

One Map, Three Descriptions

Marked roots of binary cubics and a three-dimensional Keller map

Repository compilation, exposition, and independent audit
Bartosz Naskrecki

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Abstract

An explicit polynomial map $F: \mathbb{C}^3 \rightarrow \mathbb{C}^3$ has constant Jacobian determinant -2 but sends three distinct points to one point. The formula looks mysterious; its geometry is not. A general binary cubic has three distinct projective roots. Mark one simple root and then forget which root was marked. A special affine slice makes both the space of cubics and the space of marked cubics isomorphic to affine three-space. A pencil-bundle description explains geometrically why the marked space is affine three-space. This note unifies three public presentations of that map: the formula posted by Levent Alpöge, with credit to Fable; the symmetric-product description posted by Andy Jiang (@davikrehalt), explicitly presented there as GPT output; and the factorization-resultant derivation of Aaron Lou (@aaron_lou). We give the exact coordinate changes, a short proof of étaleness, the complete set-theoretic fiber counts, and the mechanism by which inverse branches escape to infinity.

Exact claim. Define $F = (F_1, F_2, F_3)$ by

$$\begin{aligned} F_1 &= (1 + xy)^3 z + y^2(1 + xy)(4 + 3xy), \\ F_2 &= y + 3x(1 + xy)^2 z + 3xy^2(4 + 3xy), \\ F_3 &= 2x - 3x^2 y - x^3 z. \end{aligned}$$

Then

$$\det JF = -2$$

and

$$F\left(0, 0, -\frac{1}{4}\right) = F\left(1, -\frac{3}{2}, \frac{13}{2}\right) = F\left(-1, \frac{3}{2}, \frac{13}{2}\right) = \left(-\frac{1}{4}, 0, 0\right).$$

Thus F is locally invertible everywhere but is not globally injective. These are exact polynomial identities, not numerical approximations.

The Jacobian conjecture says that a polynomial self-map of $\mathbb{C}^n$ with nonzero constant Jacobian determinant must have a polynomial inverse. By the standard complex Jacobian conjecture, the two identities in the box therefore constitute a counterexample in dimension three. They are independently reproducible, but at the date of this note the announcement had not yet passed through the usual refereeing process.

1 Provenance and the three viewpoints

The public record used here, from 20–21 July 2026, separates naturally into the following roles.

- **Alpöge–Fable: the coordinate map.** Levent Alpöge’s post at 02:19:17 UTC gives the formula, determinant, and three-point fiber [1]. The post credits Akhil for asking about the problem and Fable for working during the World Cup final. We use the label “Alpöge–Fable presentation” without trying to infer a more detailed private division of authorship.
- **Jiang–GPT: the projective geometry.** At 12:02:19 UTC, Andy Jiang (@davikrehalt) posted the symmetric-product construction below and explicitly introduced it in the post as GPT output [2].
- **Lou: the resultant chart.** Aaron Lou (@aaron_lou) gave a self-contained derivation by factoring a cubic into a linear and a quadratic factor, normalizing their resultant, and finding a global affine chart [3].

Daniel Litt (@littmath) subsequently explained the affine source through a family of pencils of lines, exhibiting it as an affine-line bundle over an affine plane [4]. Litt reported substantial ChatGPT assistance with that geometric gloss. We use it below to illuminate the same Jiang–GPT/Alpöge–Fable construction, not as a fourth construction.

This repository compiles, compares, and independently checks those public descriptions. Its maintainer does not claim discovery authorship for any of them. The three descriptions fit into one diagram:

$$\begin{array}{ccccccc}
 \mathbb{A}^3 & \xrightarrow{\sim} & M_{\omega,\tau} & \xrightarrow{\sim} & X_\tau \simeq X_{\text{proj}} \\
 \downarrow \begin{smallmatrix} F \text{ or } G \\ \mathbb{A}^3 \end{smallmatrix} & \xrightarrow{\sim} & \downarrow \begin{smallmatrix} (\ell,q) \mapsto \ell q \\ H_\tau \end{smallmatrix} & \xrightarrow{\sim} & \downarrow \begin{smallmatrix} \text{forget the marked root} \\ Y_{\text{proj}} \end{smallmatrix}
 \end{array} \tag{1.1}$$

From left to right, these are the explicit formula, Lou’s normalized factor space, the affine marked-root incidence, and Jiang’s projective open set. The horizontal arrows are isomorphisms, but they require choices of bases, signs, and affine levels. Thus they are isomorphic presentations of the same abstract algebraic morphism; they are not literally the same coordinate formulas.

For a nonexpert, four translations are useful. A *binary cubic* is a homogeneous degree-three polynomial in two variables. Its roots are points of the projective line $\mathbb{P}^1$. A root is *simple* when it has multiplicity one. An *etale* map has no local folding: its derivative is invertible everywhere. A *nonproper* map may nevertheless lose an inverse branch because source points can run to infinity while their images remain bounded.

2 Jiang–GPT: mark a root, then forget it

An effective divisor of degree d on $\mathbb{P}^1$ is an unordered list of d points, counted with multiplicity. Equivalently, it is the zero set of a binary form of degree d , up to nonzero scale. Hence

$$\text{Sym}^d(\mathbb{P}^1) \simeq \mathbb{P}^d.$$

Consider the projective morphism

$$\mu: \mathbb{P}^1 \times \text{Sym}^2(\mathbb{P}^1) \longrightarrow \text{Sym}^3(\mathbb{P}^1), \quad (p, \{q, r\}) \longmapsto \{p, q, r\}. \tag{2.1}$$

It remembers a cubic divisor but forgets which point was distinguished. If p, q, r are distinct, its fiber consists of the three possible markings

$$(p, \{q, r\}), \quad (q, \{p, r\}), \quad (r, \{p, q\}).$$

This is the conceptual source of generic degree three and noninjectivity.

Let R be the ramification divisor of μ . It consists exactly of pairs for which the marked point also occurs in the residual quadratic:

$$R = \{(p, \{q, r\}) : p = q \text{ or } p = r\}. \quad (2.2)$$

The condition $q = r$ by itself causes no ramification when p is different.

The small diagonal $p \mapsto 3p$ is a twisted cubic in $\text{Sym}^3(\mathbb{P}^1) \simeq \mathbb{P}^3$. Choose a hyperplane H_∞ tangent but not osculating to that curve: its intersection divisor with the curve has type $2p + q$, where $p \neq q$. Set

$$\begin{aligned} Y_{\text{proj}} &= \text{Sym}^3(\mathbb{P}^1) \setminus H_\infty, \\ X_{\text{proj}} &= (\mathbb{P}^1 \times \text{Sym}^2(\mathbb{P}^1)) \setminus (R \cup \mu^{-1}(H_\infty)). \end{aligned}$$

Since the complement of a hyperplane in $\mathbb{P}^3$ is affine space, $Y_{\text{proj}} \simeq \mathbb{A}^3$. The target is therefore visibly affine three-space. Section 3 explains geometrically why the source is an affine-line bundle over $\mathbb{A}^2$, and then gives an explicit global trivialization.

Here is an intrinsic affine version. Let W be two-dimensional and choose a tensor $\tau \in \text{Sym}^3 W$ of multiplicity type $(2, 1)$. The projective hyperplane H_∞ is defined by $\langle \tau, C \rangle = 0$. Its complement is identified with the affine slice

$$H_\tau = \{C \in \text{Sym}^3 W^\vee : \langle \tau, C \rangle = 1\} \simeq \mathbb{A}^3, \quad [C] \mapsto \frac{C}{\langle \tau, C \rangle}. \quad (2.3)$$

Define the simple-root incidence

$$X_\tau = \{(C, [r]) \in H_\tau \times \mathbb{P}(W) : C(r) = 0, dC_r \neq 0\}. \quad (2.4)$$

Deleting R says precisely that the marked root is simple, so (2.3) gives natural isomorphisms $X_{\text{proj}} \simeq X_\tau$ and $Y_{\text{proj}} \simeq H_\tau$, under which μ becomes $(C, [r]) \mapsto C$.

3 Lou: fix the factorization gauge by a resultant

Fix a nonzero alternating form $\omega \in \wedge^2 W^\vee$. A marked root of C is the same as a factorization

$$C = \ell q, \quad \ell \in W^\vee, \quad q \in \text{Sym}^2 W^\vee,$$

where ℓ vanishes at the marked point. This factorization has a scaling ambiguity

$$(\ell, q) \mapsto (s\ell, s^{-1}q), \quad s \in \mathbb{G}_m. \quad (3.1)$$

Let $v_\ell \in W$ be defined by $\iota_{v_\ell} \omega = \ell$, and put

$$\text{Res}_\omega(\ell, q) = q(v_\ell). \quad (3.2)$$

This is the linear-quadratic resultant. It is nonzero exactly when the marked root is simple, and it has weight one under (3.1):

$$\text{Res}_\omega(s\ell, s^{-1}q) = s \text{Res}_\omega(\ell, q).$$

Therefore every coprime scaling orbit has a unique representative with resultant one. Define

$$M_{\omega, \tau} = \{(\ell, q) : \text{Res}_\omega(\ell, q) = 1, \langle \tau, \ell q \rangle = 1\}. \quad (3.3)$$

Multiplication gives an isomorphism

$$M_{\omega, \tau} \xrightarrow{\sim} X_\tau, \quad (\ell, q) \mapsto (\ell q, \mathbb{P}(\ker \ell)). \quad (3.4)$$

The same normalization has a useful root-gradient form. If $r = v_\ell$, then

$$d(\ell q)_r = q(r)\ell = \text{Res}_\omega(\ell, q) \iota_r \omega. \quad (3.5)$$

Thus resultant one means $dC_r = \iota_r \omega$. This single equation says both that r is a root, by Euler's identity, and that it is simple.

Proposition 3.1 (Why there is no local folding). *The forgetful map $X_\tau \rightarrow H_\tau$ is étale.*

Proof. Consider

$$\Psi(\ell, q) = (\ell q, \text{Res}_\omega(\ell, q)).$$

If ℓ and q are coprime and $\dot{\ell}q + \ell\dot{q} = 0$, unique factorization gives $(\dot{\ell}, \dot{q}) = a(\ell, -q)$. This is exactly the infinitesimal direction of the gauge (3.1). Along that direction the derivative of the resultant is $a \text{Res}_\omega(\ell, q)$, which is nonzero unless $a = 0$. Hence $d\Psi$ is an isomorphism. The map in the proposition is its base change to the two levels in (3.3), so it is étale. $\square$

The pencil bundle behind the affine chart

The following pencil-bundle picture makes explicit a geometric observation of Litt [4]. It explains the global affine chart rather than merely verifying its formulas. Choose an adapted basis T, S and write

$$\ell = xT + \beta S, \quad q = \gamma T^2 + \delta TS + \epsilon S^2.$$

Before fixing the two affine levels in (3.3), project to

$$\lambda = [\ell] = [x : \beta] \in \mathbb{P}^1, \quad [q] = [\gamma : \delta : \epsilon] \in \mathbb{P}^2.$$

For fixed λ , the pullback of the hyperplane at infinity and the resultant divisor are the two lines

$$\begin{aligned} \mathcal{H}_\lambda(q) &= \beta\gamma + x\delta = 0, \\ \mathcal{R}_\lambda(q) &= \beta^2\gamma - x\beta\delta + x^2\epsilon = 0. \end{aligned}$$

Put $\mathcal{K}_\lambda(q) = -2\beta\delta + x\epsilon$. The elementary identity

$$\mathcal{R}_\lambda = \beta\mathcal{H}_\lambda + x\mathcal{K}_\lambda$$

contains the geometry of the chart. In particular, $\mathcal{H}_\lambda(\ell^2) = 3x^2\beta$: $x = 0$ is the double, tangent contact, while $\beta = 0$ is the remaining transverse contact. At $x = 0$ the two deleted lines coincide, but $\mathcal{H}_\lambda$ and $\mathcal{K}_\lambda$ retain the limiting pencil.

Write a pencil member as

$$n\mathcal{H}_\lambda + m\mathcal{K}_\lambda = 0, \quad [m : n] \in \mathbb{P}^1.$$

The line $\mathcal{H}_\lambda$ is the horizontal section $[m : n] = [0 : 1]$, while $\mathcal{R}_\lambda$ is the diagonal $[m : n] = [x : \beta]$. Thus the allowed pencil parameters form

$$B = (\mathbb{P}^1 \times \mathbb{P}^1) \setminus (\text{horizontal section} \cup \text{diagonal}) \simeq \mathbb{A}^2.$$

An explicit isomorphism and its inverse are

$$\begin{aligned} (x, y) &\longmapsto ([x : 1 + xy], [1 : y]), \\ ([a : b], [m : n]) &\longmapsto \left(\frac{am}{bm - an}, \frac{n}{m} \right). \end{aligned}$$

The denominators do not vanish precisely because the horizontal section and diagonal were removed. This is the geometric origin of $\beta = 1 + xy$.

All lines in the pencil over λ meet at

$$b_\lambda = [x^2 : -x\beta : -2\beta^2] \in \mathbb{P}^2.$$

Every quadratic outside the two deleted lines lies on a unique allowed pencil member; that member, with b_λ removed, is $\mathbb{A}^1$. Consequently

$$X_{\text{proj}} \longrightarrow B \simeq \mathbb{A}^2, \quad \text{fiber} \simeq \mathbb{A}^1,$$

is an affine-line bundle, so it is locally a product with $\mathbb{A}^1$. The formulas below show directly that it is globally $\mathbb{A}^2 \times \mathbb{A}^1$.

Return now to the normalized equations. The two equations in (3.3) become

$$x\delta + \beta\gamma = 1, \quad x^2\epsilon - x\beta\delta + \beta^2\gamma = 1. \quad (3.6)$$

In other words, $\mathcal{H}_\lambda(q) = \mathcal{R}_\lambda(q) = 1$. If

$$y = -\mathcal{K}_\lambda(q) = 2\beta\delta - x\epsilon,$$

then $\mathcal{R}_\lambda = \beta\mathcal{H}_\lambda + x\mathcal{K}_\lambda$ immediately gives $\beta = 1 + xy$. The base coordinates are therefore exactly $([x : \beta], [m : n]) = ([x : 1 + xy], [1 : y])$. It remains only to choose a coordinate along the affine-line fiber. The following formulas choose it globally:

$$\begin{aligned} \beta &= 1 + xy, & \gamma &= \frac{2 - 3xy + x^2z}{2}, \\ \delta &= \frac{y + 3xy^2 - xz - x^2yz}{2}, & \epsilon &= -z(1 + xy)^2 + y^2(4 + 3xy). \end{aligned} \quad (3.7)$$

For fixed (x, y) , the dependence on z is exactly in the common-base-point direction:

$$\frac{\partial(\gamma, \delta, \epsilon)}{\partial z} = \frac{1}{2}(x^2, -x\beta, -2\beta^2).$$

Thus x, y are coordinates on B and z is the affine coordinate along its fiber. The inverse is polynomial and is evaluated in this order:

$$x = x, \quad y_* = 2\beta\delta - x\epsilon, \quad z_* = 4y_*\delta + 2\gamma y_*^2 - \epsilon. \quad (3.8)$$

Substitution verifies (3.6). Conversely, (3.6) first gives $\beta = 1 + xy_*$ and then recovers the remaining three formulas in (3.7). Thus both source and target of the forgetful map are globally affine three-space; no divisor was silently removed from the chart.

4 Alpöge–Fable: recovering the displayed formula

The direct projective derivation uses a point $p = [u : v]$, its linear factor, and a residual quadratic

$$L = vT - uS, \quad Q = AT^2 + BTS + CS^2.$$

The two projective scalings have unique representatives satisfying

$$h := vB - uA = -2, \quad r := Q(u, v) = 2. \quad (4.1)$$

This is exactly Lou's normalization in different signs and units. Indeed set

$$\ell(T, S) = L(T, -S), \quad q(T, S) = \frac{1}{2}Q(T, -S). \quad (4.2)$$

Then

$$x = v, \quad \beta = u, \quad \gamma = \frac{A}{2}, \quad \delta = -\frac{B}{2}, \quad \epsilon = \frac{C}{2}, \quad (4.3)$$

and the two left sides of (3.6) are respectively $-h/2$ and $r/2$.

In the direct chart one writes

$$\begin{aligned} u &= 1 + xy, & v &= x, \\ A &= 2 - 3xy - x^2z, \\ B &= -y - xz - 3xy^2 - x^2yz, \\ C &= 2((1 + xy)^2z + y^2(4 + 3xy)). \end{aligned} \quad (4.4)$$

Equations (4.1) hold identically. Comparing (4.4) with (3.7) shows that $z_{\text{Lou}} = -z_{\text{direct}}$. Moreover,

$$LQ = cT^3 - 2T^2S + bTS^2 - 2aS^3, \quad a = \frac{uC}{2}, \quad b = vC - uB, \quad c = vA. \quad (4.5)$$

Substitution in (4.5) is precisely the map F displayed on the first page.

If G denotes Lou's target order, define

$$S_0(x, y, z) = (x, y, -z), \quad R_0(a, b, c) = (c, b, a).$$

Then the exact relation is

$$\boxed{G = R_0 \circ F \circ S_0.} \quad (4.6)$$

Thus Jiang's projective morphism is not a second counterexample: it is the Alpöge–Fable map in different coordinates, as Litt also emphasized [5]. More precisely, Jiang's projective morphism, Lou's normalized factor map, and F are the same abstract morphism under the source and target identifications displayed above. This is left-right linear equivalence: source and target coordinates change independently. In particular, the displayed identification is not a dynamical conjugacy by one common coordinate change.

Proposition 3.1 shows that $\det JF$ never vanishes. A nowhere-zero polynomial on $\mathbb{C}^3$ is constant, and the linear part of F at the origin is

$$(x, y, z) \longmapsto (z, y, 2x).$$

Its determinant is -2 , proving $\det JF = -2$ without a large expansion. The three substitutions on the first page then prove noninjectivity.

There is also a compact certificate that the formula really marks a root. For $F(x, y, z) = (a, b, c)$ set

$$Q_{a,b,c}(U, V) = cU^3 - 2U^2V + bUV^2 - 2aV^3, \quad (U, V) = (1 + xy, x).$$

Direct substitution gives

$$Q_{a,b,c}(U, V) = 0, \quad (Q_U, Q_V)(U, V) = (2V, -2U). \quad (4.7)$$

The marked projective point is therefore a simple root, normalized by its gradient. After the sign and scale changes in (4.2)–(4.6), equation (4.7) is the coordinate form of (3.5).

5 Fibers, the discriminant, and escape to infinity

Identify a target point with its binary cubic as in (4.7). Its fiber is then read directly from the cubic's roots:

$$\#F^{-1}(C) = \begin{cases} 3, & C \text{ has three simple roots,} \\ 1, & C \text{ has a double root and a distinct simple root,} \\ 0, & C \text{ has a triple root.} \end{cases} \quad (5.1)$$

In the middle case only the simple root can be marked; at a triple root every possible marking lies in the deleted ramification divisor.

Before deleting ramification, the full incidence

$$\overline{X}_\tau = \{(C, [r]) \in H_\tau \times \mathbb{P}(W) : C(r) = 0\} \longrightarrow H_\tau$$

is finite flat of degree three: every nonzero binary cubic has a projective zero divisor of length three, counted with multiplicity. Its branch locus is the binary-cubic discriminant. In the original target coordinates a convenient normalization is

$$\Delta(a, b, c) = b^2 - cb^3 - 16a + 18abc - 27a^2c^2. \quad (5.2)$$

Off $V(\Delta)$, no point was deleted, so F is finite and therefore proper there. Every discriminant cubic has a repeated-root marking in R , and that boundary point is a limit of simple markings. A sequence of simple markings converging to it cannot remain bounded in the global $\mathbb{A}^3$ chart: otherwise it would have an affine limit, although the limiting marking was deleted. Thus an inverse branch escapes to infinity over every point of $V(\Delta)$, whereas no branch can do so off it. Hence the nonproperness set is exactly $V(\Delta)$.

The image is larger than the complement of the discriminant. Double-root cubics retain their one simple-root preimage. Only triple-root cubics are missing. In (a, b, c) coordinates this omitted curve is

$$3bc = 4, \quad 27ac^2 = 4,$$

or, equivalently,

$$(a, b, c) = \left(\frac{t^2}{3}, 2t, \frac{2}{3t} \right), \quad t \in \mathbb{C}^*. \quad (5.3)$$

This distinction is the whole mechanism: the map has no finite critical point, but some inverse branches disappear through the boundary. It is étale and quasi-finite, yet neither finite, proper, surjective, nor injective.

6 Scope, history, and reproducibility

Keller's 1939 paper is the traditional source of the conjecture [6]; Bass, Connell, and Wright established the standard reduction framework [7], and van den Essen's monograph is a broad reference [8]. Jelonek developed the modern algebraic study of nonproperness sets of polynomial maps [9]. For historical context immediately before the announcement, see Rodríguez Díaz [10]; contemporary literature still described the conjecture as open even in dimension two [11].

The construction here concerns $\mathbb{C}^3$. Adjoining identity coordinates gives $F \times \text{id}_{\mathbb{C}^{n-3}}$ in every dimension $n \geq 3$. Nothing in this argument decides the separate two-dimensional case. Visualizations in the repository show the real slice $\mathbb{R}^3 \subset \mathbb{C}^3$; the full complex space has six real dimensions, so a three-dimensional animation is an illustration rather than the whole geometry.

Exact symbolic certificates, source code, and this paper are maintained at <https://github.com/nasqret/jacobian-counterexample>. The mathematical text and original repository graphics are released under the Creative Commons Attribution 4.0 International license. Third-party posts and papers remain under their respective terms and are linked rather than republished.

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