# Could One Have Found It This Way?

## A six-question reconstruction of a three-dimensional Keller counterexample

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Synthetic dialogue, exposition, repository compilation,
and independent exact audit

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**Abstract**

This paper stages a mathematical dialogue, not a historical transcript. Its question is whether the announced three-variable Keller counterexample can be reconstructed through a short chain of ordinary decisions without guessing its nonlinear coefficients. The user asks six broad questions. The model first tests generic degree three, represents three inverse images by a cubic with one root remembered, searches standard linear affine charts, and parametrizes the only candidate among those charts by three polynomial coordinates. It then hides every nonlinear coefficient while retaining only the scaling rule, degree bounds, linearity in one variable, linear part, and two nonvanishing conditions supplied by the chart. Sixteen monomial positions remain. Two coordinate rescalings reduce the constant-Jacobian condition to sixteen exact equations in eleven unknowns. One Gröbner-basis calculation has a single complex point in normalized coordinates (algebraically, it is double); undoing the normalization recovers one rescaling family containing the displayed map, without using a collision. We finish by rewriting the answer as the operation that forgets a marked root of a cubic. This proves conditional reconstructibility within the declared cubic and linear-chart model. It does not reconstruct a private model run or show that discovery was inevitable.

## Reader's contract

The dialogue below is synthetic. It was written after the formula was public, and it is neither a quotation nor a recovered record of how Fable, GPT, or any contributor reasoned. Its purpose is narrower and testable: to find a short route whose information flow can be audited.

Every decision or convention is recorded under *Chosen*; consequences are marked *Deduced*; and each exchange states what is *Still owed.* During the coefficient calculation, all nonlinear numerical coefficients and every known collision are treated as unavailable. Computer algebra enters the derivation at one step, over exact rational numbers. The result applies to a declared degree-three, marked-cubic, linear-chart model, not to all Keller maps.

This objection-and-reply format is inspired by Lakatos's mathematical dialogues [9]; the explicit dependency checkpoints follow the spirit of Lamport's structured proofs [10]. The format is an expository device, not evidence about the historical discovery process.

## 1 The question before the conversation

A polynomial map

$$F = (F_1, F_2, F_3) : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$$

is called a *Keller map* when its Jacobian determinant $\det JF$ is a nonzero constant. Such a map has an invertible derivative at every point. The Jacobian Conjecture in dimension three states that every Keller map has a polynomial inverse.

On 20 July 2026 Levent Alpöge posted an explicit map, crediting Akhil for the question and Fable for the work, together with a constant determinant and a three-point fiber [1]. The identities can be checked exactly. The question in this paper is not whether that final check is short. It is:

> How few ordinary, text-level decisions are needed to pass from the general three-dimensional problem to a small exact calculation that recovers the announced coefficients?

The answer below has six exchanges: a decision to test degree three, followed by two substantial modeling choices and several declared coordinate conventions. The first four exchanges construct the map geometrically. The last two conduct a separate reconstruction test after its nonlinear numbers have been erased.

## 2 Question 1: what fiber size should we test first?

**USER** *If a Keller map can fail to be one-to-one, what is the first typical number of inverse images worth trying?*

**MODEL**

Start at three.

Because the derivative is invertible, the image of a Keller map is dense. Away from a lower-dimensional exceptional set its fibers have a fixed finite size, called its *generic degree*. Equivalently, this is the degree of

$$[\mathbb{C}(x, y, z) : \mathbb{C}(F_1, F_2, F_3)].$$

If this field extension is Galois—that is, normal and separable—a standard theorem says that a complex Keller map is invertible [6, 7, 8]. Degree one is immediate. In characteristic zero, every extension of degree two is separable and every separable quadratic extension is Galois. Degree three is therefore the first value not excluded by this test.

This is only a reason to try the number three. It does not predict the total degrees of the coordinate polynomials, and it does not yet select a geometric construction.

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**Checkpoint.**

**Chosen.** Test generic degree three.

**Deduced.** A counterexample cannot have generic degree one or two by the Galois case.

**Still owed.** A familiar algebraic operation that is naturally three-to-one.

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## 3 Question 2: what familiar operation has three choices?

**USER** *Degree three is only a number. What is the simplest object that already has three interchangeable choices?*

**MODEL**

Take a cubic with three distinct roots, remember one root, and then forget which root was remembered. A general cubic has exactly three possible markings. For a general cubic the full root symmetry is $S_3$; the subgroup fixing one root has order two and is not normal. Thus the resulting degree-three cover is not Galois.

Write a binary cubic as

$$C = LQ, \qquad L = xT + \beta S, \qquad Q = \gamma T^2 + \delta TS + \epsilon S^2.$$

The linear factor $L$ records the marked root and the quadratic $Q$ contains the other two. Multiplication gives

$$\begin{aligned} c_3 &= x\gamma, & c_2 &= x\delta + \beta\gamma, \\ c_1 &= x\epsilon + \beta\delta, & c_0 &= \beta\epsilon, \end{aligned} \qquad C = c_3T^3 + c_2T^2S + c_1TS^2 + c_0S^3. \tag{3.1}$$

We must remove the place where the marked root is repeated. The linear and quadratic factors have a common root exactly when their resultant

$$R = \mathrm{Res}(L, Q) = x^2\epsilon - x\beta\delta + \beta^2\gamma \tag{3.2}$$

vanishes. On the simple-root region, normalizing $R = 1$ replaces this nonvanishing condition by a fixed level and removes one scaling direction. The second normalization in Question 3 removes the remaining one.

One determinant explains why multiplication has no local folding on this region. Direct differentiation gives

$$\det \frac{\partial(c_3, c_2, c_1, c_0, R)}{\partial(x, \beta, \gamma, \delta, \epsilon)} = -R^2. \tag{3.3}$$

Hence the five displayed outputs are local coordinates wherever $R \neq 0$. Later we will fix $R$ and one linear function of the cubic coefficients. Because these are two of the five output coordinates, the remaining three-dimensional map is still locally invertible.

**Checkpoint.**

**Chosen.** Model three sheets by a cubic with one remembered root. This is the first substantial modeling choice; generic degree three does not force it.

**Deduced.** Forgetting the mark is three-to-one over a cubic with three distinct roots, and (3.3) removes local folding when $R = 1$.

**Still owed.** A way to make both the marked cubics and the unmarked cubics ordinary three-dimensional affine spaces.

**Editorial source note.** Jiang later posted the projective marked-root description and explicitly presented it there as GPT output [2]; Lou developed the factor-and-resultant form and the global chart used below [3]. These citations identify the public mathematical presentations. They are not dialogue lines or claims about the historical route.

## 4 Question 3: can both sides be ordinary three-space?

**USER** *Cubics up to scale form projective three-space. Can the marked and unmarked spaces both be cut down to copies of $\mathbb{C}^3$ in the simplest possible way?*

**MODEL**

Restrict the search to standard linear affine charts. Choose a nonzero linear function $h$ of the four cubic coefficients and impose

$$h(C) = 1.$$

The unmarked side is then a hyperplane, hence $\mathbb{A}^3$. On the marked side impose the two equations

$$R(L, Q) = 1, \qquad h(LQ) = 1. \tag{4.1}$$

This normalized source is the same as the projective pair space on which $R(L,Q)h(LQ) \neq 0$. Indeed, choose representatives $L_0, Q_0$, and put $r = R(L_0, Q_0)$, $a = h(L_0 Q_0)$. Under $(L_0, Q_0) \mapsto (sL_0, tQ_0)$, these values become $s^2tr$ and $sta$. The two normalization equations have the unique solution

$$s = \frac{a}{r}, \qquad t = \frac{r}{a^2}. \tag{4.2}$$

Thus the following projective fiber count applies exactly to the normalized affine source. It also makes the restriction in Question 2 precise: fixing the two output levels $R = h = 1$ is a base change, or equivalently an application of the implicit function theorem to two output coordinates.

Which linear chart can work? Cubics with a triple root form a curve in projective three-space. A hyperplane meets that curve in three points counted with multiplicity, so only three patterns are possible:

$$1 + 1 + 1, \qquad 2 + 1, \qquad 3.$$

They mean three ordinary intersections, one tangency plus another intersection, or third-order contact.

A small additive count distinguishes them. Denote by $e$ the count with $e(\mathbb{A}^m) = 1$, $e(\mathbb{P}^m) = m + 1$, subtraction for deleted pieces, and multiplication in locally trivial families. It is the compactly supported Euler characteristic. Fix the marked linear factor $L$. The possible quadratics form $\mathbb{P}^2$. The condition $R = 0$ deletes one line, and the condition $h(LQ) = 0$ usually deletes a different line. Thus the ordinary fiber has count

$$e(\mathbb{P}^2 \setminus (\text{two distinct lines})) = 3 - 2 - 2 + 1 = 0.$$

To see the special fibers, choose a second linear form $K$. Exact double contact says $h(L^3) = h(L^2K) = 0$, so both deleted lines have the same kernel $L\mathbb{C}[T,S]_1$. They coincide, leaving $\mathbb{A}^2$, whose count is one. At third-order contact one also has $h(LK^2) = 0$; then $h(LQ) = 0$ for every quadratic $Q$, so the fiber is empty. The remaining marked-root line can be divided into finitely many locally trivial pieces, each with ordinary fiber count zero. Additivity and multiplicativity therefore give

| contact pattern | $1+1+1$ | $2+1$ | $3$ |
|---|---|---|---|
| $e$(marked source) | $0$ | $1$ | $0.$ |

(4.3)

Only the middle pattern has the count of $\mathbb{A}^3$.

This argument is exhaustive inside the declared class of linear charts. It is only a necessary test; actual global coordinates are still required. A projective change of the variables $T, S$ carries any $2 + 1$ contact pattern to any other. We therefore choose the convenient representative

$$h(C) = c_2.$$

Indeed, on a triple-root cubic $(rT + sS)^3$, the coefficient $c_2 = 3r^2s$ has one double zero and one simple zero.

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**Checkpoint.**

**Chosen.** Search standard linear affine charts. This is the second substantial modeling choice.

**Deduced.** Within that class, the tangent but not third-order pattern $2 + 1$ is the only possible one. We may use the representative $c_2 = 1$.

**Still owed.** A polynomial coordinate system proving that the surviving marked source really is $\mathbb{A}^3$, not merely that it has the right additive count.

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# 5 Question 4: can the source be parametrized without division?

**USER** *Can we solve the two normalization equations globally, using three polynomial coordinates and without splitting into cases such as $x \neq 0$?*

**MODEL**

Yes. The global factor chart can be obtained by elimination as follows. For $h = c_2$, equations (3.2) and (4.1) become

$$x\delta + \beta\gamma = 1, \qquad x^2\epsilon - x\beta\delta + \beta^2\gamma = 1. \tag{5.1}$$

Multiply the first equation by $\beta$, subtract the second, and define

$$y = 2\beta\delta - x\epsilon.$$

The result is simply

$$\beta = 1 + xy. \tag{5.2}$$

There is one free direction after $x$ and $y$ have been fixed; then $\beta = 1 + xy$ is fixed as well. Differentiate the two equations and the definition $y = 2\beta\delta - x\epsilon$. The allowed variations satisfy

$$\beta\dot{\gamma} + x\dot{\delta} = 0, \qquad \beta^2\dot{\gamma} - x\beta\dot{\delta} + x^2\dot{\epsilon} = 0, \qquad 2\beta\dot{\delta} - x\dot{\epsilon} = 0.$$

Because $x$ and $\beta$ never vanish simultaneously on (5.1), the common solution is the line generated by the following half-scaled polynomial motion:

$$(\dot{\gamma}, \dot{\delta}, \dot{\epsilon}) = \left(\frac{x^2}{2}, -\frac{x\beta}{2}, -\beta^2\right) \tag{5.3}$$

The half-scale is chosen so that the coordinate below changes at unit speed. Along this motion,

$$\frac{d}{dt}(-\epsilon) = \beta^2, \quad \frac{d}{dt}(4y\delta) = -2x\beta y, \quad \frac{d}{dt}(2\gamma y^2) = x^2y^2.$$

Their sum is $(\beta - xy)^2 = 1$. Thus a polynomial coordinate increasing at unit speed along the remaining affine line is

$$z_0 = 4y\delta + 2\gamma y^2 - \epsilon. \tag{5.4}$$

This explains the coordinate rather than guessing it.

Now put $k = -(y\gamma + \delta)$. Using $\beta - xy = 1$, the first equation gives

$$\gamma = 1 + xk, \qquad \delta = -y - \beta k.$$

Direct substitution into the two equations gives $2k = xz_0 - 3y$, with no division by $x$. Substitution now gives

$$\boxed{\begin{aligned} &\beta = 1 + xy, \\ &\gamma = \frac{2 - 3xy + x^2z_0}{2}, \\ &\delta = \frac{y + 3xy^2 - xz_0 - x^2yz_0}{2}, \\ &\epsilon = -z_0(1+xy)^2 + y^2(4 + 3xy). \end{aligned}} \tag{5.5}$$

Conversely, $x$, $y = 2\beta\delta - x\epsilon$, and the polynomial (5.4) recover the three coordinates. Substitution in both directions proves that (5.5) is a polynomial isomorphism $\mathbb{A}^3 \to \{(5.1)\}$. No localization was used.

Multiplying $LQ$ now constructs a polynomial map. To match the public formula, replace $z_0$ by $-z$ and use the target coordinates

$$(F_1, F_2, F_3) = (c_0, 2c_1, 2c_3). \tag{5.6}$$

The sign, order, and factors of two are presentation conventions.

The structural data needed later now follow directly, before any coefficient search begins. Under

$$(x, y, z_0) \longmapsto (\lambda x, \lambda^{-1} y, \lambda^{-2} z_0),$$

formula (5.5) makes $\beta, \gamma$ unchanged and scales $\delta, \epsilon$ by $\lambda^{-1}, \lambda^{-2}$. Hence $(F_1, F_2, F_3)$ scale by $(\lambda^{-2}, \lambda^{-1}, \lambda)$. Inspection of the same formulas also resolves the final freedom in the line coordinate. A unit-speed coordinate could be replaced by $z_0 + f(x, y)$. Requiring the same scaling as $z_0$ forces $f = y^2 P(xy)$; any nonzero such term adds $x^3 f$, of total degree at least five, to $2c_3$. We choose $f = 0$, which minimizes the degree of the third component within this class. This is a presentation convention fixed before the coefficient audit.

Inspection of the same formulas gives

$$\deg(\beta, \gamma, \delta, \epsilon) \le (2, 3, 4, 5), \qquad \deg_z(\beta, \gamma, \delta, \epsilon) \le (0, 1, 1, 1).$$

Multiplication therefore gives total-degree bounds $(7, 6, 4)$ and at most linear dependence on $z$. Its linear part is $(z, y, 2x)$. Finally, the shortest component is already

$$F_3 = 2c_3 = 2x - 3x^2 y - x^3 z, \tag{5.7}$$

so the last two displayed monomials genuinely occur, even when their coefficients are subsequently hidden.

The master determinant and the restriction argument in Question 2 show that the differential is invertible everywhere. Its Jacobian determinant is therefore a polynomial with no complex zero, hence a nonzero constant: the constructed map is Keller.

At this point the geometric construction is complete. The next two questions conduct a separate test: erase the nonlinear numerical coefficients from the expansion and ask whether exact algebra can put them back.

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**Checkpoint.**

**Chosen.** The polynomial free coordinate $z_0$ in (5.4), followed by the sign and target-coordinate conventions in (5.6).

**Deduced.** The normalized marked source is globally $\mathbb{A}^3$, and multiplication already gives an everywhere locally invertible, generically three-to-one polynomial map. The chart also supplies a scaling rule, total-degree bounds $(7, 6, 4)$, linear dependence on $z$, and linear part $(z, y, 2x)$.

**Still owed.** An audit of how much coefficient search remains after the nonlinear numbers are deliberately hidden.

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## 6 Question 5: how large is the coefficient search?

**USER** *Suppose we refuse to expand the known product. What structural information may we retain without retaining a nonlinear numerical coefficient, and how many possible terms remain?*

**MODEL**

Retain only five facts proved from the chart in Question 4:

1. under $(x, y, z) \mapsto (\lambda x, \lambda^{-1} y, \lambda^{-2} z)$, the three outputs scale as $(\lambda^{-2}, \lambda^{-1}, \lambda)$;

2. the total degrees are at most $(7, 6, 4)$;
3. every component is at most linear in $z$;
4. the linear part is $(z, y, 2x)$; and
5. the coefficients of $x^2y$ and $x^3z$ in $F_3$ are nonzero.

The last nonvanishing statement is inherited from the constructed chart. It is not a consequence of the abstract Keller condition, and the boundary on which either coefficient vanishes is not being classified here.

A monomial $x^iy^jz^k$ scales by $\lambda^{i-j-2k}$. Combining this rule with the degree bounds and $k \le 1$ leaves exactly the following slots:

| component | scaling exponent | degree bound | allowed monomials |
|---|---|---|---|
| $F_1$ | $-2$ | 7 | $z, y^2, xyz, xy^3, x^2y^2z, x^2y^4, x^3y^3z$ |
| $F_2$ | $-1$ | 6 | $y, xz, xy^2, x^2yz, x^2y^3, x^3y^2z$ |
| $F_3$ | 1 | 4 | $x, x^2y, x^3z$ |

There are $7+6+3 = 16$ positions. For comparison, dense polynomials at the same degree bounds would have

$$\binom{10}{3} + \binom{9}{3} + \binom{7}{3} = 239$$

positions.

Put $u = xy$ and write the resulting form as

$$\begin{aligned} F_1 &= z(1 + A_1u + A_2u^2 + A_3u^3) + y^2(B_0 + B_1u + B_2u^2), \\ F_2 &= y(1 + C_1u + C_2u^2) + xz(D_0 + D_1u + D_2u^2), \qquad\qquad pq \neq 0. \\ F_3 &= x(2 + pu + qx^2z), \end{aligned} \tag{6.1}$$

Fixing the three linear terms leaves thirteen unknown coefficients.

Two of the thirteen coefficients can be normalized by matched source and target rescalings. For nonzero $a, b, c$, transform a map by

$$F(x, y, z) \longmapsto \Big(c^{-1}F_1(ax, by, cz), b^{-1}F_2(ax, by, cz), a^{-1}F_3(ax, by, cz)\Big). \tag{6.2}$$

This preserves the linear part and the Jacobian determinant, and sends

$$(p, q) \longmapsto (abp, a^2cq).$$

The one-parameter choice $(a, b, c) = (\rho, \rho^{-1}, \rho^{-2})$ acts trivially on every allowed monomial, so only two scale parameters are effective. Since $pq \neq 0$, the explicit choice $(a, b, c) = (1, p^{-1}, q^{-1})$ sets $p = q = 1$. Eleven unknowns remain.

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**Checkpoint.**

**Chosen.** Retain only chart-derived scaling, degree, $z$-linearity, linear part, and the inherited open condition $pq \neq 0$; then rescale $p = q = 1$.

**Deduced.** A 239-position dense search has become a sixteen-position form and, after fixing the linear part and two effective scales, an eleven-unknown problem.

**Still owed.** Determine the exact equations imposed by $\det JF = -2$, solve them, and verify that no invariant nonlinear coefficient or collision has entered.

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# 7 Question 6: can exact algebra restore the numbers?

**USER** *Now let exact algebra do the last job. Does the condition that the Jacobian be constant determine the eleven remaining coefficients?*

**MODEL**

Yes, in the retained case $pq \neq 0$.

Introduce a second abbreviation $v = x^2 z$ and write

$$F = (x^{-2}A(u,v), x^{-1}B(u,v), xC(u,v)),$$

where

$$\begin{aligned} A &= v(1 + A_1 u + A_2 u^2 + A_3 u^3) + u^2(B_0 + B_1 u + B_2 u^2), \\ B &= u(1 + C_1 u + C_2 u^2) + v(D_0 + D_1 u + D_2 u^2), \\ C &= 2 + pu + qv. \end{aligned} \tag{7.1}$$

The negative powers are only bookkeeping; (6.1) remains polynomial. The chain rule gives the two-variable identity

$$\boxed{\det JF = \det \begin{pmatrix} -2A & A_u & A_v \\ -B & B_u & B_v \\ C & C_u & C_v \end{pmatrix}.} \tag{7.2}$$

This calculation is first made on $x \neq 0$. After substituting $u = xy$ and $v = x^2 z$, both sides are polynomials in $x, y, z$; equality on the dense open set therefore extends across $x = 0$.

The linear part already has determinant $-2$. Set $p = q = 1$, expand $\det JF + 2$ as a polynomial in $u, v$, and equate its coefficients to zero. Exactly sixteen coefficients occur, giving sixteen equations in the eleven variables

$$A_1, A_2, A_3, B_0, B_1, B_2, C_1, C_2, D_0, D_1, D_2.$$

The sixteen coefficient positions in $u, v$ are

$$\begin{gathered} v, v^2, u, uv, uv^2, u^2, u^2v, u^2v^2, \\ u^3, u^3v, u^3v^2, u^4, u^4v, u^5, u^5v, u^6. \end{gathered} \tag{7.3}$$

A Gröbner basis is the polynomial analogue of row-echelon form: it makes multivariable elimination exact. Computing one over $\mathbb{Q}$ returns one complex point:

$$\begin{aligned} (A_1, A_2, A_3) &= \left(-1, \frac{1}{3}, -\frac{1}{27}\right), \\ (B_0, B_1, B_2) &= \left(-\frac{4}{9}, \frac{7}{27}, -\frac{1}{27}\right), \\ (C_1, C_2, D_0, D_1, D_2) &= (-4, 1, 9, -6, 1). \end{aligned} \tag{7.4}$$

Appendix A gives the triangular exact certificate and records that this single geometric solution is algebraically double, not two distinct solutions.

Ignoring the algebraic multiplicity recorded in the appendix, undoing the two rescalings gives every complex solution in the retained form with $pq \neq 0$:

$$\begin{aligned} (A_1, A_2, A_3) &= \left(-p, \frac{p^2}{3}, -\frac{p^3}{27}\right), \\ (B_0, B_1, B_2) &= \left(-\frac{4p^2}{9q}, \frac{7p^3}{27q}, -\frac{p^4}{27q}\right), \\ (C_1, C_2) &= (-4p, p^2), \\ (D_0, D_1, D_2) &= \left(\frac{9q}{p}, -6q, pq\right). \end{aligned} \tag{7.5}$$

This is one orbit of coordinate rescalings. Selecting a small integral representative is a later printing convention, not part of the geometric idea. For an integral map define

$$H(F) = \max\{|c| : c \text{ is one of its coefficients}\}.$$

The coefficients $p, q$ themselves force $p, q \in \mathbb{Z} \setminus \{0\}$. Integrality of $p^2/3$ gives $3 \mid p$, while the coefficient $-4p$ gives $H(F) \geq 4|p| \geq 12$. Equality forces $p = \pm 3$. Then the coefficients $-4/q$ and $-3/q$ force $q \mid 4$ and $q \mid 3$, hence $q = \pm 1$. Conversely, all four sign pairs are integral and have height twelve. Thus the minimum occurs precisely at

$$(p, q) = (\pm 3, \pm 1).$$

Choosing $(p, q) = (-3, -1)$ produces the announced signs.

No equation involving two source points was supplied. The input to this calculation was only (6.1), the removable scale normalization $p = q = 1$, and $\det JF = -2$. In particular, no coordinate-invariant nonlinear coefficient value was imposed.

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**Checkpoint.**

**Chosen.** Machine step: use exact rational Gröbner elimination. Presentation step: select the least-height integral representative only for display.

**Deduced.** On the inherited stratum $pq \neq 0$, the Keller identity recovers one orbit of complex coefficient values under coordinate rescaling. No invariant nonlinear coefficient and no collision was imposed.

**Still owed.** Translate the recovered formula back into the short geometric idea and state precisely what this reconstruction does, and does not, establish.

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# 8 The answer after the fog clears

We can now state the construction without the discovery dialogue.

**Theorem 8.1** (The marked-cubic map)**.** *Let a binary cubic be written as*

$$C = c_3T^3 + c_2T^2S + c_1TS^2 + c_0S^3.$$

*Consider the normalized pairs*

$$\mathcal{M} = \{(L, Q) : \operatorname{Res}(L, Q) = 1,\ c_2(LQ) = 1\}. \tag{8.1}$$

*Then $\mathcal{M} \simeq \mathbb{A}^3$, the target $\{C : c_2(C) = 1\} \simeq \mathbb{A}^3$, and multiplication*

$$\mathcal{M} \longrightarrow \{C : c_2(C) = 1\}, \qquad (L, Q) \longmapsto LQ,$$

*has invertible differential everywhere and is generically three-to-one.*

*Proof.* The polynomial chart (5.5) and its displayed polynomial inverse identify $\mathcal{M}$ with $\mathbb{A}^3$. The target is an affine hyperplane in the four cubic coefficients. Identity (3.3) says that

$$(L, Q) \longmapsto (LQ, R(L, Q))$$

has invertible differential wherever $R \neq 0$. Restricting to $R = 1$ and $c_2 = 1$ is the base change described in Question 2, so multiplication on $\mathcal{M}$ has invertible differential. For a cubic with three distinct roots, each root gives one projective linear factor; the normalizations $R = c_2 = 1$ fix the remaining source scalings uniquely by (4.2). Hence the general fiber has exactly three points. $\square$

Set $U = 1 + xy$, make the sign and target choices (5.6), and expand the product. The coordinate shadow of Theorem 8.1 is

$$\boxed{\begin{aligned} &F_1 = U^3 z + y^2 U(4 + 3xy), \\ &F_2 = y + 3xU^2 z + 3xy^2(4 + 3xy), \\ &F_3 = 2x - 3x^2 y - x^3 z. \end{aligned}} \tag{8.2}$$

Theorem 8.1 shows that $\det JF$ has no zero in $\mathbb{C}^3$. A nowhere-zero complex polynomial is a unit and therefore a nonzero constant. Its linear part is $(z, y, 2x)$, whose determinant is $-2$, so

$$\det JF = -2. \tag{8.3}$$

The collision is not mysterious either. The cubic

$$T^2 S - \frac{1}{4}S^3 = S\left(T - \frac{1}{2}S\right)\left(T + \frac{1}{2}S\right)$$

has three distinct linear factors and hence three possible markings. In the coordinates above they give

$$F\left(0, 0, -\frac{1}{4}\right) = F\left(1, -\frac{3}{2}, \frac{13}{2}\right) = F\left(-1, \frac{3}{2}, \frac{13}{2}\right) = \left(-\frac{1}{4}, 0, 0\right). \tag{8.4}$$

Thus $F$ is a Keller map but is not injective, and therefore has no polynomial inverse.

| **Viewpoint** | What the source remembers | The map | What it explains |
|---|---|---|---|
| **Marked roots** | a cubic and one chosen root | forget the choice | three sheets |
| **Factors** | a linear factor $L$ and a quadratic $Q$ | multiply $LQ$ | the invertible derivative |
| **Coordinates** | three numbers $(x, y, z)$ | formula (8.2) | an exact counterexample |
| **Coefficient audit** | sixteen allowed monomials | solve $\det JF = -2$ | conditional recovery of the coefficients |

The first three rows are equivalent descriptions of one morphism, after separate changes of source and target coordinates. The source and target changes may be different; the descriptions are not related by one common conjugating coordinate change. More explicitly, let $G$ denote multiplication in the factor chart before the sign change $z_0 = -z$, using source coordinate $z_0$ and target order $(2c_3, 2c_1, c_0)$. After renaming $z_0$ as $z$, this is Lou's coordinate presentation. If

$$S_0(x, y, z) = (x, y, -z), \qquad R_0(a, b, c) = (c, b, a),$$

then the exact relation is $G = R_0 \circ F \circ S_0$. The fourth row is not another construction; it is the conditional reconstruction audit performed in Questions 5 and 6.

# 9 Conclusion: what six questions establish

The numerical coefficients in (8.2) were not supplied to the reconstruction calculation. First we decided to test the smallest generic degree not covered by the Galois theorem. We then made two main geometric choices:

1. represent its three sheets by the most literal three-choice object, a cubic with one root remembered;
2. search for affine three-spaces among standard linear charts in the projective space of cubics.

After those choices, and the declared coordinate conventions, a three-case test selected the tangent chart, elementary elimination supplied global coordinates, the resulting scaling rule left sixteen monomials, two coordinate rescalings left eleven unknowns, and one exact calculation recovered a single orbit of complex coefficient values on the inherited restriction $pq \neq 0$.

The precise conclusion is conditional: once these stated choices and conventions are fixed, the remaining route to the coefficients is short and auditable. This is not a claim that degree three forces the marked-cubic model, that every affine model is a linear chart, that the boundary $pq = 0$ has been classified, that this was the historical conversation, or that discovery required no creative choice.

The exact identities in this paper have been independently replayed inside the repository, but at the date on the title page the announcement had not passed through the usual human refereeing process. The construction is over $\mathbb{C}^3$; adjoining identity coordinates extends it to every $n \geq 3$, but nothing here decides the separate two-dimensional case.

Once the route is known, the long coordinate formula condenses to one line:

$$\boxed{\text{remember one simple root of a cubic, normalize, and forget it.}}$$

# A Exact coefficient certificate

For completeness, here is a compact certificate for the only computer algebra step. Using graded reverse lexicographic order with variable order

$$(A_3, B_2, A_2, B_1, A_1, B_0, D_2, C_2, D_1, C_1, D_0),$$

the sixteen coefficient equations obtained from (7.2) with $p = q = 1$ generate the same ideal as the following triangular Gröbner basis, up to nonzero rational multiples of individual rows:

$$\begin{gathered}
(D_0 - 9)^2, \\
162A_3 + D_0 - 3, \quad 162B_2 + D_0 - 3, \quad 27A_2 - D_0, \\
108B_1 - 3D_0 - 1, \quad 18A_1 + D_0 + 9, \quad 324B_0 + 7D_0 + 81, \\
9D_2 - 2D_0 + 9, \quad 9C_2 - 2D_0 + 9, \quad D_1 + D_0 - 3, \\
18C_1 + 11D_0 - 27.
\end{gathered} \tag{A.1}$$

The first row gives the sole complex value $D_0 = 9$; the remaining rows give (7.4). They also show the nilpotent direction $\varepsilon = D_0 - 9$ with $\varepsilon^2 = 0$; equivalently, the quotient algebra is $\mathbb{Q}[\varepsilon]/(\varepsilon^2)$. Equality of the two ideals is checked in both directions by exact polynomial reduction, rather than by recognizing numerical approximations.

The independent replay is maintained at

`artifacts/sympy/verify_dialogue_reconstruction.py`.

It regenerates the master determinant, global chart and inverse, sixteen-slot support, reduced determinant, exact ideal equality, ungauged family, explicit map, Jacobian, and collision using integer and rational arithmetic.

# B Provenance of the mathematical presentations

The public sources have distinct roles. Alpöge's post supplies the coordinate formula and publicly credits Akhil and Fable [1]. Andy Jiang (@davikrehalt) posted the marked-root projective description and explicitly presented it there as GPT output [2]. Aaron Lou (@aaron_lou) supplied the factor-and-resultant derivation and global chart [3]. Daniel Litt (@littmath) explained the same marked-root source as an affine-line bundle over an affine plane, organized through a family of pencils, and separately identified the public constructions as equivalent; he reported substantial ChatGPT assistance with that explanation [4, 5].

The present author claims only this synthetic dialogue, its exposition and organization, repository compilation, and independent exact audit. No claim is made here about the private division of discovery credit beyond what the cited public posts themselves state. Mathematical text original to this repository is released under CC BY 4.0; code is released under the MIT License. Third-party sources remain under their own terms.

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