
named	Hilbert 90 for finite ideles: for every norm-one adele u of the CM field E there is a finite idele y such that the finite part of u equals $y/\bar{y}$.	J.-P. Serre, Local Fields, X §1, Proposition 2, p. 150, and cyclic corollary, p. 151. The coefficient group here is a field's multiplicative group.; J. S. Milne, Class Field Theory, VII §2, Proposition 2.5 and Corollary 2.6(a), pp. 205–206. The idèle cohomology decomposes into local groups, including the unramified integral-unit argument.; Compact finite-idèle form invoked in CharacterLevels.lean, IdeleHilbert90.lean, line 166 onward. *second reading: The quadratic finite-idèle component of idelic Hilbert 90*	`exists_divStar`

class	what it says	reference; relation	formal name
named	Local duality: L, L' potentially crystalline, perfectly paired by b into $\mathbb{Z}_\ell(1)$, $\Gamma \to G_\ell$ an open embedding, A the image of Z_f(L) in $V = L/\ell^{\wedge}m$. If A and S consist of continuous cocycles whose cup products with H bound in $\mathbb{Q}_\ell^{\wedge}\times$, then [S : S∩A] divides [Z_f(L') : H∩Z_f(L')].	J. S. Milne, Arithmetic Duality Theorems, 2nd ed., I §2, Corollary 2.3, p. 28 (perfect finite local pairing), and Theorem 2.8, p. 31 (local Euler–Poincaré characteristic).; S. Bloch and K. Kato, L-functions and Tamagawa numbers of motives, The Grothendieck Festschrift I, §3.7, pp. 353–354, Proposition 3.8, p. 354, and Corollary 3.8.4, p. 355. The groups here are H^1 with the finite local condition.; J.-P. Serre, Galois Cohomology, II §§5.2 and 5.7, sections beginning on pp. 90 and 99. The accessible preview confirms these section titles and pages. The numerical statements above were checked in Milne's full text.; Pinned period-ring identification, BdR.lean, lines 4–8, and the associated cocycle definition in Crystalline/Defs.lean, lines 51–57. *second reading: Finite Tate duality and the Bloch–Kato finite condition explain separate inputs to the integral cochain divisibility*	`IsPotCrys.relIndex_dvd_of_orthogonal`

class	what it says	reference; relation	formal name
named	Dirichlet's S-unit theorem: for a finite set S of finite places of K, the S-unit group is finitely generated of rank #S + (number of infinite places) − 1.	J. Neukirch, Algebraic Number Theory (1999), VI §1, Proposition (1.1), pp. 358–359, including the proof by valuations at finite places. The ordinary-unit case is I §7, Theorem (7.4), p. 42.; Pinned S-unit valuation and rank argument, AdeleNorm.lean 91–111. *second reading: The algebraic finite-generation and rank assertions are exactly Dirichlet's S-unit theorem.*	`SUnit.dirichlet_unit_theorem`
named	The global Artin map from the finite ideles of the number field K to the abelianisation of its absolute Galois group Γ is surjective.	Milne, Class Field Theory, v4.03, Introduction pp. 11–12, global idèle-class reciprocity; Pinned definition of the finite-idèle homomorphism in terms of the local Artin maps, lines 99–108 *second reading: surjectivity from finite idèles follows from global reciprocity and weak approximation*	`artinMap_surjective`

class	what it says	reference; relation	formal name
named	Let K be totally complex, s auxiliary primes for l^r (r ≠ 0), N and U (SplitPlaces), indexed by ι, with Frobenii φ_i; V finite, l^r-torsion, U-fixed; z central modulo U, fixing l-power roots of unity, z − 1 onto V. If U fixes each l^r-th root, fixed by all φ_i, of a U-fixed N-unit, the trivialised cocycles unramified outside N and s are finite in number, at most (#V)^{#ι+1}.	H. Darmon, F. Diamond and R. Taylor, Fermat's Last Theorem, author text dated 24 May 2000, §2.3, Theorem 2.18 for ℚ, p. 61, and Theorem 2.19 for any number field, p. 62.; A. Wiles, Modular elliptic curves and Fermat's Last Theorem, Annals of Mathematics 141 (1995), §1.2, Proposition 1.6, pp. 472–473, gives the ratio for odd-primary modules over ℚ.; J. S. Milne, Arithmetic Duality Theorems, 2nd ed., I §4, Theorem 4.10, pp. 56–57, and I §5, Theorem 5.1 (global Euler characteristic). These are cohomological inputs to the ratio.; Pinned definitions of the auxiliary primes, I and Frobenius detection, SelmerBasic.lean 91–123.; Pinned pair space and Nat.card bound, DiscreteUnits.lean 157–165. *second reading: The Greenberg–Wiles formula compares two Selmer groups in H^1. A published derivation of this absolute count of trivialised cocycles was not located.*	`SplitPlaces.hasFewCocycles_of_central`
substantial	Let V be finite, l-torsion, U-fixed, with Higman's property, z − n injective on V, and let U fix zr/r^n whenever r^l is an N-unit and all φ_i fix r. Trivialised cocycles number at most (#V)^{#ι+1}, exactly so if K is totally complex and z = n on μ_l.		`SplitPlaces.card_trivialised_of_isHigmanProjective`

Cyclotomic

class	what it says	reference; relation	formal name
substantial	For a number field E inside $\mathbb{Q}(\zeta_N)$ and a prime l there is a finite set S of elements of $\mathrm{Gal}(\bar{\mathbb{Q}}/E)$, with $\#S \le y^{20}$ whenever $N \le y$ and $l \le y^2$, such that every cocycle with values in $O/l^n(-e)$ that is unramified outside l (a Selmer class with no local condition imposed above l) and vanishes on S is zero, for all n, e and every l-adic integer ring O.		`exists_detecting_points_cocycles`
named	For a CM field E in $\mathbb{Q}(\zeta_N)$ and $t \ge 1$ there is an integer $1 \le X \le 2\hat{}\{4(N+t+3)\hat{}27\}\cdot h_E$ (h_E the class number) such that, for every prime l, every $0 \le e < t$, every n, and every characteristic-zero discrete valuation ring O over $\mathbb{Z}_l$ with finite residue rings containing the roots of unity of order $\exp((\mathbb{Z}/N)\hat{}\times)$, the odd part of the Tate–Shafarevich group with coefficients $O/l^n(-e)$ has length at most v_O(X).	Greither, Class groups of abelian fields, and the main conjecture, Theorem 3.2, p. 469; Mazur and Wiles, Class fields of abelian extensions of Q, introduction on the Iwasawa main conjecture, pp. 179–184; Pinned definition of the local conditions and cocycle module, SelmerDefs.lean:89–97; Pinned coefficient-ring class, FittingDuality.lean:32–33 *second reading: explicit finite-level estimate; main-conjecture comparison needs additional arguments*	`exists_length_sha0dd_le`

DiscreteSeries

class	what it says	reference; relation	formal name
named proof-specific	Bruhat's big cell: over a field with involution and an element of trace 1, if all leading minors of Z(γ) are nonzero, then γ = u⁻¹w_pσu' with u, u' in U, w_p a Weyl element of length ϱ and σ in U(X₀ ⊕ Az).	A. Borel and J. Tits, Groupes réductifs, Publications Mathématiques de l'IHÉS 27 (1965), §§5.13–5.15, especially Théorème 5.15, pp. 99–101; A. Knapp, Structure theory of semisimple Lie groups, Theorem 4.9, Bruhat decomposition, and its reference to Lie Groups Beyond an Introduction, Theorem 7.40 *second reading: Bruhat cells over a field are classical. An exact reference identifying these minors, unipotents and intermediate floor factors remains open.*	`Filling.exists_big_cell_decomp`
named	Plancherel's inequality at a place: for a unit η and f measurable and integrable on A^ι, ∫ |∫ ψ(η v*w) f(w) dw|² dv ≤ (c/md η)^{|ι|} ∫ |f|².	J. Tate, Fourier analysis in number fields and Hecke's zeta-functions (1950), §2.2, Theorems 2.2.1–2.2.2, self-duality and self-dual additive Haar measure; H. Cartan and R. Godement, Théorie de la dualité et analyse harmonique dans les groupes abéliens localement compacts, Annales scientifiques de l'É.N.S. 64 (1947), §§IV–VI, inversion, Plancherel and duality *second reading: A one-sided consequence of abelian Plancherel with the compact normalization. Tate's local-field calculation accounts for the local-field case.*	`Place.plancherel_inequality`
substantial	Under the hypotheses of IsBounded-Measurable.lintegral_sq_convolve, for integrable φ the function y ↦ ∫φ(a)(F*x)(ya)da has squared L²(ν) norm at most (‖x‖₂²/d)·(‖F‖²_ν/d)·‖φ‖₂².		`IsBoundedMeasurable.lintegral_sq_convolve_convolve_le`

class	what it says	reference; relation	formal name
named proof-specific	Cartan decomposition of H_ϱ = U(V_ϱ): if σ ∈ H_ϱ has Z(σ)_{nn} = c, n = ϱ+1, with |c| > 1, then σ = h w_ϱ(c) h' for some h, h' in U(X₀), w_ϱ(c) the boost cellChart.	A. Knapp, Structure theory of semisimple Lie groups, Theorem 4.10, KAK decomposition; D. Sprehn and N. Wahl, Forms over fields and Witt's lemma, Theorem 3.4, extension of subspace isometries *second reading: An explicit rank-one double-coset decomposition with the phase fixed, obtained by unitary transitivity on vectors of equal length.*	`cartan_decomposition`
substantial	Let j : S → U(J) be a proper group homomorphism, J a hermitian involution of size n ≥ 1, ν right invariant, finite on compacts, c > 8n² natural, y∈U(J). Then (2/‖j(s)y‖)^c is ν-integrable with integral ≤ 16^c·ν{‖j(s)−1‖ ≤ 1} (operator norms).	polynomial volume growth (header sources: L. K. Hua 1963; I. G. Macdonald, Invent. Math. 56 (1980); B. Gross, Invent. Math. 130 (1997))	`integral_decay_le`
substantial	Covering bound: let j: S → U(J) be a group homomorphism, ν right invariant, y∈U(J), D, R ≥ 1, K ≥ 3DR². Then ν{‖j(s)y‖ ≤ R} ≤ K^{2n²} ν{‖j(s)−1‖ ≤ 1/D} (operator norms).		`measure_ball_le_covering`
substantial proof-specific	For T ≥ 8r, r the rank, there are a compact B ⊂ {‖y‖ ≤ T^{20(ϱ+1)}} in G_1 and a measurable Ω ⊂ U^0 with 0 < ν(Ω) < ∞, Ω·{‖y‖ ≤ T} ⊂ B, and ν{u : uy ∈ B} ≤ T·ν(Ω) for all y; ν = Shape.besselHaar₀.		`exists_tube`

class	what it says	reference; relation	formal name
named	For $n \geq 1$ and gap $500n^3 + A$, the inner product on D_λ is invariant under $\mathfrak{u}(\mathrm{diag}\ \varepsilon)$, the generator ψ_λ has norm 1, and $\lvert y(g)\rvert \leq \lVert y\rVert$ for all y in D_λ and g in $U(\mathrm{diag}\ \varepsilon)$.	Harish-Chandra, Representations of semisimple Lie groups VI, American Journal of Mathematics 78 (1956), §3, Schur orthogonality relations; M. Duflo and C. Moore, On the regular representation of a nonunimodular locally compact group, Journal of Functional Analysis 21 (1976), Theorem 3, orthogonality relations; C. Moore, Square integrable primary representations, Pacific Journal of Mathematics 70 (1977), pp. 413–415, unimodular and irreducible cases and explicit orthogonality identity *second reading: Orthogonality explains the evaluation bound after identifying the concrete coefficient model. The additional Casimir assertion and the model identification need separate evidence.*	`HCParam.HasLargeGap.isUnitaryCasimirGe_dsAction`
substantial	For $n \geq 1$ and $l = 2\lambda$ of gap $500n^3 + A$, every element of $D_\lambda = U(\mathfrak{g})\psi_\lambda$ has the form $g \mapsto \int \beta(y^{-1})\ \psi_\lambda(gy)\ dy$ for some smooth compactly supported function β on $U(\mathrm{diag}\ \varepsilon)$.		`exists_smear_dsCoeffSmooth_eq`
substantial	Let $n \geq 1$ and $l = 2\lambda$ be a parameter of gap $500n^3 + B$ with all $\lvert l_a\rvert \leq L$. For x, g in $U(\mathrm{diag}\ \varepsilon)$: $\lvert\psi_\lambda(xg) - \psi_\lambda(x)\rvert \leq 100(n+L+1)^5\lVert g - 1\rVert$, in the operator norm.		`norm_dsCoeff_sub_le`

class	what it says	reference; relation	formal name
named	Reproducing formula: let the order o be injective with signs +1 first, each u_τ K-dominant, F holomorphic of weight u on U(diag ε)^Φ, and F·conj(m) integrable for the compact dual measure, m = holMinorsPi ε u. Then ∫ F·conj(m) = (∫ |m|²)·F(1).	A. Korányi, A simplified approach to the holomorphic discrete series, Indagationes Mathematicae 35 (2024), §4, formulas (4.1) and (4.4)–(4.5), and §6, condition (6.4); Harish-Chandra, Representations of semisimple Lie groups V, American Journal of Mathematics 78 (1956), §§4–5 and §9 on holomorphic function spaces and unitarity. The exact compact minors identity remains unlocated. *second reading: The usual Hilbert-space reproducing property is located. The product and integrability hypotheses in this formula require more.*	`IsHolOfWeight.integral_mul_conj_holMinorsPi`
substantial proof-specific	The measure Shape.besselHaar on U, viewed on G_1, equals the convolution $\eta_0 * \cdots * \eta_{\varrho-1}$ of the measures of the floors.	header of ComplexPlace: A. Knapp, Lie Groups Beyond an Introduction; S. Helgason, Groups and Geometric Analysis	`ascConv_heisenbergMeasure_eq_besselHaar`
substantial proof-specific	For each j there is a measurable chart g from (ℂ)^j to G_1, equal to the Weyl element w_j(p) (with the boost on the top floor beyond ϱ), such that integration against the ascending convolution of the cell laws equals integration over (ℂ)^j against the product of the cell densities.		`exists_chart_ascConv_cellLaw`
substantial	Independence of the splitting: for a parameter l = 2λ of some gap, g in U(diag ε) and a block diagonal matrix a such that $a^{-1}g$ has positive definite hermitian part, ψ_λ(g) is the normalised integral HCParam.flenstedJensen at (a, $a^{-1}g$).	M. Flensted-Jensen, Discrete series for semisimple symmetric spaces, Ann. of Math. 111 (1980)	`dsCoeff_eq_flenstedJensen`
substantial	For l = 2λ of gap h, a natural number c ≤ h and g in U(diag ε): |ψ_λ(g)|²·|det a_g|^c ≤ 1, where a_g is the block diagonal part of g.	M. Flensted-Jensen (1980); B. Sun, Matrix coefficients of cohomologically induced representations, Compositio Math. 143 (2007), Thm 1.1 (header of the family)	`norm_dsCoeff_sq_mul_det_pow_le_one`

class	what it says	reference; relation	formal name
named	For matching parameters l, l' (IsMatch), any two polynomials P, Q satisfying FockDualPair.IsPreJointHarmonic are proportional: P = cQ for some complex number c.	R. Howe, Transcending classical invariant theory, JAMS 2 (1989), §2 on the Fock model and §3, Lemma 3.3, (3.15) and the irreducibility and mutual-commutant assertions following it, pp. 543–544; A. Paul and P. Trapa, One-dimensional representations of U(p,q) and the Howe correspondence, §3, Theorem 3.1 and Lemma 3.2, compact type correspondence for unitary pairs *second reading: Howe's multiplicity-free compact-pair correspondence implies proportionality for this one matched row type.*	`IsMatch.unique`
named	Harish-Chandra: for λ of gap $500n^3+10n^5$ and signature (p,q), deg_λ = $\prod_{\lambda_j<\lambda_i}(\lambda_i-\lambda_j)/\prod_{i<n}$ i!; for x in the torus with distinct entries, $(-1)^{pq}$ $\Sigma_{\sigma\in S_n}$ $\int_G$ e_λ($g^{-1}(x\circ\sigma)g$) dg = p!q!·conj χ(x), χ the character of U(n) of highest weight λ−ρ.	Harish-Chandra, Discrete series for semisimple Lie groups I, Acta Mathematica 113 (1965), §18, Theorem 3, pp. 288–291; Harish-Chandra, Discrete series for semisimple Lie groups II, Acta Mathematica 116 (1966), §41, Theorem 16, pp. 96–100; P. Hochs and H. Wang, Orbital integrals and K-theory classes, Annals of K-Theory 4 (2019), proof of Corollary 4.10 and §§6A–6D, especially Theorem 6.4 *second reading: Harish–Chandra's character and degree formulas and the stable orbital identity are located. The exact Haar and coefficient comparison for the compact statement is open.*	`Orbital.harish_chandra_degree_and_orbital`

class	what it says	reference; relation	formal name
named	Holomorphic discrete series: order o with signs +1 first, w K-dominant, w_i+γ ≤ w_j for ε_i = 1, ε_j = −1, γ > 500n³+A+n+1, λ_i = w_i + (n+1)/2 − i with i the rank in o. Then λ has gap 500n³+A and ψ_λ = holMinors ε w on U(diag ε).	A. Korányi, A simplified approach to the holomorphic discrete series, §§3–4 on the representation and kernel, and §6, (6.4) and the comparison with Harish–Chandra's condition; Harish-Chandra, Representations of semisimple Lie groups V, American Journal of Mathematics 78 (1956), §§4–5 and §9 on holomorphic function spaces and unitarity. The exact compact minors identity remains unlocated. *second reading: The published holomorphic-chamber criterion is located. The equality of the two explicit normalized functions remains unlocated.*	`dsCoeff_eq_holMinors`
named	Let F ≠ 0 be rapidly decreasing, automorphic of level ς = −ς̄, Im Φ_wς ∈ (0,c], c ≥ 1, holomorphic of weight in [−B,B], B ≥ 2, gap 1. Some z has F(z) ≠ 0, |F| ≤ 2|F(z)|, and |F(g)| ≥ |F(z)| implies Space.firstMinimum(g) ≥ (Bc)^{−½}.	C. Poor and D. Yuen, Linear dependence among Siegel modular forms, Mathematische Annalen 318 (2000), §1, Theorems 1.2–1.3 and the discussion following them *second reading: Poor–Yuen's semihull control at a Siegel maximum is an analogue. The unitary near-maximum estimate and its constant remain unlocated.*	`exists_near_max`

class	what it says	reference; relation	formal name
substantial	Let l have some gap, IsUnitaryCasimirGe hold, x be K-highest of weight ν, |ν_a| ≤ L. For y in M_s (GlModule.filtration) and any matrix X: Xy ∈ M_{s+1} and ‖Xy‖ ≤ (n+1)⁴(L+1+s)‖X‖‖y‖, with ‖X‖ the operator norm.	method of the growth of the compact Casimir along the filtration; header sources: R. Parthasarathy, Represent. Theory 19 (2015), Thm 1.1; Hotta–Parthasarathy (1974); Schmid (1975); Vogan–Zuckerman (1984), Prop. 6.1; Nelson, Analytic vectors (1959)	`HCParam.IsKHighest.norm_apply_le_of_mem_filtration`
named	If the parameters l, l' match for T (IsMatch), then there is a joint harmonic polynomial Φ in the sense of IsJointHarmonic: nonzero and highest for both lowest K-types Λ and Λ'.	R. Howe, Transcending classical invariant theory, JAMS 2 (1989), §3, Lemmas 3.2–3.3, homogeneous degree and irreducibility of the joint harmonic summand; A. Paul and P. Trapa, One-dimensional representations of U(p,q) and the Howe correspondence, §3, Lemma 3.2(1)–(2), especially formulas (3.3), (3.4) and (3.6), pp. 8–9 in the authors' copy *second reading: The unitary compact-type correspondence and its minimal-degree formula give the strong harmonic highest vector for the matched parameters.*	`IsMatch.exists_isJointHarmonic`

class	what it says	reference; relation	formal name

GLn

class	what it says	reference; relation	formal name
substantial	Tensors on complementary blocks lie in the Cartan component: the mixed Cartan embedding applied to the Cartan product of y and η equals the tensor S_kp(incl_N)y $\otimes$ S_km(incl_m)η of their images under the block inclusions.		mixedCartan_mulVec_cartanProduct
substantial	Analogue of the standard basis theorem (independence half): over any commutative ring R the authors' minor products b_i (built one row at a time, indexed by chosen standard fillings) are linearly independent.	De Concini, Eisenbud, Procesi, Young diagrams and determinantal varieties, Invent. Math. 56 (1980); Akin, Buchsbaum, Weyman, Schur functors and Schur complexes, Adv. Math. 44 (1982) place not confirmed	linearIndependent_nablaBasis
substantial	Over $\mathbb{C}$, $\xi \mapsto \Sigma$_i (Gξ)_i b_i maps onto the module of Borel semi-invariant polynomials of the weight given by $\varkappa$ (the Schur module realized in polynomials on matrices).		range_toPoly
substantial	Contravariant form versus Fischer product: the integer Gram matrix G times the matrix of Fock pairings $\langle$b_i,b_j$\rangle$ of the basis minor products equals h_$\varkappa\cdot$1, h_$\varkappa$ the (positive) hook number.	Header of Lib/Adelic/-CuspForms/DeltaForms: Jantzen 2003; Goodman–Wallach 2009, ch. 5; Howe 1995 place not confirmed	weylGram_mul_fockPairing
substantial	Submultiplicativity of the Gram norm under the Cartan product: if the polynomial attached to ζ (weight $\varkappa+\varkappa'$) is the product of those attached to ξ (weight $\varkappa$) and η (weight $\varkappa'$), then $\|\zeta\|^2 \le \|\xi\|^2\|\eta\|^2$.		weylNormSq_le_mul_of_toPoly_eq
substantial	Duality of Weyl lattices: for the complementary weight $\varkappa'_\tau(h)=\varkappa_\tau(\lvert n_\tau\rvert-h)$ there is a unimodular integer matrix J with $J^{\mathsf T}G_\varkappa J = G_{\varkappa'}$ (Gram matrices), $J\cdot ev_e = ev_{e'}$ and $S_\varkappa(g^{-T})J = \prod_\tau \det(g_\tau)^{-c_\tau}\cdot J\cdot S_{\varkappa'}(g)$, $c_\tau = \Sigma_{1\le i<\lvert n_\tau\rvert}\,\varkappa_\tau(i)$.	Fulton, Harris, Representation Theory (1991); Grinberg, An Introduction to Algebraic Combinatorics (Jacobi's complementary minors) §15.5 (FH) and Thm 6.4.48 (Grinberg), as cited in the file header; authors' check: compared	exists_dual_isometry

class	what it says	reference; relation	formal name
named	Schur's lemma for the Schur (Weyl) modules under the unitary group: a complex matrix commuting with S_ϰ(g) for every tuple g of unitary matrices g_τ ∈ U(n_τ) is scalar, i.e. ∏_τ U(n_τ) acts absolutely irreducibly on the module of weight ϰ.	Goodman and Wallach, Symmetry, Representations, and Invariants, §4.1.2, Lemma 4.1.4 and Corollary 4.1.5; Akin, Buchsbaum and Weyman, Schur functors and Schur complexes, Theorem II.2.16, p. 232, the standard-tableau basis *second reading: The scalar-commutant conclusion is Schur's lemma once irreducibility of the specified action has been proved.*	`scalar_of_commute_schur`
substantial	For weights (ϰ,e) and (ϰ',e') with Harish-Chandra parameters l and −l there is an invariant integer bilinear pairing A between the two coordinate spaces with: F_η(x,g) = A(η(g), ρ(x)v⁺) for algebraic forms η of the second weight; C·A = h_ϰ'·1 for an integer matrix C (h the hook number); each A(·,v) = ⟨v',·⟩ with ‖v'‖ = ‖v‖.		`exists_dual_pairing_of_isHCParam`

GaloisRep

class	what it says	reference; relation	formal name
named	Tame local reciprocity on units: if s has tame exponent a at A over w and s is the local Artin image of a unit u, then T^a(x_χ) = χ(ū) for every character χ of k_w^× (x_χ the element naming χ).	J.-P. Serre, Local Fields, XIV §2, Proposition 6, p. 208, and §3, Proposition 8 and finite-field corollary, pp. 210–211. The Kummer ratio computes the prime-to-p norm residue symbol, with arithmetic reciprocity.; Pinned tame exponent, character evaluation and use of localArtin_tame_symbol. *second reading: Prime-to-p norm residue reciprocity expressed in the project's faithful-character coordinates*	`IsTameExponent.pow_apply_charName`
named	Clifford's theorem: over an algebraically closed field of characteristic 0, if r: G → GL_n(k) is semisimple (non-degenerate trace form on its span) and f: H → G has normal image, then r∘f is semisimple.	A. H. Clifford, Representations induced in an invariant subgroup, Ann. of Math. 38 (1937), §1, Theorem 1, pp. 534–535. Clifford uses "induced" here for restriction to H.; Pinned trace-form definition. *second reading: The complete-reducibility part of Clifford's theorem, recast in characteristic-zero trace form*	`IsTraceNondeg.comp`

class	what it says	reference; relation	formal name
named	Brauer–Nesbitt theorem: over an algebraically closed field of characteristic 0, if r is semisimple, ρ: G → B has semiprime span in a finite-dimensional algebra B and φ: B → M_n(k) is an algebra map with tr r = tr φρ, then r(g)P = Pφ(ρ(g)) for some invertible P.	R. Brauer and C. Nesbitt, On the modular characters of groups, Ann. of Math. 42 (1941), Part I, §§1, 3 and 6. The published setup starts with finite groups and studies ordinary and modular characters.; P. Etingof et al., Introduction to representation theory, §2.6, Theorem 2.17(i), p. 27, and the characteristic-zero multiplicity argument following Theorem 2.18, p. 28. Distinct irreducible characters of an associative algebra are linearly independent.; Pinned semiprime-span criterion and Wedderburn decomposition. *second reading: The characteristic-zero character criterion for semisimple representations after translating the two algebraic hypotheses*	`IsTraceNondeg.exists_conj_of_trace_eq`
substantial	Type implies label (Frobenius reciprocity): for a label c of degree |ρ| with roots scaled by x_ψ, distinct χ_l with c'(x_{χ_l}) ≠ 0, and F not orthogonal to Green's character Ξ_c, a form with that parahoric type has the product label c'·∏(X − x_{χ_l})^{t_l} (Green's character times χ_l∘det on the blocks).		`HasParahoricType.hasProductLabel`

class	what it says	reference; relation	formal name
named	Local-global compatibility on a parahoric eigenspace at split w ∤ ℓ: for lifts s of Art(ϖ) and τ of Art(u) and a nonzero y in the eigenspace with values d_i, in the Galois representation of x the elements s and τ commute, polynomials vanishing at all ι(d_i) kill τ, and $\prod$(X − ι(ϑ_i(ϖ)d_i^b)) kills sτ^b (b ≥ 0).	R. Taylor and T. Yoshida, Compatibility of local and global Langlands correspondences, Theorem A, J. Amer. Math. Soc. 20 (2007), pp. 467–468, author preprint p. 1. The local representation is Frobenius-semisimplified and carries the paper's dual and |det| twist.; M. Harris and R. Taylor, On the geometry and cohomology of some simple Shimura varieties, introduction, Theorem C, author preprint pp. 6–7. The stated comparison in the Grothendieck group has a nonzero integer multiplicity a(Π), at places away from ℓ, for a cohomological conjugate-self-dual cuspidal Π square-integrable somewhere.; T. Barnet-Lamb, T. Gee, D. Geraghty and R. Taylor, Potential automorphy and change of weight, Ann. of Math. 179 (2014), Theorem 2.1.1, p. 538, especially clauses (2)–(4).; Pinned ambient hypotheses and both complete parahoric statements. *second reading: The published away-from-ℓ comparison needs a local type and exponent calculation to recover these simultaneous matrix identities*	`galRep_of_parahoric_type`

class	what it says	reference; relation	formal name
named	Iwahori-level compatibility of U-operators and Frobenius: for y pro-p-Iwahori fixed at w ∤ ℓ on which the line stabiliser acts by χ of the top-left entry (χ(u) ≠ 0, χ(u)^{n!} ≠ 1), tr((τ−1)M)·G(U)y = tr(G(s)(τ−1)M)·y for every polynomial G and matrix M, U the Hecke operator of diag(ϖ,1,…,1).	R. Taylor and T. Yoshida, Compatibility of local and global Langlands correspondences, Theorem A, J. Amer. Math. Soc. 20 (2007), pp. 467–468, author preprint p. 1. The local representation is Frobenius-semisimplified and carries the paper's dual and |det| twist.; M. Harris and R. Taylor, On the geometry and cohomology of some simple Shimura varieties, introduction, Theorem C, author preprint pp. 6–7. The stated comparison in the Grothendieck group has a nonzero integer multiplicity a(Π), at places away from ℓ, for a cohomological conjugate-self-dual cuspidal Π square-integrable somewhere.; Pinned ambient hypotheses and both complete parahoric statements. *second reading: A matrix-weighted identity for a specific Hecke operator requires additional parahoric linear algebra*	`hecke_eq_frob_on_parahoricEigenspace`

class	what it says	reference; relation	formal name
named	Local-global compatibility at tame level: if r is attached to an eigenform f on the definite U(n) (standard space, Weil data) with product label c·∏(X−x_{χ_l})^{t_l} at a split place w, then above w r has finite type, Hodge–Tate jumps of the weight ϰ, and these polynomials kill the inertia/Weil elements with tame exponents (with det = product of roots away from ℓ).	T. Barnet-Lamb, T. Gee, D. Geraghty and R. Taylor, Local-global compatibility for ℓ=p, II, Ann. Sci. ENS 47 (2014), §1, Theorem 1.1 and Corollary 1.2, p. 170. The ordering gives equal semisimplifications and bounds monodromy. Full Frobenius-semisimplified equality is stated at Shin-regular weight.; R. Taylor and T. Yoshida, Compatibility of local and global Langlands correspondences, Theorem A, J. Amer. Math. Soc. 20 (2007), pp. 467–468, author preprint p. 1. The local representation is Frobenius-semisimplified and carries the paper's dual and |det| twist.; Pinned definition of HasProductLabel, including separability and the block ranks.; IsTameExponent requires s in inertia and IsLabelledAt quantifies through this predicate.; Pinned definitions of good type, finite type, jumps and PolyKillsAt in the two residue-characteristic cases. *second reading: The books and papers give parameter compatibility; the compact squarefree-label predicate needs a separate identification*	`Attached.isLabelledAt`

class	what it says	reference; relation	formal name
named	ℓ-adic avatar of an algebraic Hecke character and its local behaviour: for a stably unramified CM field E, the avatar μ of an algebraic Hecke character ϑ has at every place A over every finite w finite type, Hodge–Tate jumps −n(ι⁻¹τ), and each s of degree m ≤ 1 is Art(x) with ord x = m and X − ιϑ_w(x) killing s.	J.-P. Serre, Abelian ℓ-adic Representations and Elliptic Curves, III §1.2, pp. III-5–III-7, theorem on p. III-7: local algebraicity is equivalent to Hodge–Tate type together with semisimplicity on inertia for abelian representations.; B. Conrad, Lifting global representations with local properties, Appendix B, Definition B.1, p. 32, Proposition B.3, p. 33, and Proposition B.4(i)–(ii), p. 35 of the author text. The Artin map in that appendix has arithmetic normalization.; J.-P. Serre, Local Fields, XIV §2, Proposition 6, p. 208, and §3, Proposition 8 and finite-field corollary, pp. 210–211. The Kummer ratio computes the prime-to-p norm residue symbol, with arithmetic reciprocity.; Pinned predicate AvatarCompatAt and the Frobenius convention.; Pinned definitions of good type, finite type, jumps and PolyKillsAt in the two residue-characteristic cases. *second reading: Rank-one local character theory with CM and root-of-unity restrictions, a convention change, and two Weil degrees*	`avatarCompatAt_of_stablyUnram`

class	what it says	reference; relation	formal name
named	Galois representations attached to forms on definite unitary groups: for a CM field E with Weil data, every eigenform f on the definite U(n) of the standard form has an attached ℓ-adic representation; if f is fixed by the maximal compact at w, each attached one is unramified ($w \nmid \ell$) or crystalline ($w \mid \ell$) at w with the Hodge–Tate jumps of f's weight, and every Hecke polynomial at w kills geometric Frobenius.	G. Chenevier and M. Harris, Construction of automorphic Galois representations, II, Camb. J. Math. 1 (2013), General Hypotheses 1.1, p. 57, Theorem 3.2.3(a′)–(d), pp. 66–67. At a spherical place above ℓ the theorem identifies the characteristic polynomial of crystalline Frobenius.; T. Barnet-Lamb, T. Gee, D. Geraghty and R. Taylor, Potential automorphy and change of weight, Ann. of Math. 179 (2014), Theorem 2.1.1, p. 538, especially clauses (2)–(4).; J.-P. Labesse, Changement de base CM et séries discrètes, Corollaire 5.3, pp. 43–44 of the author's text. Weak base change gives an isobaric sum and local matching at unramified and split places.; Pinned definitions of Attached and SphericalCompat.; Pinned definitions of good type, finite type, jumps and PolyKillsAt in the two residue-characteristic cases. *second reading: Standard definite-unitary specialization of automorphic attachment, with polynomial annihilation at spherical places*	`sphericalCompat_std`

General

class	what it says	reference; relation	formal name
substantial	O a DVR, p != 0, p Phi(O^n) in N in R^iota, V a module of functions G -> O/p. Some K >= N contains Phi(v) whenever every linear Psi: O^n -> O/p with Psi o z in V kills v, and some c in O with v(c) <= length(formsMod / V) satisfies c Fitt(K) in Fitt(N).		`fittingIdeal. exists_cost_le_ relLength`
substantial	For l with a gap, u supported on pairs (i, j) with l_i < l_j and ε_i = ε_j, X and (1 − u)X accretive: Flensted-Jensen's normalised integral satisfies FJ_λ(1, ((1 − u)X)*) = conj FJ_λ(1, X).		`flenstedJensen_ shift_eq_conj`
named	Schur orthogonality for discrete series: if l = 2λ has gap ≥ 500n³ + A, then e(g) = d_λψ_λ(p⁻¹gp) on U(J) is continuous, integrable, e(g⁻¹) = conj e(g), e(1) ≠ 0 and ∫ e(z)e((zx)⁻¹y) dz = e(x⁻¹)e(y)/e(1).	Harish-Chandra, Discrete series for semisimple Lie groups II, pp. 87–88, definition and square-integrable orthogonality *second reading: Formal-degree orthogonality accounts for the integral identity. The explicit integrability threshold and normalization require separate work.*	`isIdempotentKernel_ psiInFrame`
substantial	For T hermitian, the measures δ_Γ (images of dt dw dν(a) under the flat charts) form a disintegration of the volume of A² ⊕ A^ι along v ↦ h(v, v), for the form hyperbolic plane ⊕ T (Leray measures on the quadrics).		`Place. isDisintegration_ quadricMeasure`
named	Amitsur's formula, integrality consequence: if ρ: Λ → M_n(A) is a B-algebra map and Λ is B-spanned by elements s with charpoly ρ(s) having coefficients in (the image of) B, then every ρ(w) has this property.	Chenevier, The p-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings, §1, equation (1.5), Amitsur's formula *second reading: Coefficient descent on a spanning family follows from the integral Amitsur formula.*	`amitsur_charpoly_ mem_of_span`

class	what it says	reference; relation	formal name
named	Hua's integral: for M > 0, S invertible k×k, c ≥ 0, r ≥ k, y_i = x_iS + a_i: ∫ c|det(M + Σ_{i∈s} y_i*y_i)|^{−(|s|+r)} over rows x_i ∈ ℂ^k (i ∈ s) is at most c(π^k/(k!|det S|²))^{|s|}|det M|^{−r}.	NIST Digital Library of Mathematical Functions, §5.12, equations 5.12.1 and 5.12.3; Hua, Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains, bibliographic reference *second reading: The stated upper bound follows from an exact iterated beta integral with translations and a linear change in each row.*	hua_integral_le
named	Ostrowski-Taussky inequality, in the form used: if H is positive semidefinite and Theta + Theta* - H is positive semidefinite (n x n complex), then |det H| <= 2^n |det Theta|.	Ostrowski and Taussky, On the variation of the determinant of a positive definite matrix, Indagationes Mathematicae 13 (1951), 383–385. The inequality is identified and restated in the introduction on p. 331 of this paper *second reading: Equivalent to the determinant inequality for a matrix with positive Hermitian part, including its semidefinite endpoint.*	ostrowski_taussky
named	Dedekind's different theorem (ramified primes divide the different), in the form: a prime P of O_L whose square contains a rational prime p divides the different of L over Q.	Keith Conrad, The different ideal, Theorem 4.13 and its proof *second reading: A divisibility consequence of the lower bound on the exponent of the different at a prime.*	dvd_differentIdeal_of_mem_sq
substantial	R in A an adelic pair, J hermitian unimodular, F integral, n primitive with F^H J n = 0 and complementary sizes, N = Gram matrix of (F|p) for J plus c on the p-block. If F is left invertible or [det N] != 0: n^perp lies in the span of F and [det n^H(J + c^-1 J p p^H J) n] i^2 [c]^|rho| = [det N], i = [{y : Fy integral} : R^sigma].		Filling.jacobi_perp_discriminant

class	what it says	reference; relation	formal name
named	Product formula (finite part): for x in a number field E, the generalised index of x in the finite adeles of E equals |N_{E/Q}(x)| (0 for x = 0).	Milne, Algebraic Number Theory, Chapter 7, Theorem 7.15, product formula *second reading: The number-field product formula evaluated on a principal finite adèle, together with the convention at zero.*	genIndex_algebraMap
named	Auerbach's lemma: if linear forms ℓ_a (a in a compact space) on a finite-dimensional K-space have no common zero but 0, and a ↦ φ(ℓ_a(v)) is continuous (φ: K → L normed), some ℓ_{S(k)} are the dual coordinates of a basis z with |φ(ℓ_a(z_k))| ≤ 1 for all a, k.	Schlumprecht, Course notes for Functional Analysis I, Theorem 1.5.1, the classical unit-basis statement and proof *second reading: A determinant-maximization extension of Auerbach's lemma with a prescribed compact separating family.*	exists_auerbach
named	Steinitz's theorem: for injective ring maps f: K → ℂ and g: K → L from a nonzero countable commutative ring K, with L algebraically closed of the cardinality of ℂ, there is a field isomorphism σ: ℂ → L with σ∘f = g.	Stacks Project, Fields, Lemma 9.26.3, existence and cardinality of transcendence bases; Stacks Project, Fields, Lemma 9.10.6, uniqueness over the base of the algebraic closure *second reading: Relative classification of algebraically closed fields over a countable common subring.*	exists_ringEquiv_extend
named	Uniqueness of Fourier coefficients on a compact abelian quotient B/B0 whose characters are a 'dual family' chi_i: a continuous B0-periodic F vanishes if its coefficients mean(chi_i F) vanish for every i with chi_i(a) = c whenever c F(x+a) = F(x) for all x.	Rudin, Fourier Analysis on Groups, §1.3, Theorem 1.3.6 (uniqueness), p. 17 *second reading: Compact-quotient Fourier uniqueness with an explicit reduction to the compatible characters.*	IsDualFamily.eq_zero_of_coeff
named	Schwarz lemma, Lipschitz form: if f from a finite-dimensional complex normed space to a complex normed space is holomorphic with |f| <= B on the open ball of radius r about c, then |f(z) - f(c)| <= (2B/r)|z - c| on that ball.	Rudin, Real and Complex Analysis, §12.2, Schwarz's lemma *second reading: A centre-to-point bound derived by restricting to a complex line and applying the scalar Schwarz lemma.*	norm_sub_le_of_holomorphic_bdd

class	what it says	reference; relation	formal name
substantial	R: G -> GL_n(Qbar_l) with an O-lattice model u: G -> M_n(O) (O integers of a finite extension of Q_l). If R(gamma)^q = R(phi^-1 gamma phi), R(gamma^{q^n}) = R(gamma) and R(gamma) has an eigenvalue zeta in O with zeta^{q^i} (i < n) pairwise distinct mod the maximal ideal, then u(G) spans M_n(O) over O and u(phi^-n) is scalar.		`IsLatticeIn.span_eq_top_of_tame_pair`
named	Functional equation of Dirichlet L-functions (consequence): if $L(s, \chi) = 0$ with $\mathrm{Re}\, s \neq 0$ and $s \neq 1$, then s is a pole of the gamma factor of χ (where Mathlib sets it to 0) or $L(1 - s, \chi^{-1}) = 0$.	Montgomery and Vaughan, Multiplicative Number Theory I, Chapter 10, Corollaries 10.3 and 10.9 *second reading: A zero-reflection corollary for arbitrary Dirichlet characters with an explicit gamma-factor exception.*	`LFunction_one_sub_eq_zero`

LocalGL

class	what it says	reference; relation	formal name
named	Green's orthogonality: for any finite set s of tuples (c_w) of polynomials over the residue fields k_w, Σ_{c∈s} ⊗_w e_{c_w} is a central self-adjoint idempotent of ℂ[∏_w GL_n(k_w)].	J. A. Green, The characters of the finite general linear groups, Trans. Amer. Math. Soc. 80 (1955), §8, Theorems 13–14, pp. 441–444, especially Theorem 14 and the degree formula on pp. 443–444. The parameters assign a partition to each Frobenius orbit of multiplicative characters.; Pinned squarefree labels, character signs and the zero branch of labelIdem.; Pinned definition of IsCentralProj as two conditions on the group-algebra element and its coefficients. *second reading: Finite sums of pairwise orthogonal central idempotents for the restricted Green labels*	`isCentralProj_sum_tupleIdem`

class	what it says	reference; relation	formal name
named	Green's irreducibility: for each tuple (c_w), ⊗_w e_{c_w} is a central self-adjoint idempotent of ℂ[∏_w GL_n(k_w)] (equivalently ±Ξ_{c_w} irreducible, or e_{c_w} = 0 when c_w is not a label).	J. A. Green, The characters of the finite general linear groups, Trans. Amer. Math. Soc. 80 (1955), §8, Theorems 13–14, pp. 441–444, especially Theorem 14 and the degree formula on pp. 443–444. The parameters assign a partition to each Frobenius orbit of multiplicative characters.; I. G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., IV §6, pp. 284–286, including (6.5) and the hook-length computation (6.6).; Pinned squarefree labels, character signs and the zero branch of labelIdem.; Pinned definition of IsCentralProj as two conditions on the group-algebra element and its coefficients. *second reading: Central idempotence for squarefree character labels, extended by zero and tensor products*	`isCentralProj_tupleIdem`
substantial	If x_χ is not a root of c and l ≠ 0, then e_c annihilates Σ_h χ(det h)$^{-1}$·h over the invertible h = 1 + u·l^{T}: the Ξ_c-isotypic part has no vector on which that group acts by χ∘det.		`reflectionSum_mul_labelIdem`

class	what it says	reference; relation	formal name
named	Taylor–Wiles freeness at auxiliary primes as Higman's criterion up to c: for a finite group Δ embedded in $\prod$_j GL_n(k_{w_j}), some lattice-preserving operator Ψ on forms of principal level has Σ_{g∈Δ} $g^{-1}\Psi g = c$, c a common multiple of the stabiliser orders.	L. Clozel, M. Harris and R. Taylor, Automorphy for some ℓ-adic lifts of automorphic mod ℓ Galois representations, Publ. Math. IHÉS 108 (2008), §3.3, Lemma 3.3.1, p. 99. Lemma 3.1.3, p. 84, concerns unramified GL_n-representations.; R. Taylor and A. Wiles, Ring-theoretic properties of certain Hecke algebras, Theorem 2 and Proposition 1, author copy pp. 5, 7. The former is freeness of the Hecke algebra over the auxiliary group ring and the latter treats localized cohomology.; M. Broué, Higman criterion revisited, p. 1, classical criterion stated in the abstract. Relative trace equal to the identity characterizes projectivity over a field.; Pinned BaseLevel hypotheses and the divisor c. *second reading: Classical freeness assumes trivial stabilizers; the declaration constructs a trace equal to a stabilizer multiple*	`BaseLevel.exists_higman_operator`
substantial	The Shintani lift is multiplicative for parabolic induction: if F lifts to F' and G to G' (a common Z-virtual character of GL_m(k')⋊Gal(k'/k) descends to both, for every m), then F*G lifts to F'*G'.		`Lift.mul`

class	what it says	reference; relation	formal name
substantial	Chain lemma: for a nonzero spherical Hecke eigenvector x with Hecke polynomial P in a smooth unitary representation of GL_N(E_w): a root c of P with P(qc) ≠ 0 has |c|² ≥ q^{N−2}; and if P(dX) contains the string (X−1)⋯(X−q^k) times R, then x ⊥ (U − q^k)R(U)y (in the variable dX) for every pro-p-Iwahori-fixed y, U the operator of diag(ϖ,1,…,1).	Tadić, Ann. ENS 19 (1986); Bernstein, LNM 1041 (1984) place not confirmed	chain_lemma
substantial	Strings of Satake roots: for x as above, if P(c) = 0 and |c|²q^k = q^N then c, cq, …, cq^{k−1} are all roots of P.	Tadić, Ann. ENS 19 (1986); Bernstein, LNM 1041 (1984) place not confirmed	isRoot_string_of_norm
named	Degree of Green's characters (lower bound): for a label c of degree r = |n| over F_q, Re Ξ_c(1) ≥ 0 and the coefficient at 1 of e_c, Ξ_c(1)²/|GL_r(F_q)|, has real part ≥ (1 − r/q)·q^{−r}.	J. A. Green, The characters of the finite general linear groups, Trans. Amer. Math. Soc. 80 (1955), §8, Theorems 13–14, pp. 441–444, especially Theorem 14 and the degree formula on pp. 443–444. The parameters assign a partition to each Frobenius orbit of multiplicative characters.; I. G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., IV §6, pp. 284–286, including (6.5) and the hook-length computation (6.6).; Pinned squarefree labels, character signs and the zero branch of labelIdem. *second reading: A quantitative lower bound obtained from Green's exact degree*	labelChar_degree_bounds
substantial	Over a finite field k, a cuspidal class function F on endomorphisms (all block averages over block-triangular endomorphisms with two nonzero blocks vanish) is zero at the class of g unless the characteristic polynomial of g is a power of an irreducible polynomial.		Hall.IsCuspidal.exists_charpoly_eq_pow

class	what it says	reference; relation	formal name
named	Comparison of simple trace formulas: for E ⊇ μ_5040 with an even number of infinite places, a level K, h such that every point of hK has irreducible characteristic polynomial at some finite place, B ≥ 16[E:ℚ](n+1)³ and gap H ≥ 500n³+10n⁵, the trace Σ_i⟨b_i, h·b_i⟩ over an orthonormal basis of K-fixed forms of type e_l (e_l a cusp kernel of order B, l of gap H) is the same for all signatures s and hermitian spaces V.	J.-P. Labesse, Changement de base CM et séries discrètes, Théorème 5.1 (trace identity, p. 41) and §5.2, Théorème 5.4 with proof (pp. 45–46 of author copy). The latter gives cuspidal weak base change and conditional multiplicity one.; J. Arthur and L. Clozel, Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula, I §2, Lemmas 2.4–2.6, pp. 13–15. Ordinary and twisted simple trace formulas and compactness of a kernel under the stated supercuspidal/regular elliptic test-function conditions.; The ambient bounds and the quantification of a single trace over signatures and spaces. *second reading: Labesse and Arthur–Clozel supply trace-formula inputs with different test functions and transfer statements*	`Space.exists_trace_eq`

class	what it says	reference; relation	formal name

PadicComparison

class	what it says	reference; relation	formal name
substantial	W a DVR with uniformiser π, X a derived π-complete W-module of finite rank, W derived π-complete, S a derived a-complete W-algebra (a the image of π): then S ⊗_W X is derived a-complete.	Stacks Project (tags 091N, 091P, 0G1U, 09BA, 0CQY); B. Bhatt, Torsion completions are bounded, JPAA (2019) place not confirmed	`IsDerivedComplete.baseChange`
named	Transfer for a finite Galois covering: if c: W → Y is a covering of proper R-schemes with finite group Δ, then c^*: V^n(Y) → V^n(W) on rational ℓ-adic cohomology over Ω = ℚ is injective and its image contains every Δ-invariant class.	Grothendieck, Sur quelques points d'algèbre homologique, Chapter V, §§5.2–5.3 *second reading: related descent formalism located; comparison with this cover remains open*	`ellAdicCohom.transfer`

PadicHodge

class	what it says	reference; relation	formal name
named	Artin–Schreier–Witt over the tilt: for M = W(C^♭) ⊗ Z/p^m(1) and S = W(𝔪^♭)·(1 ⊗ 1): p^m kills M, φ − 1 is onto M, φd − d ∈ S ⇒ d ∈ Z/p^m(1) + S, S ∩ Z/p^m(1) = 0; and L(hv − v) ∈ S for h near 1 when ρ is trivial mod p^m near 1.	L. Herr, Sur la cohomologie galoisienne des corps p-adiques, Bulletin de la Société Mathématique de France 126 (1998), §2, Théorème 2.1, p. 572; L. Illusie, Complexe de de Rham-Witt, Astérisque 63 (1979), §2.7, Artin–Schreier–Witt sequence (2.7.1) *second reading: The Artin–Schreier–Witt sequence explains Frobenius surjectivity. The complete small-ideal and continuity package is unlocated in Herr.*	`Lift.TateTwist.smallPart_spec`
substantial	For O a domain and L potentially crystalline: modulo forms of type u − 1, the forms of type u at the letter i are spanned over $\bar{\mathbb{Q}}$_p by at most Σ_{γ(ζ) = ϑ(ω(i))} jump_γ(u) of them.	L. Berger, Limites de représentations cristallines, Compositio Math. 140 (2004), §III; N. Wach, Bull. SMF 124 (1996) place not confirmed	`Lift.KummerChart.JumpLattice.exists_span_formsOfType_mod_pred`

class	what it says	reference; relation	formal name
substantial	If j(Γ) is open in G_p there is a Kummer chart over Λ = W(F_{p^f}) with ef = [G_p : j(Γ)], j(Γ) fixing Λ and ϑ(z), containing the fixer of ϑ(S), a p^f-Frobenius and elements with gz = [ε]^l z (l ≥ 0); e = 1, E = z + p if j(Γ) contains inertia.	M. Kisin, Crystalline representations and F-crystals, Progr. Math. 253 (2006), §1.1 place not confirmed	`Lift.exists_kummer_chart`
named	Fontaine's formalism, stability under subobjects: if finitely many jointly injective Z_p-linear maps f_i: T' → S_i are equivariant along F (F coarser than 'near 1') and each S_i is admissible along F, then T' is admissible along F.	J.-M. Fontaine, Représentations p-adiques semi-stables, Astérisque 223, Exposé III (1994), §§1.5.1–1.5.3, Proposition 1.5.2, pp. 124–125; O. Brinon and B. Conrad, CMI Summer School Notes on p-adic Hodge Theory, Theorem 5.2.1, the finite-dimensional B-admissible category *second reading: Fontaine proves subobject stability in the finite-dimensional representation category. The compact filter and module generality remains unlocated.*	`IsAdmissible.of_jointly_injective`
named	Fontaine's formalism, duals: if T (finite free over R) is admissible along F and T" ≅ Hom_R(T, R(c)) equivariantly, then T" is admissible along F; and on inertia, a polynomial killing (s, 0) on T kills (s^{-1}, 0) on T".	J.-M. Fontaine, Représentations p-adiques semi-stables, Exposé III, §1.5, Proposition 1.5.2; P. Colmez, Théorie d'Iwasawa des représentations de de Rham d'un corps local, §III.4, formula for D_cris(V(k)); J.-M. Fontaine, Représentations ℓ-adiques potentiellement semi-stables, Exposé VIII, §2.3.7, Weil action *second reading: The classical dual and Tate-twist rules explain the field-valued case. The complete filtered-ring assertion remains unlocated.*	`IsAdmissible.of_pairing`

class	what it says	reference; relation	formal name
named	Weil–Deligne Frobenius of a potentially crystalline representation: for a Weil pair (j(s), m), the chosen B_max-linear β_{s,m} agrees with φ^m j(s) ⊗ ρ(s) on every element of D_pcris(T).	J.-M. Fontaine, Représentations ℓ-adiques potentiellement semi-stables, Astérisque 223, Exposé VIII (1994), §§2.3.2–2.3.3 and §2.3.7, especially pp. 339–340 *second reading: Fontaine gives the signed Weil action on classical periods. The stated B_max-linear extension on arbitrary modules remains unlocated.*	`linearFrob_apply`
named	Potentially crystalline representations are Hodge–Tate: for R algebraic over Z_p and T finite free over R and potentially crystalline, at each γ: R → C_p the jumps (with multiplicity) form a multiset of cardinality rk T.	J.-M. Fontaine, Représentations p-adiques semi-stables, Exposé III, Theorem 3.8(i), Remark 3.9(ii) and §5.1.7, pp. 142–143 and 157–158 *second reading: The transition from potential crystallinity to Hodge–Tate weights is classical for field coefficients. The arbitrary coefficient-fibre formulation remains unlocated.*	`IsPotCrys.exists_hasJumps`
substantial	For f: A_inf → S S-linear, r_l ∈ A_inf and v_l ∈ $\mathcal{K}$_a (finitely many): every B_dR^+-submodule of B_dR containing Σ φ^(−a)(r_l) v_l also contains Σ φ^(−a)(f(r_l)) v_l.		`Lift.KummerChart.sum_map_mem_of_valueField`
named	Tate: invariants of B_dR modulo a B_dR^+-submodule S come from K: if j(g)x − x ∈ S for all g (j of open image, K its fixed field), then x − a ∈ S for some a ∈ K.	J. Tate, p-Divisible Groups, Proceedings of a Conference on Local Fields (1967), §3.3, Theorems 1 and 2, p. 176; P. Colmez, Théorie d'Iwasawa des représentations de de Rham d'un corps local, §III.1, B_{dR}^+, its residue and the action on t *second reading: A filtered consequence of the degree-zero parts of Tate's two theorems.*	`BdR.exists_sub_mem_of_smul_sub_mem`

class	what it says	reference; relation	formal name
substantial	If g ∈ G_p fixes Λ = W(F_{p^f}) pointwise and y lies in the bicommutant S_B ⊂ B^+, then z divides g y − y in B^+.		`Lift.Chart.z_dvd_sub`
named	Characters with finite image on inertia are potentially crystalline of weight 0: a continuous ℓ-adic character μ with finite image on inertia at A is of finite type at A, has the single jump 0, and X − μ(s) kills every s.	B. Conrad, Lifting global representations with local properties, Appendix B, Definition B.1 and Propositions B.3 and B.4(i)–(ii), pp. 32–36; J.-M. Fontaine, Représentations ℓ-adiques potentiellement semi-stables, Exposé VIII, §2.3.7, Weil action and Frobenius convention *second reading: The weight-zero potentially unramified case of Conrad's local character calculation, with its scalar Weil polynomial.*	`isFiniteTypeAt_char`
substantial	Hodge position: for T potentially crystalline with Z_p-basis e and jumps k_i over Z_p, there is a Z_p-linear automorphism N of B_dR^n turning the action of Γ on B_dR ⊗ T into the coordinatewise action and carrying (B_dR^+)^n onto ⊕ t^(−k_i) B_dR^+, given by a Galois-invariant matrix W t^k.	J.-M. Fontaine, Représentations p-adiques semi-stables, Astérisque 223 (1994), Exp. III; O. Brinon, B. Conrad, CMI notes on p-adic Hodge theory (2009) place not confirmed	`IsPotCrys.exists_hodge_basis`
substantial	Automatic continuity: for T finite free over Z_p and potentially crystalline, every crystalline cocycle c and every m, c(g) ∈ p^m T for all g with j(g) near 1.		`crysCocycles.eventually_eq_pow_smul`

class	what it says	reference; relation	formal name
named	Fontaine's criterion: if T is potentially crystalline and every s with j(s) in inertia acts trivially on D_pcris(T) (β_{s,0} = 1), then T is crystalline.	J.-M. Fontaine, Représentations p-adiques semi-stables, Exposé III, §§1.8.5–1.8.6 and §5.6.8(iii), unramified descent and potentially crystalline periods; J.-M. Fontaine, Représentations ℓ-adiques potentiellement semi-stables, Exposé VIII, §§2.3.2–2.3.3 and §2.3.7; O. Brinon and B. Conrad, CMI Summer School Notes on p-adic Hodge Theory, Proposition 9.3.1, invariance under unramified extension *second reading: Unramified descent and Weil inertia account for the classical crystalline criterion. The arbitrary-module descent remains unlocated.*	`IsPotCrys.isCrystalline_of_inertia_fixes`
named	Bloch–Kato's computation of H^1_f for Tate twists: if Γ acts on T (finite free over Z_p) through $\widehat{\chi}(-a)$, a ≥ 0, every crystalline cocycle c vanishes at each s with ρ(s) = 1, provided j(s) is in inertia when a = 0.	S. Bloch and K. Kato, L-functions and Tamagawa numbers of motives, The Grothendieck Festschrift I (1990), §3.7 and Example 3.9, p. 359; P. Colmez, Théorie d'Iwasawa des représentations de de Rham d'un corps local, §III.4, D_cris and Tate twists *second reading: Evaluation vanishing follows from the calculation of the finite local condition for twists.*	`crysCocycles.apply_eq_zero_of_cyclotomicGp`

class	what it says	reference; relation	formal name
named	Bloch–Kato rank formula: for T finite free over Z_p and potentially crystalline with Hodge–Tate jumps J over Z_p, Z_f(T) is finitely generated over Z_p of rank rk T + [G_p : j(Γ)]·#{negative jumps in J} (with multiplicity).	S. Bloch and K. Kato, L-functions and Tamagawa numbers of motives, The Grothendieck Festschrift I (1990), §3.7.2–3 and Corollary 3.8.4, pp. 353–355; P. Colmez, Théorie d'Iwasawa des représentations de de Rham d'un corps local, §§III.2 and III.4, equality of crystalline period spaces; J.-M. Fontaine, Représentations p-adiques semi-stables, Exposé III, Theorem 3.8(i), graded de Rham comparison *second reading: The Bloch–Kato local dimension formula applied to the integral module of actual cocycles.*	`crysCocycles.finrank_eq`
substantial	For a Breuil–Kisin chart (z = [π^♭]) and g ∈ G_p fixing Λ = W(F_{p^f}) and ϑ(z), and y in the chart's bicommutant S_B ⊂ B^+, the element τz divides g·y − y in B^+, where τ = [ε^(1/p)] − 1.		`Lift.Chart.tau_mul_z_dvd_smul_sub`

Shimura

class	what it says	reference; relation	formal name
named	Neatness (Minkowski–Serre lemma): for a CM field E with at least two infinite places, distinct finite places w_i with (2n)^{[E:ℚ]} < $\prod$ q_{w_i}, and K ⊂ G_f ∩ GL_n($\hat{O}$) reducing to 1 mod every w_i: Γ_f ∩ gKg^{-1} = {1} for all g, Γ_f the rational points of the definite unitary group U(I_n).	J.-P. Serre, Rigidité du foncteur de Jacobi d'échelon n ≥ 3, appendix to A. Grothendieck, Séminaire Cartan 13 (1960–61), exposé 17, pp. 17-18–17-20, theorem at p. 17-19. The input is a finite-order automorphism of an abelian variety congruent to 1 modulo an integer.; The trace, norm divisibility and final positive-definite argument in the pinned proof. *second reading: A quantitative trace-norm neatness lemma for CM ideal levels, requiring an additional argument beyond integral integer level*	`classAut_eq_bot`
substantial	Complex points s, u of a fibre power of the family X, attached to moduli points a, b with their $\mathfrak{m}$-torsion, are equal iff a and b are isomorphic at level K($\mathfrak{N}$) ∩ K($\mathfrak{m}$) (moduli interpretation of the Kuga–Sato fibre powers).		`ThetaFamily.IsPointOf.eq_iff`

class	what it says	reference; relation	formal name
named	Fourier coefficients of theta lifts as periods: if Y on U(I_n), n = n_1 + n_2, is continuous, left-invariant under rational points, of level K_0 above T ∪ $\bar{T}$, and all its theta descents to the split group of n_1 hyperbolic planes (character ∪^{n_1}, data vacuum at T ∪ $\bar{T}$) vanish, then ∫_{[U(I_{n_2})]} Y(h_1u) du = 0 for every h_1.	P. Walls, The theta correspondence, periods of automorphic forms and special values of standard automorphic L-functions, §§1.1–1.2 and §2.1, Proposition 2.1.1, equation (2.1.5), pp. 5–6. The group pair is symplectic–orthogonal and the right side is the stabilizer period of a convolved cusp form.; S. Rallis, On the Howe duality conjecture, Compositio Math. 51 (1984), introduction pp. 333–335. The formulation concerns the symplectic–orthogonal theta correspondence and its towers.; Unitary setup and explicit stabilizer in the compact proof. *second reading: Walls's stabilizer-period identity is for a different dual pair; the project proves the unitary vanishing separately*	`period_eq_zero`
substantial	Let S be reduced, locally noetherian, with a dense finite family of points; A, B flat and locally of finite type over S, A reduced. If complex points y_w of B are dense in B (B reduced) and, for each w, points e_w(v) are dense in the reduced fibre product A ×_S T_w, then the points (e_w(v), y_w) are dense in A ×_S B and A ×_S B is reduced.		`IsSeparatingFamily.prod`

SiegelWeil

class	what it says	reference; relation	formal name
substantial	Along an edge family with polynomial P, for every polynomial Q there is a compactly supported alpha on G_infinity such that the theta lift with datum Q at infinity equals the real convolution by alpha of the lift with datum P, uniformly in i, S, T, the modulus and the finite datum.		IsEdgeFamily.exists_lift_eq_realConv
substantial	Along an edge family, the theta datum with harmonic polynomial P at infinity and any finite part Psi mod m lifts forms of type k(d) on S to forms of type k'(d) on T, and the swapped kernel descends forms of the conjugate types from T to S.		IsEdgeFamily.isArrow_modDatum
substantial	Along an edge family with polynomial P, each polynomial Q has a constant kappa, independent of spaces and finite data, with <z, conj(Q-lift of y-bar)>_T = kappa <P-descent of z, y>_S (adjointness of lift and descent up to kappa).		IsEdgeFamily.petersson_descent_eq
substantial	Reversal of a pair: conj(theta^{S,T}_{a,b}[D,P](g,h)) = c theta^{T,S}_{b,a}[D',P'](h, g u), with D' the finite Fourier transform of D-bar, P' the signed transpose of P-bar, u = flipScalar, |c|^2 = N(m)^{-nm} and c independent of D, P, g, h.		exists_conj_modKernel_eq_swap
substantial	Rank-one unramified doubling sum: for nu with trivial tame part at w, w-bar and k > 1, over representatives supported at w, w-bar of U(1)(A_f) modulo the elements integral at w, the vacuum coefficients for (2k, nu upsilon^k) sum to prod_{u in {w, w-bar}} (1 - eps_u q_u^{-k})/(1 - nu~(pi_u)^{-1} q_u^{-k}).		exists_hasSum_vacuumCoeff_at_place
substantial	For a sign u, S of rank n and k integral: there are an additive character psi of O_E/m and c with |c|^2 = N(m)^{-n} such that theta^{<-u>,S}_{1,b}[D,P](k,h) = c theta^{S,<u>}_{a',upsilon}[D^,P'](h, k u_m) for all P, D, h (D^ the psi-Fourier transform, P' = flipPoly P).		exists_modKernel_flip_line

class	what it says	reference; relation	formal name
named	Rallis inner product formula, doubling form: if rk T > 2 rk S, f is clean, f' arbitrary and y, y' are rapidly decreasing automorphic forms on U(S), then g -> <f', omega(g,1)f><y', R(g)y> is integrable on U(S)(A) and <theta_{f'} y', theta_f y>_{U(T)} = vol([U(T)]) times its integral.	Li, Non-vanishing theorems for the cohomology of certain arithmetic quotients, case 2 on pp. 177–178 and §2, Theorem 2.1, (24)–(27), p. 184; Liu, Arithmetic theta lifting and L-derivatives for unitary groups I, §2B, Proposition 2.3, p. 862; Eischen, Harris, Li and Skinner, p-adic L-functions for unitary groups, §4.1.4, Zeta integrals *second reading: The exact shape and rank range of the classical identity are verified. The test spaces and volume normalization differ.*	`inner_product_formula`
substantial	If the Harish-Chandra parameters of s and +-l have the large-gap property at every infinite place, then the kernel pair (type kernels of l and -l) is a conjugate cusp kernel pair.		`isConjKernelPair_kernelPair`
substantial	If the parameters (gamma, l) of S and (gamma', l') of T correspond under the theta correspondence of discrete-series parameters (ParamsCorrespond), then the kernel pairs with the joint harmonic polynomial form an edge family for any family of characters with weights gamma, gamma'.		`isEdgeFamily_of_paramsCorrespond`
substantial	If the signs of S agree or m != (1), b has trivial tame part off m, and F != 0 on matrices mod m satisfies L y = 1 on its support (L fixed), then F (x) vacuum is a sharp datum for principal level m at the places dividing m m-bar, with constant <F,F>/|(O/m)^{mn}|/|GL_n(O/m)|.		`isSharp_modDatum`

class	what it says	reference; relation	formal name
named	Iwasawa decomposition: for g in U(W)(A) there are kappa in the maximal compact K_infinity K_0 and b in GL_p(A), block upper triangular for a given ordering, such that g kappa acts on the isotropic frame k by b and g kappa = (element of the unipotent radical) times (Levi element of (b, u)) for some u.	Arthur, A trace formula for reductive groups I, §1, p. 917, conditions (i)–(iii) and G(A)=P(A)K; First, Witt's extension theorem for quadratic spaces over semiperfect rings, Corollary 4.9 *second reading: Adelic Iwasawa decomposition for the parabolic preserving a prescribed isotropic flag, with the compact subgroup identified in coordinates.*	`Space.Splitting.iwasawa`
substantial	Let f be automorphic on U(W), fixed by integral points on S, radical-invariant along an extra line, det T != 0. If the T-coefficient of f vanishes at all points acting on the frame by gamma on S with delta^-1 gamma^H T gamma unimodular (primitive over the integers at S cup S-bar), it vanishes at all points supported on S cup S-bar.	E. Freitag, Singular Modular Forms and Theta Relations, LNM 1487 (1991); J.-S. Li, Automorphic forms with degenerate Fourier coefficients, Amer. J. Math. 119 (1997) place not confirmed	`freitag`
substantial	Removing a plane on the large side: for standard data, the constant term along A of the theta integral of a split tame character for (W', V') is gamma times the theta integral for (W, V) with a cone integral as real datum, where |gamma|^2 Pi(e) = C Pi(e+1), e = rk V - rk W/2, C independent of the character.	S. Rallis, On the Howe duality conjecture, Compositio Math. 51 (1984); C. Wu, Irreducibility of theta lifting for unitary groups, J. Number Theory 133 (2013) place not confirmed	`exists_constant_term_thetaLarge`
named	Siegel's formula (one vector) for the norm form x*x = sum x_i x_i-bar, n >= 3, over a CM field E: for an open level K and K-invariant Schwartz-Bruhat Phi_0 >= 0 and alpha = alpha-bar totally positive, the representation sums converge and their K-class sum equals (class sum of 1) |D_E|^{1/4} prod_v pi^n alpha_v^{n-1}/(n-1)! times the adelic Fourier integral int int Phi_0(x) psi_f(z(x*x - alpha)/2) dx dz.	Weil, Sur la formule de Siegel dans la théorie des groupes classiques, §52, Theorem 5, equation (41), pp. 76–77, referring to §44 for the gauge; Pinned conversion of measures, finite class sum and archimedean integral *second reading: Weil's mean-value identity on a nonzero index fiber, with the one-vector constants worked out in the formal source.*	`classSum_norm_eq`

class	what it says	reference; relation	formal name
named	Strong approximation, constancy form: let D be continuous on [U(W)], W split along a line, right invariant under the points supported on S cup S-bar (S nonempty) and under the integral points off S. If invariance under the unipotent radical makes D on the Levi independent of the torus, then D is constant.	Kneser, Starke Approximation in algebraischen Gruppen I, §1, (1.1), (1.4) and Satz 2, pp. 190–191; Harari, Galois Cohomology and Class Field Theory, Theorem 6.5 (Hilbert 90) and Proposition 13.1(b) (idèle cohomology as the sum of local groups); Pinned block action of the Levi on the hyperbolic pair *second reading: Constancy follows from strong approximation for the special unitary group together with the stated torus hypothesis.*	`IsAutFunction.apply_eq_apply_one`

Weil

class	what it says	reference; relation	formal name
substantial	Isometry of the theta map: ⟨f, f′⟩ equals the mean over $\widehat{\mathbb{A}}$n modulo the rational vectors of conj(f(x))·f′(x), data read as functions f(x) = ϑ(ϱ(x)f).	A. Weil, Sur certains groupes d'opérateurs unitaires, Acta Math. 111 (1964) place not confirmed	`Space.dataInner_eq_quotMean`
substantial	Poisson-type sum: for f, f′ ∈ $\mathcal{S}$(V), Σ_{γ rational} ⟨f′, ϱ(γ)f⟩ converges to ϑ(f)·conj(ϑ(f′)).	A. Weil, Sur certains groupes d'opérateurs unitaires, Acta Math. 111 (1964) place not confirmed	`Space.hasSum_dataInner`
named	Commutativity of the spherical Hecke algebra of GL_n: for split places w, u and v spherical at both, T_{w,i}T_{u,j}v = T_{u,j}T_{w,i}v.	Satake, Theory of spherical functions on reductive algebraic groups over p-adic fields, Chapter II, §5.1, Theorem 1 and the integer subring preceding it *second reading: Spherical Hecke commutativity for the specified double-coset operators, with an integral coefficient argument.*	`heckeSplit_comm`

class	what it says	reference; relation	formal name
named	Landherr's theorem (Hasse principle for hermitian forms), as used: for two spaces V, W with the same signs, some g ∈ U(s)(𝔸) carries the rational vectors of V into those of W.	Landherr, Äquivalenz Hermitescher Formen über einem beliebigen algebraischen Zahlkörper, pp. 246–247, classification by invariants and the local-global statement on p. 247; Landherr, Äquivalenz Hermitescher Formen über einem beliebigen algebraischen Zahlkörper, original journal article, volume 11, pp. 245–248 *second reading: Landherr's local-to-global isometry for two adelically framed hermitian spaces.*	`Space.exists_mapsTo_ratVectors`
named	Kashiwara–Vergne (harmonic highest vectors give holomorphic vectors): if at each infinite place P_w is a joint harmonic highest vector of weight u_w for the standard pair, then Q = ∏_w P_w is nonzero and its Fock vector is holomorphic of weight u.	Kashiwara and Vergne, On the Segal–Shale–Weil representations and harmonic polynomials, Chapter III, Theorems 6.3 and 7.2, pp. 44–45; Davidson, The harmonic representation of U(p,q), §3.3 and Theorems 3.5–3.6, pp. 47–48 *second reading: Local holomorphic highest-weight realization is located. The coefficient test on the project's normed model remains a separate assertion.*	`prod_mem_holomorphicData`

class	what it says	reference; relation	formal name
named	Kashiwara–Vergne (products of minors are joint harmonic highest vectors for (U(p,q), U(n))): the product of k(η,p) copies of the p×p minor on the first p columns of sign η and the first (η = 1) or last p rows is, if the sizes fit, nonzero, harmonic and highest on both sides, of explicit weights.	Kashiwara and Vergne, On the Segal–Shale–Weil representations and harmonic polynomials, Chapter III, Proposition 6.1, formula (6.2) and Theorem 6.3, p. 44; Davidson, The harmonic representation of U(p,q), §2, Theorem 2.9, p. 45 *second reading: The forward highest-vector construction by products of minors, in ordered and determinant-twisted coordinates.*	`staircase_harmonic_highest`
named	Gauss sums of a character of the determinant over matrix rings: for primitive ψ, χ and S on square matrices with S(gd) = χ(det g)S(d) (g invertible), $\hat{S}(y) = \chi^{-1}(\det y)\hat{S}(1)$, and $\hat{S}(1)\cdot\mathrm{conj}(\Sigma_v \chi(\det v)\psi(\mathrm{tr}\ v)) = S(1)\cdot\#M_a(R)$.	Kondo, On Gaussian sums attached to the general linear groups over finite fields, pp. 244–245, and §2.3, pp. 253–254 *second reading: The finite-field identity for invertible frequencies is located. The full finite-ring statement has a different scope.*	`matrix_gauss_sum`
substantial	Moderate growth of theta kernels: there are C and A with $\lvert\vartheta(\omega(g,h)\varphi)\rvert \le C\lVert\varphi\rVert\cdot(\mathrm{ht}(g)\cdot\mathrm{ht}(h))\hat{}A$ for all g, h and all data φ.		`ThetaKernel.moderate_growth`
substantial	For split S, $P \cap \bar{P} = \varnothing$, Ψ Schwartz–Bruhat above P supported on maps of full rank (after a ring map φ: $\mathbb{A}_f \to R \neq 0$), v with all moments finite: the datum v⊗Ψ⊗vac has absolutely convergent unfolded theta integral over [U(T)].		`Space.Splitting.hasIntegrableTerms_integrated_name`
substantial	Definite pair U(n)×U(n′), n<n′, fitting weights, corresponding parameters l, l′, scale c≠0: there is β with: for every theta kernel at level K and algebraic form φ, the adelic theta lift (joint harmonic ⊗ f) of F_φ equals $\beta[K_0:K]^{-1}\ F_{\varphi'}$, φ′ the class-sum lift of φ.	B. Gross, Algebraic modular forms, Israel J. Math. 113 (1999); Kashiwara–Vergne (1978); Howe, Transcending classical invariant theory, JAMS 2 (1989) place not confirmed	`exists_lift_ofAlgForm_eq_smul`

class	what it says	reference; relation	formal name
named	Rallis's injectivity of the theta lift in the stable range (step): rk S = n, rk T = m, 2n < m, φ = v⊗F, F sharp at J, y J-invariant rapidly decreasing with zero lift: ∫_{real x} ⟨v, Ω(x)v⟩⟨y′, R(x)y⟩ dx = 0 for every rapidly decreasing y′.	Rallis, Injectivity properties of liftings associated to Weil representations, Introduction, §1, especially Proposition 1.1 and Theorem 1.6; Li, Non-vanishing theorems for the cohomology of certain arithmetic quotients, pp. 177–178, case 2, and §2, Theorem 2.1, equation (26), p. 184 *second reading: analogue in cited paper; exact published inference open*	integral_eq_zero_of_lift_eq_zero
named	Rallis inner product formula (doubling): rk T > 2 rk S, v an edge from type e: there is A ≠ 0 such that for F sharp at J of mass c the lift L by v⊗F is injective on J-invariant forms y and ⟨Ly′, Ly⟩ = A c Σ_γ ρ(γ)⟨y′, R(γ)y⟩ (sum over cosets off S_0).	Rallis, Injectivity properties of liftings associated to Weil representations, §1 and Theorem 1.6; Li, Non-vanishing theorems for the cohomology of certain arithmetic quotients, §2, Theorem 2.1, equation (26), p. 184 *second reading: analogue in cited paper; claimed special case unsupported by that citation*	rallis_inner_product_euler
named	Seesaw identity (Kudla): for edges S → T and T′ → S′ whose kernels agree up to moderate factors on a spanning set of Fock polynomials, if finite sums of periods of f against lifts of y are proportional to (c_o) at every real translate, so are the sums of periods of y against descents of f.	Kudla, Seesaw dual reductive pairs, §1, adjointness (1.9) and seesaw identity (1.14) *second reading: application of a located classical identity with additional hypotheses*	seesaw_periods
substantial	Rapid decay for modular data: if 𝔪 ≠ (1) and Ψ is supported on matrices left invertible mod 𝔪, the kernel ϑ of v⊗Ψ satisfies, for each M, |ϑ(g,h)|·ht(g)^M ≤ C·mom_N(v)·ht(h)^B (some B, N, C) for all g, h and v with finite N-th moment.		pairTheta_modDatum_decay

class	what it says	reference; relation	formal name
named	Eichler commutation relation (Hecke transfer through the theta correspondence): w split, ranks n of S and n+k of T, a, b unramified, f the vacuum at w, Λ linear with Λ(ω(g,h)f) = π(g)ρ(h)Λf; then Λf is K_w-fixed for π and ρ and P^ρ_w(X)Λf = ∏_{l<k}(1 − q^{l+n/2}χ_S(w)X)·P^π_{w̄}(q^{k/2}χ_S(w)X)Λf. Pinned definition: the uniformizer scalar including the half-power of q *second reading: The split local parameter correspondence is verified. The universal operator identity and its splitting-character normalization need a further comparison.*	Mínguez, Correspondance de Howe explicite : paires duales de type II, introduction, Theorem 1 and the parameter formula following it, p. 718, and Corollary 6.3; Pinned definition: the uniformizer scalar including the half-power of q	hecke_transfer_of_equivariant
substantial	At a split place w with f the vacuum at w and a, b unramified: the Hecke sum at X over U(T) (resp. U(S)) applied to f equals f named at w by x ↦ trivialHecke(d − rank(x mod w), νX) on integral x, with (d, ν) = (m, frobValue of a) resp. (n, frobValue of b).		sum_heckePolyCoeff_smul_pairWeil
substantial	Decay estimate: for Ψ Schwartz–Bruhat above P (P ∩ P̄ = ∅), f₀ ∈ 𝒲_f, there is C with |L(v⊗Ψ⊗f₀)|·H^N ≤ C·mom_{N+A}(v)·ρ^{N+2A} for L = ℓ∘ω(g,h), under growth bounds on L and, if H > 2, short lattice vectors (one isotropic) giving invariances or non-vanishing conditions.	decay of theta kernels: Kudla–Rallis, A regularized Siegel–Weil formula: the first term identity, Ann. Math. 140 (1994) (by Siegel sets) place not confirmed	exists_norm_name_le
substantial	Rapid decay: for Ψ Schwartz–Bruhat above P (P ∩ P̄ = ∅) supported on matrices of full rank (after a ring map φ), every N has B, M, C with |ϑ_{a,b}(v⊗Ψ⊗vac)(g,h)|·ht(h)^N ≤ C·mom_M(v)·ht(g)^B for all g, h and v with mom_M(v) < ∞.	Kudla–Rallis, A regularized Siegel–Weil formula: the first term identity, Ann. Math. 140 (1994) place not confirmed	pairTheta_decay_of_onto

class	what it says	reference; relation	formal name
named	Kashiwara–Vergne (harmonic highest vectors for (U(p,q), U(n)) give holomorphic vectors): if $\varkappa$ has no column longer than the number of signs c, the highest vector of weight $\varkappa$ in the variables of sign c is a nonzero polynomial whose Fock vector is holomorphic of the explicit (staircase) weight.	Davidson, The harmonic representation of U(p,q), Theorem 2.9, p. 45, and Theorem 3.6, p. 48; Kashiwara and Vergne, On the Segal–Shale–Weil representations and harmonic polynomials, Chapter III, Proposition 6.1 and (6.2), p. 44 *second reading: The one-sign highest polynomial specializes the published minor construction. The full analytic conclusion needs the normed-model comparison.*	`letterPoly_highestVector_holomorphic`
substantial	For signature (p,q) and a definite second space of rank n', the product over infinite places of the d-th power of a p×p minor in the positive columns is a nonzero polynomial whose Fock vector is holomorphic of weight −d on positive and n' on negative indices.		`prod_letterMinor_mem_holomorphicData`
substantial	If the polynomial P (in the variables of sign +1) is scaled under u by ∏_w (principal minor of u_w on +)^ν_w·(on −)^c_w, then its matrix coefficient ⟨v_P, ω(u,1)v_P⟩ equals ⟨v_P,v_P⟩·M_v(u)·conj(M_k(u)), with M the explicit products of powers of principal minors (holMinorsPi).		`realInner_realPairWeil_eq_holMinorsPi`
named	Basis problem in the stable range, as a reproducing formula: for $2n < n'$, level $\mathfrak{m} = (N) \neq (1)$, unramified c, b, $Q_0 \neq 0$ holomorphic of dominant weight u with integrable coefficient m, $\psi \mapsto \int \psi(xh)m(h)dh$ on rapid level-N forms is a fixed finite linear combination of pairings with theta kernels ϑ_{c,b}[1_e, Q_i] to the definite U(n').	Böcherer, Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen, §1, Theorem III, p. 22; Li, Non-vanishing theorems for the cohomology of certain arithmetic quotients, §2, Theorem 2.1, p. 184 *second reading: The classical Siegel basis theorem and the unitary stable-range pairing are located. The finite unitary reproducing formula is distinct.*	`reproducing_formula_modKernel`

class	what it says	reference; relation	formal name
named	Joint harmonics and global theta lifts between discrete series types: for n, $n' \geq 1$ and corresponding parameters l, l', the Fock vector of the product of chosen joint harmonics is an edge from the type of l to the type of l' for the real Weil representation of weights γ, γ'.	Ichino, Theta lifting for discrete series representations of real unitary groups, Theorem 4.1 and Lemma 7.8, pp. 35–36 for the compact types; Pinned choice of local polynomials and proof of the edge for the chosen product *second reading: The formal matching condition selects witnesses for a two-sided analytic statement beyond the published joint-type criterion.*	`ParamsCorrespond.isEdge_jointHarmonic`
named	Joint harmonics and global theta lifts between discrete series types: if each Φ_w is a joint harmonic with parameters l'_w, −l_w, the Fock vector of $\prod$_w Φ_w is an edge between the discrete-series types of l and l' (both directions) for the real Weil representation of weights γ, γ'.	Ichino, Theta lifting for discrete series representations of real unitary groups, §7.2, Proposition 7.2, and §7.4, Lemma 7.8 and the convention discussion in its proof, pp. 35–36 of the preprint; Howe, Transcending classical invariant theory, Theorem 1 and Theorem 1A; Pinned definitions of IsHalfEdge and IsEdge *second reading: Joint compact types occur in the published real correspondence. The two-sided analytic conclusions need additional arguments.*	`isEdge_of_isJointHarmonic`

class	what it says	reference; relation	formal name
named	Witt decomposition: every isotropic frame k (E-matrix with a left inverse and k*Jk = 0) of a hermitian space V over E is the frame of a splitting V = (lE^p ⊕ kE^p) ⊥ iV₀ with l*Jk = 1, V₀ a hermitian space with a standard adelic point.	Landherr, Äquivalenz Hermitescher Formen über einem beliebigen algebraischen Zahlkörper, p. 247, equivalence at all completions; First, Witt's extension theorem for quadratic spaces over semiperfect rings, Remark 2.1 and Corollary 4.9; Witt, Theorie der quadratischen Formen in beliebigen Körpern, Satz 5 *second reading: Combination of hermitian local-global classification and Witt extension with a fixed isotropic frame.*	`Space.exists_splitting`
substantial	V standard, d ∈ O_E, dd̄ = N ≠ 0: every entire function S on ℂ^{Φ×n} invariant under the normalised (theta-multiplier) translations by −r(d̄γ), for all γ ∈ E^n with α^{−1}γ integral, equals Θ_F for some F in the finite part $\mathcal{W}$_f admitting vacuum-expansion data for (d, N).		`Space.exists_entireTheta_eq`
named	Bargmann–Fock model of the oscillator representation of U(p,q): for a continuous homomorphism f: U(ε) → U(ε′) with ‖f(g)‖ ≤ a‖g‖, g ↦ det(g)^γ·ω₀(f(g)) on the Fock space is strongly continuous, of norm O(‖g‖^|C|), and preserves the positive definite form ∫c̄c′.	Davidson, The harmonic representation of U(p,q), formulas (1.9a–b) and Theorem 1.12, pp. 39–40 *second reading: Strongly continuous Hilbert unitarity is verified. The degree bound is for a different operator norm.*	`isUnitaryRep_detTwist`
substantial	In a mixed Schrödinger–Fock model, for τ smooth with compact support, u(x) unitary depending smoothly on x, and ξ rapid, there is a rapid Fock vector Φ whose slice at every x is τ(x)·ω₀(u(x))ξ.		`MixedModel.exists_section`

Record of checks

what	by whom	result	source
Build of every file without sorry; #print axioms of the five theorems: propext, Classical.choice, Quot.sound; export accepted by nanoda, con-leche, con-ron; Comparator against Challenge.lean	the authors (reported)	reported as passed	lsernau/grh-lean, README of grh-compact at 72fead1
Continuous integration of every commit of the readable edition, failing on sorry or on any axiom beyond the standard three	leanprover-community/riemann-hypothesis	the runs that had finished on 26 Nov 2026 passed	the repository's Actions page
Source scan of both editions: no sorry, axiom, opaque, unsafe, native_decide, custom syntax or compile-time code	us (riemann-rev)	none found	riemann-rev, scripts/measure_condensed.py and the public summary note, section 9
Independent build of grh-compact at 72fead1 with Lean v4.35.0 and Mathlib 56a00585 on a cluster	us	interrupted at 9,788 of 9,843 build steps (no error line) by the cluster's scheduled full-system downtime of 26–27 Nov 2026; requeued, resumes from the kept build	riemann-rev, private/replays/grh-compact-72fead1.json (job 27004713; Lean 4.35.0, Mathlib manifest unchanged, clone HEAD 72fead1)
Kernel-level dependency export of the five theorems (which declarations they use) and Lean's own argument list of every declaration, compared with the dossiers	us, on the same build	pending: queued behind the build (job 27006280)	export/ExportDeps.lean, export/Signatures.lean
Structured proofs of every unit of New compared with the Lean source at the pinned commit (quotations, line ranges, hashes, the account of every argument)	us, by program (tools/rb.py check)	see the state of each unit in its chapter	reading/units/<id>/check.json
A mathematician's reading of the argument	nobody yet	this handout is a first step: notes written from the structured proofs	-